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Individual and collective analyses of the genesis of student reasoning regarding the invertible matrix theorem in linear algebra

Author
Wawro, Megan Jean

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Individual and Collective Analyses of the Genesis of Student Reasoning Regarding the Invertible Matrix Theorem in Linear Algebra

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy

in

Mathematics and Science Education

by

Megan Jean Wawro

Committee in Charge:

University of California, San Diego

Professor John Czworkowski
Professor Guershon Harel
Professor Rafael Núñez

San Diego State University

Professor Chris Rasmussen, Chair
Professor Joanne Lobato
Professor Michelle Zandieh

2011

The Dissertation of Megan Jean Wawro is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

University of California, San Diego

San Diego State University

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VITA

2000	Bachelor of Arts, Mathematics Specialization and Certification, 7-12 Mathematics Teaching Cedarville University, Cedarville, Ohio
2000-2001	Teacher, 9-12 Visual Arts, 10^{th} Grade Geometry Portsmouth East High School, Sciotoville, Ohio
2001-2003	Teacher, 7-12 Mathematics Institut auf dem Rosenberg, St. Gallen, Switzerland
2003-2005	Graduate Teaching Assistant, Department of Mathematics & Statistics Miami University, Oxford, Ohio
2005	Master of Arts, Mathematics Miami University, Oxford, Ohio
2005-2007	Visiting Instructor, Department of Mathematics & Statistics Miami University, Oxford, Ohio Teacher, 7-12 Mathematics, Summer in Switzerland program Leysin American School, Leysin, Switzerland
2007-2009	Research Assistant, San Diego State University, San Diego, California
2009-2010	Graduate Teaching Assistant, Department of Mathematics & Statistics San Diego State University, San Diego, California
2010-2011	Research Assistant, San Diego State University, San Diego, California
2011	Doctor of Philosophy, Mathematics and Science Education University of California, San Diego, San Diego, California San Diego State University, San Diego, California

PUBLICATIONS

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ABSTRACT OF THE DISSERTATION

Individual and Collective Analyses of the Genesis of Student Reasoning Regarding the Invertible Matrix Theorem in Linear Algebra

by

Megan Jean Wawro

Doctor of Philosophy in Mathematics and Science Education

University of California, San Diego, 2011
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Professor Chris Rasmussen, Chair

In this study, I considered the development of mathematical meaning related to the Invertible Matrix Theorem (IMT) for both a classroom community and an individual student over time. In this particular linear algebra course, the IMT was a core theorem in that it connected many concepts fundamental to linear algebra through the notion of equivalency. As the semester progressed, the IMT took form and developed meaning as students came to reason about the ways in which key ideas involved were connected. As such, the two research questions that guided my dissertation work were:

1. How did the collective classroom community reason about the Invertible Matrix Theorem over time?
2. How did an individual student, Abraham, reason about the Invertible Matrix Theorem over time?

Data for this study came from the third iteration of a semester-long classroom teaching experiment (Cobb, 2000) in an inquiry-oriented introductory linear algebra course. Data sources were video and transcript of whole class and small group discussion. To address the second research question, data from two individual semi-structured interviews, as well as written work, were also analyzed. The overarching analytical structure of my methodology was influenced by a framework of genetic analysis through the notion of cultural change, using two interrelated strands of microgenesis and ontogenesis (Saxe, 2002). I utilized two analytical tools, adjacency matrices and Toulmin's Model of argumentation, to analyze the structure of explanations related to the IMT both in isolation and as they shift over time.

In addition to an in-depth analysis of the complex ways in which Abraham and the classroom community reasoned about the IMT throughout the semester, this dissertation presents methodological contributions regarding the two analytical tools. First, it necessitates the definition of four new argumentation schemes that are expanded versions of Toulmin's model. Second, it further adapts and refines the use of adjacency matrices within mathematics education research to analyze student thinking, both at static moments and over time. Finally, the present study lays a strong foundation for a 2-fold analytical coordination. The first coordinates results from adjacency matrix analysis with those from Toulmin's Model to demonstrate they were often compatible, with the tools varying in their respective strengths and limitations. Second, the present study lays a foundation for coordinating the mathematical development of both the individual and the collective units of analysis.

CHAPTER ONE: INTRODUCTION

There is an increasing need in our society for more individuals with creative and capable problem solving, reasoning, logic, and argumentation skills—skills that are developed in undergraduate mathematics. With this skill set in mind, the improvement of a particular undergraduate mathematics course—that of linear algebra—is of particular interest. Linear algebra is often seen as a gateway course for many mathematics and science majors in that success in this course is critical for student continuation in a STEM major. Furthermore, a first course in linear algebra serves not only as an introduction to a new level of abstract and formal reasoning, which is necessary in order to pursue a degree in mathematics, but also as a set of tools for many applications, such as computer science and engineering. Thus, the investigation how students learn the mathematical ideas central to linear algebra is of particular relevance in our current global society.

In this chapter I discuss the goals of the study. The chapter begins with two illustrative examples through which I introduce my two research questions. The remainder of the chapter discusses why both the chosen field of study and research questions are of interest, as well as why answering the research questions is a valuable contribution to the field of mathematics education research.

Consider the following segment of transcript. The class had been asked to investigate the following properties of specific transformations and the associated standard matrices: domain, codomain, one-to-one, onto, and invertibility. Towards the end of a whole class discussion regarding this investigation, the instructor initiated the following conversation:

Instructor: So let's make a note here of what we said so far about invertible...So what was the thing that you guys just said, how did you know it was invertible? Let's say that again.
Bill: It's linearly independent.
Instructor: If 'it's' meaning…?

Jesse: The vectors of the matrix.

Instructor: ...If the column vectors of *A* are linearly independent, then you guys are saying it's invertible because of that. Are you guys able to explain why that should give invertibility? I think you said something, or go ahead, Josiah.

Josiah: When they're linearly independent, there's only one path you can take to get to it, so in order to get back, there can only be one answer to get back. Whereas if they're dependent on each other, then depending on how you got there, would determine how you get back, what vector changed, so you don't have the right information again.

Instructor: Jesse...Yeah?

Jesse: Also, if they're dependent, in the RREF, you'll have a zero row, so it will be like you're losing information when you're trying to go back...If they are dependent, then their RREF will have a zero row.

Instructor: Okay, columns dependent [writes] and so you're saying that the RREF of *A* has a zero or row of zeros.

Jesse: Right, so then if you try and, if you invert that, you can't, because it's like you're losing that information from that row.

Within this transcript, both Josiah and Jesse associated invertibility with the notion of "going back," but they connected this idea to different concepts from linear algebra. Josiah indicated that if the columns of a matrix were a linearly dependent set of vectors, then one would not have the proper information to "get back," whereas Jesse's statements indicated that particular forms of the row-reduced echelon form (RREF) of a matrix would indicate whether or not you could "go back." How did these interpretations come to be? What had occurred previously in the semester that allowed, for instance, the notions of "path," "not losing information," and "going back" to become associated with invertibility? Furthermore, how did these notions come to function as if the classroom community shared a common understanding of them? What other ways of reasoning about the concepts related to the Invertible Matrix Theorem, a set of over a dozen equivalent statements, developed for this classroom community? What ideas got taken up and utilized as key ideas for this community?

In addition to considering how the classroom community reasoned about the Invertible Matrix Theorem over time, I also conducted a parallel investigation at the individual level. Consider the following excerpt with Henry. This transcript comes from a semi-structured interview (Bernard, 1988) in which students were asked to explain how they related various key ideas, such as linear independence, invertibility, determinants, etc. In particular, question one began with the following prompt:

> Suppose you have a 3 by 3 matrix *A*, and you know that matrix *A* is invertible. Decide if each of the following statements is true or false and explain your answer: The column vectors of *A* are linearly independent.

After Henry read the prompt, the following conversation ensued:

Henry: And, let's see, because it says 'it's invertible,' and it's only invertible if they're independent; otherwise, you get a singular matrix. So that would be true. Otherwise you get the divide by zero errors.

After the interviewer prompted Henry to explain in more detail this first statement, Henry connected what he meant by "the divide by zero errors" to the concept of determinants. The interviewer then directed the conversation back to the task at hand, which was about invertibility and linear independence.

Interviewer: And then you said something about the determinant, so how do you see the determinant playing in?

Henry: Invertible. So if it's invertible, the determinant isn't zero. And if the determinant is zero, then they're [the column vectors of the matrix] dependent. If they're independent, then it's not zero, so that means it's invertible, so (a) is true.

The above transcript illuminates one way that Henry reasoned about linear independence of a set of vectors (the columns of a matrix *A*) given that *A* was invertible. What other concepts did he mention as he explained how he connected these two ideas? How did this connection develop for Henry? As previously stated, this interview took place post-instruction, at the end of the semester. How did the ways in which Henry reasoned about these two ideas change over time? These two concepts—those of linear independence and invertibility—are but two of over a dozen key concepts in linear algebra related through what is sometimes known as the Invertible Matrix Theorem (IMT).

The following two research questions bring together and extend the previous inquiries that the two illustrative examples bring to light:

1. How did the collective classroom community reason about the Invertible Matrix Theorem over time?
2. How did an individual student reason about the Invertible Matrix Theorem over time?

Within this dissertation, I analyzed the ways of reasoning about the IMT that developed for one particular linear algebra class, as well as for one particular student, Abraham, within that linear algebra class. These two units of analysis addressed my two research questions, respectively. For this classroom community, as the semester progressed, what became known as the IMT was sixteen concept statements developed individually and related to one another through the notion of equivalence. Due to the complex and ever-changing nature of the theorem itself, investigating the class's and Abraham's ways of reasoning about these concepts and how they related to each other was a multi-faceted, complex, intriguing endeavor.

The significance of this study is three-fold. The first two aspects directly regard the significance of the results of the two research questions, whereas the third speaks to methodological significance of this work. First, investigating the ways in which a particular classroom community reasoned about the IMT is analogous to investigating what classroom mathematics practices (Cobb & Yackel, 1996) relevant to the IMT developed for that community. The practical constraints placed on a teacher in any given classroom are such that she must operate as if the ways of reasoning about the mathematics at the collective level within that classroom are shared by all individual members of that community. Thus, reporting on classroom mathematics practices that developed for a particular linear algebra class contributes to the field because it makes other linear algebra instructors aware of normative ways of reasoning that may develop in their own classroom. It is a conjecture that the normative ways of reasoning presented in this study (see Chapter 4) would also develop in other linear algebra courses taught in a similar manner. Because my plans for future work involve sharing the various aspects of our instructional sequences (see, for example, Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2011), good documentation of the collective production of meaning is necessary to help other instructors implement the curriculum in their own classrooms.

Second, just as documenting classroom mathematics practices provides insight into the possible social production of mathematical meaning of other classroom communities, investigating the ways of reasoning of one individual student provides a solid foundation for understanding the possible conceptual milestones, blocking points, and potentially helpful and inhibitive ways of reasoning about the concept statements (and their equivalence) within the IMT of other linear algebra students. In other words, what is learned about Abraham's ways of reasoning—the concept statements that were most central to his reasoning, the structure of his argumentation, specific interpretations of concept statements that were particularly salient for him, and the like—serve a conjectural role regarding what may be true for other individuals' ways of reasoning. Furthermore, conducting a case study of an individual student's ways of reasoning provided a unit of analysis that was appropriate for both of my analytical tools, Toulmin's model and adjacency matrices. This allowed me to analyze the utility of adjacency matrices, a rather novel analytical tool within mathematics education research. More detail is given in the subsequent explanation of the methodological significance of this study.

My rationale for choosing the subject of my case study, Abraham, was two-fold. First, he was a member of a high-functioning small group within the linear algebra classroom. Here, I define a "high-functioning small group" as one in which its members (a) interacted well with each other by pushing one another to explain their thinking and expecting participation from each member of the group; and (b) interacted well with the mathematics by staying on task, working through the given problems together, and displaying a genuine interest in exploring and understanding the content at hand. Second, Abraham was a high-functioning individual within the classroom community and his small group. Here, I define "high-functioning individual" to mean that Abraham (a) was academically prepared, in that he had successfully completed the requisite courses that positioned him for success within linear algebra; (b) was personally engaged and interested in the mathematics; and (c) was a key informant (Tremblay, 1989) in that he possessed a unique ability to articulate his thinking and was willing to do so in whole class, small group, and interview settings. More detail regarding Abraham and his small group is given in Chapter 3.

Finally, my approach to investigating these two research questions provided the opportunity to extend existing and develop new methodologies; thus, this dissertation contributes three main methodological results as well. First, this study extended the use of Toulmin's Model as an analytical tool in mathematics education research by not only analyzing argumentation in linear algebra, often a transition into more proof-oriented mathematics courses, but also by expanding the 6-part Toulmin scheme for the analysis of more complex arguments. Second, this study demonstrates the utility of adjacency matrices as analytic tools when documenting students' ways of reasoning at both the individual and collective levels, within specific arguments and over time. I compare these results with those of Toulmin's model in order to compare and contrast the strengths, limitations, points of compatibility, and points of distinction between the two tools. Third, this study lays a solid foundation for methodological contributions regarding the coordination of individual and collective-level analyses. As such, conducting a rich case-study analysis of one student, while also investigating the development of mathematical meaning at the collective level, provided the depth necessary to lay a solid foundation from which to investigate the mathematical ways of reasoning at both the individual and collective level, as well as ways to coordinate those results across the two units of analysis.

These research questions—addressing how the ways that students reason about the IMT change over time—investigates the connections that are made, on both the individual and the collective level, between the various equivalent statements in the IMT. Thus, the objects of inquiry for the research questions are the created relationships between the constituent parts of the IMT. Furthermore, the research questions essentially developed out of four fundamental lines of inquiry, each of which has the potential to contribute to the field of mathematics education research in a variety of ways. The four fundamental lines of inquiry are:

1. The study linear algebra;
2. The Invertible Matrix Theorem and it as a line of research;
3. Students' evolving interactions with the Invertible Matrix Theorem; and
4. Analyses that examine change over time on both the individual and collective levels.

In the following sections I address each of these four aspects of inquiry and then revisit my two overarching research questions. I conclude the chapter with a short description of the organization layout of the subsequent chapters.

1.1 The Study of Linear Algebra

As detailed in the following paragraphs, there are two main reasons that the field of linear algebra is a significant and worthwhile content domain for educational research: (a) linear algebra is commonly seen as an extremely pivotal yet rather difficult mathematics course for university students; and (b) very little research has been conducted regarding student thinking in linear algebra, so much remains to be done in order to extend our knowledge of student thinking in this area.

Linear algebra is one of the most useful fields of mathematical study because of its unifying power within the discipline as well as its applicability to areas outside of pure mathematics (Dorier, 1995b; Strang, 1988). According to Harel (1989b), linear algebra is an important subject matter at the college level in that it: (a) can be applied to many different content areas, such as engineering and statistics, because of its power to model various situations; and (b) can be studied in its own right as "a mathematical abstraction which rests upon the pivotal ideas of the postulational approach and proof" (p. 139). Thus, when considering the mathematical development of undergraduate students, a first course in linear algebra plays an important, transitional role. In many universities, a first course in linear algebra follows immediately after a calculus series and, most likely, prior to an introduction to proof course. According to Carlson (1993), the majority of students' mathematical experiences up to this point in their education have been primarily computational in nature. However, the content of linear algebra can be highly abstract and formal, a stark contrast to students' previous computationally oriented coursework. This shift in the nature of the mathematical content being taught can be rather difficult for students to handle smoothly. Carlson (1993) posits that concepts are often taught without substantial connection to students' previously learned mathematical ideas, as well as without examples or applications. Thus, students struggle with connecting familiar concepts to prematurely formalized, unfamiliar ones. Dorier (1995b) contends that the serious difficulties students have in learning linear algebra are a result of the content's abstract and formal nature. Indeed, Robert and Robinet (1989) showed that prevalent student criticisms of linear algebra relate to its "use of formalism, the overwhelming amount of new definitions and the lack of connection with what they already know in mathematics" (as cited in Dorier et al., 2000, p. 86).

As the above quote indicates, the mathematical activities central to linear algebra frequently involve mathematical ideas with which students may have little experience, such as vector spaces or linear transformations. How do students try to understand these new topics? What imagery do they develop as they encounter new abstractions and formalities? Questions such as these are an integral part of basic research that investigates student thinking about fundamental mathematical ideas. Of the research that has been conducted on the learning of linear algebra, much of it has focused on student difficulties and suggestions to alleviate these difficulties (e.g., Hillel, 2000; Sierpinska, 2000; Stewart & Thomas, 2009). Relatively few studies have focused on students' creative and productive ways of reasoning about key concepts of linear algebra (e.g., Possani, Trigueros, Preciado, & Lozano, 2010; Wawro, 2009). This lack of foundational research into the generative aspects of student thinking constitutes a second reason that investigation into the learning and teaching of linear algebra is both a valid and necessary line of research.

1.2 The Invertible Matrix Theorem and It as a Line of Research

The Linear Algebra Curriculum Study Group (Carlson, Johnson, Lay, & Porter, 1993) named the following as topics necessary to be included in any syllabus for a first course in undergraduate linear algebra:

1. Matrix addition and multiplication,
2. Systems of linear equations,
3. Determinants,
4. Properties of $\mathbf{R}^n$, and
5. Eigenvectors and eigenvalues.

A few of the specific concepts involved in the aforementioned topics are: (a) span, (b) linear independence, (c) pivots, (d) row equivalence, (e) determinants, (f) existence and uniqueness of solutions to systems of equations, (g) transformational properties of one-to-one and onto, (h) invertibility, (i) basis, and (j) dimension of column space and null space. These concepts, in addition to others, are the very ones addressed and linked together in what is sometimes known as the Invertible Matrix Theorem (see Figure 1.1). It is for this reason that studying the development of the Invertible Matrix Theorem is a significant contribution to what is currently known about how students learn linear algebra.

The Invertible Matrix Theorem is a core theorem for a first course in linear algebra in that it connects the fundamental concepts of the course. Because of this, it represents a main thread that is present throughout the entire course. This thread has at least two main layers of investigation—that of the concepts involved, and that of the way in which these concepts are connected to one another. First, as previously stated, the twenty equivalencies of the IMT (those listed as (a)-(t) in Figure 1.1) highlight the concepts considered central to any linear algebra course. They are not all covered on the same day of class—rather, the twenty concepts are a large part of what is covered over the course of the entire semester. This is clear from the recommendations of the Linear Algebra Curriculum Study Group (Carlson, Johnson, Lay, & Porter, 1993), as well as from an investigation into many of today's most utilized textbooks (e.g., Lay, 2003; Strang, 1988). Thus, for students to understand the main ideas of the Invertible Matrix Theorem, they necessarily would have to be conversant with the key concepts that were developed over the course of the semester.

The Invertible Matrix Theorem

Let A be a square $n \times n$ *matrix.* Then the following are equivalent:

a) A is an invertible matrix.
b) A is row equivalent to the $n \times n$ identity matrix.
c) A has n pivot positions.
d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e) The columns of A form a linearly independent set.
f) The linear transformation $\mathbf{x} \rightarrow A\mathbf{x}$ is one-to-one.
g) The equation $A\mathbf{x} = \boldsymbol{b}$ has exactly one solution for each $\mathbf{b}$ in R^n.
h) The columns of A span R^n.
i) The linear transformation $\mathbf{x} \rightarrow A\mathbf{x}$ maps R^n onto R^n.
j) There is an $n \times n$ matrix C such that $CA = I$.
k) There is an $n \times n$ matrix D such that $AD = I$.
l) A^T is an invertible matrix.
m) The columns of A form a basis of R^n.
n) $Col\ A = R^n$.
o) dim $Col\ A = n$
p) $rank\ A = n$
q) $Nul\ A = \{\mathbf{0}\}$
r) dim $Nul\ A = 0$
s) The number 0 is not an eigenvalue of A.
t) The determinant of A is not zero.

Figure 1.1. The Invertible Matrix Theorem

The second thread of the Invertible Matrix Theorem is the connections among the constituent parts. This theorem marries a variety of mathematical concepts from a variety of contexts; some concept statements within the IMT are vector-oriented (such as span and linear independence), whereas others relate more appropriately to linear transformations (such as one-to-one and onto). Furthermore, the nature of equivalence and the importance of A being an n x n matrix are also contributing factors as students develop ways of reasoning about connections among the various mathematical ideas. Reasoning—in the sense of making connections across ideas, representations, and contexts, as well as in terms of argumentation and justification—is a valuable skill and part of the practice of mathematics. Thus, investigating the development of how ideas are connected is a valuable contribution to the field of undergraduate mathematics education research.

1.3 Students' Evolving Interactions with the Invertible Matrix Theorem

Because the Invertible Matrix Theorem weaves together most of the central ideas in an introductory course in linear algebra, no one single class session could encompass all the equivalences that constitute the IMT. Rather, new equivalencies were added to the theorem as the semester progressed. Thus, students' understanding of the theorem changed in two dimensions over the progression of the semester: (a) knowledge of the theorem changed as more aspects of the theorem were 'revealed' or layered on over the course of the semester, and (b) understanding of previously known aspects of the theorem (presumably) changed as the semester progressed. Both of these reasons warranted an investigation into how the ways in which students—both collectively and individually—reasoned about the IMT change over time.

Other researchers, from both educational and anthropological fields, have investigated this progression through what is sometimes referred to as *genetic analysis.* Ercikan and Roth (2006) state that, when dealing with a genetic explanation, "a certain fact is not derived from antecedent conditions and laws (deduction) or observations and antecedents (induction) but rather is shown to be the endpoint of a longer development, the individual stages (phases) of which can be followed" (p. 19). This type of analysis has been valued by the likes of anthropological researchers such as Saxe and Esmonde (2005), who studied, over a course of more than twenty years, how the function of a particular word form shifted for a central New Guinea tribe. In studying this word form, which dealt with the tribe's counting system, Saxe and Esmonde asked, "How do new collective systems of representation and associated mathematical ideas arise in the social history of a social group?" (p. 172).

My inquiry into how students reasoned about the Invertible Matrix Theorem over time was an analogous investigation. To study how students interact with a theorem that itself is developing over the course of the semester necessarily deals with how mathematical ideas arise in the social setting of a classroom. In a study dealing with students' developing ideas regarding equivalent fractions, Saxe et al. (2009) elaborate by stating,

> Teachers pose problems, solicit students' solutions and orchestrate discussions to guide students to make their mathematical ideas public and reflect on relationships among their ideas. In the social context of classroom lessons, students' ideas may be taken up or rejected, valued or devalued, and interpreted in various ways, and, in this process, students' mathematical ideas become elaborated and often transformed. (p. 203)

In addition to Saxe and his colleagues, other researchers have recently taken up research agendas that investigate the emergence, development, and spread of ideas in a classroom community over time (e.g., Cobb, Stephan, McClain, & Gravemeijer, 2001; Schwarz, Dreyfus & Hershkowitz, 2009; Stephan & Rasmussen, 2002; Tiberghien & Malkoun, 2009). The results of this study contribute to this emerging research agenda.

1.4 Analyses that Examine Change over Time on Both Individual and Collective Levels

Recall the two research questions under consideration: (a) how the collective classroom community reasoned about the Invertible Matrix Theorem over time, and (b) how an individual student reasoned about the Invertible Matrix Theorem over time. As indicated in the previous section, the emergence and development of mathematical ideas occurs not only for each individual student but also for the classroom as a collective whole. I contend, as others have (e.g., Cobb, 1999; Izsák, Tillema, & Tunç-Pekkan, 2008; Hershkowitz, Hadas, Dreyfus, & Schwarz, 2007; Rasmussen & Stephan, 2008), that these two forms of knowledge genesis—on an individual and on a collective level—are inextricably bound together in their respective developments. As Saxe (2002) posits,

> In collective practices, joint tasks are accomplished...through the interrelated activities of individuals. In such joint accomplishments, individual and collective activities are reciprocally related. Individual activities are constitutive of collective practices. At the same time, the joint activity of the collective gives shape and purpose to individuals' goal-directed activities. (p. 276-277)

Therefore, in order to gain the most fully developed understanding of the emergence, development, and spread of ideas in a particular classroom, I contend that analysis along both individual and collective levels, over the course of the semester, is warranted and necessary.

1.5 Conclusion

Investigating the learning and teaching of linear algebra, commonly seen as an extremely pivotal yet rather difficult mathematics course for university students, will be a valuable contribution to the field of mathematics education research. In this chapter I introduced two particular research questions regarding a central theorem in any first course in linear algebra:

1. How did the collective classroom community reason about the Invertible Matrix Theorem over time?
2. How did an individual student reason about the Invertible Matrix Theorem over time?

I then developed a rationale for the interest and validity of these questions through the exploration of four fundamental lines of inquiry: 1) the study of linear algebra; 2) the Invertible Matrix Theorem and a focus on it as a line of research; 3) a focus on students' evolving interactions with the Invertible Matrix Theorem; and 4) analyses that examine change over time on both the individual and the collective level.

Within Chapter 2, I review the relevant literature that helped to shape and inform my study, and Chapter 3 provides extensive detail of my methodological approach to both data collection and data analysis. Within Chapter 4 I present the results from the first reason questions, the ways in which the collective reasoned about the Invertible Matrix Theorem over time. Chapter 5 presents the analogous results through individual-level analysis. Finally, Chapter 6, in addition to summarizing the results, discusses (a) various strengths and weaknesses of my analytical tools as used within this study, (b) various implications for teaching, and (c) avenues for future research.

CHAPTER TWO: LITERATURE REVIEW

Chapter 1 briefly highlighted various aspects of the existing literature regarding educational research in linear algebra. In the first section of the present chapter, I explore this body of research in more depth. In particular, I focus my detailed survey on what I find in the literature regarding the teaching and learning of the foundational concepts of linear algebra—those involved in the Invertible Matrix Theorem. I also provide information on the sparse amount of research conducted regarding how students understand and build connections between these core ideas. Section two of this chapter is an in depth review of the literature regarding the analytical distinction regarding the mathematical development of an individual learner versus that of a collective group of learners. Finally, section three addresses the existing literature regarding the analysis of students' changing ways of reasoning over time.

2.1 The Teaching and Learning of Linear Algebra

Compared to other areas of mathematics education research—such as research on the teaching and learning of fractions, functions, or calculus—the area of linear algebra is relatively underdeveloped. Over the past twenty years, there have been a few key studies, namely those by Dorier and colleagues in France, Sierpinska and colleagues in Canada, and a handful by other researchers, such as Harel, Carlson, and Stewart. Although this body of literature as a whole is rather limited, what does exist covers a wide spectrum of topics. The variety of work done in linear algebra ranges from the historical development of linear algebra as a field in mathematics (Dorier, 1995a; Kleiner, 2007; Moore, 1995); discussions about the abstract, formal, and unifying nature of linear algebra as a mathematical field (Dorier, Robert, & Rogalski, 2003); the various modes of representation and modes of thinking possible within linear algebra (Harel, 1999; Hillel, 2000; Sierpinska, 2000; Stewart & Thomas, 2008); investigations into and categorizations of difficulties that students have with linear algebra (Harel, 1989b; Sierpinska, Nnadozie, & Okta, 2002); various recommendations and approaches for teaching linear algebra (Carlson, Johnson, Lay, & Porter, 1993; Harel, 1997; Rogalski, 2000; Uhlig, 2003); and descriptions of various creative and productive student work done in linear algebra (Possani, Trigueros, Preciado, & Lozano, 2010; Larson, Zandieh, & Rasmussen, 2008; Wawro, 2009). While all of these avenues of pursuit are interesting in their own right and are important to the development and enrichment of the field of linear algebra education in general, not all are of utmost relevance to the proposed study regarding the Invertible Matrix Theorem. Thus, the literature that I review in this chapter regards the nature of linear algebra and associated modes of representation and thinking, as well as research conducted regarding students' interactions with the particular concepts associated with the IMT—this includes research on student difficulties with and creative ways of interacting with these concepts. I conclude by investigating what research has been conducted regarding how students *connect* the ideas of the Invertible Matrix Theorem to one another.

2.1.1 The nature of linear algebra

The investigations by Dorier, Robert, and Rogalski (2003) into the difficulties with the formal nature of linear algebra led them to an extensive survey of the historical development of this content area. From this work they developed the notion of a *unifying and generalizing concept.* Regarding this notion, they state:

> A unifying and generalizing concept (or theory) is characterized by the fact that it did not emerge essentially to solve a new type of problems in mathematics (like the derivative or the integral for instance). Its creation and its use by mathematicians were motivated rather by the necessity to unify and generalize methods, objects and tools, which had been independently developed in various fields. Therefore the formalism attached to a unifying and generalizing concept is constitutive of its existence and creation...In other words, formalism cannot be avoided when learning linear algebra...This does not mean that unifying and generalizing concepts have no intuitive background. In fact they have several such backgrounds which result from an abstraction of the common characteristics of various objects of a less formal nature. (p. 186)

This quote indicates that the content of linear algebra is formal and unifying by nature. That is, the notion of connecting ideas together is inherent in the way in which the modern field took form. This is yet another justification for investigations into the development of reasoning regarding the Invertible Matrix Theorem which, by its very nature, is the unification and generalization of a variety of concepts, forms, and functions in linear algebra.

According to Hillel (2000), linear algebra is often the first mathematics course that students see that is a mathematical theory, systematically built and reliant upon definitions, explicit assumptions, justifications, and formal proofs. Research has shown that proof-related difficulties are not unique to a first course in linear algebra (Selden & Selden, 1987; Weber, 2001). Thus, Hillel focused instead on what he proposes are student difficulties that are *specific* to linear algebra: the existence of several modes of description, the problem of representations, and the applicability of the general theory (p. 192). Hillel claimed there exists three different modes of representation in linear algebra. The *abstract mode* utilizes language of generalized theory, including terms such as dimension, span, linear combination, and subspace. The *algebraic mode* uses concepts more particular to the vector space $\mathbf{R}^n$, such as matrix, rank, and systems of linear equations. Finally, the *geometric mode* uses language that is familiar from our lived experiences, such as point, line, plane, and geometric transformation (p. 192). In addition to detailing difficulties students have with the geometric mode (such as confusion potentially caused by describing vectors as both arrows and points), Hillel described a few difficulties that students have with moving between the three modes of representation. One particularly difficult aspect for students is to move between the abstract and the algebraic when the underlying vector space is $\mathbf{R}^n$ for both. This would occur when tackling change of basis problems. To think of a string of numbers as a representation of a vector rather than the vector itself (which is represented according to different bases but is still the same vector) is difficult for students to grasp or even see the need for grasping. In reference to Hillel's distinction of abstract, algebraic, and geometric modes of representation, Larson (2010) noted that students often seem to blend these modes of representation. For instance, she gives examples such as "span of a matrix" or "linearly dependent matrix" (blending abstract and algebraic).

In conclusion, the difficulties students have when encountering linear algebra is a result not only of their relative inexperience with formal mathematics, but also of the formal and theoretical nature of the linear algebra content itself. What are the 'content objects' in the domain of linear algebra? What are some of the specific difficulties and successes that students have with these fundamental concepts? It is to addressing these questions that the next section turns. In particular, I summarize studies that report on various aspects of student difficulty, as well as on particular concepts such as vectors and scalars, linear combination, linear independence, linear transformation, span, basis, and eigenvectors and eigenvalues.

2.1.2 Students' interactions with the ideas of the Invertible Matrix Theorem

From work spanning over the last twenty years, Harel (1989a; 1997; 2000) offers a variety of reasons which factor into why linear algebra is often such a difficult course for students. He posited one difficulty is that students are asked to treat the objects of linear algebra—such as matrices and systems of equations—as entities or objects that can be manipulated. This is consistent with research about student thinking in other areas of mathematics, such as student conceptions of the derivative (Zandieh, 2000) or function (Carlson, 1998). For example, Harel (2000) contended that one reason students have difficulty determining if the set $A=\left\{x,x^2,x^3,x^4\right\}$ is linearly independent is that the "concept of function as a vector is not concrete to them. That is to say, these students have not formed the concept of function as a mathematical object, as an entity in a vector space" (p. 181). Another reason that linear algebra is traditionally difficult for students is the change in symbolism (Harel, 2000). From high school courses or single-variable calculus, students are accustomed to equations such as $cx = d$, where all three variables take their values from the real numbers. However, consider the complexity in the matrix equation $R\vec{x} = \vec{0}$, where *R* is a matrix and both $\vec{x}$ and $\vec{0}$ are multi-component columns vectors. To find the solution (if one exists) to the aforementioned equation means to determine the values for each component of the vector $\vec{x}$ such that the associated system of linear equations, embedded with this matrix equation, have solutions. As this demonstrates, even the commonly used symbolism in linear algebra entails a new level of complexity for students.

Harel (1989b) discussed three more possible reasons for student difficult in linear algebra: attempting to solve problems in unfamiliar domains of application, using abstract setting to solve real problems, and trying to understand algebraic systems that do not have an easily accessible geometric interpretation. Related to the latter, Harel (1997) posits that a reason for students' struggle with linear independence is not necessarily solely dependent upon if they comprehend its definition, but rather in trying to use linear independence to solve problems with generalized (rather than specific) vectors. Furthermore, Harel reports that students are able to correctly determine if, for instance, a set of five specific vectors in $\mathbf{R}^5$ are linearly independent but are unable to address questions such as deciding if mutually orthogonal vectors are linearly independent, or if elementary operations on rows of matrices affect their linear independence.

In a work that investigates student understanding of linear independence and other key ideas, Stewart and Thomas (2009) utilized a framework that combines APOS theory (Dubinsky & McDonald, 2001) and Tall's three worlds of mathematics (Tall, 2004) in order to categorize student understanding of basic concepts of linear algebra. APOS theory delineates basic cognitive thinking through the categories of action, process, and object and claims that this is the order in which students should encounter new ideas in order to develop correct understanding. Tall, on the other hand, describes three interrelated worlds—embodied, symbolic, and formal—that encompass the totality of mathematical learning. Thus, Stewart and Thomas, based on the dissertation work of Stewart (2008), report on students' interactions with fundamental concepts—namely vectors and scalars, linear combination, linear independence/dependence, span, basis, and eigenvectors and eigenvalues—by using a descriptive framework utilizing the categories of action, process, object, embodied, symbolic, and formal. From six different case studies ranging over a few years, the authors reported a variety of results, such as finding little evidence of students' understanding of definitions. They also reported that the majority of students "had little embodied world thinking and instead tending to show an action/process view of the symbolic world" (p. 960). For instance, when asked to describe eigenvectors or eigenvalues, students seemed unfamiliar with embodied views of these notions, instead describing them mostly in terms of actions and symbols. In another report of the same study, Stewart and Thomas (2008) posited that the concept of basis can be seen in three different representations (explained here in $\mathbf{R}^3$ for ease): embodied as a set of three non-coplanar vectors; symbolically as the column vectors of a matrix with three pivot positions; and formally as a set of linearly independent vectors that span $\mathbf{R}^3$. The authors found that students were most likely to work symbolically with basis, feeling more comfortable with finding a basis for a particular space (procedural-symbolic) than with giving a definition or describing basis geometrically.

In a study regarding students' interactions with transformational geometry in linear algebra, one research question that Portnoy, Grundmeier, and Graham (2006) investigated was how participants in their study viewed transformations, as processes or as objects. The authors found that participants displayed an operational view of transformations, as "processes that map geometric objects onto other geometric objects" (p. 201). The authors provided a variety of data to support this claim, namely statements made by a student they refer to as 'Peter.' Two of Peter's statements that evidence his operational view of transformation are (a) "taking something and putting it in another place in the plane;" and (b) "...you can take a set of points, then flip them, slide them, or rotate them such that their measure in relation to each other remains constant after the isometry is performed." (p. 201). Thus, an operational view of transformations meant that students saw transformations as processes that map geometric objects onto other geometric objects. The authors further conjectured this one view might have contributed to the difficulty students had with utilizing transformations as objects in geometric proofs.

In another study regarding student understanding of linear transformations, Dreyfus, Hillel, and Sierpinska (1998) reflected upon a teaching experiment that heavily relied on students' interaction with coordinate-free geometry on a dynamic geometric software program. The authors reported on students' apparent understanding of the term "transformation." Students seemed to equate the term "transformation" with the vector "T(**v**)," as if transformation, rather than describe a relation between **v** and T(**v**), describes an object T(**v**) that depended on **v**. The language students used further confirmed this hypothesis, reading "T(**v**)" as "the transformation of **v**," rather than as "the image of **v** under the transformation T." The authors posited that this language might be a carry over from earlier familiarity with functions or a case of metonymy, or was quite possibly a result of interactions the dynamics geometry software itself. Because the software was designed to be "coordinate free," a given vector represented any general vector. Thus, what would it mean for two vectors or two transformations to be equal? How was that addressed in the geometric software? Thus, the authors concluded that while their research made contributions towards a dynamic representation of vector and transformation, unseen difficulties arose and further iterations of studying how students interact with the dynamic software were needed.

In addition to research that either categorizes what students struggle with doing or explains what an expert thinks is necessary to understand an idea (e.g., the notion of genetic decompositions), researchers have also looked into productive and creative ways that students are able to interact with the ideas of linear algebra. For example, Possani et al. (2010) analyzed the use of two educational theories in conjunction—those of APOS Theory (Dubinsky & McDonald, 2001) and the Models and Modeling perspective (Lesh & Doerr, 2003)—and their utility and compatibility in designing a teaching sequence that starts with a "real life" problem. This problem related to analyzing traffic flow of various configurations of open and closed streets for a city grid, and it elicited the used of systems of linear equations. The authors reported on student progress throughout the task as they progressed through different solution strategies. For instance, they state that the study shows that:

> Students learn what they are supposed to learn and they can even do more than what is normally expected from them when given the opportunity. In this experience, the geometrical analysis of the solution space and the parameters came out as a direct result of students' work on the model. (p. 15)

Larson, Zandieh, and Rasmussen (2008) reported on ways that students were conceptualizing mathematical objects (such as vectors and matrices) as the class developed algebraic methods to solve for eigenvectors and eigenvalues of a given matrix. In particular, the authors present a key idea that emerged as a central and powerful way in which students came to reason. This key idea had to do with the relationship between the determinant of a matrix and the dependence relationships among its column vectors. The class explored the question, "Given a matrix B, is there some vector $\mathbf{w}$ that, when multiplied by B, results in a vector $B\mathbf{w}$ that points in the same direction as the original vector $\mathbf{w}$?" One student, Karl, explained his thinking in the following way:

Karl: When you look at the, uh, vectors, what does the determinant give us? It gives us the area between any two given vectors. And if, if our determinant equals zero, that basically means that the vectors that we're solving for have no area in between. So therefore they lie along the same line.

As solution methods were pursued in this class, the approach of solving $\det\left[\mathbf{w}\ B\mathbf{w}\right]=0$ surfaced as one approach that resonated with Karl's geometric description. This method, which became known as the "eigenvector first" method, proved to be a creative and effective solution strategy for students when working in $\mathbf{R}^2$. In larger dimensions, however, analogous geometric interpretations are either non-existent or hard to develop in the same manner. Thus, an intellectual need (Harel, 2007) for developing the standard approach of solving $\det(B - \lambda\mathbf{w}) = 0$ for the eigenvalues first and then the associated eigenvectors developed out of student ways of reasoning.

As Larson, Zandieh, and Rasmussen (2008) noted, familiarity with geometric interpretations of linear dependence and determinants afforded the students great insight and creativity when introduced to concepts of eigen theory. Furthermore, upon reviewing the classroom video, our research team noted several instances in which the students used geometric interpretations of particular ideas as taken-as-if-shared (Stephan & Rasmussen, 2002) in both small group and whole class discussions. I agree with Sierpinska (2000), who claimed that geometric, arithmetic, and structural reasoning and the ability to move between these are fundamentally important in learning and understanding the core ideas of linear algebra. Because of this, I began to think about ways to foster other salient connections such as these in preparation for the data collection effort for this dissertation. One such way was to create tasks that build off of students' geometric understanding of the plane in order to develop rich conceptualizations of various linear algebra ideas, such as vector, matrix, linear independence, and basis.

In Wawro (2009), I reported on the development of one such instructional sequence with the particular goal of fostering a geometric conceptualization of linear transformation and change of basis in $\mathbf{R}^2$. In particular, the instructional sequence was developed with the intent of leading to a student-driven reinvention of the well-known equation $A = PDP^{-1}$, for the case where *A*, *P*, P^{-1} and *D* are 2 x 2 matrices. The sequence of tasks was first piloted in a problem-solving interview setting, in which three different pairs of students were asked to work through the tasks together. The tasks centered on the necessity of finding a convenient way to determine a 2 x 2 matrix for a particular linear transformation. For the expert, this 'convenient way' would be to use the two eigenvectors as the basis through which the matrix for the transformation is determined. However, as detailed by Wawro (2009), the solution strategy of one particular pair of students differed substantially from the expert approach, yet was mathematically correct and powerful for them. Throughout the activity, one of the students, Sophie, seemed to be thinking about the actual *action* of the transformation and how a convenient coordinate system would make the geometric transformation easier to visualize, whereas the other student, Marissa seemed focused on the matrix *representation* of the transformation and how a convenient coordinate system would make the matrix easier to ascertain. Marissa and Sophie were able to reinvent an informal version of the matrix equation $AP = PD$, and Marissa—in her manipulation of this equation in order to solve for matrix *A*—developed the equation $A = PDP^{-1}$ en route to her solution. Upon reflection, I found that this particular pair was very reliant upon not only algebraic solution techniques related to invertibility and determinants, but also upon geometric interpretations of linear transformations. Although it has yet to be analyzed, the instructional sequence, known as the 'Rubber Sheet Task' and the 'Blue to Black Task,' has been adapted and implemented in whole class instruction over the last two semesters and will be used in the upcoming teaching experiment as well.

As has been demonstrated in this section, a variety of work has been done regarding students' interaction with the individual ideas that contribute to the Invertible Matrix Theorem. Larson, Rasmussen, and Zandieh (2008) investigated students' intuitive notions related to the concepts of eigenvectors and eigenvalues. Hillel (2000) explored the difficulties that students have in coordinating both abstract and algebraic representations for basis, while Sierpinska

(2000) explored the link being theoretical and practical modes of thinking regarding students' developing notions of linear transformation. In addition, studies have been conducted regarding student difficulties with the notions of rank (Dorier, Robert, Robinet, & Rogalski, 2000), linear independence (Bogomolny, 2006; Harel, 1997), and span (Stewart & Thomas, 2009). However, I have found very little research that addresses how students *connect* these core ideas. This are of investigation—how students connect those fundamental ideas of linear algebra involved in the Invertible Matrix Theorem—that is the focus of the next section.

2.1.3 Connecting the ideas of the Invertible Matrix Theorem

The fundamental ideas of linear algebra that are involved in the formulation of the Invertible Matrix Theorem are: span, linear independence, invertibility, row equivalence, pivots, one-to-one and onto linear transformations, unique solutions, basis, determinants, and eigenvalues (see Figure 1.1). As demonstrated in the previous section, there is a growing body of research addressing student difficulties and successes with these concepts, but very little has been conducted explicitly regarding how students understand the equivalence of these concepts. A basic premise of this study was that the Invertible Matrix Theorem is a powerful tool that, for students, unifies the main concepts of the course and provides them with multiple avenues through which to forge these connections. Over the course of the various iterations of our classroom teaching experiment in linear algebra, we have consistently used the same textbook for the course (Lay, 2003), which highlights the Invertible Matrix Theorem throughout the text. Upon investigation of the origination and formation of the Invertible Matrix Theorem, the textbook author informed me that, to the best of his knowledge, he coined the title "Invertible Matrix Theorem." He elaborated by stating the following:

> I created the IMT as a framework to connect many important ideas that coincide in the case of a square matrix. The connections among these ideas, however, had been well-known for many years before my time. The first edition of my linear algebra text (in 1994) contained 15 statements in the

> Invertible Matrix Theorem. Over the years, with the addition of new material, the number of statements in the IMT has grown to 22. I believe that mental possession of a framework such as the IMT increases the length of time a student retains useful information from the course, after the course is over. (David C. Lay, personal communication, January 5, 2010).

As Lay stated, the connections among the concepts in the Invertible Matrix Theorem were well-known before the common usage of that title developed (see, for instance, originals works by Cayley (1858) or Hamilton (1853)). If educational research is concerned with creating instruction that assists students in developing from their current ways of reasoning into more complex and formal mathematical reasoning—a development that is imperative in a course such as linear algebra—then more research into how students reason about how formal, theoretical ideas is needed. Such research, like that in this dissertation, is foundational to designing instructional sequences that build on student concepts and reasoning as the starting point from which more complex and formal reasoning develops.

As previously stated, the body of literature regarding how students reason about how these key concepts of linear algebra are connected is sparse. While many of the studies previously considered in this chapter address more than one of the key concepts, few if any of them explicitly investigate how students build relationships between the ideas as a main object of inquiry. In what follows, I report on a handful of studies that I have found that address students' connections explicitly. For instance, Stewart and Thomas (2008) studied the necessary genetic decompositions of linear independence and span that students must have in order to construct an understanding of basis. The study participants were asked to create concept maps linking the notions of span, linear combination, basis, linear independence, and subspace. Some students were able to create concept maps that connected basis with span and/or linear independence, and some were missing one or both of these links. By analyzing these concepts maps in comparison to a genetic decomposition of basis, the authors were able to posit potential pedagogical changes, such as (a) focusing on the notion of linear combination more or (b) trying to develop a more "embodied view" of basis.

In a study guided by "the belief that better understanding of students' difficulties leads to improved instructional methods," Bogomolny (2007) investigated students' understanding of linear independence and how example-generation tasks can elucidate how students understand linear algebra. Bogomolny asked students to generate an example of a 3 x 3 matrix *A*, with nonzero entries, such that the column vectors of *A* formed a linearly dependent set. She found, in order to construct their examples, that many students utilized the connections "linear dependence ↔ free variables / pivot positions / zero row in echelon form, and linear independence ↔ no free variables / vectors not multiples of each other" (p. 271). She further hypothesized that the connections students relied upon were indicators as to whether they had an action view of linear dependence (e.g., a matrix should be row-reduced and then examined for a zero row) or an object view of linear dependence (e.g., the vectors have, as a property, that they are not linear combinations of each other). Thus, Bogomolny contended that example-generations tasks such as this one have the potential to be not only effective assessment tools for instructors but also helpful learning tools for students.

As has been demonstrated in this section, very little research has been conducted explicitly investigating students' reasoning about how the ideas in the Invertible Matrix Theorem are connected. How do students justify for themselves how one of these statements' validity for any given n x n matrix implies another statement's validity? How do students use aspects of the Invertible Matrix Theorem as tools to reason about a novel problem? My research fills this gap by addressing the questions, of how students—both individually and collectively—reason about the ideas in the IMT. Inherent in these two questions is the analytical distinction between "individual" and "collective." A review of the existing literature regarding this distinction is presented in the following section.

2.2 Analysis of the Individual and of the Collective

Recall that the two research questions deal with (a) how the classroom community reasoned about the Invertible Matrix Theorem over time, and (b) how an individual student reasoned about the Invertible Matrix Theorem over time. In any given classroom, the emergence and development of mathematical ideas occurs not only for each individual student but also for the classroom as a collective whole. Thus, the individual and collective forms of knowledge genesis are inextricably bound together in their respective developments. This study was not the first in mathematics education research to consider the analysis of mathematical development at both the individual and collective level. In this section, I review a portion of this body of literature. I begin with a discussion of work by Inagaki, Hatano, and Morino (1998) and Sfard (2007). I then go into particular detail summarizing the relevant work of Cobb and his colleagues (Cobb, 1999; Cobb & Yackel, 1996; Stephan, Cobb, & Gravemeijer, 2003) because of its broad impact on the work of other researchers (Elbers, 2003; Hershkowitz, Hadas, Dreyfus, & Schwarz, 2007; Stephan & Rasmussen, 2002) and its relevance to the proposed study.

