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L-Functions for Symmetric Products of Kloosterman Sums

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Abstract

The classical Kloosterman sums give rise to a Galois representation of the function field unramified outside 0 and ∞. We study the local monodromy of this representation at ∞ using l-adic method based on the work of Deligne and Katz. As an application, we determine the degrees and the bad factors of the L-functions of the symmetric products of the above representation. Our results generalize some results of Robba obtained through p-adic method.

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0. Introduction

Let $\mathbf{F}_q$ be a finite field of characteristic p with q elements and let $\psi : \mathbf{F}_q \to \overline{\mathbf{Q}}_l^*$ be a nontrivial additive character. Fix an algebraic closure $\mathbf{F}$ of $\mathbf{F}_q$. For any integer k, let $\mathbf{F}_{q^k}$ be the extension of $\mathbf{F}_q$ in $\mathbf{F}$ with degree k. If λ lies in $\mathbf{F}_{q^k}$, we define the $(n-1)$-variable Kloosterman sum by

$$\mathrm{Kl}_n(\mathbf{F}_{q^k}, \lambda) = \sum_{x_1\cdots x_n=\lambda,\ x_i\in\mathbf{F}_{q^k}} \psi(\mathrm{Tr}_{\mathbf{F}_{q^k}/\mathbf{F}_q}(x_1+\cdots+x_n)).$$

For any $\lambda \in \mathbf{F}^*$, define $\deg(\lambda) = [\mathbf{F}_q(\lambda) : \mathbf{F}_q]$. The L-function $L(\lambda, T)$ associated to Kloosterman sums is defined by

$$L(\lambda, T) = \exp\left(\sum_{m=1}^{\infty} \mathrm{Kl}_n(\mathbf{F}_{q^{m\deg(\lambda)}}, \lambda)\frac{T^m}{m}\right).$$

One can show that $L(\lambda, T)^{(-1)^n}$ is a polynomial of degree n with coefficients in $\mathbf{Z}[\zeta_p]$, where ζ_p is a primitive p-th root of unity. This follows from 7.4 and 7.5 in [D1]. Write

$$L(\lambda, T)^{(-1)^n} = (1 - \pi_1(\lambda)T)\cdots(1 - \pi_n(\lambda)T).$$

Then for any positive integer m, we have

$$\mathrm{Kl}_n(\mathbf{F}_{q^{m\deg(\lambda)}}, \lambda) = (-1)^{n-1}(\pi_1(\lambda)^m + \cdots + \pi_n(\lambda)^m).$$

We have a family of L-functions $L(\lambda, T)$ parameterized by the parameter λ. Let $|\mathbf{G}_m|$ be the set of Zariski closed points in $\mathbf{G}_m = \mathbf{P}^1 - \{0, \infty\}$. This is the parameter space for λ. For a positive integer k, the L-function for the k-th symmetric product of the Kloosterman sums is defined by

$$L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T) = \prod_{\lambda \in |\mathbf{G}_m|} \prod_{i_1+\cdots+i_n=k} (1 - \pi_1(\lambda)^{i_1} \cdots \pi_n(\lambda)^{i_n} T^{\deg(\lambda)})^{-1}.$$

This is a rational function in T by Grothendieck's formula for L-functions, see 3.1 of [Rapport] in [SGA $4\frac{1}{2}$]. In the proof of Lemma 2.2, we shall see that $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)$ actually has coefficients in $\mathbf{Z}$. Thus, this rational function is geometric in nature, that is, it should come from the zeta function of some varieties or motives. What are these varieties and motives? From arithmetic point of view, our fundamental problem here is to understand this sequence of L-functions with integer coefficients parameterized by the arithmetic parameter k and its variation with k, from both complex as well as p-adic point of view. There is still a long way to go toward a satisfactory answer of this basic question, most notably from p-adic point of view. We shall make some remarks at the end of this introduction section.

In [R], Robba studied the L-function $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_2), T)$ in the special case $n = 2$ using Dwork's p-adic methods. He conjectured a degree formula, obtained the functional equation and the bad factors of the L-function $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_2), T)$. On the other hand, Kloosterman sums have also been studied in great depth by Deligne and Katz using l-adic methods. The purpose of this paper is to use their fundamental results to derive as much arithmetic information as we can about this sequence of L-functions $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)$ for $n > 1$ not divisible by p. We have

Theorem 0.1. Suppose $(n, p) = 1$. Let ζ be a primitive n-th root of unity in $\mathbf{F}$. For a positive integer k, let $d_k(n, p)$ denote the number of n-tuples $(j_0, j_1, \ldots, j_{n-1})$ of non-negative integers satisfying $j_0 + j_1 + \cdots + j_{n-1} = k$ and $j_0 + j_1\zeta + \cdots + j_{n-1}\zeta^{n-1} = 0$ in $\mathbf{F}$. Then, the degree of the rational function $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)$ is

$$\frac{1}{n}\left(\binom{k+n-1}{n-1} - d_k(n, p)\right).$$

Note that in most cases, the L-function $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)$ is a polynomial, not just a rational function. This is the case for instance if pn is odd or $n = 2$. This follows from Katz's global monodromy theorem for the Kloosterman sheaf. Recall that the Kloosterman sheaf is a lisse $\overline{\mathbf{Q}}_l$-sheaf Kl_n on $\mathbf{G}_m$ such that for any $x \in \mathbf{G}_m(\mathbf{F}_{q^k}) = \mathbf{F}_{q^k}^*$, we have

$$\mathrm{Tr}(F_x, \mathrm{Kl}_{n,\bar{x}}) = (-1)^{n-1}\mathrm{Kl}_n(\mathbf{F}_{q^k}, x),$$

where F_x is the geometric Frobenius element at the point x. For any Zariski closed point λ in $\mathbf{G}_m$, let $\pi_1(\lambda), \ldots, \pi_n(\lambda)$ be all the eigenvalues of the geometric Frobenius element F_λ on $\mathrm{Kl}_{n,\bar{\lambda}}$. Then

$$\begin{aligned}
\mathrm{Kl}_n(\mathbf{F}_{q^{m\deg(\lambda)}}, \lambda) &= (-1)^{n-1}\mathrm{Tr}(F_\lambda^m, \mathrm{Kl}_{n,\bar{\lambda}}) \\
&= (-1)^{n-1}(\pi_1(\lambda)^m + \cdots + \pi_n(\lambda)^m).
\end{aligned}$$

So we have

$$\begin{aligned}
L(\lambda, T)^{(-1)^n} &= \exp\left((-1)^n \sum_{m=1}^{\infty} \mathrm{Kl}_n(\mathbf{F}_{q^{m\deg(\lambda)}}, \lambda)\frac{T^m}{m}\right) \\
&= \exp\left(\sum_{m=1}^{\infty} (-(\pi_1(\lambda)^m + \cdots + \pi_n(\lambda)^m))\frac{T^m}{m}\right) \\
&= (1 - \pi_1(\lambda)T)\cdots(1 - \pi_n(\lambda)T).
\end{aligned}$$

We have

$$\prod_{i_1+\cdots+i_n=k} (1 - \pi_1(\lambda)^{i_1}\cdots\pi_n(\lambda)^{i_n}T^{\deg(\lambda)}) = \det(1 - F_\lambda T^{\deg(\lambda)}, \mathrm{Sym}^k(\mathrm{Kl}_{n,\bar{\lambda}})).$$

Therefore the L-function for the k-th symmetric product of the Kloosterman sums

$$L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T) = \prod_{\lambda\in|\mathbf{G}_m|}\prod_{i_1+\cdots+i_n=k} (1 - \pi_1(\lambda)^{i_1}\cdots\pi_n(\lambda)^{i_n}T^{\deg(\lambda)})^{-1}$$

is nothing but the Grothendieck L-function of the k-th symmetric product of the Kloosterman sheaf:

$$L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T) = \prod_{\lambda\in|\mathbf{G}_m|} \det(1 - F_\lambda T^{\deg(\lambda)}, \mathrm{Sym}^k(\mathrm{Kl}_{n,\bar{\lambda}}))^{-1}.$$

The Kloosterman sheaf ramifies at the two points $\{0, \infty\}$. We would like to determine the bad factors of the L-function $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)$ at the two ramified points. Assume $n|(q-1)$. Then we can explicitly determine the bad factor at ∞. The key is to determine the local monodromy of the Kloosterman sheaf at ∞, that is, to determine Kl_n as a representation of the decomposition group at ∞. However, the bad factor at 0 seems complicated to determine for general n. In the case $n = 2$, it is easy. Thus, we have the following complete result for $n = 2$, which is the conjectural Theorem B in [R]. (Our notations are different from those in Robba).

Theorem 0.2. Suppose $n = 2$, $q = p$, and p is an odd prime. Then $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_2), T)$ is a polynomial. Its degree is $\frac{k}{2} - [\frac{k}{2p}]$ if k is even, and $\frac{k+1}{2} - [\frac{k}{2p} + \frac{1}{2}]$ if k is odd. Moreover, we have the decomposition

$$L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_2), T) = P_k(T)M_k(T)$$

with

$$P_k(T) = \begin{cases} 1 - T & \text{if } 2 \not| k, \\ (1-T)(1 - p^{\frac{k}{2}}T)^{m_k} & \text{if } 2|k \text{ and } p \equiv 1 \bmod 4, \\ (1-T)(1 + p^{\frac{k}{2}}T)^{n_k}(1 - p^{\frac{k}{2}}T)^{m_k - n_k} & \text{if } 2|k \text{ and } p \equiv -1 \bmod 4, \end{cases}$$

where

$$m_k = \begin{cases} 1 + [\frac{k}{2p}] & \text{if } 4|k, \\ [\frac{k}{2p}] & \text{if } 4 \not| k, \end{cases}$$

and $n_k = [\frac{k}{4p} + \frac{1}{2}]$, and M_k is a polynomial satisfying the functional equation

$$M_k(T) = ct^{\delta} M_k(\frac{1}{p^{k+1}T}),$$

where c is a nonzero constant (depending on k) and $\delta = \mathrm{deg} M_k$.

Note that the slightly different formula for n_k in the conjectural Theorem B of [R] is incorrect.

Remarks. Theorem 0.2 only addresses the L-function $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_2), T)$ and the polynomial $M_k(T)$ from the complex point of view. A more interesting arithmetic problem is to understand their p-adic properties. For example, the first basic question would be to determine the p-adic Newton polygon of the polynomial $M_k(T)$ with integer coefficients. This is expected to be a difficult problem. A weak but already non-trivial version is to give an explicit quadratic lower bound for the p-adic Newton polygon of $M_k(T)$, which is uniform in k. Such a uniform quadratic lower bound is known [W1] in the geometric case of the universal family of elliptic curves over $\mathbf{F}_p$, where one considers the L-function of the k-th symmetric product of the first relative ℓ-adic cohomology which is lisse of rank two outside the cusps. This latter L-function, which is an analogue of the above $M_k(T)$, is essentially (up to some trivial bad factors) the Hecke polynomial of the U_p-operator acting on the space of weight $k+2$ cusp forms. From this point of view, we can also ask for an explicit automorphic interpretation of the polynomial $M_k(T)$ with integer coefficients. The question on the slope variation of $M_k(T)$ as k varies p-adically is related to the Gouvêa-Mazur type conjecture, see Section 2 in [W3] for a simple exposition.

