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## V. Conclusions

This correspondence presents the extension of bispectral analysis from one-dimensional random processes (e.g., time series) to two-dimensional random processes. Use of the symmetry properties of the bispectrum reduces the number of computations considerably. Numerical simulations demonstrate the ability of 2-D bispectral estimation to detect quadratic phase coupling between waves traveling in different directions. By windowing the data, bicoherence leakage can be reduced.

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# On the Total Least Squares Linear Prediction Method for Frequency Estimation

YINGBO HUA AND TAPAN K. SARKAR

***Abstract*—The total least squares (TLS) linear prediction (LP) method recently presented by Rahman and Yu and the equivalent improved Pisarenko's (IP) method by Kumaresan are reviewed and generalized by the whitening approach. The resulting whitened-TLS-LP method yields higher estimation accuracy than the TLS-LP.**

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## I. The TLS-LP Method

The TLS-LP method was recently presented by Rahman and Yu [1] for frequency estimation from a short data sequence. They derived this method along the line of linear prediction (LP) [5] and compared it to the modified Prony's (MP) method by Tufts and Kumaresan [6], [8], [10]. In both the above methods, a set of prediction coefficients, denoted by the elements of the vector $\boldsymbol{c} = [c_0, \cdots, c_{L-1}]^T$, is chosen to predict a noisy vector $-\boldsymbol{y}_0$ from a set of noisy vectors, denoted by the columns of $\boldsymbol{Y} = [\boldsymbol{y}_L\, \boldsymbol{y}_{L-1} \cdots \boldsymbol{y}_1]$. (The superscript "$T$" denotes the transpose.) In the MP method, $Y$ is perturbed by $\boldsymbol{E}_1$ such that rank $[\boldsymbol{Y} + \boldsymbol{E}_1]$ is equal to a given number, say $M(\leq L)$, and $\|\boldsymbol{E}_1\|_F$ (Frobenius norm) is minimum. The prediction vector $\boldsymbol{c}$ is then obtained as the minimum-norm least square error solution to (the exact solution may not exist since the noisy vector $\boldsymbol{y}_0$ may not be in the span $[\boldsymbol{Y} + \boldsymbol{E}_1]$):

$$(\boldsymbol{Y} + \boldsymbol{E}_1)\boldsymbol{c} = -\boldsymbol{y}_0. \tag{1.1}$$

That solution is (analytically)

$$\boldsymbol{c} = -(\boldsymbol{Y} + \boldsymbol{E}_1)^+ \boldsymbol{y}_0. \tag{1.2}$$

The superscript " + " denotes the Moore–Penrose pseudoinverse [11].

In the TLS-LP method, both $\boldsymbol{Y}$ and $\boldsymbol{y}_0$ are perturbed by $\boldsymbol{E}_2$ and $\boldsymbol{e}_2$, respectively, such that rank $[\boldsymbol{Y} + \boldsymbol{E}_2, \boldsymbol{y}_0 + \boldsymbol{e}_2]$ is equal to the given number $M(\leq L)$ and $\|\boldsymbol{E}_2, \boldsymbol{e}_2\|_F$ is minimum. Then, the prediction vector $\boldsymbol{c}$ is obtained as the minimum-norm solution to (the exact solutions always exist since $\boldsymbol{y}_0 + \boldsymbol{e}_2$ is in the span $[\boldsymbol{Y} + \boldsymbol{E}_2]$)

$$(\boldsymbol{Y} + \boldsymbol{E}_2)\boldsymbol{c} = -(\boldsymbol{y}_0 + \boldsymbol{e}_2). \tag{1.3}$$

That solution is (analytically)

$$\boldsymbol{c} = -(\boldsymbol{Y} + \boldsymbol{E}_2)^+ (\boldsymbol{y}_0 + \boldsymbol{e}_2). \tag{1.4}$$

The term "TLS" in the TLS-LP is due to the fact that both $\boldsymbol{Y}$ and $\boldsymbol{y}_0$ are perturbed in the minimum way (as opposed to the fact that in the MP method, only $\boldsymbol{Y}$ is perturbed in the minimum way).

