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ON L^p RESOLVENT ESTIMATES FOR ELLIPTIC OPERATORS ON COMPACT MANIFOLDS

KATSIARYNA KRUPCHYK AND GUNTHER UHLMANN

Abstract. We prove uniform L^p estimates for resolvents of higher order elliptic self-adjoint differential operators on compact manifolds without boundary, generalizing a corresponding result of [3] in the case of Laplace– Beltrami operators on Riemannian manifolds. In doing so, we follow the methods, developed in [1] very closely. We also show that spectral regions in our L^p resolvent estimates are optimal.

1. Introduction and statement of results

The purpose of this paper is to extend the result of [3], see also [1], for the Laplace-Beltrami operator Δ_g on a compact Riemannian manifold (M, g) without boundary of dimension $n \geq 3$, to the case of higher order elliptic self-adjoint differential operators, and specifically to show how the methods of [1] apply in this context.

In [3] it was established that given $\delta > 0$ small, there exists a constant $C = C(\delta) > 0$ such that for all $u \in C^\infty(M)$ and all $\zeta \in \mathcal{R}_\delta$, the following L^p resolvent bound holds,

$$\|u\|_{L^{\frac{2n}{n-2}}(M)} \leq C\|(-\Delta_g - \zeta)u\|_{L^{\frac{2n}{n+2}}(M)}, \tag{1.1}$$

where

$$\mathcal{R}_\delta = \{\zeta \in \mathbb{C} : (\operatorname{Im}\zeta)^2 \geq 4\delta^2(\operatorname{Re}\zeta + \delta^2)\}.$$

Notice that $\mathcal{R}_\delta$ is the exterior of a parabolic region, containing the spectrum of $-\Delta_g$, see Figure 1. We observe that the bound (1.1) cannot hold if $\mathcal{R}_\delta$ intersects the spectrum of $-\Delta_g$, as the latter is discrete. The interesting question, posed in [3] and subsequently studied in [1], is how close $\mathcal{R}_\delta$ can come to the spectrum of $-\Delta_g$ near infinity, while still having the uniform estimate (1.1).

Thanks to the work [1], we know that the region $\mathcal{R}_\delta$ is in general the maximal possible for the uniform estimate (1.1) to hold. Indeed, in [1] it is shown that the region cannot be improved when M is the standard sphere, or more generally, a Zoll manifold, due to a cluster structure of the spectrum of $-\Delta_g$ on such manifolds, [17]. As explained in [1], any sharpening in the spectral region is related to improvements in estimates for the remainder term in the sharp Weyl law for $-\Delta_g$, which measures how uniformly its spectrum is distributed. Consequently,

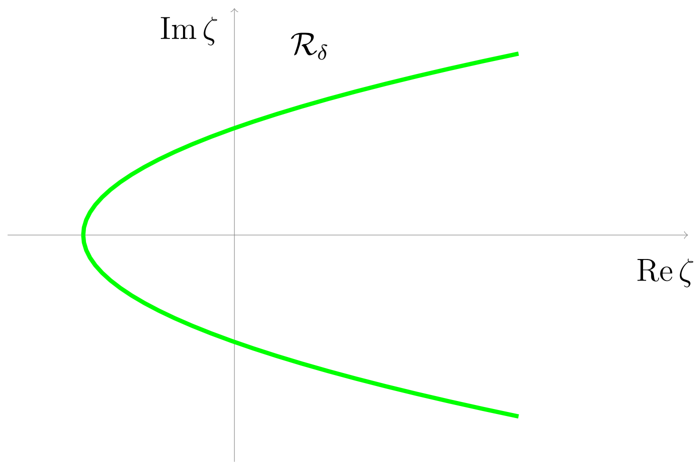

FIGURE 1. Spectral region $\mathcal{R}_\delta$ in the uniform resolvent bound (1.1).

improvements in the spectral region $\mathcal{R}_\delta$ are available for manifolds of nonpositive curvature and in the case of the torus with a flat metric, see [1], and also [13].

The corresponding uniform L^p resolvent estimates for the standard Laplacian on $\mathbb{R}^n$, $n \ge 3$, were obtained in [9]. Here in contrast to the case of a compact manifold, the estimates are valid for all values of the complex spectral parameter ζ. In [5] the results of [9] were generalized to the case of non-trapping asymptotically conic manifolds.

To formulate our results let us begin by fixing some notation. Let M be a compact connected C^∞ manifold without boundary of dimension $n \ge 2$, equipped with a strictly positive C^∞ volume density $d\mu$. Let P be a differential operator on M of order $m \ge 1$ with C^∞ coefficients. We assume that P is elliptic and formally self-adjoint with respect to $d\mu$,

$$\int_M Pu\overline{v}d\mu = \int_M u\overline{Pv}d\mu, \quad u, v \in C^\infty(M).$$

Let $p(x,\xi) \in C^\infty(T^*M)$ be the principal symbol of P, which is a real-valued homogeneous polynomial in ξ of degree m. Since $p(x,\xi) \neq 0$ for $\xi \neq 0$ and $T^*M \setminus \{0\}$ is connected, without loss of generality we shall assume, as we may, that $p(x,\xi) > 0$ for $\xi \neq 0$. The order m of the operator P is therefore even.

If we equip the operator P with the domain $C^\infty(M)$, P becomes an unbounded symmetric essentially self-adjoint operator on $L^2(M)$, i.e. P has a unique self-adjoint extension, which we shall denote again by P. The domain of the self-adjoint extension is $\mathcal{D}(P) = H^m(M)$, the standard Sobolev space on M.

An application of Gårding's inequality implies that there exists a constant $C > 0$ such that $P \ge -CI$ in the sense of self-adjoint operators. Thus, after replacing P by $P + CI$, we assume, as we may, that $P \ge 0$.

The spectrum of P is discrete, consisting only of real eigenvalues, where each eigenvalue is isolated and of finite multiplicity. Let $0 \le \lambda_1 \le \lambda_2 \le \dots$ be the eigenvalues of P repeated according to their multiplicity, and let $e_1, e_2, \dots \in L^2(M)$ be the corresponding orthonormal basis of eigenfunctions.

Seeking to generalize (1.1), our goal is to find a region $\mathcal{R} \subset \mathbb{C}$, for which there holds a uniform L^p bound of the form,

$$\|u\|_{L^q(M)} \le C_{\mathcal{R}} \|(P-\zeta)u\|_{L^p(M)}, \quad u \in C^\infty(M), \quad \zeta \in \mathcal{R}, \tag{1.2}$$

for suitable p and q. Motivated by the classical Sobolev inequalities, we shall be interested in the estimate (1.2) for pairs (p, q) belonging to the Sobolev line

$$\frac{1}{p} - \frac{1}{q} = \frac{m}{n}, \tag{1.3}$$

assuming that $p < n/m$. Following [1, 3], we shall also require the pairs (p, q) to be on the duality line,

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{1.4}$$

The restrictions (1.3) and (1.4) imply that

$$p = \frac{2n}{n+m}, \quad q = \frac{2n}{n-m}, \quad n > m.$$

It is clear that the estimate (1.2) can only hold away from the spectrum of P. Similarly to the case of $-\Delta_g$, when establishing the estimate (1.2), we shall in fact be concerned with the case of ζ away from all of $[0,\infty)$. Given $\zeta \in \mathbb{C} \setminus [0,\infty)$, it will then be convenient to write $\zeta = z^m$ with $z \in \Xi$, where

$$\Xi = \{z \in \mathbb{C} : \arg(z) \in (0, 2\pi/m)\}.$$

This is due to that fact that the map

$$f = f_m : \Xi \to \mathbb{C} \setminus [0,\infty), \quad z \mapsto z^m,$$

is a conformal isomorphism. This map extends continuously to $f : \overline{\Xi} \to \mathbb{C}$ with $f(\partial \Xi) = [0, \infty)$.

Notice that the region $\mathcal{R}_\delta$ in the uniform bound (1.1) satisfies

$$\mathcal{R}_\delta = f_2(\Xi_\delta), \quad \Xi_\delta = \{z \in \mathbb{C} : \operatorname{Im} z \ge \delta\},$$

By analogy with this, it is natural to try to establish the estimate (1.2) for $\zeta = z^m$, where

$$z \in \Xi_\delta = \{z \in \Xi : \operatorname{dist}(z, \partial \Xi) \ge \delta\},$$

with $\delta > 0$ small but fixed. We have

$$\Xi_\delta = \{z \in \mathbb{C} : \arg(z) \in (0, 2\pi/m), \operatorname{Im} z \ge \delta, -\operatorname{Im}(z e^{-2\pi i/m}) \ge \delta\}.$$

Associated with the principal symbol $p(x,\xi)$ of the operator P is the cosphere

$$\Sigma_x = \{\xi \in T_x^* M : p(x,\xi) = 1\}, \quad x \in M.$$

We may notice that for each $x \in M$, the cosphere Σ_x is a C^∞ compact connected hypersurface in $\mathbb{R}^n$, see the discussion before Lemma 2.9 below. The cosphere Σ_x is called strictly convex if the second fundamental form is definite at each point of Σ_x. This is equivalent to the fact that the Gaussian curvature of Σ_x is non-vanishing.

The following theorem is the main result of this paper, which is a generalization of the uniform estimate (1.1), obtained in [3], to the case of higher order elliptic self-adjoint differential operators.

Theorem 1.1. *Assume that $n > m \geq 2$ and that for each $x \in M$, the cosphere Σ_x is strictly convex. Then given $\delta > 0$ small, there is a constant $C = C(\delta) > 0$ such that for all $u \in C^\infty(M)$ and all $z \in \Xi_\delta$, the following estimate holds*

$$\|u\|_{L^{\frac{2n}{n-m}}(M)} \leq C\|(P - z^m)u\|_{L^{\frac{2n}{n+m}}(M)}. \tag{1.5}$$

In the case of an elliptic operator P of order $m \geq 4$, letting $\mathcal{R}_\delta = f(\Xi_\delta)$, a straightforward computation show that for $R > 0$ sufficiently large, we have

$$\mathcal{R}_\delta \cap \{\zeta \in \mathbb{C} : |\zeta| \geq R\} = (\mathcal{R}_\delta^+ \cup \mathcal{R}_\delta^-) \cap \{\zeta \in \mathbb{C} : |\zeta| \geq R\},$$

where

$$\begin{aligned}\mathcal{R}_\delta^+ :=&\{\zeta \in \mathbb{C} : \operatorname{Im} \zeta \geq (\operatorname{Re} \zeta)^{\frac{m-1}{m}} m\delta + \mathcal{O}((\operatorname{Re} \zeta)^{\frac{m-3}{m}}), \operatorname{Re} \zeta \geq 0\}\\ &\cup \{\zeta \in \mathbb{C} : \operatorname{Im} \zeta \leq -(\operatorname{Re} \zeta)^{\frac{m-1}{m}} m\delta - \mathcal{O}((\operatorname{Re} \zeta)^{\frac{m-3}{m}}), \operatorname{Re} \zeta \geq 0\},\end{aligned}$$

and

$$\mathcal{R}_\delta^- := \{\zeta \in \mathbb{C} : \operatorname{Re} \zeta \leq 0\}.$$

Thus, for $|\zeta|$ sufficiently large, similarly to the case of $-\Delta_g$, the region $\mathcal{R}_\delta$ is the exterior of a parabolic neighborhood of the spectrum of the operator P, see Figure 2.

As an example of an operator P to which Theorem 1.1 applies, one can consider $P = (-\Delta_g)^k$, $2k < n$, where $-\Delta_g$ is the Laplace–Beltrami operator on a compact Riemannian manifold (M, g).

Our proof of Theorem 1.1 relies on the approach, developed in [1]. The main ingredients here are the spectral cluster estimates, obtained in [15] in the case of the Laplace–Beltrami operator on a compact Riemannian manifold, and in [11] in the case of higher order elliptic operators, the method of stationary phase, as well as the Hörmander–Lax parametrix for the operator $e^{it\sqrt[m]{P}}$ for small times.

Let us remark that the strict convexity of the cospheres Σ_x in Theorem 1.1 guarantees that the Fourier transform of the surface measure on Σ_x has essentially the same decay at infinity, as that of the surface measure on the sphere, thanks to the method of stationary phase, see [14, Theorem 1.2.1, p. 50]. This assumption also plays a crucial role in the derivation of the spectral cluster estimates for higher order elliptic operators in [11].

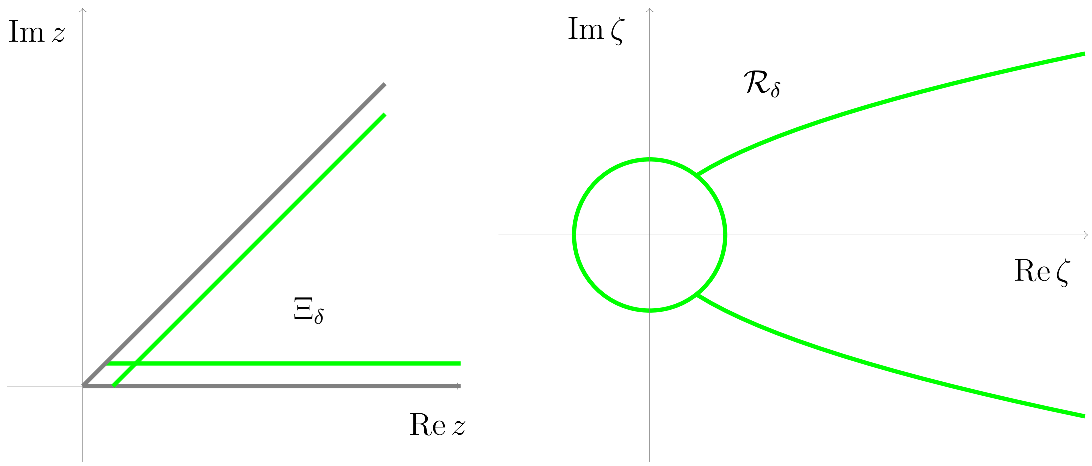

FIGURE 2. The spectral regions Ξ_δ and $\mathcal{R}_\delta = f(\Xi_\delta)$ in the uniform estimate (1.5).

We may also notice that the a priori estimate (1.5) implies that the L^2 resolvent of P, $(P-\zeta)^{-1}$, $\zeta \in \mathbb{C} \setminus [0,\infty)$, is a bounded operator: $L^{\frac{2n}{n+m}}(M) \to L^{\frac{2n}{n-m}}(M)$, see Proposition 2.10 below.

Our next result shows that the region Ξ_δ in (1.5) is in general optimal for higher order elliptic operators, since it cannot be improved for an operator whose principal symbol has a periodic Hamilton flow. This is due to the fact that the spectrum of such an operator is distributed in a non-uniform fashion, displaying a cluster structure, see [2] and [17].

Theorem 1.2. *Assume that $n > m \geq 2$ and that for each $x \in M$, the cosphere Σ_x is strictly convex. Assume furthermore that the subprincipal symbol of the operator P vanishes, and that the Hamilton flow of the principal symbol p is periodic, with a common minimal period on $p^{-1}(1)$. Then there exist*

(i) a sequence $z_k \in \Xi$ such that $\operatorname{Re} z_k \to \infty$, $0 < \operatorname{Im} z_k \to 0$ as $k \to \infty$, and

$$\|(P - z_k^m)^{-1}\|_{L^{\frac{2n}{n+m}}(M) \to L^{\frac{2n}{n-m}}(M)} \to \infty, \quad k \to \infty,$$

and

(ii) a sequence $z_k \in \Xi$ such that $\operatorname{Re}(z_k e^{-2\pi i/m}) \to \infty$, $0 < -\operatorname{Im}(z_k e^{-2\pi i/m}) \to 0$ as $k \to \infty$, and

$$\|(P - z_k^m)^{-1}\|_{L^{\frac{2n}{n+m}}(M) \to L^{\frac{2n}{n-m}}(M)} \to \infty, \quad k \to \infty.$$

As an example of the operator P in Theorem 1.2 we can take $P = (-\Delta_g)^k$, $2k < n$, on a Zoll manifold M, similarly to the case when $k = 1$ in [1]. To prove Theorem 1.2 we shall also use the methods of [1].

