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Bridging the Gap between First- and Second-Price Auctions with Withdrawable Winning Bids

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Abstract

This paper formulates and analyzes game-theoretic and decision-theoretic models of auctions in which bidders may submit multiple bids and, perhaps at cost, withdraw bids that are more aggressive than necessary to win. While such withdrawal strategies are currently surreptitious, legitimization would create market mechanisms intermediate between first-price and second-price auctions. We describe a particular auction in which a winning bid was withdrawn and fit one of our models to data from it.

1. Introduction

Bidders sometimes try to "beat the system". Occasionally a winning bidder will try to improve its situation by withdrawing its bid and allowing success of the second best bid, one offered by a formally separate but economically related bidder. Such a practice is likely to be surreptitious and may involve fraud. Recently, however, we have been able to document a situation, described below, in which a winning bidder successfully and, apparently profitably, withdrew a bid.

We are not aware of any models of auctions in which a bidder can submit more than one bid and, perhaps at a price, withdraw his more aggressive bids in order to win the auction with one of his less aggressive bids. It is the purpose of this paper to present several such models, some decision theoretic and some game theoretic. But before describing the particular models and results, we want to point out that the motivation for studying them goes well beyond analyzing unusual occurrences of the type described below. There is substantial literature dating back to Vickrey [1961], contrasting "second-price" auctions (including both second-price sealed procedures suggested by Vickrey and oral progressive auctions) with "first-price" auctions (including both standard sealed bidding and oral Dutch auctions). This literature includes discussions of "discriminatory" vs. "nondiscriminatory" procedures for Treasury bill auctions [Reiber, 1963] [Reiber, 1964] [Friedman, 1963], discussion of alternate oil lease procedures [Robinson, 1984] and of timber sale policies [Mead, 1967] [Weiner, 1979], as well as purely theoretical analyses, e.g. [Milgrom and Weber, 1982]. These alternate procedures have different strengths and weaknesses under different conditions. Nowhere in the literature, however, are we aware of any discussion of any procedure for single object auctions that is intermediate, in any sense, between first-price and second-price procedures.

The withdrawable bid procedures we are about to discuss are such intermediate procedures. As the number of allowed bids per bidder is decreased to one, or as the penalty for withdrawing a bid becomes large, the procedures become first-price auctions.

On the other hand, as the penalty for bid withdrawal is reduced and the number of bids allowed is made large, the auction approaches a second-price auction. In the limit with such rules, each bidder could submit a bid at every price at which he would prefer winning to losing and then, after all bids are revealed, withdraw all of his bids that are more aggressive than a bid that would allow him to win. When this happens, the winning bidder pays marginally more (or receives marginally less) than would be acceptable to the second most aggressive bidder.

Table 1 contains descriptions of the principal models in this paper. In Section 2, we develop an independent value, high-bid-wins, decision-theoretic model. We consider this model when the single strategic bidder may submit up to K bids and may withdraw a bid for either a fixed or a proportional penalty. For $K = 2$, we obtain an analytic solution for the optimal strategy and the resulting expected profit as a function of the size of the penalty, and of the parameters of the uniform distribution assumed for the best competitive bid. We obtain similar results for arbitrary K when there is no penalty. We also examine the decision theoretic model when there is a triangular distribution for the best competitive bid. Finally, this section presents completely analagous results for low bid wins auctions.

The third section contains a description of an auction in which a bidder withdrew a winning bid and a brief application a model from Section 2 to it.

Section 4 considers game-theoretic common value models with multiplicative bidding strategies and proportional penalties. We present results for high-bid-wins and low-bid-wins models in which two bids are allowed. We also present models in which there is no penalty and no limit on the number of bids.

The paper concludes with a brief discussion of the results and of "legitimization" of withdrawable bid auctions.

Table 1

Description of Principal Models

Section Number	High- or Low-Bid-Wins	Withdrawal Penalty Type	Model Type	Bid Distribution	Number of Bids
2.1	High	Fixed	Decision Theory	Uniform	2
	High	Proportional	Decision Theory	Uniform	2
	High	None	Decision Theory	Uniform	K
	High	Fixed	Decision Theory	Triangular	2
	High	Proportional	Decision Theory	Triangular	2
2.2	Low	Fixed	Decision Theory	Uniform	2
	Low	Proportional	Decision Theory	Uniform	2
	Low	None	Decision Theory	Uniform	K
4.1	Low	Proportional	Common Value Game with Multiplicative Strategies	Weibull	2
	High	Proportional	Common Value Game with Multiplicative Strategies	Gumbel	2
4.2	Low	None	Common Value Game with Multiplicative Strategies	Weibull	Infinite
	High	None	Common Value Game with Multiplicative Strategies	Gumbel	Infinite

2. Decision Theoretical Models

2.1 *High-Bid-Wins Models*

This section analyzes a situation in which a bidder, bidding for an object of known value, v, to himself is allowed to make different bids for the object and may withdraw a higher bid for a price after he sees the best competitive bid. To simplify the analysis, we assume that the bidder believes the best competitive bid is distributed uniformly on the interval (a,b), where $0 \leq a < v$ and $b \geq v$ (or at least $b \geq$ than the highest bid the bidder contemplates). We consider two slightly different cases. In the first, the penalty for withdrawing a bid is a fixed amount q. In the second, it is a fraction p of the bid withdrawn. (q for quantity; p for proportion.) In both cases, we assume that there is no legal or moral stigma associated with withdrawing a bid, and that the bidder wishes to bid so as to maximize his expected return.

