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ABSTRACT

A definition of causality, different from the one currently employed in field theory, is introduced. Its relation to the Lorentz group and to relativity theory is clarified. It leads to a reduction of the S matrix, appropriate to dispersion theory, that is not subject to some limitations hitherto encountered. The mathematical constructs appearing in the course of this reduction are discussed.

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INTRODUCTION

Among the more promising recent attempts in field theory are those that exploit the principle of causality. The matrix element for elastic scattering is represented as a Fourier transform of the expectation value of a retarded commutator. The latter is then required to vanish for fields with spacelike separation. This, together with the assumption of a complete set of positive-frequency timelike eigenstates for the displacement operator, leads to interesting analytic properties of the scattering amplitude.[1] In order to deal with processes of greater complexity the single-commutator representation is generalized to one with multiple commutators.[2] This extension is subject to two limitations: there must be no more than two particles in one of the states, and the one-particle states are taken to be stable.

In this article we formulate the principle of causality in a fashion that appears to us more apposite to the context in which it is used. Basing ourselves on this definition, we proceed to discuss the relation between causality and the Lorentz group. The discussion suggests a reduction of the S matrix essentially different from the multiple-commutator representations. It is not subject to limitations on the number or stability of particles entering into or emerging from a reaction. Causality so formulated is then shown to imply microcausality, a condition on a bilinear form in the Bethe-Salpeter amplitudes appropriate to the initial and final states.

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CAUSALITY IN CLASSICAL PARTICLE MECHANICS

Let a simultaneous configuration of a system of n particles, subject to a dynamics D, be specified by $z_{(s)}^{\mu}$ (μ=0, 1, 2, 3; $g_{\mu\nu}$=1, -1, -1, -1, ; s=1, 2, ··· n). Consider two distinct simultaneous configurations $z(\tau_1)$, $z(\tau_2)$ which may evolve from each other according to D. Let the D be thought of as including the mutual interactions of the particles and the effects of external fields. We associate with it a D_0 which is thought of as excluding these and representing the free dynamics of the system. We call the external fields and the mutual interactions causal if $z(\tau_1)$ and $z(\tau_2)$ are also accessible from each other by the motion D_0. Respecting the nature of the latter, this definition has nothing to say.

Applied to a single particle, whose D_0 is a forward motion in time controlled by Newton's equations, the definition would rule as acausal those external fields that result in a later position outside the forward light cone of the original. For a D_0 that admits a forward as well as a backward motion in τ, the class of causal external fields might be broader.

It is difficult to harmonize this denotation of causality with the connotations and associations the word has collected. A more felicitous term, reflecting our anxiety lest certain facets of free motion--to which we are addicted--be effaced by interactions, would be desirable. The more so, since the range of applicability of the word is now limited to theories with a prepossession for sundering interaction effects from the entire motion.

Does the word, with its common denotative value, have any meaning in physics? For a closed dynamical system, unaffected by external forces and without hidden degrees of freedom, we believe not. The central problem for these systems is that of determinism, rather than causality. We are given the law of evolution of the system in the small and inquire what additional conditions (initial, final, boundary) must be imposed for the determination in the large. More precisely the question--purely mathematical--is whether with a certain type of conditions the problem is well set (in the sense of Hadamard) and not whether these refer to the past or future. On the other hand, in considering systems subject to external agencies we are averse to basing our solution on promises, and account only the past performance of the agents. Since the latter can be comprehended in a larger closed system and codetermined with the system under consideration, the concept may well be only ancillary to classical physics. Quantum theory may present a somewhat different picture, since an observer cannot be codetermined with the system he observes and still have any knowledge of it.

Historically, the theory of relativity was the carrier of an idea, somewhat less general and explicit, that comes under the head of our definition of causality. We shall give now a crude operational picture of Lorentz invariance, unembarrassed by it. Within its frame we shall also fit the definitions and discussions of the following sections.

A physicist has somehow come by an album of sketches of orbits of particles. He may have obtained it from observation, by integrating equations of motion, or drawn them from his imagination. To a rather exclusive exhibit of these he invites that public which sees eye to eye with him on certain things. To qualify for an invitation, each one must share his opinions on how to calculate lengths of vectors from their components and agree with him on other things to be discussed in the next section. He exposes one of his sketches (say a circle) to their view. Each of them reports back his distorted impression of this object (to each of them it may appear as a different ellipse). If he can match each of their distortions with some sketch of his album ("I have one like it, but it isn't the one I'm now showing") and if he can do this for every sketch of his album, then his collection is invariant under the distortion of this group of viewers. In this formulation the invariance of the collection is contingent on the wealth of possibilities of motion the album contains within it rather than on any dynamical details. This picture is made more precise in the next section.

CAUSALITY IN QUANTUM MECHANICS

The homologues of free and interacting classical motions between two configurations are the free $(x_1 \cdots x_m \left| s \right| y_1 \cdots y_n)$ and interacting $(x_1 \cdots x_m \left| S \right| y_1 \cdots y_n)$ transition amplitudes. They are not expected to exhibit the specific features of point mechanics. Even for a single free field the transition amplitude

$$(x \left| s \right| y) = (\Omega, A^{-}(x) \, A^{+}(y)\Omega) = i \, \Delta^{+} (xy) \tag{1}$$

does not vanish outside the light cone. A general feature of the classical situation, closely relating to the Lorentz group, is, however, reflected in quantum theory. It is that a free particle can be saddled with a Lorentz observer. Our causality requirement restricted the admissible external fields open for the particle to those in which it still may be tamed. We express this in quantum language by saying that the free-transition amplitude $(x \left| s \right| y)$ admits the Lorentz group and we restrict the admissible interactions to those in which $(x \left| S \right| y)$ also admits it. In terms of the operational statement of the previous section: all those who agree among themselves on how to compute lengths of vectors do not distort s. To preserve this unanimity, we refrain from introducing controversial interactions that would lead to a distortion of S.

We now proceed to give mathematical form to the idea of distortion as applied to objects of field theory, to list the classes of object on which the Lorentz public must agree by the very nature of relativity theory, and to show that S is not included among these classes.

The representation of a distortion is well known in the theory of groups of transformations.[3] Let g be some element of such a group. Let F(x) be a function of the coordinates x whose numerical value is specified in a certain set of frames of reference. It may be a tensor, spinor, Hilbert space vector, or some combination of these. The F as distorted by g, F_g, may be expressed symbolically in terms of the undistorted F through the relation $F_g = g\,Fg^{-1}$, understood to mean that we take g on the left of F in those representations that are appropriate to the tensor or spinor character of F, and g^{-1} on the right in those appropriate to the realization of g on the coordinate variable. For a one-parameter (t) subgroup we denote the latter by x_t and its inverse by x_{-t}. The infinitesimal representation associated with tensor indices is then the matrix $J_t{}^{\mu}{}_{\nu}(x) = \dfrac{\partial x_t{}^{\mu}}{\partial x^{\nu}}$; the representations s_t associated with spin and U_t with Hilbert space indices are taken, for the moment, as independent of J. The distortion of a c-function tensor is then

$$g^{\mu\nu}(x) \rightarrow g_t{}^{\mu\nu}(x) \equiv J_t{}^{\mu}{}_{\mu'}\, J_t{}^{\nu}{}_{\nu'}\, g^{\mu'\nu'}(x_{-t}) , \qquad (2)$$

a spinor c function is

$$\psi(x) \rightarrow \psi_t(x) \equiv s_t\, \psi(x_{-t}) , \qquad (3)$$

a spin operator is

$$\gamma^{\mu}(x) \rightarrow \gamma_t{}^{\mu}(x) \equiv J_t{}^{\mu}{}_{\nu}\, s_t \gamma^{\nu}(x_{-t})\, s_t{}^{-1} , \qquad (4)$$

a Hilbert space vector is

$$\Omega(x) \rightarrow \Omega_t(x) = U_t \Omega(x_{-t}) , \qquad (5)$$

a scalar field variable (Hilbert space operator) is

$$\phi(x) \to \phi_t(x) = U_t \; \phi(x_{-t}) \; U_t^{-1} . \tag{6}$$

The quantities s and U, in general independent of J and the finite transformation on x, are in field theory related to these by restricting the class of admissible transformation to those which do not distort certain objects.