Inagaki, Hatano, and Morino (1998) investigated various aspects of individual student interaction within whole class discussion. Students in elementary classes were asked to solve a novel word problem for which they were given four possible solutions. The authors were particularly interested in the structure of whole class discussion regarding solution strategies for the given problem and, in particular, students' engagement on a student-student basis for these discussions. They were also interested in how students were able to incorporate an aspect of the whole class discussion into their own solution strategies. Although the authors stressed the importance of the social aspects of the classroom—participation, hearing others' ideas, etc.—their analysis was unapologetically focused on the individual as learner and constructor of knowledge. As they stated, "Although participants influence and are influenced by others, they are not supposed to construct a collective understanding as a product of a series of negotiations" (p. 524). The authors did hypothesize about how an analysis of both collective problem solving and individual understanding could take shape. However, their notions seemed heavily weighted towards how the classroom is a catalyst for generating and refining individual comprehension, stating that what is constructed collectively occurs first and then is incorporated by the individual classroom members into their ways of problem-solving. While Inagaki, Hatano, and Morino (1998) did pay attention to the occurrences at the collective level, their analyses seemed to merely pay credit to the classroom collective as a situational influence on learners, rather than consider the development of the collective as a unit of analysis in and of itself.

Sfard (2007), on the other hand, acknowledged in a much more reflexive manner the role of the collective on the mathematical development of a learner and vice versa. She made use of a participationist view of learning, which considers the activities and patterned processes of individuals and the collective, rather than considering the transformation in an individual. Furthermore, she offered two processes that result in the transformation of human activity which are consistent with a participationist viewpoint: that of *individualization of the collective* and *communalization of the individual.* Sfard posited that

> The processes of individualization and communalization are reflexively interrelated: The collective activities are primary models for individual forms of acting, whereas individual variations feed back into the collective forms of doing, acquire permanence, and are carried in space and time from one community of actors to another. (p. 569)

Through this viewpoint, the interrelatedness of the individual and the collective come to the fore, highlighting how the activity of one necessarily affects the activity of the other. This sentiment, which is focused on practices and activities as the objects of analysis, is consistent with that of Saxe (2002), who stated that:

> In collective practices, joint tasks are accomplished...through the interrelated

> activities of individuals. In such joint accomplishments, individual and collective activities are reciprocally related. Individual activities are constitutive of collective practices. At the same time, the joint activity of the collective gives shape and purpose to individuals' goal-directed activities. (p. 276-277)

Sfard further investigated the relationship between the individual and the collective level of activity in the classroom by suggesting the constructs of discourse and identity as potentially fruitful avenues through which this seemingly divisive distinction could be linked in a productive manner analytically. Not pursuing these possibly illuminating links but noting the complexity of the matter, Sfard left those as avenues for future research.

Along rather complimentary notes to the work of Sfard (2007) and Saxe (2002), the work of Cobb and his colleagues (Cobb, 1999; Cobb & Yackel, 1996; Stephan, Cobb, & Gravemeijer, 2003) took into consideration the socially and situated nature of mathematical activity, drawing influence from sociocultural theory, discourse theory, and symbolic interactionism. Their work analyzed students' mathematical reasoning as "acts of participation in the mathematical practice established by the classroom community" (p. 5). Originally, however, their work integrated the social aspect of the classroom merely as a catalyst for purely psychological development. Their view of the role of social interaction was severely challenged while watching a teacher struggle with getting students to express their own ideas and justifications in class discussion, in comparison with sharing what they thought the teacher expected to hear. It was through this struggle that the researchers began to consider the inextricable role of social and cultural factors in the mathematical development of students. Thus, in accordance with their goal of supporting students' mathematical development, Cobb and colleagues began considering the importance of what they referred to as social norms and sociomathematical norms (Cobb, Yackel, & Wood, 1989; Cobb & Yackel, 1996). As stated by McClain (2002), "Students' actions and constructions are influenced by a teacher's actions and constructions, and vice versa. But neither is totally dependent on or independent of the other"

(p. 217). The constructs of social norms and sociomathematical norms grow naturally out of this very belief—that the individual actions and social interactions of a community, including both students and teachers, affect all members of the community.

Social norms, by definition, are the agreed upon social interactions and practices of a community that become routine (Rasmussen, Yackel, & King, 2003). These are not discipline or situation specific; it may be the case that ‘defending a claim with evidence’ is seen as a normative behavior in multiple circles, such as an academic classroom or a court of law. Cobb and Yackel (1996) stated they are not “psychological processes or entities that can be attributed to any particular individual. Instead, they characterize regularities in communal or collective classroom activity” (p. 178). Sociomathematical norms share the same ideologies as that of social norms, but they are specific to a mathematics environment. For instance, a social norm could be the value that all community members need to justify any claim made inside the community. A sociomathematical norm, then, could be that this justification needs to be grounded in a defendable argument, such as a proof, an example, or a counterexample. Thus, the way in which this justification was specifically given inside a *mathematics* classroom served as the sociomathematical norm. Cobb and colleagues, who originally considered individual intellectual development purely in terms of psychological aspects, changed their view to one that necessarily viewed individual development as a characteristic of participation in the classroom community. Thus, the constructs of social norms and sociomathematical norms, which live at the collective level of analysis, are an integral aspect in understanding and encouraging individual students’ mathematical development.

A third tool to aid in the analysis of mathematical learning of the classroom community is that of classroom mathematical practice, which investigate the “taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas” (Cobb, 1999, p. 9). Classroom mathematical practices, then, do not illuminate any information about a particular individual student's mathematical development, nor should it be assumed that the two must overlap. Thus, an analysis utilizing the theoretical lenses of social norms, sociomathematical norms, and classroom mathematics practices and investigating their development takes the collective classroom community as the unit of analysis, documenting the development of practices and the establishment of meanings at the collective level. One example of such an analysis is the work of Stephan and Rasmussen (2002) towards describing and analyzing the emergence of classroom mathematical practices of a university-level mathematics course. Their methodology (Rasmussen & Stephan, 2008) for documenting the collective activity of a classroom is detailed in Chapter 3.

Elbers (2003), in a tribute to Streefland's work regarding interaction and collaborative learning, focused on how collective reasoning at the collective level may influence individual students' learning, as well as how individual students contribute to whole class discussion. Defining learning as the changing participation of students in the classroom discourse, Elbers analyzed student learning by comparing individual work to whole class discussion in order to "get a view of the progress of mathematical understanding both in the classroom as a whole and in individual children's minds" (p. 82). Elbers claimed that a clear separation or hierarchical relationship between individual and collective levels of activity is not possible. Given that individuals' work within the class is influenced by the formation of social and sociomathematical norms, Elbers claimed that students' learning is best understood through the light of what occurs at the collective level, which he refers to as the "discursive structure" (p. 93). He noted that students often utilize the same types of explanations in their individual work that they have used or seen used in whole class discussion. Thus, "the individual work is to be considered as an anticipation of a class discussion, or, following a class discussion, as a reconstruction of it. Given this discursive structure, there is no priority for individual or collective work: they are two sides of the same coin" (pp. 93-94).

In conclusion, the work of Cobb and colleagues regarding the coordination of individual and social lenses—described above and influential to many researchers—led to a theoretical framework known as the emergent perspective (Cobb & Yackel, 1996), and many researchers (e.g., Izsák, Tillema, & Tunç-Pekkan, 2008; Hershkowitz, Hadas, Dreyfus, & Schwarz, 2007; Stephan & Rasmussen, 2002) have found this an inspirational framework through which to analyze the mathematical development of a given classroom and its members. For instance, Hershkowitz et al. (2007) cite the emergent perspective (and the associated views of the individual and the collective) as foundational to their work in analyzing individual and group construction of knowledge. However, they claim their work integrates more of a longitudinal analysis, more details of which are provided in the next section.

2.3 Genetic Analysis of Knowledge Development

The Invertible Matrix Theorem is a powerful tool for a first course in linear algebra. As the semester progresses, the Theorem takes form and develops meaning as students come to reason about the ways in which key ideas in linear algebra are connected. The developmental nature of both the mathematical content in and the students' understanding of the IMT is consistent with a belief foundational to the teaching experiment—the belief that mathematics is a human activity (Freudenthal, 1991). Steffe and Thompson (2000), while detailing what is foundational in establishing teaching experiment methodology, took a similar position, stating that viewing mathematics as "a product of functioning human intelligence defines mathematics as a living subject." Furthermore, Steffe and Thompson posit that

> Looking behind what students say and do in an attempt to understand their mathematical realities is an essential part of a teaching experiment...Teaching experiment methodology is based on the necessity of providing an ontogenetic justification of mathematics; that is, a justification based on the history of its generation by individuals. (p. 269)

This quote stresses the importance of investigating the progressive construction of mathematical ideas by considering how students create, correlate, refine, utilize these ideas over the course of time. Furthermore, this longitudinal analysis is also informative regarding students' developing ways of reasoning regarding the mathematical ideas. The research questions for the proposed study are founded upon similar beliefs. Thus, conducting a genetic analysis of knowledge development for both individuals and the collective is appropriate and necessary. In this section, I review the works of various researchers who have either utilized or developed methodology for genetic analyses of knowledge development.

In a paper focused on the process of constructing knowledge among a group of three students, Hershkowitz, Hadas, Dreyfus, and Schwartz (2007) analyzed the construction, interaction, and consolidation of knowledge, taking individual diversity into consideration. The authors referred to this as *shared knowledge*, defined as "a common basis of knowledge which allows the students in the group to continue together the construction of further knowledge in the same topic" (p. 42). Their work detailed the high level of diversity in individual students' ways of interacting with the collective during the construction of shared knowledge and how students seemed to integrate this shared knowledge for themselves in their own work. While this work clearly relates to the individual-collective distinction previously discussed, Hershkowitz et al. (2007) claimed it goes beyond that work by Cobb et al. (2001) by adding a perspective on continuity of knowledge. They stated that their data analyses from a time span on several lessons are considered as a whole. The purpose for this is to observe and analyze students as they construct new mathematical knowledge in one activity, as well as to observe and analyze in detail if and how students use this knowledge in further activities. While the authors' point is heard, I contend that some of the work of Cobb and colleagues *does* consider a longitudinal aspect. In particular, the documentation of classroom mathematics practices through the use of Toulmin's model (1969) of argumentation has as an object of inquiry how the ways in which ideas function in the collective shift over time. I contend that the work of Hershkowitz et al. (2007) and Cobb et al. (2001) are, at least somewhat, considering the same phenomenon—the development of mathematical knowledge over time—from differing lenses and through different analytical techniques. In any case, the data corpus for an analysis of this magnitude is great, and no clearly defined methodologies for analyzing these data exist.

Other researchers have also focused on the genetic analysis of individuals over the course of the classroom. For instance, Simon, Saldanha, McClintock, Akar, Watanabe, and Zembat (2010) emphasized the utility of extremely fine-grained microgenetic analyses of individuals in order to develop an understanding of the process of conceptual change for an individual. However, the work by Saxe (2002) and Tiberghien and Malkoun (2009) delineate multiple levels of investigation through which to analyze the development of knowledge over the course of a semester at both the individual and the collective layer. It is with a rather detailed review of their relevant work that I conclude this chapter.

Approaching the same phenomenon of genetic analysis but through a foundation in anthropological research, Saxe (2002) and his colleagues (Saxe & Esmonde, 2005; Saxe, Gearhart, Shaughnessy, Earnest, Cremer, Sitabkhan, et al., 2009) investigated knowledge development through the notion of cultural change, asking questions such as, "How does the activity of individuals shape historical change, and, reciprocally, how does historical change affect individuals' practices?" (Saxe & Esmonde, 2005, p. 174). Particular to development in the classroom, the authors investigated how researchers could collect data (how much, from what sources, etc.) and conduct analyses that would allow them to make descriptions of how individuals' ideas develop in the classroom over time, given that the classroom is also changing over time. As a response, they suggested analyzing human development over time from three different strands, providing researchers a way to account for some of the complex factors of development. The three analytical strands of *microgenesis, ontogenesis,* and *sociogenesis* are offered as integrated yet distinct aspects of collective practices. The authors defined *microgenesis* as the short-term process by which individuals construct meaningful representations in activity, *ontogenesis* as the shifts in patterns of thinking over the development of individuals, and *sociogenesis* as the reproduction and alteration of representational forms that enable communication among participants in a community (Saxe et al., 2009, p. 208). As these descriptions imply, each of the three strands of development (microgenesis, ontogenesis, and sociogenesis) are a line of inquiry in and of themselves. This study utilized the strands of microgenetic and ontogenetic analysis in order to investigate students' ways of reasoning about the Invertible Matrix Theorem.

In a complementary vein of research, Tiberghien and Malkoun (2009) presented a methodological framework for reconstructing the life of knowledge in teaching sequences, where the authors state that knowledge is understood through the metaphor of life in that knowledge "lives" within groups (Chevellard, 1991). For Tiberghien and Malkoun, the driving question was what methodologies could researchers develop in order to both distinguish between individual and collective analyses of the classroom as well as relate fine-grain and broader analysis over time? To address this, the authors utilized a collective perspective to describe taught knowledge and an individual perspective to describe students' developing knowledge. The methods of analysis are similar for both and involve three different scale sizes and two different reference points. The authors labeled the collectively oriented reference point as *conventional* and the individually oriented reference point as *situational.* The three scale sizes are microscopic, mesoscopic, and macroscopic. The scale sizes correspond to analyzing key words or utterances (microscopic), themes over a few lessons (mesoscopic), and longer sequences such as an entire semester (macroscopic).

Tiberghien and Malkoun (2009) also offered up various possibilities of coordination between these analytical distinctions. For instance, coordination between the microscopic and macroscopic level from a conventional perspective could involve the notions of density and continuity. Density is the number of key words or utterances in relation to the duration of a theme or sequence, whereas continuity is the distribution of key words or utterances that are most reused over the duration of a theme or sequence.

CHAPTER THREE: RESEARCH METHODOLOGY

In this chapter, I detail the data I collected in order to address my research questions and the analytical methods by which I analyzed these data. The research questions are:

1. How did the collective classroom community reason about the Invertible Matrix Theorem over time?
2. How did an individual student reason about the Invertible Matrix Theorem over time?

The data for this study came from a semester-long classroom teaching experiment. I begin this chapter by expanding upon what is entailed in conducting a classroom teaching experiment, both theoretically and practically. The remainder of the chapter is divided into three main sections: (a) the setting and participants for the teaching experiment, (b) the data I collected and the methods by which I collected them, and (c) the analyses that I conducted on these data.

3.1 Theoretical Background

Over the past twenty years, mathematics education research has begun to shift away from an exclusively cognitive focus to one that acknowledges the situatedness of student activity within the classroom and broader communities of practice (e.g., Lave, 1988; Wenger, 1998; Yackel, Rasmussen, & King, 2000). There has also been an emphasis in the reflexivity of theory and practice, in that practicing teachers—rather than being passive recipients of research into learning and teaching—can be active participants in research that then affects the classroom. The design research methodology laid out by Cobb (2000) takes these aspects into consideration in what is referred to as a classroom-based teaching experiment. The remainder of this section elaborates (a) the underlying theoretical perspective that marriages the importance of both psychological and social processes, (b) the reflexivity of theory and practice, and (c) methodology for conducting classroom-based teaching experiments.

The theoretical perspective on learning that undergirds the classroom teaching experiment that was conducted is the emergent perspective (Cobb & Yackel, 1996), which coordinates psychological constructivism (von Glasersfeld, 1995) and interactionism (Forman, 2003; Vygotsky, 1987). With regards to mathematical development, the former places a high emphasis on individual psychological products with the social atmosphere merely being a catalyst for learning, whereas the latter perspective deemphasizes psychological processes in favor of the social organizational structure is which learners participate. The emergent perspective honors the importance of both psychological and social processes. Thus, according to the emergent perspective,

> The basic relationship posited between students' constructive activities and the social processes in which they participate in the classroom is one of reflexivity in which neither is given preeminence over the other...A basic assumption of the emergent perspective is, therefore, that neither individual students' activities nor classroom mathematical practices can be accounted for adequately except in relation to the other. (Cobb, 2000, p, 310)

From the perspective that learning is both an individual and a social process, investigating the mathematical development of students necessarily involves considering the classroom community of which they are a part. This theoretical relationship between the individual and the social has immediate implications for the relationship between theory and practice. Mathematical development happens in situ, during and through participation and interaction in the classroom. Whereas in previous approaches to the implementation of educational research the teachers were seen as passive consumers of research results, more recent research has embraced a shift away from this notion in favor of a reflexive relationship between theory and practice. From this latter perspective, "theory is seen to emerge from practice and to feed back to guide it" (Cobb, 2000, p. 308). Thus, with the goal of developing innovative ways to positively affect mathematics education, classroom-based design research is seen as a justified and valuable contribution to the improvement of education.

Classroom teaching experiments consist of the following three components: instructional design and planning, the ongoing analysis of classroom events, and the retrospective analysis of all data sources generated in the course of the teaching experiment (Cobb, 2000). The research team of which I was a member while conducting this study has and continues to integrate these three aspects in our work. Within our work in linear algebra, we have conducted three semester-long classroom teaching experiments (Fall 2007, Spring 2009, and Spring 2010), which took place at two different universities with three different teacher-researchers. The data for this dissertation comes from the Spring 2010 iteration, for which I was the teacher-researcher. Throughout the duration of the teaching experiments, the teacher-researcher met with the research team approximately three times a week in order to debrief after class, discuss impressions of student work and mathematical development, and plan the following class. The products of the teaching experiments include revised and refined sequences of instructional tasks and basic research about students thinking regarding fundamental ideas in linear algebra (Henderson, Rasmussen, Sweeney, Wawro, & Zandieh; 2010; Larson, Zandieh, & Rasmussen, 2008; Larson, Zandieh, Rasmussen, & Henderson, 2009; Wawro, 2009; Wawro, Sweeney, & Rabin, 2011; Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2011).

3.2 Setting and Participants

The semester-long classroom teaching experiment was conducted in an introductory linear algebra course during the spring semester 2010 at a large southwestern public university. This is the same course in which the research team has previously conducted iterations of the classroom teaching experiment (Fall 2007 and Spring 2009). Throughout the mathematics education literature, there exist various recommendations regarding pedagogical approaches and classroom environments. Consistent with our past work, our research team created an inquiry-oriented classroom for the 2010 teaching experiment. By the term *inquiry*, we mean both teacher-inquiry and student-inquiry (Rasmussen & Kwon, 2007). For our team, an *inquiry-oriented classroom* entails (a) students learning mathematics though inquiry by participating in mathematical discussions, posing arguments and explaining their thinking, and solving novel problems; and (b) teachers inquiring into their students' mathematical thinking and reasoning (Rasmussen, Kwon, & Marrongelle, 2008). Furthermore, our work draws on the instructional design theory of Realistic Mathematics Education (Freudenthal, 1991), which begins with the tenet that mathematics is a human activity. We endeavored to create a linear algebra course that built on student concepts and reasoning as the starting point from which more complex and formal reasoning develops. Thus, as Gravemeijer (2004) states, "For the instructional designer this implies a change in perspective from decomposing ready-made expert knowledge as the starting point for design to imagining students elaborating, refining, and adjusting their current ways of knowing" (p. 106).

3.2.1 Classroom Setting

The linear algebra course took place in one of the university's most technologically advanced classrooms. A simple schematic of the room is given in Figure 3.1.

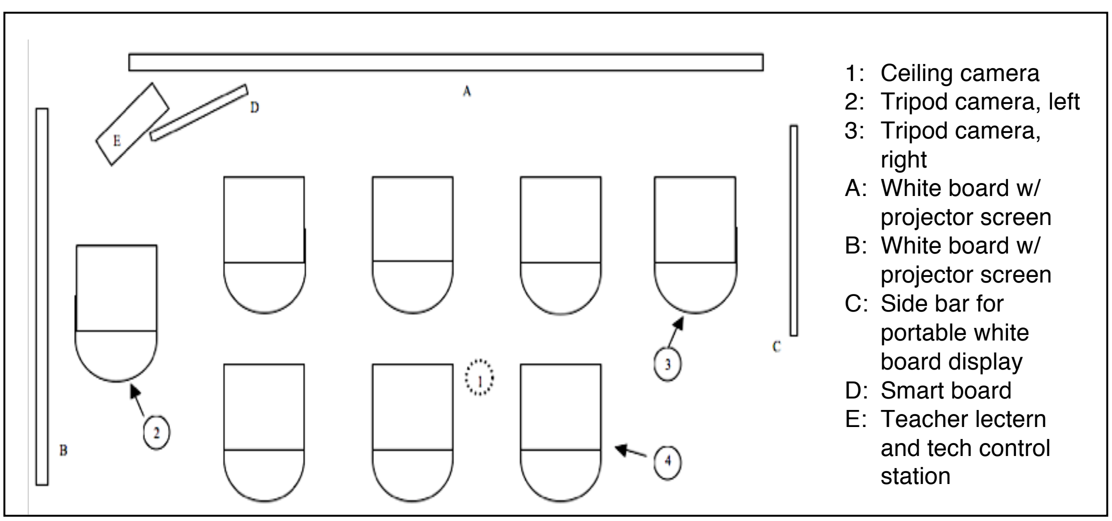


Figure 3.1. Schematic of the classroom

In particular, this classroom was equipped with the following:

- Two large whiteboards (labeled ‘A’ and ‘B’ in Figure 3.1) that covered two of the four classroom walls.
- A “smart station” (labeled ‘E’ in Figure 3.1) that was equipped with: a computer (both PC and Mac), a connection for a laptop computer, and an external document camera—all of which could be projected onto one or both of the projector screens.
- A smart board (labeled ‘D’ in Figure 3.1).
- Eight portable, double-sided whiteboards, approximately 3’ x 4’ each. These portable white boards were used during small group work as a place for students to collaborate on tasks and share their ideas and conclusions. Furthermore, the classroom was equipped with a place to hang the whiteboards on the third wall (labeled ‘C’ in Figure 3.1), or could be placed along the whiteboards ‘A’ or ‘B’ for display. In our previous work, we found these portable white boards were advantageous for group work. These boards provided a large space for collaboration and the joint production of ideas. Furthermore, the whiteboards provided clear images of student work that were easily captured on the video recordings.
- Student workspace configured as eight tables, each capable of accommodating up to five students. We have found this particular configuration—one that is already arranged to be conducive to small group interactions and collaborations—to be particularly beneficial in promoting such an atmosphere.

3.2.2 Participants

Students enrolled in the linear algebra course had generally completed three semesters of calculus (at least two semesters was required). Approximately half had also completed a discrete mathematics course, and 75% of students were in their second or third year of university. The population of the course, categorized by declared major, was: Computer Engineering (10); Computer Science (7); Mathematics (4); Statistics (3); Mathematics Single Subject (2); and other science or business field (5).

In order to address the second research question regarding how an individual student reasoned about the Invertible Matrix Theorem over time, I conducted an in-depth case study of one student enrolled in the linear algebra course. I used three main criteria by which I will chose this student: (a) given consent to be videotaped during class, (b) willingness to participate in individual interviews, and (c) perceived ability to articulate his/her thinking during whole class discussion. The importance of the first two criteria is obvious. The third criterion—perceived ability to articulate thinking—is relevant because it is important to study a student who is willing able to be "a good sharer" in order to study student thinking and mathematical development in situ. By the end of the second week of the semester, the research team had an idea of who some these "key informants" were (Tremblay, 1989) and arranged the seating chart so that they were among the students seated at the three tables that were videorecorded during class. For my dissertation, I focused on one of the three tables that were filmed, and five students sat there together for the duration of the semester. By filming their table each class day and conducting two individual interviews with each of them, I collected a wide swath of rich data from which to choose the subject for my case study. Because of his willingness to participate in class, ability to communicate his thinking and reasoning with others, and his genuine interest and in the course and the material, I chose Abraham as my case-study subject. He was a junior statistics major, and his small group was comprised of a senior business major, a sophomore mathematics major, a junior chemical physics major, and a senior computer engineering major. I plan to return to the rich set of data I collected regarding the other four students in subsequent studies, as is detailed in Chapter 6.

3.3 Data Collection

In order to collect data relevant to the *collective* establishment of meaning regarding the Invertible Matrix Theorem, I collected the following four sources of data: video and transcript of whole class discussion, video and transcript of small group work, photos of whiteboard work, and written work on in-class activities. In order to collect data relevant to *individual* establishment of meaning regarding the IMT, I collected the following four sources of data: video and transcript of whole class discussion, video and transcript of small group work, video and transcript from individual interviews, and various written work. The written work was in the form of in-class worksheets, exams, reflections, homework, and portfolios, as well as Abraham's written work from the interviews. Whole class and small group video data came from ten target class days. Individual interview data came from two interviews, one conducted midway through the semester and one conducted at the end of the semester. Each of these data sources is described in more detail in the remainder of this section.

3.3.1 Classroom data collection for collective analysis

The teaching experiment was "semester-long" in that we collected data on every class day of the semester, barring the three exam days, for a total of 29 data-collection days. We had four video cameras (labeled *1, 2, 3*, and *4* in Figure 3.1) filming the entirety of each class day, except exam days. The classroom had one permanent ceiling camera (labeled *1* in Figure 3.1), operated by a graduate student from the control room, which followed the instructor at all times in order to capture her actions and words for the entire duration of each class. The instructor wore a lapel microphone that was synced to this particular camera in order to maximize the quality of the recording.

Three cameras, on tripods, were placed in the far-left, back-middle, and far-right regions of the classroom. The left camera (labeled *2* in Figure 3.1) was set up near one particular table and captured the work and conversation of that table during times of small group work, as well the whole class discussions of the classroom on video. During whole class discussions, the instructor did not do all the talking. Rather, students contributed their ideas as well as commented on and questioned other students about their ideas. Because of that, it was the job of the left camera to capture on video the faces of whoever was speaking during class. The right camera (labeled *3* in Figure 3.1) captured the work of a second particular small group, as well as the classroom interactions—who was up front, what was written on the board, etc. The middle camera (labeled *4* in Figure 3.1) collected video data for a third small group and captured a more holistic view of the classroom during whole class discussions.

After every class session, a member of the research team was responsible for uploading the video data from the four video cameras onto the research team's desktop computer. These data served as discussion points during our weekly research meetings regarding ongoing analysis of student thinking. Furthermore, the video from 18 of the class days was transcribed during the semester by a professional transcriber in preparation for in-depth retrospective analysis. These 18 days were chosen according to my discretion regarding which days I, with input from the research team, deemed to be relevant to my study of the Invertible Matrix Theorem.

In addition to video data, the research team retained copies of all written work created during class activities. The work was collected at the end of every class and photocopied immediately by a member of the research team. The students then had the opportunity to collect the work from the instructor's office prior to the next class section; if not collected, it was returned during the next class period. Finally, the research team also took photos of collective student work at the end of each class period. One research team member was in charge of taking photos of the portable white boards every time they were used during class. This proved useful because the photos enabled us to quickly examine student work and inform teaching for the following day. Finally, of the 29 class days that were filmed, portions of ten of them served as data for this study. Specifics regarding the data reduction phase are given in Section 3.4.

3.3.2 Individual Data Collection

For the research question concerning the ways in which Abraham reasoned about the IMT, I collected data from the following four sources: video and transcript of whole class discussion, video and transcript of small group work, video and transcript from individual interviews, and various written work. The collection of video data of whole class and small group work is the same as that described in the previous section, so the description is not be repeated here. Rather, descriptions of the individual student interviews and of the relevant written work data sources are provided.

3.3.2.1 Interview data. Abraham participated in two semi-structured individual interviews (Bernard, 1988); one occurred the middle of the semester (between Days 16 and 17), and one took place the week after final exams. Both interviews were piloted on 1-2 students from the class. This allowed me to alter or edit the interview protocol as needed, as well as discuss with the interviewer any clarifications or concerns regarding the interview prior to interviewing Abraham. The other four members of Abraham's small group also took part in both individual interviews. These data sources will be integral to a future study that investigates both the individual ways of reasoning about the IMT for the small group members as well as that of the "mini- collective."

The interview protocols were based on previous iterations of our work but also grew organically in response to the development of this particular class. The first interview was conducted during the middle of the semester, before the Invertible Matrix Theorem was fully developed in an explicit manner. Thus, this interview was composed mostly of questions that asked students to relate the ideas he/she has already been introduced to, to solve a given problem, or to generate examples or non-examples. The post-semester interview explicitly asked students to reason about concept statements that comprise the Invertible Matrix Theorem and how they saw them as equivalent. The interview questions can be found in Appendix 3.1.

The data sources for analysis consisted of video recordings of the interviews, as well as copies of all student work. Additionally, each interview was transcribed completely. Although both of Abraham's interviews lasted approximately ninety minutes, only a portion of each interview was explicitly relevant to this study. More detail about the data reduction process is provided in the Data Analysis Section.

3.3.2.2 Written work. A variety of written data sources over the course of the semester were used as secondary data sources. Every assignment that students turned in—weekly homework assignments, exams, daily reflections, and associated portfolios—was scanned and stored electronically before the work is returned to the students. First, weekly homework assignments were given to the students, for which they had one week to complete. The homework questions originated both from the course textbook (Lay, 2003), as well as from what the researchers developed in order to investigate how students were making sense of particular class discussions. From the textbook, a variety of questions were assigned that necessitated connecting various ideas in the IMT in order to reach a solution. For exam data, there were two in-class exams during the semester and one final exam. The question format on exams included strictly calculational questions, example-generation questions, explanation questions, as well as true/false with justification questions.

I also collected reflection question data from the daily class sessions. Every class period (except on exam days), I saved the last five minutes of class for a reflection question. Most reflections served as an avenue to reflect on ideas that students were in the midst of learning about. They were often their first "alone time" to try and reason through ideas and make connections. Furthermore, reflections often served as catalysts for very lively debates and discussions at the start of the next class period. Finally, the course portfolio was a selection of materials that each student chose from the course that documented what he felt represented his mathematical development. This provided a mechanism for students to reflect on their previous work as they prepared for each exam. For each of the problems they selected to be in the portfolio, students developed a rationale that explained why they selected the problem and what mathematics learned they through their work on it. Expected length of rationale statement was at least 100 words, and portfolios were turned in at the time of each exam, including the final.

These data—the student written work in the form of homework, exam, reflection, and portfolio responses—served as secondary data sources for the research questions. The main data sources for analysis were the video recordings and associated transcripts from whole class discussion and small group work during class, as well as video recordings and associated transcripts from Abraham's individual interviews. The written data sources detailed above served to triangulate claims made about student thinking regarding the research questions.

3.4 Data Analysis

The instructional design of the classroom teaching experiment was highly influenced by the notions that mathematics is a human activity (Freudenthal, 1991) and the importance of engaging learners in mathematical activities such as symbolizing, defining, algorithmatizing, and justifying (Rasmussen, Zandieh, King, & Teppo, 2005). The construction of the Invertible Matrix Theorem occurred throughout the semester as students and the instructor participated in mathematical activities such as these. Because of this, a natural approach to analyze the ways in which students reasoned about the Invertible Matrix Theorem was to analyze the arguments that were made involving the IMT. In other words, because of the inherent structure of the IMT—a list of 16 equivalent statements—much of the theorem's "coming to be" is reflected in the ways in which students and teachers constructed arguments about how the different equivalencies were related. When considering the analysis of such mathematical arguments produced by students and mathematicians, Inglis, Mejia-Ramos, and Simpson (2007) stated that, "Generally these types of analysis are of two kinds: those that concentrate on the argument's *content* and those that concentrate on the argument's *structure*" (p. 3). These two aspects of an argument—content and structure—are somewhat reflexive and inseparable and were analyzed through the use of both adjacency matrices and Toulmin (1969) model of argumentation. The results of the two analytical tools address both the content and the structure of the relevant arguments, so by coordinating the two analyses, I created a more full, rich, and accurate depiction of the ways in which students reasoned about the Invertible Matrix Theorem than would have been achieved through using only one of the two analytical tools.

There were six phases of analysis, and each had both an individual and a collective level, where I refer to the analysis of my in-depth case study of Abraham as the "individual level." The six phases were:

1. Reduction of the data set
2. Construction of Toulmin schemes of argumentation
3. Construction of adjacency matrices
4. Ontogenetic analysis of constructed Toulmin schemes
5. Ontogenetic analysis of constructed adjacency matrices
6. Coordination of analysis across the two analytical tools.

Consider Table 3.1, which organizes the main aspects of both levels (individual and collective) of the six phases of the analysis. During Phase One, I reduced the data to include all data relevant to the research questions, including relevant whole class discussion, small group work, written sources, and interview data. Data for the collective analysis came from whole class discussion video data, whereas data for the individual level came from whole class discussion video data, small group work video data, and interview data, with written sources as secondary data sources.

Table 3.1. Summary of the six phases of analysis.

	Phase 1	Phase 2	Phase 3	Phase 4	Phase 5	Phase 6
	Reduction of the data set	*Construction of Toulmin schemes*	*Construction of adjacency matrices*	*Ontogenetic analysis of Toulmin schemes*	*Ontogenetic analysis of adj. matrices*	*Coordination of analyses*
Collective	Relevant WCD video & transcript	AM AM . . . AM AM	TS 1 TS 3 TS 2 . . . TS *n*	AM 1–*n* AM 1–*i* AM *d*–*k* AM *j*–*n*	TS 1–*n* TS 1–*i* TS *d*–*k* TS *j*–*n*	AM 1–*n* TS 1–*n* AM *j*–*n* TS *j*–*n*
Individual	Relevant SGW video & transcript Relevant written work Relevant Interview data	S1 AM S1 AM S1 AM S2 AM S2 AM *j* S3 AM S3 AM S4 AM S4 AM S4 AM	S1 TS S1 TS S1 TS S2 TS S2 TS *j* S3 TS S3 TS *k* S4 TS S4 TS S4 TS *r*	S1 AM 1–*n* S1 AM *d*–*k* S1 AM *j*–*n* S3 AM 1–*k* S3 AM *1*–*d* S3 AM *d*–*j* S3 AM *d*–*k* S4 AM 1–*r* S4 AM *d*–*k*	S1 TS 1–*n* S1 TS *d*–*k* S1 TS *j*–*n* S3 TS 1–*k* S3 TS *1*–*d* S3 TS *d*–*j* S3 TS *d*–*k* S4 TS 1–*r* S4 TS *d*–*k*	S1 AM 1–*n* S1 TS 1–*n* S2 AM *4* S2 TS *4* S3 AM 1–*k* S3 TS 1–*k* S4 AM 1–*r* S4 TS 1–*r* S4 AM *d*–*k* S4 TS *d*–*k*

As an illustrative example, consider the "Collective" row of Table 3.1. For analysis at the collective level, there were 117 distinct exchanges to analyze—these were the products of Phase 1 of the analysis. There were labeled according to the day on which they occurred, so, for instance, if Day 6 had 3 arguments, they were labeled Argument 6.1, Argument 6.2, and Argument 6.3. For each exchange, there was a distinct Toulmin scheme created to represent that argument. Thus, there were 117 distinct Toulmin schemes, and these are denoted in the Phase 2 column as "TS 1" through "TS n." Furthermore, for every one of these n-many exchanges identified in Phase One and used in Phase Two for Toulmin schemes, there was a distinct adjacency matrix coding created to represent that exchange. These are denoted in Phase 3 as "AM 1" through "AM n." Thus, for every i, $1 \leq i \leq n$, the Toulmin scheme "TS i" and the adjacency matrix "AM *i"* are created from the same exchange from the reduced data set in Phase 1.

While Phases 2 and 3 are comprised of many small-scale, discrete analyses—what Saxe (2002) refers to as *microgenetic analyses*—Phases 4 and 5 analyzed the shifts in form of function of those discrete arguments. This is consistent with what Saxe refers to as *ontogenetic analysis.*. That is, In Phases 4 and 5 I compiled from the results of Phases 2 and 3 and considered the body of constructed Toulmin schemes and adjacency matrices, respectively (denoted, for example, as "AM *d–k*" and "TS *1–j"* in Table 3.1), and I analyzed characteristics and shifts in students' arguments over the course of the semester. I used two primary analytic lenses in Phases 4 and 5. In Phase 4, I consider the individually constructed Toulmin schemes from Phase 2 as a whole in order to document, for the collective level, normative ways of reasoning and classroom mathematics practices. This analysis is consistent with the work of Rasmussen and Stephan (2008) and Cole, Becker, Towns, Rasmussen, Wawro, and Sweeney (2010).

During Phase 5, on the other hand, I investigated shifts in form and function of how students reason about and reason with the various concepts in the IMT over time by considering qualitative changes in constructed adjacency matrices from Phase 2. While some of this analysis of adjacency matrices was compatible with the analysis from Phase 4, it also involved macrogenetic, holistic analysis of the adjacency matrices' content and structure. For instance, which concepts from the Invertible Matrix Theorem were used most often overall by students as they reasoned about novel problems? Were there some concepts from the IMT that were used consistently over the course of the semester, whereas others dropped off? This analysis is inspired by the work of Tiberghien and Malkoun (2009) who consider the notions of density and continuity of ideas as analytical frames in genetic analyses.

Finally, Phase 6 combined the work done in parallel with Toulmin schemes and adjacency matrices on both the microgenetic level (comparing the results of Phases 2 and 3) and the ontogenetic level (Phases 4 and 5). This comparison of parallel analyses is denoted in the last column of Table 3.1 by boxes such as "AM *j–n* TS *j–n*." In other words, Phase 6 consisted of cross-comparative analyses, for any given argument or collection of arguments, of the results from both analytical tools (adjacency matrices and Toulmin schemes). This cross-comparison of results from the two analytical tool provides a rich way to investigate the content and structure of arguments offered by both individuals and the collective. Thus, by coordinating the two analyses, I obtained a rich and full depiction of the ways in which students were reasoning about the Invertible Matrix Theorem.

While the previous description accompanying Table 3.1 provides a broad overview of the proposed plan for data analysis, the remainder of this section elaborates upon the six phases of analysis in greater detail. Again, these phases are (a) reduction of the data set, (b) construction of adjacency matrices, (c) construction of Toulmin schemes, (d) ontogenetic analysis of constructed adjacency matrices, (e) ontogenetic analysis of constructed Toulmin schemes, and (f) coordination of analyses across the two analytical tools.

3.4.1 Phase One: Reduction of the Data Set

For analysis at the collective level, the data sources were video of whole class discussion. Although every day of the semester (except exam days) was video recorded, portions of the ten days that are described in Table 3.2 served as the main data sources for the analysis at the collective level. First, Figure 3.2 contains what became known as the Invertible Matrix Theorem for the class during the study. One may note that this statement of the IMT in Figure 3.2 varies slightly from that given in Figure 1.1 in Chapter 1. The variations between the two presentations are: (a) The equivalencies are in a different order, and (b) The presentation in

Figure 1.1 includes five statements that Figure 3.2 does not include: "A^{T} is an invertible matrix" (statement *l*), "dim *Col A* = *n*" (statement *o*), "rank *A* = *n*" (statement *p*), "dim *Nul A* = 0" (statement *r*), and "the columns of *A* form a basis for $\mathbf{R}^n$" (statement *m*). The change in statement ordering and omission of these five statements is to reflect the actual development of the IMT during class.

The Invertible Matrix Theorem

Let *A* be a square $n \times n$ matrix. Then the following are equivalent:

1. The columns of *A* span $\mathbf{R}^n$
2. The matrix *A* has *n* pivots
3. For every **b** in $\mathbf{R}^n$, there exists a solution **x** to $A\mathbf{x} = \mathbf{b}$
4. For every **b** in $\mathbf{R}^n$, there is a way to write **b** as a linear combination of the columns of *A*
5. *A* is row equivalent to the $n \times n$ identity matrix
6. The columns of *A* are linearly independent
7. The only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution
8. *A* is invertible
9. There exists an *n* x *n* matrix *C* such that $CA = I$
10. There exists an *n* x *n* matrix *D* such that $AD = I$
11. The transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one
12. The transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ maps $\mathbf{R}^n$ onto $\mathbf{R}^n$
13. The column space of *A* is all of $\mathbf{R}^n$
14. The null space of *A* contains only the zero vector
15. The determinant of *A* is nonzero
16. The number zero is not an eigenvalue of *A*

Figure 3.2. The Invertible Matrix Theorem, as developed in class.

In order to determine which days were necessary to analyze, I first established that a day or portion of a day would be deemed appropriate for analysis if the classroom discourse for a given portion was either implicitly or explicitly involved the class members actively engaging in developing ways of reasoning about two or more concepts from the Invertible Matrix Theorem in conjunction with each other. To analyze everything ever said about a given topic, such as linear independence, would not be fruitful, given that the data set would be too vast to analyze anything in depth. Thus, as planned, I selected portions of whole class discussion when students were developing generalizations about ways in which the concepts within the IMT related to one another (such as equivalence between span and column space). Brief summaries of those ten class days in given in Table 3.2.

Table 3.2. Chronological development of the Invertible Matrix Theorem.

Class Day	**Chronological Development of the IMT**
Day 5	Students explored the concepts of span, linear dependence, and linear independence
Day 6	The class worked to develop generalizations about linear independence and dependence
Day 9	The groundwork for the IMT began with connecting various aspects (span, pivots, linear combinations, and solutions) for non-square matrices. This was known as "Theorem 4" in class. Theorem 4: Suppose A is an m x n matrix. The following are equivalent: a. The columns of A span $\mathbf{R}^m$ b. A has a pivot in each row c. For every vector $\mathbf{b}$ in $\mathbf{R}^m$, $\mathbf{b}$ can be written as a linear combo of the columns of A. d. For every vector $\mathbf{b}$ in $\mathbf{R}^m$, there exists a solution to the equation $A\mathbf{x} = \mathbf{b}$.
Day 10	While investigating relationships between span, linear independence, and existence of unique solutions, the necessity for square matrices arose. Theorem 4 was altered to address square matrices, and new equivalencies were added. This became known as the "New Theorem" in class. New Theorem: Let A be an $n \times n$ matrix. Then the following are equivalent: 1. The columns of A span $\mathbf{R}^n$. 2. The matrix A has n pivots. 3. For every $\mathbf{b}$ in $\mathbf{R}^n$, there is a unique solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$. 4. For every $\mathbf{b}$ in $\mathbf{R}^n$, there is a unique way to write $\mathbf{b}$ as a linear combination of the columns of A. 5. A is row equivalent to the $n \times n$ identity matrix. 6. The columns of A form a linearly independent set. 7. The only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution. Note that the New Theorem is identical to the first 7 equivalencies of the Invertible Matrix Theorem in Figure 3.2.
Day 17	The class worked to develop conjectures about which types of matrices were invertible. Determined that matrices with rows or columns of zeroes, or any version of linearly dependence column vectors, were not invertible. Also determined that only square matrices could be invertible. Began the proof of "if the columns of A are linearly independent, then A is invertible," but did not complete it.
Day 18	The instructor began class by finishing the proof that had not been completed the previous class period. Thus, the equivalence between a matrix A being invertible and the columns of A forming a linearly independent set developed. The following three equivalencies (were added to the New Theorem. Furthermore, the New theorem was renamed 'The Invertible Matrix Theorem.' The Invertible Matrix Theorem: Let A be a square $n \times n$ matrix. Then the following are equivalent. ... 8. A is invertible 9. There exists an n x n matrix C such that $CA = I$ 10. There exists an n x n matrix D such that $AD = I$

Table 3.2, continued. Chronological development of the Invertible Matrix Theorem.

Class Day	Chronological Development of the IMT
Day 19	Further explorations into linear transformations led to the notions of one-to-one and onto. The class defined the terms domain, codomain, range, onto, and one-to-one. They began working through examples and nonexamples of both onto and one-to-one transformations.
Day 20	After conferring with their group members, the students presented work they had done over spring break about generating examples of transformations that were and were not onto and one-to-one for the cases of $m < n$, $m = n$, and $m > n$. Upon investigating how it would be possible to have a transformation that was both one-to-one and onto, 11 and 12 were added to the IMT. The Invertible Matrix Theorem: Let A be a square $n \times n$ matrix. Then the following are equivalent: ... ... 11. The transformation $\mathbf{x} \rightarrow A\mathbf{x}$ is one-to-one 12. The transformation $\mathbf{x} \rightarrow A\mathbf{x}$ maps R^n onto R^n.
Day 24	Determinants had been introduced the previous class day as a way to think about the area or volume of images after transformations. On Day 24 students investigated how determinants was related to linear independence and invertibility. 15 was added to the IMT. The Invertible Matrix Theorem: Let A be a square $n \times n$ matrix. Then the following are equivalent: ... ... 15. Det $A \neq 0$
Day 31	Students discussed the Invertible Matrix Theorem in small group as well as in whole class. They were given a set of 16 cards, each with one of the statements from the IMT, and asked to do a variety of sorting tasks together, such as arrange them into piles that "went together," pick the most obvious and least obvious pair of equivalent statements, etc.

For each of those ten days, I watched the video from Abraham's small group (which also captured work in whole class) created a chronological summary of that class day's events. This included time-stamped information about the topic of discussion, the tasks or problems being investigated, and instances of small group work for that day. These descriptions allowed me to know which sections of the video data were relevant for the research topics regarding the IMT.

The video for each of the seven days was transcribed completely by a professional transcriber. After checking the relevant sections of the transcript for accuracy and completeness, I created an argumentation log that identified each claim and supporting justification given. Each argument was numbered and then compiled sequentially into an argumentation log for that day. The creation of argumentation logs followed the methodology developed by Rasmussen and Stephan (2008) for the documentation of collective activity. Their methodology, which was based on the analytical tool of Toulmin's model of argumentation, is altered here to account for the additional use of adjacency matrices, which will also utilize these argumentation logs as data sources. These argumentation logs are listed as appendices within the appropriate chapters.

For reliability purposes, I had a member of the research team independently create argumentation logs for portions of the first three days of analysis. We then discussed the argumentation log by comparing and defending each coding. This method of checking for disconfirming evidence for what constitutes an argument can be referred to as investigator triangulation (Denzin, 1978). Through this process, a high level of reliability was reached, and I completed the remainder of Phase 1 analysis on my own, reporting my findings back to the research team on a regular basis. The small group discussion and individual interview data served to address the second research question only. The process for data reduction of these data sets was similar to the process of data reduction of whole class discussion video data.

Once the data sets of whole class discussion, small group discussion, and individual interviews were reduced according to the aforementioned steps, each data set was considered twice—once for the construction of Toulmin schemes and once for adjacency matrices analysis. Utilizing these analytical tools constitute Phase Two and Phase Three of the data analysis process, respectively. In the following two sections, I specify Phase Two and Phase Three in detail by (a) defining what the two tools are, (b) giving an example of each, (c) detailing how I will construct adjacency matrices and Toulmin schemes, and (d) explaining how each tool will be used on both the individual and collective levels of both research questions.