The TLS approach has recently been applied to ESPRIT [12], [13] where two matrices, say $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$, are perturbed by $\boldsymbol{E}_3$ and $\boldsymbol{E}_4$ such that rank $[\boldsymbol{Y}_1 + \boldsymbol{E}_3, \boldsymbol{Y}_2 + \boldsymbol{E}_4]$ is equal to a given number and $\|\boldsymbol{E}_3, \boldsymbol{E}_4\|_F$ is minimum. It should be noted that in [13], the term TLS was originally due to a different formulation though the above interpretation of TLS (as adopted in [11]) yields the equivalent result [14].

As shown in [11], the TLS approach can be implemented by using singular value decomposition (SVD). Clearly, the vector $\boldsymbol{c}'^T = [\boldsymbol{c}^T, 1]$ given by (1.4) is the minimum norm vector (with the last element to be one) from the null space of $[\boldsymbol{Y} + \boldsymbol{E}_2, \boldsymbol{y}_0 + \boldsymbol{e}_2]$. This null space is simply [11] the span of all the right singular vectors, except the $M$ principal ones, of the noisy matrix $\boldsymbol{Y}'$ defined by $[\boldsymbol{Y}, \boldsymbol{y}_0]$. Let the right singular vectors of $\boldsymbol{Y}'$ be $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_M, \boldsymbol{v}_{M+1}, \cdots, \boldsymbol{v}_{L+1}$, corresponding to the $L + 1$ singular values in decreasing order. Then, null space of $[\boldsymbol{Y} + \boldsymbol{E}_2, \boldsymbol{y}_0 + \boldsymbol{e}_2]$ = span $[\boldsymbol{v}_{M+1}, \cdots, \boldsymbol{v}_{L+1}]$. As shown in [1], the minimum norm vector $\boldsymbol{c}'$ is

$$\boldsymbol{c}' = \begin{bmatrix} \boldsymbol{c} \\ 1 \end{bmatrix} = \frac{\Sigma_{i=M+1,L+1} v_{i,L+1}^* \boldsymbol{v}_i}{\Sigma_{i=M+1,L+1} |v_{i,L+1}|^2} \tag{1.5}$$

where $v_{i,j}$ denotes the $j$th element of the vector $\boldsymbol{v}_i$, and the superscript "$*$" denotes the complex conjugation. Equation (1.5) requires the $L + 1 - M$ right singular vectors of $Y'$.

## II. The LP Method

The LP method was proposed by Kumaresan in [10] as an improvement of Pisarenko's method. Pisarenko's method [15] is based on the structure of the autocorrelation matrix of a stationary data sequence. For the short data sequence problem, Kumaresan replaced the autocorrelation matrix by the self-product (i.e., the covariance matrix $Y'^H Y'$; the superscript "$H$" denotes the conjugate transpose) of the data matrix $Y'$. Then, he chose such a vector that has minimum norm (with its last element to be one) and is orthogonal to the $M$ principal eigenvectors of $Y'^H Y'$ (i.e., the $M$ principal right singular vectors of $Y'$). This minimum norm vector is then used as linear prediction coefficients to retrieve the frequencies as in the TLS-LP and the MP.

Since the span $[\boldsymbol{v}_1, \cdots, \boldsymbol{v}_M]$ is the orthogonal complement of the span $[\boldsymbol{v}_{M+1}, \cdots, \boldsymbol{v}_{L+1}]$, the minimum norm vector used in the LP is equivalent to that used in the TLS-LP. But the difference is computational. In terms of the $M$ principal singular vectors, $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_M$, the minimum norm vector $\boldsymbol{c}$ is given [10] by

$$\begin{bmatrix} c_0 \\ \vdots \\ \vdots \\ c_{L-1} \end{bmatrix} = -\begin{bmatrix} v_{1,1}^* & \cdots & v_{1,L}^* \\ v_{2,1}^* & \cdots & v_{2,L}^* \\ \vdots & \cdots & \vdots \\ v_{M,1}^* & \cdots & v_{M,L}^* \end{bmatrix}^+ \begin{bmatrix} v_{1,L+1}^* \\ v_{2,L+1}^* \\ \vdots \\ v_{M,L+1}^* \end{bmatrix}. \tag{2.1}$$

Equation (2.1) requires the $M$ right singular vectors of $Y'$ and an $M \times M$ matrix inverse. (In the above equation, the pseudoinverse is the right inverse since the matrix has independent rows.) Comparing (2.1) and (1.5) implies that if $M$ is much smaller than $L - M$ then the LP is preferred, or otherwise the TLS-LP may be more efficient in computation.