We shall list the types of objects that are required to be undistortible by Lorentz invariance and show that the transition amplitude is in general not included among them. The primary object of this character is the metric tensor. Its undistortibility defines the representation of the Lorentz group on tensor indices through the requirement $g_t^{\mu\nu}(x) = g^{\mu\nu}(x)$. From now on we confine ourselves to Minkowski frames in which g is diagonal and independent of x. To correlate s to J we also require that the spin operators γ^{μ} remain undistorted, thus defining the spin representations of the group. The Hilbert space representation U_t is defined by the requirement that all field variables $\phi(x)$ remain undistorted. Since these are the three types of representation we need in field theory, Lorentz invariance alone does not compel us to postulate any further undistortible objects. In this sense causality, although intimately connected with the mappings under the Lorentz group, is quite independent of the requirements of relativity theory.

Let us now consider the state vector

$$\Omega(x, \sigma) = \int_{\sigma} d\sigma^{\mu}(\xi) \; \Delta^{+}(x-\xi) \; \overset{\leftrightarrow}{\partial}_{\mu}(\xi) \; \phi(\xi) \, \Omega , \tag{7}$$

where Ω is the vacuum state, $f \overset{\leftrightarrow}{\partial}_{\mu} g = f \partial_{\mu} g - (\partial_{\mu} f) g$, and ϕ a scalar field operator. For a free field this integral is independent of σ and may be evaluated with x on σ to yield $\omega(x) = \phi^{+}(x) \, \Omega$. We then have, by (5),

$$\omega_t(x) = U_t \omega(x_{-t}) = U_t \phi^{+}(x_{-t}) \; U_t^{-1} \; \Omega = \phi_t^{+}(x) \; \Omega = \omega(x) , \tag{8}$$

from the requirement that ϕ not be distorted. Relation (8) does not hold for a general $\Omega(x, \sigma)$. Geometric considerations indicate that $\Omega_t(x\sigma)$, regarded as a functional of the pseudonormal to the surface σ, is different from $\Omega(x, \sigma)$. Since the transition amplitude is a scalar product of state vectors of this type with $\sigma \rightarrow \pm \infty$, it need not admit the group for transitions involving interaction.

We shall accordingly reduce S by expanding it in terms of s. The exacting demand of admitting the group is then passed on to the coefficients of the expansion. These turn out to be bilinear forms in the BS amplitudes. To carry out this project, we need three lemmata. The next three sections are devoted to their derivation. The first lemma is also of some intrinsic interest. It permits us to compare the reduction scheme of dispersion theory with those employed in other approaches to field theory. The second is a device for generating elements of s. The third relates to generalized retarded commutators and implies a certain reciprocity relation for the internal orderings of factors in a product of fields. These in turn suggest a view on dispersion theory, discussed briefly in the final section.

DIRAC IDENTITIES

The object of the work in this section is to derive the formula on which our reduction of the S matrix is based. It is an extension of Dirac's $A^{out} = A^{in} + A^{rad}$ to second quantized systems. In the extended version A^{out} and A^{in} are products of outgoing and incoming operators as they occur in the construction of the outgoing and incoming state vectors. Two sets of quantities corresponding to A^{rad} are defined and have in common with it an essential property. A notation requiring somewhat eleborate definitions is introduced to make the final expression compact and explicit.

We consider a specific matrix element of the scattering operator with $m \geq 1$ incoming and $n \geq 1$ outgoing, not necessarily distinct, scalar fields. Let these be arbitrarily ordered as in

$$\Omega_i \cdot (x', x'', x''', \cdots) = \psi_i^+(x')\, \phi_i^+(x'')\, \chi_i^+(x''')\, \chi_i^+(x^{iv}) \cdots \Omega ,$$

$$\Omega_o (y', y'', y''', \cdots) = \Sigma_o^+(y')\, \Delta_o^+(y'')\, \Theta_o^+(y''')\, \Delta_o^+(y^{iv}) \cdots , \Omega ,$$

where the superscript (+) denotes the positive frequency part of the incoming (i) or outgoing (o) field and the vacuum state is assumed to be stable. The ordered sequence with repetition in the field variables is now placed in a one-to-one correspondence with sequences of field variables without repetition. These are $A_{i\,1}^{+}(x_1)$, $A_{i\,2}^{+}(x_2)$, $\cdots$ $A_{i\,m}(x_m)$ and $B_{o\,1}^{+}(y_1)$, $B_{o\,2}^{+}(y_2)$, $\cdots$ $B_{o\,n}^{+}(y_n)$ for the field operators of Ω_i and Ω_o respectively. A permutation group $\underset{\sim}{p}$ on m letters is now associated with the A sequence, and $\underset{\sim}{q}$ on n letters with B. The domains of the permutations are the identical subscripts of the field operator and coordinate variable. A particular permutation p may be defined by

$$p\, [\, A_1(x_1),\ A_2(x_2),\ A_3(x_3), \cdots A_m(x_m)\,] = [\, A_3(x_3), A_1(x_1), A_m(x_m), \cdots , A_2(x_2)\,] . \qquad (9)$$

The definition of a multifield operator constructed on a sequence of commuting operators (like A_i^+) is given by the formulas

$$^{sp1}A(x) = 1 \text{ for } s = 0 ,$$

$$^{sp1}A(x) = {}^{p1}A(x_1, x_2, \cdots x_s) = A_{p1}(x_{p1})A_{p2}(x_{p2})\cdots A_{ps}(x_{ps}) \text{ for } 1 \leq s \leq m. \quad (10)$$

The first of these is a convention. In the second, the first equality sign translates a compact into a more prolix notation; the second equality explicates these in terms of elementary field variables. In words: to construct $^{sp1}A(x)$ permute the standard sequence by p and form the product of the first s elements of the permuted sequence. It will be useful to have a notation $^{sp1}A'(x)$ for the complement of $^{sp1}A(x)$:

$$^{sp1}A'(x) = {}^{p1}A(x_{s+1}, x_{s+2}, \cdots x_m) = A(x_{p(s+1)}, x_{p(s+2)}, \cdots x_{pm}). \quad (10')$$

The identity permutation is not indicated; thus $^{s1}A(x) = A_1(x_1) \cdots A_s(x_s)$. We also omit the s for s = 1; thus $^{p1}A(x) = A_{p1}(x_{p1})$. Consistently, then, $A(x) = A_1(x_1)$. We note $^{mp1}A(x) = {}^{m}A(x)$ for all p. By way of example, with p of (9) we have

$$^{p1}A(x) = A_3(x_3),$$

$$^{2p1}A(x) = A_3(x_3)\, A_1(x_1) ,$$

$$^{3p1}A(x) = A_3(x_3)\, A_1(x_1)\, A_m(x_m) .$$

In order to extend these definitions to sequences of noncommuting operators we introduce a system of θ functions.

$$^{sp1}\theta(x) = 1 \text{ for } s = 0 ,$$

$$= 1 \text{ for } s = 1,$$

$$= \theta(x_{p1} | x_{p2} | \cdots | x_{ps}) = {}^{p}\theta(x_1 | x_2 | \cdots x_s) \text{ for } s > 1, \quad (11)$$

where

$$^{p}\theta_{+}(x_1|\cdots|x_s) = 1 \text{ for } x^0_{p1} < \cdots < x^0_{ps},$$
$$= 0 \text{ otherwise},$$
$$^{p}\theta_{-}(x_1|\cdots|x_s) = 1 \text{ for } x_{p1}^{0} > \cdots > x_{ps}^{0},$$
$$= 0 \text{ otherwise}.$$

These are direct generalizations of $^2\theta(x) = \theta(x_1 - x_2)$, and may be expressed in terms of this symbol; thus $^3\theta(x) = \theta(x_1 - x_2)\,\theta(x_2 - x_3)$. We associate with this set a permutation group $\underset{\sim}{\pi}$ on s letters whose domain consists of the subscripts of x in θ:

$$^{\pi sp1}\theta(x) = \theta(x_{\pi p1} \mid x_{\pi p2} | \cdots | x_{\pi ps}).$$

From the meaning of these symbols it is readily seen that

$$\sum_{\pi} {}^{\pi sp1}\theta_{\pm}(x) = 1. \tag{12}$$

We now define the multifield operator $^{sp}A_{\pm}(x)$ on the ordered set of simple operators $[A_1(x_1), A_2(x_2), \cdots A_m(x_m)]$ by the formula

$$^{sp}A_{\pm}(x) = 1 \quad \text{for} \quad s = 0,$$

$$^{sp}A_{\pm}(x) = \sum_{\pi} {}^{\pi sp1}\theta_{\pm}(x) \quad {}^{\pi sp1}A(x) \quad \text{for} \quad m \geq s \geq 1. \tag{13}$$

In words: to construct ^{sp}A apply a p permutation to the sequence, form the product ^{sp1}A of the first s elements from left to right, multiply by an appropriate θ, and sum over all possible temporal arrangements. We note in particular the relation $^1A_+ = {}^1A_-$. If the operators in the sequence commute we have $^{\pi sp1}A(x) = {}^{sp1}A(x)$ and, by (12), $^{sp}A(x) = {}^{sp1}A(x)$. Definition (13) then includes (10) as a special case. It is the p bracket of Dyson, the $\pm$ brackets of Schwinger, the T product of Wick, the T function of LSZ,[2] etc., defined on the sp subset of the set of m operators. We enriched this collection with one more notation, which is handled more smoothly in algebraic manipulations.

