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**A Note on the Finiteness of the Set
of Equilibria in an Exchange Economy
with Constrained Endowments**

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Abstract

Debreu (1970) proved that for smooth finite exchange economies, the set of endowment assignments which result in a finite number of price equilibria is “large” in the sense that it has null closed complement. We show that if the individual endowment space of each agent does not include *all* the commodities, then this conclusion need not hold. In particular, we consider the case where each agent’s endowment space includes only one good. We construct a class of examples in which every economy in an open subset of the constrained endowment space gives rise to a continuum of equilibria.

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In his celebrated paper, Debreu (1970) developed the idea that to have explanatory power, the equilibria of an economy should be locally unique. He showed that for fixed smooth preferences over consumption in R^n , the set of endowment assignments which result in a finite, and hence locally unique, set of equilibria is generic in the strong sense that it is an open and dense set whose complement has Lebesgue measure zero. One of the assumptions of his model is that agents are endowed with strictly positive amounts of all commodities. In contrast, casual empiricism suggests that agents are typically endowed with only a small number of the available commodities. The fisherman only brings fish to the market; the farmer only brings wheat. If we allow for zero endowment of some commodities, Debreu's question becomes whether the set of endowments which give rise to a finite number of equilibria is a generic subset of the constrained endowment space.

Mas-Colell (1975; 1985, pp. 336-341) first addressed this issue for a model in which the endowments *and* preferences are chosen from a generic set in the product space of endowment-preference pairs. Using a different notion of genericity for this expanded setting, he proved generic finiteness with constrained endowment spaces. Specifically, he showed that economies lying in an open and dense set of the constrained endowment-preference space have a finite number of equilibria. However, as Mas-Colell (1985, pp. 317-318) observes, in the infinite dimensional endowment-preference space there is no natural measure to play the role that Lebesgue measure plays in R^n . Consequently, there is no natural parallel to Debreu's result that the set of endowments which give rise to an infinite number of equilibria has Lebesgue measure zero. Since in R^n open and dense sets may have arbitrarily small positive Lebesgue measure, the attributes of openness and denseness alone are not fully satisfying as a notion of genericity. A result using fixed preferences and Debreu's stronger definition of genericity in the constrained endowment space would thus also be desirable.

This note shows that an extension of Debreu's theorem to a model with constrained endowment spaces and fixed preferences is not attainable without further assumptions on

the model. We construct a class of counterexamples in which every economy in an open subset of the constrained endowment space gives rise to a continuum of equilibrium prices and allocations.

Consider the following N agent, N good economy. Consumption takes place in $\Omega = R_+^N \setminus \{0\} = \{(x_1, \dots, x_N) \in R^N \mid x_i \geq 0, i = 1, \dots, N, x_i \gg 0 \text{ at least one } i\}$. Each agent initially holds exactly one of the N goods. The constrained endowment space of the i th agent is the i th coordinate axis in R_+^N , $\Omega_i = \{x \in \Omega \mid x_j = 0, j \neq i, j = 1, \dots, N\}$. The total constrained endowment space is then denoted by $\Omega_C = \Omega_1 \times \dots \times \Omega_N$. We take the initial endowment, $\omega^i \in \Omega_i$, of the i th agent to be:

$\{(\succeq^i, \omega^i) : i = 1, \dots, N\}$ are given by:

$$\begin{aligned}\omega_j^i &= 0, j \neq i, \\ \omega_i^i &= 2N.\end{aligned}$$

We define preferences of the agents in two steps. First, for $i = 1, \dots, N$, consider the function $u^i : R_{++}^N \longrightarrow R$,

$$u^i(x_1, \dots, x_N) = x_i \left(1 - \frac{1}{\prod_{j=1}^N x_j}\right).$$

This represents an underlying preference relation on R_{++}^N which is C^∞ , differentiable strictly monotone, and differentiable strictly convex. Second, though we suppress the details here, this preference relation is easily modified near the boundary of R_{++}^N and then extended to a preference relation, $\succeq^i$, which is C^∞ , differentiable strictly monotone, and differentiable strictly convex on all of $\Omega = R_+^N \setminus \{0\}$. The modification has no effect on the subsequent arguments using the representation u^i of $\succeq^i$ which holds away from the boundary of R_{++}^N , because these arguments are local in nature. The preferences, $\succeq^i$, are in the same class as