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1 while Section 3 contains the proof of Theorem 1.2.

2. Proof of Theorem 1.1

2.1. **Formula for the resolvent $(P-z^m)^{-1}$ based on a half wave group for $P^{1/m}$.** We shall denote by $\Psi^{\mu}_{\mathrm{cl}}(M)$ the space of classical pseudodifferential operators of order μ on M. Let $Q=P^{1/m}$ be defined by the spectral theorem. According to Seeley's theorem, see [14, Theorem 3.3.1], we have $Q\in\Psi^1_{\mathrm{cl}}(M)$ with the principal symbol $q=p^{1/m}$. Furthermore, $\mathcal{D}(Q)=H^1(M)$, and the eigenvalues of Q are $\mu_j=\lambda_j^{1/m}$, $j=1,2,\dots$.

Letting $z\in\Xi$ and following [1], let us derive a natural formula for the L^2 resolvent $(P-z^m)^{-1}$. To that end, we write $(P-z^m)^{-1}=m_z(Q)$, where the multiplier $m_z(Q)$ is given by $m_z(\tau)=(\tau^m-z^m)^{-1}$. Using the inverse Fourier transform, we get

$$m_z(\tau)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\widehat{m_z}(t)e^{it\tau}dt,\quad \widehat{m_z}(t)=\int_{-\infty}^{+\infty}\frac{1}{\tau^m-z^m}e^{-it\tau}d\tau.$$

We shall need the following result.

Lemma 2.1. *Let $z\in\Xi$. Then for any $t\in\mathbb{R}$, we have*

$$\int_{-\infty}^{+\infty}\frac{1}{\tau^m-z^m}e^{-it\tau}d\tau=\frac{2\pi i}{mz^{m-1}}\sum_{k=0}^{m/2-1}e^{2\pi ki/m+i|t|\tau_k},\tag{2.1}$$

where $\tau_k=ze^{2\pi ki/m}$, $k=0,1,\dots,m/2-1$. Here $\operatorname{Im}\tau_k>0$, $k=0,1,\dots,m/2-1$.

Proof. To show (2.1) we shall use the residue calculus. To that end writing $z=|z|e^{i\varphi}$, $0<\varphi<2\pi/m$, we obtain that the poles of the rational function $\mathbb{C}\ni\tau\mapsto(\tau^m-z^m)^{-1}$ are given by

$$\tau_k=|z|e^{i(m\varphi+2\pi k)/m}=ze^{2\pi ki/m},\quad k=0,\dots,m-1.$$

Notice that the poles are simple, none of them are on the real line, the poles τ_k, $k=0,\dots,m/2-1$, are in the upper half plane, and the poles τ_k, $k=m/2,\dots,m-1$, are in the lower half plane.

We have $|e^{-it\tau}|=e^{t\operatorname{Im}\tau}$. Let first $t\le 0$. Deforming the contour of integration in the upper half plane, we get

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{1}{\tau^m-z^m}e^{-it\tau}d\tau&=2\pi i\sum_{k=0}^{m/2-1}\operatorname{Res}\Big(\frac{e^{-it\tau}}{\tau^m-z^m};\tau_k\Big)=2\pi i\sum_{k=0}^{m/2-1}\frac{e^{-it\tau_k}}{m\tau_k^{m-1}}\\&=\frac{2\pi i}{mz^{m-1}}\sum_{k=0}^{m/2-1}e^{2\pi ki/m-it\tau_k},\quad t\le 0.\end{aligned}$$

Let now $t > 0$. Then by deforming the contour of integration in the lower half plane, we conclude that

$$\int_{-\infty}^{+\infty} \frac{1}{\tau^m - z^m} e^{-it\tau} d\tau = -2\pi i \sum_{k=m/2}^{m-1} \operatorname{Res}\left(\frac{e^{-it\tau}}{\tau^m - z^m}; \tau_k\right) = -2\pi i \sum_{k=m/2}^{m-1} \frac{e^{-it\tau_k}}{m\tau_k^{m-1}}$$

$$= -\frac{2\pi i}{mz^{m-1}} \sum_{k=m/2}^{m-1} e^{2\pi k i/m - it\tau_k} = -\frac{2\pi i}{mz^{m-1}} \sum_{k=0}^{m/2-1} e^{\pi i} e^{2\pi k i/m - it\tau_{m/2+k}}$$

$$= \frac{2\pi i}{mz^{m-1}} \sum_{k=0}^{m/2-1} e^{2\pi k i/m + it\tau_k}, \quad t > 0.$$

Thus, (2.1) follows. The proof of Lemma 2.1 is complete. □

Let $z \in \Xi$. Then by (2.1), we obtain that

$$m_z(\tau) = \frac{i}{mz^{m-1}} \sum_{k=0}^{m/2-1} e^{2\pi k i/m} \int_{-\infty}^{+\infty} e^{i|t|\tau_k + it\tau} dt.$$

Therefore, we have the following formula for the resolvent of P,

$$(P - z^m)^{-1} = m_z(Q) = \frac{i}{mz^{m-1}} \sum_{k=0}^{m/2-1} e^{2\pi k i/m} \int_{-\infty}^{+\infty} e^{i|t|\tau_k} e^{itQ} dt. \tag{2.2}$$

Here $\tau_k = ze^{2\pi k i/m}$ and $\operatorname{Im} \tau_k > 0$, $k = 0, 1, \dots, m/2 - 1$.

2.2. **Consequences of the spectral projection estimates.** Assume that, for each $x \in M$, the cosphere $\Sigma_x = \{\xi \in T_x^*M : q(x, \xi) = 1\}$ is strictly convex. Consider the k'th spectral cluster of the operator Q,

$$\{\mu_j \in \operatorname{spec}(Q) : \mu_j \in [k - 1, k)\},$$

and denote by χ_k the spectral projection operator on the space, generated by the eigenfunctions, corresponding to the kth spectral cluster,

$$\chi_k f = \sum_{\mu_j \in [k-1,k)} E_j f, \quad f \in C^\infty(M).$$

Here $E_j : L^2(M) \to L^2(M)$ is the orthogonal projection onto the space, spanned by e_j, i.e.

$$E_j f(x) = \left(\int_M f(y)\overline{e_j(y)} d\mu(y)\right) e_j(x).$$

It was shown in [11], see also [14, Theorem 5.1.1], that for $p \geq \frac{2(n+1)}{n-1}$, we have

$$\|\chi_k\|_{L^2(M) \to L^p(M)} \leq Ck^{\sigma(p)}, \quad \sigma(p) = n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}, \tag{2.3}$$

where $C > 0$ is a constant, and the dual estimate,

$$\|\chi_k\|_{L^{p'}(M)\to L^2(M)} \le Ck^{\sigma(p)}, \quad p' = \frac{p}{p-1}. \tag{2.4}$$

Similarly to [1, Lemma 2.3], we have the following consequence of the spectral clusters estimates (2.3) and (2.4).

Lemma 2.2. *Assume that, for each $x \in M$, the cosphere $\Sigma_x = \{\xi \in T_x^*M : q(x,\xi) = 1\}$ is strictly convex. Let $\alpha \in C([0,\infty),\mathbb{C})$ and define the operators $\alpha_k(Q)$ as follows,*

$$\alpha_k(Q)f = \sum_{\mu_j\in[k-1,k)} \alpha(\mu_j)E_j f, \quad f \in C^\infty(M),$$

$k = 1, 2, \dots$. *Then if $p \ge \frac{2(n+1)}{n-1}$, we get*

$$\|\alpha_k(Q)f\|_{L^p(M)} \le Ck^{2\sigma(p)}(\sup_{\tau\in[k-1;k)} |\alpha(\tau)|)\|f\|_{L^{\frac{p}{p-1}}(M)}, \ \sigma(p) = n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}, \tag{2.5}$$

where $C > 0$ is a constant independent of the function α.

Proof. First notice that $\alpha_k(Q) = \chi_k \circ \alpha_k(Q)$. Let $p \ge \frac{2(n+1)}{n-1}$. Then using the spectral clusters estimates (2.3) and (2.4), we obtain that

$$\begin{aligned}
\|\alpha_k(Q)f\|_{L^p(M)} &\le Ck^{\sigma(p)}\|\alpha_k(Q)f\|_{L^2(M)} \\
&= Ck^{\sigma(p)}\left(\sum_{\mu_j\in[k-1,k)} |\alpha(\mu_j)|^2\|E_j f\|^2_{L^2(M)}\right)^{1/2} \\
&\le Ck^{\sigma(p)}(\sup_{\tau\in[k-1,k)} |\alpha(\tau)|)\left(\sum_{\mu_j\in[k-1,k)} \|E_j f\|^2_{L^2(M)}\right)^{1/2} \\
&= Ck^{\sigma(p)}(\sup_{\tau\in[k-1,k)} |\alpha(\tau)|)\|\chi_k f\|_{L^2(M)} \\
&\le Ck^{2\sigma(p)}(\sup_{\tau\in[k-1,k)} |\alpha(\tau)|)\|f\|_{L^{\frac{p}{p-1}}(M)}.
\end{aligned}$$

□

Lemma 2.3. *Assume that for each $x \in M$, the cosphere $\Sigma_x = \{\xi \in T_x^*M : q(x,\xi) = 1\}$ is strictly convex. Let $\alpha \in C([0,\infty),\mathbb{C})$ be such that*

$$A = \sup_{\tau\in[0,\infty)} (1 + \tau^m)|\alpha(\tau)| < \infty. \tag{2.6}$$

Then we have

$$\|\alpha(Q)f\|_{L^{\frac{2n}{n-m}}(M)} \le CA\|f\|_{L^{\frac{2n}{n+m}}(M)}, \tag{2.7}$$

where $\alpha(Q)$ is the operator defined by

$$\alpha(Q)f = \sum_{j=1}^{\infty} \alpha(\mu_j) E_j f, \quad f \in C^\infty(M),$$

and $C > 0$ is a constant independent of the function α.

Proof. To establish (2.7), we shall follow [1, Lemma 2.3], see also [9], and use the one dimensional Littlewood–Paley theory. To that end, let

$$\chi(t) = \begin{cases} 1, & t \in [1/2, 1), \\ 0, & t \notin [1/2, 1), \end{cases}$$

be the characteristic function of the interval $[1/2, 1)$. Setting $\chi_j(\tau) = \chi(2^{-j}\tau)$, we obtain the dyadic partition of unity in $[0, \infty)$, $\chi_0(\tau) + \sum_{j=1}^{\infty} \chi_j(\tau) = 1$, where $\chi_0(\tau) = 1$ when $\tau \in [0, 1)$, and $\chi_0(\tau) = 0$ otherwise.

Define $\alpha_j(\tau) = \alpha(\tau)\chi_j(\tau)$, $j = 0, 1, \dots$. Assume that we have proved that

$$\|\alpha_j(Q)f\|_{L^{\frac{2n}{n-m}}(M)} \le S\|f\|_{L^{\frac{2n}{n+m}}(M)}, \quad j = 0, 1, \dots, \tag{2.8}$$

with some constant $S > 0$. By the Littlewood–Paley theorem and Minkowski's inequality, we conclude from (2.8) that

$$\|\alpha(Q)f\|_{L^{\frac{2n}{n-m}}(M)} \le C_{q,p} S\|f\|_{L^{\frac{2n}{n+m}}(M)}, \tag{2.9}$$

where $C_{q,p} > 0$ depends on q and p only, see [9] and [10]. Let us present these arguments for the convenience of the reader. We shall write $p = \frac{2n}{n+m}$ and $q = \frac{2n}{n-m}$. Then $1 < p < 2 < q$. As $q > 1$, by Littlewood–Paley theorem, we get

$$\begin{aligned} \|\alpha(Q)f\|_{L^q(M)} &\le C_q \Big\| \Big(\sum_{j=0}^{\infty} |\alpha_j(Q)f|^2 \Big)^{1/2} \Big\|_{L^q(M)} \\ &= C_q \Big\| \sum_{j=0}^{\infty} |\alpha_j(Q)f|^2 \Big\|_{L^{q/2}(M)}^{1/2} := I_1. \end{aligned}$$

As $q/2 \ge 1$, we may write from Minkowski's inequality that

$$I_1 \le C_q \Big(\sum_{j=0}^{\infty} \||\alpha_j(Q)f|^2\|_{L^{q/2}(M)} \Big)^{1/2} = C_q \Big(\sum_{j=0}^{\infty} \|\alpha_j(Q)f\|^2_{L^q(M)} \Big)^{1/2} := I_2.$$

As $\chi_j = \chi_j^2$, $j = 0, 1, \dots$, it follows from (2.8) that

$$\begin{aligned} I_2 &\le C_q S \Big(\sum_{j=0}^{\infty} \|\chi_j(Q)f\|^2_{L^p(M)} \Big)^{1/2} \\ &= C_q S \Big(\Big\| \Big\{ \int_M |\chi_j(Q)f(x)|^p d\mu(x) \Big\} \Big\|_{l^{2/p}} \Big)^{1/p} := I_3, \end{aligned}$$

where $\|\{a_j\}\|_{l^{2/p}}$ denotes the $l^{2/p}$–norm of the sequence $\{a_j\}$. Since $2/p > 1$, by Minkowski's inequality,

$$I_3 \le C_q S\bigg(\int_M \|\{|\chi_j(Q)f|^p\}\|_{l^{2/p}} d\mu\bigg)^{1/p} = C_q S\bigg\|\bigg(\sum_{j=0}^{\infty} |\chi_j(Q)f|^2\bigg)^{1/2}\bigg\|_{L^p(M)}$$
$$\le C_q C_p S\|f\|_{L^p(M)},$$

which shows (2.9).