The Dirac definition of the radiation field $\mathcal{Q}(x)$ given by the expression

$$A_o(x) = A_i(x) + \mathcal{Q}(x) \tag{14}$$

may be regarded as a formal identity:

$$A_o(x) = \int_{+\infty} d\sigma_\mu(\xi)\ \Delta(x-\xi)\ \delta_\mu(\xi)\ A(\xi)$$

$$= \left(\int_{+\infty} - \int_{-\infty}\right) d\sigma^\mu(\xi)\ \Delta(x-\xi)\delta_\mu(\xi)A(\xi)$$

$$+ \int_{-\infty} d\sigma^\mu(\xi)\,\Delta(x-\xi)\delta_\mu(\xi)\ A(\xi)$$

$$= A_i(x) + \int d\sigma(\xi)\ \Delta(x-\xi)\ K(\xi)\ A(\xi) = A_i(x) + \mathcal{Q}(x)\ .$$

In this expression $\Delta(x)$ is the radiation kernel appropriate to the A field and K the Gordon-Klein operator that annihilates it. To have a more compact notation we write

$$\int_\sigma d\sigma^\mu(\xi)\,\Delta(x-\xi)\ \delta_\mu(\xi)\cdots = \int_\sigma d\ H(x-\xi)\cdots\ ,$$

$$\int d\sigma(\xi)\,\Delta(x-\xi)\ K(\xi)\cdots = \int dK(x-\xi)\cdots\ ,$$

where H is, essentially, the Huygens kernel and K relates the radiation field to its source. In this notation, an abbreviated form of that in LSZ, Green's theorem for a c function f(x) is

$$\int_{\sigma_1}^{\sigma_2} dK(x-\xi) f(\xi) = \left(\int_{\sigma_2} - \int_{\sigma_1} \right) dH(x-\xi)\, f(\xi) \tag{15_0}$$

if $f(x)$ vanishes sufficiently rapidly on the timelike remote surfaces. We adapt now one of their principal techniques to extend (15_0) to cases in which f is replaced by the ordered operators $A_\pm$ of (13). Clearly (15_0) still holds for ${}^sA$ with $s = 1$. For ${}^sA$ with $s > 1$ we need the following versions of Green's theorem:

$$\int dK(x_n-\xi_n)\, A_+(x_1 \cdots x_{n-1}\xi_n) = A_+(x_1 \cdots x_{n-1}) \underset{o}{A}(x_n) - \underset{i}{A}(x_n) A_+(x_1 \cdots x_{n-1}) \tag{15_+}$$

$$\int dK(x_n-\xi_n) A_-(x_1 \cdots x_{n-1}\xi_n) = \underset{o}{A}(x_n) A_-(x_1 \cdots x_{n-1}) - A_-(x_1 \cdots x_{n-1}) \underset{i}{A}(x_n) . \tag{15_-}$$

By the argument leading to (15_0) we have

$$\int dK(x_n-\xi_n) A_+(x_1 \cdots x_{n-1}\xi_n) = \left(\int_{\xi_n^0 = +\infty} - \int_{\xi_n^0 = -\infty} \right) A_+(x_1 \cdots x_{n-1}\xi_n) dH(x_n-\xi_n),$$

and according to (13),

$$A_+(x_1 \cdots x_{n-1} x_n) = \sum_\pi \theta_+(x_{\pi 1} | \cdots | x_{\pi(n-1)} | x_{\pi n})\, {}^{\pi 1}A(x) \cdots {}^{\pi(n-1)}A(x)\, {}^{\pi n}A(x) .$$

We see then that only permutations with fixed $\pi(x) = n$ and $\pi(1) = n$ contribute to the upper and lower limits respectively. Equation (15_+) then follows from the definitions of $\underset{o}{A}$ and $\underset{i}{A}$. Similar considerations lead to (15_-).

The definitions

$$\underset{o\,(i)}{A}(x) = \int_{+\infty\,(-\infty)} dH(x-\xi) A(\xi), \tag{16a1}$$

$$\mathcal{A}_\pm(x) = \int dK(x-\xi) A_\pm(\xi) \tag{16b1}$$

of the quantities entering into Dirac's expression are now regarded as particular cases of the more general

$$ {}^{sp}\underset{o(i)}{A}(x) = \int_{+\infty\,(-\infty)} {}^{sp}dH(x-\xi)\,{}^{sp}A(\xi), \tag{16a}$$

$$ {}^{sp}\mathcal{Q}_{\pm}(x) = \int {}^{sp}dK\,(x-\xi)\;{}^{sp}A_{\pm}(\xi)\,. \tag{16b}$$

We take the usual asymptotic conditions to hold and therefore do not provide the A on the right-hand side of (16a) with an ordering (±).

Definition (16a) is a direct consequence of (16a1). The designation of (16b) as a radiation operator is warranted on the ground that, as also in (16b1), its positive and negative frequency projections vanish in the vacuum state. These projections

${}^{s}\underset{o(i)}{A}^{\pm}$, ${}^{s}\mathcal{Q}_{\pm}^{\;\pm}$ are obtained through the integral transforms (16) with kernels $H^{\pm}$, $K^{\pm}$ in which all the Δ functions are replaced by Δ^{+} or all the Δ functions by Δ^{-}. Since Δ is $\Delta^{+} + \Delta^{-}$, we have

$$ {}^{s}dH = {}^{s}dH^{+} + {}^{3d}H^{-} \quad \text{for} \quad s = 1, $$

$$ {}^{s}dH \neq {}^{s}dH^{+} + {}^{s}dH^{-} \quad \text{for} \quad s > 1, $$

with similar relations for K. The statements

$$ \Omega^{-s}\mathcal{Q}_{+}^{\;+}\,\Omega^{+} = 0,\; \Omega^{-s}\mathcal{Q}_{-}^{\;+}\Omega^{+} = 0,\; \Omega^{-s}\mathcal{Q}_{+}^{\;-}\Omega^{+} = 0,\; \Omega^{-s}\mathcal{Q}_{-}^{\;-}\Omega^{+} = 0 \tag{17}$$

obviously hold for $s = 1$ (with the assumed stable vacuum). The (-) + on Ω in (17) designates a (co) contravariant vector and thus obviates the necessity of brackets in denoting scalar products. We shall adhere to this convention. Applying K integral transforms to the variables $x_1 \cdots x_{n-1}$ in (15), we obtain

$$ {}^{s}\mathcal{Q}_{+}(x) = {}^{s-1}\mathcal{Q}_{+}(x)\,\underset{o}{A}(x_s) - \underset{i}{A}(x_s)\,{}^{s-1}\mathcal{Q}_{+}(x)\,, \tag{18$_+$}$$

$$ {}^{s}\mathcal{Q}_{-}(x) = \underset{o}{A}(x_s)\,{}^{s-1}\mathcal{Q}_{-}(x) - {}^{s-1}\mathcal{Q}_{-}(x)\,\underset{i}{A}(x_s)\,. \tag{18$_-$}$$

The positive-frequency projection of (18_+), recursively expanded and simplified by $\Omega^{-+} \underset{i}{A} = 0$, yields

$$\Omega^{-s}\mathfrak{a}_+{}^+(x)\,\Omega^+ = \Omega^{-\,s-1}\mathfrak{a}_+{}^+(x)\,\underset{o}{A}^+(x_s)\Omega^+ = \Omega^{-\,s-2}\mathfrak{a}_+{}^+(x)\,\underset{o}{A}^+(x_{s-1}x_s)\Omega^+$$

$$= \cdots = \Omega^-\mathfrak{a}^+(x_1)\underset{o}{A}^+(x_2\cdots x_n)\Omega^+ = 0.$$

The remaining three assertions of (17) may be deduced in a similar fashion.