Before analyzing the withdrawable bid situation, however, we develop for comparison the single bid case. In this case, if the bidder submits a bid of b_1 where $a \leq b_1 \leq v$, his expected profit $E(b_1)$, is $v-b_1$, times the probability he wins, $(b_1-a)/(b-a)$. Thus,

$$E(b_1) = (b_1-a)(v-b_1)/(b-a). \tag{1}$$

Setting $dE/db_1 = 0$ and solving for b_1 gives the optimal bid b_1^*:

$$b_1^* = (v+a)/2. \tag{2}$$

The bidder's maximized expected profit is

$$E(b_1^*) = (v-a)^2/4(b-a). \tag{3}$$

Now, suppose that faced with a fixed penalty $q \geq 0$ for withdrawing a bid, the bidder is permitted to and decides to submit two bids, b_1 and b_2, where $a \leq b_2$ and $b_2 + q \leq b_1 \leq v$. Then, the bidder's profit is $v-b_1$ with probability $(b_1-b_2)/(b-a)$ and is $v-b_2-q$ with probability $(b_2-a)/(b-a)$. Hence,

$$E_q(b_1,b_2) = \left[(b_2-a)(v-b_2-q) + (b_1-b_2)(v-b_1) \right]/(b-a). \tag{4}$$

Setting $\partial E_q/\partial b_1 = \partial E_q/\partial b_2 = 0$ and solving for b_1 and b_2 gives the optimal strategies:

$$b_1^* = (2v+a-q)/3, \tag{5}$$

$$b_2^* = (v+2a-2q)/3, \tag{6}$$

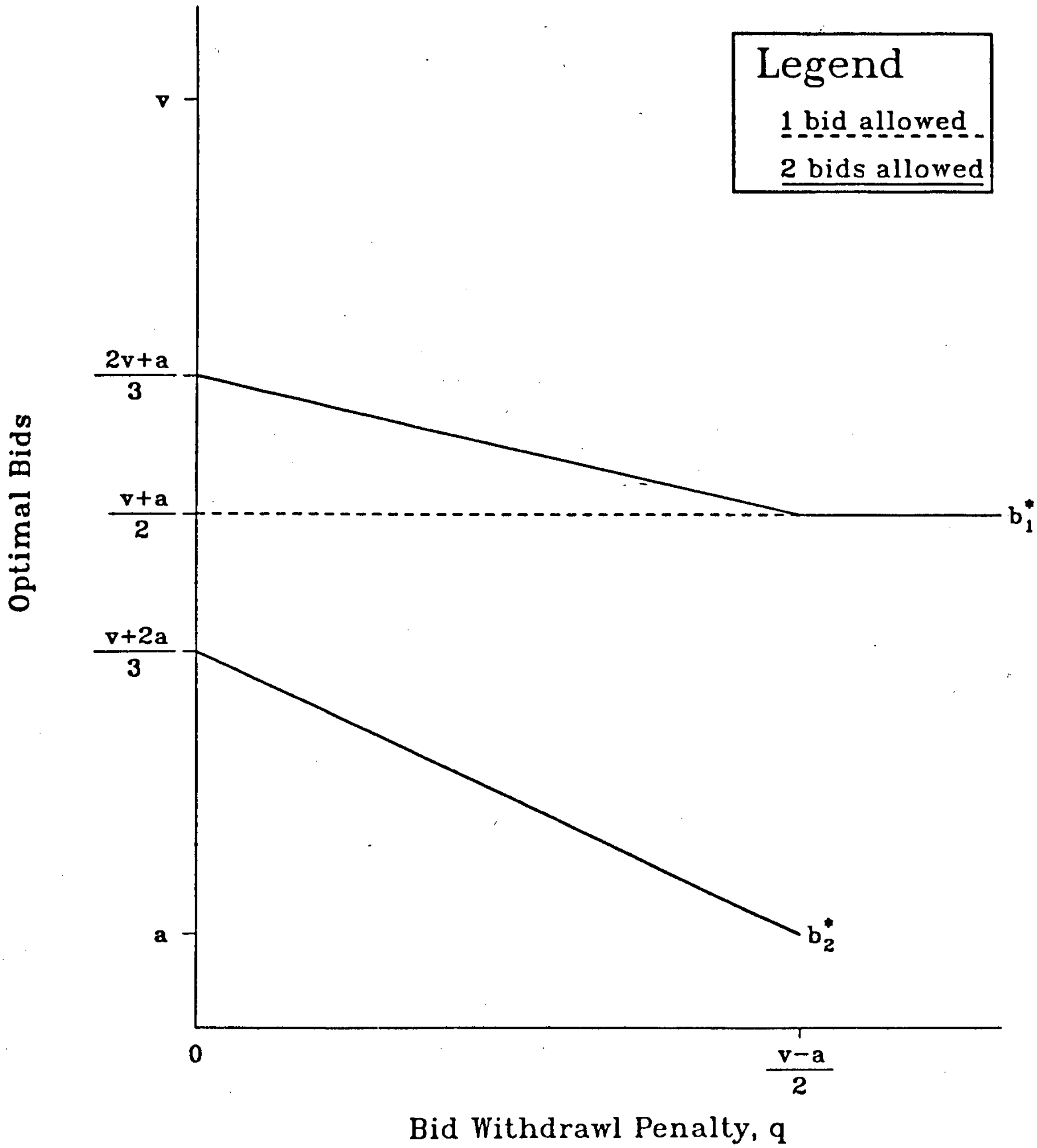
provided $q \leq (v-a)/2$. Otherwise, a single bid is optimal. The expected profit of the optimal pair of bids is

$$E_q(b_1^*,b_2^*) = \left[(v-a-q)^2 + q(v-a) \right]/3(b-a). \tag{7}$$

Figure 1 illustrates the effect on optimal strategy of adding the possibility of a second bid with a fixed bid withdrawal penalty, while holding competitive behavior constant. Figure 2 shows the effect of this on expected profit. Note that b_1^* in (5) is greater than b_1^* in (2) and that b_2^* in (6) is less than b_1^* in (2), provided $0 \leq q \leq (v-a)/2$. In this range, $E_q(b_1^*,b_2^*) \geq E(b_1^*)$. When $q=0$, $E_q(b_1^*,b_2^*) = 4/3\, E(b_1^*)$. As q increases towards $(v-a)/2$, the difference between $E_q(b_1^*,b_2^*)$ and $E(b_1^*)$ decreases towards 0. As q increases toward $(v-a)/2$, both b_2^* and b_1^* decrease linearly towards a and b_1^* as given in (2), respectively.