More generally, for any fixed n, the sequence of L-functions $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)$ is p-adically continuous in k as a formal power series in T with p-adic coefficients. In fact, viewed as p-adic integers, it is easy to show that the numbers $\pi_i(\lambda)$ can be re-ordered such that $\pi_1(\lambda)$ is a 1-unit and all other $\pi_i(\lambda)$ are divisible by p (the exact slopes of the $\pi_i(\lambda)$ are determined by Sperber [Sp]). From this and the Euler product definition of $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)$, one checks that if $k_1 = k_2 + p^m k_3$ with k_2 and k_3 non-negative integers, we have the following congruence:

$$L(\mathbf{G}_m, \mathrm{Sym}^{k_1}(\mathrm{Kl}_n), T) \equiv L(\mathbf{G}_m, \mathrm{Sym}^{k_2}(\mathrm{Kl}_n), T) \pmod{p^m}.$$

For any p-adic integer $s \in \mathbf{Z}_p$, let k_i be an infinite sequence of strictly increasing positive integers which converge p-adically to s. Then, the limit

$$L(n; s, T) = \lim_{i \to \infty} L(\mathbf{G}_m, \mathrm{Sym}^{k_i}(\mathrm{Kl}_n), T)$$

exists as a formal power series in T with p-adic integral coefficients. It is independent of the choice of the sequence k_i we choose. This power series is closely related to Dwork's unit root zeta function. It follows from [W2] that for any p-adic integer s, the L-function $L(n; s, T)$ is the L-function of some infinite rank nuclear overconvergent σ-module over $\mathbf{G}_m$. In particular, $L(n; s, T)$ is p-adic meromorphic in T. In fact, $L(n; s, T)$ is meromorphic in the two variables (s, T) with $|s|_p \leq 1$. Grosse-Klönne [GK] has extended the p-meromorphic continuation of $L(n; s, T)$ to a larger disk of s with $|s|_p < 1 + \epsilon$ for some $\epsilon > 0$. Presumably, this two variable L-function $L(n; s, T)$ is related to some type of p-adic L-functions over number fields. It would be very interesting to understand the slopes of the zeros and poles of these p-adic meromorphic L-functions. Some explicit partial results were obtained in [W2], see [W4] for a self-contained exposition of such L-functions in the general case.

This paper is organized as follows. In §1, we study the local monodromy of the Kloosterman sheaf at ∞. The main result is Theorem 1.1 which determines Kl_n as a representation of the decomposition subgroup at ∞. In §2, we calculate the bad factors at ∞ of the L-functions of the symmetric products of the Kloosterman sheaf. Using these results, we can then complete the proof of Theorem 1.1. In §3, we calculate the degrees of the L-functions of the symmetric products of the Kloosterman sheaf. In particular, Theorem 0.1 is proved and some examples are given. Finally in §4, we study the special case $n = 2$ and prove Theorem 0.2.

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1. Local Monodromy at ∞

In [D1], Deligne constructs a lisse $\overline{\mathbf{Q}}_l$-sheaf Kl_n on $\mathbf{G}_m = \mathbf{P}^1 - \{0, \infty\}$, which we call the Kloosterman sheaf. It is lisse of rank n, puncturely pure of weight $n-1$, tamely ramified at 0, totally wild at ∞ with Swan conductor 1, and for any $x \in \mathbf{G}_m(\mathbf{F}_{q^k}) = \mathbf{F}_{q^k}^*$, we have

$$\mathrm{Tr}(F_x, \mathrm{Kl}_{n,\bar{x}}) = (-1)^{n-1}\mathrm{Kl}_n(\mathbf{F}_{q^k}, x),$$

where F_x is the geometric Frobenius elements at the point x. Since Kl_n is a lisse sheaf on $\mathbf{G}_m$, it corresponds to a galois representation of the function field $\mathbf{F}_q(t)$ of $\mathbf{G}_m$. In this section, we give a detailed study of Kl_n as a representation of the decomposition subgroup at ∞. This result will then be used to study L-functions of symmetric products of Kl_n.

Before stating the main theorem of this section, let's introduce some notations. Fix a separable closure $\overline{\mathbf{F}_q(t)}$ of $\mathbf{F}_q(t)$. Let x be an element in $\overline{\mathbf{F}_q(t)}$ satisfying $x^q - x = t$. Then $\mathbf{F}_q(t, x)$ is galois over $\mathbf{F}_q(t)$. We have a canonical isomorphism

$$\mathbf{F}_q \stackrel{\cong}{\to} \mathrm{Gal}(\mathbf{F}_q(t,x)/\mathbf{F}_q(t))$$

which sends each $a \in \mathbf{F}_q$ to the element in $\mathrm{Gal}(\mathbf{F}_q(t,x)/\mathbf{F}_q(t))$ defined by $x \mapsto x + a$. For the additive character $\psi : \mathbf{F}_q \to \overline{\mathbf{Q}}_l^*$, let $\mathcal{L}_\psi$ be the galois representation defined by

$$\mathrm{Gal}(\overline{\mathbf{F}_q(t)}/\mathbf{F}_q(t)) \to \mathrm{Gal}(\mathbf{F}_q(t,x)/\mathbf{F}_q(t)) \stackrel{\cong}{\to} \mathbf{F}_q \stackrel{\psi^{-1}}{\to} \overline{\mathbf{Q}}_l^*.$$

It is unramfied outside ∞, and totally wild at ∞ with Swan conductor 1. This galois representation defines a lisse $\overline{\mathbf{Q}}_l$-sheaf on $\mathbf{A}^1 = \mathbf{P}^1 - \{\infty\}$ which we still denote by $\mathcal{L}_\psi$.

Let $\mu_m = \{\mu \in \mathbf{F} | \mu^m = 1\}$ be the subgroup of $\mathbf{F}^*$ consisting of m-th roots of unity. Suppose $m|(q-1)$. Then μ_m is contained in $\mathbf{F}_q$. Let y be an element in $\overline{\mathbf{F}_q(t)}$ satisfying $y^m = t$. Then $\mathbf{F}_q(t, y)$ is galois over $\mathbf{F}_q(t)$. We have a canonical isomorphism

$$\mu_m \stackrel{\cong}{\to} \mathrm{Gal}(\mathbf{F}_q(t,y)/\mathbf{F}_q(t))$$

which sends each $\mu \in \mu_m$ to the element in $\mathrm{Gal}(\mathbf{F}_q(t,y)/\mathbf{F}_q(t))$ defined by $y \mapsto \mu y$. For any character $\chi : \mu_m \to \overline{\mathbf{Q}}_l^*$, let $\mathcal{L}_\chi$ be the galois representation defined by

$$\mathrm{Gal}(\overline{\mathbf{F}_q(t)}/\mathbf{F}_q(t)) \to \mathrm{Gal}(\mathbf{F}_q(t,y)/\mathbf{F}_q(t)) \stackrel{\cong}{\to} \mu_m \stackrel{\chi^{-1}}{\to} \overline{\mathbf{Q}}_l^*.$$

It is unramified outside 0 and ∞, and tamely ramified at 0 and ∞. This galois representation defines a lisse $\overline{\mathbf{Q}}_l$-sheaf on $\mathbf{G}_m$ which we still denote by $\mathcal{L}_\chi$.

Let $\theta : \mathrm{Gal}(\mathbf{F}/\mathbf{F}_q) \to \overline{\mathbf{Q}}_l^*$ be a character of the galois group of the finite field. Denote by $\mathcal{L}_\theta$ the galois representation

$$\mathrm{Gal}(\overline{\mathbf{F}_q(t)}/\mathbf{F}_q(t)) \to \mathrm{Gal}(\mathbf{F}/\mathbf{F}_q) \xrightarrow{\theta} \overline{\mathbf{Q}}_l^*.$$

It is unramified everywhere, and hence defines a lisse $\overline{\mathbf{Q}}_l$-sheaf on $\mathbf{P}^1$ which we still denote by $\mathcal{L}_\theta$.

Finally let $\overline{\mathbf{Q}}_l\left(\frac{1-n}{2}\right)$ be the sheaf on $\mathrm{Spec}\mathbf{F}_q$ corresponding to the galois representation of $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q)$ which maps the geometric Frobenius to $q^{\frac{n-1}{2}}$. For any scheme over $\mathbf{F}_q$, the inverse image of $\overline{\mathbf{Q}}_l\left(\frac{1-n}{2}\right)$ on this scheme is also denoted by the same notation.

Now we are ready to state the main theorem of this section.

Theorem 1.1. Suppose $n|(q-1)$. As a representation of the decomposition subgroup D_∞ at ∞, the Kloosterman sheaf Kl_n is isomorphic to

$$[n]_*(\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\chi) \otimes \mathcal{L}_\theta \otimes \overline{\mathbf{Q}}_l\left(\frac{1-n}{2}\right),$$

where $[n] : \mathbf{G}_m \to \mathbf{G}_m$ is the morphism defined by $x \mapsto x^n$, ψ_n is the additive character

$$\psi_n(a) = \psi(na),$$

χ is trivial if n is odd, and χ is the (unique) nontrivial character

$$\chi : \mu_2 \to \overline{\mathbf{Q}}_l^*$$

if n is even, and $\theta : \mathrm{Gal}(\mathbf{F}/\mathbf{F}_q) \to \overline{\mathbf{Q}}_l^*$ is a character of $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q)$ with the following properties:

(1) If n is odd, then θ is trivial.

(2) If n is even, then θ^2 can be described as follows: Let ζ be a primitive n-th roots of unity in $\mathbf{F}_q$. Fix a square root $\sqrt{\zeta}$ of ζ in $\mathbf{F}$. We have a monomorphism

$$\mathrm{Gal}(\mathbf{F}_q(\sqrt{\zeta})/\mathbf{F}_q) \hookrightarrow \mu_2$$

defined by sending each $\sigma \in \mathrm{Gal}(\mathbf{F}_q(\sqrt{\zeta})/\mathbf{F}_q)$ to $\frac{\sigma(\sqrt{\zeta})}{\sqrt{\zeta}} \in \mu_2$. The character θ^2 is the composition

$$\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q) \to \mathrm{Gal}(\mathbf{F}_q(\sqrt{\zeta})/\mathbf{F}_q) \hookrightarrow \mu_2 \xrightarrow{\chi^{\frac{n}{2}}} \overline{\mathbf{Q}}_l^*.$$

Remark. For even n, the above description of θ^2 shows that $\theta^2 = 1$ if $n \equiv 0 \pmod 4$. If $n \equiv 2 \pmod 4$ and $4|(q-1)$, then since $n|(q-1)$, we must have $2n|(q-1)$. So $(\sqrt{\zeta})^{q-1} = \zeta^{\frac{q-1}{2}} = 1$ and hence $\sqrt{\zeta} \in \mathbf{F}_q$. So $\theta^2 = 1$ if $n \equiv 2 \pmod 4$ and $4|(q-1)$. In the remaining case that $n \equiv 2 \pmod 4$ and $4 \not| (q-1)$, we have $\sqrt{\zeta} \notin \mathbf{F}_q$ and θ^2 is a primitive quadratic character. We don't know how to determine θ itself completely in the case where n is even.

Throughout this section, we assume $n|(q-1)$. Then the group $\mu_n = \{\mu \in \mathbf{F} | \mu^n = 1\}$ of n-th roots of unity in $\mathbf{F}$ is contained in $\mathbf{F}_q$. Fix an algebraic closure $\overline{\mathbf{F}_q(t)}$ of $\mathbf{F}_q(t)$. The aim of this section is to determine as much as possible Kl_n as a representation of the decomposition subgroup D_∞ at ∞. The proof of Theorem 1.1 will be completed in Section 2. Before that, we need a series of Lemmas. Our starting point is the following result of Katz:

Lemma 1.2. As a representation of the wild inertia subgroup P_∞ at ∞, the Kloosterman sheaf Kl_n is isomorphic to $[n]_*\mathcal{L}_{\psi_n}$.

Proof. This follows from Propositions 10.1 and 5.6.2 in [K]. **Lemma 1.3.** Let G be a group, H

a subgroup of G with finite index, and $\rho : H \to \mathrm{GL}(V)$ a representation. Suppose H is normal in G. For any $g \in G$, let ρ_g be the composition

$$H \stackrel{\mathrm{adj}_g}{\to} H \stackrel{\rho}{\to} \mathrm{GL}(V),$$

where $\mathrm{adj}_g(h) = g^{-1}hg$. Then the isomorphic class of the representation ρ_g depends only on the image of g in G/H, and

$$\mathrm{Res}_H \mathrm{Ind}_H^G \rho \cong \bigoplus_{g \in G/H} \rho_g.$$

Proof. This is a special case of Proposition 22 on Page 58 of [S].