In [9], the LP method is generalized for a multiple-measurement problem (i.e., using the multiple snapshots of an array output for wave direction finding). That method is often referred to as the Mini-Norm method.

## III. The Whitened TLS-LP Method

To present the W-TLS-LP, we write the measured data sequence as

$$y_k = \sum_{i=1}^{M} b_i z_i^k + n_k \tag{3.1}$$

where $k = 0, 1, \cdots, N-1$ and $z_i = \exp(\alpha_i + jw_i) = \exp(\alpha_i + j2\pi f_i)$. $z_i$'s are called the signal poles, $\alpha_i$'s the damping factors, $f_i$'s the frequencies, and $b_i$'s the amplitudes. $n_k$ is the noise. According to the linear prediction approach, one tries to fit $y_k$ into a polynomial of degree $L \geq M$ as follows:

$$\sum_{m=0}^{L} c_m y_{k-m} = 0 \tag{3.2}$$

for $k = L, L+1, \cdots, N-1$. $N - L \geq M$ is assumed. But due to the noise $n_k$, (3.2) cannot be true in general. The prediction errors can be put into the matrix form

$$\begin{bmatrix} e_L \\ \vdots \\ e_{N-1} \end{bmatrix} = \begin{bmatrix} y_L & \cdots & y_0 \\ \vdots & \cdots & \vdots \\ y_{N-1} & \cdots & y_{N-L-1} \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_L \end{bmatrix} \tag{3.3}$$

or equivalently

$$\boldsymbol{e} = [\boldsymbol{y}_L, \cdots, \boldsymbol{y}_0]\,\boldsymbol{c}' = \boldsymbol{Y}'\boldsymbol{c}' \tag{3.4}$$

(3.3) and (3.4) can also be written as

$$\boldsymbol{e} = \boldsymbol{C}\boldsymbol{y} \tag{3.5}$$

where

$$\underset{(N-L)\times N}{\boldsymbol{C}} = \begin{bmatrix} c_L & \cdots & c_0 & & \\ & & \cdots & & \\ & & c_L & \cdots & c_0 \end{bmatrix} \tag{3.6}$$

$$\boldsymbol{y} = [y_0, y_1, \cdots, y_{N-1}]^T. \tag{3.7}$$

If the $L$-degree polynomial

$$p(z) = \sum_{m=0,L} c_m z^{-m} \tag{3.8}$$

has the $M$ roots at $\{z_i; i = 1, \cdots, M\}$ then $\boldsymbol{e} = \boldsymbol{C}\boldsymbol{n}$ where $\boldsymbol{n} = [n_0, n_1, \cdots, n_{N-1}]^T$. With the assumption that $n_k$ is the white noise (i.e., $E\{\boldsymbol{n}\} = 0$ and cov $\{\boldsymbol{n}, \boldsymbol{n}\} = 2\sigma^2 I$ where $I$ is the identity matrix). $\boldsymbol{e}$ is not white because cov $\{\boldsymbol{e}, \boldsymbol{e}\} = 2\sigma^2 \boldsymbol{C}\boldsymbol{C}^H$. Now it is natural to whiten $\boldsymbol{e}$ by weighting $\boldsymbol{e}$ with $(\boldsymbol{C}\boldsymbol{C}^H)^{-1/2}$ to have

$$\boldsymbol{e}_w = (\boldsymbol{C}\boldsymbol{C}^H)^{-1/2}\boldsymbol{C}\boldsymbol{y} = (\boldsymbol{C}\boldsymbol{C}^H)^{-1/2}\boldsymbol{Y}'\boldsymbol{c}'. \tag{3.9}$$