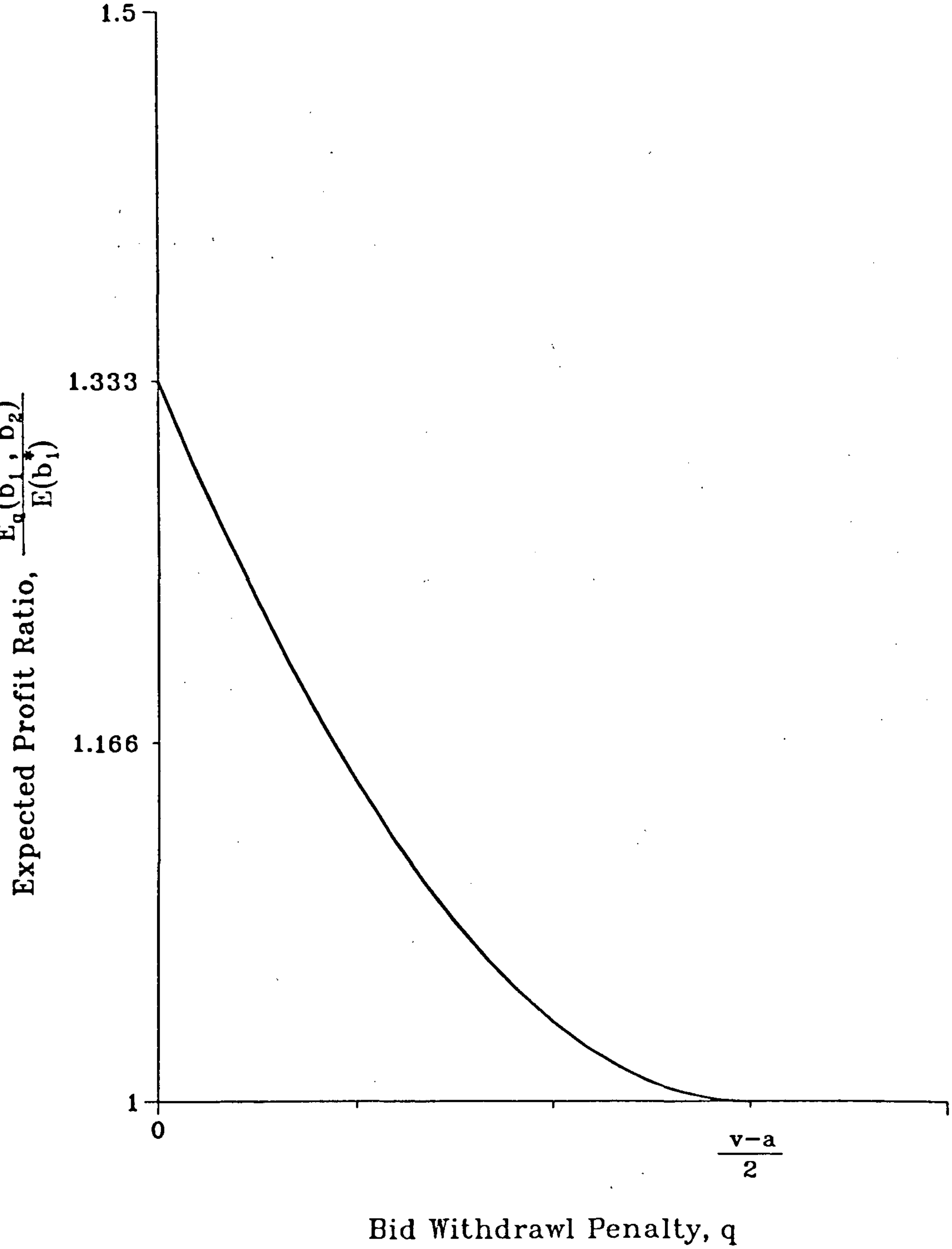
Next, consider a variation of this model in which the penalty q for withdrawing bid b_1 is replaced by a penalty pb_1, a fraction of b_1. Now,

Figure 1



Comparison of optimal bidding strategies with one and two bids allowed; independent private values model with uniform distribution of best competitive bid.

Figure 2



Ratio of expected profit with a pair of optimal bids to expected profit with one optimal bid, assuming the same uniform distribution of best competitive bid.

$$E_p(b_1,b_2) = \Big[(b_2-a)(v-b_2-pb_1) + (b_1-b_2)(v-b_1) \Big]/(b-a). \tag{8}$$

Setting $\partial E_p/\partial b_1 = \partial E_p/\partial b_2 = 0$ and solving for the optimal strategies gives:

$$b_1^* = (2v + a + ap)/(1 + p)(3 - p), \tag{9}$$

$$b_2^* = (v+2a + ap - pv - ap^2)/(1 + p)(3-p), \tag{10}$$

provided $0 \leq p \leq (v-a)/(v+a)$. If p is greater than $(v-a)/(v+a)$, a single bid is optimal. Over the entire range $0 < p < (v-a)/(v+a)$, b_1^* given by (9) is less than b_1^* given by (5) with q replaced by pb_1^*. This result confirms the intuition arising from the observation that reducing b_1 lowers the expected bid withdrawal penalty paid when the penalty is proportional but not when it is fixed. The expression for expected profit from the optimal pair of bids given by (9) and (10) is

$$E_p(b_1^*,b_2^*) = \frac{3(v-a)^2 + p(2v^2-7av+5a^2) - p^2(v^2-3av-a^2) + p^3a(3v-a) - p^4av}{(b-a)(1+p)^2(3-p)^2}. \tag{11}$$

Its value with $p=0$ is the same as the value of $E_q\ (b_1^*,b_2^*)$ with $q=0$. When $p = (v-a)/(v+a)$, b_1^* is given by (2), $b_2^* = a$, and $E_p(b_1^*,b_b^*)\ = E(b_1^*)$.