Lemma 1.4. We have

$$[n]^*[n]_*\mathcal{L}_{\psi_n} \cong \bigoplus_{\mu^n=1} \mathcal{L}_{\psi_{n\mu}},$$

where $\psi_{n\mu}$ is the additive character $\psi_{n\mu}(x) = \psi(n\mu x)$.

Proof. Let y, z be elements in $\overline{\mathbf{F}_q(t)}$ satisfying $y^n = t$ and $z^q - z = y$. Note that $\mathbf{F}_q(z)$ and $\mathbf{F}_q(y)$ are galois extensions of $\mathbf{F}_q(t)$. Let $G = \mathrm{Gal}(\mathbf{F}_q(z)/\mathbf{F}_q(t))$ and $H = \mathrm{Gal}(\mathbf{F}_q(z)/\mathbf{F}_q(y))$. Then H is a normal subgroup of G. Let ρ be the representation

$$H = \mathrm{Gal}(\mathbf{F}_q(z)/\mathbf{F}_q(y)) \stackrel{\cong}{\to} \mathbf{F}_q \stackrel{\psi_n^{-1}}{\to} \overline{\mathbf{Q}}_l^*.$$

Then $[n]^*[n]_*\mathcal{L}_{\psi_n}$ is isomorphic to the composition of $\mathrm{Res}_H \mathrm{Ind}_H^G \rho$ with the canonical homomorphism $\mathrm{Gal}(\overline{\mathbf{F}_q(t)}/\mathbf{F}_q(y)) \to \mathrm{Gal}(\mathbf{F}_q(z)/\mathbf{F}_q(y)) = H$. We have canonical isomorphisms

$$G/H \stackrel{\cong}{\to} \mathrm{Gal}(\mathbf{F}_q(y)/\mathbf{F}_q(t)) \stackrel{\cong}{\to} \mu_n.$$

For each $\mu \in \mu_n$, let $g_\mu \in G = \mathrm{Gal}(\mathbf{F}_q(z)/\mathbf{F}_q(t))$ be the element defined by $g_\mu(z) = \mu z$. Then the images of g_μ ($\mu \in \mu_n$) in G/H form a family of representatives of cosets. By Lemma 1.3, we have

$$\mathrm{Res}_H \mathrm{Ind}_H^G \rho \cong \bigoplus_{\mu^n=1} \rho_{g_\mu},$$

where ρ_{g_μ} is the composition

$$H \stackrel{\mathrm{adj}_{g_\mu}}{\to} H \stackrel{\cong}{\to} \mathbf{F}_q \stackrel{\psi_n^{-1}}{\to} \overline{\mathbf{Q}}_l^*.$$

One can verify that we have a commutative diagram

$$\begin{array}{ccc} H & \stackrel{\mathrm{adj}_{g_\mu}}{\to} & H \\ \cong\downarrow & & \downarrow\cong \\ \mathbf{F}_q & \stackrel{a\mapsto \mu a}{\to} & \mathbf{F}_q, \end{array}$$

where the vertical arrows are the canonical isomorphism $H = \mathrm{Gal}(\mathbf{F}_q(z)/\mathbf{F}_q(y)) \stackrel{\cong}{\to} \mathbf{F}_q$. So ρ_{g_μ} is the composition

$$H \stackrel{\cong}{\to} \mathbf{F}_q \stackrel{\psi_{n\mu}^{-1}}{\to} \overline{\mathbf{Q}}_l^*.$$

This proves the lemma.

Before stating the next lemma, let us introduce some notations. Let η_∞ be the generic point of the henselization of $\mathbf{P}^1$ at ∞ and let $\overline{\eta}_\infty$ be a geometric point located at η_∞. Fix an embedding of $\overline{\mathbf{F}_q(t)}$ into the residue field $k(\overline{\eta}_\infty)$ of $\overline{\eta}_\infty$. This defines a monomorphism $\mathrm{Gal}(k(\overline{\eta}_\infty)/k(\eta_\infty)) \hookrightarrow \mathrm{Gal}(\overline{\mathbf{F}_q(t)}/\mathbf{F}_q(t))$ whose image is the decomposition subgroup D_∞ of $\mathrm{Gal}(\overline{\mathbf{F}_q(t)}/\mathbf{F}_q(t))$ at ∞. We identify $\mathrm{Gal}(k(\overline{\eta}_\infty)/k(\eta_\infty))$ with D_∞ through this monomorphism. The category of lisse $\overline{\mathbf{Q}}_l$-sheaves on η_∞ is equivalent to the category of $\overline{\mathbf{Q}}_l$-representations of D_∞. For convenience, we denote a lisse sheaf on η_∞ and the corresponding representation of D_∞ by the same symbol. The morphism $[n] : \mathbf{G}_m \to \mathbf{G}_m$ induces a morphism $\eta_\infty \to \eta_\infty$ which we still denote by $[n]$.

Lemma 1.5. As a representation of the decomposition subgroup D_∞ at ∞, the Kloosterman sheaf Kl_n is isomorphic to $[n]_*(\mathcal{L}_{\psi_n} \otimes \phi)$ for some tamely ramified one dimensional representation $\phi : D_\infty \to \overline{\mathbf{Q}}_l^*$.

Proof. Let V be the stalk of the Kloosterman sheaf Kl_n at $\overline{\eta}_\infty$. Then D_∞ acts on V. Since the Swan conductor of Kl_n at ∞ is 1, V is irreducible as a representation of the inertia group I_∞ and hence irreducible as a representation of D_∞. Since n is relatively prime to p, the morphism $[n] : \mathbf{G}_m \to \mathbf{G}_m$ induces an isomorphism $[n]_* : P_\infty \stackrel{\cong}{\to} P_\infty$ on the wild inertia subgroup P_∞. By Lemma 1.2, V has a one-dimensional subspace L, stable under the action of P_∞, and isomorphic to

$\mathcal{L}_{\psi_n}$ as a representation of P_∞. By Lemma 1.4, the restriction of V to P_∞ is not isotypic. Let D'_∞ be the subgroup of D_∞ consisting of those elements leaving L stable. Then by Proposition 24 on page 61 of [S], V is isomorphic to $\mathrm{Ind}_{D'_\infty}^{D_\infty}(L)$ as a representation of D_∞. Since the rank of V is n, D'_∞ is a subgroup of D_∞ with index n. It defines a finite extension of degree n over the residue field $k(\eta_\infty)$ of η_∞. Since D'_∞ contains P_∞, this finite extension is tamely ramified. The degree of inertia of this finite extension is necessarily 1. Otherwise, the family of double cosets $I_\infty\backslash D_\infty/D'_\infty$ would contain more than one elements and $\mathrm{Res}_{I_\infty}\mathrm{Ind}_{D'_\infty}^{D_\infty}(L)$ would not be irreducible by Proposition 22 on page 58 of [S]. This contradicts to the fact that V is irreducible as a representation of I_∞. These facts imply that the above finite extension is just $[n]:\eta_\infty\to\eta_\infty$ and D'_∞ is the image of the monomorphism $[n]_*:D_\infty\hookrightarrow D_\infty$ induced by the morphism $[n]$. Since L is isomorphic $\mathcal{L}_{\psi_n}$ as a representation of P_∞, L is isomorphic to $\mathcal{L}_{\psi_n}\otimes L'$ as a representation of D'_∞, where L' is a representation of D'_∞ of rank 1 which is trivial when restricted to P_∞, that is, L' is tamely ramified. Composing with the isomorphism $[n]_*:D_\infty\xrightarrow{\cong}D'_\infty$, L' defines a tamely ramified one-dimensional representation $\phi:D_\infty\to\overline{\mathbf{Q}}_l^*$, and V is isomorphic to $[n]_*(\mathcal{L}_{\psi_n}\otimes\phi)$.

Lemma 1.6. Keep the notation of Lemma 1.5. As a representation of the inertia subgroup I_∞, ϕ is isomorphic to $\mathcal{L}_\chi$, where χ is trivial if n is odd, and χ is the unique nontrivial character $\chi:\mu_2\to\overline{\mathbf{Q}}_l^*$ if n is even.

Proof. First recall that making the base extension from $\mathbf{F}_q$ to $\mathbf{F}$ has no effect on the inertia subgroup I_∞. So we can work over the base $\mathbf{F}$. By the Appendix of [ST], the representation $\phi:D_\infty\to\overline{\mathbf{Q}}_l^*$ is quasi-unipotent when restricted to I_∞, and hence has finite order when restricted to I_∞ (since the representation is one-dimensional). Since ϕ is tamely ramified, there exists a positive integer m relatively prime to p so that as a representation of I_∞, ϕ is isomorphic to the restriction to I_∞ of the galois representation

$$\mathrm{Gal}(\mathbf{F}(\sqrt[m]{y})/\mathbf{F}(y))\xrightarrow{\cong}\mu_m\xrightarrow{\chi^{-1}}\overline{\mathbf{Q}}_l^*$$

for some primitive character $\chi:\mu_m\to\overline{\mathbf{Q}}_l^*$.

Let us calculate $\det([n]_*(\mathcal{L}_{\psi_n}\otimes\mathcal{L}_\chi))$. Let y, z, w be elements in $\overline{\mathbf{F}(t)}$ satisfying $y^n=t$, $z^q-z=y$, and $w^m=y$. Then $\mathbf{F}(z,w)$ and $\mathbf{F}(y)$ are galois extensions of $\mathbf{F}(t)$. Let $G=\mathrm{Gal}(\mathbf{F}(z,w)/\mathbf{F}(t))$ and $H=\mathrm{Gal}(\mathbf{F}(z,w)/\mathbf{F}(y))$. Then H is normal in G, and we have canonical isomorphisms

$$G/H\xrightarrow{\cong}\mathrm{Gal}(\mathbf{F}(y)/\mathbf{F}(t))\xrightarrow{\cong}\mu_n.$$

We have an isomorphism

$$\mathbf{F}_q \times \mu_m \stackrel{\cong}{\to} H = \mathrm{Gal}(\mathbf{F}(z,w)/\mathbf{F}(y))$$

which maps $(a,\mu) \in \mathbf{F}_q \times \mu_m$ to the element $g_{(a,\mu)} \in \mathrm{Gal}(\mathbf{F}(z,w)/\mathbf{F}(y))$ defined by $g_{(a,\mu)}(z) = z+a$ and $g_{(a,\mu)}(w) = \mu w$. Let $\omega : H \to \overline{\mathbf{Q}}_l^*$ be the character defined by

$$\omega(g_{(a,\mu)}) = \psi_n(-a)\chi(\mu^{-1}).$$

Then $[n]_*(\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\chi)$ is just the composition of $\mathrm{Ind}_H^G(\omega)$ with the canonical homomorphism $\mathrm{Gal}(\overline{\mathbf{F}(t)}/\mathbf{F}(t)) \to \mathrm{Gal}(\mathbf{F}(z,w)/\mathbf{F}(t)) = G$. Let ζ be a primitive n-th root of unity in $\mathbf{F}$. Choose an m-th root $\sqrt[m]{\zeta}$ of ζ. (Then $\sqrt[m]{\zeta}$ is a primitive mn-th root of unity). Let g be the element in $G = \mathrm{Gal}(\mathbf{F}(z,w)/\mathbf{F}(t))$ defined by $g(z) = \zeta z$ and $g(w) = \sqrt[m]{\zeta}w$. Then the image of g in G/H is a generator of the cyclic group G/H. So G is generated by $g_{(a,\mu)} \in H$ $((a,\mu) \in \mathbf{F}_q \times \mu_m)$ and g. By Lemma 1.3, we have

$$\mathrm{Res}_H \mathrm{Ind}_H^G(\omega) = \bigoplus_{i=0}^{n-1} \omega_{g^i},$$

where ω_{g^i} is the composition

$$H \stackrel{\mathrm{adj}_{g^i}}{\to} H \stackrel{\omega}{\to} \overline{\mathbf{Q}}_l^*.$$