those in Mas-Colell's model.¹

For an interior allocation $x = (x^1, \dots, x^N) \in R_{++}^N \times \dots \times R_{++}^N$ which assigns consumer i the bundle x^i and for a strictly positive, normalized $p = (1, p_2, \dots, p_N) \in R_{++}^N$, the strict convexity and differentiability of u^i imply the following familiar sufficient conditions for (x, p) to be a Walrasian equilibrium:

$$p = \alpha^i [\partial u^i(x^i)], \quad \alpha^i \in R_{++}, \quad (i = 1, \dots, N), \quad (1)$$

$$p \cdot (x^i - \omega^i) = 0, \quad (i = 1, \dots, N), \quad (2)$$

$$\sum_{i=1}^N x^i = \sum_{i=1}^N \omega^i, \quad (3)$$

where $\partial u^i(x^i)$ is the gradient of u^i at x^i . Of course, there may be other Walrasian allocations, for instance along the boundary in $\Omega \times \dots \times \Omega$, as well.

As a concrete example consider the Edgeworth box in Figure 1. Here $u^1(x_1, x_2) = x_1 - 1/x_2$, $u^2(x_1, x_2) = x_2 - 1/x_1$; $\omega^1 = (4, 0)$, $\omega^2 = (0, 4)$. Rearranging, we have $u^2(4 - x_1, 4 - x_2) = (4 - x_2) - 1/(4 - x_1)$. So agent 1 has indifference curves $u^1(x_1, x_2) = a$ or $x_2 = 1/(x_1 - a)$, and agent 2 has indifference curves $u^2(4 - x_1, 4 - x_2) = b$ or $x_2 = (4 - b) - 1/(4 - x_1)$. As is easily calculated, there is a continuum of Walrasian equilibrium allocations indexed by the curve $c : [0, 1] \rightarrow R_{++}^2 \times R_{++}^2$,

¹Mas-Colell also assumes an indecomposability condition which states that there is a single coordinate axis contained in each Ω_i . This assumption is made to rule out the possibility that the agents can be decomposed into disjoint groups which buy and sell totally distinct sets of goods. In that case, the relative price level between the two sets of goods is indeterminate. In the above examples, agents have disjoint endowment spaces, but at the equilibria which we demonstrate, each agent consumes a positive amount of each good so the indeterminacy of equilibrium here is not a result of decomposability.

$$c(t) = [(4 - 1/(1+t), 1+t), (1/(1+t), 4 - (1+t))],$$

with corresponding equilibrium prices $p : [0, 1] \rightarrow R_{++}^2$, $p(t) = (1, 1/(1+t)^2)$.

Figure 1

Now, suppose we perturb ω^1 and ω^2 by $(\varepsilon_1, 0)$ and $(0, \varepsilon_2)$, respectively. For $\varepsilon_1 > 0$ ($\varepsilon_1 < 0$), the first perturbation translates the upper right corner of the box and agent 2's indifference curves to the right (left) by ε_1 . Since agent 1's indifference curves are invariant under shifts to the right (left), the curve of equilibrium allocations merely shifts to the right (left). Similarly the second perturbation translates agent 1's indifference curves down (up) by $\varepsilon_2 > 0$ ($\varepsilon_2 < 0$), shifting the curve of equilibrium allocations down (up). For small enough $\varepsilon_1, \varepsilon_2$, the shifts of $c(t)$ define new curves which lie entirely in $R_{++}^2 \times R_{++}^2$, thus giving a continuum of equi-

librium allocations for the perturbed economy, $\{u^1, u^2; \omega^1 = (4 + \varepsilon_1, 0), \omega^2 = (0, 4 + \varepsilon_2)\}$.

More generally, for our original economy we have the following result.

Theorem. Consider the economy $\{(u^i, \omega^i), i = 1, \dots, N\}$ defined above. Let $x(p) = (x^1(p), \dots, x^N(p)) \in R^N \times \dots \times R^N$ be given as follows for $p = (1, p_2, \dots, p_N) \in R_{++}^N$:

$$x_j^i(p) = \sqrt[N]{\frac{\prod_{k \neq j} p_k}{(p_j)^{N-1}}}, \quad i \neq j,$$

$$x_i^i(p) = 2N - (N-1) \sqrt[N]{\frac{\prod_{k \neq i} p_k}{(p_i)^{N-1}}}.$$

Then there exists $\epsilon > 0$ such that for all $p \in R_{++}^N$ with $p_1 = 1, p_i \in [(2/3)^N - \epsilon, (2/3)^N + \epsilon], i = 2, \dots, N$, $(x(p), p)$ is a Walrasian equilibrium. Thus the set of equilibrium allocations and prices contains an $(N - 1)$ -dimensional manifold.