Next, consider a similar situation, except that this time the bidder is allowed to submit K bids and, if, after the auction, his highest bid is the overall highest bid, he may withdraw without penalty any of his bids provided that the lowest of his bids that is higher than the best competitive bid is left in force. As before, we assume that the object is worth v and that the best competitive bid is uniformly distributed on (a,b) with

$$0 \leq a \leq b_K \leq b_{K-1} \leq \cdots \leq b_1 \leq v \leq b, \tag{12}$$

where b_k, $k = 1,2,\ldots,K$ are his K bids. The bidder's expected profit is given by

$$E(b_1,b_2...,b_K) = \frac{1}{b-a}\left[\sum_{k=1}^{K-1}(v-b_k)(b_k-b_{k+1}) + (v-b_K)(b_K-a)\right] \quad . \tag{13}$$

Setting $\partial E/\partial b_k = 0$, k=1,2,...,K yields the equations:

$$b_1 = \frac{v + b_2}{2} \quad , \tag{14}$$

$$b_k = \frac{b_{k-1} + b_{k+1}}{2}, \; k = 2,3,...,K-1 \quad , \tag{15}$$

$$b_K = \frac{b_{K-1} + a}{2} \; . \tag{16}$$

When solved, these give the optimal bids

$$b_k^* = v - k\,\frac{v-a}{K+1} \; , \; k = 1,2...,K. \tag{17}$$

The bidder's expected profit with the optimal bids is given by

$$E(b_1^*, b_2^*,...,b_K^*) = \frac{(v-a)^2}{2(b-a)} \cdot \frac{K}{K+1} . \tag{18}$$

The ratio of $E(b_1^*,b_2^*,...,b_k^*)$ to $E(b_1^*)$ is $2K/(K+1)$. This ratio increases monotonically from 1 for K=1 to 4/3 for K=2 and towards 2 as K gets large.

The assumption that the best competitive bid is uniform on (a,b) is not only convenient, it is also less restrictive than it might at first seem. If there is an announced minimum price of a, the bidder does not care what the form of the distribution of the best competitive bid takes except on the interval (a,v). Thus, if the part of the distribution lying on (a,v) is roughly uniform given that the best bid does lie in that interval, the remainder of the distribution can be of any form whatever without disturbing the above analysis. Furthermore, it is worth noting that all of the results above for optimal bid level, i.e. equations (2), (5), (6), (9), (10), and (17), depend only on a and not upon b, the other parameter we have used to define the uniform distribution (except for the requirement that $b \geq b_K^*$). Hence, these optimum bid levels are independent of the conditional probability that the best competitive bid is in the interval (a,v).

The strength of the argument about the robustness of the assumption of a uniform distribution for the best competitive bid depends in part upon the existence of a minimum bid. If there is no minimum bid, then it may be more likely that a bidder will believe that the probability density of the best competitive bid increases smoothly over a range below his value. To study this situation, we consider a model in which the probability density of the best competitive bid is triangular on the range (a,b), increasing linearly with slope $2/(b-a)^2$ from zero at a to $2/(b-a)$ at some value of $b \geq v$. With one bid, b_1, allowed, the bidder's expected profit is given by

$$E(b_1) = (v-b_1)\,(b_1-a)^2/(b-a)^2 . \tag{19}$$

Setting $dE/db_1 = 0$ yields the optimum bid,

$$b_1^* = (2v+a)/3 = a + 2(v-a)/3 . \tag{20}$$

This yields expected profit

$$E\,(b_1^*) = 4\,(v-a)^3/27\,(b-a)^2 . \tag{21}$$

When a second bid, b_2, $a \leq b_2 \leq b_1$, is allowed with a fixed bid withdrawal penalty of q, the expected profit is given by

$$E_q(b_1,\, b_2) = (v-b_1)(b_1-a)^2/(b-a)^2 + (b_1-b_2-q)\,(b_2-a)^2/(b-a)^2 . \tag{22}$$

Setting $\partial E/\partial b_1 = \partial E/\partial b_2 = 0$ yields the optimum bids

$$b_1^* = a + \frac{9(v-a) - 4q + [81\,(v-a)^2 - 72(v-a)q + 108q^2]^{1/2}}{23}\,, \tag{23}$$

and

$$b_2^* = a + \frac{6(v-a) - 18q + [36\,(v-a)^2 - 32(v-a)q + 48q^2]^{1/2}}{23}\,, \tag{24}$$

provided $q \leq 2(v-a)/3$. Otherwise, using only one bid is optimal. Note that as with the uniform distribution, the optimal bids depend only upon a and not upon b. When $q = 0$, these expressions simplify to

$$b_1^* = a + 18(v-a)/23 \tag{25}$$

and

$$b_2^* = a + 12(v-a)/23. \tag{26}$$

With the rectangular distribution for the best competitive bid and no withdrawal penalty, the likelihood the best bid lies below b_2^* equals the likelihood it lies between b_2^* and b_1^*, which in turn equals the likelihood that it lies between b_1^* and v. With the triangular distribution considered here, the relative values of these likelihoods change moderately from 1 : 1 : 1 to .817 : 1.021 : 1.163.

With the triangular distribution, the expected profit with optimal bidding is

$$E(b_1^*,b_2^*) = 108(v-a)^3/529(b-a)^2. \tag{27}$$

This is $(27/23)^2 \approx 1.378$ times the expected profit in the one bid situation. This compares with the corresponding ratio of $4/3 \approx 1.333$ with the uniform distribution.

When there is a proportional penalty, p for bid withdrawal, the expected profit is given by

$$Ep(b_1, b_2) = (v-b_1)(b_1-a)^2/(b-a)^2 + (b_1-b_2-b_1p)(b_2-a)^2/(b-a)^2, \tag{28}$$

and the optimum bids are given by

$$b_1^* = a + \frac{9(v-a) - 4ap(1-p) + [9(v-a)^2 - 8(v-a)ap(1-p) + 108a^2p^2]^{1/2}}{27 - 4(1-p)^2} \tag{29}$$

and

$$b_2^* = a + \frac{6(v-a) - 18ap + 2(1-p)[9(v-a)^2 - 8(v-a)ap(1-p) + 12a^2p^2]^{1/2}}{27 - 4(1-p)^2} \tag{30}$$

provided p is small enough that the expression for b_2^* exceeds a. If it isn't, only one bid is used.