The two pairs of expansion formulas for ${}^m\underset{o}{A}$ and ${}^m\underset{i}{A}$ are written down as

$$ {}^m\underset{o}{A}(x) = \frac{1}{m!}\sum_\sigma \binom{m}{\sigma} \sum_p {}^{\sigma p}\underset{i}{A}(x)\,{}^{\sigma p}\mathfrak{a}'_+(x), \qquad (D_1a)$$

$$ {}^m\underset{i}{A}(x) = \frac{1}{m!}\sum_\sigma \binom{m}{\sigma} \sum_p {}^{\sigma p}\underset{o}{A}(x)\,(-1)^{\sigma'}\,{}^{\sigma p}\mathfrak{a}'_-(x); \qquad (D_1b)$$

$$ {}^m\underset{o}{A}(x) = \frac{1}{m!}\sum_\sigma \binom{m}{\sigma} \sum_p {}^{\sigma p}\mathfrak{a}_-(x)\,{}^{\sigma p}\underset{i}{A'}(x), \qquad (D_2a)$$

$$ {}^m\underset{i}{A}(x) = \frac{1}{m!}\sum_\sigma \binom{m}{\sigma} \sum_p (-1)^\sigma\,{}^{\sigma p}\mathfrak{a}_+(x)\,{}^{\sigma p'}\underset{o}{A}(x). \qquad (D_2b)$$

The grouping into pairs is motivated by the fact that from a more detailed version of D (with expansions for ${}^{sp}A$) it is possible to obtain a completeness relation--a bilinear form involving only $\mathfrak{a}_\pm$--for each pair. Let $p(\sigma)$ be the stability subgroup of p that leaves $(1, 2, \cdots \sigma)$ invariant and p/σ the set of equivalence classes of p defined by $p(\sigma)$. Since A and $\mathfrak{a}$ are symmetric in their arguments, we may convert the sum over all the elements of the group into one over the equivalence classes by the corresponding stability subgroup. For (D_1a) we then obtain

$$ {}^{m}\underset{o}{A}(x_1 \cdots x_m) = \sum_{\sigma=0}^{m} \sum_{p/\sigma} {}^{p}\underset{i}{A}(x_1, \cdots, x_\sigma)\, {}^{p}\mathcal{Q}_{+}(x_{\sigma+1}, \cdots x_m) \qquad (D'_1 a). $$

This reduces for $m = 1$ to $\underset{o}{A}(x_1) = \underset{i}{A}(x_1) + \mathcal{Q}_{+}(x_1)$, the correct identity. Assume that $(D_1' a)$ holds for $m = n - 1$ and multiply both sides from the right by $A(x_n)$:

$$ {}^{n}\underset{o}{A}(x_1 \cdots x_n) = \sum_{\sigma=0}^{n-1} \sum_{p/\sigma} {}^{p}\underset{i}{A}(x_1, \cdots x_\sigma)\, {}^{p}\mathcal{Q}_{+}(x_{\sigma+1}, \cdots x_{n-1})\, \underset{o}{A}(x_n). \tag{19} $$

In (19) p is a permutation on $n - 1$ letters. With the aid of (18_+):

$$ {}^{n}\underset{o}{A}(x_1 \cdots x_n) = \sum_{\sigma=0}^{n-1} \sum_{p/\sigma} \Big[{}^{p}\underset{i}{A}(x_i \cdots x_\sigma)\, {}^{p}\mathcal{Q}_{+}(x_{\sigma+1}, \cdots x_{n-1}\, x_n) + $$

$$ + {}^{p}\underset{i}{A}(x_1, \cdots x_\sigma x_n)\, {}^{p}\mathcal{Q}_{+}(x_{\sigma+1}, \cdots x_{n-1}) \Big]. \tag{20} $$

The summation of the first term in the bracket covers the subset of p/σ on n letters that leaves x_n alone; of the second, its complement. The two may be combined into a single term with p/σ on n letters and represented in the form $(D'_1 a)$ with σ running from zero to n. The remaining three expansions are deduced in a similar fashion.

FREE-FIELD COMMUTATORS

In this sections we derive expansions for commutators of free multifield operators acting on the vacuum. These expressions identified with either incoming or outgoing fields are used in a later section.

Our starting point is the observation that $[A^-(x), B^+(y)]$ may be regarded as an element of the free-particle s matrix $(x|s|y)$, and the familiar expansion

$$[A^-(x), B_1^+(y_1)B_2^+(y_2)\cdots B_n^+(y_n)] = [A^-(x), B_1^+(y_1)]B_2^+(y_2)\cdots B_n^+(y_n) + \cdots$$

may therefore be rewritten as

$$[A^-(x), {}^nB^+(y)] = \sum_{q/1} (x|s|y)^q \cdot {}^qB^{+\prime}(y) = \sum_{\sigma=q0}^{n} \sum_{q/\sigma} (x|s|y)^{q\sigma}\, {}^{\sigma q}B^{+\prime}(y) \,. \qquad (21)$$

The second equality is valid because the s operator between states with different numbers of fields vanishes. This, and a corresponding relation, may therefore be expressed as

$$[A^-(x), {}^nB^+(y)] = \frac{1}{n!} \sum_{\sigma} \binom{n}{\sigma} \sum_{q} (x|s|y)^{q\sigma}\, {}^{\sigma q}B^{+\prime}(y), \qquad (E_+^0)$$

$$[{}^mA^-(x), B^+(y)] = \frac{1}{m!} \sum_{\sigma} \binom{m}{\sigma} \sum_{p} {}^{\sigma p}A^{-\prime}(x)\, {}^{\sigma p}(x|s|y) \,. \qquad (E_-^0)$$

Contracting E_+^0 with $\omega_-(x_2\cdots x_n)$, ω_+ and E_-^0 with ω_-, $\omega_+(y_2\cdots y_m)$, we get the recurrences

$$^n(x|s|y)^n = \frac{1}{n!} \sum_{\sigma} \binom{n}{\sigma} \sum_{q} (x|s|y)^{q\sigma}\; {}'(x|s|y)'^{\,q\sigma} \,, \qquad (22a)$$

$$^n(x|s|y)^n = \frac{1}{n!} \sum_{\sigma} \binom{n}{\sigma} \sum_{p} {}^{\sigma p'}(x|s|y)'\; {}^{\sigma p}(x|s|y) \,, \qquad (22b)$$

or--without redundant summations--

$$ {}^{n}(x\,|s\,|y)^{n} = \sum_{q/1} (x\,|s\,|y)^{q}\;{}'(x\,|s\,|y)'^{q} \qquad (22a') $$

$$ {}^{n}(x\,|s\,|y)^{n} = \sum_{p/1} {}^{p'}(x\,|s\,|y)'\;{}^{p}(x\,|s\,|y). \qquad (22b') $$

The contraction of $E_+^{\,0}$ with ω_+ and $E_-^{\,0}$ with ω_- gives

$$ [\,{}^{m}A^{-}(x),\ {}^{n}B^{+}(y)\,]\,\omega_+ = \frac{1}{n!}\sum_{\sigma}\binom{n}{\sigma}\sum_{q}{}^{m}(x\,|s\,|y)^{q\sigma}\,\omega_+{}'^{\sigma q}(y) - \delta^{m0}\,{}^{n}\omega_+(y), (E_+) $$

$$ \omega_-\,[\,{}^{m}A^{-}(x),\ {}^{n}B^{+}(y)\,] = \frac{1}{m!}\sum_{\sigma}\binom{m}{\sigma}\sum_{p}\omega'_-{}^{p\sigma}(x)\,{}^{\sigma p}(x\,|s\,|y)^{n} - \delta^{n0}\,{}^{m}\omega_-(x), (E_-) $$

where $m = 1$ in E_+ and $n = 1$ in E_-. We shall now show that E is valid without this restriction. To insure its validity for $m = 0$, $n = 0$ we inserted the terms with the Kronecker deltas.