Thus, we are left with proving (2.8). Let $f \in C^\infty(M)$. For $j = 1, 2, \dots$, we write

$$\alpha_j(Q)f = \sum_{l=1}^{\infty} \alpha_j(\mu_l)E_l f = \sum_{\mu_l \in [2^{j-1}, 2^j)} \alpha_j(\mu_l)E_l f$$
$$= \sum_{r=1}^{2^j - 2^{j-1}} \sum_{\mu_l \in [2^{j-1}+r-1, 2^{j-1}+r)} \alpha_j(\mu_l)E_l f = \sum_{r=1}^{2^{j-1}} \alpha_{j,2^{j-1}+r}(Q)f,$$

where the truncated operator $\alpha_{j,k}(Q)$ is given by

$$\alpha_{j,k}(Q)f = \sum_{\mu_l \in [k-1,k)} \alpha_j(\mu_l)E_l f.$$

Since $\frac{2n}{n-m} \ge \frac{2(n+1)}{n-1}$, by (2.5) and the fact that $\sigma(2n/(n-m)) = (m-1)/2$, we get

$$\|\alpha_j(Q)f\|_{L^{\frac{2n}{n-m}}(M)} \le \sum_{r=1}^{2^{j-1}} \|\alpha_{j,2^{j-1}+r}(Q)f\|_{L^{\frac{2n}{n-m}}(M)}$$
$$\le C\sum_{r=1}^{2^{j-1}} (2^{j-1}+r)^{m-1} \Big(\sup_{\tau \in [2^{j-1}+r-1, 2^{j-1}+r)} |\alpha(\tau)|\Big)\|f\|_{L^{\frac{2n}{n+m}}(M)}, \quad j = 1, 2, \dots.$$

Now using (2.6), we obtain that

$$\begin{aligned}\|\alpha_j(Q)f\|_{L^{\frac{2n}{n-m}}(M)} &\le CA\sum_{r=1}^{2^{j-1}} (2^{j-1}+r)^{m-1}\frac{1}{(2^{j-1}+r-1)^m}\|f\|_{L^{\frac{2n}{n+m}}(M)} \\ &\le CA\sum_{r=1}^{2^{j-1}} \frac{(2^{j-1}2)^{m-1}}{(2^{j-1})^m}\|f\|_{L^{\frac{2n}{n+m}}(M)} \le CA\|f\|_{L^{\frac{2n}{n+m}}(M)},\end{aligned} \tag{2.10}$$

for $j = 1, 2, \dots$. We also have

$$\alpha_0(Q)f = \sum_{\mu_l \in [0,1)} \alpha(\mu_l)E_l f,$$

and therefore, it follows from (2.5) that

$$\|\alpha_0(Q)f\|_{L^{\frac{2n}{n-m}}(M)} \le C\Big(\sup_{\tau \in [0,1)} |\alpha(\tau)|\Big)\|f\|_{L^{\frac{2n}{n+m}}(M)} \le CA\|f\|_{L^{\frac{2n}{n+m}}(M)}. \tag{2.11}$$

We obtain (2.8) as a consequence of (2.10) and (2.11). The proof of Lemma 2.3 is complete. □

2.3. **Derivation of the resolvent estimate with bounded $|z|$.** Let us first prove the resolvent estimate (1.5) for all $z \in \Xi_\delta$ when $|z|$ is bounded by a fixed constant, i.e. $z \in \Xi_\delta \cap \{z \in \mathbb{C} : |z| \le D\}$. To that end, consider the multiplier

$$m_z(\tau) = \frac{1}{\tau^m - z^m}, \quad \tau \in [0, \infty).$$

First notice that $\tau^m - z^m \neq 0$ for all $\tau \ge 0$ and all $z \in \mathbb{C}$ with $\arg(z) \in (0, 2\pi/m)$. Then by continuity of $|\tau^m - z^m|$ on a compact set, we have that for any $A, D, \delta > 0$, there exists a constant $C > 0$ such that $|\tau^m - z^m| \ge 1/C$ for $\tau \in [0, A]$ and $z \in \Xi_\delta \cap \{z \in \mathbb{C} : |z| \le D\}$. For τ large and $z \in \Xi_\delta \cap \{z \in \mathbb{C} : |z| \le D\}$, we have $|\tau^m - z^m| \sim \tau^m$, and therefore, we conclude that

$$|m_z(\tau)| \le C_{\delta,D}(1 + \tau^m)^{-1}$$

uniformly in $z \in \Xi_\delta \cap \{z \in \mathbb{C} : |z| \le D\}$. By appealing to Lemma 2.3, we obtain the resolvent estimate (1.5) for $z \in \Xi_\delta \cap \{z \in \mathbb{C} : |z| \le D\}$.

Remark 2.4. *Notice that applying Lemma 2.3, we can immediately obtain the (non-uniform) estimate*

$$\|u\|_{L^{\frac{2n}{n-m}}(M)} \le C_\zeta \|(P - \zeta)u\|_{L^{\frac{2n}{n+m}}(M)},$$

for all $\zeta \in \mathbb{C} \setminus [0, \infty)$ and $u \in C^\infty(M)$.

2.4. **Uniform bounds for a local term in the case of unbounded $|z|$.** Let $z \in \Xi_\delta \cap \{z \in \mathbb{C} : |z| \ge 1\}$. Here it will be convenient to use the representation (2.2) for the multiplier $m_z(Q)$. To define the localized version of $m_z(Q)$, we fix a function $\rho \in C^\infty(\mathbb{R})$ satisfying

$$\rho(t) = \begin{cases} 1, & |t| \le \varepsilon/2, \\ 0, & |t| \ge \varepsilon, \end{cases} \tag{2.12}$$

where $0 < \varepsilon < 1/2$ will be specified later. In view of (2.2), the localized version of $m_z(Q)$ is given by

$$m_z^{\mathrm{loc}}(Q)f = \frac{i}{mz^{m-1}} \sum_{k=0}^{m/2-1} e^{2\pi k i/m} \int_{-\infty}^{+\infty} \rho(t) e^{i|t|\tau_k} e^{itQ} f\, dt, \quad f \in C^\infty(M). \tag{2.13}$$

Here $\tau_k = z e^{2\pi k i/m}$ and $\operatorname{Im} \tau_k > 0$, $k = 0, 1, \ldots, m/2 - 1$.

To prove the resolvent estimate (1.5) for $z \in \Xi_\delta \cap \{z \in \mathbb{C} : |z| \ge 1\}$, let us first establish this estimate for $m_z^{\mathrm{loc}}(Q)$, i.e.

$$\|m_z^{\mathrm{loc}}(Q)f\|_{L^{\frac{2n}{n-m}}(M)} \le C\|f\|_{L^{\frac{2n}{n+m}}(M)}. \tag{2.14}$$

When doing so we shall use a dyadic partition of the t–interval in the definition (2.13) of $m_z^{\text{loc}}(Q)$. To that end let $\psi \in C_0^\infty(\mathbb{R})$ be such that supp $(\psi) \subset [-2,2]$, $\psi = 1$ on $[-1,1]$, and ψ is even. Define $\beta(t) = \psi(t) - \psi(2t)$. Thus,

$$\beta(t) = 0, \quad |t| \notin [1/2, 2],$$

and

$$\sum_{j=-\infty}^{+\infty} \beta(2^{-j}t) = 1, \quad t \neq 0.$$

It will be convenient to write,

$$\widetilde{\rho}(t) = 1 - \sum_{j=0}^{+\infty} \beta(2^{-j}t) \in C_0^\infty(\mathbb{R}).$$

Notice that $\widetilde{\rho}(t) = 0$ when $|t| \geq 1$.

For a given $z \in \Xi_\delta \cap \{z \in \mathbb{C} : |z| \geq 1\}$, we define the multipliers

$$S_{z,j}(\tau) = \frac{i}{mz^{m-1}} \sum_{k=0}^{m/2-1} e^{2\pi k i/m} \int_{-\infty}^{+\infty} \beta(2^{-j}|z|t)\rho(t) e^{i|t|\tau_k} e^{it\tau} dt, \quad j = 0, 1, 2, \dots, \tag{2.15}$$

and

$$\widetilde{S}_z(\tau) = \frac{i}{mz^{m-1}} \sum_{k=0}^{m/2-1} e^{2\pi k i/m} \int_{-\infty}^{+\infty} \widetilde{\rho}(|z|t)\rho(t) e^{i|t|\tau_k} e^{it\tau} dt. \tag{2.16}$$

We have

$$S_{z,j} = 0 \quad \text{if} \quad 2^{-j}|z| \leq 1. \tag{2.17}$$

Indeed, if $|t| \leq \varepsilon$, then $2^{-j}|z||t| < 1/2$ and therefore, $\beta(2^{-j}|z|t) = 0$.

The bound (2.14) follows once we show that there is a uniform constant C so that for all $z \in \Xi_\delta \cap \{z \in \mathbb{C} : |z| \geq 1\}$, we have

$$\|S_{z,j}(Q)f\|_{L^{\frac{2n}{n-m}}(M)} \leq C 2^{j\frac{2n-m-nm}{2n}} \|f\|_{L^{\frac{2n}{n+m}}(M)}, \quad j = 0, 1, \dots, \tag{2.18}$$

and

$$\|\widetilde{S}_z(Q)f\|_{L^{\frac{2n}{n-m}}(M)} \leq C\|f\|_{L^{\frac{2n}{n+m}}(M)}. \tag{2.19}$$

Let us start with establishing the estimate (2.19). When doing so, we shall follow [12] and obtain the following result.

Lemma 2.5. *The multiplier $\widetilde{S}_z$ belongs to the symbol class $S^{-m}(\mathbb{R})$ uniformly in $z \in \mathbb{C}$, $|z| \geq 1$, i.e.*

$$|d_\tau^j \widetilde{S}_z(\tau)| \leq C_j (1 + |\tau|)^{-m-j}, \quad j = 0, 1, 2, \dots, \tag{2.20}$$

with the constants C_j independent of z.

Proof. Recall first that $\widetilde{\rho}(|z|t) = 0$ when $|t| \geq 1/|z|$. Furthermore, as $\operatorname{Im} \tau_k > 0$, $k = 0, 1, \dots, m/2 - 1$, we conclude that $|e^{i|t|\tau_k}| \leq 1$.

Let $|\tau| \leq 1$. Then for $j = 0, 1, \dots,$ we have

$$|d_\tau^j \widetilde{S}_z(\tau)| \leq \frac{C}{|z|^{m-1}} \int_{-1/|z|}^{1/|z|} |t|^j dt \leq \frac{C}{|z|^{m+j}} \leq C,$$

uniformly in z, $|z| \geq 1$, which shows the estimate (2.20) in the case $|\tau| \leq 1$.

Assume now that $|\tau| > 1$. Let us first prove the estimate (2.20) for $j = 0$. To that end we shall integrate by parts m times in the expression (2.16) for $\widetilde{S}_z$.

Let us first explain that all boundary terms vanish when we integrate by parts $m - 1$ times in (2.16). Indeed, integrating by parts once in (2.16), we obtain the following boundary terms,

$$\frac{i}{i\tau m z^{m-1}} \sum_{k=0}^{m/2-1} e^{2\pi k i/m} \Big(\widetilde{\rho}(|z|t)\rho(t) e^{-it\tau_k} e^{it\tau} |_{t=-\infty}^{t=0} + \widetilde{\rho}(|z|t)\rho(t) e^{it\tau_k} e^{it\tau} |_{t=0}^{t=+\infty} \Big)$$
$$= \frac{i}{i\tau m z^{m-1}} \sum_{k=0}^{m/2-1} e^{2\pi k i/m} \Big(1 - 1 \Big) = 0.$$

Here we have used the fact that $\widetilde{\rho}$ and ρ are compactly supported, and $\widetilde{\rho}(0) = \rho(0) = 1$.

Furthermore, since all the derivatives of $\widetilde{\rho}$ and ρ vanish at the origin, when integrating by parts m times in (2.16), the only possible contribution to the boundary terms may be written in the form $\sum_{l=1}^m B_l$, where

$$B_l = \frac{i}{(i\tau)^l m z^{m-1}} \sum_{k=0}^{m/2-1} e^{2\pi k i/m} (-1)^{l-1} \Big(\widetilde{\rho}(|z|t)\rho(t)(-i\tau_k)^{l-1} e^{-it\tau_k} e^{it\tau} |_{t=-\infty}^{t=0}$$
$$+ \widetilde{\rho}(|z|t)\rho(t)(i\tau_k)^{l-1} e^{it\tau_k} e^{it\tau} |_{t=0}^{t=+\infty} \Big)$$
$$= \frac{i}{(i\tau)^l m z^{m-1}} \sum_{k=0}^{m/2-1} e^{2\pi k i/m} (-1)^{l-1} ((-i\tau_k)^{l-1} - (i\tau_k)^{l-1}).$$

When l is odd, it is clear that $B_l = 0$. Recall now that m is even. When l is even and $l \neq m$, we also have $B_l = 0$ due to the fact that

$$\sum_{k=0}^{m/2-1} e^{2\pi k i/m} (\tau_k)^{l-1} = z^{l-1} \sum_{k=0}^{m/2-1} (e^{2\pi l i/m})^k = z^{l-1} \frac{1 - e^{\pi l i}}{1 - e^{2\pi l i/m}} = 0.$$

Here we have used that $\tau_k = z e^{2\pi k i/m}$ and the fact that $e^{2\pi l i/m} \neq 1$ when $2 \leq l \leq m - 2$. Hence, when integrating by parts m times in (2.16), the only possible contribution to the boundary terms is of the form,

$$B_m = \frac{2}{\tau^m m z^{m-1}} \sum_{k=0}^{m/2-1} e^{2\pi k i/m}(\tau_k)^{m-1} = \frac{2}{\tau^m m} \sum_{k=0}^{m/2-1} e^{2\pi k i} = \frac{1}{\tau^m}. \tag{2.21}$$

Let us explain how to estimate the integrals arising after having integrated by parts m times in (2.16). The worst case scenario occurs when no derivatives fall on $\rho(t)$, and the corresponding contribution can be estimated by a constant times

$$\left| \frac{1}{\tau^m} \int_{-1/|z|}^0 |z|^{l_1} (d_t^{l_1} \widetilde{\rho})(|z|t)\rho(t)(-i\tau_k)^{l_2} e^{-it\tau_k} e^{it\tau} dt \right| \le C \frac{|z|^{m-1}}{|\tau|^m}. \tag{2.22}$$

Here $l_1 + l_2 = m$. Then it follows from (2.16), (2.22) and (2.21) that

$$|\widetilde{S}_z(\tau)| \le \frac{C}{|\tau|^m},$$

which shows (2.20) for $j = 0$ in the case $|\tau| > 1$.

To establish (2.20) for $j = 1, 2, \dots$ in the case $|\tau| > 1$, we write

$$\begin{aligned} d_\tau^j \widetilde{S}_z(\tau) = \frac{i}{m z^{m-1}} \sum_{k=0}^{m/2-1} e^{2\pi k i/m} \bigg(\int_{-\infty}^0 \widetilde{\rho}(|z|t)\rho(t) e^{-it\tau_k} (it)^j e^{it\tau} dt \\ + \int_0^{+\infty} \widetilde{\rho}(|z|t)\rho(t) e^{it\tau_k} (it)^j e^{it\tau} dt \bigg), \end{aligned} \tag{2.23}$$

and integrate by parts $(m+j)$ times in (2.23). Due to the appearance of the terms t^j in the integrands in (2.23), no boundary terms arise when integrating by parts the first j times. Integrating by parts further, the contributions to the boundary terms that one has to consider would be similar to those in the case $j = 0$, and therefore, we need only to discuss the integrals obtained after an integration by parts $m + j$ times in (2.23). The worst case scenario here occurs when no derivatives fall on $\rho(t)$, and the corresponding contribution to the integrals can be bounded by a constant times

$$\left| \frac{1}{\tau^{m+j}} \int_{-1/|z|}^0 |z|^{l_1} (d_t^{l_1} \widetilde{\rho})(|z|t)\rho(t)(-i\tau_k)^{l_2} e^{-it\tau_k} t^{j-l_3} e^{it\tau} dt \right| \le C|z|^{m-1} \frac{1}{|\tau|^{m+j}}.$$

Here $l_1 + l_2 + l_3 = m + j$, $0 \le l_3 \le j$. Together with (2.23) this implies (2.20). The proof is complete. □

Combing Lemma 2.5 with the fact that $Q \in \Psi^1_{\mathrm{cl}}(M)$ is elliptic and self-adjoint, we conclude from [14, Theorem 4.3.1] that $\widetilde{S}_z(Q)$ is a pseudodifferential operator of order $-m$, with the symbol seminorms uniformly bounded in $z \in \mathbb{C}$, $|z| \ge 1$.

Let $\widetilde{S}_z(Q)(x, y) \in \mathcal{D}'(M \times M)$ be the Schwartz kernel of the operator $\widetilde{S}_z(Q)$. Then $\widetilde{S}_z(Q)(x, y)$ is C^∞ away from the diagonal $\{(x, x) : x \in M\}$. By [16, Proposition

1, p. 241], since $n-m>0$, we have near the diagonal, in local coordinates,

$$|\widetilde{S}_z(Q)(x,y)|\le C|x-y|^{m-n},$$

uniformly in $z\in\mathbb{C}$, $|z|\ge 1$. An application of the Hardy-Littlewood-Sobolev inequality gives the estimate (2.19).