3.4.2 Phase Two: Construction of Toulmin Schemes of Argumentation

The Toulmin model of argumentation (Toulmin, 1969) is based upon a distinction between logical arguments and substantial arguments. Toulmin claimed that arguments presented in discourse in some socio-historical context for a particular purpose (what he called substantial arguments), such as a justification for an already established claim in the court of law, often are structurally distinct from the formal, deductive structure of logical arguments thought to be inherent and absolute. In other words, some aspects of socially presented arguments are context dependent yet still function as acceptable justifications for particular claims. Toulmin described six main components of a substantial argument: claim, data, warrant, backing, qualifier, and rebuttal. The first three of these—claim, data, and warrant—are seen as the core of an argument. According to this scheme, the claim is the conclusion that is being justified, whereas the data is the evidence that demonstrates that claim's truth. The warrant is seen as the explanation of how the given data supports the claim, and the backing, if provided, demonstrates why the warrant has authority to support the data-claim pair.

This work has been adapted by many in the fields of mathematics and science education research (Krummheuer, 1995; Rasmussen & Stephan, 2008; Yackel, 2001) as a tool to assess the quality or structure of a specific mathematical or scientific argument and to analyze students' evolving conceptions by documenting their collective argumentation (Erduran, Simon, & Osborne, 2004; Inglis, Mejia-Ramos, & Simpson, 2007; Weber, Powell, & Lee, 2008). Noting that the particular statements that students use in an argument are situation-specific, Yackel (2001) states that, "what constitutes data, warrants, and backing is not predetermined but is negotiated by the participants as they interact," and the Toulmin model of argumentation is

> useful as a methodological tool for documenting the collective learning of a class because it provides a way to demonstrate changes that take place over time. Further it helps to clarify the relationship between the individual and the collective, that is, between the explanations and justifications that individual students give in specific instances and the classroom mathematical practices that become taken-as-shared. (p. 7)

My study utilized Toulmin's model to analyze structure of individual and collective exchanges both in isolation and as they shift over time. These analyses are the subjects of the first half of both Chapter 4 and 5, so I provide only an example here.

3.4.2.1 An example. To illustrate the Toulmin model of argumentation, consider the following segment of transcript, which was also seen in Chapter 1. Recall that the class had been asked to investigate the following properties of specific matrices and the associated transformations: domain, codomain, one-to-one, onto, and invertibility. Towards the end of a whole class discussion regarding this, the instructor initiated the following conversation:

Instructor: So let's make a note here of what we said so far about invertible...So what was the thing that you guys just said, how did you know it was invertible? Let's say that again.
Bill: It's linearly independent.
Instructor: If 'it's' meaning…?
Jesse: The vectors of the matrix.
Instructor: ...If the column vectors of *A* are linearly independent, then you guys are saying it's invertible because of that. Are you guys able to explain why that should give invertibility? I think you said something, or go ahead, Josiah.
Josiah: When they're linearly independent, there's only one path you can take to get to it, so in order to get back, there can only be one answer to get back. Whereas if they're dependent on each other, then depending on how you got there, would determine how you get back, what vector changed, so you don't have the right information again.
Instructor: Jesse...Yeah?
Jesse: Also, if they're dependent, in the RREF, you'll have a zero row, so it will be like you're losing information when you're trying to go back...If they are dependent, then their RREF will have a zero row.
Instructor: Okay, columns dependent [writes] and so you're saying that the RREF of *A* has a zero or row of zeros.
Jesse: Right, so then if you try and, if you invert that, you can't, because it's like you're losing that information from that row.

In lines 2-3, the instructor repeated a claim the class had stated (the matrix *A* is invertible) as she asked how they knew that claim was true (i.e., asked for data to support the claim). This exchange is documented in Figure 3.3. Upon hearing Bill's reply, the instructor pushed for clarification by what he meant by "it." After Jesse explained that "it" means the vectors of the matrix *A*, the instructor restated this data-claim pair and asked the students for a warrant—*how* did they know the data supported their claim? This exchange is in Figure 3.4.

	Example, Argument 1
Claim	The matrix *A* is invertible (Instructor, lines 2-3)
Data	It's linearly independent (Bill, line 4)

Figure 3.3. Example 1 of a Toulmin scheme.

	Example, Argument 2
Claim	Matrices with an RREF with a row of zeroes are not invertible (Jesse, lines 23-24)
Data	It's like you're losing that information from that row (Jesse, line 24)
Warrant	When they're linearly independent, there's only one path you can take to get to it, so in order to get back, there can only be one answer to get back (Josiah, lines 11-13)
Backing	Whereas if they're dependent on each other, then depending on how you got there, would determine how you get back, what vector changed, so you don't have the right information again." (Josiah, lines 13-16)

Figure 3.4. Example 2 of a Toulmin scheme.

Next, the instructor did not comment on Josiah's justification but rather called on another student. Jesse, in his justification, also mentioned the notion of "getting back," but added in information about the RREF of the matrix *A*.

	Example, Argument 3
Claim	The matrix *A* is invertible (Instructor, line 8)
Data	The column vectors of *A* are linearly independent (Instructor, restating students' statements, line 7)
Warrant	If they are dependent, then their RREF will have a zero row, so it will be like you're losing information when you're trying to go back (Jesse, lines 18-20)

Figure 3.5. Example 3 of a Toulmin scheme.

Notice that Jesse did not elaborate upon what Josiah might have meant by "getting back." Rather, Jesse offered up yet another connection between a linearly dependent set and a noninvertible matrix—that of an RREF with a row of zeroes. In fact, the idea of "getting back" went from being a warrant in Josiah's argument to being part of the warrant in Jesse's. Does this imply that the idea of "getting back" was functioning as-if the class community understood it to be synonymous with a matrix being invertible? With such a short data set in the transcript above, such strong claims cannot be substantiated here. Rather, this serves as an abbreviated example of how such analysis—how the collective reasoning about the IMT changes over time—took shape within the study.

3.4.2.2 Constructing Toulmin schemes. The data sources for constructing Toulmin schemes were each of the exchanges from the argumentation logs created in Phase One. For analysis at the collective level, I used exchanges from the argumentation logs created from whole class discussion, whereas analysis at the individual level arose out of the argumentation logs created from small group work and individual interviews, as well as a small number from whole class discussion. For instance, the previous example with the whole class discussion would be one exchange from the collective argumentation log.

In order to construct a Toulmin scheme of a given exchange, I followed Phase One of the methodology detailed by Rasmussen and Stephan (2008). For reliability purposes, at least one member of the research team also constructed Toulmin schemes for a sample of the arguments from the argumentation log. As Rasmussen and Stephan state, this process is used to verify and/or refute the argumentation log for "each instance of a claim or a conclusion. We [the researchers] come to agreement on the argumentation scheme by presenting and defending our identification of the argumentation elements (i.e., the data, conclusion, warrant, and backing) to each other" (p. 199). Finally, each Toulmin scheme was labeled according to the day on which it occurred and which argument it was (e.g., Argument 20.1, 20.2, etc.).

3.4.2.3 Relevance to the research questions. As previously stated, the Toulmin model has proven a useful tool for documenting mathematical development at a collective level. It was also an illuminating tool for documenting the ways in which Abraham reasoned about the Invertible Matrix Theorem as well. As was discussed in the individual Data Collection section, many opportunities arose throughout the semester asking a student to justify (that is, provide data, warrant, and possibly backing) for a particular pre-determined claim about the IMT.

3.4.3 Phase Three: Construction of Adjacency Matrices

Adjacency matrices are representational tools from graph theory used to depict how the vertices of a particular graph are connected (Chartrand & Lesniak, 2005). These matrices can be used to represent data from a variety of graph forms. For instance, an undirected graph is just as it sounds—it only tallies whether or not an edge between vertices is present. A directed graph, on the other hand, not only indicates the presence of an edge, it also displays the directionality of the vertices' connection. In this study, I created adjacency matrices that correspond to directed graphs in which the vertices are the statements in the Invertible Matrix Theorem (or students' explanations of those statements) and the edges are directed in such a way as to match the implication offered by the student. The developed adjacency matrices were n x n, where n is the number of recorded relevant yet distinct statements made by students in any given explanation. The rows were the 'p' and the columns were the 'q' in statements of the form "p implies q" or "another way to say p is q." For example, the statement, "I know that because the determinant of A is zero, matrix A is not invertible," is of the form *p implies q* (determinant of A equals 0 implies A is not invertible), whereas a statement such as "the determinant of A is zero means that the parallelogram determined by columns of A has no area" is of the form *another way to say p is q*.

Each matrix entry a_{ij}, where a_{ij} is the entry in the i^{th} row and j^{th} column of a given adjacency matrix, was determined by $a_{ij} = c$, where c was the frequency of the connected statement. Thus, the possible values for c—that is, for each component in an adjacency matrix—ranged from zero through the positive integers. Furthermore, the rows and columns will be comprised of the same statements; in other words, $p_i = q_i \ \forall i,\ 0 \le i \le n$. As previously stated, each p_i represents the statements used by students as they reason about the IMT.

3.4.3.1 An example. For instance, consider the following excerpt from an interview with Henry, which was also considered in Chapter 1. Henry was responding to the following question prompt:

> Suppose you have a 3 x 3 matrix *A*, and you know that *A* is invertible. Decide if each of the following statements is true or false, and explain your answer.
> a. The column vectors of *A* are linearly independent.

Henry: So if it's invertible, the determinant isn't 0. And if the determinant is 0, then they're dependent. If they're independent, then it's not 0, so that means it's invertible, so (a) is true.

	A is invertible	det *A* = 0	det *A* ≠ *0*	The columns of *A* are linearly independent	The columns of *A* are linearly dependent
A is invertible			1		
det *A* = 0					1
det *A* ≠ *0*	1				
The columns of *A* are linearly independent			1		
The columns of *A* are linearly dependent					

Figure 3.6. Adjacency matrix for Henry's response to the interview prompt.

What can be noticed from an adjacency matrix such as this? Taking a closer look, there are some patterns inherent in the matrix itself because the columns and the rows are the same. We can see from the symmetry of a_{31} and a_{13} both being highlighted that Henry treated those two statements as equivalent. That is, his argument displayed the notion that *A* is invertible if and only if the determinant of *A* is nonzero. Furthermore, he used the notion of determinants two other times—that nonzero determinants implied linear independence and that linear dependence implied a zero determinant (the contrapositive of the first statement). Finally, if the total frequency across the columns is considered, one can see that 'det *A ≠ 0'* has the highest density across the columns, being used in two of the four statements.

As evidenced in the above adjacency matrix, the rows and columns are composed of statements that Henry made as he reasoned about two statements in the IMT—a matrix *A* is invertible and the columns of *A* form a linearly independent set. Mathematically, these two statements are, in fact, equivalent for an n x n matrix *A*. However, the purpose of this analysis was not to determine whether or not a student such as Henry could correctly report that the statements are equivalent. Rather, the purpose of the analysis was to chart *how* students such as Henry explained that the truth of the one statement guarantees the truth of another. What was the content of the argument he constructed as he reasoned about with the ideas in the IMT?

3.4.3.2 Constructing adjacency matrices. The data sources for constructing adjacency matrices was each of the arguments from the argumentation logs created in Phase One. For analysis at the collective level, I used arguments from the argumentation logs created from whole class discussion, whereas analysis at the individual level arose out of the argumentation logs created from small group work and individual interviews. For instance, the previous example with Henry would be one argument from Henry's individual interview argumentation log. In order to construct an adjacency matrix, an argument is broken down into clauses of separate ideas. Again, considering Henry's example above, the clauses would be: (a) It's invertible, (b) The determinant isn't zero, (c) The determinant is zero, (d) They're dependent, (e) They're independent, (f) It's not zero, (g) It's invertible, and (h) (a) is true. Furthermore, the clauses are connected through conjunctions such as *if, then, so,* and *that means.* Next, each clause became an ordered row and column for a given adjacency matrix. Next, the original transcript was considered, both clauses and conjunctions, to determine where frequency indicators should be placed in the matrix. As previously stated, the rows were the '*p*' and the columns were the '*q*' in statements of the form "*p* implies *q*" or "another way to say *p* is *q*." Note that, if taken alone, the argument requires a small level of inference to analyze. Henry's argument contains many pronouns such as *it's* and *they're*. For instance, in (g), does Henry mean the matrix *A* is invertible? Is it the determinant that is not zero in (f)? He uses the same pronoun in both cases. Taken alone, this argument requires a small level of inference on the part of the researcher to infer what Henry most likely meant. Thus, in order to establish construct validity and reliability of relevant data, I used multiple sources of evidence. Using these multiple data sources—classroom video, transcript, researcher field notes of direct observation, post-lesson debriefing notes, and student written work—allowed for data triangulation (Mathison, 1998). Data triangulation leads to more reliable results and helps address potential problems with construct validity because multiple sources of evidence essentially provide multiple measures of the same phenomenon.

3.4.3.3 Relevance to the research questions. Adjacency matrices were also used to analyze arguments made at the collective level during whole class discussion. These arguments were comprised of statements from one or many students in the class as meaning was negotiated collectively through participation in the classroom. This is a relatively novel analytical tool in mathematics education research in a similar fashion as within this study. In Selinski, Rasmussen, Zandieh, and Wawro (2011), we report on the power of adjacency matrices for analyzing student responses during one interview. Thus, the present study furthers the use of adjacency matrices as an analytic tool by using them at the collective level as well as to document change over time.

What has been described in Phases 2 and 3 is a very static, snapshot-view type of analysis of individual and collective arguments that occurred throughout the semester. A coordination of these snapshot views, on both the individual and the collective level, constituted Phases 4 and 5 of this proposed study's analysis. Whereas Phases 2 and 3 investigated the content and structure of each particular instance of reasoning about the Invertible Matrix Theorem, Phases 4 and 5 investigated how the content and structure of these arguments compared to each other and/or shifted over time. In Phase 4, the primary means of coordination were inspired by the methodology for documentation of classroom mathematics practices, as detailed in Rasmussen and Stephan (2008) and Cole et al. (2010). In Phase 5, the primary means of coordination were inspired by the genetic analytical work of Saxe (2002), Saxe et al. (2009), and Tiberghien and Malkoun (2009).

3.4.4 Phase Four: Ontogenetic Analysis of Toulmin Schemes

In Phase 2, the various exchanges identified in Phase 1 were analyzed individually through the use of Toulmin's (1969) model of argumentation. In Phase 4, these snapshot views were coordinated and compared within both the individual and within the collective level. In particular, Phase 4 was based on the work of Rasmussen and Stephan (2008), who have developed a rigorous methodology for documenting the collective mathematical development of a classroom community. The authors define collective activity as "a social phenomenon in which mathematical or scientific ideas become established in a classroom community through patterns of interaction" (p. 195). Furthermore, they discuss documenting collective activity through two evidences for knowing when an idea functions in the collective as if its understanding is shared by the individuals in the collective: (a) when the backings and/or warrants for an argument no longer appear in students' explanations, and (b) when students use a previously justified claim as unchallenged justification for future arguments (p. 200). In addition, Cole et al. (2011) posit another way to identify when reasoning with a particular idea functions as-if shared: if that idea appears repeatedly as the data for a variety of different claims. This last aspect is different than the first two in that it takes note of repeated use and consistency in arguments rather than focusing on the shifting content and structure of arguments. This was particularly useful when analyzing the ways in which students reasoned about the Invertible Matrix Theorem because of the Theorem's inherent nature as a unifying structure linking sixteen equivalent statements. Of these sixteen statements, were there any that are repeatedly used at data for a variety of claims? The ease and diversity of the repeated use of that concept would indicate that the concept functions as-if shared in the collective community.

The notion of using Toulmin's model to document when ideas function as-if shared in the classroom was somewhat analogous to Saxe's notion of ontogenetic analysis, which is described in Phase 5. Ontogenetic analysis concerns shifts in patterns of thinking over time, and these shifts are documented by analyzing a learner's microgenesis process at various stages of development. In particular, Saxe et al. (2009) state that microgenesis is "the process of moment-to-moment construction of representations as individuals work to turn representational forms into means to serve mathematical functions" (p. 209). Thus, each individual Toulmin scheme created is analogous to a microgenetic analysis regarding an explanation offered in order to serve the function of justifying a connection or solving a novel problem, and the compilation and comparison over time of these Toulmin schemes is an ontogenetic analysis of students' development regarding the Invertible Matrix Theorem. Ontogenetic analysis considers the shifts in patterns of thinking over time, and the three aforementioned criteria for establishing when ideas function as-if shared accomplish that same purpose. All three criteria analyze the changing structure of explanations that accompany claims made by students regarding the Invertible Matrix Theorem. This ontogenetic analysis of how students reason about the Invertible Matrix Theorem facilitated investigating aspects such as what ideas in the IMT were used most frequently as data to support particular claims, how the structure of these explanations changed over time, and the like.

While the aforementioned three criteria were developed by Rasmussen and Stephan (2008) and Cole et al. (2011) to document when concepts function as-if shared in the *collective*, I adapted these three criteria to analyze the ontogenetic progress of Abraham as well. In this aspect, the notion of an idea functioning as-if shared loses meaning; rather, utilizing the three criteria facilitated the documentation of the shifting structure of Abraham's explanations regarding the Invertible Matrix Theorem. This ontogenetic analysis allowed me to develop a rich description of, for instance, what ideas from the IMT were the most salient for Abraham, when he became confident enough in a particular equivalency that a warrant dropped off of an argument, etc. This, along with the individual level of analysis of Phase 5, contributes to developing a deep understanding of (a) how the ways in which individual students reason about the IMT change over time, and (b) the various ways that students reason with the IMT to solve novel problems over the course of the semester.

3.4.5 Phase Five: Ontogenetic Analysis of Adjacency Matrices

Researchers from both educational and anthropological fields (Ercikan & Roth, 2006; Saxe, 1999; Saxe et al., 2009) have investigated the progression of student development through what is sometimes referred to as *genetic analysis.* For instance, Saxe and Esmonde (2005), who studied, over a course of more than twenty years, how the function of a particular word form shifted for a central New Guinea tribe. In studying this word form, which dealt with the tribe's counting system, Saxe and Esmonde asked, "How do new collective systems of representation and associated mathematical ideas arise in the social history of a social group?" (p. 172). The proposed inquiry into how students' use of the Invertible Matrix Theorem as a tool for reasoning changes over time is an analogous investigation. To study how students interacted with a theorem that itself was developing over the course of the semester necessarily dealt with how mathematical ideas arose in the social setting of a classroom. In addition to Saxe and his colleagues, other researchers have recently taken up research agendas that investigate the emergence, development, and spread of ideas in a classroom community over time (e.g., Cobb, Stephan, McClain, & Gravemeijer, 2001; Schwarz, Dreyfus & Hershkowitz, 2009; Stephan & Rasmussen, 2002; Tiberghien & Malkoun, 2009). The primary means of genetic analysis conducted in Phase 5 were inspired by the analytical work of Saxe (2002), Saxe et al. (2009), and Tiberghien and Malkoun (2009).

Saxe et al.'s (2009) methodology for the genetic analysis of children's mathematical development considers three distinct developmental processes: microgenesis, ontogenesis, and sociogenesis. They define *microgenesis* as the short-term process by which individuals construct meaningful representations in activity; *ontogenesis* as the shifts in patterns of thinking over the development of individuals; and *sociogenesis* as the reproduction and alteration of representational forms that enable communication among participants in a community (p. 208). Phases 2/3 and 4/5 of the study made explicit use of the notions of microgenetic and ontogenetic analysis, respectively. For instance Phase 2 analysis—the construction of adjacency matrices for every exchange related to the IMT—could be seen as a microgenetic analysis of every exchange.

Microgenetic analysis concerns the construction of representational forms that serve a particular function in order to accomplish a particular goal (Saxe, 2002). For Saxe et al. (2009), an example of a microgenetic construction is when a student turns forms such as the number line into a particular meaning and uses it to accomplish a goal in activity. In the present study, the constructed representational forms witness their analog in students' given explanations or justifications regarding the IMT, which function to explain or justify connections within the IMT and uses for it in novel situations. Furthermore, the compilation of these microgenetic analyses served as the data points for ontogenetic analysis. Saxe's work with developmental analysis using microgenetic, ontogenetic, and sociogenetic distinctions has mainly considered one given class day as a unit of analysis. The study extended this work by analyzing larger sections of time—anywhere from a section of one day to the entirety of the semester as the unit of analysis for ontogenetic analysis.

Furthermore, the inherent structure of adjacency matrices provided a convenient way to draw quantifiable analyses, such as the density (Tiberghien & Malkoun, 2009) of the various ideas related to the Invertible Matrix Theorem used over the course of the semester. Thus, the other main type of ontogenetic analysis in Phase 5 was a macrogenetic, holistic analysis of how the adjacency matrices compared and changed over time. For instance, which concepts from the Invertible Matrix Theorem were used most often overall by students as they reasoned about novel problems? Were there some concepts from the IMT that were used consistently over the course of the semester, whereas others dropped off? This analysis was inspired by the work of Tiberghien and Malkoun (2009) who consider the notions of density and continuity of ideas as analytical frames in genetic analysis.

3.4.6 Phase Six: Coordination of Analyses Across the Two Analytical Tools

The final phase of analysis concerned comparing and coordinating the analytical results from Phases 2 and 3, as well as from Phases 4 and 5 (see Table 3.1). One layer of comparison—between the results of Phases 2 and 3—occurred at the discrete level. This comparison illuminated aspects of students' ways of reasoning that may not be apparent by considering that same argument through only one of the analytical lenses. The other layer of comparison—between the results of Phases 4 and 5—occurred by comparing and contrasting the aspects of ontogenetic progress that were illuminated by these two phases. This coordination between the analytical results from adjacency matrices and Toulmin schemes that was the focus of Phase 6 was possible because the data set was the same for both veins of analysis. That is, because the objects of inquiry (i.e., the reduced data set from Phase 1) were the same for both analytical tools, a comparative analysis between them was facilitated. Various adjacency matrices and Toulmin schemes could be compared directly for any given argument by comparing results of

Phases 2 and 3, whereas developmental comparisons between the two analytical tools could be accomplished by directly comparing the results of Phases 4 and 5.

3.5 Conclusion

In this chapter, I detailed the methods by which I will analyze the proposed research questions. I began the chapter by expanding upon what is entailed in conducting a classroom teaching experiment, both theoretically and practically. The remainder of the chapter was divided into three main sections: (a) the setting and participants for the teaching experiment, (b) the data I collected and the methods by which I collected them, and (c) the methodology for the analyses I conducted on these data.

Chapter 4 presents the results of my analysis at the collective level. In the first half of the chapter, I present results of the microgenetic and ontogenetic analysis via Toulmin's Model, and the second half contains results from the microgenetic and ontogenetic analysis via adjacency matrices. Chapter 5 follows a parallel structure but concerning results of how Abraham reasoned about the IMT over time. Chapter 6 discusses a comparison of the two analytical tools, including limitations and affordances of both in carrying out the presented analysis. It also includes implications for teaching and concludes with avenues for future research.

CHAPTER FOUR: ANALYSIS AT THE CLASSROOM LEVEL

This chapter discusses the results of the first research question: How did the classroom community reason about the Invertible Matrix Theorem (IMT) over time? Specifically, what was the nature of argumentation structure, and what ways of reasoning functioned as-if shared in whole class discussion? To address these questions, I utilized two different analytical tools, Toulmin's model and adjacency matrices, on relevant classroom discourse throughout the semester. I also utilized the cultural change notions of microgenesis and ontogenesis as strands through which I coordinated the structure of argumentation within the classroom discourse (microgenetic level) and how the forms and functions of these various arguments shifted over time (ontogenetic analysis).

This chapter has two main sections. In the first section, I detail results from both the microgenetic and ontogenetic analyses from using Toulmin's model. The microgenetic analysis revealed, in addition to Toulmin's classic 6-part model of argumentation, four distinct expanded structures of Toulmin's model that were utilized when reasoning about the IMT during whole class discussion. I provide 1-2 examples of each of these expanded structures. The ontogenetic analysis revealed ten normative ways of reasoning that characterize student reasoning about the IMT during class discussion. In the second section, I detail both the microgenetic and ontogenetic analyses from utilizing adjacency matrices as an analytical tool. The analysis revealed that the use of concept statements from the IMT versus interpretations of those concepts statements depended on the purpose of the particular argument. Microgenetic analysis also revealed the importance of reasoning about non-square matrices in order to develop ways of reasoning about the equivalence of concepts in the IMT (which are for square matrices). In addition to how the structure of arguments changed over time, ontogenetic analysis revealed the concepts most central to the class community's development as a whole. I conclude with a discussion of the results and a reflection on and comparison of the separate analytical methods.

4.1 Toulmin's Model of Argumentation

Toulmin's 6-part scheme for substantial argumentation consists of *claim, data, warrant, backing, qualifier,* and *rebuttal*, each with its explicit role in any given argument (see Figure 4.1). The operational definition for *argument* in this present study on student reasoning in linear algebra is "an act of communication intended to lend support to a claim" (Aberdein, 2009, p. 1). To address the research question of how the classroom community reasoned about the IMT throughout the semester, I coded arguments that explicitly involved developing meaning for the concepts in the IMT or for relationships between the ideas in the IMT, and each of these arguments occurred during whole class discussion.

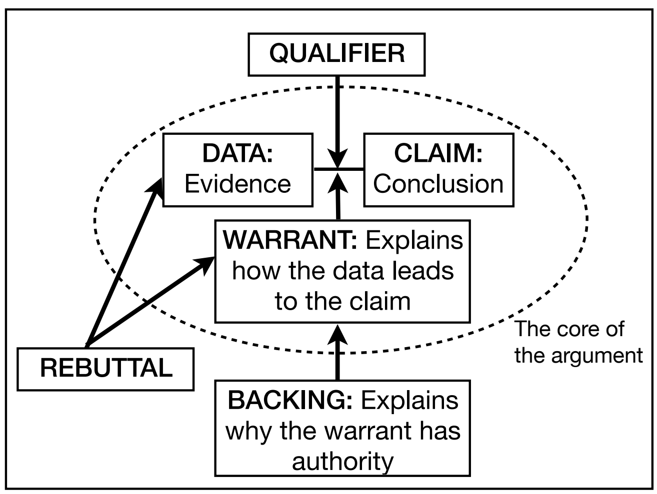


Figure 4.1. Toulmin's Model of Argumentation

The claims made about these arguments occur at two different levels. First, microgenetic analysis considers the structural layout as well as the mathematical content of each distinct argument that occurred at different moments of whole class discussion throughout the semester. Ontogenetic analysis, by considering the collection of these distinct arguments and their relationship to one another, considers the mathematical development of the collective, thus addressing how the classroom community reasoned about the IMT over time. A meta-analysis of the argument structures as a whole, as well of as shifts in the form and function of the argumentation, are also part of the ontogenetic analysis. For brevity and clarity, the results presented in the Microgenetic Analysis section are restricted to explanations and examples of the four types of expanded Toulmin scheme structures found in my analysis, rather than of all arguments. The Ontogenetic Analysis section considers a wider swath of these arguments and thus provides a comprehensive account of the mathematical development over time. In doing so, it necessarily involves microgenetic analysis of the arguments involved. Thus, while the Microgenetic Analysis section focuses on novel argumentation structure as its results, analysis at the microgenetic strand is given in the Ontogenetic Analysis section as well. The entirety of the argumentation logs can be found in Appendix 4.1.

Over the course of the ten days of classroom data analyzed, 118 arguments were coded using Toulmin's model of argumentation. Microgenetic analysis of these 118 arguments revealed four different complex structures of argumentation that were utilized when reasoning about the IMT during whole class discussion, each of which is an expansion of Toulmin's classic 6-part scheme: (a) Embedded structure, (b) Linked structure, (c) Proof by Cases structure, and (d) Sequential structure. Ontogenetic analysis of the arguments, with a focus on span, linear independence, one-to-one, and onto, revealed ten ways of reasoning that functioned as-if shared in the classroom. These ways of reasoning focused on drawing conclusions about the four aforementioned concepts across various scenarios. Furthermore, these normative ways of reasoning were gathered around two common themes to constitute two classroom mathematics practices. To highlight both of these results—the various argumentation structures and the ideas that functioned as-if shared, I present a selection of data from the following ten class days. A description of relevant moments from each of these days is given in Chapter 3.

1. Day 5, when students explored the concepts of span, linear dependence, and linear independence;
2. Day 6, when the class worked to develop generalizations about linear independence and dependence;
3. Day 9, when the class investigated the relationships between linear independence, span, and the row-reduced echelon form of matrices;
4. Day 10, when the class focused on equivalent statements for square matrices and developed what was known as the New Theorem;
5. Day 17, when invertibility was explored in conjunction with other concepts;
6. Day 18, when the New Theorem was renamed the IMT;
7. Days 19, when the concepts of one-to-one and onto transformations were introduced and explored;
8. Day 20, when the class related one-to-one and onto with ideas in the IMT;
9. Day 24, when "*A* has a nonzero determinant" was added to the IMT; and
10. Day 31, when the IMT was an explicit focus of reflection for the day.

4.1.1 Microgenetic Analysis via Toulmin's Model

After narrowing the data set to those class sessions most instrumental to the development of the Invertible Matrix Theorem, I coded 118 different arguments using Toulmin's model of argumentation, all of which occurred during whole class discussion during one of the ten class sessions. Of these coded arguments, 81 of them were of a form that consisted of some subset of the six parts of the layout (see Table 4.1). While permutations of a given layout were traced in analysis (for instance, C-D-Q-W and C-D-W-Q), they were counted in the same category of layout types in Table 4.1.

Table 4.1. Frequency of the various 6-part Toulmin scheme layouts over the 10 class days.

	Day 6	**Day 9**	**Day 10**	**Day 17**	**Day 18**	**Day 19**	**Day 20**	**Day 24**	**Day 31**	**Total**
C-C							1			1
C-D	1		1	2			6	2	1	13
C-D-W	2	5	2	6	4	5	3	2	2	31
C-D-W-B		3	1			1	7	2	3	17
C-D-W-B-Q	1	1						1		3
C-D-W-B-R	1						1			2
C-D-Q	1	1		1		1			1	5
C-D-W-Q		1	3	1		1		1		7
C-R			1							1
C-D-W-R-Q				1						1
										81

Additionally, there were another 15 arguments in which the instructor played a unique role in the development of the arguments. Rather than being a contributor to the argument directly (as seen in Figure 4.2, where she provides the warrant to someone's claim, for example), she called for data, warrants, or backing to be provided by either the speaker or another member of the class. This speaks to the teacher's unique role to move the mathematical agenda forward as well as to push for developing the social norms of explaining one's thinking and justifying one's claims (Rasmussen & Marrongelle, 2006; Rasmussen, Zandieh, & Wawro, 2009). An expanded version of Table 4.1 is given in Table 4.2, with the second number in a given cell indicating the additional layouts of this type, but within the particular structure indicated by the row. Thus, the last column containing the total count increased by 15 from Table 4.1. Information regarding when during the argumentation the teacher called for elaboration, although traced in the analysis, is omitted in Table 4.2.

Table 4.2. Frequency of the various 6-part Toulmin scheme layouts over the 10 class days, with arguments containing a push for D, W, or B by the instructor noted with a secondary number.

	Day 6	**Day 9**	**Day 10**	**Day 17**	**Day 18**	**Day 19**	**Day 20**	**Day 24**	**Day 31**	**Total**
C-C							1			1
C-D	1		1	2 / 1			6	2	1	13 / 1
C-D-W	2	5 / 2	2 / 1	6	4	5 / 1	3 / 2	2	2	31 / 6
C-D-W-B	0 / 1	3 / 1	1			1	7	2 / 1	3	17 / 3
C-D-W-B-Q	1	1					0 / 1	1		3 / 1
C-D-W-B-R	1						1			2
C-D-Q	1	1		1 / 1		1			1	5 / 1
C-D-W-Q		1	3	1		1		1 / 1		7 / 2
C-R			1							1
C-D-W-R-Q				1 / 1						1 / 1
										81 / 15

Within the remaining 22 arguments, the 6-part layout, with each part occurring at most once, seemed insufficient to capture the complexity of the arguments that transpired during whole class discussion. Some layouts were a string of the six components (such as C-D1-Q-D2-W2-Q-D3-W) and occurred, for instance, when multiple members of the classroom were working together to justify a relatively new claim (such as why if the determinant of a matrix is zero, then the column vectors of that matrix have to be linearly dependent). Other arguments were structurally complex (such as a student proving a claim by presenting justifications for all possible cases) in ways that necessitated an expansion of some aspect of the original Toulmin's model of argumentation. The four varieties of this that I encountered in my analysis were:

1. *Embedded structure*: When data or warrants for a specific claim were so complex, they had minor embedded arguments within them;
2. *Proof by Cases structure*: When claims were justified using cases within the data and/or warrants;
3. *Linked structure*: When data or warrants for a specific claim had more than one aspect that were linked by words such as "and" or "also"; and
4. *Sequential structure*: When data for a specific claim contained an embedded string of if-then statements, where a claim became data for the next claim.

These four structures were adapted from work by Aberdein, a researcher in the fields of logic and humanities, who has done a variety of research regarding argumentation in mathematics. With respect to both Toulmin and mathematics, Aberdein notes that Toulmin presented a lone example of a mathematical proof diagrammed with his scheme: a proof that there are exactly five platonic solids (Toulmin, Rieke, & Janik, 1979, as cited in Aberdein, 2006). This argument, however, is a relatively concise mathematical proof. Aberdein makes the following observation considering Toulmin's model of argumentation:

> Toulmin's focus is on argumentation in natural language, not mathematics, although he is satisfied that the layout applies there as well ... One substantial shortcoming that this proof has as a model for how Toulmin's layout may be applied more generally is that it has only one step. Most mathematical proofs have many. (2006, p. 5)

Aberdein continues to explain how he expanded Toulmin's basic framework to include more complex proof structures, such as induction or proof by contradiction. My research and analysis focuses on whole-class discussion and argumentation ("in the natural language," to use Aberdein's term) that occurs in a college-level linear algebra class. Thus, much of the argumentation lies in a tension point between these two. Often, the complexity of their justifications did not seem adequately captured with only the "data-claim-warrant-backing" structure. As such, I adapted Aberdein's notion of the expanded layout in order to characterize these more complex structures. Consideration of the arguments in relation to each other is done during the subsequent section, Ontogenetic Analysis, where I discuss the mathematical development of the class over the course of the semester. The focus of this section, however, is to analyze individual arguments independent of how they relate to others.

4.1.1.1 Embedded Structure. I define an *embedded structure* as a Toulmin scheme within which one or more of the data, warrant, or backing is itself composed of a Toulmin scheme, minimally a C-D pair. A simple example of an embedded structure is provided in Figure 4.2. This argument occurred on Day 24 of the semester, during which, as described in Chapter 3, the class was investigating determinants and their connections to other ideas in the Invertible Matrix Theorem. The instructor was making explicit how the formula for a 2x2 matrix A, $\det A = ad - bc$, connected to the row-reduced echelon form of A. In previous class sessions, the class members had discussed why row-reducing an n x n matrix A augmented with the n x n identity matrix not only was a valid method to determine if A was invertible, but it also allowed you to compute A^{-1}. In other words, the class discussed why $\left[A \mid I\right] \sim \left[I \mid A^{-1}\right]$ "worked." On Day 24, the teacher revisited this computational method with the generalized matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The class had reached the point $\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} ac & bc & c & 0 \\ 0 & ad-bc & -c & a \end{array}\right]$ in their work when Argument 9 occurred (Figure 4.2).

The original 6-part Toulmin scheme is not sufficient to capture the complexity of this argument. In a Toulmin scheme, a warrant serves to explain why the data is relevant to the claim. For Argument 24.9, the warrant explains why having a zero in the bottom corner (the data) relates to not row-reducing to the identity matrix (the claim). The data-claim pair, seen in

Figure 4.2 as being part of the data, does not serve this purpose of connecting the data to the claim; rather, it serves to explain the data in more detail.

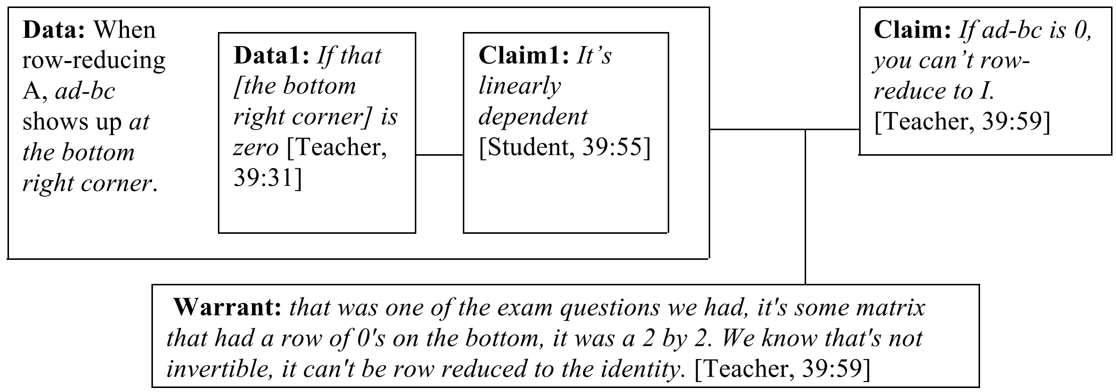


Figure 4.2. Argument 24.9, Embedded Structure: Explanation of why if the determinant of a 2x2 matrix is 0, the matrix cannot row-reduce to the 2x2 identity matrix.

Purely embedded structures are the least sophisticated or interesting of the four complex structures, yet they are foundational to the other three. In each of the remaining three complex structures—Proof by Cases, Linked, and Sequential—an embedded structure takes a more specific form within either the data, warrant, or backing of a given Toulmin scheme.

4.1.1.2 Proof by Cases Structure. I define a *proof by cases structure* as a Toulmin scheme within which the data is comprised of multiple scenarios that, when considered as a unit, create that data. This definition draws on Aberdein's (2006) Proof by Cases layout (see Figure 4.3). In his layout, there are multiple date-claim-warrant units embedded within the overall data, D_n, for the scheme. The relationship between these embedded schemes is that each is a distinct yet integral case that, when considered together, comprise the data for the claim C_n. The warrant, W_n, is the support that all of the minor data considered as an entity is a logically valid form of justification. I conjecture that this structure could also exist in such a way that the warrant or backing is composed of cases, rather than or in addition to the data. I did not, however, see examples of those in the data set for this study.

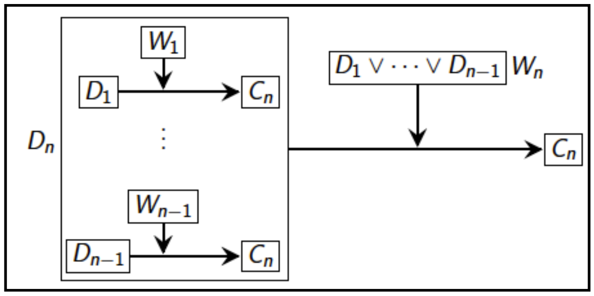


Figure 4.3. The general "Proof by Cases" layout (Aberdein, 2006, p. 9).

I provide two examples of the Proof by Cases structure. The first comes from Day 5, when students worked together in small groups to create examples of both linearly independent and linearly dependent sets of vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$ and to make generalizations regarding linear independence and dependence based on their work (see Figure 4.4 for the task).

LINEAR INDEPENDENCE AND DEPENDENCE: CREATING EXAMPLES

Fill in the following chart with the requested sets of vectors.

	Linearly dependent set	Linearly independent set
A set of 2 vectors in $\mathbf{R}^2$		
A set of 3 vectors in $\mathbf{R}^2$		
A set of 2 vectors in $\mathbf{R}^3$		
A set of 3 vectors in $\mathbf{R}^3$		
A set of 4 vectors in $\mathbf{R}^3$		

Write at least 2 generalizations that can be made from this table.

Figure 4.4. Task from Day 5: Create examples and make generalizations about linearly independent and linearly dependent sets in $\mathbf{R}^2$ and $\mathbf{R}^3$.

The argument presented here occurred when Justin explained his group's rationale for how they determined there was "no solution" for the cell in the chart that asked them to create an example of a set of three linearly independent vectors in $\mathbf{R}^2$ (see Figure 4.5). On the surface, it may seem like this argument is not directly related to how the classroom reasoned about the Invertible

Matrix Theorem; however, it is a rather relevant argument because it is one of the first instances in whole class discussion during which the concepts of span and linear independence (which *are* key concepts in the IMT) were discussed in conjunction with each other.

Instructor: So Justin, do you want to come up and explain your group's explanation for how you got 'no solution' on this?

Justin: So we're still in $\mathbf{R}^2$. So basically, let's just start with any random vector, let's call it that one [draws a vector in the first quadrant]. Now after we have 1 vector down, there's only basically two situations we could have. We can either have a vector that is parallel with this one, either another multiple or going the other way or whatever. Or we can have one that is not parallel, it doesn't have to be perpendicular, it can be anywhere. But it's either parallel or not. So if it's parallel, we already said that if we have two vectors that are parallel, we have a, they're dependent. But when we did our magic carpet-hoverboard, we had two that weren't parallel [draws a vector in the fourth quadrant], and we said the span of any two that aren't parallel, is all of $\mathbf{R}^2$. So if we have two that aren't parallel, we can get anywhere in $\mathbf{R}^2$, no matter where we throw in our third vector, we can get there with a combo of these two and make it back on that third one. So there can't be any solution, so there's no, as long as we have three vectors in $\mathbf{R}^2$, it has to be linearly dependent. Does that make sense, any questions?

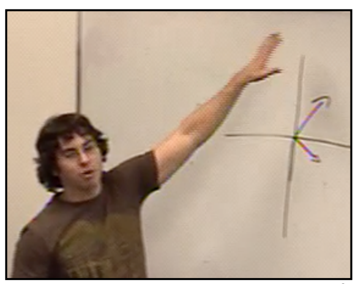


Figure 4.5. Justin explains why any set of 3 vectors in $\mathbf{R}^2$ must be linearly dependent.

Justin began his explanation by stating there were two only two cases to consider: when two of the three vectors in the set were or were not parallel ("parallel" was a term some students used to describe vectors that were scalar multiples of each other). In the first case, the set was already linearly dependent; in the second case, the addition of any third vector would cause the set to be linearly dependent. In the Toulmin analysis of Justin's explanation (see Figure 4.6), the claim is that any set of 3 vectors in $\mathbf{R}^2$ must be linearly dependent. Justin set up the structure of his data by stating, "there's only basically two situations." His data then was composed of two embedded arguments: a data-claim-warrant for the first case (labeled Data1, Claim1, and Warrant1 in Figure 4.6) and a data-claim-warrant for the second case (labeled Data2, Claim2, and Warrant2 in Figure 4.6). I refer to these sub-arguments as D_1 and D_2 in order to reference that they are embedded within the data of the main argument. Justin's warrant, finally, occurred when he stated why his data supported the claim. Justin's use of the word "so" in the statement, "so there can't be any solution," which he stated immediately after the two sub-arguments, indicates that, for him, the joint consideration of the sub-arguments constituted a valid data for the claim.

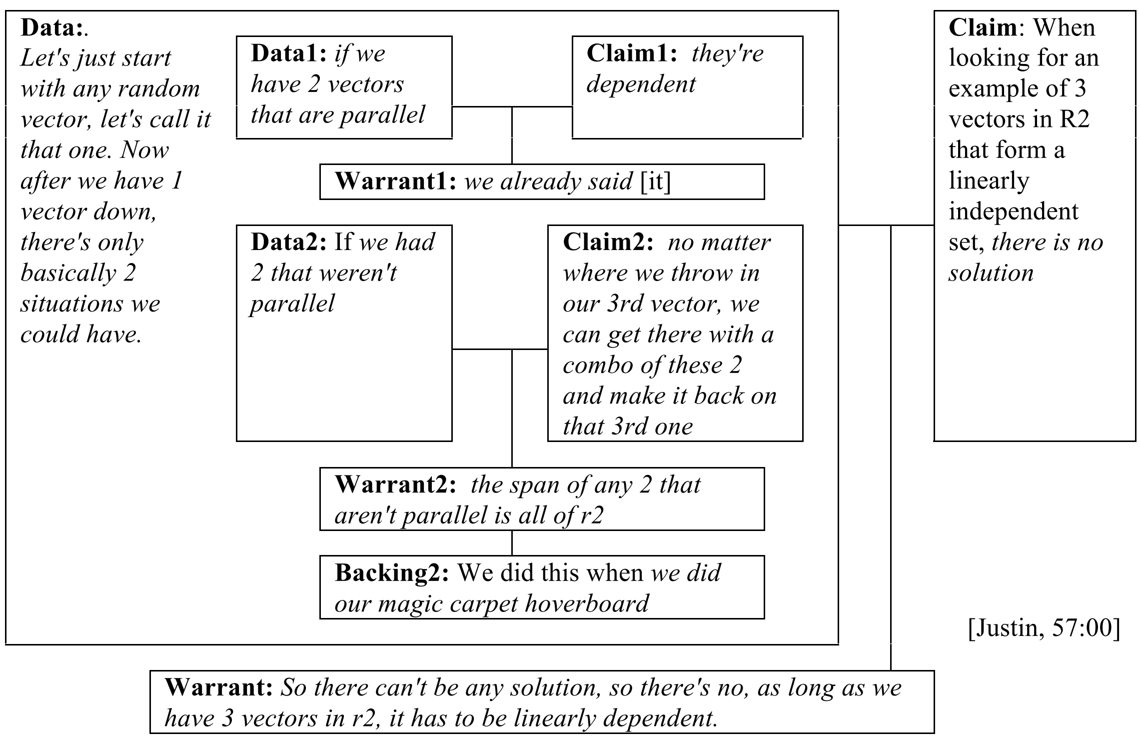


Figure 4.6. Argument 5.1, Proof by Cases structure: Justin explains why any set of 3 vectors in $\mathbf{R}^2$ must be linearly dependent.

A second example of the Proof by Cases argumentation structure is Argument 5 from Day 6. At the end of Day 5, the class had developed a list of four generalizations about linear independence and dependence. On Day 6, the class spent a considerable amount of time discussing one of these generalizations: If a set of vectors in $\mathbf{R}^n$ contains more than n vectors, then the set is linearly dependent. Prior to arguments about $\mathbf{R}^n$, the class discussed specific examples in $\mathbf{R}^2$, as well as generalizations in $\mathbf{R}^2$. Argument 5 from this day (see Figure 4.7) comes from the lattermost category.

With his claim, Aziz explicitly related the ideas of span and linear dependence by stating that a vector in the span of two other vectors (thus, a set of three vectors) would allow one to "reach the origin, get back to the origin." Recall from Chapter 3 that the notion of "getting back to the origin" was one the class had developed within the context of the Magic Carpet Ride Problem as a concept image for linear dependency of sets of vectors.