In the alternative version, E_+ states

$$ [\,{}^{m}A^{-}(x),\ {}^{n}B^{+}(y)\,]\,\omega_+ = \sum_{\sigma=0}^{n}\sum_{q/\sigma}{}^{m}(x\,|s\,|y)^{q\sigma}\,\omega_+{}'^{\sigma q}(y) = 0 $$

$$ \text{for } m > n \qquad (23\alpha) $$

$$ = \sum_{q/m}{}^{m}(x\,|s\,|y)^{qm}\,\omega'_+{}^{mq}(y) $$

$$ \text{for } 0 < m \le n \qquad (23\beta) $$

where we made use of the orthogonality of initial and final states with different numbers of fields. Assertion (23α) is obviously correct. This is evident from the fact that, because of the presence of ω_+, we can construct a multiple commutator of ${}^{n}B^{+}(y)$ with single $A^{-}(x)$, of which there are enough to devour it. To investigate (23β) we introduce $\nu \ge 0$, $n = m + \nu$ and rewrite the expression as

$$ [\,{}^{m}A^{-}(x),\ {}^{(m+\nu)}B^{+}(y)\,]\,\omega_+ = \sum_{q/m}{}^{m}(x\,|s\,|y)^{qm}\,\omega_+{}'^{mq}(y)\,, \qquad (23\beta') $$

valid for $m = 1$ by (21). We assume it to hold for $m = n$ and use recurrence (22a') to prove its validity for $m = n + 1$. In extenso, this is

$$(x_1, \cdots, x_{n+1}|s|y_1, \cdots, y_{n+1}) = \sum_{\pi/1} (x_1|s|y_{\pi 1})(x_2, \cdots x_{n+1}|s|y_{\pi 2}, \cdots y_{\pi(n+1)}) \tag{22a''}$$

$$= \sum_{\pi/n} (x_1, \cdots, x_n|s|y_{\pi 1}, \cdots, y_{\pi n})(x_{n+1}|s|y_{\pi(n+1)})$$

by an obvious change in variables, or

$$(x_1, \cdots, x_{n+1}|s|y_{q1}, \cdots y_{q(n+1)}) = \sum_{\pi/n} (x_1, \cdots, x_n|s|y_{\pi q 1}, \cdots, y_{\pi q n})(x_{n+1}|s|y_{\pi q(n+1)}), \tag{24}$$

where q is any permutation appearing in (23β'). Substituting this into the right-hand member of the latter for $m = n + 1$, we show, by a sequence of steps, that it is equal to the left:

$$\sum_{q/(n+1)} {}^{(n+1)}(x|s|y)^{q(n+1)} \omega'_+{}^{(n+1)}\, {}^{q}(y)$$

$$= \sum_{q/(n+1)} (x_1, \cdots, x_{n+1}|s|y_{q1}, \cdots y_{q(n+1)})\omega_+(y_{q(n+2)}, \cdots y_{q(n+\nu+1)}) =$$

$$= \sum_{q/(n+1)} \sum_{\pi/n} (x_1, \cdots, x_n|s|y_{\pi q 1}, \cdots y_{\pi q n})(x_{n+1}|s|y_{\pi q(n+1)}) \times \omega_+(y_{q(n+2)}, \cdots, y_{q(n+\nu+1)})$$

$$= \sum_{\pi/n} \sum_{q/\pi(n+1)'} (x_1 \cdots x_n|s|y_{\pi 1}, \cdots, y_{\pi n})\ (x_{n+1}|s|y_{q\pi(n+1)}) \times \omega_+(y_{q\pi(n+2)}, \cdots y_{q\pi(n+\nu+1)})$$

$$= \sum_{\pi/n} (x_1, \cdots x_n|s|y_{\pi 1}, \cdots y_{\pi n}) \sum_{q/\pi(n+1)'} \times (x_{n+1}|s|y_{q\pi(n+1)})\ \omega_+(y_{q\pi(n+2)}, \cdots y_{q\pi(n+\nu+1)})$$

$$= A^{-}_{n+1}(x_{n+1}) \sum_{\pi/n} (x_1, \cdots, x_n|s|y_{\pi 1}, \cdots y_{\pi n})\omega_+(y_{\pi(n+1)}, \cdots y_{\pi(n+1+\nu)})$$

$$= A^{-}_{n+1}(x_{n+1}) \left[{}^{n}A^{-}(x),\ B^{n+1+\nu}(y) \right] \omega_{+} = \left[{}^{n+1}A^{-}(x),\ {}^{n+1+\nu}B(y) \right] \omega_{+} .$$

We go from the first member to the second by simply expanding the notation; from the second to the third, by substituting (24). A simple consideration shows the validity of the interchange of the order of summations involved in passing from the third to the fourth member. We can see that the q summation is carried out as indicated in the fifth by noting its equivalence to the action of a destruction operator on a state vector, which leads to the sixth. In passing to the seventh we use the hypothesis that the statement holds for $m = n$. The transition to the eighth involves an elementary manipulation with operators. The same may be done for E_{-}.

RETARDED COMMUTATORS AND RECIPROCITIES

Our goal is to express an arbitrary element of the scattering operator by single retarded commutators of multiple fields. One of the factors of the commutator is constructed on Heisenberg operators of the initial, and the other, of the final state. The Dirac identities provide the link between the incoming and outgoing fields and these factors. In arriving at the Dirac identifies we introduced into the expansions (D) for ${}^{m}A_{i}, {}^{m}A_{o}$ internal orderings ($\pm$) of the fields pertaining to the initial and final states. These have no physical basis and should not appear in any physically significant context. In this section we derive reciprocity relations which indicate that this is indeed so, and that the only relevant temporal order is that of the initial relative to the final state. To express this order we need another θ symbol,

$$
{}^{m}(x|\theta|y)^{n} = (x_1, \cdots, x_m|\theta|y_1, \cdots, y_n) = 1, \text{ if } x_i^0 \text{ is less than } y_j^0 \quad \text{for all } 1 \le i \le m,\ 1 \le j \le n,
$$
$$
= 0, \text{ if not,}
$$

with the obvious composition law

$$
{}^{m}(x|\theta|y)^{n} = (x_1|\theta|y)^{n}(x_2|\theta|y)^{n} \cdots (x_m|\theta|y)^{n} = {}^{m}(x|\theta|y_1)\,{}^{m}(x|\theta|y_2) \cdots {}^{m}(x|\theta|y_n). \tag{25}
$$

We start with the identities

$$
[\,{}^{m}A_{i}(x), {}^{n}F(y)\,] = (-1)^{\sigma}\int {}^{m-\sigma}dK'\,(x-\xi)^{\sigma'}(\xi|\theta|y)^{n}\,[\,{}^{m-\sigma}A_{i}(x)^{m-\sigma}A'_{+}(\xi), {}^{n}F(y)\,], \tag{26i+}
$$

$$
[\,{}^{m}A_{i}(x), {}^{n}F(y)\,] = (-1)^{\sigma}\int {}^{\sigma}dK(x-\xi)^{\sigma}(\xi|\theta|y)^{n}\,[\,{}^{\sigma}A(\xi)\,{}^{\sigma}A'_{i}(x), {}^{n}F(y)\,], \tag{26i-}
$$

$$
[\,{}^{m}F(x), {}^{n}B_{o}(y)\,] = \int {}^{\sigma}dK(y-\eta)\,{}^{m}(x|\theta|\eta)^{\sigma}\,[\,{}^{m}F(x), {}^{\sigma}B_{+}(\eta)^{\sigma}B'_{o}(y)\,], \tag{26o+}
$$

$$[\,{}^{m}F(x), {}^{n}\underset{o}{B}\;(y)\,] = \int^{n-\sigma} dK'(y-\eta)\; {}^{m}(x\,|\theta|\,\eta)'^{\sigma}\; [\,{}^{m}F(x), {}^{n-\sigma}\underset{o}{B}(y)^{n-\sigma}B'_{-}(\eta)\,]\;, \tag{26o-}$$

where ${}^{n}F(x) = F(x_1, \cdots x_n)$ is an operator depending on the indicated coordinates. They express the commutator of an incoming or outgoing field with an arbitrary field as an integral transform of a retarded commutator. One of its factors has a part in which the initial or final fields are internally (±) ordered. This order depends on the position the fields occupy relative to the unordered part. With the devices of LSZ their derivation is quite simple. We illustrate it by indicating the essential steps that lead to (26i-):

$$[\,{}^{m}\underset{i}{A}(x), {}^{n}F(y)\,] = \int_{-\infty} dH(x_1-\xi_1)\,[\,\underset{i}{A}(\xi_1)\,\underset{i}{A}'(x), {}^{n}F(y)\,] =$$

$$= -\int dK\,(x_1-\xi_1)\,(\xi_1|\theta|y)^{n}\,[\,A(\xi_1)\,\underset{i}{A}'(x), {}^{n}F(y)\,]\,. \tag{27}$$