Let us now prove the estimate (2.18). By the Riesz–Thorin interpolation theorem, (2.18) follows, if we show that that there is a constant $C=C(\delta)$ so that for all $z\in\Xi_\delta\cap\{z\in\mathbb{C}:|z|\ge 1\}$, we have

$$\|S_{z,j}(Q)f\|_{L^2(M)}\le C|z|^{-m}2^j\|f\|_{L^2(M)},\quad j=0,1,\dots,\tag{2.24}$$

and

$$\|S_{z,j}(Q)f\|_{L^\infty(M)}\le C|z|^{n-m}2^{-\frac{(n-1)}{2}j}\|f\|_{L^1(M)},\quad j=0,1,\dots.\tag{2.25}$$

Here the interpolation parameter $\theta=\frac{n-m}{n}$, and

$$(|z|^{-m}2^j)^\theta(|z|^{n-m}2^{-\frac{(n-1)}{2}j})^{1-\theta}=2^{j\frac{2n-m-nm}{2n}}.$$

When proving the estimate (2.24), we use the identity $\|e^{itQ}f\|_{L^2(M)}=\|f\|_{L^2(M)}$, $t\in\mathbb{R}$, the fact that $\beta(2^{-j}|z|t)=0$ when $|t|\notin[2^{j-1}/|z|,2^{j+1}/|z|]$, and Minkowski's inequality, to get

$$\|S_{z,j}(Q)f\|_{L^2(M)}\le\frac{C}{|z|^{m-1}}\int_{|t|\in[2^{j-1}/|z|,2^{j+1}/|z|]}\|e^{itQ}f\|_{L^2(M)}dt\le\frac{C}{|z|^m}2^j\|f\|_{L^2(M)},$$

uniformly in z, which shows (2.24).

Now we are left with proving (2.25). Let us denote by $K_{z,j}(x,y)$ the Schwartz kernel of the operator $S_{z,j}(Q)$. The estimate (2.25) is implied by the estimate

$$|K_{z,j}(x,y)|\le C|z|^{n-m}2^{-\frac{(n-1)}{2}j},\quad x,y\in M,\tag{2.26}$$

for all $z\in\Xi_\delta\cap\{z\in\mathbb{C}:|z|\ge 1\}$, uniformly in z. By (2.15), we have

$$K_{z,j}(x,y)=\frac{i}{mz^{m-1}}\sum_{k=0}^{m/2-1}e^{2\pi ki/m}\int_{-\infty}^{+\infty}\beta(2^{-j}|z|t)\rho(t)e^{i|t|\tau_k}e^{itQ}(x,y)dt,\tag{2.27}$$

where $e^{itQ}(x,y)$ is the Schwartz kernel of the half-wave operator e^{itQ}. To proceed, we shall make use of the Hörmander–Lax parametrix for the the half-wave operator e^{itQ}, see [6], [14, Theorem 4.1.2].

Lemma 2.6. *Let $Q\in\Psi^1_{\mathrm{cl}}(M)$ be elliptic and self-adjoint with respect to a positive C^∞ density $d\mu$, and $q(x,\xi)$ be the principal symbol of Q. Then there is $\varepsilon>0$ small, depending on M and Q, so that if $|t|<\varepsilon$,*

$$e^{itQ}=G(t)+R(t),$$

where the remainder $R(t)$ has the kernel $R(t,x,y)\in C^\infty([-\varepsilon,\varepsilon]\times M\times M)$, and the kernel $G(t,x,y)$ is supported in a small neighborhood of the diagonal in $M\times M$, for $|t|<\varepsilon$. Furthermore, suppose that local coordinates are chosen in a patch

$\Omega\subset M$ so that $d\mu$ agrees with the Lebesque measure in the corresponding open subset of $\mathbb{R}^n$. If $\omega\subset\Omega$ is relatively compact, $G(t,x,y)$ has the form,

$$G(t,x,y)=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{i[\varphi(x,y,\xi)+tq(y,\xi)]}g(t,x,y,\xi)d\xi$$

when $(t,x,y)\in[-\varepsilon,\varepsilon]\times M\times\omega$. Here $g\in S^0_{1,0}$, i.e.

$$|\partial_\xi^\alpha\partial_t^{\beta_1}\partial_x^{\beta_2}\partial_y^{\beta_3}g(t,x,y,\xi)|\le C_{\alpha,\beta_1,\beta_2,\beta_3}(1+|\xi|)^{-|\alpha|},$$

for all multi-indices α, β_1, β_2, β_3, and g is supported in a small neighborhood of the diagonal in $\omega\times\omega$, and φ is a real function which is homogeneous of degree one in ξ, C^∞ for $\xi\ne0$, and satisfies

$$\varphi(x,y,\xi)=\langle x-y,\xi\rangle+\mathcal{O}_{S^1}(|x-y|^2|\xi|), \tag{2.28}$$

i.e.

$$|\partial_\xi^\alpha(\varphi(x,y,\xi)-\langle x-y,\xi\rangle)|\le C_\alpha|x-y|^2|\xi|^{1-|\alpha|},$$

for all multi-indices α.

In what follows, we shall make the choice of ε in the definition (2.12) of the function $\rho(t)$ so that Lemma 2.6 is applicable.

We assume that $2^{-j}|z|>1$, as otherwise $S_{z,j}=0$, cf. (2.17). Let us write

$$K_{z,j}(x,y)=K^{(1)}_{z,j}(x,y)+K^{(2)}_{z,j}(x,y),$$

where

$$K^{(1)}_{z,j}(x,y)=\frac{i}{mz^{m-1}}\sum_{k=0}^{m/2-1}e^{2\pi ki/m}\int_{-\infty}^{+\infty}\beta(2^{-j}|z|t)\rho(t)e^{i|t|\tau_k}G(t,x,y)dt,$$

$$K^{(2)}_{z,j}(x,y)=\frac{i}{mz^{m-1}}\sum_{k=0}^{m/2-1}e^{2\pi ki/m}\int_{-\infty}^{+\infty}\beta(2^{-j}|z|t)\rho(t)e^{i|t|\tau_k}R(t,x,y)dt.$$

Since $R(t,x,y)\in C^\infty([-\varepsilon,\varepsilon]\times M\times M)$, we have

$$|K^{(2)}_{z,j}(x,y)|\le\frac{C}{|z|^{m-1}}\left|\int_{|t|\in[2^{j-1}/|z|,2^{j+1}/|z|]}dt\right|\le\frac{2^jC}{|z|^m}. \tag{2.29}$$

As $2^{-j}|z|>1$, the estimate (2.29) is better than the desired bound (2.26) for $K_{z,j}$.

Let us now estimate $K^{(1)}_{z,j}$. Setting

$$r=\frac{2^j}{|z|},\quad \frac{1}{|z|}\le r<1,$$

and assuming that the local coordinates are chosen as in Lemma 2.6, we write

$$\begin{aligned} K^{(1)}_{z,j}(x,y) = & \frac{i}{mz^{m-1}} \sum_{k=0}^{m/2-1} e^{2\pi ki/m} \frac{1}{(2\pi)^n} \\ & \int_{\mathbb{R}^n} \int_{-\infty}^{+\infty} \beta(t/r)\rho(t) e^{i|t|\tau_k} e^{i[\varphi(x,y,\xi)+tq(y,\xi)]} g(t,x,y,\xi)\,dtd\xi, \end{aligned} \tag{2.30}$$

for $(x,y) \in M \times \omega$. We would like to replace φ by the Euclidean phase function $\varphi_0 = \langle x-y,\xi\rangle$. In doing so, we shall follow [11] and notice that both φ and φ_0 parametrize the trivial Lagrangian manifold $\{(x,\xi,x,\xi)\}$. This is due to the fact that when (x,y) is in a neighborhood of the diagonal, we have $\varphi'_\xi = 0$ precisely when $x=y$, and then $\varphi'_x = -\varphi'_y = \xi$. Following [11], we can use the following result of [7, Theorem 3.1.6].

Lemma 2.7. *Suppose that φ is as in Lemma 2.6, i.e. φ satisfies* (2.28). *Then, for (x,y) close to the diagonal, there is a C^∞ for $\xi \neq 0$ homogeneous of degree one change of coordinates*

$$\eta = \kappa_{x,y}(\xi)$$

so that

$$\varphi(x,y,\kappa^{-1}_{x,y}(\eta)) = \langle x-y,\eta\rangle.$$

The transformation $\kappa_{x,y}$ depends smoothly on the parameters x, y, and in addition,

$$\kappa_{x,y} = \text{Identity}, \quad \textit{when} \quad x=y. \tag{2.31}$$

Lemma 2.7 implies that (2.30) can be rewritten as

$$\begin{aligned} K^{(1)}_{z,j}(x,y) = & \frac{i}{mz^{m-1}} \sum_{k=0}^{m/2-1} e^{2\pi ki/m} \frac{1}{(2\pi)^n} \\ & \int_{\mathbb{R}^n} \int_{-\infty}^{+\infty} \beta(t/r)\rho(t) e^{i|t|\tau_k} e^{i[\langle x-y,\eta\rangle + t\widetilde{q}(x,y,\eta)]} \widetilde{g}(t,x,y,\eta)\,dtd\eta, \end{aligned} \tag{2.32}$$

where

$$\widetilde{g}(t,x,y,\eta) = g(t,x,y,\kappa^{-1}_{x,y}(\eta)) \left| \frac{D(\kappa^{-1}_{x,y})(\eta)}{D\eta} \right|,$$

with $\frac{D(\kappa^{-1}_{x,y})(\eta)}{D\eta}$ being the Jacobian of the transformation $\kappa^{-1}_{x,y}$, has the same properties as g, in particular $\widetilde{g} \in S^0_{1,0}$. Also,

$$\widetilde{q}(x,y,\eta) = q(y,\kappa^{-1}_{x,y}(\eta))$$

depends smoothly on x, y. Furthermore, since strict convexity is preserved under diffeomorphisms that are sufficiently close to the identity in the C^∞ sense, the surface

$$\widetilde{\Sigma}_{x,y} = \{\eta \in \mathbb{R}^n : \widetilde{q}(x,y,\eta) = 1\}$$

is strictly convex.

Making the change of variables $t \mapsto t/r$ in (2.32), we get

$$\begin{aligned} K^{(1)}_{z,j}(x,y) =& \frac{ir}{mz^{m-1}} \sum_{k=0}^{m/2-1} e^{2\pi k i/m} \frac{1}{(2\pi)^n} \\ & \int_{\mathbb{R}^n}\int_{-\infty}^{+\infty} \beta(t)\rho(rt) e^{ir|t|\tau_k} e^{i\langle x-y,\eta\rangle} e^{itr\widetilde{q}(x,y,\eta)}\widetilde{g}(rt,x,y,\eta)dtd\eta. \end{aligned} \tag{2.33}$$

As q and $\kappa_{x,y}$ are homogeneous of degree one, we have

$$r\widetilde{q}(x,y,\eta) = q(x,y,r\kappa_{x,y}^{-1}(\eta)) = \widetilde{q}(x,y,r\eta).$$

Making further change of variables $\eta \mapsto r\eta$ in (2.33), we obtain that

$$\begin{aligned} K^{(1)}_{z,j}(x,y) =& \frac{ir^{1-n}}{mz^{m-1}} \sum_{k=0}^{m/2-1} e^{2\pi k i/m} \frac{1}{(2\pi)^n} \\ & \int_{\mathbb{R}^n}\int_{-\infty}^{+\infty} \beta(t)\rho(rt) e^{ir|t|\tau_k} e^{i\langle \frac{x-y}{r},\eta\rangle} e^{it\widetilde{q}(x,y,\eta)}\widetilde{g}(rt,x,y,\eta/r)dtd\eta. \end{aligned} \tag{2.34}$$

As $\widetilde{q}(x,y,\eta)$ is not smooth at $\eta = 0$, it will be convenient to write

$$\begin{aligned} J_1(x,y,t,r) &= \int_{\mathbb{R}^n} e^{i[\langle \frac{x-y}{r},\eta\rangle + t\widetilde{q}(x,y,\eta)]}\chi(\eta)\widetilde{g}(rt,x,y,\eta/r)d\eta, \\ J_2(x,y,t,r) &= \int_{\mathbb{R}^n} e^{i[\langle \frac{x-y}{r},\eta\rangle + t\widetilde{q}(x,y,\eta)]}(1-\chi(\eta))\widetilde{g}(rt,x,y,\eta/r)d\eta, \end{aligned}$$

where $\chi \in C_0^\infty(\mathbb{R}^n)$ and $\chi = 1$ when $|\eta| \le 1$. Here $|t| \in [1/2, 2]$ and $0 < r \le 1$.

As $\widetilde{g} \in S^0_{1,0}$, we see that

$$|J_1(x,y,t,r)| \le C, \tag{2.35}$$

for all $x, y \in \omega$, $|x-y|$ small enough, uniformly in r.

Let us now estimate the absolute value of the oscillatory integral $J_2(x,y,t,r)$ when $|t| \in [1/2, 2]$. To that end, consider

$$\nabla_\eta[\langle \frac{x-y}{r},\eta\rangle + t\widetilde{q}(x,y,\eta)], \quad |t| \in [1/2,2].$$

As $\widetilde{q}(x,y,\eta)$ is homogeneous of degree one in η, by the Euler homogeneity relation, we have

$$\eta\cdot\nabla_\eta\widetilde{q}(x,y,\eta) = \widetilde{q}(x,y,\eta).$$

This and the ellipticity of $\widetilde{q}$ imply that $\nabla_\eta\widetilde{q}(x,y,\eta) \ne 0$ for all $\eta \in \mathbb{R}^n \setminus \{0\}$. Thus, there is a constant $A > 1/2$ such that $|\nabla_\eta\widetilde{q}(x,y,\eta)| \ge A^{-1}$ for all $\eta \in \mathbb{S}^{n-1}$, and therefore, by the fact that $\nabla_\eta\widetilde{q}$ is homogeneous of degree zero, we conclude that

$$|\nabla_\eta\widetilde{q}(x,y,\eta)| \ge A^{-1} \quad \text{for all} \quad \eta \in \mathbb{R}^n \setminus \{0\}.$$

On the other hand, since $\nabla_\eta \widetilde{q} \in S^0_{1,0}$, for $|\eta| \geq 1$, we have

$$|\nabla_\eta \widetilde{q}(x, y, \eta)| \leq A.$$

Hence, for $|t| \in [1/2, 2]$, if x, y are such that

$$\frac{|x-y|}{r} \notin [A^{-1}/4, 4A], \tag{2.36}$$

then

$$|\nabla_\eta [\langle \frac{x-y}{r}, \eta\rangle + t\widetilde{q}(x, y, \eta)]| \geq A^{-1}/2. \tag{2.37}$$

Assume first that (2.36) holds. Then we shall integrate by parts in the oscillatory integral J_2, see [7, Lemma 1.2.1]. To that end, setting

$$\psi(t, x, y, \eta) = \langle \frac{x-y}{r}, \eta\rangle + t\widetilde{q}(x, y, \eta),$$

we consider the operator

$$L = \sum_{j=1}^n a_j \partial_{\eta_j}, \quad a_j = \frac{\partial_{\eta_j}\psi}{i|\nabla_\eta \psi|^2}.$$

We have $L^N(e^{i\psi(\eta)}) = e^{i\psi(\eta)}$ for any $N \in \mathbb{N}$, and the transpose L' of L is given by

$$L' = -\sum_{j=1}^n a_j \partial_{\eta_j} - \operatorname{div} a, \quad a = (a_1, \dots, a_n). \tag{2.38}$$

Hence, we get

$$J_2(x, y, t, r) = \int_{\mathbb{R}^n} e^{i\psi(\eta)} (L')^N ((1 - \chi(\eta))\widetilde{g}(rt, x, y, \eta/r)) d\eta.$$

We observe that

$$(1 - \chi(\eta))\widetilde{g}(rt, x, y, \eta/r) \in S^0_{1,0} \tag{2.39}$$

uniformly in $0 < r \leq 1$. This follows from the facts that when $|\eta| \geq 1$,

$$|\partial_\eta^\alpha \partial_t^{\beta_1} \partial_x^{\beta_2} \partial_y^{\beta_3} \widetilde{g}(rt, x, y, \eta/r)| \leq \frac{r^{\beta_1}}{r^{|\alpha|}} C_{\alpha,\beta_1,\beta_2,\beta_3} (1+|\eta|/r)^{-|\alpha|} \leq C_{\alpha,\beta_1,\beta_2,\beta_3}(1+|\eta|)^{-|\alpha|},$$

for all $\beta_1 \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and all $\alpha, \beta_2, \beta_3 \in \mathbb{N}_0^n$, and

$$|\partial_\eta^\alpha \chi(\eta)| \leq C_{\alpha,N}(1 + |\eta|)^{-N},$$

for all $\alpha \in \mathbb{N}_0^n$ and all $N > 0$.