It is understood that for any noise sequence of known covariances, the whitening approach can be similarly applied. Following the approach of the TLS-LP or the LP, we look for the minimum norm vector $\boldsymbol{c}'$ in the "near" null space of $R_w = \boldsymbol{Y}'^H(\boldsymbol{C}\boldsymbol{C}^H)^{-1}\boldsymbol{Y}'$. (The "near" null space of the matrix $\boldsymbol{R}_w$ is the null space of the perturbed matrix $\boldsymbol{R}_w + \boldsymbol{\delta R}_w$ of rank $M$ where $\|\delta R_w\|_F$ is minimized. The number $M$ of the complex poles can be estimated by the number of the dominant singular values of $Y'$ or the eigenvalues of $\boldsymbol{Y}'^H\boldsymbol{Y}'$.) The W-TLS-LP method performs iteratively as follows.

1) $\boldsymbol{c}'$ is initialized as $[0, \cdots, 0, 1]^T$. (This is just a natural choice. Other initializations would also work. Intuitively, the closer to the true minimum norm vector is $\boldsymbol{c}'$, the faster the iteration converges. Good initial vector $\boldsymbol{c}'$ could be obtained from *a priori* information about the signal frequencies. In the noiseless case, any initial $\boldsymbol{c}'$ leads to the exact solution at the end of the first iteration.)

2) By using (1.5) or (2.1) in terms of the $L + 1$ eigenvectors of $\boldsymbol{R}_w$, find the minimum norm vector $\boldsymbol{c}_i'$ (with its last element equal to one) in the "near" null subspace (or called the noise subspace) of $R_w$. For the undamped sinusoids, replace $\boldsymbol{R}_w$ by the forward-backward version [6], [7] $\boldsymbol{R}_{w,FB} = \boldsymbol{Y}'^H(\boldsymbol{C}\boldsymbol{C}^H)^{-1}\boldsymbol{Y}' + \boldsymbol{P}\boldsymbol{Y}'^T(\boldsymbol{C}\boldsymbol{C}^H)^{-1}\boldsymbol{Y}'^*\boldsymbol{P}$ to obtain higher accuracy. $\boldsymbol{P}$ is the permutation matrix with ones on its antidiagonal axis and zeros in other places.

3) If $\|\boldsymbol{c}_i' - \boldsymbol{c}_{i-1}'\| < \epsilon$ (a small number), go to (4). Otherwise, go to (2).

4) Find the $M$ signal poles from the $L$ roots of $p(z)$ as in the TLS-LP, and take the angles of the signal poles as the estimates of $2\pi f_i$. (In the noiseless case, $p(z)$ has $M$ roots exactly equal to the signal poles and $L - M$ extraneous roots of magnitude larger than one [6], [8], [10].)

For the first iteration, $\boldsymbol{R}_w = \boldsymbol{Y}'^H\boldsymbol{Y}'$ (or $\boldsymbol{R}_{w,FB} = \boldsymbol{Y}'^H\boldsymbol{Y}' + \boldsymbol{P}\boldsymbol{Y}'^T\boldsymbol{Y}'^*\boldsymbol{P}$ for the undamped signal) and hence the solution vector $\boldsymbol{c}'$ from step 2 is the TLS-LP or the LP solution vector. Also note that if $L = M$, this algorithm is the same as the iterative quadratic maximum likelihood (IQML) algorithm [2]–[4] without additional constraints on $\boldsymbol{c}'$. The constraints introduced in [2]–[4] do not apply to the damped signals. Hence, the W-TLS-LP is a generalized algorithm of both the TLS-LP and the IQML.