2.2 *Low-Bid Wins Models*

Completely analgous results hold for the low-bid-wins auction in which a bidder can make different bids for a job of known cost, c, and withdraw a lower bid for a price when he sees the best competitive bid. For low-bid-wins auctions, we assume as before a uniform distribution on (a,b) for the best competitive bid, K bids, $b_1, b_2, \ldots, b_K$, and a penalty for withdrawing a bid either of p times the bid or of q. This time, we assume $c \leq b_1 \leq b_2 \leq \cdots b_K \leq b$ and $a \leq c$ (or at least below the value we will calculate for b_1^*, the optimum value of the lowest bid). Since all derivations are completely analagous to those above, we supply just the results.

When K=1,

$$b_1^* = (b+c)/2 \tag{31}$$

and

$$E(b_1^*) = (b-c)^2/4(b-a) \ . \tag{32}$$

When K=2 and the penalty for withdrawing a bid is fixed quantity $q \leq (b-c)/2$,

$$b_1^* = (2c+q+b)/3, \tag{33}$$

$$b_2^* = (c+2q+2b)/3, \tag{34}$$

and

$$E_q(b_1^*, b_2^*) = \left[(b-c-q)^2 + q(b-c) \right] / 3(b-a) \ . \tag{35}$$

When K=2 and the penalty for withdrawing a bid is a proportion, $p \leq (b-c)/(b+c)$, of that bid,

$$b_1^* = (b-pb+2c)\,/\,(3+p)\,(1-p), \tag{36}$$

$$b_2^* = (2b-pb-p^2b+c+pc)\,/\,(3+p)(1-p)\ , \tag{37}$$

and

$$E_p\,(b_1^*,b_2^*) = \frac{3(b-c)^2 - p(5b^2-7bc+2c^2) + p^2(b^2+3bc-c^2) + p^3b(b-3c) - p^4bc}{(b-a)\,(3+p)^2\,(1-p)^2} \quad . \tag{38}$$

When K bids are allowed and there is no penalty for bid withdrawal,

$$b_k^* = c + k(b-c)\,/\,(K+1),\ k=1,2,\ldots,K, \tag{39}$$

and

$$E(b_1^*,b_2^*,\ldots,b_K^*) = (b-c)^2K/2(b-a)\,(K+1) \quad . \tag{40}$$

3. An Auction with a Withdrawn Winning Bid

The California State Lands Commission acquired in an exchange 134 acres of land leased for cultivation of brussel sprouts. (The land was adjacent to a state park near Santa Cruz and intended, ultimately, for acquisition by the park.) When the lease expired in 1979, the Commission solicited sealed bids for a new five-year lease even though the practice of the previous owners and on neighboring land under lease for brussel sprout cultivation had long been negotiated renewals. Based on comparable leases, the Commission staff set a minimum bid of \$16,750 per year. Apparently, brussel sprout farming was substantially more profitable than absentee land owners in the area realized. A bid of \$23,584 per year was received from the corporation that had previously farmed the land, and one of \$30,420 per year from an individual. After the bid opening, however, that individual indicated in writing an unwillingness to go through with the deal. When the justifications he raised were rejected as groundless, he still insisted on withdrawing his bid "for personal reasons". The Commission voted unanimously to forfeit his bid deposit of \$3,042 and to award the lease to the corporation. One Commision member indicated that he might not have voted to forfeit the deposit had the individual shown up at the Commission meeting and answered some questions.

[California State Lands Commission, 1979] Commission staff members believed, however, that the answers to questions about his economic relationship to the corporation would have proved embarrassing.

We now use the first proportional penalty model developed in Section 2 above to analyze this brussel sprout farm lease auction on the assumption that the withdrawn bid and the winning bid were from a single party. This analysis assumes that that party expected that the best competitive bid would be distributed uniformly on some range beginning at the minimum bid of $16,750 per year and extending to some value greater than $30,420 (or, in the final calculation, $31,638) per year. We also assume a 15% per year interest rate so that the bid withdrawal penalty of 10% of the first year's rental on a 5-year lease corresponds, on a present value basis, to $p = 0.027$. With these assumptions, we can insert the observed bids of $30,420 and $23,584 per year into equations (9) and (10) and use each of them to calculate a value for the bidder's true valuation, v. These values, $37,839 per year and $39,125 per year, are close to each other. They average $38,482 per year. Using this average value for v in (9) and (10) yields theoretical values for b_1^* and b_2^* within about 1% of the observed values. Thus, the model is reasonably consistent with the data.

Using $v =$ $38,482 per year, the model predicts that p would have to be increased to .393, 146% of the first year's rental, to make the bidder's use of a two bid strategy completely unprofitable. If the penalty had been this high, the model would predict a single bid of $27,616 per year. This would exceed the State Lands Commission's actual receipts by $990 the first year and $4,032 in each of the next four years. If there were no penalty for withdrawal of a second bid, the model would predict a withdrawn bid of $31,238 per year and an accepted bid of $23,994 per year. The State Lands Commission's actual receipts exceeded $23,994 by $2,632 the first year and were $410 less in the second through fifth years.

4. Game-Theoretic Models

4.1 *Two-Bid Models*

Next we consider the effect of adding a second bid possibility to a game theoretic model of bidding first considered by Rothkopf [1969a] and since discussed and extended by Oren and Rothkopf [1975] and by Rothkopf [1980a, 1980b]. The model is an n bidder, symmetric, lowest-bid-wins, common value model with multiplicative bidding strategies. In it, each bidder gets an independent cost estimate down from an unbiased Weibull distribution. The i^{th} bidder, $i=1,2,\ldots,n$, then multiplies his estimate by his strategy, P_i, to obtain his bid, x_i, with distribution

$$F_i(x_i) = 1-e^{-a_i x_i^m}, x_i \geq 0 \ , \tag{41}$$

and density

$$f_i(x_i) = m a_i x_i^{m-1} e^{-a_i x_i^m} \ , \ x_i > 0 \ , \tag{42}$$

where

$$a_i = \left[\Gamma\,(1+1/m)/P_i c \right]^m , \tag{43}$$

$c>0$ is the common cost and $m>0$ is the parameter of the Weibull distribution that determines its mean to standard deviation ratio. Into this model, we insert a second, less aggressive bid by bidder i at α_i times the first bid and the possibility of withdrawing the lower, more aggressive bid for a penalty of p times that bid, $i=1,2,\ldots,n$. Note that this second bid is only of interest if $\alpha_i \geq 1+p$. We assume that each bidder knows that all bidders are aware of the withdrawable bid possibility.