Let us now show that

$$a_j(\eta) \in S^0_{1,0}, \quad |\eta| \geq 1, \tag{2.40}$$

uniformly in r, x, y and t satisfying (2.36). Indeed, first using (2.37), we have

$$|a_j(\eta)| = \frac{|\partial_{\eta_j}\psi|}{|\nabla_\eta \psi|^2} \leq 2A. \tag{2.41}$$

Let $\alpha\in\mathbb{N}^n$ be such that $|\alpha|\ge 1$. Then by Leibniz formula, we get

$$\partial_\eta^\alpha a_j(\eta)=\sum_{\beta+\gamma=\alpha}c_{\beta,\gamma}\partial_\eta^\beta(\partial_{\eta_j}\psi)\partial_\eta^\gamma\bigg(\frac{1}{|\nabla_\eta\psi|^2}\bigg), \tag{2.42}$$

with constants $c_{\beta,\gamma}$. Here

$$\partial_{\eta_j}\psi=\frac{x_j-y_j}{r}+t\partial_{\eta_j}\widetilde{q}(x,y,\eta),$$

and hence, for $|\beta|\ge 1$, we have

$$|\partial_\eta^\beta(\partial_{\eta_j}\psi)|\le C_\beta(1+|\eta|)^{-|\beta|}, \tag{2.43}$$

uniformly in r. To estimate the absolute value of $\partial_\eta^\gamma(1/|\nabla_\eta\psi|^2)$ for $|\gamma|\ge 1$, we shall use the Faà di Bruno formula, see [18, p. 94],

$$\partial_\eta^\gamma\bigg(\frac{1}{b}\bigg)=\frac{1}{b}\sum_{\substack{1\le k\le|\gamma|\\ |\gamma|=|\gamma^1|+\dots+|\gamma^k|\\ |\gamma^j|\ge 1}}C_{\gamma^1,\dots,\gamma^k}\prod_{j=1}^k\frac{\partial_\eta^{\gamma^j}b}{b}. \tag{2.44}$$

For $|\gamma^j|\ge 1$, using Leibniz formula and (2.43), we have

$$|\partial_\eta^{\gamma^j}(|\nabla_\eta\psi|^2)|\le C_{\gamma^j}|\nabla_\eta\psi|(1+|\eta|)^{-|\gamma^j|}.$$

Therefore, (2.44) implies that for $\gamma\in\mathbb{N}_0^n$,

$$\bigg|\partial_\eta^\gamma\bigg(\frac{1}{|\nabla_\eta\psi|^2}\bigg)\bigg|\le C_\gamma\frac{1}{|\nabla_\eta\psi|^2}(1+|\eta|)^{-|\gamma|} \tag{2.45}$$

uniformly in r. We conclude from (2.42) with the help of (2.43) and (2.45) that for all $a\in\mathbb{N}^n$, $|\alpha|\ge 1$,

$$|\partial_\eta^\alpha a_j(\eta)|\le C_\alpha(1+|\eta|)^{-|\alpha|}, \tag{2.46}$$

uniformly in r. Hence, (2.40) follows from (2.41) and (2.46).

Using (2.46), we obtain that

$$\operatorname{div}a\in S^{-1}_{1,0},\quad |\eta|\ge 1, \tag{2.47}$$

uniformly in r, x, y and t satisfying (2.36). Thus, it follows from (2.38) with the help of (2.40), (2.47) and (2.39) that

$$(L')^N((1-\chi(\eta))\widetilde{g}(rt,x,y,\eta/r))\in S^{-N}_{1,0}$$

uniformly in r, x, y and t satisfying (2.36).

Hence, choosing N sufficiently large, we conclude that

$$|J_2(x,y,t,r)|\le C. \tag{2.48}$$

Therefore, it follows from (2.34), (2.35) and (2.48) that

$$|K_{z,j}^{(1)}(x,y)| \leq C\frac{r^{1-n}}{|z|^{m-1}} = 2^{j(1-n)}|z|^{n-m}, \tag{2.49}$$

when x, y are such that $\frac{|x-y|}{r} \notin [A^{-1}/4, 4A]$. The estimate (2.49) is better than the desired estimate (2.26).

Assume now that $\frac{|x-y|}{r} \in [A^{-1}/4, 4A]$ and let us estimate the absolute value of $K_{z,j}^{(1)}(x,y)$ in this case. As above, we only need to estimate the absolute value of

$$\begin{aligned} K_{z,j}^{(1,2)}(x,y) =& \frac{ir^{1-n}}{mz^{m-1}} \sum_{k=0}^{m/2-1} e^{2\pi ki/m} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{-\infty}^{+\infty} \beta(t)\rho(rt)e^{ir|t|\tau_k} \\ & e^{i\langle \frac{x-y}{r},\eta\rangle} e^{it\widetilde{q}(x,y,\eta)}(1-\chi(\eta))\widetilde{g}(rt,x,y,\eta/r)dtd\eta, \end{aligned}$$

where $\chi \in C_0^\infty(\mathbb{R}^n)$ is such that $\chi = 1$ when $|\eta| \leq 1$. Using (2.1), we get

$$\begin{aligned} K_{z,j}^{(1,2)}(x,y) =& \frac{r^{1-n}}{(2\pi)^{n+1}} \int_{-\infty}^{+\infty} \int_{\mathbb{R}^n} \int_{-\infty}^{+\infty} \frac{e^{it(-r\tau+\widetilde{q}(x,y,\eta))}}{\tau^m - z^m} d\tau \\ & \beta(t)\rho(rt)e^{i\langle \frac{x-y}{r},\eta\rangle}(1-\chi(\eta))\widetilde{g}(rt,x,y,\eta/r)d\eta dt. \end{aligned} \tag{2.50}$$

Making the change of variables $\tau \mapsto -r\tau + \widetilde{q}(x,y,\eta)$, we obtain that

$$K_{z,j}^{(1,2)}(x,y) = \frac{r^{-n}}{(2\pi)^n} \int_{-\infty}^{+\infty} \int_{\mathbb{R}^n} \frac{h_r(\tau,x,y,\eta)e^{i\langle \frac{x-y}{r},\eta\rangle}}{(\frac{\widetilde{q}(x,y,\eta)-\tau}{r})^m - z^m} d\eta d\tau, \tag{2.51}$$

where

$$h_r(\tau,x,y,\eta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{it\tau}\beta(t)\rho(rt)(1-\chi(\eta))\widetilde{g}(rt,x,y,\eta/r)dt \tag{2.52}$$

is the inverse Fourier transform of the compactly supported smooth function $t \mapsto \beta(t)\rho(rt)(1-\chi(\eta))\widetilde{g}(rt,x,y,\eta/r)$.

We have

$$|\partial_\eta^\gamma h_r(\tau,x,y,\eta)| \leq C_{N,\gamma}(1+|\tau|)^{-N}(1+|\eta|)^{-|\gamma|}, \tag{2.53}$$

uniformly in r, for all $N > 0$ and $\gamma \in \mathbb{N}_0^n$. This can be seen by using (2.39) in the case $|\tau| \leq 1$, and by integrating by parts N times in (2.52) and using (2.39) in the case $|\tau| \geq 1$.

We write

$$\Big(\frac{\widetilde{q}(x,y,\eta)-\tau}{r}\Big)^m - z^m = \prod_{k=0}^{m-1}\Big(\frac{\widetilde{q}(x,y,\eta)-\tau}{r} - ze^{2\pi ki/m}\Big),$$

and using a partial fraction decomposition, we get

$$\frac{1}{(\frac{\widetilde{q}(x,y,\eta)-\tau}{r})^m - z^m} = \frac{r}{z^{m-1}}\sum_{k=0}^{m-1}\frac{A_k}{\widetilde{q}(x,y,\eta)-\tau-rze^{2\pi ki/m}},$$

where

$$A_k = \Big(\prod_{\substack{l=0\\ l\neq k}}^{m-1}(e^{2\pi ki/m}-e^{2\pi li/m})\Big)^{-1}.$$

Thus, it follows from (2.51) that

$$K_{z,j}^{(1,2)}(x,y) = \frac{r^{1-n}}{(2\pi)^n z^{m-1}}\sum_{k=0}^{m-1}A_k\int_{-\infty}^{+\infty}\int_{\mathbb{R}^n}\frac{h_r(\tau,x,y,\eta)e^{i\langle\frac{x-y}{r},\eta\rangle}}{\widetilde{q}(x,y,\eta)-(\tau+rze^{2\pi ki/m})}d\eta d\tau. \tag{2.54}$$

Recalling that $\arg(z)\in(0,2\pi/m)$, we see that $\tau+rze^{2\pi ki/m}$ avoids the real axis, for $k=0,\dots,m-1$. To proceed further, we shall need the following result, similar to [1, Proposition 2.4].

Lemma 2.8. *Let $n\geq 2$ and let $h\in C^\infty(\mathbb{R}^n\setminus\{0\})$ satisfy the Mihlin-type condition,*

$$|\partial_\xi^\alpha h(\xi)|\leq H_\alpha|\xi|^{-|\alpha|},\quad \xi\neq 0,\quad \alpha\in\mathbb{N}_0^n. \tag{2.55}$$

Let $a\in C^\infty(\mathbb{R}^n\setminus\{0\})$ be homogeneous of degree one. Assume that $a(\xi)>0$ for all $\xi\in\mathbb{R}^n\setminus\{0\}$ and that the cosphere $\Sigma=\{\xi\in\mathbb{R}^n : a(\xi)=1\}$ is strictly convex. Then there is a constant $C>0$ such that for all $x\in\mathbb{R}^n$, $x\neq 0$, and all $w\in\mathbb{C}\setminus[0,\infty)$, we have

$$\Big|\int_{\mathbb{R}^n}\frac{h(\xi)e^{i\langle x,\xi\rangle}}{a(\xi)-w}d\xi\Big|\leq C(|x|^{1-n}+(|w|/|x|)^{\frac{n-1}{2}}). \tag{2.56}$$

Proof. First notice that since $a\in C^\infty(\mathbb{R}^n\setminus\{0\})$ is homogeneous of degree one, we have

$$|\partial_\xi^\alpha a(\xi)|\leq A_\alpha|\xi|^{1-|\alpha|},\qquad \xi\neq 0,\quad \alpha\in\mathbb{N}_0^n.$$

Let $b\in C^\infty(\mathbb{R}^n\setminus\{0\})$ be such that

$$|\partial_\xi^\alpha b(\xi)|\leq B_\alpha|\xi|^{-1-|\alpha|},\quad \xi\neq 0,\quad \alpha\in\mathbb{N}_0^n.$$

Then it follows from [16, p. 245] that the Fourier transform of $b(\xi)$ satisfies

$$\Big|\int_{\mathbb{R}^n}b(\xi)e^{-i\langle x,\xi\rangle}d\xi\Big|\leq C|x|^{1-n},\quad x\neq 0. \tag{2.57}$$

Assume first that w is outside of a small but fixed conic neighborhood of the positive real axis $[0,\infty)$, i.e. $\arg w\in[\theta,2\pi-\theta]$ for some $\theta>0$ small but fixed, and $|w| = 1$. Let us establish that

$$b_w(\xi) = \frac{h(\xi)}{a(\xi) - w} \in C^\infty(\mathbb{R}^n \setminus \{0\}),$$

satisfies

$$|\partial_\xi^\alpha b_w(\xi)| \le B_\alpha |\xi|^{-1-|\alpha|}, \quad \xi \ne 0, \quad \alpha \in \mathbb{N}_0^n, \tag{2.58}$$

uniformly in w.

To that end, let us show that

$$|a(\xi) - w| \ge \frac{1}{C_\theta}(|\xi| + 1), \tag{2.59}$$

with a constant $C_\theta > 0$ uniformly in w. When doing so, we notice there is a constant $\delta > 0$ such that $a(\xi) \ge \delta|\xi|$, and then (2.59) follows for all $|\xi|$ large enough. It remains to consider the case when $|\xi|$ is bounded. Then if $\arg w \in [\theta, \pi - \theta] \cup [\pi + \theta, 2\pi - \theta]$, we get

$$|a(\xi) - w| \ge |\mathrm{Im}(w)| \ge \frac{1}{C_\theta}.$$

If $\arg w \in (\pi - \theta, \pi + \theta)$, we write $\arg w = \pi + \mathcal{O}(\theta)$. Then $w = -1 - \mathcal{O}(\theta)$, and therefore,

$$|a(\xi) - w| = |a(\xi) + 1 + \mathcal{O}(\theta)| \ge \frac{1}{2},$$

for θ small enough. The bound (2.59) follows.

By the Leibniz formula we write

$$\partial_\xi^\alpha(b_w(\xi)) = \sum_{\beta+\gamma=\alpha} C_{\beta,\gamma} \partial_\xi^\beta(h(\xi)) \partial_\xi^\gamma \left(\frac{1}{a(\xi) - w}\right), \tag{2.60}$$

with constants $C_{\beta,\gamma}$. It follows from the Faà di Bruno formula (2.44) and (2.59) that for $|\gamma| \ge 0$,

$$\left|\partial_\xi^\gamma \left(\frac{1}{a(\xi) - w}\right)\right| \le C_{\gamma,\theta} |\xi|^{-1-|\gamma|}, \quad \xi \ne 0, \tag{2.61}$$

uniformly in w. Hence, we conclude from (2.60), with the help of (2.55) and (2.61), that (2.58) holds.

Thus, applying (2.57) for b_w, we obtain that

$$\left| \int_{\mathbb{R}^n} \frac{h(\xi) e^{i\langle x, \xi\rangle}}{a(\xi) - w} d\xi \right| \le C|x|^{1-n}, \quad x \ne 0, \tag{2.62}$$

uniformly in $w \in \mathbb{C}$, $\arg w \in [\theta, 2\pi - \theta]$ with $\theta > 0$ small but fixed, and $|w| = 1$.

Assume now that $w\in\mathbb{C}$, $\arg w\in[\theta,2\pi-\theta]$ with $\theta>0$ small but fixed, and $|w|\ne 1$. Letting $\widetilde{w}=w/|w|$, we have

$$\int_{\mathbb{R}^n}\frac{h(\xi)e^{i\langle x,\xi\rangle}}{a(\xi)-w}d\xi=\frac{1}{|w|}\int_{\mathbb{R}^n}\frac{h(\xi)e^{i\langle x,\xi\rangle}}{a(\xi/|w|)-\widetilde{w}}d\xi=|w|^{n-1}\int_{\mathbb{R}^n}\frac{h(|w|\xi)e^{i\langle |w|x,\xi\rangle}}{a(\xi)-\widetilde{w}}d\xi.$$

Since the dilate $h(|w|\xi)$ of $h(\xi)$ satisfies exactly the same bounds as in (2.55), as above, we obtain the uniform estimate (2.62), for all $w\in\mathbb{C}$, $\arg w\in[\theta,2\pi-\theta]$ with $\theta>0$ small but fixed.