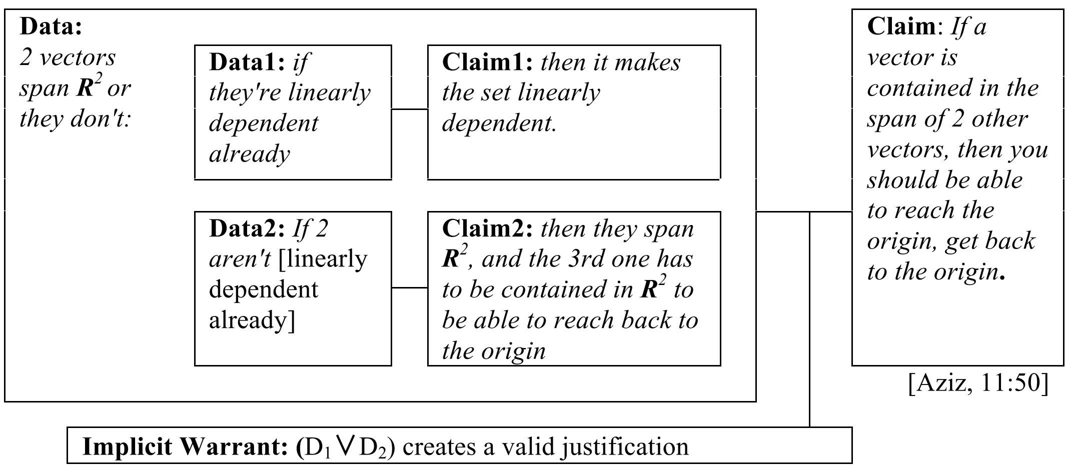


Figure 4.7. Argument 6.5, Proof by Cases structure: Aziz explains why any set of 3 vectors in $\mathbf{R}^2$ must be linearly dependent.

Aziz's data is composed of two cases: if the first two vectors are or are not linearly dependent. Each of these cases is explored with a sub-argument (labeled Data1-Claim1 and Data2-Claim2, respectively). Unlike the example in Figure 4.6, which detailed the expanded Toulmin scheme for Justin's argument about a claim very similar to Aziz's, Aziz provided no warrants for any of his data-claim pairs. The warrant for the main argument is noted as implicit because he did not verbalize anything but behaved in a manner consistent with the logical conclusion that the union of the two sub-arguments within the data qualified as a valid justification for the claim.

4.1.1.3 Linked Structure. I define a *linked structure* as a Toulmin scheme within which the data and/or warrant for the claim are composed of more than one embedded sub-argument that are linked by words such as "and" or "also." This differs from the Proof by Cases structure in that the sub-arguments are not related in the same manner. Furthermore, this structure goes beyond, for instance, a Toulmin scheme with multiple data. As Aberdein points out, Toulmin himself allowed for multiple data but that the linked structure expands upon Toulmin by "permitting multiple propositions within a node to be distinguished as separate nodes...However, this is necessary unless the propositions are individually attached to other nodes" (Aberdein, 2006, p.7). In other words, the difference in the Linked structure is that the multiple data are actually sub-arguments themselves.

An example of a Linked structure comes from Argument 16 on Day 20 of the semester. On Days 19 and 20, the class investigated notions related to one-to-one and onto transformations: examples of each, non-examples of each, and other concepts to which they were similar. The students were parsing out the relationship between one-to-one and onto (which are properties of linear transformations) and linear independence and span (which are properties of sets of vectors). On Day 20, students began to explore the connections between onto and span, as well as between one-to-one and linear independence. In Figure 4.8, Abraham explains how the claim of "being linearly independent is the same as being onto" if the matrix is square made sense to him.

Abraham, after he made his claim, began by qualifying his claim by stating he "just remembers" the data he was about to share with the class. He then stated two sub-arguments, which are from "the n x n theorem": "If a matrix is square and linearly independent" (Data1) then "it also spans" (Claim1) and "if it spans" (Data2) then "it's also linearly independent"

(Claim2). One may notice the metonymic nature (Lakoff & Johnson, 1980) of these statements: a matrix does not, in the strict mathematical sense, have the properties of linear independence or span, but rather the column vectors of that matrix do. The wording of the claims and data reflect this in order to be as true as possible to students' original utterances. Furthermore, these four statements together comprise the data for the original claim. They are separated into sub-arguments because, mathematically, they are quite different and the validity of those data-claim pairs had to be established previously in the semester. Furthermore, this Toulmin scheme is not an example of the Proof by Cases structure because it is not appropriate for this particular claim. Explaining each possible scenario that would lead to linear independence and onto being equivalent for square matrices is not a possible method of proof here, whereas explaining how each possible scenario leads to linear dependency of a set, for instance in Figure 6, was sensible.

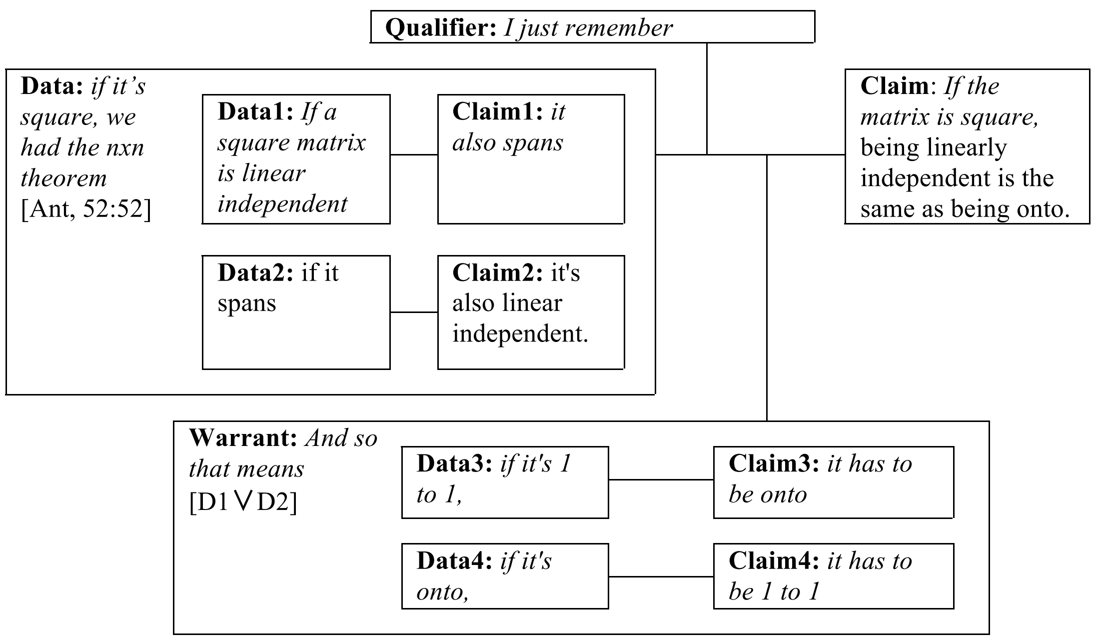


Figure 4.8. Argument 20.16, Linked structure: Abraham explains a connection between linear independence and onto.

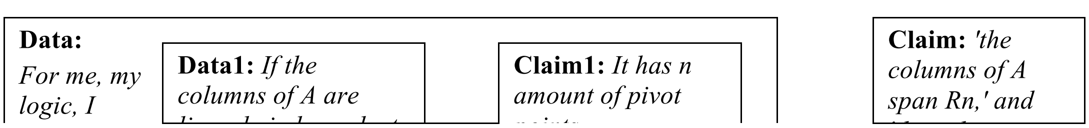
Abraham's warrant in Argument 16 (see Figure 4.8), which began with "and so that means," has a structure similar to that of his data. What was left unsaid in his explanation is why his warrant connected the data to the claim. His claim discussed the possible equivalency of linear independence and onto, he stated (without support) the equivalency of linear independence and span in his data, and he stated (again, without support) the equivalency of one-to-one and onto in his warrant. How, then, is the equivalency mentioned in the claim supported? One solution is to analyze the relevant mathematical activities that occurred in this classroom through ontogenetic analysis of the collection of all Toulmin schemes. That would help explain why, for instance, the warrants may have dropped off from the sub-arguments. These results are given in the forthcoming ontogenetic analysis section of the chapter.

4.1.1.4 Sequential Structure. I define a *sequential structure* as a Toulmin scheme within which the data for a specific claim contains an embedded string of Data-Claim pairs, such that claim C_k is data D_{k+1} for the next claim C_{k+1}. As with the last two expanded Toulmin structures, Proof by Cases and Linked, the Sequential structure assumes an Embedded structure as well, and, as before, the distinction lies within the nature of the relationship between the sub-arguments of the main data or warrant.

An example of a sequential structure comes from Argument 8 on Day 31, which was the last day of class. The students had worked in their small groups with 17 cards, each one containing one of the statements from the Invertible Matrix Theorem. Prior to Argument 8, the students had discussed in their small groups which pairs of statements were, to them, most obviously equivalent and least obviously equivalent. The instructor asked someone from their group to explain their choice. Nate volunteered and stated:

Nate: For me, my logic, I think if the columns of *A* are linear independent, then it has n amount of pivot points. Then if it has *n* amount of pivot points, and it's an *n* x *n* matrix assuming that, then it spans all of $\mathbf{R}^n$.

The expanded Toulmin scheme of his response is given in Figure 4.9.



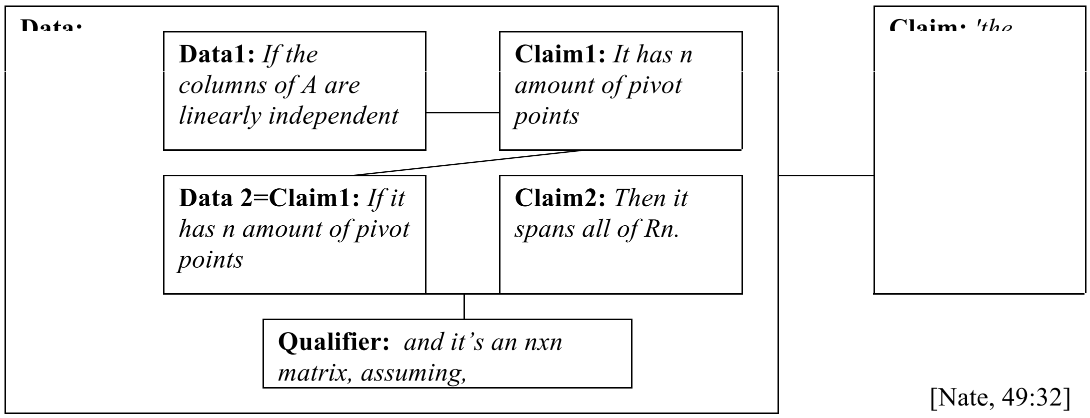


Figure 4.9. Argument 31.8, Sequential structure: Nate explains why two concepts are

Nate claimed that for his group, the concepts of 'the columns of *A* span $\mathbf{R}^n$" and "the columns of *A* are linearly independent" were obviously equivalent. He volunteered data for this claim with two sub-arguments such that the claim for the first sub-argument ("it has *n* amount of pivots") was explicitly re-stated as the data for the subsequent claim. One may think of this as a "chain of reasoning," in that the nodes of the various sub-arguments are linked together explicitly. Another example of the Sequential structure occurred on Day 18. The instructor began class by picking up on a proof that they had not completed the previous class period. The class was working with the New Theorem (what the IMT was known as before the concept of invertibility was added to it), and the class was working to prove the direction of, given a square matrix *A*, if the columns of *A* are linearly independent, then the matrix *A* is invertible. During class, the full development of the proof took ten minutes and was composed of twelve sub-arguments. Below is transcript of the teacher initiating the proof.

Instructor: So last class we were like, Wow, how are we going to get there? And so I started with, since we know *A* is square and has linear independent column vectors, from the New Theorem, we know that *A* is row equivalent to the identity. All right? So we had that last time, we were like, What does that mean? And Nate reminded us, if we row reduce to the identity which meant that there existed a sequence of elementary row operations that turns *A* into *I* and vice versa...So let's think about this sentence here for a second. If you had elementary row operations that turn *A* into *I*, that means there's something

> acting on *A* and changing it. So *A*, there's something that's turning *A* into *I*, there's something that's acting on *A* and changing it. So we're going to say each elementary row operation can be thought of as a linear transformation.

The expanded Toulmin scheme for this portion of the argument is given in Figure 4.10. The main claim for Argument 1 is “matrix *A* is invertible.” What are provided in Figure 4.10 are the first four sub-arguments for the main data in Argument 1. In every data-claim pair, the claim for a given sub-argument serves as the data for the immediately subsequent sub-argument. The pattern continued, with some variation, for the remainder of the main argument (for the complete Toulmin scheme, see Appendix 4.1).

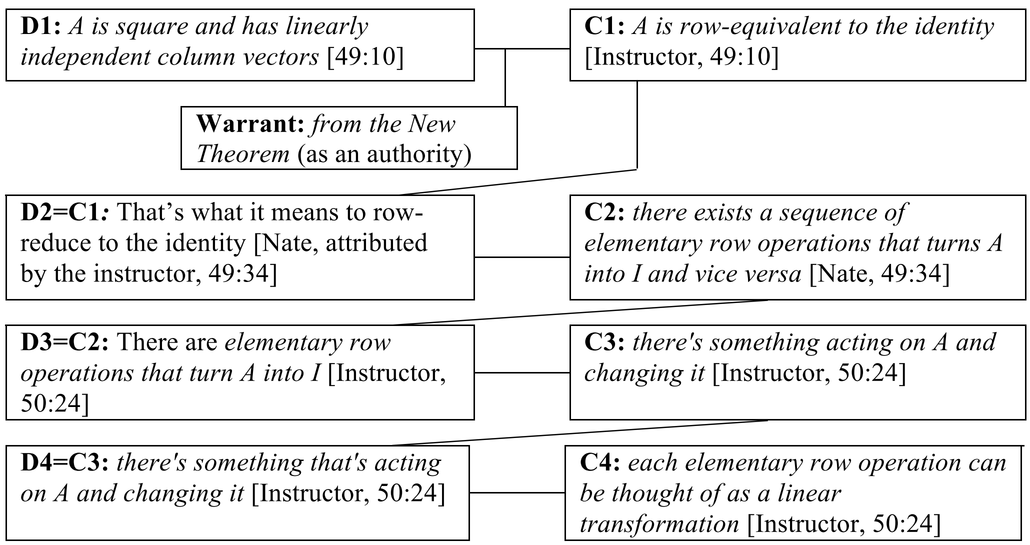


Figure 4.10. Day 18, portion of Argument 1, Sequential structure: A section of the data for the claim that if *A* is square and its column vectors are linearly independent, then *A* is invertible.

In both examples given in Figures 4.9 and 4.10, the speaker explicitly restated a claim from a sub-argument as data for the subsequent sub-argument. This is in contrast to the Linked structure in which two sub-arguments are connected through words such as “and” or “also,” as well as in contrast to the Proof by Cases structure in which the sub-arguments constitute all possible outcomes.

4.1.1.5 Conclusion. In this section, I presented four types of argumentation schemes that are an expanded version of Toulmin's model: Embedded, Proof by Cases, Linked, and Sequential. These argumentation structures were developed out of examining the transcript and video from one particular linear algebra class as its members reasoned about the Invertible Matrix Theorem and the mathematical concepts involved in that theorem. These four expanded structures were adapted from and are compatible with the expanded Toulmin schemes presented by Aberdein (2006, 2009). His work in informal logic and argumentation in mathematics presented analysis of logical structure and proof; however, it does not seem that his work analyzed the argumentation practices that occurred in actual discourse. For instance, Aberdein (2006) provided two examples of how an expanded structure could be used to map out a proof. One was a proof by induction (that involved the embedded structure) that every natural number greater than one has a prime factorization, and the other was a proof of the Intermediate Value Theorem. While expanded Toulmin structures were quite useful in mapping out the proofs' structures, the source of the proof was left unstated. Was the proof given in a textbook and mapped out by Aberdein? Did he himself develop the proof and, if so, in what form? Was it written or communicated verbally to others, and then analyzed via the expanded structures? Thus, Aberdein's use of Toulmin's model is distinct from its use in the work presented here. The research in linear algebra presented here investigated the ways in which the members of a classroom reasoned about the Invertible Matrix Theorem. The analysis in this section focused on whole class discussion and examined the structure of arguments given by members of a classroom as they justified claims in situ. Thus, this study contributes by investigating the "argumentation of natural language" in an inquiry-oriented mathematics classroom and found it beneficial to adapt Aberdein's notion of the expanded Toulmin layout to do so.

This section presented one type of microgenetic analysis of argumentation that occurred in whole class discussion during ten days over the course of one particular linear algebra course.

These results do not concern the mathematical development of the collective as a unit of analysis per se, but rather the content and structure of argumentation at the collective level at various moments during the semester. Through ontogenetic analysis via Toulmin's model of how these various individual arguments shift in form and function over time, various aspects of how the classroom community came to reason about the IMT over time comes to light.

4.1.2 Ontogenetic Analysis via Toulmin's Model

Recall that microgenesis is the short-term process by which meaningful representations are constructed in activity, and ontogenesis is the developmental shifts in relations between the forms used and the functions that they serve (Saxe, 2002; Saxe et al., 2009). The previous section on microgenetic analysis investigated the content and structure of particular instances of reasoning about the Invertible Matrix Theorem; this was referred to as "Phase 3" in Chapter 3. The current section considers the shifts in form and function of how the classroom reasoned about the various concepts in the Invertible Matrix Theorem over time; this was referred to as "Phase 5" in Chapter 3. In Phase 5, the primary means of coordination were inspired by the methodology for documentation of both normative ways of reasoning and classroom mathematics practices, as detailed in Rasmussen and Stephan (2008) and Cole, Becker, Towns, Rasmussen, Wawro, and Sweeney (in 2011).

Rasmussen and Stephan (2008) provide a methodology for documenting when ways of reasoning become normative in a classroom community. An abbreviated description of these two criteria is:

- *Criterion One:* When the backings and/or warrants for an argumentation no longer appear in students' explanations, no member of the community challenges the argumentation, or a challenge to the argument is rejected
- *Criterion Two*: When any of the four parts of an argument (data, claim, warrant, or backing) shift position within subsequent arguments and is unchallenged (or a challenge is rejected)

Further work with documenting ways of reasoning that function as-if shared in classroom communities in the discipline of physical chemistry revealed a third criterion (Cole et al., 2011):

- *Criterion Three:* When a particular idea is repeatedly used as either data or warrant for different claims across multiple days

Through the use of these three criteria, ten normative ways of reasoning were identified from the aforementioned 118 Toulmin schemes:

1. Two vectors that are parallel are linearly dependent;
2. The span of two non-parallel vectors in $\mathbf{R}^2$ is all of $\mathbf{R}^2$;
3. A set of vectors being linearly dependent means the same thing as being able to return to your original position;
4. For a given set of n vectors in $\mathbf{R}^m$, if $m < n$, the set must be linearly dependent;
5. For a given set of n vectors in $\mathbf{R}^m$, if $m > n$, the set cannot span all of $\mathbf{R}^m$;
6. If A is n x n and the columns of A span $\mathbf{R}^n$, then the columns of A are linearly independent;
7. If A is n x n and the columns of A are linearly independent, then the columns of A span $\mathbf{R}^n$;
8. A transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ is onto $\mathbf{R}^m$ when you can "get everything" or "get everywhere" and not onto $\mathbf{R}^m$ when you can not;
9. "A transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ is onto $\mathbf{R}^m$" is interchangeable with "the column vectors of the associated matrix A span all of $\mathbf{R}^m$; and
10. For a given transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$, if $m > n$, the transformation is not onto $\mathbf{R}^m$.

The first three (henceforth referred to as NWR #1-3) receive an abbreviated treatment in analysis. Normative ways of reasoning #4-7 from above (henceforth referred to as NWR 1.1-1.4) and #8-10 (henceforth referred to as NWR 2.1-2.3) each constitute a classroom mathematics practice. The bulk of this section provides the evidence for these latter seven as normative ways of reasoning and the classroom mathematics practices that they constitute.

As defined by Rasmussen and Stephan (2008), a classroom mathematics practice is "a collection of as-if shared ideas that are integral to the development of a more general mathematical activity" (p. 201). The "as-if shared ideas" in this definition are what is referred to as normative ways of reasoning in the present analysis. Thus, classroom mathematics practices are collections of normative ways of reasoning around a general mathematical activity, such as reasoning about equivalencies in linear algebra. Through this focus on span, linear independence, one-to-one, and onto, this section presents results for two classroom mathematics practices: Reasoning about span and linear independence as equivalent ideas for square matrices (referred to as CMP #1), and Determining whether or not a transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ is onto $\mathbf{R}^m$ by considering the span of the column vectors of the associated matrix A (CMP #2).

4.1.2.1 CMP #1: Reasoning about Span and Linear Independence as Equivalent Ideas for Square Matrices. One classroom mathematics practice that developed in this linear algebra course is "Reasoning about Span and Linear Independence as Equivalent Ideas for Square Matrices." Here, *equivalent* takes on the mathematical definition that, for an n x n matrix A, the column vectors of A span $\mathbf{R}^n$ if and only if the column vectors of A are linearly independent. The development of this practice involved various ways of reasoning during the semester. These normative ways of reasoning, per the three criteria for ideas functioning as-if shared are given in Figure 4.11.

Classroom Mathematics Practice #1:
Reasoning about Span and Linear Independence as Equivalent Ideas for Square Matrices

Associated Normative Ways of Reasoning:

1.1 For a given set of n vectors in $\mathbf{R}^m$, if $m < n$, the set must be linearly dependent.
1.2 For a given set of n vectors in $\mathbf{R}^m$, if $m > n$, the set cannot span all of $\mathbf{R}^m$.
1.3 If A is n x n and the columns of A span $\mathbf{R}^n$, then the columns of A are linearly independent.
1.4 If A is n x n and the columns of A are linearly independent, then the columns of A span $\mathbf{R}^n$.

Figure 4.11. The normative ways of reasoning that comprise class mathematics practice #1, "Reasoning about Span and Linear Independence as Equivalent Ideas for Square Matrices."

Given that the concepts of span and linear independence were ideas introduced very early in the semester, the relationship between these two ideas developed gradually over the entire semester. Furthermore, Stephan and Rasmussen (2002) noted that normative ways of reasoning might not develop in a linear, non-overlapping manner. That is true in this case as well. To keep the results organized, however, I present the normative ways of reasoning one by one and make note of when one shows up in context of the other. Each section includes information regarding all relevant arguments and explanation for a subset of those arguments.

4.1.2.1.1 *NWR 1.1: For a given set of n vectors in R^m, if m < n, the set must be linearly dependent.* This idea first surfaced as a generalization conjectured by a small group on Day 5, became explicitly debated and discussed on Day 6, and then used for data or warrant for other claims on Days 9 and 20. The idea became identified in the class with the phrase, "if there are more vectors than dimensions, the vectors have to be linearly dependent." The first argument relevant to NWR 1.1 occurred on Day 5, when Justin explained his group's rationale for why there was "no solution" for the task of generating an example of a set of three linearly independent vectors in $\mathbf{R}^2$ (see Figure 4.12). This argument was presented in the previous section as an example of a Proof by Cases structure (Figure 4.6); it is given here to help analyze how the concepts of span and linear independence developed during whole class discussion.

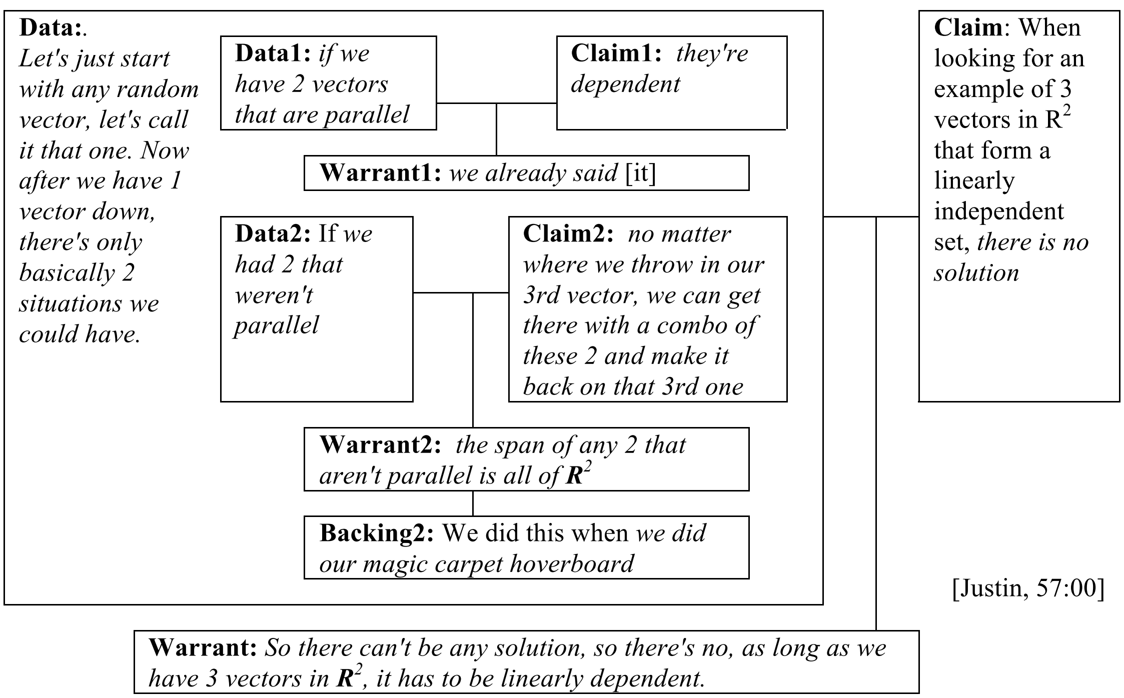


Figure 4.12. Argument 5.1: Justin's argument for why it is impossible to have a linearly independent set of three vectors in $\mathbf{R}^2$.

Whereas in the previous section on Microgenetic Analysis the focus was on the structure of Justin's argument, I now present this argument to consider the mathematics in Justin's statements and how these ideas shift in form and function within the various goals of justifying claims throughout the semester. Note that Justin's explanation referenced previous activities in the class: In Warrant 1 he stated "we already said," and Backing 2 was from when "we did our magic carpet hoverboard," which was the task sequence from the first few days of the course. The Data1-Claim1 pair, as well as Data2-Claim2 pair, both previously served as claims that needed justification within the class (on Days 5 and 3, respectively; the details are omitted here). In Argument 5.1, they functioned without debate to serve as part of the Data (the Warrant and Backing for the two sub-arguments within the Data) in order to help Justin justify his claim. Thus, according to Criterion Two, there is evidence that the following two ways of reasoning were normative in this classroom community:

NWR #1: Two vectors that are parallel are linearly dependent

NWR #2: The span of two non-parallel vectors in $\mathbf{R}^2$ is all of $\mathbf{R}^2$

The main claim in Argument 5.1, that it is impossible to have a linearly independent set of three vectors in $\mathbf{R}^2$, is relevant to developing NWR 1.1 because it is a specific case of it—the generalization of the case of three vectors in $\mathbf{R}^2$ to more than n vectors in $\mathbf{R}^n$ was a major point of conversation on Day 6.

On Day 6, the class members discussed various generalizations regarding linear independence and linear dependence that they, in their small groups, had worked on the previous day. As a result of their work on the Example Generation Task (see Figure 4.4), four generalizations were brought out during whole class discussion:

1. For any set of vectors in $\mathbf{R}^n$ where two of the vectors are multiples of each other, the set is linearly dependent.

2. If any vector in a set can be written as a linear combination of the other vectors, then the set is linearly dependent.
3. If the zero vector is included in a set of vectors, then the set is linearly dependent.
4. If a set of vectors in $\mathbf{R}^n$ contains more than n vectors, then it is linearly dependent.

From the reflections students completed on Day 5, the research team conjectured that the fourth generalization was the most problematic for students, so the instructor began class on Day 6 discussing this statement. The following transcript provides the data from which Arguments 6.1 and 6.2 are drawn. The coded transcript of Day 6, along with the original Toulmin coding, can be found in Appendix 4.2.

Instructor: As I was walking around, I saw more puzzled faces than I expected, it is Monday and I actually heard someone say, 'How are we supposed to know how to make a generalization if we don't know what it's about, or we don't understand what it's about?' I think that's a great question. So how about, can I have someone who understands #4 restate #4 in their own words, what do we mean by this generalization?

Justin: If you have more vectors than dimensions, you'll always be able to return to your original position.

Instructor: Could you say that louder? If you have more vectors than dimensions?

Justin: Then you can always return to your original position.

Instructor: Does that resonate with anyone else's way of thinking about this problem? Nate, can you say anything about the way you understand what #4 is about?

Nate: It's saying if you have, there's no other way of putting it, if there's more vectors than. I don't know another way to say it.

The first argument coded occurred after the instructor asked Justin to restate Generalization #4 in his own words (see Argument 6.1 in Figure 4.13). Justin claimed that his statement, "if you have more vectors than dimensions, you'll always be able to return to your original position," is a way to restate Generalization #4 in his own words. After he did not provide any justification of why it is valid to restate the generalization in this way, the instructor asked Nate if Justin's restatement resonated with his own thinking. This type of move, often demonstrated by the instructor, can be referred to as "calling for data," which serves to push the argument further. Nate's utterance served as data for Justin's claim because it functioned to provide validity to the claim. It did so because Nate confirmed the claim's validity by saying he couldn't think of any other way to restate the generalization.

Data: *There's no other way of putting it ... I don't know any other way to say it* [Nate, 05:26]	**Claim:** Generalization #4 in my own words is *if you have more vectors than dimensions, you'll always be able to return to your original position* [Justin, 04:54]

Figure 4.13. Argument 6.1: Justin describes Generalization #4 in his own words, and Nate comments.

Within this data-claim pair, Justin and Nate treated the concept of a set of vectors being linearly dependent the same as being able to return to your original position. Furthermore, we see from the transcript that neither the instructor nor other members of the classroom asked for more justification regarding the validity of Justin's claim. This is distinct from classroom behavior on Day 3, when the term 'linearly dependent' was provided as a term for a set of vectors with this property. When the formal definition of linear dependence, that a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2,...,\mathbf{v}_n\}$ is linearly dependent if there exists a nonzero solution to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 +...+ c_n\mathbf{v}_n = \mathbf{0}$, was introduced, a substantial portion of whole class discussion and small group work was spent correlating this contextual situation with the formal definition. This analysis from previous days is beyond the scope of the focus of this section; however, this transition from being a claim that needed justification to one stated without such support satisfies Criterion One that the given idea was functioning as-if all members of the classroom shared an understanding of it. Thus, another idea that functioned as-if shared in this particular classroom is the following:

NWR #3: A set of vectors being linearly dependent means the same thing as being able to return to your original position

As the discussion continued, the instructor attempted to elicit justification of why the generalization "if there are more vectors than dimensions, the vectors are linearly dependent" would be a valid generalization:

Instructor: That's fine. So that's the problem statement, now the question would be, how can we explain why that is true? Jerry, did your table get to talk about a reason why #4 is going to make sense to your table?
Jerry: Not really.
Instructor: Not really. Saul, how did your table talk about #4?
Saul: We were confused on how to prove it, we didn't know where to start, where to go with a proof.

As shown in Figure 4.14, the generalization was the claim in Argument 6.2 and the instructor pushed for data by asking, "how can we explain why that is true?" She asked one student, Jerry, if he could give a reason why this generalization made sense for his table, but he replied "not really." After an extended pause during which she looked around the room at the other students, the instructor called on Jerry, who was also unable to give a justification for the claim. Instead, Saul provided a qualifier that voiced his group's confusion and difficulty in forming a proof.

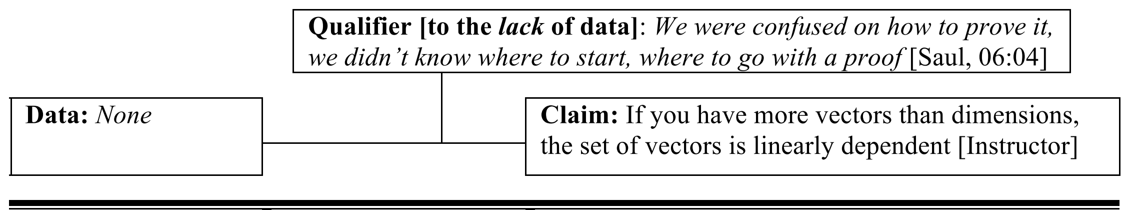


Figure 4.14. Argument 6.2: Saul states his table was unsure how to prove Generalization 4.

Of note here is the difficulty of the classroom, when considered an entity comprised of all members, to provide a justification for the generalization offered the previous class day. The instructor noted this difficulty and decided to shift focus from the generalized claim towards specific examples of the claim (Arguments 6.3-6.7).

The instructor suggested that they start with an example and wrote $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}3\\5\end{bmatrix},\begin{bmatrix}-9\\7\end{bmatrix}\right\}$ on the board. She gave the students time in their small groups to consider the example by saying, "Let's think about why if we had 3 vectors in $\mathbf{R}^2$, that would mean I should always be able to get back home." Notice in the instructor's statement that interpreting being able to "get back home" as being the same as the vectors forming a linearly dependent set is functioning as-if shared.

Note the instructor did not ask them to work in on the claim in Argument 6.2; rather, she asked students to work on an example in $\mathbf{R}^2$ and, even in her working of the task ("should be able to get back home") prompted them to refer back to their previous work to help them on this task. After discussing in their small groups, Lawson came to the front of the class to explain how his group was thinking. The structure of his explanation is shown in Figure 4.15.

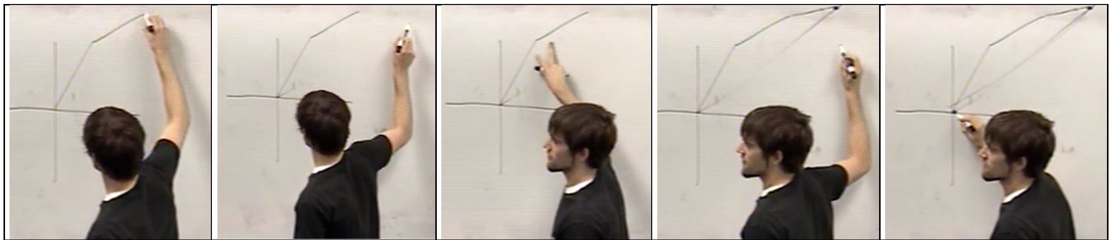
Data: *Graphically think about it. You say you have a vector over here, and then plus a vector that's not parallel to it [draws two vectors tip-to-tail on the board]. And we have just any other random vector, say it goes this way. If you add any 2 nonparallel vectors that at some point these vectors are going to come across this other vector, and hit it at a point.* [Lawson, 08:58]

Claim: *If we had 3 nonparallel vectors in* $\boldsymbol{R}^2$*, I should always be able to get back home* [Instructor, 6:46]

Warrant: *So if that happens, then you can just take the last vector and multiply it by some scalar to get you back to the origin. That's how graphically we got it.* [Lawson, 08:58]

Figure 4.15. Argument 6.3: Lawson justifies why three nonparallel vectors in $\mathbf{R}^2$ can always "get back home."

In Argument 6.3 (see Figure 4.15), we see a repeat of the claim specific to three vectors in $\mathbf{R}^2$. The claim is attributed to the instructor, and the data is attributed to Lawson because he came up and shared his group's thoughts from their work on the example. Notice that his explanation did not actually address the specific example of $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}3\\5\end{bmatrix},\begin{bmatrix}-9\\7\end{bmatrix}\right\}$ written on the board; rather, he addressed the claim by using a generic example (Mason & Pimm, 1984). He started his explanation by stating, "graphically think about it." He then presented a justification that relied on three generic vectors in $\mathbf{R}^2$ and sketched his explanation as he spoke (see Figure 4.16).



Graphically think about it. You say you have a vector over here, and then plus a vector that's not parallel to it	And we have just any other random vector, say it goes this way.	If you add any two nonparallel vectors	that at some point these vectors are going to come across this other vector, and hit it at a point.	So if that happens, then you can just take the last vector and multiply it by some scalar to get you back to the origin. That's how graphically we got it.

Figure 4.16. Lawson's justification for why any 3 vectors in $\mathbf{R}^2$ are linearly dependent.

The instructor, who noticed that Lawson's justification dealt with three "non-parallel" vectors, posed the question, "What if the set of vectors were $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\4\end{bmatrix},\begin{bmatrix}-9\\7\end{bmatrix}\right\}$?" In Argument 6.4 (see Figure 4.17), Lawson claimed that this set is also linearly dependent and provided the data that "You can just ignore this last vector, take one, like $\begin{bmatrix}1\\2\end{bmatrix}$, which would be, and just take the $\begin{bmatrix}2\\4\end{bmatrix}$ back." Lawson then provided information regarding how his data was relevant to his claim by providing the warrant that because $\begin{bmatrix}2\\4\end{bmatrix}$ is a multiple of $\begin{bmatrix}1\\2\end{bmatrix}$, the set is linearly dependent. Cayla's backing came after the instructor asked the class if Lawson's explanation made sense. This again speaks to the role of the instructor in soliciting warrants and backings. Cayla's backing is notable because she commented on whether Lawson's explanation justifies that $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\4\end{bmatrix},\begin{bmatrix}-9\\7\end{bmatrix}\right\}$ is a linearly dependent set by attempting to address the more general case of a set of three vectors in $\mathbf{R}^2$. Toulmin (1969) noted that warrants and backing often are of this more general form, versus data that often are more specific. Her explanation, however, was unclear and possibly incorrect: by saying, "ignore the last one," did she mean that the set $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}3\\5\end{bmatrix}\right\}$ is linearly dependent?

Data: *You can just ignore this last vector, take 1, like* $\begin{bmatrix}1\\2\end{bmatrix}$*, which would be, and just take the* $\begin{bmatrix}2\\4\end{bmatrix}$ *back*

Claim: *If you have the set of vectors* $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\4\end{bmatrix},\begin{bmatrix}-9\\7\end{bmatrix}\right\}$, it's linearly dependent [Lawson, 09:57]

Warrant: *it's a multiple of* $\begin{bmatrix}1\\2\end{bmatrix}$*, and you can just not use this one. So it's linear dependent.* [Lawson, 09:57]

Backing: *It makes sense that pretty much if you have 3 vectors in* R^2*, even in the 1st example, you could technically ignore the last one, because if you take* $\begin{bmatrix}1\\2\end{bmatrix}$ *out, then you can still use any combination of the* $\begin{bmatrix}1\\2\end{bmatrix}$ *and the* $\begin{bmatrix}3\\5\end{bmatrix}$ *to get back* [Cayla, 10:28]

Figure 4.17. Argument 6.4: Lawson provides evidence that a particular set of three vectors in $\mathbf{R}^2$ is linearly dependent.

Immediately after Cayla spoke and without the instructor's prompting, Aziz volunteered a new idea (see Argument 6.5 in Figure 4.18). Aziz claimed that if a third vector was in the span of two other vectors, then the set of three vectors is linearly dependent. This is coded as a new claim rather than another support for Lawson's original explanation because Aziz mentioned span in conjunction with linear dependence, and this was the first time that day that these two concepts had been linked together in some way. This argument was presented in detail, but with regard to its Proof by Cases structure, in the section on Microgenetic Analysis (Figure 4.7). It is presented here to highlight the mathematical content of Aziz's argument. First, Aziz's claim, in line with Cayla's backing, generalized to all sets of three vectors in $\mathbf{R}^2$. To do so, he used a proof structure very similar to Justin's from Day 5 (see Figure 4.12). He also made use of the aforementioned normative ways of reasoning but with slightly different vocabulary.

In his data-claim pair of his second sub-argument, he stated that two vectors that aren't linearly dependent span $\mathbf{R}^2$. In Argument 5.1 (Figure 4.12), Justin used this statement (except with the words "not parallel" instead of "not linearly dependent") as a warrant for his claim. Here, Aziz used the same concept for his data but without the additional backing for why this is valid.

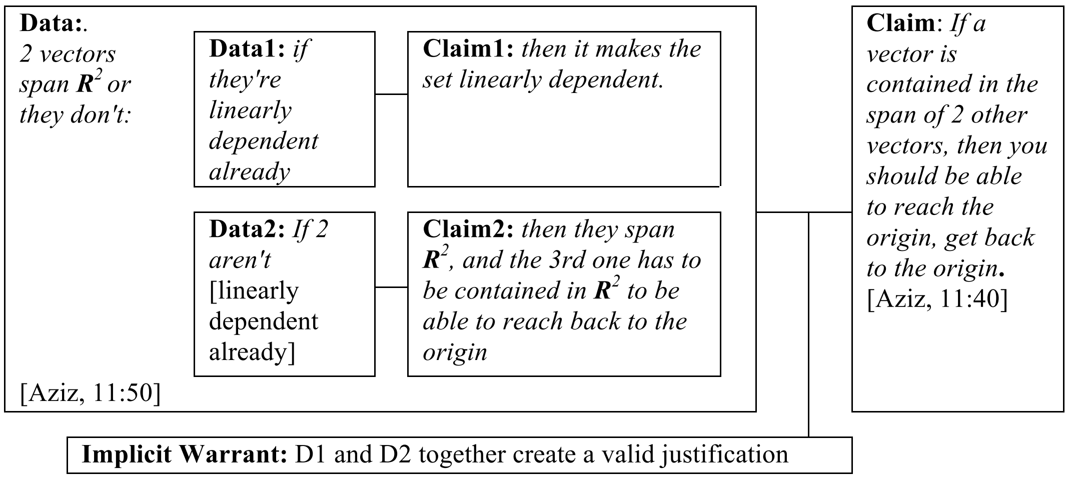


Figure 4.18. Argument 6.5: Aziz argues that if a vector is in the span of two other vectors, you can get back to the origin.

After Cayla's explanation, in which the distinction between span and linear independence was unclear, and Aziz's argument that explicitly tried to relate the two concepts, the instructor made this distinction the focus of conversation. Reminding the students of Cayla's claim that if they were to remove one of the vectors from the set $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}3\\5\end{bmatrix},\begin{bmatrix}-9\\7\end{bmatrix}\right\}$, the span would still be all of $\mathbf{R}^2$, she asked, "Now the question would be, how does this relate to linear independence or dependence, are they both linearly independent, are they both linearly dependent, one of each?" She then wrote two sets of vectors on the on the board, $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}3\\5\end{bmatrix},\begin{bmatrix}-9\\7\end{bmatrix}\right\}$ and $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}3\\5\end{bmatrix}\right\}$, and facilitated a discussion of linear independence or dependence. Details of these two arguments, 6.6 and 6.7, can be found in Appendix 4.2.

Finally, the last relevant argument from Day 6 occurred when the instructor asked if someone from Justin's group would present a justification for their original generalization #4: If a set of vectors in $\mathbf{R}^n$ contains more than n vectors, then the set is linearly dependent. Justin volunteered, and his explanation is given in Figure 14.9.

Qualifier: *Unless you have 2 vectors that are lying on the same line, then you won't have the span of all of your dimension, but it's negligible because those 2 will give you a linearly dependent set. Does that make sense?* [Justin, 17:19]

Data: *So if you start in any Rn, and you just start with 1 vector and keep adding more. So let's do R3, just for an example. So we start with 1 vector. So either, we have 2 choices: The next vector we add can either be on the same line, which means it's already linearly dependent, so we don't want that, so we're going to put it off somewhere else. Now the span of that is a plane in 3 dimensions. So now we're going to add another vector in. Our 3rd vector, now it can either be in that span or out of that span. And we want it to be linearly independent, so we're going to put it out of that span. But now that we have that going off of that plane, we just extended our span to all of r3. So our 4th vector, when we put it in, no matter where we put it, it's going to get us back home. Because just like in this case, we have to have the last one to get back home, we can get anywhere with those 1st 3 that we put in, but we have to have to have that 4th one to come back.* [Justin, 17:19]

Claim: If you have more vectors than dimensions, the set of vectors is linearly dependent [Justin, 17:19]

Warrant: *And so it works like that in any dimension, because the more you, if you keep adding, eventually you're going to get the span of your dimensions, and then you're going to have that extra one bringing you back.* [Justin, 17:19]

Backing: *I'd like to add on to this, if you're going from the plane to the volume of space, that's with 3 vectors in 3 dimensions. With the 4th one, you can cross all those dimensions back to the origin, that's what makes the 4th dimension, the 4th vector, it makes you able to go back to the beginning. Because each vector takes up a different point in space. You're going from the plane to the volume, gives you those 3 dimensions but you can't get back to origin, unless you have a 4th one that can cross all 3.* [Aziz, 19:46]

Figure 4.19. Argument 6.8: Justin's argument for why a set with more than n vectors in $\mathbf{R}^n$ has to be linearly dependent.

Justin built his explanation by choosing $\mathbf{R}^3$ as a generic vector space through which to explain his reasoning. His data was dense, going through each possibility with the various combinations of vectors, starting with only one and continuing up to four. He continually spoke about span and linear independence in relation to one another; for example, he stated, “Now the span of that [the two vectors] is a plane in three dimensions. So now we're going to add another vector in. Our third vector, now it can either be in that span or out of that span. And we want it to be linearly independent, so we're going to put it out of that span.” After he spoke about a fourth vector in $\mathbf{R}^3$, he changed the focus of his explanation and stated, “it would work like that in any dimension, because...eventually you’re going to get the span of your dimensions, and then you’re going to have that extra one bringing you back.” It functioned without justification that “bringing you back” was synonymous with linear dependence. Finally, Justin ended his justification with a qualifier that, in essence, his entire justification did not matter in the case that two of the given vectors were “lying on the same line” because that made the entire set of vectors linearly dependent. Within this one complex argument, we see the ways of reasoning identified as NWR #1-3 function not only without need for justification but also as justification for a new claim.

On Day 9 of the semester, the class investigated the row-reduced echelon form of various matrices (e.g., different types of 2 x 2, 3 x 2, and 5 x 3 matrices) and how those forms provided information about the span or linear independence of the column vectors in the various matrices. The teacher started with examples that were similar to others that the class had previous experience with in regard to determining span or linear independence, one of which was the set $\left\{\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}2\\7\end{bmatrix},\begin{bmatrix}10\\5\end{bmatrix}\right\}$. After Aziz offered why this set was linearly dependent, the class row-reduced the matrix $\left[\begin{array}{ccc|c}1 & 2 & 10 & 0\\4 & 7 & 5 & 0\end{array}\right]$ in order to inquire into how free variables and pivots played a role in helping to determine the dependency relationship that existed between the three vectors. Of interest for establishing how the way of reasoning labeled #1.1 functioned as-if shared for this classroom, is Aziz’s explanation, labeled Argument 9.1 in Figure 4.20. Here Aziz claimed the three vectors are linearly dependent, and the data he provided was "If there are more vectors than there are columns, then it's linearly dependent." This is very nearly the claim that was debated on Day 6 in Arguments 6.1-6.8 above. Aziz did say, however, "more vectors than columns," which the instructor corrected in her warrant by saying "more vectors than dimensions." The instructor also affirmed that what Aziz was arguing was consistent with the class's previous work. Thus, this idea that previously was the claim in multiple arguments served here as data for a new claim, so we have our first evidence of this idea functioning as-if shared by the members of the classroom.

Data: *If there are more vectors than there are columns, then it's linearly dependent.* [Aziz, 28:50]

Claim: The set $\left\{\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}2\\7\end{bmatrix},\begin{bmatrix}10\\5\end{bmatrix}\right\}$ is linearly dependent. [Aziz, 28:50]

Warrant: *So Aziz is reminding us of the thing we decided a while back, if we have more vectors than dimensions, it has to be linearly dependent* [Instructor, 28:56]

Figure 4.20. Argument 9.1: An argument for why a given set of three vectors in $\mathbf{R}^2$ is linearly dependent.

After many conversations about pivots, row-reduction, span, and linear independence, the teacher brought the discussion to a close by tying all of their ideas together into the semester's first theorem, which consisted of four equivalent statements (Figure 4.21). This theorem became known as "Theorem 4," to match the name it was given in the textbook (Lay, 2003).