One can verify that

$$\omega_{g^i}(g_{(a,\mu)}) = \psi_n(-\zeta^i a)\chi(\mu^{-1}).$$

So

$$\begin{aligned}
(\det(\mathrm{Ind}_H^G(\omega)))(g_{(a,\mu)}) &= \prod_{i=0}^{n-1}(\psi_n(-\zeta^i a)\chi(\mu^{-1})) \\
&= \psi_n(-(\sum_{i=0}^{n-1}\zeta^i)a)\chi(\mu^{-n}) \\
&= \psi_n(0)\chi(\mu^{-n}) \\
&= \chi(\mu^{-n})
\end{aligned}$$

We have

$$\begin{aligned}
(\det)(\mathrm{Ind}_H^G(\omega)))(g) &= \det\begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ \omega(g^n) & & & \end{pmatrix} \\
&= (-1)^{n+1}\omega(g^n).
\end{aligned}$$

One can verify $g^n = g_{(0,(\sqrt[m]{\zeta})^n)}$. So $\omega(g^n) = \chi((\sqrt[m]{\zeta})^{-n})$ and hence

$$(\det)(\mathrm{Ind}_H^G(\omega)))(g) = (-1)^{n+1}\chi((\sqrt[m]{\zeta})^{-n}).$$

By Lemma 1.5, $\det(\mathrm{Kl}_n)$ is isomorphic to $\det([n]_*(\mathcal{L}_{\psi_n} \otimes \phi))$ as a representation of I_∞. By [K] 7.4.3, $\det(\mathrm{Kl}_n)$ is geometrically constant. On the other hand, as representations of I_∞, we have $\det([n]_*(\mathcal{L}_{\psi_n} \otimes \phi)) \cong \det([n]_*(\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\chi))$, and $\det([n]_*(\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\chi))$ is isomorphic to the composition of $\det(\mathrm{Ind}_H^G(\omega))$ with the canonical homomorphism $\mathrm{Gal}(\overline{\mathbf{F}(t)}/\mathbf{F}(t)) \to \mathrm{Gal}(\mathbf{F}(z,w)/\mathbf{F}(t)) = G$. So $\det(\mathrm{Ind}_H^G(\omega))$ is trivial as a representation of I_∞. Hence $\chi(\mu^{-n}) = 1$ for all $\mu \in \mu_m$ and $\chi((\sqrt[m]{\zeta})^{-n}) = (-1)^{n+1}$. Since ζ is a primitive n-th root of unity, $(\sqrt[m]{\zeta})^{-n}$ is a primitive m-th root of unity. This implies that the order m of χ is at most 2. If n is odd, the relation $\chi((\sqrt[m]{\zeta})^{-n}) = (-1)^{n+1} = 1$ implies that χ is trivial. If n is even, the relation $\chi((\sqrt[m]{\zeta})^{-n}) = (-1)^{n+1} = -1$ implies that the order m of χ is exactly 2. This finishes the proof of the lemma.

Lemma 1.7. As a representation of D_∞, the Kloosterman sheaf Kl_n is isomorphic to

$$[n]_*(\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\chi) \otimes \mathcal{L}_\theta \otimes \overline{\mathbf{Q}}_l\left(\frac{1-n}{2}\right),$$

where χ is trivial if n is odd, and χ is the nontrivial character $\chi : \mu_2 \to \overline{\mathbf{Q}}_l^*$ if n is even, and $\theta : \mathrm{Gal}(\mathbf{F}/\mathbf{F}_q) \to \overline{\mathbf{Q}}_l^*$ is a character of $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q)$.

Proof. By Lemma 1.5, as a representation of D_∞, Kl_n is isomorphic to $[n]_*(\mathcal{L}_{\psi_n} \otimes \phi)$ for some character $\phi : D_\infty \to \overline{\mathbf{Q}}_l^*$. By Lemma 1.6, ϕ is isomorphic to $\mathcal{L}_\chi$ when restricted to I_∞. So as a representation of D_∞, ϕ is isomorphic to $\mathcal{L}_\chi \otimes \mathcal{L}_\theta \otimes \mathbf{Q}_l\left(\frac{1-n}{2}\right)$ for some character $\theta : \mathrm{Gal}(\mathbf{F}/\mathbf{F}_q) \to \overline{\mathbf{Q}}_l^*$. So as a representation of D_∞, Kl_n is isomorphic to

$$[n]_*\left(\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\chi \otimes \mathcal{L}_\theta \otimes \overline{\mathbf{Q}}_l\left(\frac{1-n}{2}\right)\right) \cong [n]_*(\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\chi) \otimes \mathcal{L}_\theta \otimes \overline{\mathbf{Q}}_l\left(\frac{1-n}{2}\right).$$

Lemma 1.7 is proved.

Lemma 1.8. Keep the notation in Lemma 1.7. The character $\theta : \mathrm{Gal}(\mathbf{F}/\mathbf{F}_q) \to \overline{\mathbf{Q}}_l^*$ has the following properties:

(1) If n is odd, then θ^n is trivial.

(2) If n is even, then θ^2 can be described as follows: Let ζ be a primitive n-th roots of unity in $\mathbf{F}_q$. Fix a square root $\sqrt{\zeta}$ of ζ in $\mathbf{F}$. We have a monomorphism

$$\mathrm{Gal}(\mathbf{F}_q(\sqrt{\zeta})/\mathbf{F}_q) \hookrightarrow \mu_2$$

defined by sending each $\sigma \in \mathrm{Gal}(\mathbf{F}_q(\sqrt{\zeta})/\mathbf{F}_q)$ to $\frac{\sigma(\sqrt{\zeta})}{\sqrt{\zeta}} \in \mu_2$. The character θ^2 is the composition

$$\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q) \to \mathrm{Gal}(\mathbf{F}_q(\sqrt{\zeta})/\mathbf{F}_q) \hookrightarrow \mu_2 \xrightarrow{\chi^{\frac{n}{2}}} \overline{\mathbf{Q}}_l^*,$$

where $\chi : \mu_2 \to \overline{\mathbf{Q}}_l^*$ is the nontrivial character on μ_2.

(3) If n is odd and $p = 2$, then θ is trivial.

Proof. Suppose n is odd. By Lemma 1.7, Kl_n is isomorphic to $[n]_*\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\theta \otimes \overline{\mathbf{Q}}_l\left(\frac{1-n}{2}\right)$ as a representation of D_∞. Using the same method as in the proof of Lemma 1.6 (but working over the base $\mathbf{F}_q$), one can show that $\det([n]_*\mathcal{L}_{\psi_n})$ is trivial. So

$$\det\left([n]_*\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\theta \otimes \overline{\mathbf{Q}}_l\left(\frac{1-n}{2}\right)\right) = \mathcal{L}_{\theta^n} \otimes \overline{\mathbf{Q}}_l\left(\frac{n(1-n)}{2}\right).$$

By [K] 7.4.3, we have $\det(\mathrm{Kl}_n) = \overline{\mathbf{Q}}_l\left(\frac{n(1-n)}{2}\right)$. So θ^n is trivial. This proves part (1) of Lemma 1.8.

Suppose n is even. In this case, Kl_n is isomorphic to $[n]_*(\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\chi) \otimes \mathcal{L}_\theta \otimes \overline{\mathbf{Q}}_l\left(\frac{1-n}{2}\right)$ as a representation of D_∞, where χ is the nontrivial character $\chi : \mu_2 \to \overline{\mathbf{Q}}_l^*$. By [K] 4.1.11 and 4.2.1, there exists a perfect skew-symmetric pairing $\mathrm{Kl}_n \times \mathrm{Kl}_n \to \overline{\mathbf{Q}}_l(1-n)$, and any such pairing invariant under the action of I_∞ coincides with this one up to a scalar. So the I_∞-coinvariant space $(\bigwedge^2 \mathrm{Kl}_n)_{I_\infty}$ of $\bigwedge^2 \mathrm{Kl}_n$ is isomorphic to $\overline{\mathbf{Q}}_l(1-n)$ as a representation of $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q)$. Hence $(\bigwedge^2[n]_*(\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\chi))_{I_\infty} \otimes \mathcal{L}_{\theta^2} \otimes \overline{\mathbf{Q}}_l(1-n)$ is isomorphic to $\overline{\mathbf{Q}}_l(1-n)$. Note that $\bigwedge^2[n]_*(\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\chi)$ is a semisimple representation of I_∞ since it has finite monodromy. So the canonical homomorphism

$$(\bigwedge^2[n]_*(\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\chi))^{I_\infty} \to (\bigwedge^2[n]_*(\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\chi))_{I_\infty}$$

is an isomorphism. We will show that as a representation of $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q)$, $(\bigwedge^2[n]_*(\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\chi))^{I_\infty}$ is isomorphic to the composition

$$\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q) \to \mathrm{Gal}(\mathbf{F}_q(\sqrt{\zeta})/\mathbf{F}_q) \hookrightarrow \mu_2 \xrightarrow{\chi^{\frac{n}{2}}} \overline{\mathbf{Q}}_l^*.$$

This will prove part (2) of Lemma 1.8.

Let ζ be a primitive n-th root of unity in $\mathbf{F}_q$. Fix a square root $\sqrt{\zeta}$ of ζ in $\mathbf{F}$. Let y, z, w be elements in $\overline{\mathbf{F}_q(t)}$ satisfying $y^n = t$, $z^q - z = y$, and $w^2 = y$. Then $\mathbf{F}_q(z, w, \sqrt{\zeta})$ and $\mathbf{F}_q(y)$ are galois extensions of $\mathbf{F}_q(t)$. Let $G = \mathrm{Gal}(\mathbf{F}_q(z, w, \sqrt{\zeta})/\mathbf{F}_q(t))$ and $H = \mathrm{Gal}(\mathbf{F}_q(z, w, \sqrt{\zeta})/\mathbf{F}_q(y))$. Then H is normal in G, and we have canonical isomorphisms

$$G/H \xrightarrow{\cong} \mathrm{Gal}(\mathbf{F}_q(y)/\mathbf{F}_q(t)) \xrightarrow{\cong} \mu_n.$$

Consider the case where $\sqrt{\zeta}$ does not lie in $\mathbf{F}_q$. We then have an isomorphism

$$\mathbf{F}_q \times \mu_2 \times \mu_2 \xrightarrow{\cong} H = \mathrm{Gal}(\mathbf{F}_q(z, w, \sqrt{\zeta})/\mathbf{F}_q(y))$$

which maps $(a, \mu', \mu'') \in \mathbf{F}_q \times \mu_2 \times \mu_2$ to the element $g_{(a,\mu',\mu'')} \in \mathrm{Gal}(\mathbf{F}_q(z, w, \sqrt{\zeta})/\mathbf{F}_q(y))$ defined by $g_{(a,\mu',\mu'')}(z) = z + a$, $g_{(a,\mu',\mu'')}(w) = \mu' w$, and $g_{(a,\mu',\mu'')}(\sqrt{\zeta}) = \mu''\sqrt{\zeta}$. (In the case where $\sqrt{\zeta}$ lies in $\mathbf{F}_q$, we have $\mathbf{F}_q(z, w, \sqrt{\zeta}) = \mathbf{F}_q(z, w)$, and we have an isomorphism

$$\mathbf{F}_q \times \mu_2 \xrightarrow{\cong} H = \mathrm{Gal}(\mathbf{F}_q(z, w, \sqrt{\zeta})/\mathbf{F}_q(y))$$

which maps $(a, \mu') \in \mathbf{F}_q \times \mu_2$ to the element $g_{(a,\mu')} \in \mathrm{Gal}(\mathbf{F}_q(z, w, \sqrt{\zeta})/\mathbf{F}_q(y))$ defined by $g_{(a,\mu')}(z) = z + a$ and $g_{(a,\mu')}(w) = \mu' w$. All the following argument works for this case with slight modification. We leave to the reader to treat this case.) Let $\omega : H \to \overline{\mathbf{Q}}_l^*$ be the character defined by

$$\omega(g_{(a,\mu',\mu'')}) = \psi_n(-a)\chi(\mu'^{-1}).$$

Then $[n]_*(\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\chi)$ is just the composition of $\mathrm{Ind}_H^G(\omega)$ with the canonical homomorphism $\mathrm{Gal}(\overline{\mathbf{F}_q(t)}/\mathbf{F}_q(t)) \to \mathrm{Gal}(\mathbf{F}_q(z, w, \sqrt{\zeta})/\mathbf{F}_q(t)) = G$. Let g be the element in $G = \mathrm{Gal}(\mathbf{F}_q(z, w, \sqrt{\zeta})/\mathbf{F}_q(t))$ defined by $g(z) = \zeta z$, $g(w) = \sqrt{\zeta} w$ and $g(\sqrt{\zeta}) = \sqrt{\zeta}$. Then the image of g in G/H is a generator of the cyclic group G/H. So G is generated by $g_{(a,\mu',\mu'')} \in H$ ($(a, \mu', \mu'') \in \mathbf{F}_q \times \mu_2 \times \mu_2$) and g. By Lemma 1.3, we have

$$\mathrm{Res}_H \mathrm{Ind}_H^G(\omega) = \bigoplus_{i=0}^{n-1} \omega_{g^i},$$

where ω_{g^i} is the composition

$$H \overset{\mathrm{adj}_{g^i}}{\to} H \xrightarrow{\omega} \overline{\mathbf{Q}}_l^*.$$