Assume now that $w\in\mathbb{C}\setminus[0,\infty)$, $\arg w\in(-\theta,\theta)$ with $\theta>0$ small but fixed, and $|w|=1$. Then $w=1+\mathcal{O}(\theta)$, and therefore,

$$|a(\xi)-w|=|a(\xi)-1-\mathcal{O}(\theta)|\ge\frac{1}{C},$$

for $\xi\notin a^{-1}([1/2,2])$, uniformly in w. Hence, letting $0\le\chi\in C_0^\infty((0,\infty))$ be such that $\chi(t)=1$ when $t\in[1/2,2]$ and $\operatorname{supp}(\chi)\subset[1/4,4]$, by the above argument, we conclude that

$$b_w(\xi):=\frac{h(\xi)(1-\chi(a(\xi)))}{a(\xi)-w}$$

satisfies the bound (2.58) uniformly in w. Therefore,

$$\left|\int_{\mathbb{R}^n}\frac{h(\xi)(1-\chi(a(\xi)))e^{i\langle x,\xi\rangle}}{a(\xi)-w}d\xi\right|\le C|x|^{1-n},$$

uniformly in $w\in\mathbb{C}\setminus[0,\infty)$, $\arg w\in(-\theta,\theta)$ with $\theta>0$ small but fixed, and $|w|=1$.

Let us now write,

$$I(x)=\int_{\mathbb{R}^n}\frac{h(\xi)\chi(a(\xi))e^{i\langle x,\xi\rangle}}{a(\xi)-w}d\xi=I_1(x)+I_2(x), \tag{2.63}$$

where

$$I_1(x):=\int_{\mathbb{R}^n}\frac{h(\xi)\chi(a(\xi))(a(\xi)-w_1)e^{i\langle x,\xi\rangle}}{(a(\xi)-w_1)^2+w_2^2}d\xi,\ I_2(x)=\int_{\mathbb{R}^n}\frac{ih(\xi)\chi(a(\xi))w_2e^{i\langle x,\xi\rangle}}{(a(\xi)-w_1)^2+w_2^2}d\xi.$$

Here $w_1=\operatorname{Re}w=1+\mathcal{O}(\mu^2)$, $w_2=\operatorname{Im}w=\mu+\mathcal{O}(\mu^2)$, and $\mu:=\arg w$, $|\mu|$ small.

Using the coarea formula in the integral $I_2(x)$, we get

$$\begin{aligned}|I_2(x)|&\le C|w_2|\int_{a^{-1}([1/4,4])}\frac{d\xi}{(a(\xi)-w_1)^2+w_2^2}\\&=C|w_2|\int_{1/4}^4\int_{a(\xi)=E}\frac{dS_E}{|\nabla_\xi a(\xi)|}\frac{dE}{(E-w_1)^2+w_2^2},\end{aligned} \tag{2.64}$$

where dS_E is the Lebesque measure on the surface $a(\xi)=E$.

Let us notice that by Euler homogeneity relations for $a(\xi)=E$, we have

$$|\nabla_\xi a(\xi)|\ge 1/C,$$

uniformly in $E \in [1/4, 4]$. Therefore,

$$|I_2(x)| \leq C|w_2| \int_{1/4}^{4} \frac{dE}{(E-w_1)^2 + w_2^2} \leq C|w_2| \int_{-\infty}^{+\infty} \frac{dE}{E^2 + w_2^2} \leq C, \tag{2.65}$$

uniformly in μ.

Appealing to the coarea formula in the integral $I_1(x)$, we get

$$\begin{aligned} I_1(x) &= \int_{a^{-1}([1/4,4])} \frac{h(\xi)\chi(a(\xi))(a(\xi)-w_1)e^{i\langle x,\xi\rangle}}{(a(\xi)-w_1)^2 + w_2^2} d\xi \\ &= \int_{1/4}^{4} \frac{(E-w_1)}{(E-w_1)^2+w_2^2} J(E,x)dE, \end{aligned} \tag{2.66}$$

where

$$J(E,x) = \chi(E) \int_{a(\xi)=E} \frac{h(\xi)e^{i\langle x,\xi\rangle}}{|\nabla_\xi a(\xi)|} dS_E = E^{n-1}\chi(E) \int_{a(\xi)=1} \frac{h(E\xi)e^{i\langle x,E\xi\rangle}}{|\nabla_\xi a(\xi)|} dS_1.$$

We see that $J(E,x)$ is C^∞ in E, x. Making the change of variables $E \mapsto E - w_1$ in (2.66), we get

$$\begin{aligned} I_1(x) &= \left(\int_{1/4-w_1}^{0} + \int_{0}^{w_1-1/4} + \int_{w_1-1/4}^{4-w_1} \right) \frac{E}{E^2+w_2^2} J(E+w_1,x)dE \\ &= \int_0^{w_1-1/4} \frac{E(J(E+w_1,x) - J(-E+w_1,x))}{E^2+w_2^2} dE \\ &+ \int_{w_1-1/4}^{4-w_1} \frac{E}{E^2+w_2^2} J(E+w_1,x)dE. \end{aligned}$$

As $f(E) = J(E+w_1,x) - J(-E+w_1,x)$ is C^∞ in E, w_1, and x, and $f(0) = 0$, it follows that $f(E) = Eg(E)$ with a function g which is C^∞ in E, w_1, and x. Hence, recalling that $w_1 = 1 + \mathcal{O}(\mu^2)$, for $|x| \leq 1$, we get

$$|I_1(x)| \leq C \int_0^2 \frac{E^2}{E^2+w_2^2} dE + C \int_{1/4}^{4} \frac{1}{E} dE \leq C, \tag{2.67}$$

uniformly in μ with $0 < |\mu| \leq \theta$, where θ is sufficiently small.

We conclude from (2.63), (2.65) and (2.67) that

$$|I(x)| \leq C,$$

for $|x| \leq 1$, uniformly in μ with $0 < |\mu| \leq \theta$, where θ is sufficiently small.

Let us now show that when $|x| \geq 1$, we get

$$|I(x)| \leq C|x|^{-\frac{(n-1)}{2}}, \tag{2.68}$$

uniformly in μ. First using the coarea formula in (2.63), we get

$$\begin{aligned} I(x) &= \int_{1/4}^{4}\int_{a(\xi)=E}\frac{h(\xi)\chi(E)e^{i\langle x,\xi\rangle}}{(E-w)}\frac{dS_E}{|\nabla_\xi a(\xi)|}dE\\ &= \int_{1/4}^{4}\frac{E^{n-1}\chi(E)}{E-w}\int_{a(\xi)=1}\frac{h(E\xi)}{|\nabla_\xi a(\xi)|}e^{i\langle Ex,\xi\rangle}dS_1 dE. \end{aligned}$$

To proceed recall that $a(\xi)$ is homogeneous of degree one, C^∞ for $\xi\neq 0$, and $a(\xi)>0$ on $\mathbb{R}^n\setminus\{0\}$. Then $\nabla_\xi a\neq 0$ along the cosphere $\Sigma=\{\xi\in\mathbb{R}^n: a(\xi)=1\}$, which is therefore is a C^∞ compact hypersurface. Furthermore, Σ is homeomorphic to the sphere $\mathbb{S}^{n-1}$ via the homeomorphism $\mathbb{S}^{n-1}\to\Sigma$, $\omega\mapsto\omega/a(\omega)$. Hence, Σ is connected. The assumption that the Gaussian curvature of Σ never vanishes implies that the Gauss map is a diffeomorphism from Σ to $\mathbb{S}^{n-1}$. Thus, given $x\in\mathbb{R}^n\setminus\{0\}$, there are exactly two points $\xi_1(x),\xi_2(x)\in\Sigma$ with normal x. Since $\xi_1(x)$, $\xi_2(x)$, are homogeneous of degree zero and smooth in $\mathbb{R}^n\setminus\{0\}$, it follows that the functions $\langle x,\xi_1(x)\rangle$, $\langle x,\xi_2(x)\rangle$ are also smooth for $x\neq 0$ and homogeneous of degree one.

We shall need the following result concerning the inverse Fourier transform of a smooth measure carried by the cosphere Σ, which is an application of the stationary phase theorem, see [14, Theorem 1.2.1, p. 50] and [14, p. 68].

Lemma 2.9. *Let $d\sigma(\xi)=\beta(\xi)dS(\xi)$ with $\beta\in C^\infty(\Sigma)$ and dS is the surface measure on Σ. Then under the above assumptions, the inverse Fourier transform of the measure $d\sigma$ satisfies*

$$(2\pi)^{-n}\int_\Sigma e^{i\langle x,\xi\rangle}d\sigma(\xi)=\frac{b_1(x)e^{i\langle x,\xi_1(x)\rangle}}{|x|^{(n-1)/2}}+\frac{b_2(x)e^{i\langle x,\xi_2(x)\rangle}}{|x|^{(n-1)/2}},\quad |x|\geq 1,$$

where the functions b_j are such that

$$|\partial_x^\alpha b_j(x)|\leq C_\alpha|x|^{-|\alpha|},\quad |x|\geq 1,\quad \alpha\in\mathbb{N}_0^n.$$

As $\xi_j(x)$ is homogeneous of degree zero, by Lemma 2.9, for $|x|\geq 1$, we get

$$I(x)=(2\pi)^n|x|^{-\frac{(n-1)}{2}}\sum_{j=1}^{2}\int_{1/4}^{4}\frac{E^{(n-1)/2}\chi(E)b_j(x,E)}{E-w}e^{iE\langle x,\xi_j(x)\rangle}dE,$$

with some functions $b_j\in C^\infty$ for $|x|\geq 1$ and $E\in[1/4,4]$, and

$$|\partial_E^l\partial_x^\alpha b_j(x,E)|\leq C_{l,\alpha}|x|^{-|\alpha|},\quad |x|\geq 1,\quad E\in[1/4,4],\quad l\in\mathbb{N}_0,\quad \alpha\in\mathbb{N}_0^n. \tag{2.69}$$

The estimate (2.68) would follow if we could show that

$$\left|\int_{1/4}^{4}\frac{E^{(n-1)/2}\chi(E)b_j(x,E)}{E-w}e^{iE\langle x,\xi_j(x)\rangle}dE\right|\leq C, \tag{2.70}$$

uniformly in μ, $0 < |\mu| \leq \theta \ll 1$. To show (2.70), we let

$$f(E,x) = E^{(n-1)/2}\chi(E)b_j(x,E), \quad \varphi(x) = \langle x, \xi_j(x)\rangle.$$

For $|x| \geq 1$, the function $f(\cdot, x)$ is C^∞ with compact support in $E \in [1/4, 4]$, and (2.69) yields that

$$|\partial_E^l f(E,x)| \leq C_l. \tag{2.71}$$

We write

$$J(x) = \int_{1/4}^4 \frac{f(E,x)e^{iE\varphi(x)}}{E-w}dE = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\widehat{f}(t,x)\int_{-\infty}^{+\infty}\frac{e^{iE(t+\varphi(x))}}{E-w_1-iw_2}dEdt$$
$$= -\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\widehat{f}(t,x)e^{iw_1(t+\varphi(x))}\int_{-\infty}^{+\infty}\frac{e^{-i\tau(t+\varphi(x))}}{w_2-i\tau}d\tau dt,$$

where $\widehat{f}(t,x)$ is the Fourier transform of $f(E,x)$. We shall use the following fact: for all $\alpha \in \mathbb{R}$, $\alpha \neq 0$,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-i\tau t}}{\alpha - i\tau}d\tau = \text{sgn}\alpha H(\alpha t)e^{-\alpha t},$$

where $H(t)$ is the Heaviside function which equals one for $t \geq 0$ and zero for $t < 0$, see [1, Lemma 2.1]. As $w_2 \neq 0$, we get

$$J(x) = \int_{-\infty}^{+\infty}\widehat{f}(t,x)ie^{iw_1(t+\varphi(x))}\text{sgn}(w_2)H(w_2(t+\varphi(x)))e^{-w_2(t+\varphi(x))}dt,$$

and therefore, using that f has compact support in E and (2.71), we obtain that

$$|J(x)| \leq C\int_{-\infty}^{+\infty}|\widehat{f}(t,x)|dt \leq C\|(1+t^2)\widehat{f}(t,x)\|_{L_t^\infty}$$
$$\leq C(\|f(E,x)\|_{L_E^1} + \|\partial_E^2 f(E,x)\|_{L_E^1}) \leq C,$$

uniformly in w. This establishes (2.70), and hence, (2.68). Thus, for $w \in \mathbb{C}\setminus[0,\infty)$, $\arg w \in (-\theta,\theta)$, $\theta > 0$ small but fixed, and $|w| = 1$, we get

$$\left|\int_{\mathbb{R}^n}\frac{h(\xi)e^{i\langle x,\xi\rangle}}{a(\xi)-w}d\xi\right| \leq C(|x|^{1-n} + |x|^{-\frac{(n-1)}{2}}), \quad x \neq 0, \tag{2.72}$$

uniformly in w. In the case when $w \in \mathbb{C}\setminus[0,\infty)$, $\arg w \in (-\theta,\theta)$, $\theta > 0$ small but fixed, and $|w| \neq 1$, the estimate (2.56) follows from (2.72) by a change of scale. The proof of Lemma 2.8 is complete. $\square$

Now using Lemma 2.8, the estimate (2.53), and the fact that $\frac{|x-y|}{r} \in [A^{-1}/4, 4A]$, we obtain that

$$\left|\int_{\mathbb{R}^n}\frac{h_r(\tau,x,y,\eta)e^{i\langle\frac{x-y}{r},\eta\rangle}}{\widetilde{q}(x,y,\eta)-(\tau+rze^{2\pi ki/m})}d\eta\right| \leq C_N(1+|\tau|)^{-N}(1+|\tau|+r|z|)^{\frac{n-1}{2}}, \tag{2.73}$$

for $k=0,1,\dots,m-1$ and $N>0$. It follows from (2.54) and (2.73) that for $N>0$ sufficiently large,

$$
\begin{aligned}
|K^{(1,2)}_{z,j}(x,y)| &\le C\frac{r^{1-n}}{|z|^{m-1}}\int_{-\infty}^{+\infty}(1+|\tau|)^{-N+\frac{n-1}{2}}(1+r|z|)^{\frac{n-1}{2}}d\tau\\
&\le Cr^{-\frac{(n-1)}{2}}|z|^{\frac{n+1-2m}{2}}.
\end{aligned}
$$

Here we have used that $r|z|\ge 1$. Recalling that $r=2^j/|z|$, the above estimate completes the proof of the estimate (2.26), and therefore, the estimates (2.25) and (2.18). As $\sum_{j=0}^\infty 2^{j\frac{2n-m-nm}{2n}}=1/(1-2^{\frac{2n-m-nm}{2n}})$, we have obtained the (2.14) for the local operator.

2.5. **Uniform estimate for the non-local operator in the case of unbounded $|z|$.** Let $\tau\in\mathbb{R}$ and consider the multipliers

$$
r_z(\tau)=m_z(\tau)-m_z^{\text{loc}}(\tau)=\frac{i}{mz^{m-1}}\sum_{k=0}^{m/2-1}e^{2\pi ki/m}\int_{-\infty}^{+\infty}(1-\rho(t))e^{i|t|\tau_k}e^{it\tau}dt, \quad (2.74)
$$

for all $z\in\Xi_\delta\cap\{z\in\mathbb{C}:|z|\ge 1\}$.

In order to prove (1.5), we are left with establishing that

$$
\|r_z(Q)f\|_{L^{\frac{2n}{n-m}}(M)}\le C\|f\|_{L^{\frac{2n}{n+m}}(M)}, \quad (2.75)
$$

for all $z\in\Xi_\delta\cap\{z\in\mathbb{C}:|z|\ge 1\}$, uniformly in z.