In this new model, bidder i's expected profit is given by

$$E_i = \int_o^{\infty} (x_i - c)\, f_i\,(x_i) \prod_{j \neq i} \left[1 - F_j(x_i)\right] dx_i \tag{44}$$

$$+ \int_o^{\infty} (\alpha_i - 1 - p)\, x_i f_i\,(x_i) \prod_{j \neq i} \left[1 - F_j(\alpha_i x_i)\right] dx_i,\ i=1,2,\ldots,n,$$

$$= a_i \left[A^{-1/m}\, \Gamma\,(1+1/m) - c\right] / A + a_i(\alpha_i - p - 1) A_i^{1-1/m}\, \Gamma\,(1+1/m)\ ,\ i=1,2,\ldots,n,$$

where $A = \sum_{i=1}^{n} a_i$ and $A_i = a_i + \alpha_i^m \sum_{j \neq i} a_j$. In the first expression for E_i, the second integral is the expected extra profit received from withdrawing the lower bid when the higher bid is still below the best competitive bid. By selecting $\alpha_i \geq 1+p$, a bidder guarantees that this second term is non-negative.

Setting

$$\partial E_i / \partial P_i = 0\ , \quad i = 1,2,\ldots,n, \tag{45}$$

$$\partial E_i / \partial \alpha_i = 0\ , \quad i = 1,2,\ldots,n, \tag{46}$$

$$P_i = P\ , \qquad i = 1,2,\ldots,n, \tag{47}$$

$$\alpha_i = \alpha\ , \qquad i = 1,2,\ldots,n, \tag{48}$$

yields a symmetric set of equilibrium strategies. Solving (45), (47) and (48) for P gives the following equilibrium common P strategy for given common α:

$$P = \frac{1}{\dfrac{m(n-1)-1}{m(n-1)n^{1/m}} + \dfrac{[\alpha-(1+p)]\ [m(n-1)\alpha^m - 1]\ n^2}{m(n-1)\ [1+(n-1)\alpha^m]^{2+1/m}}}\ . \tag{49}$$

This equilibrium for P exists only if the denominator in (49) is positive. Note that if $\alpha = 1+p$, the second term in the denominator is 0, and P is the same as the single bid equilibrium strategy derived by Rothkopf in 1969. However, if $\alpha > 1+p$, this term is positive, and the equilibrium value of P is smaller than in the single bid model.

Equations (46), (47) and (48) yield the relationship

$$m(n-1)\alpha^m - (n-1)(1+p)(m+1)\alpha^{m-1} - 1 = 0 \tag{50}$$

for the equilibrium common value for α given P. Note that P does not appear in the equations, and hence, the equilibrium value for α is independent of P. For m=1 (the exponential distribution), equation (50) specializes to

$$\alpha = 2(1+p) + 1/(n-1) \ . \tag{51}$$

For m=2 (the Rayleigh distribution), equation (50) is a quadratic with positive solution

$$\alpha = \left\{3(1+p) + \left[9(1+p)^2 + 8/(n-1)\right]^{1/2}\right\}/4 \ . \tag{52}$$

For m = 1/2, the solution of (50) is

$$\alpha = 3(1+p) + 2\left\{1 + [1+3(n-1)^2(1+p)]^{1/2}\right\}/(n-1)^2 \ . \tag{53}$$

Provided only that $m>0, n>1$, and $p>0$, the relevant solution of (50) for α approaches $(1+p)(m+1)/m$ as m, n, or p gets large. Figure 3 illustrates the equilibrium value of α for various values of m, n, and p.

The expected profit paid by a bid taker in a two bid auction as a fraction of the cost of the job up for bid when n bidders follow equilibrium strategies is given by

$$\frac{nE_i(P,\alpha)}{c} = P\left[n^{-1/m} + \frac{(\alpha-1-p)n}{\left[1+(n-1)\alpha^m\right]^{1+1/m}}\right] - 1 \ , \tag{54}$$

where α is given by the solution to (50) and P is given by (49). When m=1, the solution to (50) is given by (51) and, with this value for α, P can be expressed as

$$P = \frac{n(n-1)}{n-2}\left[1 - \frac{n^3(1+p)}{n^3(1+p) + 4(n-2)[1+(n-1)(1+p)]^2}\right] \tag{55}$$

and the bid taker's expected profit payment by

$$\frac{nE_i(P,\alpha)}{c} = \frac{n(n-1)}{n-2}\left[1 - \frac{n^3(1+p)}{n^3(1+p) + 4(n-2)[1+(n-1)(1+p)]^2}\right]\left[1 + \frac{n^2}{4(n-1)[1+(n-1)(1+p)]}\right] - 1 \ . \tag{56}$$

When bids can be withdrawn without a penalty (i.e. $p=0$), this becomes

$$\frac{nE_i(P,\alpha)}{c} = \frac{(n-1)}{(n-2)}\left[1 - \frac{n}{(n-1)(5n-8)}\right] - 1 = \frac{4}{5n-8} \quad . \tag{57}$$

This is smaller than the similar result, $1/(n-2)$, for the auction in which only one bid is allowed (or in which the penalty, p, is made large).

The probability that bidder i wins and withdraws a bid in order to win with its higher bid is give by

$$P_{Wi} \equiv \int_0^\infty f_i(x_i) \prod_{j \neq i} \left[1 - F_j(\alpha_i x_i)\right] dx_i \, . \tag{58}$$

When all bidders use the same strategy P for their lower bids,

$$P_{Wi} = 1/\left[1 + (n-1)\,\alpha_i^m\right]. \tag{59}$$

If all bidders use the same strategies P and α, then the probability that the winner withdraws its lower bid to win with its higher bid is

$$P_W = n/\left[1 + (n-1)\alpha^m\right]. \tag{60}$$

When $m=1$ and α has the equilibrium value given in equation (51),

$$P_W = n/2\left[n + (n-1)p\right]. \tag{61}$$

This is ½ when $p = 0$. For large n, it approaches $1/2\,(1 + p)$.