One can verify that

$$\omega_{g^i}(g_{(a,\mu',\mu'')}) = \psi_n(-\zeta^i a)\chi(\mu'^{-1}\mu''^{-i}).$$

Let V be the stalk of $[n]_*(\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\chi)$ at the geometric point $\overline{\eta}_\infty$. The above calculation shows that there exists a basis $\{e_0, \ldots, e_{n-1}\}$ of V such that $ge_0 = e_1$, $ge_1 = e_2, \ldots, ge_{n-2} = e_{n-1}$, and for any $g_{(a,\mu',\mu'')} \in H$ ($(a, \mu', \mu'') \in \mathbf{F}_q \times \mu_2 \times \mu_2$), we have

$$g_{(a,\mu',\mu'')}(e_i) = \omega_{g^i}(g_{(a,\mu',\mu'')})e_i = \psi_n(-\zeta^i a)\chi(\mu'^{-1}\mu''^{-i})e_i.$$

One can verify $g^n = g_{(0,(\sqrt{\zeta})^n,1)}$. So

$$g^n e_0 = \chi((\sqrt{\zeta})^{-n})e_0 = -e_0.$$

Consider the element $\sum_{i=0}^{\frac{n}{2}-1} e_i \wedge e_{i+\frac{n}{2}} \in \bigwedge^2 V$. We have

$$\begin{aligned} g(e_0 \wedge e_{\frac{n}{2}} + \cdots + e_{\frac{n}{2}-1} \wedge e_{n-1}) &= e_1 \wedge e_{1+\frac{n}{2}} + \cdots + e_{\frac{n}{2}-1} \wedge e_{n-1} + e_{\frac{n}{2}} \wedge g^n e_0 \\ &= e_0 \wedge e_{\frac{n}{2}} + \cdots + e_{\frac{n}{2}-1} \wedge e_{n-1} \end{aligned}$$

and

$$\begin{aligned} & g_{(a,\mu',\mu'')}(\sum_{i=0}^{\frac{n}{2}-1} e_i \wedge e_{i+\frac{n}{2}}) \\ = & \sum_{i=0}^{\frac{n}{2}-1} \psi_n(-\zeta^i a)\chi(\mu'^{-1}\mu''^{-i})\psi_n(-\zeta^{i+\frac{n}{2}}a)\chi(\mu'^{-1}\mu''^{-(i+\frac{n}{2})})e_i \wedge e_{i+\frac{n}{2}} \\ = & \sum_{i=0}^{\frac{n}{2}-1} \psi_n(-\zeta^i a(1+\zeta^{\frac{n}{2}}))\chi(\mu'^{-2}\mu''^{-2i-\frac{n}{2}})e_i \wedge e_{i+\frac{n}{2}} \\ = & \chi(\mu'')^{\frac{n}{2}} \sum_{i=0}^{\frac{n}{2}-1} e_i \wedge e_{i+\frac{n}{2}} \end{aligned}$$

since $\zeta^{\frac{n}{2}} = -1$ and χ is of order 2. In particular, g and $g_{(a,\mu',1)}$ act trivially on $\sum_{i=0}^{\frac{n}{2}-1} e_i \wedge e_{i+\frac{n}{2}}$. So $\sum_{i=0}^{\frac{n}{2}-1} e_i \wedge e_{i+\frac{n}{2}}$ lies in $(\bigwedge^2 V)^{I_\infty}$. Note that $\sum_{i=0}^{\frac{n}{2}-1} e_i \wedge e_{i+\frac{n}{2}}$ spans $(\bigwedge^2 V)^{I_\infty}$ as the latter space is one dimensional. On the other hand, for any $\mu'' \in \mu_2$, we have $g_{(0,1,\mu'')}(\sum_{i=0}^{\frac{n}{2}-1} e_i \wedge e_{i+\frac{n}{2}}) = \chi(\mu'')^{\frac{n}{2}} \sum_{i=0}^{\frac{n}{2}-1} e_i \wedge e_{i+\frac{n}{2}}$. So as a representation $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q)$, $(\bigwedge^2 V)^{I_\infty}$ is isomorphic to the composition

$$\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q) \to \mathrm{Gal}(\mathbf{F}_q(\sqrt{\zeta})/\mathbf{F}_q) \hookrightarrow \mu_2 \stackrel{\chi^{\frac{n}{2}}}{\to} \overline{\mathbf{Q}}_l^*.$$

This finishes the proof of part (2) of Lemma 1.8.

Finally suppose n is odd and $p = 2$. Then Kl_n is isomorphic to $[n]_* \mathcal{L}_{\psi_n} \otimes \mathcal{L}_\theta \otimes \overline{\mathbf{Q}}_l\left(\frac{1-n}{2}\right)$ as a representation of D_∞. By [K] 4.1.11 and 4.2.1, there exists a perfect symmetric pairing $\mathrm{Kl}_n \times \mathrm{Kl}_n \to \overline{\mathbf{Q}}_l(1-n)$, and any such pairing invariant under the action of I_∞ coincides with this one up to a scalar. So the I_∞-coinvariant space $(\mathrm{Sym}^2(\mathrm{Kl}_n))_{I_\infty}$ of $\mathrm{Sym}^2(\mathrm{Kl}_n)$ is isomorphic to $\overline{\mathbf{Q}}_l(1-n)$ as a representation of $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q)$. Hence $(\mathrm{Sym}^2([n]_* \mathcal{L}_{\psi_n}))_{I_\infty} \otimes \mathcal{L}_{\theta^2} \otimes \overline{\mathbf{Q}}_l(1-n)$ is isomorphic to $\overline{\mathbf{Q}}_l(1-n)$. Note that $\mathrm{Sym}^2([n]_* \mathcal{L}_{\psi_n})$ is a semisimple representation of I_∞ since it has finite monodromy. So the canonical homomorphism

$$(\mathrm{Sym}^2([n]_* \mathcal{L}_{\psi_n}))^{I_\infty} \to (\mathrm{Sym}^2([n]_* \mathcal{L}_{\psi_n}))_{I_\infty}$$

is an isomorphism. One can show that as a representation of $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q)$, $(\mathrm{Sym}^2([n]_* \mathcal{L}_{\psi_n}))^{I_\infty}$ is trivial. (Using the notation in the previous paragraph, one verifies $e_0^2 + e_1^2 + \cdots + e_{n-1}^2$ is a generator of $(\mathrm{Sym}^2 V)^{I_\infty}$ and the geometric Frobenius acts trivially on this vector.) So θ^2 is trivial. By (1), θ^n is also trivial. So θ must be trivial. This finishes the proof of part (3) of Lemma 1.8. The proof of Lemma 1.8 is complete.

2. The bad factors of the L-functions of symmetric products

Let ζ be a primitive n-th root of unity in $\mathbf{F}$. For each positive integer k, let $S_k(n,p)$ be the set of n-tuples $(j_0, \cdots, j_{n-1})$ of non-negative integers satisfying $j_0 + j_1 + \cdots + j_{n-1} = k$ and $j_0 + j_1\zeta + \cdots + j_{n-1}\zeta^{n-1} = 0$ in $\mathbf{F}$. Let σ denote the cyclic shifting operator

$$\sigma(j_0, \cdots, j_{n-1}) = (j_{n-1}, j_0, \cdots, j_{n-2}).$$

It is clear that the set $S_k(n,p)$ is σ-stable.

Let V be a $\overline{\mathbf{Q}}_l$-vector space of dimension n with basis $\{e_0, \cdots, e_{n-1}\}$. For an n-tuple $j = (j_0, \cdots, j_{n-1})$ of non-negative integers such that $j_0 + \cdots + j_{n-1} = k$, write

$$e^j = e_0^{j_0} e_1^{j_1} \cdots e_{n-1}^{j_{n-1}}$$

as an element of $\mathrm{Sym}^k V$. For such an n-tuple j, we define

$$v_j = \sum_{i=0}^{n-1} (-1)^{j_{n-1}+\cdots+j_{n-i}} e^{\sigma^i(j)}.$$

This is an element of $\mathrm{Sym}^k V$. If $k = j_0 + j_1 + \cdots + j_{n-1}$ is even, then we have $v_{\sigma(j)} = (-1)^{j_{n-1}} v_j$, and hence the subspace spanned v_j depends only on the σ-orbit of j. Let $a_k(n,p)$ be the number of σ-orbits in $S_k(n,p)$. When k is even, let $b_k(n,p)$ be the number of those σ-orbits in $S_k(n,p)$ such that the subspace spanned by the orbit is not zero.

Lemma 2.1. Suppose $(n,p) = 1$. We have

$$\dim(\mathrm{Sym}^k(\mathrm{Kl})_n)^{I_\infty} = \begin{cases} a_k(n,p) & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even and } k \text{ is odd,} \\ b_k(n,p) & \text{if } n \text{ and } k \text{ are both even.} \end{cases}$$

Proof. Since $(n,p) = 1$ and the inertia subgroup I_∞ does not change if we make base change from $\mathbf{F}_q$ to its finite extensions, we may assume that $n | (q-1)$.

We use the notations in the proof of Lemma 1.6. Recall that we have $m = 1$ if n is odd, and $m = 2$ if n is even. Let β_m be a primitive m-th root of unity in $\mathbf{Q}$. Thus $\beta_m = 1$ if n is odd, and

$\beta_m = -1$ if n is even. Let V be the stalk of $[n]_*(\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\chi)$ at the geometric point $\overline{\eta}_\infty$. There exists a basis $\{e_0, \ldots, e_{n-1}\}$ of V such that $ge_0 = e_1,\ ge_1 = e_2, \ldots, ge_{n-2} = e_{n-1}, ge_{n-1} = \beta_m e_0$, and for any $g_{(a,\mu)} \in H$ ($(a,\mu) \in \mathbf{F}_q \times \mu_m$), we have

$$g_{(a,\mu)}(e_i) = \omega_{g^i}(g_{(a,\mu)})e_i = \psi_n(-\zeta^i a)\chi(\mu^{-1})e_i.$$

A basis for $\mathrm{Sym}^k V$ is $\{e_0^{j_0} e_1^{j_1} \cdots e_{n-1}^{j_{n-1}}\}$, where j_i are non-negative integers satisfying $|j| = \sum_{i=0}^{n-1} j_i = k$. Suppose $v = \sum_{|j|=k} a_j e_0^{j_0} e_1^{j_1} \cdots e_{n-1}^{j_{n-1}}$ lies in $(\mathrm{Sym}^k V)^{I_\infty}$. Then g and $g_{(a,1)}$ ($a \in \mathbf{F}_q$) act trivially on it. We have

$$g(v) = g(\sum_{|j|=k} a_j e_0^{j_0} e_1^{j_1} \cdots e_{n-1}^{j_{n-1}}) = \sum_{|j|=k} \beta_m^{j_{n-1}} a_j e_0^{j_{n-1}} e_1^{j_0} \cdots e_{n-1}^{j_{n-2}}.$$

From $g(v) = v$, we get

$$a_j = \beta_m^{j_{n-1}} a_{\sigma(j)} = \beta_m^{j_{n-1}+j_{n-2}} a_{\sigma^2(j)} = \cdots = \beta_m^{j_{n-1}+j_{n-2}+\cdots+j_0} a_{\sigma^n(j)} = \beta_m^k a_j.$$

If n is even and k is odd, then $\beta_m^k = -1$. The above relation then shows that $a_j = 0$. So $(\mathrm{Sym}^k V)^{I_\infty} = 0$ in this case.