Let us first show that $r_z(\tau)$ is bounded for all $z\in\Xi_\delta\cap\{z\in\mathbb{C}:|z|\ge 1\}$, uniformly in z. Indeed, we have

$$
|r_z(\tau)|\le\frac{C}{|z|^{m-1}}\sum_{k=0}^{m/2-1}\bigg(\int_{-\infty}^{-\varepsilon/2}e^{t\text{Im}\tau_k}dt+\int_{\varepsilon/2}^{+\infty}e^{-t\text{Im}\tau_k}dt\bigg)\le C\sum_{k=0}^{m/2-1}\frac{1}{\text{Im}\tau_k}. \quad (2.76)
$$

Recall that $\tau_k=ze^{2\pi ki/m}$, and therefore, $0<\arg(\tau_k)<\pi$, $k=0,\dots,m/2-1$. If now $0<\arg(\tau_k)\le\pi/2$, then

$$
\frac{\text{Im}\tau_k}{|z|}=\sin(\arg(\tau_k))\ge\sin(\arg(z)),
$$

and thus, using the fact that $z\in\Xi_\delta$, we get

$$
\text{Im}\tau_k\ge\text{Im}z\ge\delta. \quad (2.77)
$$

If $\pi/2<\arg(\tau_k)<\pi$, then

$$
\frac{\text{Im}\tau_k}{|z|}=\sin(\pi-\arg(\tau_k))\ge\sin(\pi-\arg(\tau_{m/2-1}))=-\sin(\arg(z)-2\pi/m),
$$

and therefore,

$$
\text{Im}\tau_k\ge-\text{Im}(ze^{-2\pi i/m})\ge\delta. \quad (2.78)
$$

Hence, it follows from (2.76), (2.77) and (2.78) that

$$|r_z(\tau)| \le C\delta^{-1}, \tag{2.79}$$

for all $z \in \Xi_\delta \cap \{z \in \mathbb{C} : |z| \ge 1\}$, uniformly in z.

To obtain the decay of $r_z(\tau)$, let us integrate by parts N times, $N = 1, 2, \dots$, in (2.74). We have

$$\begin{aligned} r_z(\tau) = \frac{i}{mz^{m-1}} \sum_{k=0}^{m/2-1} e^{2\pi ki/m} \bigg(& \frac{(-1)^N}{i^N(-\tau_k+\tau)^N} \int_{-\infty}^0 (-\partial_t^N \rho(t)) e^{it(-\tau_k+\tau)} dt \\ & + \frac{(-1)^N}{i^N(\tau_k+\tau)^N} \int_0^{+\infty} (-\partial_t^N \rho(t)) e^{it(\tau_k+\tau)} dt \bigg). \end{aligned}$$

Notice that all the boundary terms disappear when integrating by parts due to the presence of the term $(1 - \rho(t))$ in (2.74) and the fact that $\mathrm{Im}\tau_k > 0$. As

$$\begin{aligned} |\pm \tau_k + \tau| = \sqrt{|\pm \operatorname{Re}\tau_k + \tau|^2 + |\operatorname{Im}\tau_k|^2} &\ge \sqrt{|\pm \operatorname{Re}\tau_k + \tau|^2 + \delta^2} \\ &\ge \frac{\delta}{\sqrt{2}}(1 + |\pm \operatorname{Re}\tau_k + \tau|), \end{aligned}$$

where $\delta < 1$, we obtain that

$$|r_z(\tau)| \le \frac{C}{|z|^{m-1}} \sum_{k=0}^{m/2-1} ((1 + |-\operatorname{Re}\tau_k + \tau|)^{-N} + (1 + |\operatorname{Re}\tau_k + \tau|)^{-N}),$$

uniformly in z. Thus, for $\tau \ge 0$, we get

$$|r_z(\tau)| \le \frac{C}{|z|^{m-1}} \Bigg(\sum_{\substack{k=0,\dots,m/2-1 \\ \operatorname{Re}\tau_k \ge 0}} (1+|-\operatorname{Re}\tau_k+\tau|)^{-N} + \sum_{\substack{k=0,\dots,m/2-1 \\ \operatorname{Re}\tau_k < 0}} (1+|\operatorname{Re}\tau_k+\tau|)^{-N} \Bigg) \tag{2.80}$$

We have

$$r_z(Q)f = \sum_{j=1}^\infty r_z(\mu_j)E_j f = \sum_{l=1}^\infty r_z^l(Q)f, \quad f \in C^\infty(M), \tag{2.81}$$

where

$$r_z^l(Q)f = \sum_{\mu_j \in [l-1,l)} r_z(\mu_j)E_j f, \quad l = 1, 2, \dots.$$

Using Lemma 2.2 and (2.80) with $N=m+1$, we obtain that

$$\|r_z^l(Q)f\|_{L^{\frac{2n}{n-m}}(M)}\le Cl^{m-1}(\sup_{\tau\in[l-1,l)}|r_z(\tau)|)\|f\|_{L^{\frac{2n}{n+m}}(M)}\le \frac{Cl^{m-1}}{|z|^{m-1}}$$
$$\Big(\sum_{\substack{k=0\\ \mathrm{Re}\tau_k\ge 0}}^{m/2-1}\frac{1}{(1+|-\mathrm{Re}\,\tau_k+l|)^{m+1}}+\sum_{\substack{k=0\\ \mathrm{Re}\tau_k<0}}^{m/2-1}\frac{1}{(1+|\mathrm{Re}\,\tau_k+l|)^{m+1}}\Big)\|f\|_{L^{\frac{2n}{n+m}}(M)}. \tag{2.82}$$

Here we have used the fact that for $l-1\le\tau\le l$, we have

$$|\pm\mathrm{Re}\,\tau_k+l|\le|\pm\mathrm{Re}\,\tau_k+\tau|+|l-\tau|\le|\pm\mathrm{Re}\,\tau_k+\tau|+1.$$

Hence, (2.75) would follow from (2.81) and (2.82), if we could show that

$$\Sigma:=\frac{1}{|z|^{m-1}}\sum_{l=1}^{\infty}\frac{l^{m-1}}{(1+|-a+l|)^{m+1}}\le C,\quad a=|\mathrm{Re}\,\tau_k|, \tag{2.83}$$

with some constant $C>0$ uniform in $z\in\mathbb{C}$, $|z|\ge 1$.

Let us now show (2.83). Assume first that $a\le 1$. Then

$$\Sigma=\frac{1}{|z|^{m-1}}\sum_{l=1}^{\infty}\frac{l^{m-1}}{(1-a+l)^{m+1}}\le\frac{1}{|z|^{m-1}}\sum_{l=1}^{\infty}\frac{1}{l^2}\le C,$$

with a constant $C>0$ uniform in $z\in\mathbb{C}$, $|z|\ge1$. Consider now the case $a>1$. Then denoting $[a]$ the integer part of a, we write

$$\Sigma=\Sigma_1+\Sigma_2+\Sigma_3,$$

where

$$\Sigma_1:=\frac{1}{|z|^{m-1}}\sum_{l\le[a]-1}\frac{l^{m-1}}{(1+a-l)^{m+1}},$$
$$\Sigma_2:=\frac{1}{|z|^{m-1}}\Big(\frac{[a]^{m-1}}{(1+|-a+[a]|)^{m+1}}+\frac{([a]+1)^{m-1}}{(1+|-a+[a]+1|)^{m+1}}\Big),$$
$$\Sigma_3:=\frac{1}{|z|^{m-1}}\sum_{l\ge[a]+2}\frac{l^{m-1}}{(1-a+l)^{m+1}}.$$

Using the fact that $a\le|z|$, we see that $\Sigma_2\le C$, uniformly in $z\in\mathbb{C}$, $|z|\ge1$.

We shall next estimate Σ_3. As the function $t^{m-1}/(1-a+t)^{m+1}$ is decreasing for $t>0$, we get

$$\Sigma_3\le\frac{1}{|z|^{m-1}}\int_{[a]+1}^{+\infty}\frac{t^{m-1}}{(1-a+t)^{m+1}}dt=\frac{1}{|z|^{m-1}}\int_{2+[a]-a}^{+\infty}\frac{(t+a-1)^{m-1}}{t^{m+1}}dt$$
$$\le\frac{C_m}{|z|^{m-1}}\Big(\int_1^{+\infty}\frac{dt}{t^2}+(a-1)^{m-1}\int_1^{+\infty}\frac{dt}{t^{m+1}}\Big)\le C,$$

uniformly in $z \in \mathbb{C}$, $|z| \geq 1$.

Let us now estimate Σ_1. Since the function $t^{m-1}/(1+a-t)^{m+1}$ is increasing for $t > 0$, we obtain that

$$\begin{aligned}\Sigma_1 \leq \frac{1}{|z|^{m-1}} \int_1^{[a]} \frac{t^{m-1}}{(1+a-t)^{m+1}} dt \leq \frac{1}{|z|^{m-1}} \int_{1+a-[a]}^{a} \frac{|1+a-t|^{m-1}}{t^{m+1}} dt \\ \leq \frac{C_m}{|z|^{m-1}} \left((1+a)^{m-1} \int_1^{+\infty} \frac{dt}{t^{m+1}} + \int_1^{+\infty} \frac{dt}{t^2} \right) \leq C,\end{aligned}$$

uniformly in $z \in \mathbb{C}$, $|z| \geq 1$. This completes the proof of (2.83) and hence, of Theorem 1.1.

Finally let us remark that the a priori estimate (1.5) implies the following simple result concerning the L^2 resolvent of P, $(P-\zeta)^{-1}$.

Proposition 2.10. *Let $\zeta \in \mathbb{C} \setminus [0,\infty)$. Then the resolvent $(P-\zeta)^{-1}$ is a bounded operator: $L^{\frac{2n}{n+m}}(M) \to L^{\frac{2n}{n-m}}(M)$.*

Proof. Let $\zeta \notin \{\lambda_1, \lambda_2, \dots\}$ so that $(P-\zeta)^{-1} : L^2(M) \to L^2(M)$ is bounded. By elliptic regularity, we have $(P-\zeta)^{-1} C^\infty(M) \subset C^\infty(M)$, and therefore, the linear continuous operator $P - \zeta : C^\infty(M) \to C^\infty(M)$ is bijective. By the open mapping theorem, $(P-\zeta)^{-1} : C^\infty(M) \to C^\infty(M)$ is continuous.

We have next the linear continuous map $P - \zeta : \mathcal{D}'(M) \to \mathcal{D}'(M)$ given by

$$\langle (P-\zeta)u, \varphi \rangle = \langle u, \overline{(P-\overline{\zeta})\overline{\varphi}} \rangle, \quad \varphi \in C^\infty(M),$$

which is bijective, with continuous inverse $(P-\zeta)^{-1} : \mathcal{D}'(M) \to \mathcal{D}'(M)$.

By Remark 2.4, when $\zeta \in \mathbb{C} \setminus [0,\infty)$, we have the following a priori estimate

$$\|u\|_{L^{\frac{2n}{n-m}}(M)} \leq C_\zeta \|(P-\zeta)u\|_{L^{\frac{2n}{n+m}}(M)},$$

for all $u \in C^\infty(M)$. Thus, for any $f \in C^\infty(M)$, we get

$$\|(P-\zeta)^{-1} f\|_{L^{\frac{2n}{n-m}}(M)} \leq C_\zeta \|f\|_{L^{\frac{2n}{n+m}}(M)}. \tag{2.84}$$

Now let $f \in L^{\frac{2n}{n+m}}(M)$. Then there is a sequence $f_j \in C^\infty(M)$, converging to f in $L^{\frac{2n}{n+m}}(M)$ as $j \to \infty$. It follows from (2.84) that $(P-\zeta)^{-1} f_j$ is a Cauchy sequence in $L^{\frac{2n}{n-m}}(M)$, and therefore, it converges in $L^{\frac{2n}{n-m}}(M)$. As $(P-\zeta)^{-1} : \mathcal{D}'(M) \to \mathcal{D}'(M)$ continuous, we have $(P-\zeta)^{-1} f \in L^{\frac{2n}{n-m}}(M)$ and $(P-\zeta)^{-1} f_j$ converges to $(P-\zeta)^{-1} f$ in $L^{\frac{2n}{n-m}}(M)$ as $j \to \infty$. Hence, (2.84) is valid for any $f \in L^{\frac{2n}{n+m}}(M)$, which shows the claim of Proposition 2.10. □

3. Saturation of the resolvent estimates. Proof of Theorem 1.2

We shall need the following Bernstein type inequality, established in [1, Lemma 3.1].

Lemma 3.1. *Let $\beta \in C_0^\infty(\mathbb{R})$ be such that $0 \notin supp\,(\beta)$. Then if $1 \le q \le r \le \infty$, there is a constant $C = C(r, q)$ so that*

$$\|\beta(Q/\alpha)f\|_{L^r(M)} \le C\alpha^{n(\frac{1}{q}-\frac{1}{r})}\|f\|_{L^q(M)}, \quad \alpha \ge 1.$$

In Theorem 1.1 we obtained the uniform estimate (1.5) for all z in the sector Ξ of the complex plane such that $\text{dist}(\partial\Xi, z) \ge \delta$ for some $\delta > 0$. The next result shows that removing the eigenvalues of the operator $Q = P^{1/m}$ in some interval $[\alpha - 1, \alpha + 1]$ allows us to obtain the uniform estimate (1.5) for all $z \in \Xi$ with $\text{Re}\, z = \alpha \gg 1$ or $\text{Re}\,(ze^{-2\pi i/m}) = \alpha \gg 1$.

Lemma 3.2. *Let*

$$\chi_{[\alpha-1,\alpha+1)}f = \sum_{\mu_j \in [\alpha-1,\alpha+1)} E_j f.$$

Then we have the uniform estimate:

$$\|(I - \chi_{[\alpha-1,\alpha+1)}) \circ (P - z^m)^{-1} f\|_{L^{\frac{2n}{n-m}}(M)} \le C\|f\|_{L^{\frac{2n}{n+m}}(M)}, \tag{3.1}$$

with $z \in \Xi$, $\text{Re}\ z = \alpha \gg 1$, and $0 < \text{Im}\ z \le 1$, and the uniform estimate:

$$\|(I - \chi_{[\alpha-1,\alpha+1)}) \circ (P - z^m)^{-1} f\|_{L^{\frac{2n}{n-m}}(M)} \le C\|f\|_{L^{\frac{2n}{n+m}}(M)}, \tag{3.2}$$

with $z \in \Xi$, $\text{Re}\ (ze^{-2\pi i/m}) = \alpha \gg 1$, and $0 < -\text{Im}\ (ze^{-2\pi i/m}) \le 1$.

Proof. Let us start by proving (3.1). Let $z \in \Xi$, $\text{Re}\ z = \alpha \gg 1$, and assume first that $\delta \le \text{Im}\ z = \beta \le 1$ for some $\delta > 0$. We write

$$\chi_{[\alpha-1,\alpha+1)} \circ (P - z^m)^{-1} f = \sum_{\mu_j \in [\alpha-1,\alpha+1)} (\mu_j^m - z^m)^{-1} E_j f.$$

By (2.5), we get

$$\|\chi_{[\alpha-1,\alpha+1)} \circ (P - z^m)^{-1} f\|_{L^{\frac{2n}{n-m}}(M)} \le C\alpha^{m-1}(\sup_{\tau \in [\alpha-1,\alpha+1)} |(\tau^m - z^m)^{-1}|)\|f\|_{L^{\frac{2n}{n+m}}(M)}, \tag{3.3}$$

Writing

$$z^m = (\alpha + i\beta)^m = \alpha^m(1 + mi\beta/\alpha + \mathcal{O}(\beta^2/\alpha^2)),$$

we have

$$\text{Im}\ z^m = m\beta\alpha^{m-1} + \mathcal{O}(\beta^2\alpha^{m-2}) \ge \frac{m}{2}\beta\alpha^{m-1} \ge \frac{m}{2}\delta\alpha^{m-1}, \tag{3.4}$$

for α sufficiently large. Therefore, it follows from (3.3), (3.4) and (1.5) that

$$\|(I - \chi_{[\alpha-1,\alpha+1)}) \circ (P - z^m)^{-1} f\|_{L^{\frac{2n}{n-m}}(M)} \le C\|f\|_{L^{\frac{2n}{n+m}}(M)}, \tag{3.5}$$

for all $z \in \Xi$, Re $z = \alpha \gg 1$, and $\delta \le \text{Im } z \le 1$, uniformly in z.