For $m=2$ and α given by equation (52),

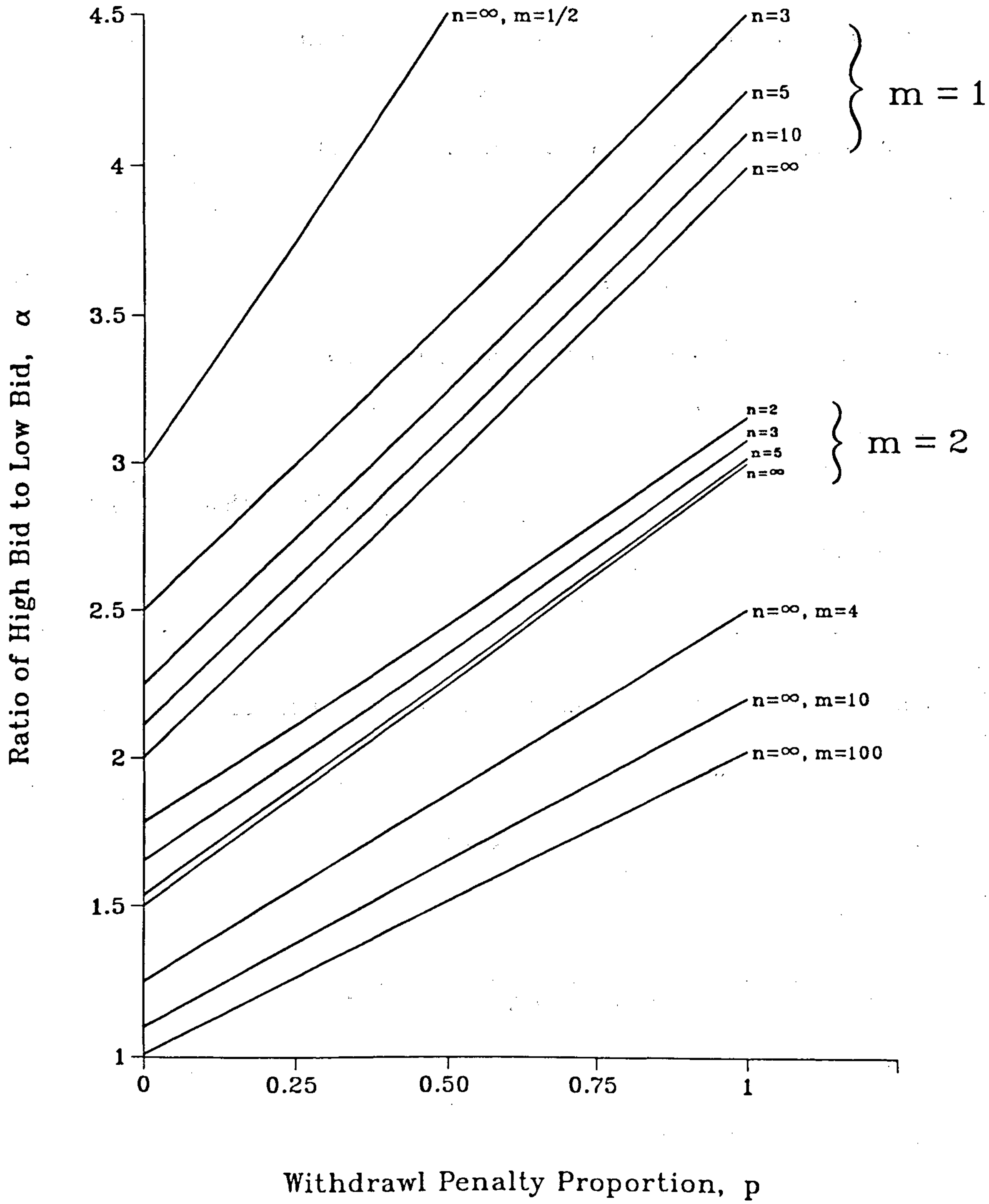
$$P_W = \frac{8n}{12 + 9(1+p)^2(n-1) + 3(1+p)\left[9(1+p)^2(n-1) + 8(n-1)\right]^{1/2}} \; . \tag{62}$$

When $p=0$, this is

$$P_W = \frac{8n}{12 + 9(n-1) + 3[(9n-1)(n-1)]^{1/2}} \tag{63}$$

which takes on values .475, .465, .459, .456, and .450 for $n = 2, 3, 4, 5$ and 10. For large n, P_W approaches $4/9(1 + p)^2$.

Figure 3
Ratio of Higher Bid to Lower Bid in Two Bid Symmetric Equilibrium



In general, in equilibrium as n gets large $\alpha \rightarrow (1+p)(1+m)/m$ and $P_W \rightarrow [m/(m+1)(1+p)]^m$. For large m, $\alpha \rightarrow 1+p$ and $P_W \rightarrow 0$ if $p>0$ and ½ if $p=0$.

Analagous results obtained by analagous means hold for the high-bid-wins variant of this model described in Rothkopf's 1969 paper. In the variant, each bidder receives an independent unbiased estimate of the common value, v, drawn from Gumbel's third asymptotic distribution. The i^{th} bidder, $i = 1,2,\ldots,u$, then multiplies his estimate by his strategy, Q_i, to obtain his bid, y_i, with distribution

$$G_i(y_i) = e^{-b_i y_i^{-m}},\ y_i \geq 0, \tag{64}$$

with density

$$g_i(y_i) = m\, b_i y_i^{-m-1} e^{-b_i y_i^{-m}},\ y_i > 0\ , \tag{65}$$

where

$$b_i = \left[vQ_i/\Gamma(1-1/m)\right]^m \tag{66}$$

and m is the parameter of the estimating distribution that determines its mean-to-standard-deviation ratio. Into this model, we insert a second, less aggressive bid by bidder i at β_i times the first bid, $0 \leq \beta_i < 1$, and the possibility of withdrawing the higher, more aggressive bid for a penalty of p times that bid.