In Figs. 1–4, we compare the 200-run white noise simulation results of the TLS-LP and the W-TLS-LP. $R_{w,FB}$ was used. The signal parameters were chosen to be the same as those in [2], i.e., $N = 25$, $n = 2$, $b_1 = 1$, $b_2 = \exp(j\pi/4)$, $f_1 = 0.52$, and $f_2 = 0.5$. Note that in these figures, the performances of the W-TLS-LP using only two iterations are shown. We should point out that in this simulation the W-TLS-LP algorithm with the forward-and-backward weighted covariance matrix $\boldsymbol{R}_{w,FB}$ converged for almost all runs within 20 iterations for all choices of $L$, and the converged results were found to be better than the nonconverged most of the time. Since the computation time is linearly proportional to the number of iterations, two iterations cases as shown in Figs. 1–4 should be of more significance for comparison to the TLS-LP. One can see that the W-TLS-LP has higher frequency estimation accuracy both in bias and variance than the TLS-LP over all prediction order $L$, and that the weighting (or whitening) has more effect on lower order prediction than higher order prediction.

Compared to the IQML as in [2] where conjugate symmetry constraint was applied on $\boldsymbol{c}'$ at each iteration, the W-TLS-LP with any choice of $L$ does not perform that well. (The conjugate symmetry

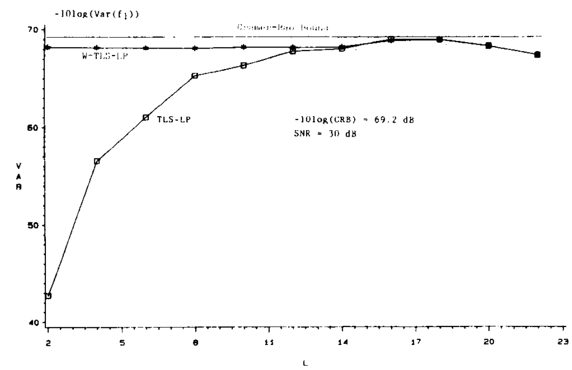


Fig. 1. 10 $\log_{10}$ ($1/\text{var}$ ($f_1$)) versus $L$, where SNR = 10 $\log_{10}$ ($\| b_1 \|^2 / 2\sigma^2$) = 30 dB. The stars are for the W-TLS-LP with two iterations, and the squares for the TLS-LP.

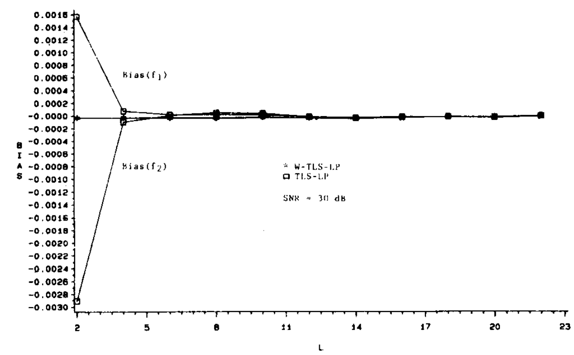


Fig. 2. Bias ($f_1$) and bias ($f_2$) versus $L$, where SNR = 30 dB. The stars are for the W-TLS-LP with two iterations, and the squares for the TLS-LP. The bias ($f_1$) and bias ($f_2$) for the W-TLS-LP are overlapped in this plot.

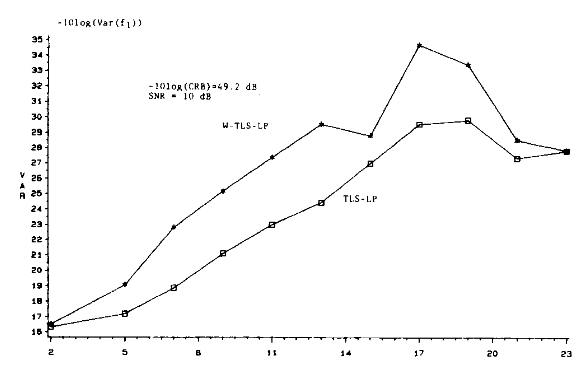


Fig. 3. 10 $\log_{10}$ ($1/\text{var}$ ($f_1$)) versus $L$, where SNR = 10 dB which is in the threshold region for this example. The CRB would be drawn way above this plot.