In going from the second to the third member we ordered ξ_1 relative to the set of y's by means of the θ symbol. With this symbol in, we could with impunity add a surface integral at $+\infty$ and apply Green's theorem. The next step is

$$[\,A(\xi_1)\,\underset{i}{A}'(x), {}^{n}F(y)] = \int_{-\infty} dH(x_2-\xi_2)\,[\,A(\xi_1)A(\xi_2)\,{}^{2}\underset{i}{A}'(x), {}^{n}F(y)\,] =$$

$$\int_{-\infty} dH(x_2-\xi_2)\,(\xi_2|\theta|y)^{n}\,[\,A_1(\xi_1)A_2(\xi_2)\,{}^{2}\underset{i}{A}'(x), {}^{n}F(y)\,]$$

$$-\int_{+\infty} dH(x_2-\xi_2)\,(\xi_2\,|\theta|y)^{n}\,[\,A_2(\xi_2)A_1(\xi_1)^{m-2}\underset{i}{A}'(x), {}^{n}F(y)]$$

$$= -\int dK\,(x_2-\xi_2)\,(\xi_2|\theta|y)^{n}\,[\,A_{-}(\xi_1\xi_2)^{m-2}\underset{i}{A}'(x), {}^{n}F(y)\,]\,, \tag{28}$$

where the third member is obtained from the second by equipping it with a θ factor and subtracting a null surface integral at $+\infty$, with the order of the $A(\xi)$ reversed. The fourth member is obtained from the third on the basis of the observation that the order of the $A(\xi)$'s in the surface integrals corresponds to that of $A_-(\xi_1\xi_2)$. Substituting (28) into the last member of (27), we notice that the θ factors combine according to (25), and the resulting identity is (26i) with $\sigma=2$.

We now take $\sigma = m$ in (26i) and $\sigma = n$ in (26o). From this we deduce

$$\int {}^m dK(x-\xi)^n(\xi\,|\theta|\,y)^n\,[\,{}^mA_+(\xi),{}^nF(y)\,] = \int {}^m dK(x-\xi)^m(\xi\,|\theta|\,y)^n\,[\,{}^mA_-(\xi),{}^nF(y)] \tag{29}$$

and a similar equation involving B with identical content; the $(\pm)$ ordering is immaterial for the transform of the retarded commutator. Identities (26) may then by rewritten as

$$(-1)^m\,[\,{}^m\underset{i}{A}(x),{}^nF(y)] = \int {}^m dK(x-\xi)^m(\xi\,|\theta|\,y)^n\,[\,{}^mA(\xi),{}^nF(y)]\,, \tag{30i}$$

$$[\,{}^mF(x),\,{}^n\underset{o}{B}(y)\,] = \int {}^n dK(y-\eta)^m(x\,|\theta|\,\eta)^n\,[\,{}^mF(x),{}^nB(\eta)]\,, \tag{30o}$$

where we omitted the otiose subscripts of ${}^mA(\xi)$ and ${}^nB(\eta)$. This accords with the fact that $\underset{i}{A}$ and $\underset{o}{B}$ are unordered. Substituting ${}^mF(x) = {}^mA_\pm(x)$, ${}^nF(y) = {}^nB_\pm(y)$ into (30) and subjecting both sides to the K transform whose domain is y in (30i) and x in (30, o), we obtain, with (16b),

$$(-1)^m\,[\,{}^m\underset{i}{A}(x),{}^n\mathcal{B}_\pm(y)\,] = \int {}^m dK(x-\xi)^n dK(y-\eta)^m(\xi\,|\theta|\,\eta)^n\,[\,{}^mA(\xi),{}^nB(\eta)]\,, \tag{31,i}$$

$$[\,{}^m\mathcal{Q}_\pm(x),{}^n\underset{o}{B}(y)\,] = \int {}^m dK(x-\xi)^n dK(y-\eta)^m(\xi\,|\theta|\,\eta)^n\,[\,{}^mA(\xi),{}^nB(\eta)\,]\,, \tag{31, o}$$

where the $(\pm)$ in the retarded commutator has been dropped by invoking (29).

The right-hand member of (31,i), identical with that of (31, o), and independent of the internal ordering of the A's and B's, plays a significant role in the next section. Of direct physical interest will be

$$ {}^{m}(x\,|\tau|\,y)^{n} = \int {}^{m}dK^{-}(x-\xi)^{n}\, dK^{+}(y-\eta)^{n}(\xi\,|\theta|\,\eta)^{n}\,\Omega_{-}\,[\,{}^{m}A(\xi),\,{}^{n}B(\eta)\,]\;\Omega_{+}\,. \quad (32) $$

This integral transform (with K^{-} on the initial, K^{+} on the final state) of the retarded commutator in the vacuum state is referred to as its projection unto the mass shell. Here we eliminate it between the (i) and (o) to obtain two sets of reciprocity relations. The first,

$$ [\,{}^{m}\underset{i}{A}(x),\,{}^{n}\mathcal{B}_{+}(y)\,] = [\,{}^{m}\underset{i}{A}(x),\,{}^{n}\mathcal{B}_{-}(y)\,]\,, $$

$$ [\,{}^{m}\mathcal{A}_{+}(x),\,{}^{n}\underset{o}{B}(y)\,] = [\,{}^{m}\mathcal{A}_{-}(x),\,{}^{n}\underset{o}{B}(y)\,]\,, \quad (33) $$

states that the unordered character of $\underset{i}{A}$ and $\underset{o}{B}$ renders the ordering of the radiation operators with which they are commuted immaterial. A single denotation in the commutator thendoes for both orderings. The second,

$$ [\,{}^{m}\mathcal{A}(x),\,{}^{n}\underset{o}{B}(y)\,] = (-1)^{m}\,[\,{}^{m}\underset{i}{A}(x),\,{}^{n}\mathcal{B}(y)\,]\,, \quad (34) $$

relates the commutator of a radiation operator with an outgoing field to that of an incoming.

REDUCTION OF S

With the material of the preceding three sections this is done quite simply. We put

$$ {}^{m}(x|S|y)^{n} = \Omega_{-}\, {}^{m}A_{i}^{-}(x)\, {}^{n}B_{o}^{-}(y)\Omega_{+} = \Omega_{-}[\, {}^{m}A_{i}^{-}(x),\, {}^{n}B_{o}^{+}(y)\,]\,\Omega_{+} + \delta^{m0}\,\delta^{n0}, \tag{35} $$

where we added a Kronecker delta term to make the equality sign hold for all nonnegative integers m, n. Substituting (D_1a) for B_o, we have

$$ {}^{m}(x|S|y)^{n} = \frac{1}{n!}\sum_{\rho}\binom{n}{\rho}\sum_{q}\Omega_{-}[\, {}^{m}A_{i}^{-}(x),\, {}^{\rho q}B_{i}^{+}(y)\,]\, {}^{\rho q}\mathcal{B}'^{+}_{+}(y)\Omega_{+} $$

$$ + \frac{1}{n!}\sum_{\rho}\binom{n}{\rho}\sum_{q}\Omega_{-}\, {}^{\rho q}B_{i}^{+}(y)[\, {}^{m}A_{i}^{-}(x),\, {}^{\rho' q}B'^{+}(y)\,]\Omega_{+} + \delta^{m0}\delta^{n0}, $$

where (33) was used to drop the + ordering subscript from the $\mathcal{B}$ in the commutator of the second term of the right-hand member. We now identify A_i, B_i with the free fields of E_- and use this relation to substitute for the commutator in the first term, with the simple result

$$ {}^{n}(x|S|y)^{n} = \frac{1}{m!}\,\frac{1}{n!}\sum_{\sigma\rho}\binom{m}{\sigma}\binom{n}{\rho}\sum_{pq}{}^{\sigma p}(x|s|y)^{q\rho}\,\Omega_{-}\, {}^{\sigma p}A_{i}'^{-}(x)\, {}^{\rho q}\mathcal{B}'^{+}_{+}(y)\Omega_{+}, \tag{37} $$

because we have

$$ \frac{1}{n!}\sum_{\rho}\binom{n}{\rho}\sum_{q}\Omega_{-}\, {}^{\rho q}B_{i}^{+}(y)[\, {}^{m}A_{i}^{-}(x),\, {}^{\rho' q'}\mathcal{B}^{+}(y)\,]\Omega_{+} $$

$$ + \delta^{m0}\delta^{n0} - \frac{1}{n!}\sum_{\rho}\binom{n}{\rho}\sum_{q}\delta^{\rho 0}\Omega_{-}\, {}^{m}A_{i}^{-}(x)^{\rho' q}B'^{+}_{+}(y)\Omega_{+} = 0\,. \tag{38} $$

The last term of (38) comes from the "boundary term" from E_-. To verify (38), observe that the only contribution to the ρ sum of the first and last term is from ρ = 0; the middle combines with the last to give a commutator that cancels the first. The A in (37) is now commuted with the $\mathcal{B}$ (with the addition of another Kronecker delta term) and this commutator $[\, A_i^{-}(x), \mathcal{B}^{+}(y)\,]$ is just the τ defined by (32). The expansion

$$^{n}(x|S|y)^{n} = {}^{m}(x|s|y)^{n} + \frac{1}{m!\,n!} \sum_{\sigma\rho p,q} \Sigma(-1)^{\sigma} \binom{m}{\sigma} \binom{n}{\rho} {}^{\sigma p}(x|\tau|y)^{q\rho\,\sigma p'}(x|s|y)'^{q\rho} \tag{39a}$$

is obtained after some trivial changes in the summation variables. An alternative procedure is to first express the incoming field by means of $(D_2 b)$ in terms of the outgoing and then have recourse to E_+. In this scheme one has, in the final stage, vacuum expectation values of commutators of the type $[\mathfrak{a}^-(x), B_o^+(y)]$. Reciprocity (34) then indicates that the result is the same by virtue of an operator identity.