Let $z \in \Xi$, Re $z = \alpha \gg 1$, and $0 < \text{Im } z = \beta \le 1/2$. Then using the fact that $\alpha + i \in \Xi$ for α sufficiently large and (3.5), we see that (3.1) follows once we establish that

$$\|(I - \chi_{[\alpha-1,\alpha+1)}) \circ ((P - z^m)^{-1} - (P - (\alpha+i)^m)^{-1})f\|_{L^{\frac{2n}{n-m}}(M)} \le C\|f\|_{L^{\frac{2n}{n+m}}(M)}, \tag{3.6}$$

uniformly in z. We have

$$\begin{aligned}(I - \chi_{[\alpha-1,\alpha+1)}) &\circ ((P - z^m)^{-1} - (P - (\alpha+i)^m)^{-1})f \\ &= \Big(\sum_{\mu_j \in [0,\alpha-1)} + \sum_{\mu_j \in [\alpha+1,+\infty)}\Big)\Big(\frac{1}{\mu_j^m - z^m} - \frac{1}{\mu_j^m - (\alpha+i)^m}\Big)E_j f \\ &= \Big(\sum_{\mu_j \in [0,\alpha-1)} + \sum_{k=2}^{\infty} \sum_{\mu_j \in [\alpha+k-1,\alpha+k)}\Big)\Big(\frac{1}{\mu_j^m - z^m} - \frac{1}{\mu_j^m - (\alpha+i)^m}\Big)E_j f.\end{aligned} \tag{3.7}$$

By (2.5), for $k = 2, 3\ldots$, we get

$$\begin{aligned}\|\sum_{\mu_j \in [\alpha+k-1,\alpha+k)} \Big(\frac{1}{\mu_j^m - z^m} - \frac{1}{\mu_j^m - (\alpha+i)^m}\Big)E_j f\|_{L^{\frac{2n}{n-m}}(M)} \le C(\alpha+k)^{m-1} \\ \sup_{\tau \in [\alpha+k-1,\alpha+k)} \Big|\frac{z^m - (\alpha+i)^m}{(\tau^m - z^m)(\tau^m - (\alpha+i)^m)}\Big| \|f\|_{L^{\frac{2n}{n+m}}(M)}.\end{aligned} \tag{3.8}$$

We have, for α sufficiently large, that

$$z^m - (\alpha+i)^m = \alpha^{m-1} mi(\beta - 1) + \mathcal{O}(\alpha^{m-2}),$$

and therefore,

$$|z^m - (\alpha+i)^m| \le C\alpha^{m-1}. \tag{3.9}$$

As Re $z^m = \alpha^m + \mathcal{O}(\alpha^{m-2})$, we obtain that

$$\begin{aligned}&|\tau^m - z^m| \ge |\tau^m - \alpha^m - \mathcal{O}(\alpha^{m-2})| \\ &= |(\tau - \alpha)(\tau^{m-1} + \tau^{m-2}\alpha + \cdots + \tau\alpha^{m-2} + \alpha^{m-1}) - \mathcal{O}(\alpha^{m-2})| \\ &\ge (k-1)(\tau^{m-1} + \alpha^{m-1}) - |\mathcal{O}(\alpha^{m-2})| \ge (k-1)\tau^{m-1} \ge (k-1)(\alpha+k)^{m-1}/C,\end{aligned} \tag{3.10}$$

for $\tau \in [\alpha + k - 1, \alpha + k)$, $k = 2, 3, \ldots$, and α sufficiently large. Thus, it follows from (3.8), (3.9), and (3.10) that

$$\begin{aligned}\|\sum_{\mu_j \in [\alpha+k-1,\alpha+k)} \Big(\frac{1}{\mu_j^m - z^m} - \frac{1}{\mu_j^m - (\alpha+i)^m}\Big)E_j f\|_{L^{\frac{2n}{n-m}}(M)} \\ \le \frac{C}{(k-1)^2}\|f\|_{L^{\frac{2n}{n+m}}(M)},\end{aligned} \tag{3.11}$$

for $k = 2, 3, \dots$. Using (2.5) and rescaling, we get

$$\| \sum_{\mu_j \in [0, \alpha - 1)} \Big(\frac{1}{\mu_j^m - z^m} - \frac{1}{\mu_j^m - (\alpha + i)^m} \Big) E_j f \|_{L^{\frac{2n}{n-m}}(M)} \le C \|f\|_{L^{\frac{2n}{n+m}}(M)}. \quad (3.12)$$

Hence, (3.6) follows from (3.7), (3.11), and (3.12). The proof of (3.1) is complete.

Let us now show (3.2). To that end, letting $w = ze^{-2\pi i/m}$, we have $w^m = z^m$, and therefore, (3.2) is a consequence of the uniform estimate,

$$\|(I - \chi_{[\alpha-1, \alpha+1)}) \circ ((P - w^m)^{-1} - (P - (\alpha + i)^m)^{-1}) f\|_{L^{\frac{2n}{n-m}}(M)} \le C \|f\|_{L^{\frac{2n}{n+m}}(M)},$$

with $z \in \Xi$, $w = ze^{-2\pi i/m}$, $\mathrm{Re}\, w = \alpha \gg 1$, and $0 < -\mathrm{Im}\, w \le 1$. This is obtained similarly to the derivation of (3.6). The proof of Lemma 3.2 is complete. □

Let

$$N(\alpha) = \#\{j : \mu_j < \alpha\}$$

be the counting function for the eigenvalues of the operator Q. We have

$$N(\alpha) = \int_M S_\alpha(x, x) d\mu(x), \quad (3.13)$$

where

$$S_\alpha(x, y) = \sum_{\mu_j < \alpha} e_j(x) \overline{e_j(y)}$$

is the spectral function.

Similarly to [1, Theorem 1.2] we obtain the following result which gives a sufficient condition for the optimality of the region Ξ_δ in the uniform resolvent estimate (1.5) for operators of order m, in terms of the density of eigenvalues in shrinking intervals of the form $[\alpha_k - \beta_k, \alpha_k + \beta_k)$, $\alpha_k \to \infty$, $0 < \beta_k \to 0$ as $k \to \infty$.

Lemma 3.3. *Assume that there exist sequences* $\alpha_k \to \infty$ *and* $0 < \beta_k \to 0$ *as* $k \to \infty$ *such that*

$$(\beta_k \alpha_k^{n-1})^{-1} [N(\alpha_k + \beta_k) - N(\alpha_k - \beta_k)] \to \infty, \quad k \to \infty. \quad (3.14)$$

Let $z_k^{(1)} = \alpha_k + i\beta_k$ *and* $z_k^{(2)} = e^{2\pi i/m}(\alpha_k - i\beta_k)$. *Then we have*

$$\|(P - (z_k^{(j)})^m)^{-1}\|_{L^{\frac{2n}{n+m}}(M) \to L^{\frac{2n}{n-m}}(M)} \to \infty, \quad k \to \infty, \quad j = 1, 2. \quad (3.15)$$

Proof. In what follows we shall only establish (3.15) for $j = 1$, the proof in the other case being similar. We shall then write $z_k = z_k^{(1)}$. Let us notice that $z_k \in \Xi$ for k large enough.

By (3.1), we know that for large k,

$$\|(I - \chi_{[\alpha_k - 1, \alpha_k + 1)}) \circ (P - z_k^m)^{-1}\|_{L^{\frac{2n}{n+m}}(M) \to L^{\frac{2n}{n-m}}(M)} \le C,$$

uniformly in k. Thus, we only need to show that

$$\|\chi_{[\alpha_k-1,\alpha_k+1)}\circ(P-z_k^m)^{-1}\|_{L^{\frac{2n}{n+m}}(M)\to L^{\frac{2n}{n-m}}(M)}\to\infty,\quad k\to\infty. \tag{3.16}$$

Let $g\in C_0^\infty(\mathbb{R})$ be such that $0\notin$ supp (g) and $g(\tau)=1$ for $\tau\in[1/2,2]$. Then for large k, we have

$$\chi_{[\alpha_k-1,\alpha_k+1)}=g(Q/\alpha_k)\circ\chi_{[\alpha_k-1,\alpha_k+1)}\circ g(Q/\alpha_k). \tag{3.17}$$

Using (3.17) and Lemma 3.1, we obtain

$$\begin{aligned}
&\|\chi_{[\alpha_k-1,\alpha_k+1)}\circ(P-z_k^m)^{-1}f\|_{L^\infty(M)}\\
&\quad=\|g(Q/\alpha_k)\circ\chi_{[\alpha_k-1,\alpha_k+1)}\circ(P-z_k^m)^{-1}\circ g(Q/\alpha_k)f\|_{L^\infty(M)}\\
&\quad\le C\alpha_k^{\frac{n-m}{2}}\|\chi_{[\alpha_k-1,\alpha_k+1)}\circ(P-z_k^m)^{-1}\|_{L^{\frac{2n}{n+m}}(M)\to L^{\frac{2n}{n-m}}(M)}\|g(Q/\alpha_k)f\|_{L^{\frac{2n}{n+m}}(M)}\\
&\quad\le C\alpha_k^{n-m}\|\chi_{[\alpha_k-1,\alpha_k+1)}\circ(P-z_k^m)^{-1}\|_{L^{\frac{2n}{n+m}}(M)\to L^{\frac{2n}{n-m}}(M)}\|f\|_{L^1(M)}.
\end{aligned}$$

Thus, in order to show (3.16) it suffices to check that

$$\alpha_k^{-(n-m)}\|\chi_{[\alpha_k-1,\alpha_k+1)}\circ(P-z_k^m)^{-1}\|_{L^1(M)\to L^\infty(M)}\to\infty,\quad k\to\infty. \tag{3.18}$$

The kernel of the operator $\chi_{[\alpha_k-1,\alpha_k+1)}\circ(P-z_k^m)^{-1}$ is given by

$$K(x,y)=\sum_{\mu_j\in[\alpha_k-1,\alpha_k+1)}\frac{1}{\mu_j^m-z_k^m}e_j(x)\overline{e_j(y)}.$$

We have

$$\begin{aligned}
\alpha_k^{-(n-m)}\|\chi_{[\alpha_k-1,\alpha_k+1)}\circ(P-z_k^m)^{-1}\|_{L^1(M)\to L^\infty(M)}&=\alpha_k^{-(n-m)}\sup_{x,y\in M}|K(x,y)|\\
&\ge\alpha_k^{-(n-m)}\sup_{x\in M}\left|\sum_{\mu_j\in[\alpha_k-1,\alpha_k+1)}\frac{1}{\mu_j^m-z_k^m}|e_j(x)|^2\right|\\
&\ge\alpha_k^{-(n-m)}\sup_{x\in M}\left|\operatorname{Im}\sum_{\mu_j\in[\alpha_k-1,\alpha_k+1)}\frac{\mu_j^m-\overline{z_k}^m}{|\mu_j^m-z_k^m|^2}|e_j(x)|^2\right|\\
&\ge\alpha_k^{-(n-m)}|\operatorname{Im}(-\overline{z_k}^m)|\sup_{x\in M}\sum_{\mu_j\in[\alpha_k-\beta_k,\alpha_k+\beta_k)}\frac{1}{|\mu_j^m-z_k^m|^2}|e_j(x)|^2:=L_k,
\end{aligned}$$

for k sufficiently large. Writing $\overline{z_k}^m=(\alpha_k-i\beta_k)^m$, we get

$$\operatorname{Im}(-\overline{z_k}^m)=m\beta_k\alpha_k^{m-1}+\mathcal{O}(\beta_k^2\alpha_k^{m-2})\ge m\beta_k\alpha_k^{m-1}/2, \tag{3.19}$$

for k sufficiently large. Using the fact that $\mu_j\in[\alpha_k-\beta_k,\alpha_k+\beta_k)$ in the last sum, we obtain that

$$|\mu_j^m-z_k^m|=|\mu_j-z_k||\mu_j^{m-1}+\mu_j^{m-2}z_k+\cdots+\mu_jz_k^{m-2}+z_k^{m-1}|\le C\beta_k\alpha_k^{m-1}, \tag{3.20}$$

for k sufficiently large. It follows from (3.13), (3.19), (3.20) and (3.14) that

$$\begin{aligned} L_k &\ge \frac{1}{C}(\beta_k\alpha_k^{n-1})^{-1} \sup_{x\in M} \sum_{\mu_j\in[\alpha_k-\beta_k,\alpha_k+\beta_k)} |e_j(x)|^2 \\ &\ge \frac{1}{C}(\beta_k\alpha_k^{n-1})^{-1} \frac{1}{\mathrm{Vol}(M)} \int_M \sum_{\mu_j\in[\alpha_k-\beta_k,\alpha_k+\beta_k)} |e_j(x)|^2 d\mu(x) \\ &= \frac{1}{C}(\beta_k\alpha_k^{n-1})^{-1} \frac{1}{\mathrm{Vol}(M)} [N(\alpha_k+\beta_k)-N(\alpha_k-\beta_k)] \to \infty, \end{aligned}$$

as $k\to\infty$. Hence, we get (3.18), which completes the proof of (3.15). The proof of Lemma 3.3 is complete. □

Notice that the Weyl law, see [6],

$$N(\alpha) = C\alpha^n + \mathcal{O}(\alpha^{n-1}), \quad C = (2\pi)^{-n} \iint_{\{(x,\xi)\in T^*M: q(x,\xi)\le 1\}} dx d\xi,$$

implies that

$$N(\alpha_k+1) - N(\alpha_k-1) = \mathcal{O}(\alpha_k^{n-1}).$$

Consequently, to find sequences $\alpha_k\to\infty$ and $0<\beta_k\to 0$ as $k\to\infty$ satisfying (3.14), we would like to exhibit a situation when the spectrum of the operator Q is distributed in a non-uniform fashion, clustering around the sequence α_k.

To verify the assumption (3.14) in Lemma 3.3, we shall need the following result concerning the spectrum of Q, when the Hamilton flow of q is periodic, due to [17] and [2], see also [8, Theorem 29.2.2].

Theorem 3.4. *Let $Q\in\Psi^1_{\mathrm{cl}}(M)$ be positive elliptic self-adjoint operator with principal symbol q and zero subprincipal symbol. Assume that the Hamilton flow* $\exp(tH_q)$*, generated by the principal symbol q, is periodic with a common minimal period T on $q^{-1}(1)$. Then there is a constant $C>0$ such that all eigenvalues of Q, except finitely many, belong to the intervals $I_k := [\frac{2\pi}{T}(k+\frac{\alpha}{4})-\frac{C}{k}, \frac{2\pi}{T}(k+\frac{\alpha}{4})+\frac{C}{k}]$, $k=1,2\dots,$ where $\alpha>0$ is a constant. Furthermore, the number of eigenvalues of Q in I_k, denoted by d_k, is a polynomial in k of degree $n-1$ of the form*

$$d_k = nk^{n-1}T^{-n}\iint_{q<1} dx d\xi + \mathcal{O}(k^{n-2}).$$

To prove Theorem 1.2, let $Q=P^{1/m}$ and observe that the subprincipal symbol of Q vanishes, see [4, Section 1]. It follows from Theorem 3.4 that the assumptions of Lemma 3.3 are satisfied with $\alpha_k = \frac{2\pi}{T}(k+\frac{\alpha}{4})$ and $C/k<\beta_k\to 0$ as $k\to\infty$. The proof of Theorem 1.2 is complete.

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