Theorem: Let A be an m x n matrix (where m is # of rows and n is # of columns). The following are equivalent:

1) The columns of A span $\boldsymbol{R}^m$.
2) There exists a pivot in every row (There are m pivots).
3) For every vector $\mathbf{b}$ in $\boldsymbol{R}^m$, there is a way to write $\mathbf{b}$ as a linear combination of the columns of A.
4) For every $\mathbf{b}$ in $\boldsymbol{R}^m$, there is a solution to the equation $A\mathbf{x} = \mathbf{b}$.

Figure 4.21. Theorem 4 from Day 9. This was the first occurrence during the semester of equivalent statements being explicitly connected through a theorem.

The class discussed that one way to interpret a theorem composed of equivalencies was that if one of the statements is true, then they are all true, and if one is false, all of the statements must be false. As the last conversation that day, the instructor asked the class to think about various possibilities for span and linear independence based on whether $m > n$ or vice versa.

Instructor: This theorem doesn't say a lot about linear independence or dependence, right? Yeah, so for instance, *m* by *n*, what if *m* is less than *n*, or *m* is greater than *n*? Cases like that we haven't gotten to too much yet. So let's think for a second, if *m* is less than *n*, that means we have less rows and more columns.

Randall: Dependent.

Instructor: Did you say dependent? He didn't even say much, he just said dependent, how can you say that so quickly?

Randall: If *n* is greater than *m*, that means there's going to be, say, 3 vectors in 2 space, and 2 of those are going to span, the last one makes it dependent.

Instructor: Yeah, so we could do that example from before, if we had 3 vectors in $\mathbf{R}^2$, we try to row reduce it, we could have a pivot here, a pivot here, and can we get a 3rd pivot? [Randall: No.] Yeah, so we'll start to look at pivots this way for linear independence. So this guy [the third column] has no pivot, so it has to be a linearly dependent set.

In Argument 9.15 (see Figure 4.22), Randall quickly said "dependent" in response to the teacher's initial statement of "if *m* is less than *n*." The instructor asked Randall to provide data for his claim, which she then supported by adding a warrant and backing.

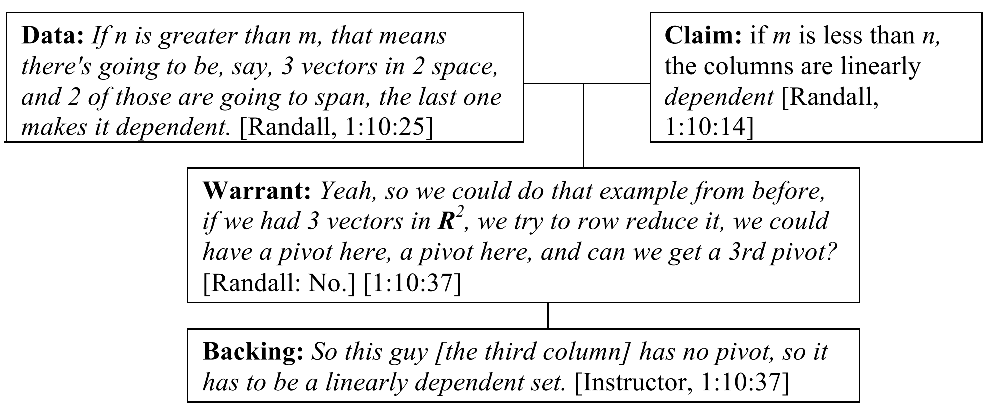


Figure 4.22. Argument 9.15: Randall and the teacher discuss if $m < n$, the columns are linearly dependent.

The instructor ended the class period by saying they would return to these ideas in homework and in class the following week. She drew explicit attention to the concepts of linear independence and span, stating, "as you saw in that last example, maybe we can get linear independence and dependence, but we can't [get] span. Maybe we can span but not have independence, etc." A discussion of what could occur when *m* equals *n* did not occur during this class. However, Abraham stayed after class that day to speak to the instructor. While doing so, he conjectured on his own that the only way a set of vectors in a matrix could both span and be linearly independent is if $m = n$. Details of this conjecture are given in Chapter 5.

The final example of reasoning that for a given set of *n* vectors in R^m, if $m < n$, the set must be linearly dependent occurred on Day 20 of the semester. On Day 20, the class investigated the notions of one-to-one and onto transformations, and how those concepts were related to ones with which they were already familiar, such as span and linear independence. In Argument 20.4 (see Figure 4.23), Mitchell claimed that a transformation T: $\mathbf{R}^m \rightarrow \mathbf{R}^n$ could not be one-to-one if *m* was less than *n*. His data for this claim, however, was "it's just not possible." Unsatisfied with this justification, the instructor asked Abraham if he could provide for the class some reason why it wouldn't be possible. Abraham's data for Mitchell's claim, notated as Data2 in Figure 4.23, was that having more vectors than dimensions means the set has to be linear dependent. Abraham went on to provide a reason why that data had anything to do with Mitchell's claim by stating, "In order for it to be 1-1, it has to be linear independent." This is the warrant in Argument 20.4.

Again, in Argument 20.4, as was true in Arguments 9.1 and 9.2, the idea of "having more vectors than dimensions implies the vectors are linearly dependent" shifted from being the claim that needed justified (as seen, in some version, in Arguments 5.1, 6.1-6.5, and 6.8) to the data for new claims. Thus, per Criterion 2, this idea functioned as-if shared in this particular classroom community. The activity of using linear independence as a way of reasoning about one-to-one transformations is described in more detail in the forthcoming section regarding the second classroom mathematics practice.

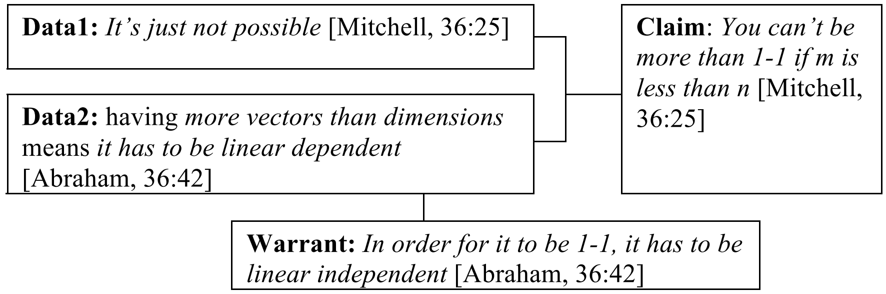


Figure 4.23. Argument 20.4: Mitchell and Abraham explain why $m < n$ implies the transformation is not 1-1.

The purpose of this above description was to document the development of NWR 1.1, which was one of the four normative ways of reasoning that comprised the first classroom mathematics practice (CMP 1), "Reasoning about Span and Linear Independence as Equivalent Ideas for Square Matrices." A much briefer treatment of the development of NWR 1.2 follows.

4.1.2.1.2 ***NWR #1.2: For a given set of n vectors in R^m, if m > n, the set cannot span all of R^m.*** The second normative way of reasoning associated with the CMP "Reasoning with Span and Linear Independence as Equivalent Ideas for Square Matrices" was initiated within the first few days of the semester. It was a way of reasoning compatible in style with the first normative way of reasoning associated with this CMP; that is, NWR 1.1, which dealt with defining situations in which when linear independence was impossible. NWR 1.2 can be seen as a correlate of that for the idea of span, and there were seeds of it in Justin's argument for why a set with "more vectors than dimensions" is linearly dependent (see Argument 6.8 in Figure 4.19). Within his argument, he stated there are two possible cases for the span of two vectors in $\mathbf{R}^3$: If two are linearly dependent, "you won't have the span of all of your dimension" (qualifier), and if the two are linearly independent, "the span of that is a plane in three dimensions" (data).

Another argument of interest occurred immediately after Argument 9.15, which was detailed in the previous section (see Figure 4.22). The instructor asked what could be said, if anything, about span or linear independence in the case opposite to that explored in Argument 9.15, when *m* was less than *n.* Abraham responded immediately (see Figure 4.24) by stating, "If *m* is greater than *n,* you won't be able to span (pause)." As he paused, Justin chimed in with "span the entire dimension," and Abraham agreed with him. This interaction is summarized as the claim in Argument 9.16. The instructor asked Abraham if he could give a reason why his and Justin's claim would be true, to which he replied: "I don't know, it just makes sense, you don't have enough vectors for the dimensions. So it seems you wouldn't be able to go to all the directions in those dimensions."

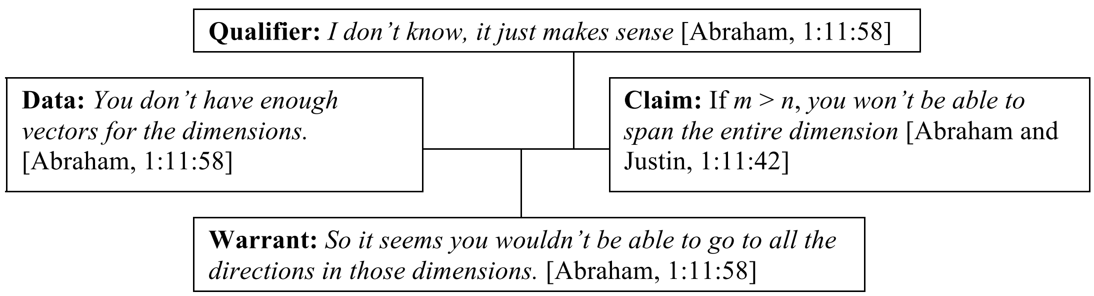


Figure 4.24. Argument 9.16: Abraham and Justin's argument for why if $m > n$, a set of n vectors won't span $\mathbf{R}^m$.

In an m x n matrix, m corresponds to the number of components in the vectors, and n to the number of vectors. Thus, another way to say that m is greater than n is that there are less vectors than dimensions. In Argument 20.19, Edgar claimed it was not possible to have a transformation from $\mathbf{R}^n \rightarrow \mathbf{R}^m$, where $m > n$, be onto. The data he provided was, "There's only two vectors, so you can't possibly span $\mathbf{R}^3$. You simply don't have enough vectors to get anywhere you have in $\mathbf{R}^3$. Thus, the claim that if $m < n$, the columns don't span $\mathbf{R}^n$ served as data in a subsequent argument. Thus, per Criterion Two, this became a normative way of reasoning in this classroom.

4.1.2.1.3 ***NWR #1.3: If A is n x n and the columns of A span R^n^, then the columns of A are linearly independent.*** Another normative way of reasoning that developed in this classroom was the implication that, for a square matrix *A*, if the column vectors of *A* spanned all of $\mathbf{R}^n$, then the columns of the set had to be linearly independent as well. The first few explicit arguments related to this way of reasoning occurred on Day 10. First, in Argument 9.2 and 9.3 (see Figures 4.23 and 4.24), the class made initial conjectures regarding what could be said for the cases of $m < n$ and $m > n$. On Day 10, the teacher had the theorem from the previous day (see Figure 4.21) prepared as a starting point for this day's conversation. After reminding students of their progress the week before, she asked them, based on student responses to the reflections, to focus on $m = n$.

Instructor: So let's jump right in where we left off on Wednesday. We had this theorem up on the board, where *A* was an *m* x *n* matrix where *m* and *n* weren't necessarily equal. So you could have maybe a matrix, maybe it was a 3 x 5. Or you have something like this, where maybe *A* was a 5 x 3. And then we talked about this theorem and on your reflection, a lot of you talked about how you understood this. And you mostly focused on span and pivot points... So what I want to work on today, and I also [saw] this on the Reflections, a lot of you started to talk about what would happen for square matrices. So the theorem actually it could be a square matrix, but it doesn't have to be. So if I tie your arm behind your back and say, let's only talk about square matrices, what in this theorem would change, if anything? And then, are there other things that we also know about linear independence, span, row reduced echelon form, etc. that we could add on to this theorem?

TASK: MAKING CONNECTIONS

What about when $m = n$? What connections are there between the various concepts when the matrix is square? How do you justify these equivalencies? Develop a set of equivalent statements for square matrices (if $m = n$). Write them out as a new theorem, in the following form:

Let *A* be an *n* x *n* matrix. The following are equivalent:
1.
2.
3.
Etc.

Figure 4.25. The main task that students worked on in their small groups on Day 10.

She then posed the task shown in Figure 4.25 for students to work on in their small groups. As part of constituting the task, the instructor facilitated a discussion about what would change in statement 1 from Theorem 4 if the matrix was now *n* x *n*. The class decided the new statement should read, “the columns of *A* span $\mathbf{R}^n$.” This seemed to influence the students to word their conjectures such as, “if *A* is square and the columns span $\mathbf{R}^n$, then...” During the subsequent whole class discussion, four arguments were given for completing this conjecture with “then the columns of *A* are linearly independent” (Arguments 10.2, 10.3, and 10.5).

In Argument 10.2 (see Figure 4.26), Justin, after being asked to share a thought from his table, volunteered the claim that “If we have an *n* x *n* matrix that spans $\mathbf{R}^n$, then the set of vectors in *A* is linear independent.” The instructor immediately called for Justin to provide data for his claim. Before he did so, he qualified his claim by stating, “that's required, [copying #1-4] because if we don't require there to be a pivot in every row, this doesn't have to be true.” His data and warrant provide information regarding how having “as many pivots as we have dimensions” and having “ones along the diagonal” gives “us the span” (Figure 4.26).

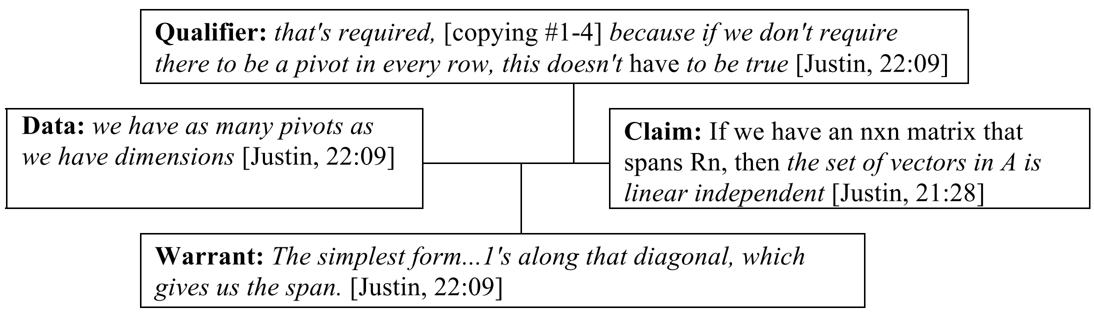


Figure 4.26. Argument 10.2: Justin’s argument for if the columns vectors of an *n* x *n* matrix *A* span $\mathbf{R}^n$, then the column vectors of *A* are linearly independent.

One may notice that, in Justin’s argument, his claim was knowing that *n* columns spanned $\mathbf{R}^n$ implied that those vectors were linearly independent, but that his warrant stated “...which gives us the span.” Justin’s support for the relevance of the data to the claim, which contained mention of “1’s along the diagonal,” was unclear. The instructor, who may have possibly noticed that, asked Nigel if he agreed with Justin. Nigel claimed his group arrived at the same new equivalent statement as Justin (Argument 10.3 in Figure 4.27). Nigel also talked about pivots in his data but added “you only use one vector once.” His warrant, then, served to explain why using vectors once had anything to do with his claim. He stated, "you use each vector to go in a certain dimension, but you can never get back."

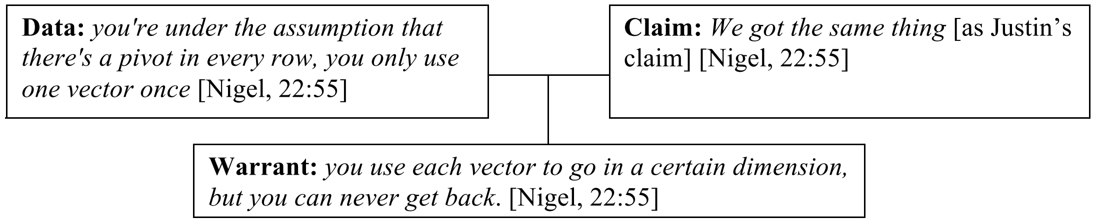


Figure 4.27. Argument 10.3: Nigel's argument for if the columns vectors of an n x n matrix A span $\mathbf{R}^n$, then the column vectors of A are linearly independent.

Nigel's warrant referenced back to the Magic Carpet Ride problem, when the concept of linear independence was first understood in terms of modes of transportation (i.e., the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$) with which you were unable to complete a journey that began and ended at home (had a nontrivial solution to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_n = \mathbf{0}$). Thus, although both Justin and Nigel first tried to use having a pivot in every row for their identical claims, Nigel was able to explain in his warrant, through the language of the Magic Carpet Ride problem, why that implied the columns of A had to be linearly independent.

When asked if his table had any statements they would like to add, Matthew responded, "Any system that spans $\mathbf{R}^n$ would be linearly independent, because the number of variables correspond to the number of rows, so it also corresponds to the number of dimensions." When asked to repeat himself, Matthew stated, "A system that spans all of $\mathbf{R}^n$ would be linearly independent, because it has the same number of rows and columns, and the number of variables corresponds to the number of rows, and also corresponds to the number of dimensions in the system" (Argument 10.4 in Figure 4.28). Matthew, in his explanation of relating span and linear independence, relied first on the distinction of having the same number of rows and columns. Next, in his warrant, Matthew tried to relate having the same number of rows and columns to the number of variables and dimensions. The instructor then supported his effort by noting that, if the matrix were in a system, the number of columns would provide the number of variable and the number of rows would provide the number of equations. The points I draw from Arguments 10.2, 10.3, and 10.5 is that, in each argument, the claim that if the column vectors of an *n* x *n* matrix *A* span $\mathbf{R}^n$, then the column vectors of *A* are linearly independent was under debate and necessitated a high level of justification, and that justification itself was not obvious for members of the class.

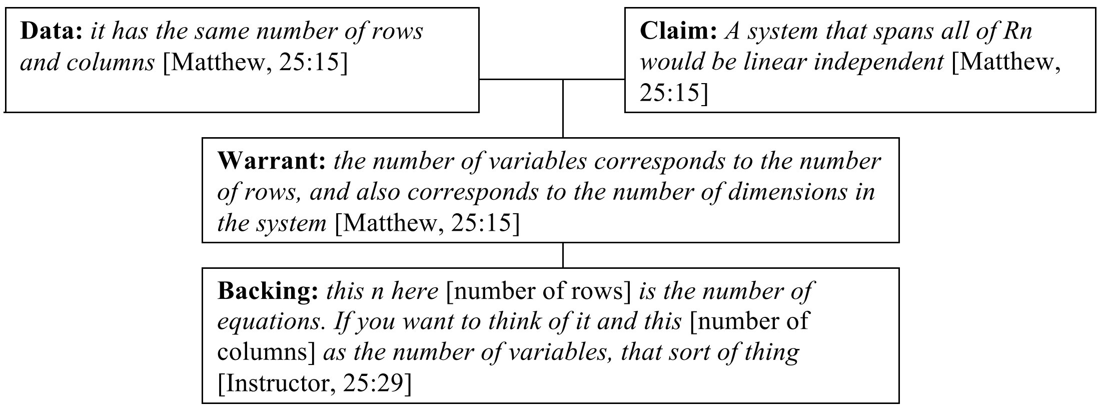


Figure 4.28. Argument 10.5: An argument by Matthew and the instructor for any [square] system that spans all of $\mathbf{R}^n$ would be linearly independent.

The final argument presented for NWR 1.3 comes from Day 20. The class was investigating one-to-one and onto and how those concepts relate to linear independence and span. Within Argument 20.16 (see Figure 4.29) consider Abraham's data. To support his claim that if the matrix is square, being linearly independent is the same as being onto, he mentioned the "*n* x *n* theorem." By Day 20 of the semester, the "*n* x *n* theorem," already known as the Invertible Matrix Theorem, consisted of twelve equivalent statements. In Argument 20, Abraham mentioned two of these equivalencies by name. In the Data1-Claim1 pair, he stated, "If a square matrix is linearly independent, it also spans, and vice versa." The "vice-versa" is inferred to mean, "If a square matrix spans, it also is linearly independent (Data2-Claim2 pair in the data). Focusing on this Data2-Claim2, we see Abraham use this pair (as a unit) as data to support his claim. He did not provide any justification for why this Data2-Claim2 pair is true, nor was any requested by the class members. This is distinct from Arguments 10.2, 10.3, and 10.5 during which justification for this data-claim pair was needed. Thus, per Criterion Two, "If A is n x n and the columns of A span $\mathbf{R}^n$, then the columns of A are linearly independent" functioned as a normative way of reasoning in this classroom.

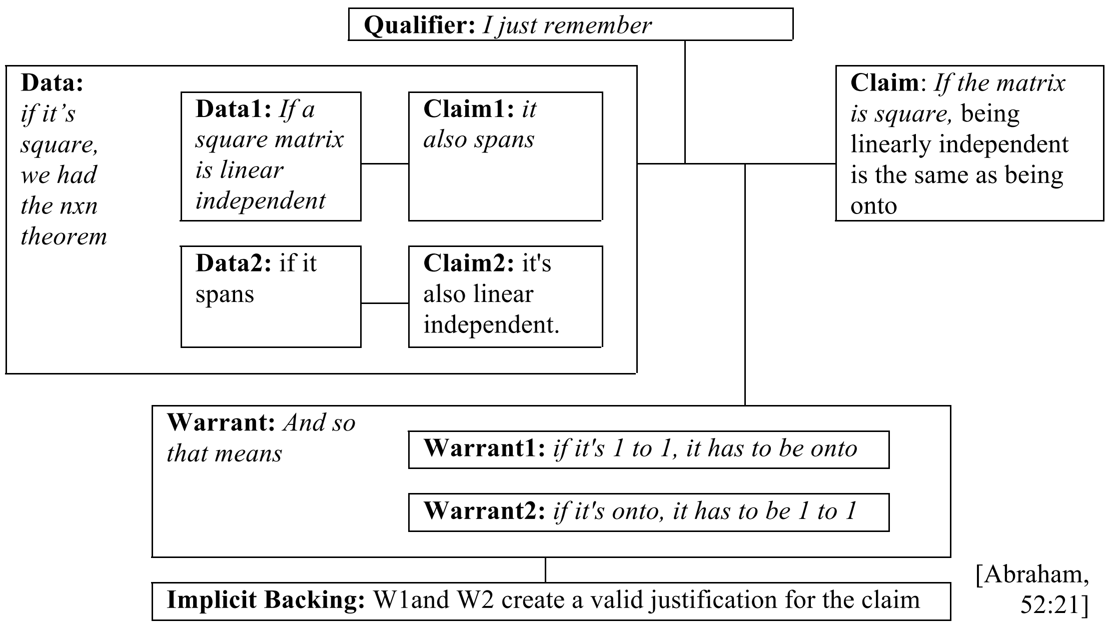


Figure 4.29. Argument 20.16: Abraham's argument for if the matrix is square, being linearly independent is the same as being onto.

4.1.2.1.4 ***NWR 1.4: If A is n x n and the columns of A are linearly independent, then the columns of A span R^n.*** The final normative way of reasoning for the first classroom mathematics practice was established via Criterion 3: it was used as data or warrants in support of various new claims or conjectures related to the IMT. One example of this was just seen in Argument 20.16 (Figure 4.29). Abraham used his understanding of NWR 1.3 and 1.4 as ways of reasoning about the claim, "If the matrix is square, being linearly independent is the same as being onto." In another instance, NWR 1.4 was used as data to support a new claim about determinants. On Day 24, the class discussed determinants and the implications that nonzero and zero determinants had for span, linear independence or dependence, and invertibility. First the class worked to justify the claim that, for a 2 x 2 matrix *A*, if the columns of *A* are linearly dependent, then the determinant of *A* is zero. Next, the teacher posed the question, “Alright, what about the opposite direction? Could I say, 'If the determinant is 0, then the column vectors have to be linear dependent?'” Abraham claimed the assertion was true, but qualified his response with “I don’t know, it just seems...” Without the teacher asking for justification for Abraham’s claim, Randall offered, “It’s a square matrix and it’s independent, so it’s going to span” (see Argument 24.4 in Appendix 4.1). Although Randall seemed uncertain how the data he provided was connected to Abraham’s claim, the data itself was not under scrutiny. Thus, we see this way of reasoning, which previously needed justified during whole class discussion, become used as data for yet another new claim. Thus, for the classroom mathematics practice “Reasoning about Linear Independence and Span as Equivalent Ideas for Square Matrices,” the notion, “If *A* is n x n and the columns of *A* are linearly independent, then the columns of *A* span R^n became a normative way of reasoning for this classroom community per Criterion 3.

4.1.2.1.5 *Conclusion regarding CMP 1.* As defined by Rasmussen and Stephan (2008), a classroom mathematics practice is “a collection of as-if shared ideas that are integral to the development of a more general mathematical activity” (p. 201). The first classroom mathematics practice is the collection of four normative ways of reasoning that were integral to the development of reasoning about span and linear independence as equivalent ideas for square matrices. As these are two of the foundational concepts in the Invertible Matrix Theorem, documenting the development of this classroom mathematics practice addresses the research question of investigating how the classroom reasoned about the IMT over time.

The four normative ways of reasoning involved in this practice were: (a) For a given set of n vectors in $\mathbf{R}^m$, if $m < n$, the set must be linearly dependent; (b) For a given set of n vectors in $\mathbf{R}^m$, if $m > n$, the set cannot span all of $\mathbf{R}^m$; (c) If A is n x n and the columns of A span $\mathbf{R}^n$, then the columns of A are linearly independent; and (d) If A is n x n and the columns of A are linearly independent, then the columns of A span $\mathbf{R}^n$. Each NWR became established as such through the use of one of the three criteria for doing so (Cole et al., 2011; Rasmussen & Stephan, 2008). These normative ways of reasoning encapsulated other normative ways of reasoning; furthermore, they did not develop mutually exclusively or sequentially. The same can be true for classroom mathematics practices. They can emerge in a non-sequential structure, and a CMP itself may be embedded as a normative way of reasoning within a different classroom mathematics practice (Rasmussen, Zandieh, & Wawro, 2009). We see hints of this scenario in Argument 20.16, detailed above in Figure 4.29. Abraham began to conjecture about the existence of an equivalency between one-to-one and onto transformations from $\mathbf{R}^n$ to $\mathbf{R}^n$ (the warrant in Argument 20.16) because of reasoning about linear independence and span as equivalent ideas for n vectors in $\mathbf{R}^n$ (in his data). This argument is explored further in the following section.

As a final illustration of the first CMP, consider Argument 31.8 (see Figure 4.30). This argument was presented in the previous section because of the sequential structure of the argument (see Figure 4.9); here it is presented because of its mathematical content. On Day 31, the last day of class, students worked in their small groups to discuss which ideas, for them, they saw as "most obviously equivalent." Nate responded that "the columns of A span $\mathbf{R}^n$" and "the columns of A are linear independent" are an obviously equivalent pair:

Nate: For me, my logic, I think if the columns of A are linear independent, then it has n amount of pivot points. Then if it has n amount of pivot points, and it's an n x n matrix assuming that, then it spans all of $\mathbf{R}^n$.

This argument is a nice example of a student reflecting on the semester and of what ways of reasoning were established as related to one another. Out of numerous choices of concepts for "obviously equivalent," Nate articulately explained why, for him, span and linear independence were equivalent ideas for n vectors in $\mathbf{R}^n$.

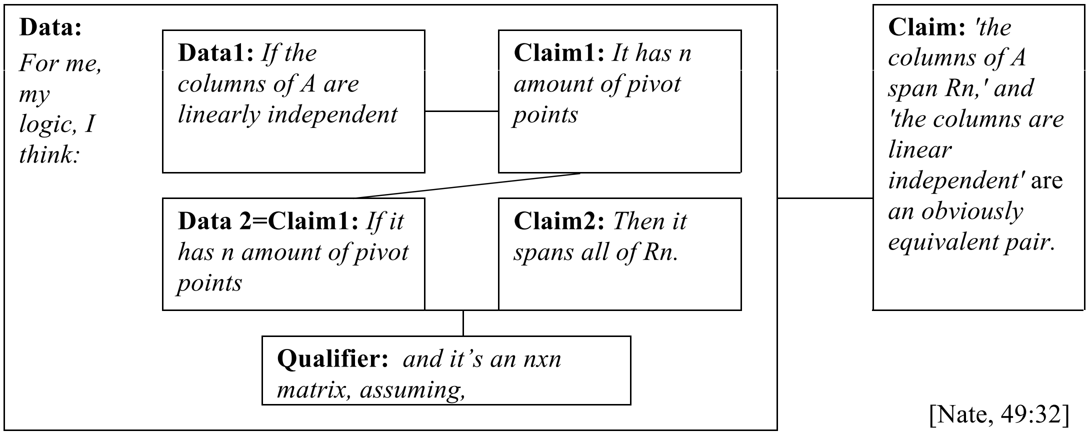


Figure 4.30. Argument 31.8: Nate explains why 'the columns of A span $\mathbf{R}^n$,' and 'the columns are linear independent' are an obviously equivalent pair for him.

4.1.2.2 CMP #2: Determining whether or not a transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ is onto $\mathbf{R}^m$ by considering the span of the column vectors of the associated matrix A. Through analysis of the 118 Toulmin schemes of whole class discussion from the semester, a second classroom mathematics practice was documented. This practice involved a collection of three normative ways of reasoning around the general activity of reasoning with the span of a set of vectors in order to make conclusions regarding if a related transformation was onto. NWR 2.1 is established here via Criterion 3, whereas NWR 2.2 and 2.3 are established via Criterion Two.

CMP #2:
Determining whether or not a transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ is onto $\mathbf{R}^m$ by considering the span of the column vectors of the associated matrix A.

Associated Normative Ways of Reasoning:

2.1 A transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ is onto $\mathbf{R}^m$ when you can "get everything" or "get everywhere" and not onto $\mathbf{R}^m$ when you cannot.
2.2 "A transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ is onto $\mathbf{R}^m$" is interchangeable with "the column vectors of the associated matrix A span all of $\mathbf{R}^m$."
2.3 For a given transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$, if $m > n$, the transformation is not onto $\mathbf{R}^m$.

Figure 4.31. The second classroom mathematics practice and associated ways of reasoning.

4.1.2.2.1 ***NWR 2.1: A transformation T: R^n → R^m is onto R^m when you can "get everything" or "get everywhere" and not onto R^m when you cannot.*** On Day 19, the class session was devoted to defining both onto and one-to-one transformations and working through examples non-examples of each. The class began by considering the transformation associated with the matrix $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$. Students worked in their small groups to figure out what happens geometrically to the unit square under the transformation and then explained their findings in whole class discussion. For example, Randall explained that his group found that the "transformation squeezes the box and stretches it, so it makes it into just that line [$y = 2x$]." Other students made similar comments. The instructor, after saying it seemed like all inputs end up getting sent to that line, went through the calculation $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = (x+3y)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ to show that for any input vector $\begin{bmatrix} x \\ y \end{bmatrix}$, the image of that vector after the transformation is a scalar multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. From this, the instructor defined the term "range" as the set of all images under a given transformation *T*. In addition, based on their work in the previous example, she defined the term *onto* to describe instances in which the range of a transformation was not all the codomain.

Instructor: And our codomain was $\mathbf{R}^2$, so our range doesn't end up being all of $\mathbf{R}^2$, only part of it...So since this doesn't happen, there is a special name for this kind of transformation where this thing seems to fail. So since the range isn't all the codomain, the terms is that *T* is not onto $\mathbf{R}^2$. And let's think about what we just had, we said it would be onto if the range equaled the codomain. Another way to say that would be, if every **b** in the codomain is the image of at least one **x** from the domain.

After considering the function $y = x + 5$ as an example of a function from high school algebra that was onto, the instructor asked students to work in their groups and generate examples of functions from high school algebra that were not onto. Nigel, for example, argued that $y = x^2$ was not onto (Argument 19.1, see Figure 4.32) and supported this claim with the data of "for all your *x* values, you're going to have values, you're going to have *y* values, but you're not going to have these *y* values [gestures to the area below the *x*-axis] at all." When the instructor asked how he knew it was impossible to get a *y* value of negative five, Nigel responded, " because there's no *x* value that would get you to that." In this warrant, Nigel made use of the phrase "get you to" to describe what input values for the given function could do; there did not exist any real-values numbers that, when input for the function $y = x^2$, would "get you to" $y = -5$.

Data: *for all your x values, you're going to have y values, but you're not going to have these y values at all* [Nigel 16:35]

Claim: the function $y = x^2$ is not onto [Nigel, 16:35]

Warrant: For instance, the y-value of -5 is impossible *because there's no x value that would get you to that.* [Nigel & Instructor, 17:03-17:11]

Backing: *from the graph you can see, the graphical representation of the function never crosses at y equals -5* [Instructor, 17:14]

Figure 4.32. Argument 19.1: Nigel's argument for why the function $y = x^2$ is not onto.

Data: [draws graph of function] *you can not, you can't even get* [draws a dash on the *y*-axis and circles his pen over it] [Nate, 18:17]

Claim: the function $y = \sqrt{x}$ is not onto [Nate, 17:40]

Warrant: *Yeah, so altitude up and down, the y values, there's no way to put in something, take the square root of it and get a negative number, that would be a real-valued solution, right?* [Instructor, 18:22]

Figure 4.33. Argument 19.2: Nate's argument for why the function $y = \sqrt{x}$ is not onto.

Immediately following Nigel's explanation, Nate volunteered that the function $y = \sqrt{x}$ is not onto. He came to the board and sketched the graph of the function, and then seemed hesitant about what other explanation was needed. He said, "you can't even get" in his data, but did not finish his sentence; rather, he circled a hash on the *y*-axis below the *x*-axis as he said, "you can't even get." The instructor's warrant served to help support why Nate's data made sense for the claim, mimicking his language by saying there was no way to "get a negative number." From language used in both Arguments 19.1 and 19.2, we see evidence of not being able to "get" certain outputs as evidence that the function is not onto.

Next, the instructor asked students to consider the transformation T from $\mathbf{R}^2$ to $\mathbf{R}^3$ defined by the matrix $A = \begin{bmatrix} 4 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}$, and she stated, "And so the question is really, Can we get to everywhere in $\mathbf{R}^3$? Is the range going to be all of $\mathbf{R}^3$?" Notice the instructor reworded the problem using the phrase, "can we get everywhere," which is the phrasing used in this normative way of reasoning 2.1. Arguments 19.3 and 19.4, which are detailed in the discussion of NWR 2.2, pertained to explanations of why this transformation was not onto. Argument 19.5 (see Figure 4.34) addressed the teacher's request for information about the range of T.

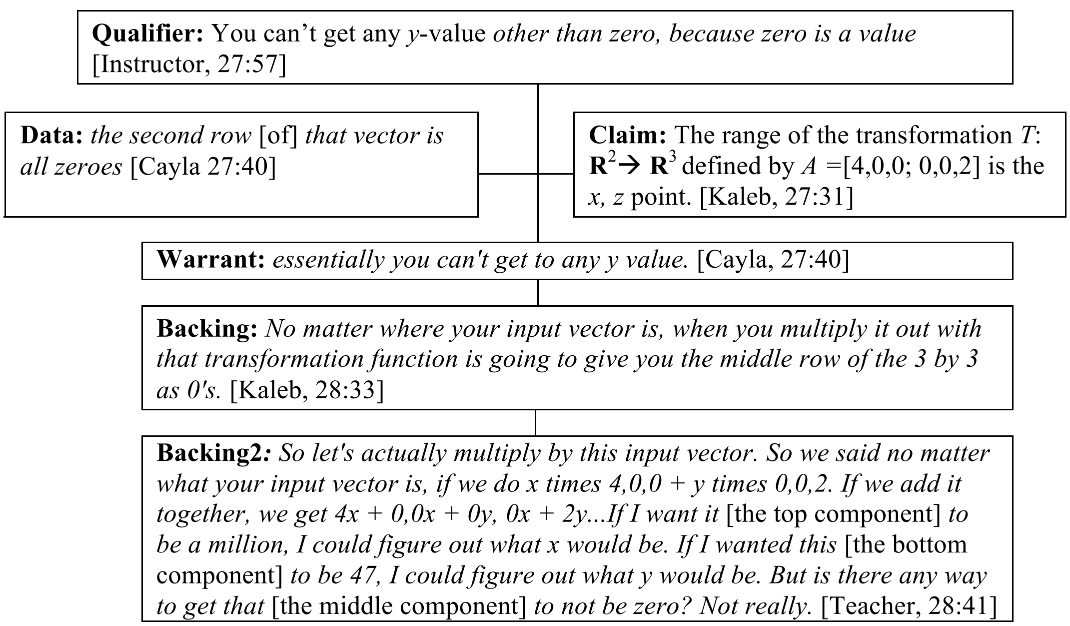


Figure 4.34. Argument 19.5: An argument regarding the range of a transformation from $\mathbf{R}^2$ to $\mathbf{R}^3$.

Of interest in Argument 19.5 for NWR 2.1 is Cayla's warrant that "essentially you can't get to any *y*-value." The instructor qualified Cayla's statement by adding, "you can't get to any *y*-value other than zero, because zero is a value." There was a slight shift in use of the word "get" in this argument. In Arguments 19.1 and 19.2, students spoke about various outputs that were impossible to "get" from the functions under discussion. In Argument 19.5, as well as in the instructor's prompt preceding Argument 19.5, the word "get" was being used to express that you cannot "get *to*" all values. This language is reminiscent of a way of speaking that developed out of the Magic Carpet Ride problem, during which students explored span as all the possible locations they could reach with their modes of transportation (i.e., the linear combinations of a given set of vectors). As such, a normative way of reasoning in the class became the equivalence of "getting everywhere" with "the span is all of $\mathbf{R}^n$."

An exemplary case of this distinction can be seen by considering Arguments 20.18 and 20.19. Both occurred on Day 20, when the class developed generalizations regarding what types of linear transformation were or were not one-to-one and/or onto. In both arguments, the claim was that it is not possible to have a transformation from $\mathbf{R}^n \rightarrow \mathbf{R}^m$, where $m > n$, be onto $\mathbf{R}^m$. These two arguments are important to the development of NWR 2.3, but what is of importance in the discussion of NWR 2.1 is the underlying imagery in each argument. In Argument 20.18, the data Nate provided for the claim was that "you'll always go up a dimension." This transformational imagery was reiterated in his warrant when he said, "that means we're going to have, more or less the way I think of it as dots in a diagram, we have more dots in our results in our codomain than in our domain," so "not every dot can be mapped to" (given in his backing). The instructor supported his argument and added that all the vectors in $\mathbf{R}^3$ "can't all get mapped to." This underlying imagery for the notion of onto is very much grounded in the transformation mapping input vectors to output vectors, consistent with Larson's "matrix acting on a vector" view of matrix multiplication (Larson, 2010).

Data: *you'll always go up a dimension* [Nate, 55:45]

Claim: It is not possible to have a transformation from $\mathbf{R}^n \rightarrow \mathbf{R}^m$, where $m > n$, be onto [Nate, 55:45]

Warrant: *that means we're going to have, more or less the way I think of it as dots in a diagram, we have more dots in our results in our codomain than in our domain* [Nate, 55:45]

Backing1: *so there's no way for every dot in R3 to be mapped to*. [Nate, 55:45]

Backing2: *I think that was really cool ... He's saying if you start out in* $\boldsymbol{R}^2$ *but we have more outputs vectors in* $\boldsymbol{R}^3$ *in a certain way, well, they can't all get mapped to*. [Instructor, 56:13]

Figure 4.35. Argument 20.18: Nate's argument for why it is impossible for T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$, $m > n$, to be onto $\mathbf{R}^m$.

In contrast, however, was Edgar's justification for the same claim (see Argument 20.19 in Figure 4.36). For his data, Edgar stated that "there's only two vectors, so you can't possibly span $\mathbf{R}^3$. You simply don't have enough vectors to get anywhere you have in $\mathbf{R}^3$." Whereas Nate's explanation focused on not being able to map to every "dot" in the codomain, Edgar's focused on not having enough vectors to get everywhere in $\mathbf{R}^3$. As stated, this is consistent with the functioning as-if shared way of reasoning about span as all places "you could get" with the linear combination of vectors. Furthermore, it is also consistent with a "vector acting on a matrix" view of matrix multiplication (Larson, 2010). Further discussion of this distinction is not pursued here, but the implications for viewing a matrix times a vector as either a transformation mapping an input vector to an output vector or as a linear combination of the column vectors of the matrix surfaces here in so little as the use of the word "get."

Data: *There's only 2 vectors, so you can't possibly span* $\boldsymbol{R}^3$. *You simply don't have enough vectors to get anywhere you have in* $\boldsymbol{R}^3$. [Edgar, 56:48]

Claim: It is not possible to have a transformation from $\mathbf{R}^n \rightarrow \mathbf{R}^m$, where $m > n$, be onto [Edgar, 56:48]

Warrant: *If we only have 2 vectors then all combinations of those 2 vectors are only going to give us a plane in* $\boldsymbol{R}^3$. [Instructor, 57:02]

Backing: *You don't have any way to get to all of* $\boldsymbol{R}^3$. *So there's no way it could be onto.* 57:02]

Figure 4.36. Argument 20.19: Edgar's argument for why it is impossible for T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$, $m > n$, to be onto $\mathbf{R}^m$.

From the repeated use of variations of the terms "get everything" or "get everywhere" over the course of Days 19 and 20, per Criterion 3, the way of reasoning with these terms in order to support various claims about transformations that were or were not onto a given $\mathbf{R}^m$ has been established as normative in this class.

4.1.2.2.2 ***NWR 2.2: "A transformation T: $R^n \rightarrow R^m$ is onto R^m" is interchangeable with "the column vectors of the associated matrix A span all of R^m."*** As alluded to in the discussion of NWR 2.1, an equivalence between determining if a transformation was onto and determining the span of the column vectors of a matrix *A* associated with the transformation developed into a way of reasoning that functioned as-if shared in the classroom. NWR 2.2 specifically dealt with how this equivalence allowed for this way of reasoning to develop.

Argument 19.3 (see Figure 4.37) was the first mention of span in relationship to onto, which had been defined in terms of range and codomain. In Argument 19.3, Justin attempted to provide a reason that the transformation defined by $A = \begin{bmatrix} 4 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}$ was not onto $\mathbf{R}^3$ by focusing on the row of zeroes in matrix *A*, saying it would "give you a column of zeroes." He continued with a warrant, saying "and that breaks it spanning all of $\mathbf{R}^3$" (see Figure 4.37). Justin's warrant was the first mention of span in conjunction with onto during whole class discussion. The instructor called for elaboration, and Justin responded by stating that the columns of *A* would not span $\mathbf{R}^3$ (see Argument 19.4 in Figure 4.38).

Data: *you have a row of zeroes, so it's going to give you a column of zeroes. And so you will have two vectors with numbers in it, or two columns with numbers in it, and a column with zeroes in it* [Justin, 22:32]

Claim: The transformation T: R2→ R3 given by [4,0,0; 0,0,2] *can't be onto* [all of $\mathbf{R}^3$] [Justin, 22:32]

Warrant: *And that breaks it spanning all of* $\boldsymbol{R}^3$ [Justin, 22:32]

Figure 4.37. Argument 19.3: Justin explains why a particular transformation from $\mathbf{R}^2$ to $\mathbf{R}^3$ cannot be onto.

For his justification, Justin returned to trying to talk about zeroes in either the columns or the rows of the output, and Nigel assisted Justin by suggested in his warrant that "there's no values in the middle...[so you] can't span where the zeroes are, that dimension" (see Argument 19.4 in Figure 4.38). The instructor supported their contributions by stating there were some good ideas but qualifying that "they all involve a little more thought." From these two arguments we see the first seeds of this way of reasoning, about how understanding the span of a set of vectors in a matrix can inform whether the associated transformation is onto.

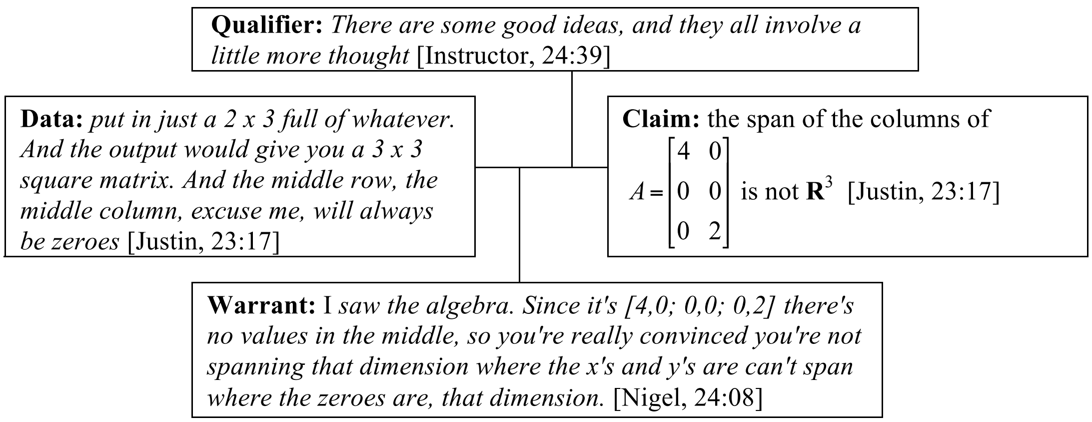


Figure 4.38. Argument 19.4: Nigel and Justin explain why the columns of a given 3x2 matrix do not span $\mathbf{R}^3$.

The next problem the class worked through was to determine the matrix associated with the transformation $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$; Saul's group determined the matrix was $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and the class agreed. When discussing whether or not *T* was onto $\mathbf{R}^2$, the instructor called on Giovanni:

Instructor: So Giovanni, I know your table was talking about this, other people might not have gotten that far, and I realize that. So let's hear a good explanation of what you guys were discussing about onto $\mathbf{R}^2$?

Giovanni: Onto $\mathbf{R}^2$, we were talking about that it has two separate vectors, the two pivot points right there. Minus, you don't even necessarily need that first vector. So with those two pivot points, that's how it can span all of $\mathbf{R}^2$.

Instructor: What's the 'it' that can span all of $\mathbf{R}^2$?

Giovanni: The vector. Not the vector, the matrix.

Instructor: The vectors in the matrix? [Giovanni: Yes.] So he's saying that these vectors can span $\mathbf{R}^2$. We haven't talked about spanning and onto together yet, so let's investigate that a little bit. So someone else who hadn't heard this before or even from Giovanni's table, how would we connect why vectors spanning $\mathbf{R}^2$ and relate it to the transformation being onto $\mathbf{R}^2$? How can we relate Giovanni's, what he noticed, which is true, since the vectors span $\mathbf{R}^2$, why would that mean the transformation is onto $\mathbf{R}^2$?

Giovanni responded with data that the matrix *A* had two pivot points, and then his warrant attested to why that was relevant to the claim by stating "so with those two pivot points, that's how it can span all of $\mathbf{R}^2$" (see Argument 19.7 in Figure 4.39). The instructor responded by directing attention to the notion of span connecting to onto, something that had surfaced in three arguments so far. She stated, "We haven't talked about spanning and onto together yet, so let's investigate that a little bit...How can we relate Giovanni's, what he noticed, which is true, since the vectors span $\mathbf{R}^2$, why would that mean the transformation is onto $\mathbf{R}^2$?"