constraint cannot be applied to $c'$ with $L > M$.) For the damped sinusoids, the conjugate symmetry constraint [2] does not apply, and hence the W-TLS-LP method with $L = M$ is the IQML method. Based on our simulation results, the W-TLS-LP method (with $L >$ M) generally has close (or slightly poorer) estimation accuracy as (or than) the IQML method (with $L = M$) in the damped case. Table I shows the 200-run simulation results for the following

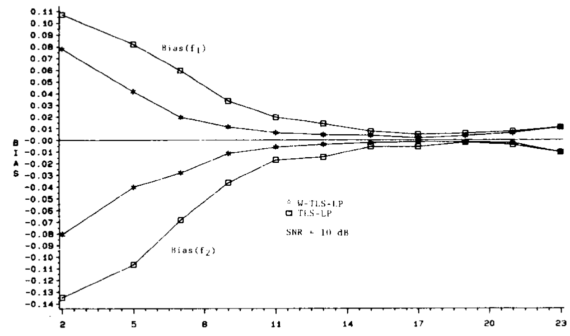


Fig. 4. Bias ($f_1$) and bias ($f_2$) versus $L$, where SNR = 10 dB.

TABLE I
200-Run Sample Biases and Deviation for the W-TLS-P Applied to a Damped Signal (Where CRB Corresponds to Deviation)

| L | Bias(f) | Bias(α) | Dev(f) | Dev(α) | CRB(f) | CRB(α) |
|---|---|---|---|---|---|---|
| 2(IQML) | 0.191D-3 | 0.168D-4 | 0.114D-2 | 0.703D-2 | 0.120D-2 | 0.670D-2 |
| 4 | 0.143D-3 | 0.536D-4 | 0.122D-2 | 0.716D-2 | \| | \| |
| 6 | 0.110D-3 | 0.160D-3 | 0.131D-2 | 0.727D-2 | \| | \| |
| 8 | 0.130D-3 | -0.120D-3 | 0.127D-2 | 0.755D-2 | \| | \| |
| 10 | 0.133D-3 | 0.116D-3 | 0.123D-2 | 0.782D-2 | \| | \| |
| 12 | 0.587D-4 | 0.288D-3 | 0.132D-2 | 0.780D-2 | \| | \| |
| 14 | 0.550D-4 | 0.383D-4 | 0.149D-2 | 0.836D-2 | \| | \| |

damped signal:

$$y_k = \exp(\alpha k) \cos(2\pi f k) + \sigma n_k \tag{3.10}$$

where $k = 0, 1, \cdots, 25$, $\alpha = -0.05$, $f = 0.2$, and $\sigma = 0.1$. $n_k$ is the pseudowhite Gaussian noise with unit deviation. $\epsilon = 10^{-6}$ was chosen and each run converged within 10 iterations. Note that this signal has $M = 2$ signal poles. So, for $L = 2$, the results are actually for the IQML. It can be observed that the sample deviations for $L \leq 12$ are quite robust to the variation of $L$. The biases of the estimated $f$ are smaller for $L > 2$ than for $L = 2$ in this example. But overall, the IQML is a better method (in noise sensitivity and computational) than the W-TLS-LP although the W-TLS-LP provides much higher accuracy than the TLS-LP advocated in [1].

## IV. Final Remarks

This correspondence has shown that a) the TLS-LP recently presented in [1] is (analytically) equivalent to the LP earlier proposed in [10] and the only difference between the above two methods is computational (either one of them may be preferred to the other, depending on whether $M \ll L - M$ or not); b) the TLS-LP provides much better estimation accuracy after one simple step of whitening; and c) the $W$-TLS-LP is a generalized algorithm of the IQML but it fails to compete against the IQML in overall performance.

Our simulation was carried out in double precision FORTRAN-77 on VAX-8810. The IMSL routines were used to perform eigendecompositions, compute the polynomial roots, and to generate the pseudo-Gaussian random numbers.