Moreover, there exists an open set in the constrained endowment space $\Omega_C = \Omega_1 \times \dots \times \Omega_N$ such that each economy with endowment in that set gives rise to a set of equilibrium allocations and prices containing an $(N - 1)$ -dimensional manifold.

PROOF: Substitution into the equations (1), (2), and (3), show $(x(p), p)$ is an equilibrium provided each agent's demand $x^i(p)$ is positive. For $p_o = (1, (2/3)^N, \dots, (2/3)^N)$ we have $x^i(p_o) >> 0, i = 1, \dots, N$. So continuity of $x^i : R_{++}^N \rightarrow R^N$ implies the existence of an $\epsilon > 0$ such that the first conclusion of the theorem holds.

Now fix i . Suppose we perturb the economy by adding $\varepsilon^i \in \Omega_i$, to i 's endowment ω^i . That is we allow ω^i to vary in agent i 's constrained endowment space. Consider the allocation $x(p)_{\varepsilon^i} = (x^1(p), \dots, x^{i-1}(p), x^i(p) + \varepsilon_i, x^{i+1}(p), \dots, x^N(p))$, where $\varepsilon_i \in R$. Clearly $x(p)_{\varepsilon^i}$ satisfies properties 2 and 3 for the perturbed economy $\{(u^j, \omega^j), j = 1, \dots, N, j \neq i; (u^i, \omega^i + \varepsilon^i)\}$ and the price p . The linearity of u^i in the i -th coordinate direction implies

that $\partial u^i(x^i(p)) = \partial u^i(x^i(p) + \varepsilon^i)$ so property 1 holds as well. In other words, provided $x(p)_{\varepsilon^i}$ lies in R_{++}^N for all $p \in \{1\} \times [(2/3)^N - \varepsilon, (2/3)^N + \varepsilon] \dots \times [(2/3)^N - \varepsilon, (2/3)^N + \varepsilon]$, the economy with endowment perturbed by ε^i also has a continuum of equilibria.

Iteration of this argument as i varies gives the result that perturbing $(\omega^1, \dots, \omega^N)$ to $(\omega^1 + \varepsilon^1, \dots, \omega^N + \varepsilon^N)$ gives a new economy with an $(N - 1)$ -dimensional manifold of equilibrium allocations, $(x^1(p) + \varepsilon^1, \dots, x^N(p) + \varepsilon^N)$, provided $x^i(p) + \varepsilon^i \in R_{++}^N, i = 1, \dots, N$, for all $p \in \{1\} \times [(2/3)^N - \varepsilon, (2/3)^N + \varepsilon] \dots \times [(2/3)^N - \varepsilon, (2/3)^N + \varepsilon]$. But clearly the openness of R_{++}^N in R^N implies that there exists $\delta > 0$ such that for each $i, \|\varepsilon^i\| < \delta$ implies that $x^i(p) + \varepsilon^i \in R_{++}^N$, for all $p \in \{1\} \times [(2/3)^N - \varepsilon, (2/3)^N + \varepsilon] \dots \times [(2/3)^N - \varepsilon, (2/3)^N + \varepsilon]$. Consequently the equilibrium set of the economy $\{(u^i, \omega^i + \varepsilon^i); i = 1, \dots, N, \|\varepsilon^i\| < \delta\}$ contains an $(N - 1)$ -dimensional manifold. The open set in Ω_C given by $U = \{(\tilde{\omega}^1, \dots, \tilde{\omega}^N) \in \Omega_C; \|\tilde{\omega}^i - \omega^i\| < \delta, i = 1, \dots, N\}$ has the desired property.

Q.E.D.

A natural question is whether Debreu's theorem also fails to hold in economies where some agents have endowments of more than one commodity. The method used here does not extend to provide a counterexample in these cases without relaxing the assumption of strict convexity of preferences. Examination of Debreu's proof reveals that the theorem holds as long as at least one of the agents has the fully unconstrained endowment space $\Omega_i = R_+^N \setminus \{0\}$ ². The intermediate cases, $1 < \dim \Omega_i < N, i = 1, \dots, N$, remain to be addressed.

²Debreu assumes that there is one agent whose demand for each commodity goes to ∞ as the price of that commodity goes to zero. It is also this agent who must have the unconstrained endowment space.

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