Now assume that either n is odd or k is even. The above vector v is a linear combination of the following vectors

$$v_j = \sum_{i=0}^{n-1} \beta_m^{j_{n-1}+\cdots+j_{n-i}} e^{\sigma^i(j)}.$$

The g-invariant subspace $(\mathrm{Sym}^k V)^g$ is thus spanned by the vectors v_j with $|j| = k$, where j runs only over σ-orbits. On the other hand, one computes that

$$\begin{aligned} g_{(a,1)}(v) &= g_{(a,1)}(\sum_{|j|=k} a_j e_0^{j_0} e_1^{j_1} \cdots e_{n-1}^{j_{n-1}}) \\ &= \sum_{|j|=k} \psi_n(-a(j_0 + j_1\zeta + \cdots + j_{n-1}\zeta^{n-1}))a_j e_0^{j_0} e_1^{j_1} \cdots e_{n-1}^{j_{n-1}} \end{aligned}$$

for all $a \in \mathbf{F}_q$. Since $g_{(a,1)}(v) = v$, if a_j is non-zero, then we must have $j_0 + j_1\zeta + \cdots + j_{n-1}\zeta^{n-1} = 0$ in $\mathbf{F}$, that is, $j \in S_k(n,p)$. Thus, we have proved that the inertia invariant $(\mathrm{Sym}^k V)^{I_\infty}$ is spanned by the vectors v_j where j runs over the σ-orbits of $S_k(n,p)$.

If n is odd, then $\beta_m = 1$ and each v_j is non-zero. As j runs over the σ-orbits of $S_k(n,p)$, the vectors v_j are clearly independent. We thus have $\dim(\mathrm{Sym}^k V)^{I_\infty} = a_k(n,p)$.

If both n and k are even, then $\beta_m = -1$. In this case, some of the vectors v_j can be zero. The remaining non-zeros vectors v_j, as j runs over the σ-orbits of $S_k(n,p)$, will be linearly independent. We thus have $\dim(\mathrm{Sym}^k V)^{I_\infty} = b_k(n,p)$. The lemma is proved.

Lemma 2.2. Suppose p is odd and either n is odd or $n = 2$. Let $j : \mathbf{G}_m \hookrightarrow \mathbf{P}^1$ be the open immersion. Then the L-functions $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)$ and $L(\mathbf{P}^1, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)), T)$ are polynomials. All the reciprocal roots of the polynomial $L(\mathbf{P}^1, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)), T)$ have weight $k(n-1)+1$. The polynomial $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)$ has coefficients in $\mathbf{Z}$.

Proof. By Grothendieck's formula for L-functions, we have

$$L(\mathbf{P}^1, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)), T) = \prod_{i=0}^{2} \det(1 - FT, H^i(\mathbf{P}^1 \otimes \mathbf{F}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)))^{(-1)^{i+1}}.$$

Under our conditions on p and n, the global monodromy group of Kl_n is $\mathrm{SL}(n)$ by [K] 11.1. In particular $\mathrm{Sym}^k(\mathrm{Kl}_n)$ is irreducible as a representation of the geometric fundamental group $\pi_1(\mathbf{G}_m \otimes \mathbf{F})$ and hence $H^i(\mathbf{P}^1 \otimes \mathbf{F}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)))$ vanishes for $i = 0, 2$. So $L(\mathbf{P}^1, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)), T) = \det(1 - FT, H^1(\mathbf{P}^1 \otimes \mathbf{F}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)))$ is a polynomial. Since Kl_n is a lisse sheaf on $\mathbf{G}_m$ puncturely pure of weight $n-1$, all the reciprocal roots of the polynomial $\det(1-FT, H^1(\mathbf{P}^1 \otimes \mathbf{F}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)))$ have weight $k(n-1)+1$ by [D2] 3.2.3. Similarly, one can show $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)$ is a polynomial. Let's prove it has coefficients in $\mathbf{Z}$. For any $a \in \mathbf{F}_p^*$, let σ_a denote the element of $\mathrm{Gal}(\mathbf{Q}(\xi_p)/\mathbf{Q})$ such that $\sigma(\xi_p) = \xi_p^a$, where $\xi_p \neq 1$ is a p-th root of unity in $\overline{\mathbf{Q}}$. Using the definition of Kloosterman sums, one checks that

$$\sigma_a(\mathrm{Kl}_n(\mathbf{F}_{q^k}, \lambda)) = \mathrm{Kl}_n(\mathbf{F}_{q^k}, a^n\lambda),$$

and thus

$$\sigma_a(L(\lambda, T)) = L(a^n\lambda, T).$$

Recall from the Introduction that

$$L(\lambda, T)^{(-1)^n} = (1 - \pi_1(\lambda)T) \cdots (1 - \pi_n(\lambda)T).$$

Write

$$\mathrm{Sym}^k(L(\lambda, T)^{(-1)^n}) = \prod_{i_1+\cdots+i_n=k} (1 - \pi_1^{i_1}(\lambda) \cdots \pi_n^{i_n}(\lambda)T).$$

Then, we deduce

$$\sigma_a(\mathrm{Sym}^k(L(\lambda, T)^{(-1)^n})) = \mathrm{Sym}^k(L(a^n\lambda, T)^{(-1)^n}).$$

Now, by definition,

$$L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T) = \prod_{\lambda \in |\mathbf{G}_m|} \frac{1}{\mathrm{Sym}^k(L(\lambda, T^{\deg(\lambda)})^{(-1)^n})}.$$

Thus,

$$\sigma_a(L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)) = \prod_{\lambda \in |\mathbf{G}_m|} \frac{1}{\mathrm{Sym}^k(L(a^n\lambda, T^{\deg(\lambda)})^{(-1)^n})}.$$

The right side is clearly the same as $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)$. The lemma is proved.

Proposition 2.3. Let F_0 be the geometric Frobenius element at 0 and I_0 the inertia subgroup at 0. The reciprocal roots of the polynomial $\det(1 - F_0T, (\mathrm{Sym}^k(\mathrm{Kl}_n))^{I_0})$ are of the form q^i, where $0 \le i \le k(n-1)/2$. In particular, $\det(1-F_0T, (\mathrm{Sym}^k(\mathrm{Kl}_n))^{I_0})$ is a polynomial in integer coefficients of weights at most $k(n-1)$. In the case $n = 2$, we have

$$\det(1 - F_0T, (\mathrm{Sym}^k(\mathrm{Kl}_n))^{I_0}) = 1 - T.$$

Proof. By [K] 7.3.2 (3) and [D2] 1.8.1, the eigenvalues of F_0 acting on $\mathrm{Sym}^k(\mathrm{Kl}_n))^{I_0}$ are of the form q^i with $0 \le i \le k(n-1)/2$. This proves the first part of the Proposition.

By [K] 7.4.3, the local monodromy at 0 of Kl_n is unipotent with a single Jordan block, and F_0 acts trivially on $(\mathrm{Kl}_n)^{I_0}$. If $n = 2$, using this fact, one can show the local monodromy at 0 of $\mathrm{Sym}^k(\mathrm{Kl}_2)$ has the same property. Proposition 2.3 follows.

Remark. We do not know a precise formula for $\det(1 - F_0T, (\mathrm{Sym}^k(\mathrm{Kl}_n))^{I_0})$ in general for $n > 2$.

Lemma 2.4. Keep the notation in Lemma 1.7. Suppose pn is odd. Then θ is trivial.

Proof. Take a positive integer k such that $(k, n) = 1$ and such that $S_k(n, p)$ is non-empty. For instance, we can take $k = n + mp$ for any positive integer m prime to n. (Recall that we always assume $(n, p) = 1$). Then $a_k(n, p) \neq 0$. By Lemma 2.2, $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)$ is a polynomial with coefficient in $\mathbf{Z}$. So each pure weight part of $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)$ also has coefficients in $\mathbf{Z}$. We have

$$\begin{aligned} & L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T) \\ = \ & L(\mathbf{P}^1, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)), T)\det(1 - F_0T, (\mathrm{Sym}^k(\mathrm{Kl}_n))^{I_0})\det(1 - F_\infty T, (\mathrm{Sym}^k(\mathrm{Kl}_n))^{I_\infty}), \end{aligned}$$

where F_∞ is the geometric Frobenius element at ∞. Since n is odd, by Lemmas 1.7 and 1.8, we have

$$(\mathrm{Sym}^k(\mathrm{Kl}_n))^{I_\infty} = (\mathrm{Sym}^k([n]_*\mathcal{L}_{\psi_n}))^{I_\infty} \otimes \mathcal{L}_{\theta^k} \otimes \overline{\mathbf{Q}}_l\left(\frac{k(1-n)}{2}\right).$$

Using the calculation in Lemmas 1.8 and 2.1, one can verify F_∞ acts trivially on $(\mathrm{Sym}^k([n]_*\mathcal{L}_{\psi_n}))^{I_\infty}$. Let $\lambda = \theta(F_\infty)$. Then $\lambda^n = 1$ by Lemma 1.8 (1), and

$$\det(1 - F_\infty T, (\mathrm{Sym}^k(\mathrm{Kl}_n))^{I_\infty}) = (1 - \lambda^k q^{\frac{k(n-1)}{2}} T)^{a_k(n,p)}.$$

So we have

$$L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T) =$$

$$\det(1 - F_0T, (\mathrm{Sym}^k(\mathrm{Kl}_n))^{I_0})(1 - \lambda^k q^{\frac{k(n-1)}{2}} T)^{a_k(n,p)} L(\mathbf{P}^1, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)), T).$$

By Lemma 2.2, $L(\mathbf{P}^1, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)), T)$ is pure of weight $k(n-1)+1$. So the part of $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)$ with weight at most $k(n-1)$ is given by

$$\det(1 - F_0T, (\mathrm{Sym}^k(\mathrm{Kl}_n))^{I_0})(1 - \lambda^k q^{\frac{k(n-1)}{2}} T)^{a_k(n,p)}.$$

It must have coefficients in $\mathbf{Z}$. The first factor also has coefficients in $\mathbf{Z}$. Working with the coefficients of T, we see that $\lambda^k q^{\frac{k(n-1)}{2}}$ must be an integer. Since $\lambda^n = 1$, we must have $\lambda^k = \pm 1$. As $(k, n) = 1$ and n is odd, we must have $\lambda = 1$. So θ is trivial.

Now Theorem 1.1 in Section 1 follows from Lemmas 1.7, 1.8, and 2.4.

In the following, we calculate the bad factors at ∞ of the L-function of the k-th symmetric product of Kl_n.

Theorem 2.5. Suppose $n|(q-1)$. Let F_∞ be the geometric Frobenius element at ∞.