Data: *It has 2 separate vectors, the 2 pivot points right there. Minus, you don't even necessarily need that 1st vector.* [Giovanni, 41:45]

Claim: The transformation $T[x_1,x_2,x_3]=[x_1,x_2]$ is onto $\mathbf{R}^2$ [Giovanni, 41:45]

So with those 2 pivot points, that's how it [the vectors in the matrix] *can span all of* $\mathbf{R}^2$. [Giovanni, 41:45, 42:05]

Figure 4.39. Argument 19.7: Giovanni's argument for why $T\,[x_1, x_2, x_3] = [x_2, x_3]$ is onto $\mathbf{R}^2$.

Justin volunteered data to the instructor's claim that span had anything to do with onto by stating "if you can't choose any value, you're not going to be able to get output. But since you can go anywhere, you can use anywhere as an, do you know what I'm trying to say?" Josh offered another explanation, stating "having the required amount of pivot points in the correct positions can span that number of dimensions." Wanting to hear more about how that data supported the claim that onto and span were connected concepts, the instructor prompted Josh for a warrant. He responded, "Because it has two pivots, and it can span $\mathbf{R}^2$. And then since those two pivots are located in the *y* and *z* column, you're trying to have it in the *y* and *z*, it just, it works out" (full analysis is seen labeled Argument 19.8 in Appendix 4.1).

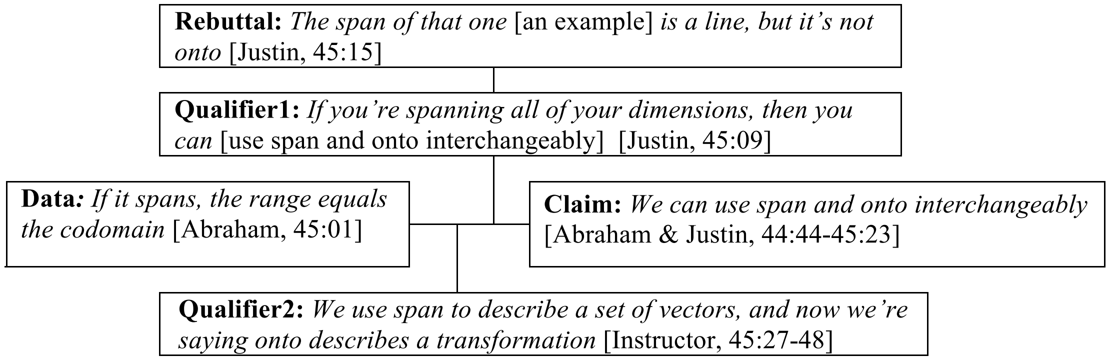


Figure 4.40. Argument 19.9: Abraham and Justin address if span and onto can be used "interchangeably."

As a response to Giovanni's, Justin's, and Josh's effort, the instructor stated how interesting it was that "span has always been a property of vectors. And now we're trying to talk about functions, so it's how do we correlate what we know about vectors and how you combine them to a certain span to get something in the transformation?" In response to this, Jerry asked the class, "Can we use the span and onto interchangeably?" (See Argument 19.9 in Figure 4.40.)

Justin and Abraham were quick to respond. Abraham provided the data that "if it [the column vectors] span, the range equals the codomain." That was exactly how "onto" was originally defined in class—a transformation is onto $\mathbf{R}^n$ if the range equals the codomain. Justin, however, offered a qualifier to this claim, stating the two concepts were only interchangeable if the span was all of $\mathbf{R}^m$. He provided an example that, without the qualifier, would serve to rebut the claim. He referenced a previous example that had been discussed in class: the transformation associated with $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$, for which he said, "the span of that one is a line, but it's not onto." Justin's qualifier and rebuttal served to add a degree of care in conclusions that were being made in the classroom. He pointed out, through the use of a counterexample, that "span" and "onto" are only interchangeable when the vectors' span is all of $\mathbf{R}^n$. Finally, the instructor added another qualifier to whether or not the two ideas were interchangeable, stating, "We use span to describe a set of vectors, and now we're saying onto describes a transformation" (see Argument 19.9 in Figure 4.40). After this argument, the classroom discussion shifted to the notion of one-to-one transformations. The discussion followed much of the same format, with students investigating examples and non-examples from both high school algebra and linear algebra.

Data: *A lot of you had also said* [it] [Instructor, 02:48]

Claim: *Onto is the same as saying the column vectors of the matrix span all of* $\boldsymbol{R}^m$ [Instructor, 02:48]

Figure 4.41. Argument 20.2: The instructor reminds students of the connection between span and onto.

On Day 20, the discussion regarding one-to-one and onto continued. The instructor began class by reminding the students of their previous agreement, that "onto is the same as saying the column vectors of the matrix span all of $\mathbf{R}^m$," and the only data she provided was "a lot of you had said it." The claim did not need supported with data or warrant (see Figure 4.41).

The remainder of Day 20 was dedicated to making generalizations regarding transformations that were one-to-one, onto, both, or neither. Throughout the day, the way of reasoning that "onto is the same as saying the column vectors of the matrix span all of $\mathbf{R}^m$" functioned as-if shared in the classroom. For instance, in Argument 20.13 (see Figure 4.42), Cayla claimed that a transformation defined by the matrix $A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$ was onto.

Data: *Columns are linearly independent.* [Cayla, 48:21]

Claim: The transformation defined by the matrix $A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$ is onto [Cayla, 48:21]

Warrant: *If you row reduce it, you would be able to get a pivot position in every row* [Cayla, 48:21]

Backing: *so it would span all of* $\boldsymbol{R}^2$. [Cayla, 48:21]

Figure 4.42. Argument 20.13: Cayla's argument for why a given transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ was onto.

Her statement, "the column vectors are linearly independent" provided data for the claim. Next, she stated the row-reduced form of *A* would have a pivot in every row and thus could deduce that the columns spanned $\mathbf{R}^2$ (warrant and backing, respectively, in Argument 20.13). Note that no one inquired into why her backing, which was about span, supported her claim, which was about onto. Furthermore, that which had been a source of debate the previous day (e.g., Arguments 19.8 and 19.9) was now being used to support a new claim. This way of reasoning also surfaced in various arguments in Day 20 as data or warrants for other new claims, namely that "If the matrix is square, being linearly independent is the same as being onto" (Argument 20.16, see Figure 4.29) and "if a matrix is square, if the associated transformation is one-to-one, it is always also onto" (Argument 20.25, see Figure 4.45). In those arguments, NWR 2.2 was used in a very key way to make these new claims about connecting, in the square case, linear independence with onto as well as one-to-one with onto. Those arguments are examined in more detail in the conclusion of this section. Thus, we can say that "a transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ is onto $\mathbf{R}^m$" is interchangeable with "the column vectors of the associated matrix *A* span all of $\mathbf{R}^m$" was a normative way of reasoning in this class.

4.1.2.2.3 ***NWR 2.3: For a given transformation T: $R^n \rightarrow R^m$, if m > n, the transformation is not onto R^m.*** This is the last normative way of reasoning that comprised the second classroom mathematics practice. This way of reasoning developed on Day 20 of the semester, when students were investigating the properties of one-to-one and onto for linear transformations. In Arguments 20.18 and 20.19, the main claim that was being justified is that it is not possible to have a transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ be onto $\mathbf{R}^m$ if $m > n$. In these two arguments, this way of reasoning about onto was new and still needed justification regarding its validity. As was seen in the development of NWR 2.1, this claim was supported in two different ways: one argument focused on where input vectors get mapped to (Argument 20.18, see Figure 4.35), and one focused on the linear combinations of the column vectors (Argument 20.19, see

Figure 4.36). Specifically, Edgar's argument in 20.19 used the idea that there aren't enough column vectors to span $\mathbf{R}^m$ if m was less than n. This way of reasoning was reminiscent of NWR 1.2, which said if $m < n$ then the column vectors of a matrix cannot span $\mathbf{R}^m$. Thus, an idea previously established as a normative way of reasoning for one classroom mathematics practice can be used as data for a claim that is relevant to a different classroom mathematics practice.

In Arguments 20.22 and 20.26, this way of reasoning shifted from being the claim to being part of the data and part of the warrant, respectively. Both of these arguments dealt with the co-existence of the properties of one-to-one and onto for transformations from $\mathbf{R}^n$ to $\mathbf{R}^n$. Argument 20.22 (see Figure 4.43) occurred in response to Argument 20.21 (see Appendix 4.1) in which Cayla supported her claim that it is possible to have a transformation that is both one-to-one and onto. In her data, Cayla said, "it had to be square" for it to be possible, so the instructor elaborated upon that in Argument 20.22. Within her justification, the instructor made use of a Proof by Cases structure. Her Data-Claim1 pair stated, "If it's [the matrix associated with a transformation] wider than tall, then 1-1 isn't possible," and her Data2-Claim2 pair was "if it's taller than wide, then onto isn't possible." The latter pair could be rephrased as, "if m is less than n, then onto isn't possible." Thus, we see what had previously served as the claim become part of the data for a new claim.

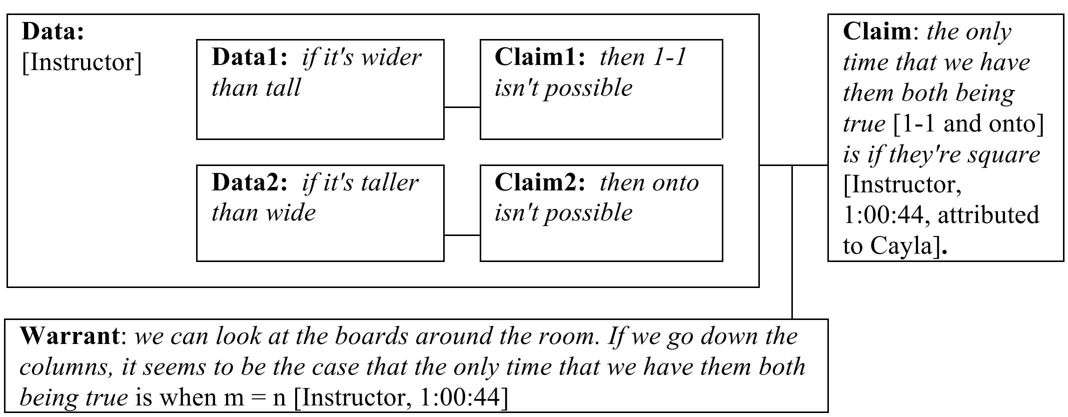


Figure 4.43. Argument 20.22: The instructor discusses that transformations can be both 1-1 and onto if the matrix is square.

In Argument 20.26 (see Figure 4.44), Abraham claimed that if a transformation is either one-to-one or onto but not both, then it has to be "non-square." Here is another example of metonymic speech (Lakoff & Johnson, 1980) in which the matrix has come to represent the transformation for students. In his warrant, Abraham explained what in the chart (something that been created in class to organize generalizations), was relevant to his claim. His second data-claim pair in his warrant, which has the Proof by Cases structure, makes use of NWR 2.3. Thus, per Criterion Two, we can say that **"**For a given transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$*,* if $m > n$*,* the transformation is not onto $\mathbf{R}^m$" was a normative way of reasoning in this class.

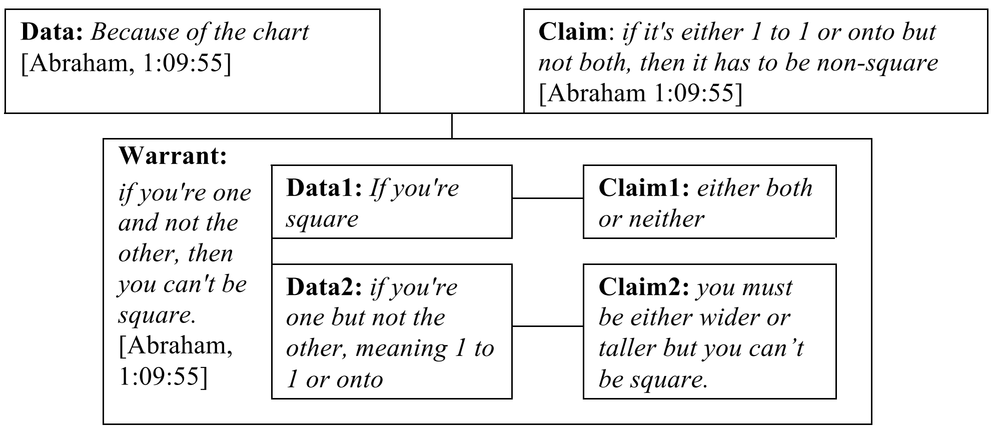


Figure 4.44. Argument 20.26: Abraham's argument for if transformations from $\mathbf{R}^n$ to $\mathbf{R}^m$ are either 1-1 or onto but not both that m cannot equal n.

4.1.2.2.4 ***Conclusion regarding CMP 2.*** Relating the span of the columns of the matrix A associated with the given transformation in order to reason about whether or not that transformation was onto became a classroom mathematics practice for this class. It is the collection of three normative ways of reasoning that were integral as the class developed a way to reason about whether a given transformation is onto by considering the span of the column vectors of the matrix associated with the transformation. The three normative ways of reasoning in this practice were: (a) a transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ is onto $\mathbf{R}^m$ when you can "get everything" or "get everywhere" and not onto $\mathbf{R}^m$ when you cannot; (b) "a transformation T: $\mathbf{R}^n$

$\rightarrow$ $\mathbf{R}^m$ is onto $\mathbf{R}^m$" is interchangeable with "the column vectors of the associated matrix A span all of $\mathbf{R}^m$"; and (c) for a given transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$, if $m > n$, the transformation is not onto $\mathbf{R}^m$. As with the first class mathematics practice, each of these were established through the three criteria for when ideas function as-if shared in a classroom setting.

4.1.3 Conclusion

I conclude this section regarding the documentation of classroom mathematics practices with two final remarks. First, CMP 2 was a collection of normative ways of reasoning about span and onto. Preliminary analysis suggests that a similar practice developed in this classroom for the concepts of linear independence and one-to-one. Especially for NWR 2.2 and 2.3, analogous normative ways of reasoning could be documented, such as: "a transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ is one-to-one" is interchangeable with "the column vectors of the associated matrix A are linearly independent;" and for a given transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$, if $m < n$, the transformation is not one-to-one. As the documentation of this proposed CMP would be very similar to that of CMP 2, doing so is delegated to future work.

Second, I reflect on and compare the two documented classroom mathematics practices. The final form of the second classroom mathematics practice was not what was expected. Given that the Invertible Matrix Theorem developed over the course of the semester, and that the ideas involved in the IMT can be grouped together by theme, I thought the CMPs would similarly be parallel by theme. For instance, if we consider the first CMP, it was a collection of functioning as-if shared ideas that led to students reasoning about span and linear independence of column vectors as equivalent ideas for square matrices. I imagined a similar classroom mathematics practice for the concepts of one-to-one and onto transformations. It is the case that there exists a few arguments that do involve reasoning about one-to-one and onto as equivalent ideas when m equals n, but the arguments necessary to substantiate normative ways of reasoning that could be collected around this common theme in order to call it classroom mathematics practice per the methodology, quite simply, do not exist within the data set.

Consider, for example, Argument 20.25 (see Figure 4.45). Gabe presented a complex justification for the claim that if a matrix is square and the associated transformation is one-to-one, it must also always be onto. The reasoning he employed in his justification, however, relied completely on the equivalence of span with onto, of linear independence with one-to-one, and of span with linear independence.

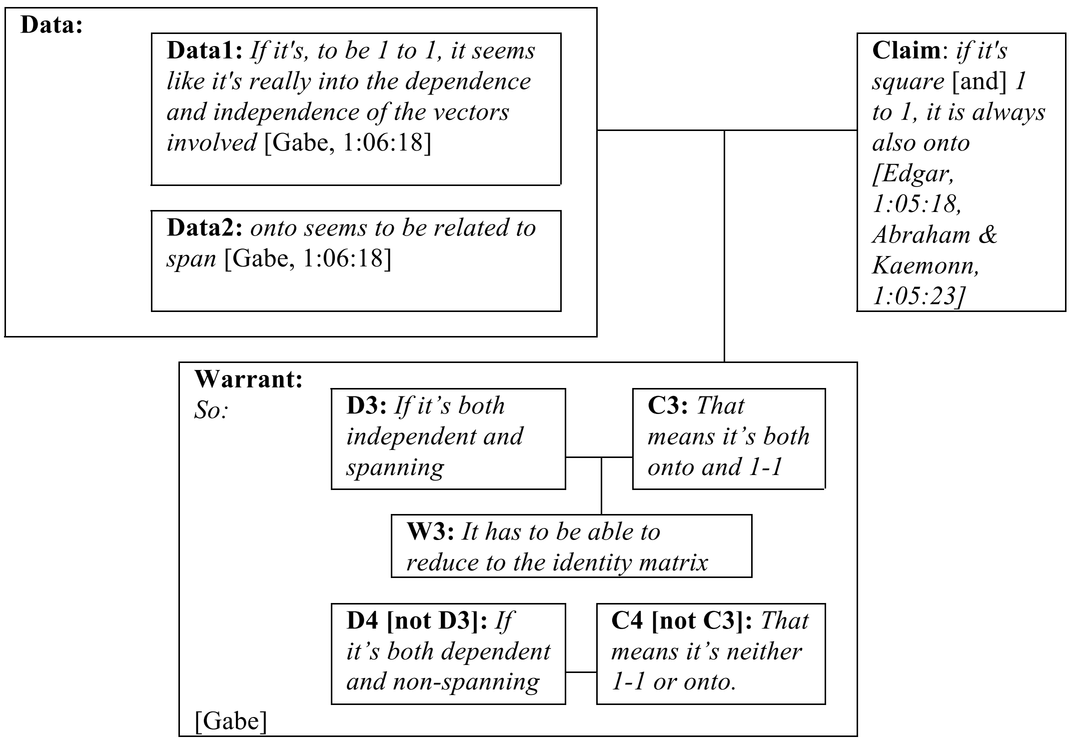


Figure 4.45. Argument 20.25: Gabe justifies why 1-1 implies onto for square matrices.

In Data1 and Data2, he stated that one-to-one "seems like it's really into the dependence and independence of the vectors involved" and onto "seems to be related to span" (NWR 2.2), respectively. He then made use of Data1 and Data2 in his warrant (Data3-Claim3) by claiming "if it's both independent and spanning, that means it's both onto and one-to-one." The statement "if it's both independent and spanning" implies that he knew such a situation was possible

(CMP 1). His Data4-Claim4 pair was the converse of Data3-Claim3. In a loose sense, Gabe's argumentation exhibited a sort of "transitivity." Consider the schematic in Figure 4.46:

Because:	
For $m = n$:	Span ⇔ Linear Independence
For all m, n:	Span ⇔ Onto
For all m, n:	Linear Independence ⇔ One-to-One
I can conclude:	
For $m = n$:	Onto ⇔ One-to-One

Figure 4.46. Simplified schematic of Gabe's justification layout in Argument 20.25.

What justified the equivalence of one-to-one and onto is firmly tied to the equivalence of span and linear independence in the case of *m* equals *n*. Other arguments presented in class (such as Argument 20.16) displayed a similar reliance upon connections to span and linear independence. Furthermore, there were no arguments that presented a clear explanation that relied upon what it means for a transformation to be onto and what it means for a transformation to be one-to-one within the justification for the equivalence. Thus, for when $m = n$, the establishment of equivalence for the concepts of onto and one-to-one was not analogous to the development of equivalence for span and linear independence. Possible implications for curriculum development and instruction are discussed in Chapter 6.

4.2 Adjacency Matrix Analysis

The remainder of this chapter continues to address results concerning the first research question: How did the classroom community reason about the Invertible Matrix Theorem (IMT) over time? In the previous section, I utilized Toulmin's model as an analytical tool at both the microgenetic and ontogenetic level to address this research question, and I presented results concerning (a) the nature of the argumentation structure, and (b) what ways of reasoning functioned as-if shared within the classroom discourse. In this section, I present results concerning the use of adjacency matrices as an analytical tool to address the same overarching research question. I detail both the microgenetic and ontogenetic analyses from utilizing adjacency matrices on the same portions of whole class discussion throughout the semester. The analysis revealed that uses of concept statements from the IMT versus interpretations of those concept statements depended on the purpose of the particular argument, as well as the prevalence of reasoning about non-square matrices in order to develop ways of reasoning about the equivalence of concepts in the IMT. In addition to how the structure of arguments changed over time, ontogenetic analysis revealed the concepts most central to the class community's development as a whole. I conclude the chapter with a discussion of the results and a reflection on and comparison of the separate analytical methods.

As stated in Chapter 3, I conducted analysis over the same data set using both Toulmin's model and adjacency matrices as analytic tools. The video and transcript of whole class discussion from all 31 days of the semester were narrowed down to portions of ten days to analyze. From this reduction of the data set, 118 arguments were analyzed using Toulmin's model. A selected portion of these results was presented in the previous Microgenetic Analysis and Ontogenetic Analysis via Toulmin's Model sections. The transcript from this reduced data set was also analyzed using adjacency matrices, and 109 of the 118 arguments were coded; the content of the other nine was inappropriate for the codes developed for the present adjacency matrix analysis. The complete list of adjacency matrix codes for each argument, as well as which arguments were not analyzed, can be found in Appendix 4.3.

An adjacency matrix depicts how the vertices of a particular directed graph (also called a digraph) are connected. For a given digraph, an adjacency matrix is an *n* x *n* matrix $[a_{ij}]$ with one row and one column for each of the *n* vertices in the digraph, and the values a_{ij} of the matrix are given by $a_{ij} = c$, where *c* is the frequency of the edges from the i^{th} vertex to the j^{th} vertex (Chartrand & Lesniak, 2005). Furthermore, a vertex *u* is *adjacent to v* (and *v* is *adjacent from u*) if there exists an arc *a* from *u* to *v*. These and other relevant definitions are given in Figure 4.47.

Mathematical Definitions relevant to Adjacency Matrix Analysis

1. **Edge**: $e = \{u, v\} = uv$ joins the vertices *u* and *v*.
2. **Arc**: an edge of a directed graph. $a = (u, v)$ is an arc of digraph *D* if it joins *u* to *v*.
3. **Incident**: If $a = (u, v)$ is an arc of digraph *D, a* is incident from *u* and incident to *v*. Also, *u* is incident to *a,* and *v* is incident from *a*.
4. **Adjacent vertices**: Vertices *u* and *v* are adjacent if there exists an arc *a* to join them. If $a = (u, v)$, *u* is adjacent to *v* but *v* is not adjacent to *u* (rather, *v* is adjacent from *u*).
5. **Adjacency matrix**: The adjacency matrix *A*(*D*) of digraph *D* with vertices $V(D) = \{v_1, v_2, ... v_n\}$ is the *n* x *n* matrix $[a_{ij}]$ defined by $a_{ij} = c$, where *c* is the number of arcs that are incident from v_i to v_j for any two vertices in *D.* Thus, the entries for any adjacency matrix *A* must be nonnegative.
6. **Size**: Total number of arcs in digraph *D.* This is notated $A(D) = s$.
7. **Order**: the number *r* of vertices in a digraph *D.*
8. **Out-degree**: The out-degree of a vertex *u* (od *u*) in digraph *D* is the number of arcs that are incident from *u*.
9. **Out-connection**: The out-connection of a vertex *u* (oc *u*) in digraph *D* is the number of distinct vertices that are adjacent to *u.*
10. **In-degree**: The in-degree of a vertex *v* (id *v*) in digraph *D* is the number of arcs that are incident to *v*.
11. **In-connection**: The in-connection of a vertex *v* (ic *v*) in digraph *D* is the number of distinct vertices that are adjacent from *v.* (in *v*)
12. **Degree**: The degree of a vertex *w* (deg *w*) of a digraph *D* is defined as deg *w* = od *w* + id *w*.

Figure 4.47. Mathematical definitions relevant to adjacency matrix analysis.

The adjacency matrices analyzed in this section correspond to directed graphs in which the vertices are statements related to the concepts of the IMT, and the edges are directed in such a way as to match the implication offered by the speaker(s). All video and transcript analyzed with adjacency matrices occurred during segments of whole class discussion that were relevant to the development of the Invertible Matrix Theorem. Furthermore, as described in Chapter 3, these data are the same segments of video and transcript that were analyzed in the previous section regarding Toulmin analysis.

The statements and interpretations used as the vertices for the directed graphs (and hence as the rows and columns of the corresponding adjacency matrices) were determined through a grounded analysis of the video and transcript of whole class discussions during which the classroom conversation was directly relevant to the development of and reasoning about the Invertible Matrix Theorem. There were 100 total vertices, arranged into 15 categories (see Figure 4.48). The main code for each of the 15 categories (except for the "S: Miscellaneous" code) are concept statements from the Invertible Matrix Theorem (for instance, code "E" is "the columns of *A* are linearly independent," code "I" is "the row-reduced echelon form of *A* has *n* pivots,' etc.), or are the negation of statements in the IMT (e.g., code "F" is "the columns of *A* are linearly dependent," and code "J" is "the row-reduced echelon form of *A* does not have *n* pivots," etc.). I arranged the remaining vertices as subcodes within these categories by documenting the various interpretations of these statements that were given during whole class discussion. For example, code F3 references being able to "get back home with the column vectors of *A*" and code F4 references vectors that "lie along the same line," both of which I inferred were interpretations of statement F, "the columns of *A* are linearly dependent."

I used this set of 100 concept statements and interpretations as the vertices by which I coded every argument. Each adjacency matrix had the same vertices; what differed between each analysis (and thus each coded adjacency matrix) was how the vertices were or were not connected.

E. Column vectors of A are linearly independent
E1. *Trivial:* Only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution
E2. *Unique:* There is a unique soln to matrix eqn/system of eqns
E3. *Travel*: Can't get back (home)/get to origin with column vectors of A
E4. *Geometric*: Vectors are not parallel/on the same line or plane
E5. *Proportional*: No vector is a scalar multiple of another
E6. *Linear combination*: No vector is a linear combo of another
E7. *Placement*: No vector is in the span of the other vectors
E8. *Extra*: Do not have an extra vector needed in order to return home

F. Column vectors of A are linearly dependent
F1. *Trivial*: Is more than one solution to $A\mathbf{x} = \mathbf{0}$.
F2. *Unique:* No unique/multiple/infinitely many solns to system/matrix eq
F3. *Travel:* Can get back home/back to a point with column vectors of A
F4. *Geometric*: Vectors are parallel/on the same line or plane
F5. *Proportional*: One vector is a scalar multiple of another
F6. *Linear combination*: One vector is a linear combo of others
F7. *Placement*–A vector is in the span of the other vectors
F8. *Zeroes*: the matrix A has a row or column of zeroes
F9. *Extra:* Have an extra vector needed in order to return home

G. Column vectors of A span $\mathbf{R}^n$
G1. *Size:* Are enough vectors to span the entire space
G2. *Geometric*: Can use vectors to get to every pt/go everywhere
G3. *Algebraic:* Is a linear combination of vectors for all pts in $\mathbf{R}^n$
G4. *Direction:* Can use each vector to go in a certain direction
G5. *Solution*: There is a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$

H. Column vectors of A do not span $\mathbf{R}^n$
H1. *Size:* Are not enough vectors to span the entire space
H2. *Geometric*: Can't use vectors to get to every pt/go everywhere in dim
H3. *Direction:* Cannot go in all directions with the vectors
H4. *Clarify*: The vectors of A span a k-dim subspace of $\mathbf{R}^n$
H5. *Partial:* Span is only a point/line/plane
H6: *Solution*: There is not a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$

I. Row-reduced echelon form of A has n pivots
I1. *Diagonal*: RREF(A) has all ones on the main diagonal
I2. *Identity:* Can row-reduce / is row equivalent to the identity
I3. *Pivot-R:* Is a pivot in each row
I4. *Pivot-C:* Is a pivot in each column
I5. *Free Variable:* Each variable is defined in system/matrix eqn
I6. *Zeroes:* RREF(A) has no rows of zeroes

J. Row-reduced echelon form of A has less than n pivots
J1. *Diagonal:* RREF(A) does not have all ones on the main diagonal
J2. *Identity:* Cannot row-reduce /not row equivalent to the identity
J3. *Pivot-R:* Is not a pivot in each row
J4. *Pivot-C:* Is not a pivot in each column
J5. *Free Variable:* Not every variable is defined in system/matrix equation
J6. *Zeroes:* RREF(A) has at least one row of zeroes

K. A is invertible
K1. *Augment with the identity:* $[A \mid I] \sim [I \mid A^{-1}]$ is possible
K2. *Formula:* Can calculat, no "divide by 0 errors"
K3. *Undo transformation:* Can undo/get back from the transformation
K4. *Inverse matrix:* Exists a C s.t. $AC=I$ and/or $CA=I$
K5. *Transform:* Exists sequence of elementary row ops that turns A into I
K6. *Necessary output*: Something gets sent to $\mathbf{e}_1$, $\mathbf{e}_2$, ...

L. A is not invertible
L1. *Augment with the identity:* $[A \mid I] \sim [I \mid A^{-1}]$ is not possible
L2. *Formula:* Can't calculate, get "divide by 0 errors"
L3. *Undo transformation:* Can't undo/get back from the transformation
L4. *Inverse matrix:* Does not exist a C s.t. $AC=I$ and/or $CA=I$
L5. *Transform:* Does not exist sequence of elem row ops to turn A into I
L6. *Necessary output*: Nothing gets sent to $\mathbf{e}_1$, $\mathbf{e}_2$, ...

M. The transformation defined by A is onto
M1. *Definition:* For every $\mathbf{b}$ there is at least one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$
M2. *Range:* Range is all of the codomain
M3. *Images:* All values in codomain get used/mapped to as outputs/images

N. The transformation defined by A is not onto
N1. *Definition:* For every $\mathbf{b}$ there is not at least one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$
N2. *Range:* Range is not all of the codomain
N3. *Images*: Not all values in codomain get used/mapped to as outputs/images
N4. *Collapse:* Transformation collapses everything to a point/line/plane
N5. *Transform:* The transformation goes up/adds a dimension

O. The transformation defined by A is 1-1
O1. *Definition:* For every $\mathbf{b}$ there is at most one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$
O2. *Input/Output:* Each output has at most one input
O3. *Reachable:* There is only one way to "get to" the output/vector

P. The transformation defined by A is not 1-1
P1. *Definition:* For every $\mathbf{b}$ there is more than one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$
P2. *Input/Output:* Each output has more than one input
P3. *Reachable:* There is more than one way to "get to" the output/vector
P4. *Multiplicity:* At least 2 inputs give the same output
P5: *Transform:* The transformation goes down/exludes a dimension

Q. det(A) $\neq$ 0: Determinant of A is nonzero
Q1. *Area:* Unit square/cube has area/volume $\neq 0$ after transformation
Q2. *Calculation*: formula to calculate determinant yields nonzero result

R. det(A) = 0: Determinant of A is equal to 0
R1. *Area:* Unit square/cube has area/volume = 0 after transformation
R2. *Calculation*: formula to calculate determinant yields zero
R3. *Measure:* No area/volume to a line/plane

S. Miscellaneous
S1. *Calculation:* get inconsistent solutions (e.g., $0x + 0y = 1$)
S2. *Column Space:* The Col A is all of $\boldsymbol{R}^n$
S3. *Column Space:* The Col A is all not of $\boldsymbol{R}^n$
S4. *Null Space:* The Nul A contains only the zero vector
S5. *Null Space:* The Nul A contains more than the zero vector
S6. *Eigen Value:* The number zero is not an eigenvalue of A
S7. *Eigen Value:* The number zero is an eigenvalue of A
S8: $m < n$: A has more vectors than dimensions
S9: $m = n$: A has less vectors than dimensions
S10: $m = n$: A has the same number of rows/columns
S11: $m \neq n$: A does not have the same number of rows/columns
S12: *Miscellaneous*

Figure 4.48. The 100 codes used as the rows and columns for the adjacency matrices.

For an example of how these codes were used in analyzing data, consider the following utterance: "If the column vectors of the matrix span $\mathbf{R}^n$ (G), that means you can go everywhere in that dimension (→G3)." In the transcript, the codes G and G3 are written in after the utterances that correspond to those codes, and an arrow is written in from of the G3 code to indicate that G was adjacent to G3. In the corresponding adjacency matrix, a "1" would be written in the G row-G3 column to indicate this edge. In some instances, it was necessary to switch the statements out of chronological order to match the implication order. In an utterance such as, "the columns are linearly independent because there is a pivot in every column," the speaker said the claim before the data. This switch was notated in the transcript with the use of two asterisks, placed after the code that was to go first and before the code that was to go second. Thus, coding for the previous example would be, "the columns are linearly independent (**→E) because there is a pivot in every column (I4→**)."

4.2.1 Microgenetic Analysis via Adjacency Matrices

In this section, microgenetic analysis is accomplished through considering the content and structure of particular instances of reasoning about the Invertible Matrix Theorem during whole class discussion. This consideration of individual arguments (and thus the corresponding individual adjacency matrices) necessarily continues in the subsequent section of Ontogenetic Analysis because, as shifts in argumentation are analyzed, the finer-grained analysis of individual arguments is also necessary. Then, considering how these shift over time, conclusions are made regarding how the classroom community was reasoning about the IMT. In this section, however, because claims cannot be made about how the development of mathematical meaning for a collective without considering reasoning over time, I focus on two results that were revealed through using adjacency matrices as an analytic tool: (a) The structure of argumentation by considering where within a block of cells an argumentation occurs; and (b) The structure of argumentation by considering the type of sub-digraph in which the argumentation occurs.

4.2.1.1 The structure of argumentation by considering where within a block of cells an argumentation occurs. As previously stated, what this particular classroom community came to know as the Invertible Matrix Theorem developed over the course of the semester. The concepts involved in the IMT were introduced at various times, and those early-developed concepts were often integral to the development of latter ones. Furthermore, the structure of argumentation was not uniform throughout the semester, and the adjacency matrix analysis highlights this. This is reasonable considering the fact that the IMT was developed throughout the semester. Two of the main structures that occurred were: (a) when the argumentation served towards developing a way of reasoning about a new concept or connection between two concepts, the argument involved multiple uses of the "interpretation" subcodes; and (b) when the argumentation made use of relatively well-established concepts or connections between concepts, the argument involved mainly the "concept statement" main codes. An example of each follows.

4.2.1.1.1 ***Arguments that involve multiple uses of the "interpretation" subcodes***. On Day 20, the class developed ways of reasoning about one-to-one and onto in conjunction with span and linear independence. In a previous argument, Abraham had volunteered, "If it's one-to-one, the columns of *A* have to be linear independent." The instructor asked David how his small group discussed how that implication might be true; the transcript for his response and subsequent discussion is given, followed by its adjacency matrix in Figure 4.49.

Instructor: David, how were you talking about it at your table, how that might be true?

David: We just went over his explanation, we're having a hard time why that works, why that makes it a multiple solution, it's not 1 to 1 **(F2→P)**.

Instructor: So he's saying it makes sense but he's having a hard time explaining it. The other tables, what did your table talk about, so Brad, can you tell me what your table talked about for this one?

Brad: When you reduce matrices linear dependent, you're going to have a free variable **(F→J5).** When you have that free variable, there has to be more than 1 input to get the same output **(→P4).**

Instructor: Yeah, we haven't talked about free variables for a while; I think that's a great start. Does that make sense? So remember those, when you took a matrix and you row reduced it to the row reduced echelon form, if you got a free variable, then that gave you some leniency for how you could put down a solution for what was in the span of the vectors **(J5→F2)**. Well, that gives you some variability as to how you're going to answer. That variability is giving you more than 1 way to get to that certain output, not a unique solution **(→P3→P1)**.

In this discussion, David struggled with a justification for “if the columns of *A* are linearly dependent” (code F) then “the transformation defined by *A* is not one-to-one” (code P), so Brad and the instructor contributed to the argument as well. The adjacency (F, P) was nascent in this community, and this is evidenced through the variety of interpretations of these concepts that were utilized in an effort to justify the proposed implication. Each of the six adjacencies [(F2, P), (F, J5), (J5, P4), (J5, F2), F2, P3), and (P3, P1)] in Argument 20.6 involved interpretations of F or P (e.g., “there has to be more than 1 input to get the same output,” coded P4), as well as the concept “there exists a free variable” (code P5), which served as the intermediary vertex between interpretations of F and interpretations of P. Thus, this is an example of an argumentation that served towards developing a way of reasoning about a new connection between concepts (namely linear dependence and not being one-to-one) that involved multiple used of the “interpretation” subcodes. Other instances of arguments of this structure are discussed in the subsequent section of Ontogenetic Analysis of arguments relevant to the development of CMP 1.

	F	F1	F2	F3	F4	F5	F6	F7	F8	F9	J	J1	J2	J3	J4	J5	J6	P	P1	P2	P3	P4	P5
F																1							
F1																							
F2																		1			1		
F3																							
F4																							
F5																							
F6																							
F7																							
F8																							
F9																							
J																							
J1																							
J2																							
J3																							
J4																							
J5			1																			1	
J6																							
P																							
P1																							
P2																							
P3																			1				
P4																							
P5																							

Figure 4.49. Partial adjacency matrix for Argument 20.6.

As a final comment, the direct adjacency (F, P) only occurred three times during the 109 analyzed arguments: in Arguments 20.11, 20.12, and 20.25. Each of these occurred on Day 20 after the aforementioned Argument 20.6, and none of the arguments contained any justification of that implication. Thus, there is an indication that the implication "if the columns of *A* are linearly dependent, then the associated transformation is not 1-1" was functioning as-if shared, but analysis of this type is left to the subsequent Ontogenetic Analysis section.

4.2.1.1.2 ***Arguments that primarily involved the "concept statement" main codes.*** A second prevalent structure of argumentation was such that when the argumentation made use of relatively well-established concepts or connections between concepts, the argument involved mainly the "concept statement" main codes, rather than substantive use of the interpretive subcodes. As an example, consider a portion of Argument 20.16.

Instructor: So Lawson was saying something about being linear independent, is that the same as being able to say something about being onto?

Abraham: It is, if the matrix is square **(E→M, M→E)**

Instructor: Yeah, I think that's something that we need to get at. So the one that we had from before was if the columns of *A* span $\mathbf{R}^n$ **(G)**, so the transformation *T* is onto $\mathbf{R}^m$ **(→M)**. So now we're saying, can we say anything about connecting onto to linear independence? And Abraham's talking about we can if they're square. So I think I agree, can you say a little bit more?

Abraham: I just remember if it's square, we had the *n* x *n* Theorem way back when. And if a square matrix is linear independent **(E)**, it also spans **(→G)**. And if it spans **(G)**, it's also linear independent **(→E)**. And so that means that if it's one-to-one **(→O)**, it has to be onto **(→M)**; if it's onto **(M)**, it has to be one-to-one **(→O)**. Do you know what I mean, like connecting the ideas?

In this argument, Abraham stated that Lawson's claim (that being linearly independent is the same as being onto) was true if the matrix is square. Thus, the entirety of this argument was coded within the $m = n$ sub-digraph and its associated adjacency matrix. Lawson's claim of "the same as" was interpreted as equivalence of the concepts, thus Abraham's agreement was coded both E→M and M→E (i.e., (E, M) and (M, E) in digraph notation). Abraham's justification of this claim depended on two more equivalencies for square matrices: between span and linear independence ((G, E), and (E, G)), as well as between onto and one-to-one ((O,

M) and (M, O)). He stated these equivalencies without support, and an equivalent between span and onto (vertices G and M) as well as linear independence and one-to-one (vertices E and O) is left implicit but is not challenged. Similar analysis of this argument occurred in the previous Microgenetic Analysis via Toulmin's Model section (see Figure 4.8).

This is a different structure of reasoning than in the example seen in Figure 4.49. Here, although the claim Lawson suggested and Abraham justified was novel (that onto and linear independence were equivalent for square matrices), the justification provided by Abraham involved previously established ways of reasoning. His justification of the equivalence of linear independence and onto was established via connecting pre-established equivalencies rather than by reasoning about why conceptually they were equivalent for square matrices. Thus, Abraham's justification was based on a deductive argument that utilized relatively well-established connections between concepts, and this argument involved mainly the "concept statement" main codes.

4.2.1.2 The structure of argumentation by considering the type of sub-digraph in which the argumentation occurs. After the 100 codes had been finalized as those by which argumentation regarding how the classroom reasoned about the IMT would be analyzed, I began to code all the arguments. Upon doing so, however, I encountered an unforeseen problem. The IMT is a theorem about equivalent statements for *square* matrices, yet a substantial amount of reasoning occurred about cases in which m was less than n or vice versa. The 100 codes are, for the most part, concept statements from the IMT, their converse, or interpretations of those concepts statements or inverses. In certain cases, utilizing the same codes in arguments that assume $m < n$, $m = n$, any $m > n$ without differentiating between them would lead to conflated conclusions regarding how the classroom community was reasoning about concepts involved in the IMT. Thus, sub-digraphs based on four scenarios: $m < n$, $m = n$, $m > n$, and *any* m, n were created in order to analyze the particular types separately. These were then combined into the total digraph *T*. More detail regarding these sub-digraphs and digraph *T,* as well as the associated adjacency matrices, is given in the subsequent section.

Utilizing four different sub-digraphs in order to most accurately code argumentation was especially revealing. For instance, Argument 6.8 (previously mentioned in Figure 4.19) was at first quite difficult to code utilizing adjacency matrices. Justin's explanation for why having more vectors than dimensions guaranteed linear dependence was quite elegant, yet his ways of reasoning were not satisfactorily captured within only one adjacency matrix. Upon separating his argumentation into the sub-types of $m < n$, $m = n$, and $m > n$, the structure of his argumentation was revealed in a way that had previously not been highlighted. The transcript and coding is given in Figure 4.50.

Within this one argument, Justin switched between different dimensions of matrices in order to build his case for the given claim. He was quite methodical, starting with fewer vectors than dimensions and made implications (F4, F), (E5, E), and (E, H5). Note that E→H5, "if the columns of *A* are linearly independent, then they span an *r*-dimensional subspace of $\mathbf{R}^n$," would not be true in other types (such as $m = n$). Justin then shifted his argument to address a case of equal vectors and dimensions, and then finally the desired case of more vectors than dimensions ($m > n$). These three types were not unrelated; Justin built the main cohesive argument through a series of well-planned sub-arguments in the three different cases that, when considered as a whole, eliminate all possibilities other than linear dependency of sets.

The particulars of Justin's reasoning were discussed previously (see Figure 4.19) and as well in the forthcoming section on adjacency matrix analysis of CMP 1. Of note in this section is the unexpected structure of argumentation that occurred in the classroom: that of switching between different types of *m* and *n* scenarios in order to make one generalization. This particular aspect of argumentation structure did not surface as a result of the Toulmin's Model analysis. Thus, it is interesting that to develop ways of reasoning about the concepts involved in a theorem about square matrices, reasoning about non-square matrices was important as well. Part of knowing what something is necessarily involves knowing what it is not.

0:16:47.2	*Instructor:*	The original question I was talking about, the generalization #4...So I think Table 4, you guys were the ones who came up with this generalization, can you guys say a little bit more in general how this makes sense to you, not just in the $\mathbf{R}^2$ case?
0:17:19.3	*Justin:*	So if you start in any $\mathbf{R}^n$, and you just start with one vector and keep adding more. So let's do $\mathbf{R}^3$, just for an example.
Explanation begins within m > n (1-2 vectors in $\mathbf{R}^3$*):*		
		So we start with one vector. So either, we have two choices: The next vector we add can either be on the same line **(F4)**, which means it's already linearly dependent **(→F)**, so we don't want that **(**→E)**, so we're going to put it off somewhere else **(E5**)**. Now the span of that is a plane in three dimensions **(→H5)**.
Explanation is now for m = n (3 vectors in $\mathbf{R}^3$*):*		
		So now we're going to add another vector in. Our third vector, now it can either be in that span or out of that span. And we want it to be linearly independent **(**→E)**, so we're going to put it out of that span **(E7**)**. But now that we have that going off of that plane **(→G4)**, we just extended our span to all of $\mathbf{R}^3$ **(→G)**.
Explanation is finally for m < n (4 vectors in $\mathbf{R}^3$):		
0:18:09.9	*Justin:*	So our 4th vector, when we put it in, no matter where we put it, it's going to get us back home **(S10→F9)**. Because just like in this case, we have to have the last one to get back home, we can get anywhere with those first three that we put in **(G2)**, but we have to have to have that fourth one to come back **(→F9)**. And so it works like that in any dimension, because the more you, if you keep adding, eventually you're going to get the span of your dimensions **(G1)**, and then you're going to have that extra one bringing you back **(→F9)**. Unless you have two vectors that are lying on the same line **(F5)**, then you won't have the span of all of your dimension **(→H1)**, but it's negligible because those two will give you a linearly dependent set **(F5→F)**. Does that make sense?

Figure 4.50. Transcript with adjacency matrix coding for Argument 6.8.

4.2.2 Ontogenetic Analysis via Adjacency Matrices

There is no pre-established methodology for documenting the development of mathematical meaning at the collective level using adjacency matrices as an analytical tool.

This is distinct from the previous Toulmin analysis section, in which existing methodology for documenting normative ways of reasoning at the collective level were utilized (Rasmussen & Stephan, 2008; Cole et al., 2011). The ontogenetic section here contains multiple references to that analysis and how it is complementary. Not only do the results here corroborate those from the previous section, but aspects of the methods here suggest points of compatibility with the three criteria from Toulmin methodology as well. Finally, the ontogenetic analysis via adjacency matrices documented aspects of the classroom's ways of reasoning that were not apparent in the Toulmin analysis, such as concepts of high and low centrality.

For a digraph of order *r*, $\sum_{i=1}^{r}\left(\text{od } v_i\right) = \sum_{i=1}^{r}\left(\text{id } v_i\right) = s$. In other words, the sum of the out-degrees in *D* is equal to the sum of the in-degrees in *D*, which is also the size (or number of edges) in *D*. In the associated adjacency matrix *A*(*D*) for digraph *D*, the out-degrees of the vertices v_i are calculated by summing the entries of *A*(*D*) for every row v_i, and the in-degrees are calculated by summing the entries for every column v_i. The out-connection of a vertex (oc *v*) is the total number of edges that are incident from *v* to distinct vertices, and the in-connection (ic *v*) of vertex *v* is the number of arcs incident to *v* from distinct vertices. These calculations are fundamental to the analysis in this section. The notion of *centrality* "indicates a node's degree of participation in the structure of the graph by measuring the relative connectivity of a node within a graph" (Strom et al, 2001, p. 752). It is a measure of how central a vertex is in a given digraph—does it serve as a sort of "hub," adjacent to and adjacent from multiple vertices, or is it just adjacent to a relatively low number of vertices but with high edge frequency? For instance, for a given vertex B, suppose od B = 7. This could appear a variety of ways; two examples are given in Figure 4.51.

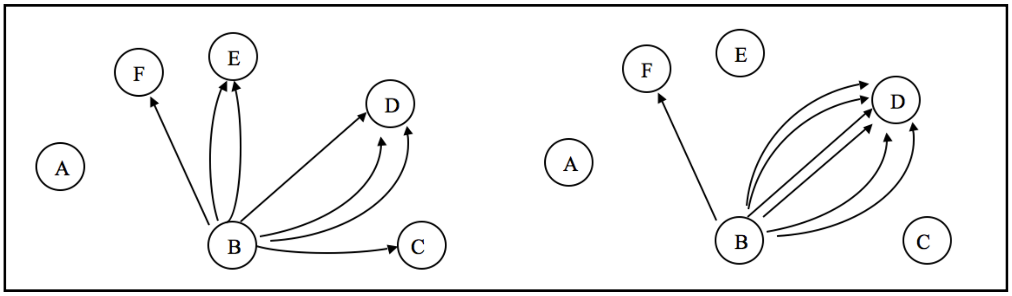


Figure 4.51. Two examples of a digraph such that the out-degree of vertex B is 7.