In this new variant, bidder i's expected profit is given by

$$E_i = \int_0^\infty (V-y_i)\, g_i(y_i) \prod_{j\neq i} G_j(y_i) dy_i \tag{67}$$

$$+ \int_0^\infty (1-\beta_i - p)\, y_i\, g_i(y_i) \prod_{j\neq i} G_j(\beta_i y_i) dy_i$$

$$= b_i\left[V - B^{1/m}\,\Gamma(1-1/m)\right]/B$$

$$+ b_i\,(1-\beta_i-p)\, B_i^{-1+1/m}\,\Gamma(1-1/m),$$

where $B = \sum_{i=1}^{n} b_i$ and $B_i = b_i + \beta_i^{-m} \sum_{j \neq i} b_j$. By selecting $\beta_i \leq 1-p$, the bidder guarantees that the expected profit from withdrawing the more aggressive bid is non-negative. Solving the first order optimality condition for Q_i along with the symmetry conditions, $Q_i = Q, i=1,2,\ldots,n$ and $B_i = \beta$, i–1,2,...,n yields

$$Q = \frac{1}{\frac{m(n-1)+1}{m(n-1)n^{1/m}} + \frac{[\beta-(1-p)]\,[m(n-1)\beta^{-m}+1]n^2}{m(n-1)[1+(n-1)\beta^{-m}]^{2-1/m}}} \tag{68}$$

When $\beta = 1-p$, the second term in the denominator is 0, and the 1969 result is repeated. Solving the first order optimality condition for β_i along with the symmetry conditions yields the equation

$$\beta^{m+1} + m(n-1)\,\beta - (1-p)\,(m-1)\,(n-1) = 0\,. \tag{69}$$

For large n, the relevant solution of (69) for β approaches $(1-p)(m-1)/m$. For $p > 0$, $n \geq 2$, and large m, it approaches $(1-p)$.

4.2 *Second Price Models*

As was argued above in the introduction, if bidders can submit as many bids as they wish and withdraw bids without a penalty, the auction becomes, effectively, a second price auction. Results generalizing the symmetric version of Rothkopf's model considered above to second price auctions have been obtained [Rothkopf 1969b], but not previously published. We summarize them here. However, it is important to realize that these results, unlike those for the first price auction, are not robust. They depend absolutely on the symmetry of the situation. If a bidder's value relative to that of his competitors is even slightly different, there is no multiplicative strategy equilibrium involving him. Furthermore, even when the situation confronting the bidders is completely symmetric, there is a continuum of nonsymmetric equibibria, all less favorable to the bid taker than the symmetric equilibrium.

For the low-bid-wins second price auction, the i^{th} bidder's expected profit is given by

$$E_i = \int_o^\alpha (x-c) f_i^* (x)\, F_i(x) dx \tag{70}$$

$$= \Gamma(1+1/m)\, A_i^{*-1/m} - c - \left[\Gamma(1+1/m) A^{-1/m} - c\right] A_i^*/A,$$

where $F_i(x)$, A, and a_i are given above, $A_i^* = A - a_i$, and $f_i^*(x)$ is the probability density of the best competitive bid given by

$$f_i^*(x) = A_i\, m x^{m-1}\, e^{-x^m A_i}, x > 0 \,. \tag{71}$$

Solving the equilibrium condition (45) with the symmetry condition (47) yields the symmetric second price equilibrium strategy

$$P = m\, n^{1/m}/(m+1)\,. \tag{72}$$

If all bidders follow this strategy, the expected profit paid by the bid taker as a multiple of c is given by

$$\frac{nE_i(P)}{c} = \frac{m}{m+1}\left[n(1-1/n)^{-1/m} - n - 1/m\right]. \tag{73}$$

This fraction decreases with increases in m or n. The ratio of this quantity to the corresponding fraction for single-bid first price auctions lies between .21 and .49 for $2 \leq n \leq 10$ and $2 \leq m \leq 100$. Within this range, the ratio increases at a decreasing rate with increasing n and with increasing m.

Completely analagous non-robust results hold for the high-bid-wins version of the model. For it, we have

$$E_i = v - \Gamma(1-1/m)\, B_i^{*\,1/m} - \left[v - \Gamma(1-1/m) B^{1/m}\right] B_i^*/B, \tag{74}$$

where $B_i^* = B - b_i$; the symmetric equilibrium strategy is given by

$$Q = \frac{m}{m-1} n^{-1/m} , \tag{75}$$

and

$$\frac{nE_i(Q)}{v} = \frac{m}{m-1} \left[n - 1/m - n(1-1/n)^{1/m}\right]. \tag{76}$$

5. Discussion

This paper has presented several simple models of auctions with withdrawable winning bids. We have discussed both decision theoretic models of the problem faced by bidders contemplating entering such auctions and game theoretic models of the auctions of possible interest to bid takers as well as bidders. We have also described a particular auction with a withdrawn high bid and fitted it to one of our models. However, such auctions are rare. Is this a good thing, or should withdrawable winning bid auctions be "legitimized," and made routine?

We're not at all sure, but we think legitimization is at least worth further study. There are sales, such as the Federal timber auction, about which there has been a substantial controversy between supporters of first-price sealed and second-price oral auctions. (For the timber auction situation, see [Weiner 1979] and the discussion that follows his paper.) This is neither the paper nor the forum in which to analyze the pros and cons of such policy debates, but their existence does suggest that some withdrawable winning bid auction format could at least serve as a compromise and, conceivably, might even provide a way of meeting the major concerns of each side.

I am indebted to Arnold Langsen for pointing out that one easy way of achieving legitimization, might be for the bid taker to auction off a short-term option with publicly opened bids. A bidder who won the option would have to pay for it, but not necessarily exercise it. If the option were not exercised, the second best bid would win the option and, if necessasry, the third, the fourth, etc. The bidding would be on the exercise price and the cost of option could be a fixed amount or a percentage of the exercise price. Of course, more direct means of legitimization, such as announcing that each bidder is entitled to two bids and could use the less aggressive one if it chose, might also work well.

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