(1) If n is odd, then for all k, we have

$$\det(1 - F_\infty T, (\mathrm{Sym}^k(\mathrm{Kl}_n))^{I_\infty}) = (1 - q^{\frac{k(n-1)}{2}} T)^{a_k(n,p)}.$$

(2) If n is even and k is odd, then we have

$$\det(1 - F_\infty T, (\mathrm{Sym}^k(\mathrm{Kl}_n))^{I_\infty}) = 1.$$

(3) Suppose n and k are both even. We have

$$\det(1 - F_\infty T, (\mathrm{Sym}^k(\mathrm{Kl}_n))^{I_\infty})$$
$$= \begin{cases} (1 - q^{\frac{k(n-1)}{2}} T)^{b_k(n,p)} & \text{if } 2n|(q-1), \\ (1 + q^{\frac{k(n-1)}{2}} T)^{c_k(n,p)} (1 - q^{\frac{k(n-1)}{2}} T)^{b_k(n,p)-c_k(n,p)} & \text{if } 2n \not|(q-1), \text{ either } 4|n \text{ or } 4|k, \\ (1 - q^{\frac{k(n-1)}{2}} T)^{c_k(n,p)} (1 + q^{\frac{k(n-1)}{2}} T)^{b_k(n,p)-c_k(n,p)} & \text{if } 2n \not|(q-1), \ 4 \not|n \text{ and } 4 \not|k. \end{cases}$$

where $c_k(n,p)$ denotes the number of σ-orbits j in $S_k(n,p)$ such that $v_j \neq 0$ and such that $j_1 + 2j_2 + \cdots + (n-1)j_{n-1}$ is odd.

Proof. By Lemmas 1.7 and 1.8, we have

$$(\mathrm{Sym}^k(\mathrm{Kl}_n))^{I_\infty} = (\mathrm{Sym}^k([n]_*(\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\chi)))^{I_\infty} \otimes \mathcal{L}_{\theta^k} \otimes \overline{\mathbf{Q}}_l\left(\frac{k(1-n)}{2}\right).$$

Suppose n is odd. Then χ and θ are trivial. One can verify F_∞ acts trivially on $(\mathrm{Sym}^k([n]_*\mathcal{L}_{\psi_n}))^{I_\infty}$. (1) then follows from Lemma 2.1.

Suppose n is even and k is odd. Then $(\mathrm{Sym}^k(\mathrm{Kl}_n))^{I_\infty} = 0$ by Lemma 2.1. (2) follows.

Suppose n and k are even. If $2n|(q-1)$, then $(\sqrt{\zeta})^{q-1} = \zeta^{\frac{q-1}{2}} = 1$ and hence $\sqrt{\zeta} \in \mathbf{F}_q$. In this case, one verifies that θ is trivial and F_∞ acts trivially on $(\mathrm{Sym}^k([n]_*(\mathcal{L}_{\psi_n} \otimes \mathcal{L}_\chi)))^{I_\infty}$. The first case of (3) then follows from Lemma 2.1. Suppose $2n \not| (q-1)$, then $\sqrt{\zeta} \notin \mathbf{F}_q$. We use the notation in the proof of Lemma 1.8. Note that $g_{(0,1,-1)}$ is a lifting of the geometric Frobenius element in $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$. Recall that $(\mathrm{Sym}^k V)^{I_\infty}$ is generated by the vectors

$$v_j = \sum_{i=0}^{n-1} (-1)^{j_{n-1}+\cdots+j_{n-i}} e^{\sigma^i(j)},$$

where j runs over the σ-orbits of $S_k(n,p)$. One checks that

$$\begin{aligned} & g_{(0,1,-1)}(v_j) \\ = & \sum_{i=0}^{n-1} (-1)^{j_{n-1}+\cdots+j_{n-i}} (-1)^{0\cdot j_{n-i}+1\cdot j_{n-i+1}+\cdots+(i-1)j_{n-1}+ij_0+(i+1)j_1+\cdots+(n-1)j_{n-i-1}} e^{\sigma^i(j)}. \end{aligned}$$

We have

$$\begin{aligned} & 0\cdot j_{n-i} + 1\cdot j_{n-i+1} + \cdots + (i-1)j_{n-1} + ij_0 + (i+1)j_1 + \cdots + (n-1)j_{n-i-1} \\ = & \; i(j_0 + j_1 + \cdots + j_{n-i-1} + j_{n-i} + \cdots + j_{n-1}) \\ & + j_1 + 2j_2 + \cdots + (n-i-1)j_{n-i-1} \\ & + (0-i)j_{n-i} + (1-i)j_{n-i+1} + \cdots + (-1)j_{n-1} \\ = & \; ik \\ & + \Big(j_1 + 2j_2 + \cdots + (n-i-1)j_{n-i-1} \\ & + (n-i)j_{n-i} + (n-i+1)j_{n-i+1} + \cdots + (n-1)j_{n-1} \Big) \\ & - n(j_{n-i} + j_{n-i+1} + \cdots + j_{n-1}). \end{aligned}$$

But n and k are even. So we have

$$g_{(0,1,-1)}(v_j) = (-1)^{j_1+2j_2+\cdots+(n-1)j_{n-1}} v_j.$$

So v_j is an eigenvector of F_∞ on $(\mathrm{Sym}^k V)^{I_\infty}$ with eigenvalue $(-1)^{j_1+2j_2+\cdots+(n-1)j_{n-1}}$. On the other hand, we have $\theta^k(F_\infty) = 1$ if either $4|n$ or $4|k$, and $\theta^k(F_\infty) = -1$ if $4 \not| n$ and $4 \not| k$. The last two cases of (3) follows.

3. The degrees of the L-functions of symmetric products

Let ζ be a primitive n-th root of unity in $\mathbf{F}$ and let k be a positive integer. Denote by $d_k(n,p)$ the number of the set $S_k(n,p)$, that is, the number of n-tuples $(j_0, j_1, \ldots, j_{n-1})$ of non-negative integers satisfying $j_0 + j_1 + \cdots + j_{n-1} = k$ and $j_0 + j_1\zeta + \cdots + j_{n-1}\zeta^{n-1} = 0$ in $\mathbf{F}$. In this section, we prove the following result, which is Theorem 0.1 in the Introduction:

Theorem 3.1. Suppose $(n,p) = 1$. The degree of $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)$ is

$$\frac{1}{n}\left(\binom{k+n-1}{n-1} - d_k(n,p)\right).$$

Proof. By Grothendieck's formula for L-functions, $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)$ is a rational function, and its degree is the negative of the Euler characteristics

$$\chi_c(\mathbf{G}_m \otimes \mathbf{F}, \mathrm{Sym}^k(\mathrm{Kl}_n)) = \sum_{i=0}^{2}(-1)^i \mathrm{dim} H_c^i(\mathbf{G}_m \otimes \mathbf{F}, \mathrm{Sym}^k(\mathrm{Kl}_n))$$

To calculate the Euler characteristic, we may replace the ground field $\mathbf{F}_q$ by its finite extensions. So we may assume $n|(q-1)$. Then ζ lies in $\mathbf{F}_q$. By Lemmas 1.2 and 1.4, $[n]^*\mathrm{Kl}_n$ is isomorphic to $\mathcal{L}_{\psi_n} \oplus \mathcal{L}_{\psi_{n\zeta}} \oplus \cdots \oplus \mathcal{L}_{\psi_{n\zeta^{n-1}}}$ as a representation of the wild inertia subgroup P_∞ at ∞, where for any $a \in \mathbf{F}_q$, ψ_a is the additive character $\psi_a(x) = \psi(ax)$. So we have

$$[n]^*(\mathrm{Sym}^k(\mathrm{Kl}_n)) \cong \bigoplus_{j_0+j_1+\cdots+j_{n-1}=k,\ j_0,j_1,\ldots,j_{n-1}\geq 0} \mathcal{L}_{\psi_{n(j_0+j_1\zeta+\cdots+j_{n-1}\zeta^{n-1})}}$$

as representations of P_∞. But the Swan conductor of $\mathcal{L}_{\psi_a}$ at ∞ is 1 if $a \neq 0$, and 0 if $a = 0$. So the Swan conductor of $[n]^*(\mathrm{Sym}^k(\mathrm{Kl}_n))$ at ∞ is the number of those n-tuples $(j_0, j_1, \ldots, j_{n-1})$ of non-negative integers satisfying $j_0 + j_1 + \cdots + j_{n-1} = k$ and $j_0 + j_1\zeta + \cdots + j_{n-1}\zeta^{n-1} \neq 0$ in $\mathbf{F}_q$. This number is exactly $\binom{k+n-1}{n-1} - d_k(n,p)$. By [K] 1.13.1, the Swan conductor of $\mathrm{Sym}^k(\mathrm{Kl}_n)$ at ∞ is $\frac{1}{n}$ of the Swan conductor of $[n]^*(\mathrm{Sym}^k(\mathrm{Kl}_n))$ at ∞. Since Kl_n is tame at 0, $\mathrm{Sym}^k(\mathrm{Kl}_n)$ is also tame at 0, and hence its Swan conductor at 0 vanishes. By the Grothendieck-Ogg-Shafarevich formula, $-\chi(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n))$ is equal to the sum of the Swan conductors of $\mathrm{Sym}^k(\mathrm{Kl}_n)$ at 0 and at ∞. Theorem 3.1 follows.

In some special cases, an explicit formula for the degree of $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)$ can be obtained. The following is one example.

Corollary 3.2. Suppose n is a prime number different from p such that p is a primitive $(n-1)$-th root of unity mod n. Let $\widetilde{(\frac{k}{n})}$ be the smallest non-negative integer so that its image in $\mathbf{F}_p$ is $\frac{k}{n}$. Then the degree of $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_n), T)$ is

$$\frac{1}{n}\left(\binom{k+n-1}{n-1} - \binom{\frac{k-n\widetilde{(\frac{k}{n})}}{p} + n - 1}{n-1}\right).$$

Proof. Let $d = [\mathbf{F}_p(\zeta) : \mathbf{F}_p]$. Then we have $\zeta^{p^d-1} = 1$. Since ζ has order n, we must have $n|p^d - 1$. On the other hand, since p is a primitive $(n-1)$-th root of unity mod n, $n-1$ is the smallest natural number with the property $p^{n-1} = 1$ in $\mathbf{Z}/n$, that is, $n|p^{n-1} - 1$. So we have $n - 1 \leq d$. Since $1 + \zeta + \cdots + \zeta^{n-1} = 0$, we have $d = [\mathbf{F}_p(\zeta) : \mathbf{F}_p] \leq n - 1$. So we must have $d = [\mathbf{F}_p(\zeta) : \mathbf{F}_p] = n-1$ and $1+X+\cdots+X^{n-1}$ is the minimal polynomial of ζ over $\mathbf{F}_p$. Therefore, if $j_0 + j_1\zeta + \cdots + j_{n-1}\zeta^{n-1} = 0$ in $\mathbf{F}$ for some integers $j_0, j_1, \ldots, j_{n-1}$, then we must have

$$j_0 \equiv j_1 \equiv \cdots \equiv j_{n-1} \ (\mathrm{mod}\ p).$$

Let us determine the number $d_k(n,p)$ of n-tuples $(j_0, j_1, \ldots, j_{n-1})$ of non-negative integers with the property $j_0+j_1+\cdots+j_{n-1} = k$ and $j_0+j_1\zeta+\cdots+j_{n-1}\zeta^{n-1} = 0$ in $\mathbf{F}$. By the above discussion, the second equation implies that $j_0 \equiv j_1 \equiv \cdots \equiv j_{n-1}$ (mod p). Substituting this into the first equation, we get that

$$j_0 \equiv j_1 \equiv \cdots \equiv j_{n-1} \equiv \widetilde{(\frac{k}{n})} \ (\mathrm{mod}\ p).$$

Write $j_i = pj_i' + \widetilde{(\frac{k}{n})}$, where j_i' are non-negative integers. Then the first equation becomes $p(j_0' + j_1' + \cdots + j_{n-1}') + n\widetilde{(\frac{k}{n})} = k$, that is,

$$j_0' + j_1' + \cdots + j_{n-1}' = \frac{k - n\widetilde{(\frac{k}{n})}}{p}.$$

Thus we have

$$d_k(n,p) = \begin{pmatrix} \frac{k-n\widetilde{(\frac{k}{n})}}{p} + n - 1 \\ n-1 \end{pmatrix}.$$

The corollary then follows from Theorem 3.1.