In the example of the left, vertex B is adjacent to four other vertices; thus oc B = 4. In the example on the right, vertex B is only adjacent to two vertices; thus oc B = 2. The participation of the vertex B in the overall graph is quite different in these two examples, and the centrality measure (Strom et al, 2001) is one such method of parsing out this distinction. It is given by, for any given vertex *v* in a digraph *D*, $C(v) = \frac{\text{ic } v + \text{oc } v}{2r}$, where *r* is the total number of nodes in the digraph. According to this measure, the graph on the left in Figure 4.51 would have a higher centrality score (*C*(B)=4/12) than the one on the right (*C*(B)=2/12).

Investigating how students reasoned about the concepts involved in the Invertible Matrix Theorem was facilitated by the centrality measure. In the following section, I present results from an adjacency matrix coding of the arguments involved in the first of the two classroom mathematics practices established in the previous section, as well as results from analyzing the entire data set of arguments.

4.2.2.1 Adjacency Matrices for coded arguments involved in CMP 1. In the previous section of Chapter 4, Toulmin analysis was used to document a classroom mathematics practice referred to as "Reasoning with Span and Linear Dependence as Equivalent Ideas for Square Matrices" (CMP 1). In demonstrating that classroom mathematics practice and the four associated normative ways of reasoning, I referenced 18 different arguments: Arguments 5.1, 6.1-6.8, 9.1-2, 10.1-4, 20.4, 20.16, and 31.8. For the present section, I used the transcript from which these argumentation schemes originated and analyzed each of them according to the coding scheme (Figure 4.48) created for adjacency matrix analysis. As stated previously, the transcript was coded based on $m < n$, $m = n$, $m > n$, or *any* m, n. For CMP 1, the utterances were divided into the first three types, resulting in three sub-digraphs for CMP 1.

The associated adjacency matrices for these three sub-digraphs are notated $A(P_1)_{m<n}$, $A(P_1)_{m=n}$, and $A(P_1)_{m>n}$, and the one for all of CMP 1 is notated $A(P_1)_{tot}$. These adjacency matrices can be found in Appendix 4.4, 4.5, 4.6, and 4.7, respectively. The sizes of the sub-digraphs are $A(P_1)_{m<n} = 38$, $A(P_1)_{m=n} = 46$, $A(P_1)_{m>n} = 3$, making the size of the digraph for CMP 1 $A(P_1)_{tot} = 38 + 46 + 3 = 87$. Furthermore, the order of the digraph was 33. Although not each of the three sub-digraphs utilized each of the 33 vertices, I used 33 as their order in my calculations of centrality for consistency purposes.

The out-degree, out-connection, in-degree, and in-connection, and centrality were calculated for each vertex for the aforementioned sub-digraphs and total digraph (see Figure 4.52). The values for each of these cells derived from each graph's adjacency matrix. The vertices in each adjacency matrix range from E to S12; for ease of presentation, only the calculations for vertices E-H6 are shown in Figure 4.52. The entire calculations are given with each respective adjacency matrix in the aforementioned appendices. The vertices below H6 were minimally used in these particular digraphs, and thus are not central to the analysis presented here.

Across the four types, different nodes had the highest centrality. The highest for each is indicated with red framing, the second highest with orange framing, and the third highest with yellow framing. The most central vertex for $A(P_1)_{m<n}$ was F9, was G for $A(P_1)_{m=n}$, was E for $A(P_1)_{m>n}$, and was G for $A(P_1)_{tot}$. Further examination of code G in $A(P_1)_{tot}$ reveals that id G =14 and od G = 15. In other words, G was incident to and incident from nearly the same amount of arcs. The in-connections and out-connections for G in $A(P_1)_{tot}$ were ic G = 9 and oc G = 6, meaning that G was adjacent to 6 vertices and adjacent from 9 vertices. These totals represent the usage of code G in all types $A(\mathrm{P}_1)_{m<n}$, $A(\mathrm{P}_1)_{m=n}$, and $A(\mathrm{P}_1)_{m>n}$. To better illustrate what these counts mean, examine the subset of G codes used in $A(\mathrm{P}_1)_{m=n}$.

CMP1 m < n

	OD	OC	ID	IC	Centrality
E	1	1			0.015
E1					
E2					
E3					
E4	2	2	1	1	0.045
E5					
E6					
E7					
E8					
F			11	6	0.091
F1			1	1	0.015
F2					
F3	1	1	3	2	
F4	2	2	2	2	0.061
F5	4	2	1	1	0.045
F6					
F7					
F8					
F9	3	3	11	6	0.136
G	4	2	1	1	0.045
G1	3	1			0.015
G2	2	1	1	1	0.030
G3	1	1			0.015
G4	2	2	1	1	0.045
G5					
H					
H1			1	1	0.015
H2					
H3					
H4					
H5					
H6					

CMP1 m = n

	OD	OC	ID	IC	Centrality
E	7	4	12	5	0.1364
E1					
E2					
E3	2	2	3	3	0.0758
E4	3	3			0.0455
E5					
E6					
E7	1	1			0.0152
E8			1	1	0.0152
F	2	2	3	3	0.0758
F1					
F2					
F3			1	1	0.0152
F4	1	1			0.0152
F5					
F6					
F7	1	1			0.0152
F8					
F9					
G	11	4	13	9	0.197
G1					
G2	1	1	1	1	0.0303
G3	1	1			0.0152
G4	1	1	1	1	0.0303
G5					
H	1	1	1	1	0.0303
H1					
H2					
H3					
H4					
H5					
H6					

CMP1 m > n

	OD	OC	ID	IC	Centrality
E	1	1	1	1	0.030
E1					
E2					
E3					
E4					
E5	1	1			0.015
E6					
E7					
E8					
F			1	1	0.015
F1					
F2					
F3					
F4	1	1			0.015
F5					
F6					
F7					
F8					
F9					
G					
G1					
G2					
G3					
G4					
G5					
H					
H1					
H2					
H3					
H4					
H5			1	1	0.015
H6					

CMP1 as combined together

	OD	OC	ID	IC	Centrality
E	9	6	13	6	0.182
E1					
E2					
E3	2	2	3	3	0.076
E4	5	4	1	1	0.076
E5	1	1			0.015
E6					
E7	1	1			0.015
E8			1	1	0.015
F	2	2	15	10	0.182
F1			1	1	0.015
F2					
F3	1	1	4	3	0.061
F4	4	3	2	2	0.076
F5	4	2	1	1	0.045
F6					
F7	1	1			0.015
F8					
F9	3	3	11	6	0.136
G	15	6	14	9	0.227
G1	3	1			0.015
G2	3	2	2	1	0.045
G3	2	2			0.030
G4	3	3	2	2	0.076
G5					
H	1	1	1	1	0.030
H1			1	1	0.015
H2					
H3					
H4					
H5			1	1	0.015
H6					

Figure 4.52. Out-degree, out-connection, in-degree, in-connection, and centrality for $A(\mathrm{P}_1)_{m<n}$, $A(\mathrm{P}_1)_{m=n}$, $A(\mathrm{P}_1)_{m>n}$, and $A(\mathrm{P}_1)_{tot}$.

4.2.2.1.1 *The m = n sub-digraph of CMP 1.* For $A(\mathrm{P}_1)_{m=n}$, od G = 11, oc G = 4, id G = 13, and ic G = 9 (see Figure 4.52). From investigating the G row and G column of $A(\mathrm{P}_1)_{m=n}$ (Appendix 4.5), a subgraph of $A(\mathrm{P}_1)_{m=n}$ of the vertices adjacent to and adjacent from G was created (Figure 4.53). In addition to displaying which vertices were adjacent to or adjacent from vertex G in $A(\mathrm{P}_1)_{m=n}$, Figure 4.52 provides information regarding (a) the distribution of the utterances that were adjacent from or adjacent to concept statement G, and (b) the arguments from which these utterances derived over the course of the semester. For instance, regarding part (a), there was a high frequency of connections between concept statement G, "the columns of A span $\mathbf{R}^n$" and concept statement E, "the columns of A are linearly independent." These 8 arcs are a subset of the 11 arcs accounted for in the “out degree” column and “G” row of Table 2 in Figure 4.52 (the other three are the arcs (G, M), (G, O), and (G, E3)). Figure 4.52 shows that id G = 13 for this subgraph; 3 of those arcs are accounted for in the arrows from E to G in Figure 4.53. Regarding (b), by considering the color of the arrows that correspond to the particular argument in which the utterance occurred, we can see how concept statement G was reasoned about over time for this small subset of utterances.

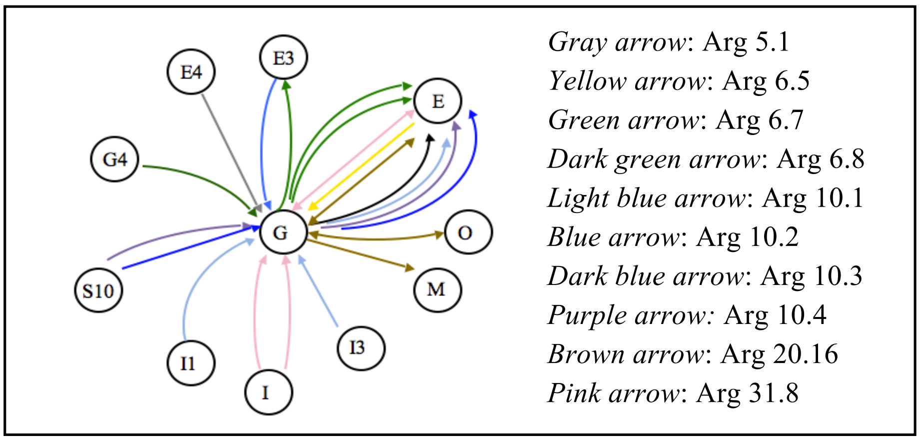


Figure 4.53. Sub-digraph of $A(P_1)_{m=n}$ for vertices adjacent to and adjacent from vertex G.

In Argument 5.4, Justin’s claim that “when we did our magic carpet-hoverboard, we had two [vectors in $\mathbf{R}^2$] that weren't parallel, and we said the span of any two that aren't parallel is all of $\mathbf{R}^2$” was captured with the grey arrow from E4 to G in Figure 4.53. Aziz, one of Justin’s small group members, repeated this sentiment in Argument 6.5 during the following class session: “if two vectors [in $\mathbf{R}^2$] aren’t [linearly dependent], then they span $\mathbf{R}^2$,” and this is represented with the yellow arrow from E to G in Figure 4.53. On Day 6, the adjacency (G, E) occurred three times (the yellow and green arrows). By examining the transcript, we see that these arguments were about either 2x2 or 3x3 matrices. This is distinct from the utterances in Argument 31.8, the pink arrows, when equivalence was stated between the two concept statements for any *n* x *n* matrix *A*. This transition is partly accounted for on Day 10, when the task in the class was to adjust Theorem 4 (See Figure 4.21), which was for *m* x *n* matrices, to one appropriate for *n* x *n* matrices. In Figure 4.53, there are three more arrows from G to E; these utterances were part of three arguments that occurred on Day 10, when the class discussed and justified the claim, "for an *n* x *n* matrix *A* whose column vectors span $\mathbf{R}^n$, the columns of *A* are linearly independent." By the end of Day 10, the statement "the columns of *A* are linearly independent" was added to the adjusted theorem for *n* x *n* matrices, thus establishing the equivalence of these ideas. Finally (for CMP 1), this equivalence of G and E is used during class on Days 20 and 31 (pink and brown arrows in Figure 4.53). The transcript reveals that the argument from Day 20 made use of the equivalence (in the *n* x *n* case) between span and linear independence to reason about an equivalence between one-to-one and onto, and the argument on Day 31 discussed how a student made sense of the equivalence between span and linear independence for the column vectors of *n* x *n* matrices. Below is the transcript and coding for this Argument 31.8.

Instructor:	So Nate's table over here had 'the columns of *A* span $\mathbf{R}^n$,' and 'the columns are linear independent' as an obviously equivalent pair **(G→E, E→ G)**. So if one of you guys could explain why you put that one as obvious to you all?
Nate:	Because if the columns are linear independent (E), then it kind of goes, to me it goes with pivots points **(→I)**. And then, if it has n pivot points, it spans all of $\mathbf{R}^n$ **(→G)**, it goes [inaud]
Instructor:	Okay, I think you can say it a little louder, can you say it one more time a little louder, please?
Nate:	For me, my logic, I think if the columns of *A* are linear independent **(E)**, then it has n amount of pivot points **(→I)**. Then if it has n amount of pivot points **(I)**, and it's an *n* x *n* matrix assuming that, then it spans all of $\mathbf{R}^n$ **(→G)**.

First, Nate's statement that those concepts were obviously equivalent was coded (E→G, G→E), and then his explanation, which he gave twice, was coded (E→I→G). After this, no member of the class pushed him for further clarification, nor did anyone show signs of disagreement. No one asked him to explain why the columns of *A* being linearly independent implied that *A* had *n* pivots, nor why *A* having *n* pivots implied the columns of *A* spanned $\mathbf{R}^n$.

On Day 9, however, these implications were not so readily accepted—they had to be investigated, explored with examples, debated, and justified.

This change in how the classroom community discussed the concepts involved in the Invertible Matrix Theorem and how they related to each other is one of many results that can be deduced from the adjacency matrix for the $m = n$ sub-digraph for CMP 1, $A(P_1)_{m=n}$. As described above, the adjacency (E4, G) was utilized in order to support a new claim; this same if-then statement was captured as a normative way of reasoning, NWR #2, "The span of two non-parallel vectors in $\mathbf{R}^2$ is all of $\mathbf{R}^2$," in the previous Toulmin analysis section of this chapter. Furthermore, the Toulmin analysis section provided evidence for CMP 1, which consisted of four normative ways of reasoning: For a given set of n vectors in $\mathbf{R}^m$, if $m < n$, the set must be linearly dependent (1.1); For a given set of n vectors in $\mathbf{R}^m$, if $m > n$, the set cannot span all of $\mathbf{R}^m$ (1.2); If A is n x n and the columns of A span $\mathbf{R}^n$, then the columns of A are linearly independent (1.3); and if A is n x n and the columns of A are linearly independent, then the columns of A span $\mathbf{R}^n$ (1.4). While the explanation of $A(P_1)_{m=n}$ just provided does corroborate with the Toulmin analysis for the second two normative ways of reasoning, which deal with square matrices, it did not provide details regarding any argumentation in the non-square cases. To parallel the analysis in the Toulmin section, I provide results concerning the adjacency matrix of the $m < n$ sub-digraph for CMP 1.

4.2.2.1.2 *The m < n sub-digraph of CMP 1*. The summary data for $A(P_1)_{m<n}$ in Figure 4.52 reveals a low usage level for concept statements E-E8 and high usage levels for vertices F-F9. When considered collectively, the codes beginning with "E" are various interpretations of concept statement E, "the columns of A are linearly independent," whereas the "F" codes regard "the columns of A are linearly dependent." This is directly opposite of the trend in $A(P_1)_{m=n}$, which has a higher connection level of "E" codes than "F" codes. Furthermore, the vertex with the highest centrality for $A(P_1)_{m<n}$ was "F9: Have an extra vector needed in order to return

home." Code G did not have a high centrality, as was the case for $A(\mathrm{P}_1)_{m=n}$; however, if the collection of G codes are considered together, they had a high centrality. Why this distinction of uses of G versus interpretations of G in the discourse for when $m < n$ and $m = n$? And why, for $m < n$, was F9 so high, whereas it wasn't used at all in the $m = n$ sub-digraph?

In order to see how the vertices in the sub-digraph of CMP 1 were connected and within which arguments the various connections occurred, consider the portion of the adjacency matrix $A(\mathrm{P}_1)_{m<n}$ in Figure 4.54, which includes the rows and columns for vertices E-G5. This portion of $A(\mathrm{P}_1)_{m<n}$ was chosen because it is the most dense section of the adjacency matrix; the entire adjacency matrix is found in Appendix 4.4.

The connections represented in this portion of the adjacency matrix all occurred within the first six days of class, with one exception on Day 9 (the vertices (G, F) are adjacent once in Argument 9.15). First, consider the adjacency (F5, F): "If one vector is a scalar multiple of another, the vectors are linearly dependent." This occurred within three different arguments: 6.4, 6.5, and 6.8, and these are indicated with the orange, yellow, and green values of one, respectively, in the (F5, F) cell. In the previous Toulmin analysis section, it was mentioned that this was a generalization that became established in the class on Day 5. Here, on Day 6, members of the class used that if-then pair to reason about various claims. In Argument 6.4 Lawson used it to support the claim that the vectors $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\4\end{bmatrix},\begin{bmatrix}-9\\7\end{bmatrix}\right\}$ were linearly dependent. It was used in arguments 6.5 and 6.8 as well, the transcripts of which can be found in Appendix 4.2. Those two arguments were of a similar structure, so only Argument 6.8 is considered here.

	E	E1	E2	E3	E4	E5	E6	E7	E8	F	F1	F2	F3	F4	F5	F6	F7	F8	F9	G	G1	G2	G3	G4	G5
E. Column vectors of A are linearly independent																									
E1. *Trivial:* Only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution																									
E2. *Unique:* There is a unique soln to matrix eqn/system of eqns																									
E3. *Travel*: Can't get back (home)/get to origin with column vectors of A																									
E4. *Geometric*: Vectors are not parallel/on the same line or plane																			1 (light green)			1 (red)			
E5. *Proportional*: No vector is a scalar multiple of another																									
E6. *Linear combination*: No vector is a linear combo of another																									
E7. *Placement*: No vector is in the span of the other vectors																									
E8. *Extra*: Do not have an extra vector needed in order to return home																									
F. Column vectors of A are linearly dependent																									
F1. *Trivial*: Is more than one solution to $A\mathbf{x} = \mathbf{0}$.																									
F2. *Unique:* No unique/multiple solns to system/matrix eqn																									
F3. *Travel:* Can get back home/back to a point with column vectors of A															1 (orange)										
F4. *Geometric*: Vectors are parallel/on the same line or plane					1 (light green)									1 (orange)											
F5. *Proportional*: One vector is a scalar multiple of another										1 (orange) 1 (yellow) 1 (dark green)															
F6. *Linear combination*: One vector is a linear combo of others																									
F7. *Placement*–A vector is in the span of the other vectors																									
F8. *Zeroes*: the matrix A has a row or column of zeroes																									
F9. *Extra:* Have an extra vector needed in order to return home										1 (light green)				1 (dark green)											
G. Column vectors of A span $\mathbf{R}^n$										1 (teal)									3 (yellow)						
G1. *Size:* Are enough vectors to span the entire space																			3 (dark green)						
G2. *Geometric*: Can use vectors to get to every pt/go everywhere																			1 (red) 1 (dark green)						
G3. *Algebraic:* Is a linear combination of vectors for all pts in $\mathbf{R}^n$																			1 (gray)						
G4. *Direction:* Can use each vector to go in a certain direction											1 (dark green)			1 (orange)											
G5. *Solution*: There is a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$																									

Argument legend:

Gray: Arg 5.1 ***Orange***: Arg 6.4 ***Light Green***: Arg 6.6 ***Teal***: Arg 9.15
Red: Arg 6.3 ***Yellow***: Arg 6.5 ***Dark green***: Arg 6.8

Figure 4.54. The rows and columns E-G5 of $A(\mathrm{P}_1)_{m<n}$.

Argument 6.8 was analyzed in detail in the previous microgenetic section of the adjacency matrix analysis because of its structure of utilizing reasoning about the types $m < n$, $m = n$, and $m > n$ in order to support the claim that any set if vectors in $\mathbf{R}^n$ with more than n vectors must be linearly dependent (see Figure 4.50). The following piece of transcript is the section of Argument 6.8 relevant to $m < n$. Justin had just explained how it was possible to have a set of three vectors in $\mathbf{R}^3$ that were linearly independent, and here he stated that, with a fourth vector, the set would be forced to be linearly dependent.

Justin: So our 4th vector, when we put it in, no matter where we put it, it's going to get us back home **(S10→F9)**. Because just like in this case, we have to have the last one to get back home, we can get anywhere with those first three that we put in **(G2)**, but we have to have to have that fourth one to come back **(→F9)**. And so it works like that in any dimension, because the more you, if you keep adding, eventually you're going to get the span of your dimensions **(G1)**, and then you're going to have that extra one bringing you back **(→F9)**. Unless you have two vectors that are lying on the same line **(F5)**, then you won't have the span of all of your dimension **(→H1)**, but it's negligible because those two will give you a linearly dependent set **(F5→F)**. Does that make sense?

In this section, Justin's explanation is coded as S10→F9, G2→F9, G1→F9, F5→H1, and F5→F. The last implication, F5→F, is represented with the dark green "1" in the (F5, F) cell of $A(P_1)_{m<n}$ in Figure 4.54. The other members of the class did not contest this aspect of his argument; furthermore, the pair of adjacent vertices (F5, F) did not occur again at all during the semester during whole class discussion (see Figure 4.56 in the subsequent section). Rather, the vertex F5 was adjacent to a variety of new claims for n x n matrices, such as J4 ("There is not a pivot in each column of A"), L ("A is not invertible"), N3 ("Not all values in the codomain get mapped to as outputs"), or P ("The transformation defined by A is not 1-1"). These vertices that were adjacent from F5 in whole class discussion were almost entirely also adjacent from F in whole class discussion during other arguments. Thus, the ideas of F5 and F were possibly functioning as equivalent ideas in the classroom. Results such as this, which involve

considering all coded arguments throughout the semester, are explored further in a subsequent section.

Another reason Argument 6.8 is noteworthy is the unexpected prevalence of the code F9 ("You have an extra vector needed in order to return home"). In Figure 4.52, in fact, we see F9 has the highest centrality of all codes used in $A(\mathrm{P}_1)_{m<n}$. The in-degree of F9 was 11 for CMP 1, and in Argument 6.8 alone, Justin made use of this way of reasoning three different times. The "travel" language that typifies the F9 code has it roots in the Magic Carpet Ride problem, during which students worked within an experientially real problems setting of travel in 2- and 3-dimensions towards developing more formal ways of reasoning about properties of vectors and vector spaces (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2011). Within that task sequence, linear dependence was first explored as a property of a certain type of modes of transportation that allowed you to complete a journey that began and ended at home (find a nontrivial solution to the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$). Thus, in retrospect, it is not surprising that the code F3 ("You can get back home/back to a point with the column vectors of A") served as an interpretation of linear dependence during the semester (it was referred to as NWR #3 in the previous Toulmin section).

This "travel language" was even more influential with the concept of span, as will be seen in the subsequent section. The code F9, however, was used to make very particular claims in the classroom: to reason about why, if a set of vectors in $\mathbf{R}^n$ contained more than n vectors (i.e., there were "more vectors than dimensions"), it had to be linearly dependent. This occurred in multiple arguments, all of which occurred on Day 5 or Day 6; thus, it did not have continuity throughout the semester. Nor did F9 occur in whole class discussion occur in any scenario except when $m < n$. Rather, F9 was used most often as students explained why, if you had n vectors that spanned $\mathbf{R}^n$, you would need that extra vector in order to "return home." It was as if they wanted a linearly dependent set that also spanned $\mathbf{R}^n$, then a minimally spanning set was insufficient and thus needed a vector to be added; in other words, if they could "get everywhere" but wanted to "be able to return home," then they needed an extra vector to "ride back." While the Toulmin analysis in the previous section did provide evidence that "if there are more vectors than dimensions, the set of vectors is linearly dependent" was a normative way of reasoning for the classroom, the adjacency matrix analysis provided a method by which to examine the specific phrases that were uttered within the development of this larger generalization, as well as information regarding how these smaller ways of reasoning, such as F9, dropped off during the semester and were only used in specific scenarios.

4.2.2.1.3 *Conclusion for adjacency matrix analysis for CMP 1.* Many aspects of the classroom's normative ways of reasoning that were established using Toulmin's model are apparent through adjacency matrix analysis as well. By computing centrality for the various concept statements, those with a high degree of participation in the overall argumentation were revealed. In the case of $A(\mathrm{P}_1)_{m=n}$, concept statement G ("the columns of A span $\mathbf{R}^n$") had the highest measure of centrality. A digraph of these connections was constructed in order to investigate more specifically how the concept of "the columns of A span $\mathbf{R}^n$" was used over the course of the arguments analyzed. This demonstrated within which arguments concept statement G was uttered in conjunction with other concept statements.

4.2.2.2 Adjacency Matrices for the compilation of all coded arguments. From the 109 coded arguments and the 100 possible vertices, 83 vertices were used at least once during whole class discussion during the ten days analyzed, and 452 total edges existed between the vertices. In other words, 83 various interpretations of the concept statements (or concept statements themselves) associated with the IMT were used during whole class discussion during the days analyzed, and 452 edges exists in such a way as to match the implication offered by the speaker(s). The digraph for all the vertices and edges is denoted T, with adjacency matrix $A(\mathrm{T})_{tot}$. The sub-digraphs of $m < n$, $m = n$, $m > n$, and any m, n and their associated orders are

$A(\mathrm{T})_{m<n} = 89$, $A(\mathrm{T})_{m=n} = 262$, $A(\mathrm{T})_{m>n} = 48$, $A(\mathrm{T})_{any}$, = 53; thus, $A(\mathrm{T})_{tot} = 89 + 262 + 48 + 53 = 452$.

The adjacency matrix $A(\mathrm{T})_{tot}$ is provided in Figures 4.56-4.59. Because of its size, the adjacency matrix was partitioned into four figures. Figure 4.56 is the upper-left quadrant of $A(\mathrm{T})_{tot}$, displaying rows E-K3 and columns E-K3; Figure 4.57 is the lower-left quadrant, displaying rows K4-S12 and columns E-K3; Figure 4.58 is the upper-right quadrant of $A(\mathrm{T})_{tot}$, displaying rows E-K3 and columns K4-S12; and Figure 4.59 is the lower-right quadrant, displaying rows K4-S12 and columns K4-S12. Because of the large number of arguments being analyzed, coding each argument in a different color, as was done for CMP 1, was not possible for $A(\mathrm{T})_{tot}$ or its sub-digraph adjacency matrices. Thus, a different color was used for each of the ten days coded (see Figure 4.55).

- Days 5 and 6: Light pink
- Day 9: Red
- Day 10: Orange
- Day 17: Green
- Day 18: Dark Green
- Day 19: Light Blue
- Day 20: Dark Blue
- Day 24: Purple
- Day 31: Brown

Figure 4.55. Key of colors used for entries in adjacency matrix $A(\mathrm{T})_{tot.}$

These colors appear in the various cells in the adjacency matrix and correspond to the days in which that connection occurred. For instance, the adjacency (F, H), “if the columns of *A* are linearly dependent, then the columns do not span $\mathbf{R}^n$,” occurred three times—twice on Day 17 (indicated with a green “2”) and once on Day 20 (indicated with a dark blue “1”).

	E	E1	E2	E3	E4	E5	E6	E7	E8	F	F1	F2	F3	F4	F5	F6	F7	F8	F9	G	G1	G2	G3	G4	G5	H	H1	H2	H3	H4	H5	H6	I	I1	I2	I3	I4	I5	I6	J	J1	J2	J3	J4	J5	J6	K	K1	K2	K3
E. Column vectors of A are linearly independent		111	1																	12211		1		1							1		2		21	3											33			
E1. *Trivial:* Only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution	11																																																	
E2. *Unique:* There is a unique soln to matrix eqn/system of eqns	21																																																	
E3. *Travel*: Can't get back (home)/get to origin with column vectors of A	1																			1																														
E4. *Geometric*: Vectors are not parallel/on the same line or plane	1																		1	1		2																												
E5. *Proportional*: No vector is a scalar multiple of another	1																			2																														
E6. *Linear combination*: No vector is a linear combo of another																																																		
E7. *Placement*: No vector is in the span of the other vectors	1																																																	
E8. *Extra*: Do not have an extra vector needed in order to return home																																																		
F. Column vectors of A are linearly dependent										1	1															21					11														1		1			
F1. *Trivial*: Is more than one solution to $A\mathbf{x} = \mathbf{0}$.										1																																								
F2. *Unique:* No unique/multiple solns to system/matrix eqn										1																																								
F3. *Travel:* Can get back home/back to a point with column vectors of A										1					1					1																														
F4. *Geometric*: Vectors are parallel/on the same line or plane					1					21			1															1																						
F5. *Proportional*: One vector is a scalar multiple of another										3			1	1													1																		1	1				
F6. *Linear combination*: One vector is a linear combo of others																																																		
F7. *Placement*–A vector is in the span of the other vectors													1																																					
F8. *Zeroes*: the matrix A has a row or column of zeroes										11																2		1															1							
F9. *Extra:* Have an extra vector needed in order to return home										1				1																																				
G. Column vectors of A span $\mathbf{R}^n$	2411			1						12		2							3			111	1		1											1		1							1		1			
G1. *Size:* Are enough vectors to span the entire space																			3	1																														
G2. *Geometric*: Can use vectors to get to every pt/go everywhere									1										2	1																														
G3. *Algebraic:* Is a linear combination of vectors for all pts in $\mathbf{R}^n$				1															1	1		2			1																									
G4. *Direction:* Can use each vector to go in a certain direction														1						13	1									1																				
G5. *Solution*: There is a solution to $A\mathbf{x}$=$\mathbf{b}$ for every $\mathbf{b}$																				1		1	1																											
H. Column vectors of A do not span $\mathbf{R}^n$																																																		
H1. *Size:* Are not enough vectors to span the entire space																										11		1		1																				
H2. *Geometric*: Can't use vectors to get to every pt/go everywhere in dim																																																		
H3. *Algebraic:* Is no linear combo of vectors for some pts in $\mathbf{R}^n$																																																		
H4. *Clarify*: The vectors of A span a k-dim subspace of $\mathbf{R}^n$																																																		
H5. *Partial:* Span is only a point/line/plane																												1																						
H6: *Solution*: There is not a solution to $A\mathbf{x}$=$\mathbf{b}$ for every $\mathbf{b}$																																																		
I. Row-reduced echelon form of A has n pivots																				221				1																										
I1. *Diagonal*: RREF(A) has all ones on the main diagonal			2																	1																														
I2. *Identity:* Can row-reduce / is row equivalent to the identity																																															1			
I3. *Pivot-R:* Is a pivot in each row																				1142				2									1	1			1	2												
I4. *Pivot-C:* Is a pivot in each column				1																																														
I5. *Free Variable:* Each variable is defined in system/matrix eqn																																							2								1			
I6. *Zeroes:* RREF(A) has no rows of zeroes																																																		
J. Row-reduced echelon form of A has less than n pivots																														1																				
J1. *Diagonal:* RREF(A) does not have all ones on the main diagonal																																												1						
J2. *Identity:* Cannot row-reduce /not row equivalent to the identity																																																		
J3. *Pivot-R:* Is not a pivot in each row										1																1				3																				
J4. *Pivot-C:* Is not a pivot in each column										1																																			3					
J5. *Free Variable:* Not every variable is defined in system/matrix equation										1		11								1	1																									1				
J6. *Zeroes:* RREF(A) has at least one row of zeroes										1																																								
K. A is invertible	321																			4															2				1									1		
K1. *Augment with the identity:* $[A \mid I] \sim [I \mid A^{-1}]$ is possible																																																		
K2. *Formula:* Can calculat, no "divide by 0 errors"																																																		
K3. *Undo transformation:* Can undo/get back from the transformation																																																		

Figure 4.56. Upper left quadrant of $A(\mathrm{T})_{tot}$.

	E	E1	E2	E3	E4	E5	E6	E7	E8	F	F1	F2	F3	F4	F5	F6	F7	F8	F9	G	G1	G2	G3	G4	G5	H	H1	H2	H3	H4	H5	H6	I	I1	I2	I3	I4	I5	I6	J	J1	J2	J3	J4	J5	J6	K	K1	K2	K3
K4. *Inverse matrix:* Exists a C s.t. $AC=I$ and/or $CA=I$																																															2 1			
K5. *Transform:* Exists sequence of elementary row ops that turns A into I																																			1															
K6. *Necessary output:* Something gets sent to $\mathbf{e}_1$, $\mathbf{e}_2$, ...																																																		
L. A is not invertible										2 1																																								
L1. *Augment with the identity:* $[A \mid I] \sim [I \mid A^{-1}]$ is not possible																																														1				
L2. *Formula:* Can't calculate, get "divide by 0 errors"																																																		
L3. *Undo transformation:* Can't undo/get back from the transformation																																																		
L4. *Inverse matrix:* Does not exist a C s.t. $AC=I$ and/or $CA=I$																																																		
L5. *Transform:* Does not exist sequence of elem row ops to turn A into I																																																		
L6. *Necessary output:* Nothing gets sent to $\mathbf{e}_1$, $\mathbf{e}_2$, ...																												1																						
M. The transformation defined by A is onto	1																			4 1																											1			
M1. *Definition:* For every $\mathbf{b}$ there is at least one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																																		
M2. *Range:* Range is all of the codomain																				1																														
M3. *Images:* All values in codomain get used/mapped to as outputs/images																				1																														
N. The transformation defined by A is not onto																										1																								
N1. *Definition:* For every $\mathbf{b}$ there is not at least one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																																		
N2. *Range:* Range is not all of the codomain																																																		
N3. *Images:* Not all values in codomain get used/mapped to as outputs/images																										1																								
N4. *Collapse:* Transformation collapses everything to a point/line/plane																																																		
N5. *Transform:* The transfromation goes up/adds a dimension																																																		
O. The transformation defined by A is 1-1	3																			1																											1			
O1. *Definition:* For every $\mathbf{b}$ there is at most one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$	1																																																	
O2. *Input/Output:* Each output has at most one input																																																		
O3. *Reachable:* There is only one way to "get to" the output/vector																																																		
P. The transformation defined by A is not 1-1																																																		
P1. *Definition:* For every $\mathbf{b}$ there is more than one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																																		
P2. *Input/Output:* Each output has more than one input																																																		
P3. *Reachable:* There is more than one way to "get to" the output/vector																																																		
P4. *Multiplicity:* At least 2 inputs give the same output																																																		
P5: *Transform:* The transformation goes down/exludes a dimension																																																		
Q. $\det(A) \neq 0$: Determinant of A is nonzero	2																																														1			
Q1. *Area:* Unit square/cube has area/volume $\neq 0$ after transformation																																																		
Q2. *Calculation:* formula to calculate determinant yields nonzero result																																																		
R. $\det(A) = 0$: Determinant of A is equal to 0	1									6																																								
R1. *Area:* Unit square/cube has area/volume $= 0$ after transformation										1 1																																								
R2. *Calculation:* formula to calculate determinant yields zero																																										1								
R3. *Measure:* No area/volume to a line/plane																																																		
S. Miscellaneous																																																		
S1. *Calculation:* get inconsistent solutions (e.g., $0x + 0y = 1$)																		1																																
S2. *Column Space:* The Col A is all of R^n																				1		1	1		1																									
S3. *Column Space:* The Col A is all not of R^n																																																		
S4. *Null Space:* The Nul A contains only the zero vector																																																		
S5. *Null Space:* The Nul A contains more than the zero vector																																																		
S6. *Eigen Value:* The number zero is not an eigenvalue of A																																																		
S7. *Eigen Value:* The number zero is an eigenvalue of A																																																		
S8: $m < n$: A has more vectors than dimensions										1 4			2											1																										
S9: $m = n$: A has less vectors than dimensions																										1	1 1				1												1							
S10: $m = n$: A has the same number of rows/columns			1							3									1	2 1																								1						
S11: $m \neq n$: A does not have the same number of rows/columns																																																		
S12: *Miscellaneous*																																																		

Figure 4.57. Lower left quadrant of $A(\mathrm{T})_{tot}$.

	K4	K5	K6	L	L1	L2	L3	L4	L5	L6	M	M1	M2	M3	N	N1	N2	N3	N4	N5	O	O1	O2	O3	P	P1	P2	P3	P4	P5	Q	Q1	Q2	R	R1	R	R3	S	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12
E				11							1						1		1		6			1							1																			
E1																																																		
E2																																																		
E3																																																		
E4																																																		
E5																																																		
E6																																																		
E7																																																		
E8																																																		
F				841														1	1						3									6		1			1											
F1																													1																					
F2																									2			1																						
F3																																																		
F4																																																		
F5				1														1							1																									
F6																			1																															
F7																																																		
F8				2			1	1							1			4																					2											
F9				1																																										1				
G											74		1								1																			2										
G1																																																		
G2											1			1																																				
G3																																								1										
G4																																																		
G5														1																										1										
H				1											23			1																																
H1																																																		
H2															1			1																																
H3																																																		
H4																		1																																
H5										2					1			1	12																1															
H6																																																		
I																																																		
I1																																																		
I2		1									1										1																													
I3											11										1																													
I4																																																		
I5																																																		
I6																																																		
J																																																		
J1																																																		
J2				1																																														
J3																		1																																
J4																																																		
J5																													1																					
J6				11																					1																									
K	211																														1																	1		
K1																																																		
K2																																																		
K3																																																		

Figure 4.58. Upper right quadrant of $A(\mathrm{T})_{tot}$.

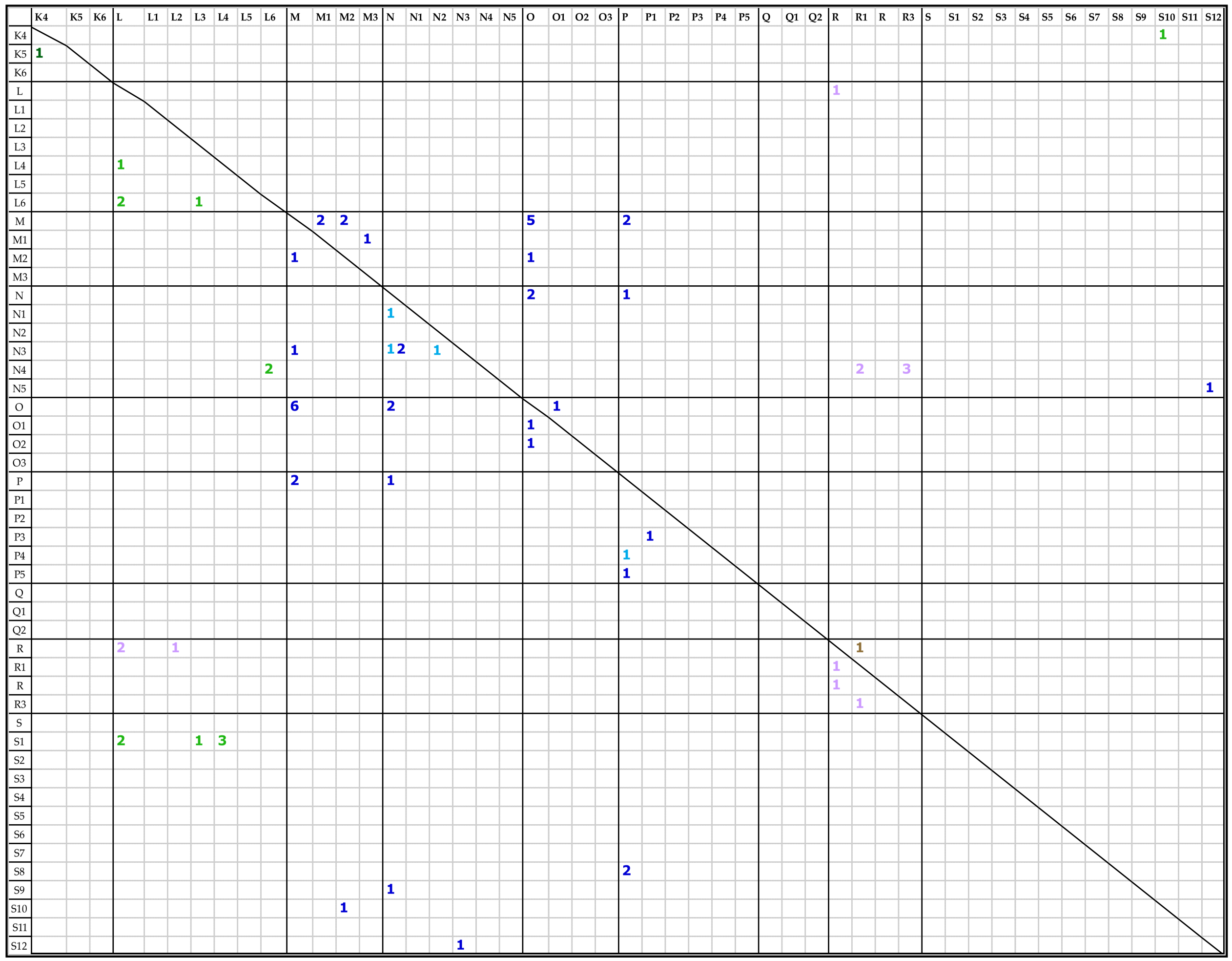


Figure 4.59. Lower right quadrant of $A(\mathrm{T})_{tot}$.

As stated, the adjacency matrix given in Figures 4.56-4.59 is a compilation of the adjacency matrices for the sub-digraphs for the types $m < n$, $m = n$, $m > n$, and *any m, n*. These correspond to whether the speaker in a given argument was reasoning about m x n matrices (or n vectors in $\mathbf{R}^m$, or a transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$) such that m was less than n, m and n were equal, m was greater than n, or no particular m, n type was specified. As such, care must be given regarding what conclusions can be made from $A(\mathrm{T})_{tot}$. For instance, consider the implication, "the columns of span R^n then the columns are linearly independent." It is a perfectly valid implication in the $m = n$ type, but it is invalid in the $m < n$ type (for instance, three vectors in $\mathbf{R}^2$ can span $\mathbf{R}^2$ but they cannot be linearly independent). My conclusions when considering $A(\mathrm{T})_{tot}$, therefore, are limited to ones that pertain to global qualities such as density and continuity of ideas over the entire semester, as well as how this usage compares and contrasts across the four types. I present three categories of results: centrality measures and implications for all m, n; frequencies and implications for individual cells, and shifts in argumentation over the course of the semester.

4.2.2.2.1 ***Category One: Centrality measures and implications for all m, n types.*** Table 4.3 provides a breakdown of how many total edges existed between vertices (i.e., the size) for each of the sub-digraphs for $A(\mathrm{T})_{m<n}$, $A(\mathrm{T})_{m=n}$, $A(\mathrm{T})_{m>n}$, $A(\mathrm{T})_{any}$ and how these edges were distributed over the ten analyzed class days. The content covered on each day is as follows: Days 5/6: generalizations about linear independence, dependence, and span; Day 9: connections between pivots, linear independence, and span; Day 10: connecting ideas for square matrices; Days 17 and 18: generalizations about invertible matrices; Days and 20: investigations into one-to-one and onto; Day 24: connections between determinants and other concepts; and Day 31: reflections on the Invertible Matrix Theorem. Given the content development, the distribution of edges across the various sub-digraphs for the types $m < n$, $m = n$, $m > n$, and *any m, n* are, are perhaps not surprising (see Table 4.3). For instance, only $A(\mathrm{T})_{m=n}$ had any vertices adjacent on

Days 18 and 24. This is not surprising because on these two days, the class reasoned about invertible matrices and determinants of matrices, which are concepts appropriate for square matrices. Thus, while the content discussed on a particular day did not alone dictate the size of particular sub-digraphs of the various types, it did influence which sub-digraphs were necessary.