Without redundant sums and symmetrizations, (39a) states

$$^{m}(x|S|y)^{n} = {}^{m}(x|s|y)^{n} + (-1)^{m} \sum_{\sigma=0}^{\mu(m,n)} (-1)^{\sigma} \sum_{p/\sigma} \sum_{q/\sigma} {}^{\sigma p}(x|s|y)^{q\sigma\,\sigma p'}(x|\tau|y)'^{q\sigma}, \tag{39b}$$

where we make use of the orthogonality of initial and final states for the free s matrix. The μ associated with the upper limit of the summation means "the lesser of." Such explicit indications of limits are not needed in the representations with binomial coefficients. It is this accident that recommends them for certain tasks. Expression (39b) is, however, more convenient for generating particular expansion of S in terms of s. We see from (32) that we have ${}^{0}\tau^{n} = {}^{m}\tau^{0} = 0$ for all m and n. As a test of the algebraic consistency of the formula, we also note that

$$^{o}(x|S|y)^{n} = \delta^{0n} \text{ and } {}^{m}(x|S|y)^{0} = \delta^{m0} \tag{40, o}$$

follow from this property of τ. For the simplest processes--decays--the expansions are

$$\begin{aligned}(x|S|y) &= (x|s|y) - (s|\tau|y), \\ (x|s|y_1\, y_2 \cdots y_n) &= -(x|\tau|y_1\, y_2 \cdots y_n) \text{ for } n > 1.\end{aligned} \tag{40$_1$}$$

With two fields in the initial state the various possibilities in the final are indicated in

$$(x_1x_2|S|y) = (x_1x_2|\tau|y),$$
$$(x_1x_2|S|y_1y_2) = (x_1x_2|s|y_1y_2) + (x_1x_2|\tau|y_1y_2) - (x_1|s|y_1)(x_2|\tau|y_2) - \cdots, \qquad (40_2)$$
$$(x_1x_2|S|y_1y_2\cdots y_n) = (x_1x_2|\tau|y_1y_2\cdots y_n) - (x_1|s|y_1)(x_2|\tau|y_2\cdots y_n) - \cdots,$$
$$\text{for } n > 2 .$$

The presence of one-to-one particle-transition terms should be noted. They do not appear in a properly renormalized theory that deals only with stable particles.

The combinational sense of the expansion is evident. A game which m particles enter and n leave is played according to all possible arrangements of spectators and participants. Throughout each engagement (each term of the expansion) the spectators maintain their number and identity; players may be lost in or new ones emerge from the scuffle. Since preponderantly spectator sports do not seem to be popular at high energies, the expansion does not signify.

We now discuss the relation of the reduction (39) to other reductions of the S matrix. The idea is an old one; its nearest classical correlative is, perhaps, found in the work of Dirac. The energy (power) radiated by a charge distribution may be calculated from the time integral (average for discrete lines) of the surface integral of the Poynting vector at a surface remote from the source. By defining a radiation field inside the charge distribution this author was able to transform the asymptotic representation into a space-time integral over the charge. This operation corresponds to dH ——> dK integration. Most of the perturbation treatments of scattering that favor the Heisenberg picture follow this impulse of Dirac rather closely. They are equivalent to the use of (D_2a) to obtain

$$^m(x|S|y)^n = \frac{1}{n!} \sum_\rho \binom{n}{\rho} \sum_q \Omega^-_i \; {}^m(x)\; {}^{\rho q}\mathcal{B}_-^{\;+}(y)\; {}^{\rho q}\, \Omega'_i + (y) , \qquad (41i)$$

where the expectation of the (-) ordered radiation fields (earlier times to the right) is taken in states constructed of incoming fields. The observation that the other choice would be just as good is equivalent to the use of (D_1b), which gives

$$ {}^{m}(x|S|y)^{n} = \frac{(-1)^{m}}{m!} \sum_{\sigma} \binom{m}{\sigma} (-1)^{\sigma} \sum_{p} \Omega_{o-}^{p\sigma}(x)^{\sigma' p} \mathfrak{a}'_{-}(x)^{n}\, \Omega_{o+}(y) \,. \tag{41o} $$

The joint use of (D_2a) and (D_1b) produces a complete reversal of fields:

$$ {}^{m}(x|S|y)^{n} = \frac{(-1)^{m}}{m!\, n!} \sum_{\sigma\rho} (-1)^{\sigma} \binom{m}{\sigma} \binom{n}{\rho} \sum_{pq} \Omega_{o-}^{p\sigma}(x)^{\sigma' p} \mathfrak{a}'_{-}(x)^{\rho q} \mathfrak{B}_{-}^{+}(y)^{\rho' q} \Omega'_{i+}(y) \,. \tag{41r} $$

We are not aware of any past use of this representation. It should be noted that the (±) order is pertinent to (41). Somewhat outside all this is the Green's-function formalism of Schwinger. Its reflection in the S-matrix context amounts to an indiscriminate single ordering of all fields, initial and final. What distinguishes the reduction used in dispersion work from the others is the reciprocity (34) that enables us to omit the (±) subscript; it is, on the other hand, decidedly emphatic in its discrimination between initial and final states.

The development of this article followed, too, the path of Dirac, but had a particular orientation: it sought to express S in terms of s. Prompted by the simple observation that a matrix element of s-- a c-number--could be identified with a commutator--a function on q numbers--of multiple fields, it proceeded to generate the former by means of the latter (E). To contrive a mutual confrontation in S of the two factors of this commutator it augmented the number of Dirac identities (D_1b, D_2a) that may be used in perturbation treatments with two more (D_1a, D_2b). The elements of τ were generated in the course of this confrontation.

MICROCAUSALITY

We assume the existence of a complete,

$$\sum_a \int d\kappa^2 \rho(\kappa, a) \int \frac{d^4k}{(2\pi)^4} \Delta^+(k, \kappa)\, \omega_+(k, a)\omega_-(k\, a) = 1, \tag{42}$$

orthonormal,

$$\rho(\kappa, a)\, \Delta^+(k, \kappa)\omega_-(k,a)\, \omega_+(k', a')\Delta^+(k', \kappa')\rho(\kappa', a')$$
$$= (2\pi)^4\, \rho(\kappa, a)\Delta^+(k, \kappa)\delta(k-k')\delta(\kappa^2 - \kappa'^2)\delta_{aa'}, \tag{43}$$

set of eigenstates $\omega(k,a)$ of the Hilbert-space representation U^D of the displacement group D,

$$U_a^D \omega_+(ka) = e^{-ika}\omega_+(k,a) \tag{44}$$

undistorted by mappings of the Lorentz group

$$\omega(k,a) \xrightarrow[\Lambda(t)]{} \omega^t(k, a) \equiv U_t^{\Lambda}\, \omega(k_t, a) = \omega(k\, a)\ . \tag{45}$$

Completeness (42) and orthonormality (43) have been stated in terms of two distributions, rather than proper functions, in order to evidence their covariant character. These are $\rho(\kappa, a)$, the spectral density of the mass) and

$$\Delta^+(k, \kappa) = 2\pi\ \theta(k)\ \delta(k^2 - \kappa^2)\ ,$$

the Fourier transform of the familiar $\Delta^+(x)$. All other quantum numbers needed for the definition of a state have been designated by a. Orthogonality in these is indicated by a dimensionless Kronecker delta to accord with the fact that in (42) ω must have the dimension of length. Assumptions (42) and (43) imply the possibility of expanding an arbitrary element $\Omega_\pm$ of the Hilbert space

$$\Omega = \sum_a \int d\kappa^2 \int \frac{d^4k}{(2\pi)^4}\, \omega(ka)\Delta^+(k,\kappa)\rho(\kappa,a)\,\psi(ka) \tag{46a}$$

with

$$\rho(\kappa a)\ \Delta^+(k\kappa)\ [\,\psi(ka) - \omega_-(ka)\ \Omega_+\,] = 0\ , \tag{46b+}$$

$$\rho(\kappa a)\ \Delta^+(k\kappa)\ [\,\psi(ka) - \Omega_-\omega_+(ka)\,] = 0 \tag{46b-}$$

for contravariant and covariant Hilbert-space vectors, respectively.