4. An example

In this section, we prove Theorem 0.2 in the Introduction. Throughout this section, we assume that $n = 2$, $q = p$ and p is an odd prime.

Lemma 4.1. The degree of $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_2), T)$ is $\frac{k}{2} - [\frac{k}{2p}]$ if k is even, and $\frac{k+1}{2} - [\frac{k}{2p} + \frac{1}{2}]$ if k is odd.

Proof. Note that -1 is a primitive square root. Let's determine the number $d_k(2,p)$ of pairs (j_0, j_1) of non-negative integers satisfying $j_0 + j_1 = k$ and $j_0 - j_1 = 0$ in $\mathbf{F}$, or equivalently,

$$\begin{aligned} j_0 + j_1 &= k, \\ j_0 - j_1 &\equiv 0 \quad (\ \mathrm{mod}\ p). \end{aligned}$$

This is equivalent to the problem of determining the number of those integers $0 \leq j_0 \leq k$ with the property

$$2j_0 \equiv k \pmod{p}.$$

Since p is an odd prime, the inverse of 2 in $\mathbf{F}_p$ is $\frac{p+1}{2}$, and hence the above equation is equivalent to the equation

$$j_0 \equiv \frac{k(p+1)}{2} \pmod{p}.$$

So we are reduced to determining the number of those integers j with the property

$$0 \leq jp + \frac{k(p+1)}{2} \leq k,$$

that is,

$$-\frac{k}{2p} \leq j + \frac{k}{2} \leq \frac{k}{2p},$$

or equivalently,

$$-\frac{k}{2p} + \frac{1}{2} \leq j + \frac{k+1}{2} \leq \frac{k}{2p} + \frac{1}{2}.$$

When k is even, the number of those integers j satisfying

$$-\frac{k}{2p} \leq j + \frac{k}{2} \leq \frac{k}{2p}$$

is $2[\frac{k}{2p}] + 1$. When k is odd, the number of those integers j satisfying

$$-\frac{k}{2p} + \frac{1}{2} \leq j + \frac{k+1}{2} \leq \frac{k}{2p} + \frac{1}{2}$$

is $2[\frac{k}{2p} + \frac{1}{2}]$. So

$$d_k(2,p) = \begin{cases} 2[\frac{k}{2p}] + 1 & \text{if } k \text{ is even,} \\ 2[\frac{k}{2p} + \frac{1}{2}] & \text{if } k \text{ is odd.} \end{cases}$$

The lemma then follows from Theorem 3.1.

Lemma 4.2. Let F_∞ be the geometric Frobenius element at ∞. We have

$$\det(1 - F_\infty T, (\mathrm{Sym}^k(\mathrm{Kl}_2))^{I_\infty}) = \begin{cases} 1 & \text{if } 2 \not| k, \\ (1 - p^{\frac{k}{2}} T)^{m_k} & \text{if } 2|k \text{ and } p \equiv 1 \pmod{4}, \\ (1 + p^{\frac{k}{2}} T)^{n_k} (1 - p^{\frac{k}{2}} T)^{m_k - n_k} & \text{if } 2|k \text{ and } p \equiv -1 \pmod{4}, \end{cases}$$

where

$$m_k = \begin{cases} 1 + [\frac{k}{2p}] & \text{if } 4|k, \\ [\frac{k}{2p}] & \text{if } 4 \not| k, \end{cases}$$

and $n_k = [\frac{k}{4p} + \frac{1}{2}]$,

Proof. The first case follows from Theorem 2.5 (2). Suppose $2|k$. We will treat the case where $p \equiv -1 \pmod{4}$ and leave to the reader to treat the other case. Note that the condition $p \equiv -1$

(mod 4) is equivalent to saying that the square root $\sqrt{-1}$ of the primitive square root of unity -1 does not lie in $\mathbf{F}_p$. We use the notation in the proof of Lemma 1.8. The proof of Lemma 2.1 shows that a basis for $(\mathrm{Sym}^k V)^{I_\infty}$ is given by

$$\{e_0^i e_1^{k-i} + (-1)^{k-i} e_0^{k-i} e_1^i | i-(k-i) \equiv 0 \ (\mathrm{mod}\ p),\ 0 \leq i \leq \frac{k}{2}\}.$$

(When $i = \frac{k}{2}$, the element $e_0^i e_1^{k-i} + (-1)^{k-i} e_0^{k-i} e_1^i = (1+(-1)^{\frac{k}{2}}) e_0^{\frac{k}{2}} e_1^{\frac{k}{2}}$ is nonzero only when $4|k$. We exclude this element from the basis if $4 \not| k$.) A calculation similar to that in the proof of Lemma 4.1 shows that this basis has $1+[\frac{k}{2p}]$ elements if $4|k$, and $[\frac{k}{2p}]$ elements if $4 \not| k$. So we have

$$\dim(\mathrm{Sym}^k(\mathrm{Kl}_2))^{I_\infty} = \begin{cases} 1+[\frac{k}{2p}] & \text{if } 4|k, \\ [\frac{k}{2p}] & \text{if } 4 \not| k. \end{cases}$$

Note that $g_{(0,1,-1)}$ is a lifting of the geometric Frobenius element in $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$. Let $e_0^i e_1^{k-i} + (-1)^{k-i} e_0^{k-i} e_1^i$ be an element in the above basis. Then

$$\begin{aligned} g_{(0,1,-1)}(e_0^i e_1^{k-i} + (-1)^{k-i} e_0^{k-i} e_1^i) &= (-1)^{k-i} e_0^i e_1^{k-i} + (-1)^{k-i}(-1)^i e_0^{k-i} e_1^i \\ &= (-1)^i (e_0^i e_1^{k-i} + (-1)^{k-i} e_0^{k-i} e_1^i). \end{aligned}$$

(Recall that k is even.) So F_∞ acts semisimply on $(\mathrm{Sym}^k V)^{I_\infty}$ with eigenvalues 1 and -1, and the dimension of the eigenspace corresponding to the eigenvalue 1 (resp. -1) is the number of those even (resp. odd) integers i satisfying $0 \leq i \leq \frac{k}{2}$ and $i-(k-i) \equiv 0$ (mod p). (If $4 \not| k$, we don't count the odd number $i = \frac{k}{2}$). We have $i \equiv \frac{k}{2}$ (mod p). So dimension of the eigenspace corresponding to the eigenvalue 1 (resp. -1) is the number of integers j such that $0 \leq jp + \frac{k}{2} \leq \frac{k}{2}$ and that $jp + \frac{k}{2}$ is even (resp. odd). (Again if $4 \not| k$, we don't count the odd number $i = 0 \cdot p + \frac{k}{2}$). Note that if $4 \not| k$, then $jp+\frac{k}{2}$ is even if and only if j is odd; if $4|k$, then $jp + \frac{k}{2}$ is odd if and only if j is odd. One can show the number of odd integers j satisfying $0 \leq jp + \frac{k}{2} \leq \frac{k}{2}$ is $[\frac{k}{4p}+\frac{1}{2}]$. So if $4 \not| k$ (resp. $4|k$), then the dimension of the eigenspace corresponding to the eigenvalue 1 (resp. -1) is $[\frac{k}{4p}+\frac{1}{2}]$.

On the other hand, by Lemma 1.8, $(\mathrm{Sym}^k(\mathrm{Kl}_2))^{I_\infty}$ is isomorphic to $(\mathrm{Sym}^k V)^{I_\infty} \otimes \mathcal{L}_{\theta^k} \otimes \overline{\mathbf{Q}}_l\left(-\frac{k}{2}\right)$ as a representation of $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$. Since k is even, we have $\theta^k = (\theta^2)^{\frac{k}{2}}$. By the description of θ^2 in Lemma 1.8, if $4|k$, then $\theta^k(F_\infty) = 1$; if $4 \not| k$, then $\theta^k(F_\infty) = -1$. Combining with the above calculation, this proves Lemma 4.2.

Finally, we discuss the functional equation for $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_2), T)$. By Lemma 2.2, $L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_2), T)$ and $L(\mathbf{P}^1, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2)), T)$ are polynomials, where $j : \mathbf{G}_m \hookrightarrow \mathbf{P}^1$ is the open immersion. By

Proposition 2.3, we have

$$\det(1 - F_0T, (\mathrm{Sym}^k(\mathrm{Kl}_2))^{I_0}) = 1 - T.$$

Combining with Lemma 4.2, we see that

$$\det(1 - F_0T, (\mathrm{Sym}^k(\mathrm{Kl}_2))^{I_0})\det(1 - F_\infty T, (\mathrm{Sym}^k(\mathrm{Kl}_2))^{I_\infty})$$

is exactly the polynomial P_k defined in the Introduction. We have

$$\begin{aligned} & L(\mathbf{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_2), T) \\ = \; & L(\mathbf{P}^1, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2)), T)\det(1 - F_0T, (\mathrm{Sym}^k(\mathrm{Kl}_2))^{I_0})\det(1 - F_\infty T, (\mathrm{Sym}^k(\mathrm{Kl}_2))^{I_\infty}). \end{aligned}$$

So $L(\mathbf{P}^1, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2)), T)$ is the polynomial M_k defined in the Introduction.

By [K] 4.1.11, we have $(\mathrm{Kl}_2)^\vee = \mathrm{Kl}_2 \otimes \overline{\mathbf{Q}}_l(1)$. So $(\mathrm{Sym}^k(\mathrm{Kl}_2))^\vee = \mathrm{Sym}^k(\mathrm{Kl}_2) \otimes \overline{\mathbf{Q}}_l(k)$ General theory then shows that we have a functional equation

$$L(\mathbf{P}^1, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2)), T) = ct^\delta L(\mathbf{P}^1, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2)), \frac{1}{p^{k+1}T}),$$

where

$$c = \prod_{i=0}^{2} \det(-F, H^i(\mathbf{P}^1 \otimes \mathbf{F}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2)))^{(-1)^{i+1}}$$

and $\delta = -\chi(\mathbf{P}^1 \otimes \mathbf{F}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2), T)$. This proves the functional equation for M_k in the Introduction.

References

[D1] P. Deligne, *Applications de la Formule des Traces aux Sommes Trigonométriques*, in Cohomologie Étale (SGA $4\frac{1}{2}$), 168-232, Lecture Notes in Math. 569, Springer-Verlag 1977.

[D2] P. Deligne, *La Conjecture de Weil II*, Publ. Math. IHES 52 (1980), 137-252.

[GK] E. Grosse-Klönne, On families of pure slope L-functions, *Documenta Math.*, 8(2003), 1-42.

[K] N. Katz, *Gauss Sums, Kloosterman Sums, and Monodromy Groups*, Princeton University Press 1988.

[R] P. Robba, *Symmetric powers of p-adic Bessel equation*, J. Reine Angew. Math. 366 (1986), 194-220.

[S] J.-P. Serre, *Linear Representations of Finite Groups*, Sringer-Verlag 1977.

[SGA $4\frac{1}{2}$] P. Deligne, et al, *Cohomologie Étale* (SGA $4\frac{1}{2}$), Lecture Notes in Math. 569, Springer-Verlag 1977.

[Sp] S. Sperber, Congruence properties of hyperkloosterman sums, Compositio Math., 40(1980), 3-33.

[ST] J.-P. Serre and J. Tate, *Good reduction of abelian varieties*, Ann. of Math. 88 (1968), 492-517.

[W1] D. Wan, *Dimension variation of classical and p-adic modular forms*, Invent. Math., 133(1998), 449-463.

[W2] D. Wan, *Rank one case of Dwork's conjecture*, J. Amer. Math. Soc., 13(2000), 853-908.

[W3] D. Wan, *A quick introduction to Dwork's conjecture*, Contemporary Math., Volume 245 (1999), 147-163.

[W4] D. Wan, *Geometric moment zeta functions*, in Geometric Aspects of Dwork Theory, Walter de Gruyter, 2004, 1113-129.