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# Elimination of Exponential Interference from Finite-Length Discrete Signals

OMRY PAISS

***Abstract*—Discrete exponential (or modal) signals satisfy homogeneous difference equations. We find matrices that when multiplied with a vector of samples of any given signal, completely eliminate from it an exponential interference that satisfies a given homogeneous difference equation (HDE). All other components of the given signal (which are orthogonal to the solutions of that HDE) are unaffected.**

## I. Introduction

Discrete exponential (or modal) signals are discrete sequences that satisfy homogeneous difference equations (HDE) and form an important class of signals often encountered in signal processing. Real exponentials, polynomials, sine waves, and their combinations are all exponential signals.

In the elimination problem we are given an observed signal $x(n)$:

$$x(n) = u(n) + e(n) \qquad (1)$$

where $u(n)$ is the desired signal and $e(n)$ is the interference. We are interested in the case where the signals are of finite extent, $e(n)$ is an exponential signal, and $u(n)$ does not have any component that satisfies the HDE corresponding to $e(n)$.

The extraction problem is practically the same where the exponential signal $e(n)$ is the desired signal. Those problems are equivalent because after eliminating some interference, it can be retrieved by subtracting the output vector from the input vector.

The traditional filtering methods, such as notch filtering, have a considerable transient effect and therefore the elimination is good for only part of the resulting filtered sequence. Here we are looking for exact solutions having no transients.

The concept of eliminating signals appears in [1], where a function elimination filter (FEF) was used to eliminate a sine-wave interference from a given signal, but at the same time alter all of the other components of that signal. This was achieved by passing the observed signal through the appropriate second-order HDE whose solution is the sine wave.

In [2], an interference that satisfies a homogeneous differential equation was separated from a signal, provided the signal (without interference) satisfied another homogeneous differential equation, when the coefficients of one of the equations were known. It was shown how the same method could be used for discrete signals with difference equations replacing differential equations.

Given the observed sequence $x(n)$ $n = 0, 1, \cdots, N-1$ and the coefficients $h_i$ $i = 1, \cdots, M$, our aim is to eliminate from $x(n)$ the interference $e(n)$ that is known to be a solution of the following HDE:

$$f(n) + h_1 f(n-1) + \cdots + h_M f(n-M) = 0 \qquad (2)$$

without affecting the other components of the sequence $u(n)$ and without any *a priori* information about these other components.

We know that such $e(n)$ is a linear combination of $M$ independent solution sequences $e_j(n)$, $j = 1, 2, \cdots, M$. These solutions are of the form of exponentials, polynomials, sine waves, or intermultiplications of them.

The basic idea is to consider all finite sequences as vectors and find the projection matrix, $\boldsymbol{A}$, onto the space orthogonal to the solutions of the HDE. Explicitly,

$$\boldsymbol{Ae} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{Au} = \boldsymbol{u} \quad \text{so } \boldsymbol{Ax} = \boldsymbol{u} \qquad (3)$$

where $\boldsymbol{e} = [e(0) \cdots e(N-1)]^T$, $\boldsymbol{u} = [u(0) \cdots u(N-1)]^T$, and $\boldsymbol{x} = [x(0) \cdots x(N-1)]^T$.

An alternative method for constructing $\boldsymbol{A}$ will be given if we know any set of basis sequences $e_j(n)$ for the interference signal. In this case there are numerical advantages in the construction process.

## II. Constructing the Elimination Matrix from the Difference Equation

Referring to (2), we see that there are actually $L = N - M$ equations $(n = M, M+1, \cdots, N-1)$ that can be written in matrix form

$$\boldsymbol{Hf} = \boldsymbol{0} \qquad (4)$$

where

$$\boldsymbol{f} = [f(0) f(1) \cdots f(M) \cdots f(N-1)]^T$$

$$\boldsymbol{H} = \begin{bmatrix} h_M & h_{M-1} & \cdots & h_1 & 1 & & & & 0 \\ & \ddots & & & & \ddots & & & \\ 0 & & & & h_M & h_{M-1} & \cdots & h_1 & 1 \end{bmatrix}. \qquad (5)$$

$\boldsymbol{H}$ is banded of dimensions $L \times N$ and of rank $L$. We denote $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_M$ the $M$ solution sequences for (2) put in vector form.

*Proposition 1:* The null space of $\boldsymbol{H}$ is spanned by $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_M$.

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