Expanding ${}^{m}A_{\pm}(x)\omega_{+}(k\alpha)$ and $\omega_{-}(k\alpha){}^{m}A_{\pm}(x)$, we denote the vacuum components of these vectors, the Bethe-Salpeter amplitudes,[4] by

$$ {}^{m}\psi_{\pm}^{\ +}(x;\, k\alpha) = \omega_{-}(k\alpha)\ {}^{m}A_{\pm}(x)\ \omega_{+}\ , \tag{47+} $$

$$ {}^{m}\psi_{\pm}^{\ -}(x;\, k\alpha) = \omega_{-}\ {}^{m}A_{\pm}(x)\ \omega_{+}\ (k\alpha)\ . \tag{47-} $$

Under a Lorentz mapping we have

$$ {}^{m}\psi_{+}^{\ -}(x, k\alpha) \xrightarrow[\Delta(t)]{} {}^{m}\psi_{+}^{\ t-}(x, k\alpha) \equiv {}^{m}\psi_{+}^{\ -}(x_{-t}, k_{t}\alpha)\ , $$

and with (47-) and (45) obtain

$$ {}^{m}\psi^{t-}_{\ +}(x, k\alpha) = \omega_{-}U_{t}^{\ \Delta}\,{}^{m}A(x_{-t})U_{-t}^{\ \Delta}\omega_{+}(k\alpha)\ . \tag{48} $$

It is evident from (48) that ψ's admit the Lorentz group trivially, for $m = 0$; by virtue of the defining equation (6) of the U^{Δ} representation, for $m = 1$ but, in general, not for higher m, we can write

$$ {}^{m}\psi_{\pm}^{\ t} = {}^{m}\psi_{\pm} \quad \text{for } m = 0,\ 1.\ (\Delta) \tag{49} $$

For the displacement group, because the fields are ordered relative to each other, we have, for all m,

$$ U_{a}^{\ D}\ {}^{m}A_{\pm}(x - a)\ U_{-a}^{\ D} = {}^{m}A_{\pm}(x) $$

and therefore

$$ {}^{m}\psi_{\pm}^{\ a} = {}^{m}\psi_{\pm} \quad \text{for all } m.\ (D) \tag{50} $$

To obtain a differential version of (50) we define

$$ {}^{m}\Box_{\mu} = \frac{\partial}{\partial x_{1}^{\ \mu}} + \frac{\partial}{\partial x_{2}^{\ \mu}} + \cdots + \frac{\partial}{\partial x_{m}^{\ \mu}}\ , $$

and deduce from (47), (48), and the well-known exponential representation of U^{D}

$$ i\,{}^{m}\Box_{\mu}\ \rho(\kappa\alpha)\ \Delta^{+}(k\kappa)\,{}^{m}\psi^{\pm}(x, k\alpha) = \pm k_{\mu}\ \rho(\kappa\alpha)\ \Delta^{+}(k,\kappa)\,{}^{m}\psi^{\pm}(x, k\alpha), \tag{51} $$

which indicates that $\rho\Delta^{+}\psi^{+}$ and $\rho\Delta^{+}\psi^{-}$ are the positive and negative frequency functions satisfying the generalized Gordon-Klein equation

$$({}^{m}\square^{2} + \kappa^{2})\,\rho(\kappa, a)\,\Delta^{+}(k, \kappa)\,{}^{m}\psi_{\pm}(x, ka) = 0\,. \qquad (52G_{o})$$

The definition of τ may now be given in terms of the B-S amplitudes:

$$^{m}(x\,|\tau|\,y)^{n} = \int^{m} dK^{-}(x - \xi)^{n} dK^{+}(y - \eta)^{m}(\xi\,|\theta|\,\eta)^{n}\,{}^{m}(\xi\,|D|\,\eta)^{n}, \qquad (53)$$

$$^{m}(x\,|D|\,y)^{n} = \sum_{a}\int d\kappa^{2}\,\rho(\kappa, a)\int\frac{d^{4}k}{(2\pi)^{4}}\,\Delta^{+}(k, \kappa)$$

$$\times\,[\,{}^{m}\psi^{-}(x, ka)\,{}^{n}\phi^{+}(y, ka) - {}^{m}\psi^{+}(x, ka)\,{}^{n}\phi^{-}(y, ka)\,] \qquad (54)$$

where ψ pertains to the initial, ϕ to the final state. The ($\pm$) ordering subscripts of ψ and ϕ have been omitted in accordance with (32). For the special case $m = n = 1$, this reduces to the well-known bilinear representation of the radiation kernel, vanishing for $(x - y)^{2} < 0$ and admitting the Lorentz group. The latter property is an immediate consequence of (49), and because of the restricted validity of this equation, need not hold for other m and n. The implications of the requirement that S admit the Lorentz group are now plain. Since s admits the group, this burden is shifted to τ. An inspection of (53) indicates that the radiation kernel D with an arbitrary number of initial and final particles must conform in its basic properties to that of a single particle:

(a) $^{m}(x\,|D|\,y)^{n}$ must admit the Lorentz group for all m, n;

(b) $^{m}(x\,|D|\,y)^{n} = 0$ if $(x_{i} - y_{j})^{2}$ is less than 0 for any $i = 1 \cdots m$, $j = 1 \cdots n$.

That the physically significant quantity, τ, evinces no interest in our nice distinctions between the + and - order is suggestive. Let us imagine that we expanded the single equation ($52\,G_{o}$) into an set ($G_{0}, G_{1}, \cdots G_{s} \cdots a_{\pm}$), whose solutions are ψ's equivalent to the definition (47) of ψ in terms of expectation values of ordered operators. The G's of this set are differential equations coupling the various amplitudes, and the $a_{\pm}$ denotes statements of asymptotic conditions, which discriminate between the orderings. Since they are irrelevant to τ, one is tempted to replace them by a set (b) of boundary conditions; that is, statements of the asymptotic behavior of the solutions ${}^{m}\psi(x)$ of the system $(G_0', \cdots G_s' \cdots b)$ for large spacelike separations of the arguments. The primes of the G indicate that these boundary conditions might be more simply expressed for certain linear combinations of ψ (the Tamm-Dancoft wave functions, say, rather than the Bethe-Salpeter amplitudes).[4] It is not unreasonable to expect that these boundary conditions could be satisfied only for certain values of κ and thus determine $\rho(\kappa)$ in terms of a few elementary masses. The wave functions obtained from solving this set could also be used to construct the bilinear form (54).

This suggests the view that it is the wave mechanics $(G'b)$ that is the substantial part of quantum theory of fields; manipulations with operators are merely an umbral calculus to generate the mechanics. This descriptive term, borrowed from combinational analysis, implies there that the analytic behavior of the generating functions is irrelevant to the final result, the correct enumeration of things. One might perhaps adopt a similar view that the precise conditions imposed on field operators are not particularly germane to the substantial part of field theory. This mechanics finds its inchoate expression, at the moment, in dispersion theory. In it, an attempt is made to exploit one of the facets of the mechanics, the vanishing of a bilinear form on the wave functions in certain regions and the existence of a mass spectrum with positive frequencies.

SUMMARY

.An arbitrary element of S may be expanded in terms of elements of s with elements of τ as coefficients.

.The elements of τ are mass-shell projections of a retarded generalized Green's function which is a product of a retardation factor ordering the final relative to the initial state, and a generalized radiation kernel **D**.

.The kernel **D** is a bilinear form in the BS amplitudes, appropriate to the initial and final states.

.The internal ordering of the fields of the initial and final states entering into the definition of the BS amplitudes in terms of matrix elements of products of operators is immaterial for **D**.

.The requirement that S follow the example set by s in admitting the Lorentz group (causality) leads to the condition that ${}^{m}D^{n}$ conform to the pattern established by the single-field kernel ${}^{1}D^{1}$ in admitting the group and in vanishing for spacelike separations of the initial and final states (microcausality).

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