Table 4.3. Sizes of $A(\mathrm{T})_{m<n}$, $A(\mathrm{T})_{m=n}$, $A(\mathrm{T})_{m>n}$, $A(\mathrm{T})_{any}$, and $A(\mathrm{T})_{tot}$ by day.

	m < n	***m = n***	***m > n***	***any m, n***	***Total (by day)***
Days 5 & 6	28	14	3		45
Day 9	39		5	7	51
Day 10		29			29
Day 17	4	61	4		69
Day 18		14			14
Day 19	3		18	12	33
Day 20	15	52	18	34	119
Day 24		57			57
Day 31		35			35
Total Size (by type)	89	262	48	53	452

The adjacency matrices for each of the four sub-digraphs are in Appendices 4.8-4.11. The in-degree, in-connection, out-degree, out-connection, and centrality were calculated for each vertex in all sub-digraphs, and those calculations are provided in entirety in the appropriate Appendix 4.8-4.11. The summary of these data (except centrality) is presented in Figures 4.60 and 4.61. The out-degree and out-connection for each vertex in $A(\mathrm{T})_{tot}$, $A(\mathrm{T})_{m<n}$, $A(\mathrm{T})_{m=n}$, $A(\mathrm{T})_{m>n}$, and $A(\mathrm{T})_{any}$ is given in Figure 4.60, and the in-degree and in-connection for each vertex in the same matrices is given in 4.61. This information for $A(\mathrm{T})_{tot}$, as well as each vertex's centrality measure, is given in Figure 4.62. The style of this table was seen previously in Figure 4.52, which summarized this same information but for arguments relevant to CMP 1.

total		m<n		m=n		m>n		any		
OD	OC	OD	OC	OD	OC	OD	OC	OD	OC	
41	17	3	2	32	14	4	4	2	1	E
2	1			2	1					E1
3	1			3	1					E2
2	2			2	2					E3
5	4	2	2	3	3					E4
3	2	2	1			1	1			E5
										E6
1	1			1	1					E7
										E8
35	13	4	2	28	10	1	1	2	1	F
2	2	1	1					1	1	F1
4	3	2	2					2	2	F2
3	3	2	2					1	1	F3
6	4	3	3	1	1	1	1	1	1	F4
11	9	5	3	4	4			2	2	F5
1	1			1	1					F6
1	1			1	1					F7
17	10			9	7	7	3	1	1	F8
4	4	3	3	1	1					F9
41	16	8	4	24	12	2	2	7	3	G
4	2	4	2							G1
6	5	2	1	2	2	2	2			G2
7	6	1	1	6	5					G3
7	4	5	4	1	1			1	1	G4
5	5			5	5					G5
7	3	1	1	3	2	2	1	1	1	H
4	3	2	2			2	2			H1
2	2					1	1	1	1	H2
										H3
1	1							1	1	H4
9	6	2	1	4	3	1	1	2	2	H5
										H6
6	2	2	2	2	1	2	1			I
3	2			3	2					I1
4	4			4	4					I2
18	8	3	2	10	7	4	2	1	1	I3
1	1			1	1					I4
3	2			3	2					I5
										I6
1	1	1	1							J
1	1	1	1							J1
1	1			1	1					J2
6	4	3	1	1	1	2	2			J3
4	2	4	2							J4
7	6	4	4	1	1			2	2	J5
4	3			4	3					J6
20	8			19	8	1	1			K
										K1
										K2
										K3

total		m<n		m=n		m>n		any		
OD	OC	OD	OC	OD	OC	OD	OC	OD	OC	
4	2			3	1	1	1			K4
2	2			2	2					K5
										K6
4	2			4	2					L
1	1			1	1					L1
										L2
										L3
1	1			1	1					L4
										L5
4	3			4	3					L6
18	7	2	1	8	4			8	3	M
1	1							1	1	M1
3	3			1	1			2	2	M2
1	1			1	1					M3
4	3			1	1	2	1	1	1	N
1	1							1	1	N1
										N2
6	4					3	3	3	2	N3
7	3			7	3					N4
1	1					1	1			N5
14	6			8	4	2	1	4	3	O
2	2							2	2	O1
1	1			1	1					O2
										O3
3	2	2	1	1	1					P
										P1
										P2
1	1							1	1	P3
1	1							1	1	P4
1	1	1	1							P5
3	2			3	2					Q
										Q1
										Q2
11	5			11	5					R
3	2			3	2					R1
2	2			2	2					R
1	1			1	1					R3
										S
7	4			7	4					S1
4	4			4	4					S2
										S3
										S4
										S5
										S6
										S7
10	4	8	4	1	1			1	1	S8
6	5			1	1	5	4			S9
10	6	6	4	4	3					S10
										S11
1	1					1	1			S12

Figure 4.60. Out-degree and out-connection for $A(\mathrm{T})_{tot}$, $A(\mathrm{T})_{m<n}$, $A(\mathrm{T})_{m=n}$, $A(\mathrm{T})_{m>n}$, and $A(\mathrm{T})_{any}$.

total		m<n		m=n		m>n		any		
ID	IC	ID	IC	ID	IC	ID	IC	ID	IC	
31	13			27	11	1	1	3	2	E
3	1	1	1	2	1					E1
4	3			4	3					E2
3	3			3	3					E3
1	1	1	1							E4
										E5
										E6
										E7
1	1			1	1					E8
39	18	18	9	16	8	1	1	4	4	F
1	1							1	1	F1
4	2	1	1	2	1			1	1	F2
5	4	3	2	1	1			1	1	F3
3	3	2	2					1	1	F4
1	1	1	1							F5
										F6
										F7
1	1			1	1					F8
11	6	11	6							F9
51	21	11	8	29	16	5	2	6	3	G
2	2	2	2							G1
10	6	2	2	7	6	1	1			G2
3	3			3	3					G3
5	4	3	3	1	1			1	1	G4
3	3			3	3					G5
11	7	1	1	3	2	6	5	1	1	H
3	2	1	1			2	1			H1
5	5	1	1	2	2	2	2			H2
										H3
6	4	6	4							H4
4	3			2	1	2	2			H5
										H6
3	2			3	2					I
1	1			1	1					I1
6	6			6	3					I2
4	4			4	2					I3
1	1			1	1					I4
3	3			3	2					I5
3	3			3	2					I6
										J
										J1
1	1			1	1					J2
2	2			1	1	1	1			J3
2	2	2	2							J4
6	4	4	2	1	1			1	1	J5
3	3			3	3					J6
16	9			16	9					K
1	1			1	1					K1
										K2
										K3

total		m<n		m=n		m>n		any		
ID	IC	ID	IC	ID	IC	ID	IC	ID	IC	
5	2			4	2	1	1			K4
1	1			1	1					K5
										K6
30	12	2	1	26	12	2	2			L
										L1
1	1			1	1					L2
3	3			3	3					L3
4	2			4	2					L4
										L5
4	2			4	2					L6
26	9	2	1	12	5	3	3	9	4	M
2	2							2	1	M1
4	3			1	1			3	2	M2
3	3			1	1	1	1	1	1	M3
16	9			4	3	7	5	5	4	N
										N1
2	2					2	2			N2
12	9	1	1	2	2	6	3	3	3	N3
6	4	2	1	4	4					N4
										N5
19	9	2	1	12	7	2	1	3	2	O
1	1							1	1	O1
										O2
1	1					1	1			O3
14	9	8	6	4	4			2	2	P
1	1							1	1	P1
										P2
1	1							1	1	P3
2	2							2	2	P4
										P5
2	2			2	2					Q
										Q1
										Q2
9	4			9	4					R
5	4			5	4					R1
1	1			1	1					R
3	1			3	1					R3
										S
3	2			3	2					S1
4	3			4	3					S2
										S3
										S4
										S5
										S6
										S7
1	1	1	1							S8
										S9
2	2			1	1	1	1			S10
										S11
1	1					1	1			S12

Figure 4.61. In-degree and in-connection for $A(\mathrm{T})_{tot,}$ $A(\mathrm{T})_{m<n,}$ $A(\mathrm{T})_{m=n}$, $A(\mathrm{T})_{m>n}$, and $A(\mathrm{T})_{any}$.

	total		total		
	OD	OC	ID	IC	centrality
E	41	17	31	13	0.1807
E1	2	1	3	1	0.0120
E2	3	1	4	3	0.0241
E3	2	2	3	3	0.0301
E4	5	4	1	1	0.0301
E5	3	2			0.0120
E6					
E7	1	1			0.0060
E8			1	1	0.0060
F	35	13	39	18	0.1867
F1	2	2	1	1	0.0181
F2	4	3	4	2	0.0301
F3	3	3	5	4	0.0422
F4	6	4	3	3	0.0422
F5	11	9	1	1	0.0602
F6	1	1			0.0060
F7	1	1			0.0060
F8	17	10	1	1	0.0663
F9	4	4	11	6	0.0602
G	41	16	51	21	0.2229
G1	4	2	2	2	0.0241
G2	6	5	10	6	0.0663
G3	7	6	3	3	0.0542
G4	7	4	5	4	0.0482
G5	5	5	3	3	0.0482
H	7	3	11	7	0.0602
H1	4	3	3	2	0.0301
H2	2	2	5	5	0.0422
H3					
H4	1	1	6	4	0.0301
H5	9	6	4	3	0.0542
H6					
I	6	2	3	2	0.0241
I1	3	2	1	1	0.0181
I2	4	4	6	6	0.0602
I3	18	8	4	4	0.0723
I4	1	1	1	1	0.0120
I5	3	2	3	3	0.0301
I6			3	3	0.0181
J	1	1			0.0060
J1	1	1			0.0060
J2	1	1	1	1	0.0120
J3	6	4	2	2	0.0361
J4	4	2	2	2	0.0241
J5	7	6	6	4	0.0602
J6	4	3	3	3	0.0361
K	20	8	16	9	0.1024
K1			1	1	0.0060
K2					
K3					

	total		total		
	OD	OC	ID	IC	centrality
K4	4	2	5	2	0.0241
K5	2	2	1	1	0.0181
K6					
L	4	2	30	12	0.0843
L1	1	1			0.0060
L2			1	1	0.0060
L3			3	3	0.0181
L4	1	1	4	2	0.0181
L5					
L6	4	3	4	2	0.0301
M	18	7	26	9	0.0964
M1	1	1	2	2	0.0181
M2	3	3	4	3	0.0361
M3	1	1	3	3	0.0241
N	4	3	16	9	0.0723
N1	1	1			0.0060
N2			2	2	0.0120
N3	6	4	12	9	0.0783
N4	7	3	6	4	0.0422
N5	1	1			0.0060
O	14	6	19	9	0.0904
O1	2	2	1	1	0.0181
O2	1	1			0.0060
O3			1	1	0.0060
P	3	2	14	9	0.0663
P1			1	1	0.0060
P2					
P3	1	1	1	1	0.0120
P4	1	1	2	2	0.0181
P5	1	1			0.0060
Q	3	2	2	2	0.0241
Q1					
Q2					
R	11	5	9	4	0.0542
R1	3	2	5	4	0.0361
R	2	2	1	1	0.0181
R3	1	1	3	1	0.0120
S					
S1	7	4	3	2	0.0361
S2	4	4	4	3	0.0422
S3					
S4					
S5					
S6					
S7					
S8	10	4	1	1	0.0301
S9	6	5			0.0301
S10	10	6	2	2	0.0482
S11					
S12	1	1	1	1	0.0120

Figure 4.62. Summary information for adjacency matrix $A(\mathrm{T})_{tot}$.

Figure 4.62 summarizes multiple aspects of the information provided in Figures 4.60 and 4.61. First, it summarizes information regarding how each vertex was used in the four sub-digraphs. The grayed out table values indicate that a particular node was used in only one type. For instance, all the Q and R codes are grayed-out values because they appeared in $A(\mathrm{T})_{m=n}$ but not the others. It makes sense that these were used in $m = n$ argumentation because they are codes related to the concept of determinant.

Some of the remaining cells, those that do not have gray table values, are shaded light blue. The light blue shaded squares in Figure 4.62 indicate non-mutually exclusive in-connections or out-connections for a particular vertex. For instance, the cell for the in-connection of vertex N4, "the transformation collapses everything to a point/line/plane," is shaded blue and has a value of 4. By taking a closer look and examining the N4 row of the table in Figure 4.61 to order to determine the in-connections for that vertex across the four types, we see the in-connection of N4 for $A(\mathrm{T})_{m<n}$ is 4, for $A(\mathrm{T})_{m=n}$ is 2, for $A(\mathrm{T})_{m>n}$ is 4, and for $A(\mathrm{T})_{any}$ is 0. If the in-connection of N4 for $A(\mathrm{T})_{tot}$ were 4+ 2 + 4 = 10, then the total vertices adjacent to N4 would all be distinct. Because the number is less than 10 (it is 4), then across the three types for which the in-connection of N4 is nonzero, at least one vertex adjacent to N4 was repeated. In contrast, neither the in-connection nor out-connection for vertex J3, "There is not a pivot in each row," is shaded light blue. This situation arises when the vertices that a given vertex is adjacent to (or adjacent from) are mutually exclusive.

Figure 4.62 shows that vertex G and vertex F had the highest and second highest centrality, respectively, in $A(\mathrm{T})_{tot}$. Figure 4.62 further shows that, for both G and F, neither their in-connections nor out-connections were mutually exclusive across m, n types. Not only were their measure of centrality the highest, indicating their participation in the structure of argumentation over the course of the semester was central by being adjacent to or adjacent from the highest number of distinct nodes, codes F and G were adjacent to or adjacent from some of the same vertices in more than one m, n type. Thus, across two measures, vertex F ("the columns of A are linearly dependent") and vertex G ("the columns of A span $\mathbf{R}^n$") were central ways of reasoning for the collective during whole class discussion over the course of the semester. More regarding these specific vertices is addressed in the subsequent Category Two results section.

The use of adjacency matrices can also provide information on when the in-connection or out-connection of a vertex across types was mutually exclusive. One such cell whose out-connection was mutually exclusive was G2, "Can use all the vectors to get to every point/to get everywhere." This vertex was adjacent to five different vertices (see Table 4.4).

Table 4.4. Out-connection details for vertex G2.

	Total OD	**Total OC**	$m < n$ **OD**	$m < n$ **OC**	$m = n$ **OD**	$m = n$ **OC**	$m > n$ **OD**	$m > n$ **OC**
	6	5	2	1	2	2	2	2
Specific instances of adjacency				(G2, F9) = 2 (Days 5/6)		(G2, E8) = 1 (Days 5/6) (G2, G) = 1 (Day 31)		(G2, M) = 1 (Day 20) (G2, M3) = 1 (Day 19)

The second row of Table 4.4 is extracted from Figure 4.60, and the specific vertices to which G2 was adjacent is extracted from the adjacency matrices for these sub-digraphs, found in Appendices 4.8- 4.10. In the "$m < n$ OC" column of Table 4.4, the adjacency (G2, F9), "if you can get everywhere with the column vectors, then you have that extra vector needed to return home" occurred twice. This only makes sense in the $m < n$ type: it would not be true in for m x n matrices where $m = n$, for instance. Also, these two adjacencies occurred in Day 5 or 6, when the class was developing a way to reason about linear dependence for when there were more than n vectors in $\mathbf{R}^n$. Also on Day 5 or 6, the adjacency (G2, E8) occurred. The code E8, "you do not have an extra vector needed to return home," could be considered the negation of F9. Although it seems that (G2, F9) and (G2, E8) could not both be sensible ways of reasoning, they are not contradictory because the former occurred was reasoning about m x n matrices where $m < n$, and the latter when reasoning about matrices where $m = n$.

The language in both F9 and E8 coincide with the travel language of G2, all of which grew out of the class's experience with the Magic Carpet Ride Problem. In that problem setting, the class first reasoned about span by considering "all the places you could get" with a linear combination of a set of given vectors. Keeping within the Magic Carpet Ride setting, they investigated journeys that began and ended at home and developed the formal definitions of linear independence and dependence out of this situational experience. Thus, it is reasonable that students would, early in the first few days of the semester, reason that a set of two vectors in $\mathbf{R}^2$ are linearly independent because they can get everywhere with the 2 vectors (G2) but don't have an extra vector needed to get back home (→E8). It is also reasonable that reasoning about span and linear independence, over the course of the semester, became less dependent on these situational ways of reasoning, moving more into general and formal activity (Gravemeijer, 1999). This can be seen through the drop-off of the E8, F9, G2, and H2 codes and using the "parents codes" E, F, G, and H to make claims about new concepts such as invertibility or one-to-one transformations. More regarding this transition is given in the third category of results given in a subsequent section.

The travel language of G2 did surface in argumentation concerning the concept of onto transformations. On Day 19, G2 was adjacent to M ("The transformation defined by *A* is onto"), and G was adjacent to M3 ("All values in the codomain get used/mapped to as outputs") on Day 20. These two days were ones in which the concept of onto was explored, so it is noteworthy to see the travel language was used to reason about transformations that were or were not onto the given codomain. In contrast, the class developed a notion of equivalence between the concepts of one-to-one and linear independence, yet there was no reference to F3, F9, E3, or E8 (the codes referring to an ability or inability to travel “back home” with given vectors) when reasoning about one-to-one transformations.

Results about the ways in which the classroom community reasoned about the Invertible Matrix Theorem presented in Category One focused on the notions of centrality for given vertices across the total adjacency matrix $A(\mathrm{T})_{tot}$, as well as within and across the various types of *m*, *n* scenarios. Vertices F and G had high centrality measures, indicating their integral participation in the argumentation structure by being adjacent to or adjacent from a high number of other vertices. Furthermore, vertices F and G were adjacent to or from other vertices in each of the *m*, *n* sub-digraphs, indicating their importance in the classroom argumentation in a variety of situations.

The results presented in Category One correlate well with certain aspects of the methodology for documenting collective activity via Toulmin’s Model. For instance, a vertex with a high out-connection means that the vertex was adjacent to a variety of other vertices. Given that the adjacency (*u*, *v*) for vertices *u* and *v* could be read as “if *u* then *v*,” a vertex with a high out-connection means that the concept statement or interpretation represented by that vertex served as data for a variety of claims. This is precisely what was presented as Criterion 3 for documenting normative ways of reasoning (Cole et al, in press). High diversity within a particular cell (such as the (G, E) cell), rather than over a particular row, indicates that the given implication (rather than a given concept) served a role in multiple arguments. How this relates to Toulmin analysis is investigated further in the Category Two results section.

As a final quantitative consideration of the centrality measure, Figure 4.63 tabulates the centrality of each concept statement and their associated subcodes as one measure.

	OD	OC	ID	IC	Centrality
Code E and subcodes:	57	28	43	22	0.3012
Code F and subcodes:	84	50	65	36	0.5181
Code G and subcodes:	70	38	74	39	0.4639
Code H and subcodes:	23	15	29	21	0.2169
Code I and subcodes:	35	19	21	20	0.2349
Code J and subcodes:	24	18	14	12	0.1807
Code K and subcodes:	26	12	23	13	0.1506
Code L and subcodes:	10	7	42	20	0.1627
Code M and subcodes:	23	12	35	17	0.1747
Code N and subcodes:	19	12	36	24	0.2169
Code O and subcodes:	17	9	21	11	0.1205
Code P and subcodes:	6	5	18	13	0.1084
Code Q and subcodes:	3	2	2	2	0.0241
Code R and subcodes:	17	10	18	10	0.1205
Code S and subcodes:	38	8	11	9	0.1024

Figure 4.63. Out-degree, out-connection, in-degree, in-connection, and centrality information for $A(\mathrm{T})_{tot}$ grouped by main code and subcodes.

Figure 4.52 showed that as a single vertex, vertex G was the most central. Figure 4.53 shows that when considering a concept statement and its associated subcodes (now called a *category*) as a unit, category F was more central than category G. Thus, although concept statement G (“the columns of A span $\mathbf{R}^n$”) was the most central way of reasoning throughout the semester, the concept statement and associated interpretations for “the columns of A are linearly dependent” were more central than that of span and its associated interpretations. One explanation for this could be that the notion of linear dependence had a wider variety of powerful interpretations for this particular classroom community than did span.

Finally, Figure 4.63 can be read as a comparison between categories and their negations. For instance, the E category involved the concept statement “the columns of A are linearly independent” and interpretations of that, whereas the F category was related to linear dependence. Upon comparing categories and their negations (E vs. F, G vs. H, I vs. J, K vs. L, M vs. N, O vs. P, and Q vs. R), we see the following:

- Three categories as stated in the IMT are more central than their negations: "The columns of *A* span Rn" (category G); "The row-reduced echelon form of *A* has *n* pivots" (category I); and "The transformation defined by *A* is one-to-one" (category O).
- Four categories as stated in the IMT were *less* central than their negations. The more central negations were: "The columns of *A* are linearly dependent" (category F); "The matrix *A* is not invertible" (category L); "The transformation defined by *A* is not onto $\mathbf{R}^n$" (category N); and "The determinant of *A* is zero" (category R).

Thus, for this classroom community, reasoning about the Invertible Matrix Theorem also involved reasoning about negations of the statements involved in the theorem. This could have occurred as the justifications for a given implication were given in the form of a contrapositive. Additionally, some concepts may lend themselves naturally to students reasoning about their negations. From this data, one such possibility could be with the idea of determinants. The centrality of "the determinant of *A* is zero" was drastically higher than "the determinant of *A* is nonzero." A deeper examination of the particular arguments regarding determinants would reveal that the class developed ways of reasoning about determinant and how it was connected to other concepts by focusing more examples of matrices with determinants of zero rather than nonzero. In Chapter 5, I report that this t preference with determinants held true for Abraham as well.

4.2.2.2.2 ***Category Two: Frequencies and implications for individual cells.*** The adjacency matrix $A(\mathrm{T})_{tot}$ is also an informative tool regarding the participation of individual cells in the classroom discourse throughout the semester. By "participation of individual cells," I mean specific pairs of adjacent vertices that occur during whole class discussion, and this information is gleaned by examining specific cells of $A(\mathrm{T})_{tot}$. To determine an area of focus for this results section, I examined the adjacency matrix of $A(\mathrm{T})_{tot}$. For any two vertices x, y, the adjacent pairs that occurred in at least three different days were:

1. (E, E1): 3 times over 3 different days
2. (E, G): 7 times over 5 different days
3. (G, E): 8 times over 4 different days
4. (F, L): 13 times over 3 different days
5. (G, G2): 3 times on 3 different days
6. (I, G): 5 times over 3 different days
7. (I3, G): 8 times over 4 different days
8. (K, K4): 4 times over 3 different days
9. (K, E): 6 times over 3 different days

From this information, the adjacency that occurred with the most frequency was (F, L), "If the column vectors of *A* are linearly dependent, then *A* is not invertible," which occurred 13 times. The adjacency that occurred on the highest number of days was (E, G), "If the columns of *A* are linearly independent, then the columns of *A* span $\mathbf{R}^n$," which occurred on five of the ten days analyzed.

First, consider the adjacency (F, L). Eight of the (F, L) adjacencies occurred on Day 17, the day in which the class worked to create generalizations regarding types of matrices that were or were not invertible. The first five arguments, students focused their presentations on why matrices with rows or columns of zeroes were not invertible. Aspects of these arguments surface in $A(\mathrm{T})_{tot}$ in the other cells of the F-F9/L-L5 block (see Figure 4.62). During Argument 17.6, Randall first proposed this if-then pair as a valid claim. The teacher elaborated on this claim by discussing its validity in the $m < n$, $m = n$, and $m > n$ cases. She then limited the notion of "invertibility" to one for square matrices, and Arguments 17.8-17.10 investigated why the 2x2 matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ failed to be invertible. In these three arguments, however, the (F, L) adjacency does not appear. In Argument 17.8, vertices F and L were the "bookends" to Edgar's explanation that the calculations in determining a matrix *C* such that $AC = I$ yielded an inconsistent system. In Arguments 17.9 and 17.10, Abraham's explanation focused on the span of the columns of *A* being only a line, as well as if *A*, as a transformation, collapsed everything to the line $y = 2x$, then the column vectors of *I* could never be outputs for that transformation. The teacher then led the class in the first few steps of a very formal proof that the adjacency (F, L) was true in all n x n cases. This series of arguments contributed the eight edges incident from F to L seen in the F-L cell of $A(\mathrm{T})_{tot}$. The adjacency (F, L) surfaced again on Day 24, during investigations into determinants. Edgar was the first to suggest that if the determinant equals *A* is zero then *A* is not invertible, and the connection (F, L) was used in his justification. This connection surfaced three more times on Day 24. In all instances on Day 24, the adjacency (F, L) was not under debate but rather was being used to justify a new claim (about determinants).

Second, consider the adjacency (E, G), "If the columns of *A* are linearly independent, then the columns of *A* span $\mathbf{R}^n$," which occurred on five of the ten days analyzed: Days 5/6, Day 17, Day 20, Day 24, and Day 31. Notice that these five days are distributed from the first to the last day analyzed, and the main concept on the days varied from span and linear independence (Day 5/6), invertibility (Day 17), one-to-one and onto (Day 20), and determinants (Day 24). While this information may not show up in the out-degree and out-connection summary data (such a high frequency in one given cell does nothing to increase a vertex's out-connection), it is fascinating to notice this if-then pair was used throughout the semester to reason about other concepts in the Invertible Matrix Theorem. Furthermore, on Day 31, Nate explained why, for him, these two concept statements were the most obviously equivalent (this argument was analyzed in the previous sections of Toulmin analysis).

A variation of considering a given individual cell's frequency is considering a particular block of cells and their frequencies. For instance, consider the cells for rows I-I6 and columns G-G5. A dense block of cells was the I-I3→ G block of cells (see Figure 4.56). Over five different days (Days 9, 10, 19, 20, and 31), there were 14 times the code I, I1, or I3 was adjacent to G. During Days 9 and 10, the number of rows with a pivot was being connected to the dimension of the subspace that the column vectors would span. This same adjacent pair was used during Days 19 and 20 to support claims that specific transformations, or transformations in general, were onto the codomain. Again, as was the case with the first two examples (F, L) and (E, G), the data-claim pair represented in the adjacency shifted in its role in whole class argumentation. It, as an entity, transitioned from being discussed as a valid if-then connection to a new role in arguments, that of provided data or warrant in new arguments.

The analysis presented in Category #2 is compatible with the Toulmin analysis presented in the first half of this chapter. In the adjacency matrix analysis of this present section, by considering individual cells of high frequency across multiple days, I was able to analyze how particular adjacent vertices functioned as data-claim pairs over the course of the semester. The adjacency matrix $A(\mathrm{T})_{tot}$ provides information regarding which days certain vertices are adjacent to one another during whole class discussion. While the specifics of the argumentation that these adjacent pairs belong to is not provided in $A(\mathrm{T})_{tot}$, knowing on which days the pairs occurred provides easily-accessible information regarding what concepts during the semester the pairs are used to reason about. This allows an analysis compatible with Criterion 2 (Rasmussen & Stephan, 2008), developed and used in conjunction with Toulmin analysis, in order to determine what normative ways of reasoning develop in a classroom. Furthermore, the adjacency matrix analysis also provides information that is difficult to glean from only the Toulmin analysis. The sub-digraphs associated with $A(\mathrm{T})_{tot}$, as well as the summary data in Figures 4.60-4.62, provide specific information regarding what type of m x n situation ($m < n$, etc.) the adjacent pairs were a part of, as well as quick quantitative data regarding on which days the data-claim pair (or pair of adjacent vertices) were utilized.

4.2.2.2.3 *Category Three: shifts in argumentation over the course of the semester.* The last category of results presented regarding how the classroom reasoned about the

Invertible Matrix Theorem focuses on shifts in argumentation patterns over the course of the semester. The primary information source for this analysis is $A(\mathrm{T})_{tot}$ (see Figures 4.56-4.59). By considering where in the adjacency matrix entries appear, in conjunction with considering which days these adjacencies appeared, information regarding shifts in argumentation patterns, thus a sense of the "travel of ideas" (Saxe et al, 2009) throughout the semester is revealed.

Recall the order of the days in association with their representative color schemes in the adjacency matrix: Days 5 and 6: Pink; Day 9: Red; Day 10: Orange; Day 17: Green; Day 18: Dark Green; Day 19: Light Blue; Day 20: Dark Blue; Day 24: Purple; and Day 31: Brown. This order dictates the way in which the shifts in argumentation patterns are detectable in $A(\mathrm{T})_{tot}$, as well as in $A(\mathrm{T})_{m<n}$, $A(\mathrm{T})_{m=n}$, $A(\mathrm{T})_{m>n}$, and $A(\mathrm{T})_{any}$. First, the colors alone in $A(\mathrm{T})_{tot}$ help illuminate the movement of argumentation throughout the semester. Through a very large grain size, there is a high concentration of pink and red in the upper left of the adjacency matrix (coinciding with days 5, 6, and 9); a concentration of orange within the I block (Day 10); a concentration of light and dark blue under the M, N, O, and P blocks (Days 19 and 20); purples occurred under the Q and R blocks (Day 24), and brown (Day 31) is scattered throughout the adjacency matrix.

Given the order of the development of mathematical meaning over the course of the semester and what the teacher posed as tasks for discussion on each day, the aforementioned concentrations are sensible. First, the high pink and red concentration in the upper left corner of $A(\mathrm{T})_{tot}$ indicates a high level of explanation within the F category. Early in the semester, the class had to develop ways of reasoning about linear dependence, but there was no need to explain or reinterpret different F codes later in the semester. Thus, the use of the various F codes moved away from being adjacent to or adjacent from other F codes towards making claims about other concepts in the IMT. Furthermore, when considering the sub-adjacency matrices, there is no pink or red in the upper left quadrant of the $A(\mathrm{T})_{m=n}$ adjacency matrix, but rather these adjacencies appear in $A(\mathrm{T})_{m<n}$. This coincides with the previous analyses of CMP 1 (via both adjacency matrices and Toulmin's model). Through this class mathematics practice, the collective's way of reasoning about linear independence and span as equivalent ideas for square matrices took time to develop, first focusing on those two concepts individually and for non-square matrices. This reliance on explanations within the F category became less prominent as the semester progressed; rather, the class used vertices within the F category to reason about other concepts. This coincides with Criterion Two for documenting normative ways of reasoning: when claims that previously needed justification shift to becoming part of the justification for new claims.

Second, consider the concentration of light and dark blue under the M, N, O, and P blocks (Days 19 and 20); these categories correspond to reasoning about whether or not transformations are onto or one-to-one. In order to facilitate comparison with the analysis of CMP 2, "Determining whether or not a transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ is onto $\mathbf{R}^m$ by considering the span of the column vectors of the associated matrix," presented in the Ontogenetic Analysis via Toulmin's Model section of this chapter, I focus the current analysis on a discussion of span and onto.

Within Days 19 and 20, there was a high concentration in the rows for the G and H categories and the columns for the M and N categories (14 and 13, respectively). This implies a high frequency of adjacencies from the interpretations of column vectors spanning or not spanning all of $\mathbf{R}^n$ leading to conclusions about whether transformations are onto or are not onto $\mathbf{R}^n$. There are far fewer adjacencies from rows for the M and N categories to columns for the G and H categories (6 and 2, respectively). This means that the concept statement of "the transformation defined by A is not onto" was the claim more often than it was the data. Considered collectively, this is compatible with the second classroom mathematics practice of using span to reason about onto. This way of reasoning, rather than the reverse, is sensible because of the structure of the course and what the class was investigating on those days. Furthermore, substantial portion of the adjacencies in the N category (the transformation is not onto) occurred while reasoning about transformations such that $m > n$ (see Figures 4.61 and 4.62). This corresponds to NWR 2.3 from the previous Toulmin analysis of CMP 2.

Finally, the adjacencies from Day 31 (the brown entries in $A(\mathrm{T})_{tot}$) are somewhat scattered through the adjacency matrix. On this class day, students were given time to discuss in their small groups various aspects of the Invertible Matrix Theorem, such as the concepts and equivalencies with which they were most and least comfortable, how they understood certain equivalencies, etc. A small portion of time was given in whole class discussion to report back about this small group work. For instance, the concentration of brown adjacencies in the G category, along with connections to S2, correspond to two arguments by Justin and Abraham (Arguments 31.3 and 31.4) in which they explain how the statements "G: the columns of A span all of $\mathbf{R}^n$," "S2: the column space of A is all of $\mathbf{R}^n$," "G3: for every $\mathbf{b}$ in $\mathbf{R}^n$, there's a way to write $\mathbf{b}$ as a linear combination of the columns of A," and "'G5: For every $\mathbf{b}$ in $\mathbf{R}^n$, there exists a solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$" are equivalent ideas for them. These explanations focus on how the involved concepts are "the same thing," or are defined in terms of each other. This is distinct from, for instance, how Nate explains how he sees linear independence and span as equivalent ideas (through a connection to pivot positions). I revisit these notions of *conceptual equivalency* and *logical equivalency,* respectively, in Chapter 6.

4.3 Conclusion

Within this chapter, I explored the ways in which the classroom community reasoned about the Invertible Matrix Theorem over the semester. I utilized two analytical tools, Toulmin's Model of Argumentation and adjacency matrices, married with the microgenetic and ontogenetic strands of analysis. I considered the mathematical development of the classroom through each analytical tool separately; in doing so, noteworthy aspects regarding the structure of argumentation were revealed.

Through Toulmin analysis, investigating the mathematical development of the classroom community revealed the existence of at least two classroom mathematics practices: (a) CMP 1: Reasoning about span and linear independence as equivalent ideas for square matrices, and (b) CMP 2: Determining whether or not a transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ is onto $\mathbf{R}^m$ by considering the span of the column vectors of the associated matrix A. The normative ways of reasoning associated with these two classroom mathematics practices involved: making use of the travel language derived from the Magic Carpet Ride problem in order to develop ways of reasoning about the formal notions of span and linear independence; exploring examples and generating conjectures about span, linear independence, one-to-one, and onto in both $m > n$ and $m < n$ scenarios in order to draw conclusions regarding those concepts when $m = n$; establishing the equivalence of " the span is all $\mathbf{R}^n$" and "the transformation is onto $\mathbf{R}^n$"; and reasoning about implications from one concept statement to the other for square matrices. In addition, the argumentation log indicates that reasoning about one-to-one and onto transformations as equivalent when $m = n$ relied heavily on associations between span and onto, one-to-one and linear independence, brought together with the equivalence of span and linear independence from CMP 1.

Beyond the mathematical results determined through the use of Toulmin's Model, I also defined four types of argumentation schemes that are an expanded version of Toulmin's model. These expanded structures developed out of necessity when the original 6-part Toulmin's scheme proved inadequate while analyzing argumentation of the classroom community. Aspects of these structures were adapted from and are compatible with those by Aberdein (2006, 2009).

Analyzing the classroom community's argumentation relevant to reasoning about the Invertible Matrix Theorem through adjacency matrices revealed a variety of results. Within the microgenetic analysis via adjacency matrices section, I presented two categories of results regarding (a) where within a category an argumentation occurred, and (b) the type of sub-digraph in which the argumentation occurred. For (a), I reported on two main structures that existed in the data. First, when the argumentation served towards developing a way of reasoning about a new concept or connection between two concepts, the argument involved multiple uses of the "interpretation" subcodes. Second, when the argumentation made use of relatively well-established concepts or connections between concepts, the argument involved mainly the "concept statement" main codes. For (b), regarding reasoning within the various m, n sub-digraphs, I provided an example from Day 9 in which Justin justified to the rest of the class, "if there are more vectors than dimensions, the set of vectors must be linearly dependent." Within that one justification, Justin first reasoned about $m > n$ matrices, then square matrices, and then reached the desired case of $m > n$ matrices.

The results of ontogenetic analysis via adjacency matrices were presented in two sections: those arguments involved in CMP 1 (from the ontogenetic Toulmin analysis), and the compilation of all coded arguments. Within the first section, analysis of the $m < n$ and $m = n$ sub-digraphs revealed high levels of compatibility with the Toulmin analysis for those same arguments. Concept statement G had the highest centrality for $A(P_1)_{m=n}$, and I constructed a digraph of these connections to investigate more specifically how the concept of "the columns of A span $\mathbf{R}^n$" was used over the course of the arguments analyzed. This demonstrated within which arguments statement G was uttered in conjunction with other concept statements.

The results from analyzing the adjacency matrices for the compilation of all coded arguments were presented according to three categories: (a) centrality measures and implications for all m, n types; (b) frequencies and implications for individual cells; and (c) shifts in argumentation over the course of the semester. For (a), I reported that vertex G ("the columns of A span $\mathbf{R}^n$") and vertex F ("the columns of A are linearly dependent") had the highest and second highest centrality, respectively, in $A(\mathrm{T})_{tot}$. Furthermore, for both G and F, neither their in-connections nor out-connections were mutually exclusive across *m*, *n* types. Not only were their measure of centrality the highest, indicating their participation throughout the semester was central by being adjacent to or adjacent from the highest number of distinct nodes, codes F and G were adjacent to or from some of the same vertices in more than one *m*, *n* type.

The second category revealed that the adjacency matrix $A(\mathrm{T})_{tot}$ was also an informative tool regarding the participation of individual cells in the classroom discourse. The adjacency that occurred most was (F, L), "If the column vectors of *A* are linearly dependent, then *A* is not invertible," and the adjacency that occurred on the most days was (E, G), "If the columns of *A* are linearly independent, then the columns of *A* span $\mathbf{R}^n$." Finally, the third category of results focused on shifts in argumentation patterns over the course of the semester. The primary information source for this analysis was $A(\mathrm{T})_{tot}$ (see Figures 4.56-4.59). By considering where in the adjacency matrix entries appear, in conjunction with which days these adjacencies appeared, information regarding shifts in argumentation patterns was revealed.

Finally, Toulmin's Model and adjacency matrix analysis were compatible across a variety of the reported result strands, as mentioned previously within the associated sections. For instance, a vertex with a high out-connection means that the concept statement or interpretation represented by that vertex served as data for a variety of claims. This is precisely what was presented as Criterion 3 for documenting normative ways of reasoning (Cole et al, 2011). High diversity within a particular cell indicates that the given implication (rather than a given concept) served a role in multiple arguments. Furthermore, knowing on which days the implication occurred (read in the matrix as a (u, v) adjacency) provides information regarding what concepts during the semester the implication is used to reason about. This allows an analysis compatible with Criterion 2 (Rasmussen & Stephan, 2008) in the establishment of normative ways of reasoning.

CHAPTER FIVE: ANALYSIS AT THE INDIVIDUAL LEVEL

This chapter presents results concerning the second research question: How did an individual student, Abraham, reason about the Invertible Matrix Theorem over time? To address this question, I utilized two different analytical tools, Toulmin's model and adjacency matrices, on classroom discourse, small group discourse, interview data, and written work from throughout the semester relevant to how Abraham reasoned about the IMT. I also utilized the cultural change notions of microgenesis and ontogenesis as strands through which I coordinated the structure and content of discrete argumentations (microgenetic level) and how the forms and functions of these various arguments shifted over time (ontogenetic analysis).

The layout of this chapter mirrors that of Chapter 4. In the first section, I detail results from both the microgenetic and ontogenetic analyses from using Toulmin's model. The microgenetic analysis revealed, in addition to instances in which the four expanded Toulmin structures introduced in Chapter 4 served well to analyze the structure of Abraham's individual argumentation, variations in the nature of equivalence between different pairs of concept statements from the IMT. Within the ontogenetic analysis, I focused on shifts in Abraham's reasoning about span and linear independence in conjunction with each other, as well as his prevalent use of reasoning about solutions to the matrix equations $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$ to make and support claims. In the second section, I detail both the microgenetic and ontogenetic analyses from utilizing adjacency matrices as an analytical tool to investigate Abraham's reasoning. The microgenetic analysis revealed Abraham's reasoning about the negation of statements regarding determinants and eigenvalues from the IMT, as well as how a difficulty with a particular generalization about linear dependence was revealed through analyzing the $m < n$ sub-digraph. Within the ontogenetic analysis section, I report results concerning the centrality measure for the concept of span, the prevalence of codes within the theme of "getting everywhere," and a comparison of arguments from identical prompts. The chapter concludes with a summary of Abraham's mathematical development, as well as a comparison of the two tools and a discussion of their affordances and limitations at the individual level.

Although the structure of the two results chapters is parallel, the theoretical underpinning and resulting types of conclusions that are made at each of the two units of analysis differ. In the previous chapter, I was restricted regarding what conclusions I could make regarding the collective in the microgenetic analysis section, given the inability to make claims about mathematical ways of reasoning that function as-if shared from the analysis of single arguments. Here, however, through microgenetic analysis, I am able to make claims about Abraham's mathematical understanding, as well as comment on aspects similar to those of the collective analysis, namely the structure of Abraham's argumentation. On the ontogenetic level, I discuss Abraham's ways of reasoning about the IMT over time—I consider how his ways of talking about how two or more concepts relate changes over time, analyze the prevalence of certain themes of mathematical concepts or metaphors, make analyses about which ideas seem most salient and most problematic for him, compare his responses to identical questions at different points in time, and discuss variations in argumentation structure with respect to the type of equivalency the concepts embody for him.

5.1 Toulmin's Model of Argumentation

Per the methodology described in Chapter 3, the following were data sources from which I looked for arguments that could be attributed to only Abraham regarding the IMT: video from whole class and small group discussion from every class day, video and transcript from two, ninety-minute individual interviews, and written work in the form of reflections, portfolios, exams, and homework. Within the verbal data sources—transcript and video of whole class discussion (WCD), small group work (SG), interactions with Abraham after class

(AC), and the two interviews (Int 1 and Int 2)—there were 105 different argumentations coded with Toulmin schemes. The breakdown of when these occurred is given in Table 5.1. The written work was selectively coded as a secondary data source for the purpose of supporting results in the ontogenetic analysis section; thus, there is no quantitative summary or breakdown of the written argumentations. The entire argumentation log is provided in Appendix 5.1.

Table 5.1. Summary of the days and circumstances in which Abraham provided argumentation regarding ideas involved in the IMT.

	Day 7	**Day 9**	**Day 10**	**Day 17**	**Day 18**	**Day 19**	**Day 20**	**Day 24**	**Day 31**	**Int 1**	**Int 2**	**Total**
WCD	1	1		3		1	4		2			12
SGW			4		1	2	7	1	7			22
AC		5										5
Int										22	44	66
												105

The ontogenetic section discusses Abraham's mathematical development and shifts in ways of reasoning over time pertaining to particular mathematical ideas. In particular, I discuss how Abraham reasoned about span and linear independence of column vectors for both square and non-square matrices, as well as his prevalent use of reasoning about solutions to the matrix equations $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$. In the present microgenetic section, I focus on ways in which Abraham reasoned at various points throughout the semester, focusing on two themes: the nature of an argument's structure, and the nature of equivalence.

5.1.1 Microgenetic Analysis via Toulmin's Model

Within this section, I examine the nature of the structure of argumentation, and the nature of the equivalence for Abraham. One way to pursue the research question relevant to this chapter, how Abraham reasoned about the Invertible Matrix Theorem over time, via a microgenetic analysis is through considering the structure of his explanation, as captured by Toulmin's model. Two results of this analysis are: (a) the standard 6-part Toulmin was often insufficient in capturing the complexity of some of Abraham's arguments (as also noticed in the collective chapter); and (b) through close examination of the data and warrants that he provided, I found that the *nature* of equivalence was not consistent across each pair of concept statements. Two types of equivalencies grounded in and resulting from an analysis of Abraham are those of conceptual equivalence and logical equivalence. The particular results presented here show that, at certain moments in the semester, Abraham (a) reasoned about the pair "the column space of A is all of $\mathbf{R}^n$" and "the columns of A span all of $\mathbf{R}^n$," as well as about the pair "the columns of A span all of $\mathbf{R}^n$" and "for every **b** there is a way to write **b** as a linear combination of the columns of A" as conceptually equivalent; and (b) reasoned about the pair "the columns of A span $\mathbf{R}^3$" and "the system $A\mathbf{x} = \mathbf{b}$ has no free variables" as logically equivalent concept statements.

I first present Abraham's response to a sorting task from the second interview, which occurred about three days after he took the final exam for the course. The interaction between Abraham, the interviewer, and the mathematical ideas in the IMT provides a rich source of investigation into the various ways Abraham interacted with the word "equivalent," as well as various structures by which he formed his arguments. Investigating the development over time of Abraham's ways of reasoning—how they shifted or remained constant in form or function—is the topic of the subsequent section of Ontogenetic Analysis via Toulmin's Model.

For the sorting task, Abraham was asked to reflect on the Invertible Matrix Theorem. After being given a stack of 16 cards, each with a concept statement from the IMT, he was asked to sort the cards in different piles based on what concepts for him went together, and explain how he made his choices. The choice of words, "which concepts go together," was purposefully vague in order to allow Abraham to choose the criterion by which he grouped them. Abraham sorted the concept statements into five different piles, which are recreated in Figure 5.1.

Pile 1	Pile 2	Pile 3	Pile 4	Pile 5
A is row equivalent to the n x n identity matrix	For every $\mathbf{b}$ in R^n, there exists a solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$	The only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution	There exists an n x n matrix C such that $CA = I$	The number zero is not an eigenvalue of A
The matrix A has n pivots	For every $\mathbf{b}$ in R^n, there is a way to write $\mathbf{b}$ as a linear combination of the columns of A	The transformation T: $R^n \rightarrow R^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one	There exists an n x n matrix D such that $AD = I$	The null space of A contains only the zero vector
	The column space of A is all of R^n	The columns of A are linearly independent	A is invertible	
	The columns of A span all of R^n			
	The transformation T: $R^n \rightarrow R^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ maps R^n onto R^n			
	The determinant of A is not zero			

Figure 5.1. Abraham's piles after sorting the 16 concept statements from the IMT into piles of cards that "go together."

During the next fifteen minutes, Abraham explained his various piles to the interviewer. Three of them he explained rather quickly (Piles 1, 3, and 4), whereas his explanations for the other two were more involved. Here I briefly present his explanations for Piles 1 and 3; the argumentation scheme for Pile 4 is in Appendix 5.1. The interviewer began by asking Abraham how he decided to make his various piles.

Interviewer: So you have, what is that, 1, 2, 3, 4, 5 piles going? So just overall, what were your criteria, how did you figure out how to make piles?

Abraham: I tried to do these ones [circles Pile 3 in the air] to put as to linear independent. For me those are together on linear independence. These ones [Pile 1], the n x n identity matrix has n pivots, so it's just I always see those ones and I

> automatically think, if it's row equivalent to that, it has *n* pivots. And I see those ones in my head, so I put those together.

To support his claim that the three cards in Pile 3 went together (see Figure 5.2), Abraham provided the data that, “for me, those are together on linear independence.” Regarding Pile 1, Abraham stated, “I always see those ones and I automatically think, if it’s row equivalent to that, it has *n* pivots,” and added on, “I see those ones in my head” (see Figure 5.3). As Abraham spoke about Pile 1, the “ones” he was referring to were those on the diagonal of the identity matrix. Thus, Abraham related the cards “*A* is row equivalent to the identity matrix” and “*A* has *n* pivots” based on how the appearance of diagonal ones in the identity made him “automatically think” of pivot positions.

	Interview 2 Q3 Argument 1 [39:22]
Claim	The three cards in Pile 3 were placed together.
Data	*For me, those are together on linear independence.*

Figure 5.2. Abraham’s justification of Pile 3.

	Interview 2 Q3 Argument 2 [39:22]
Claim	The three cards in Pile 4 go together
Data	*I always see those ones and I automatically think, if it’s row equivalent to that, it has n pivots*
Warrant	I see those ones in my head.

Figure 5.3. Abraham’s justification of Pile 1.

The interviewer did not ask Abraham for further explanation of Piles 1 and 3 at that moment but rather allowed him to continue his explanation of the other three piles. While Abraham’s argumentation regarding Piles 1 and 3 seem somewhat surface-level, his explanations of Piles 2 and 5 were more in depth and provide the data through which I introduce the notions of conceptual equivalency and logical equivalency. The mathematical concepts of linear independence and invertibility (the content involved in Piles 1 and 3) are revisited in the subsequent Ontogenetic Analysis section in order to investigate how his reasoning about those concepts shifted over time.

5.1.1.1 Abraham's Types of Equivalence. Two statements are *equivalent* if there exists a logical chain of deductions from statement A to statement B and vice versa. This leads to the conclusion that if statement A is valid (true) then statement B must be valid (and vice versa), as well as if statement B is false then statement A must be false (and vice versa). By the very nature of the Invertible Matrix Theorem, which is a collection of equivalent statements for an n x n matrix A, then every pair of statements within the IMT must, by definition, be equivalent. While this is true from an external standpoint, what is of interest in the present study is how *people*—whether it is the classroom as a collective body or an individual student—actually reason about the concepts in the IMT and the supposed equivalencies between them. Within this chapter that focuses on Abraham, I examine the nature of the ways in which he justifies the equivalence between two concept statements in the IMT. For instance, does he provide a multi-step chain of logical deductions from one statement to another, does he rely on notions of "sameness," or some mix of both? By focusing on the nature of Abraham's justification for a given equivalence, I gain more insight into how he understands the concepts whose equivalence he is justifying. It reveals aspects of his concept image (Tall & Vinner, 1981) for the various concept statements. By considering such a rich case study at such an in-depth level, it provides a valuable foundation for continued research into how students may utilize more than one notion of equivalence.

Within this section, I define two ways in which Abraham was reasoning about various concept statements in the IMT: as conceptually equivalent and as logically equivalent. I define and provide examples of each of these are in their respective sections.

5.1.1.1.1 *Conceptual Equivalence.* For the purposes of the present research, I define the term *conceptual equivalence* as the absence of differences in meaning between two or more concept statements. The term conceptual equivalence also appears in the field of translation and linguistics, with my definition adapted from a set of guidelines for linguistic validation in translation that defined the term as “the absence of differences in the meaning and content between the ... source language and the translated version” (Mapi Research Institute, 2002). My purpose in utilizing the term in this study is to capture the various ways in which Abraham interacted with the notion of equivalence when reasoning about the concepts involved in the Invertible Matrix Theorem. While there is no “source language and translated version” per se in Abraham’s interactions with the IMT, he was often presented with the task of discussing how two concepts are related—this involved considering the language with which each was worded, discussing what each meant to him personally, and relating those interpretations to each other.

Utilizing Toulmin’s Model of Argumentation at a microgenetic level was a useful in illuminating instances of conceptual equivalence in Abraham’s reasoning. In many cases, Abraham would claim that two or more concepts were equivalent, and within his data or warrant there would be an explicit indication that, for him, the concepts were indistinguishable, or “the same thing.” Note this is distinct from *using* concepts as if they are the same thing; rather, Abraham indicated in some way that the way in which he understands them is essentially “the same.” Second, for Abraham, the justifications for conceptually equivalent ideas tended to be less structurally complex than other justifications. This could possibly be attributed to “less work” having to be done in order to establish the concept statements as equivalent, or it may be that providing justification for “obvious” equivalencies proved difficult for him. The latter can also be determined through the use of qualifiers that indicated Abraham’s difficulty in providing a mathematically satisfying justification. Further detail about each of these is given within the context of the given examples.

The notion of conceptual equivalent is, of course, subjective, as not each individual will find the same statements “absent of differences in meaning.” The present study focused on

Abraham and his ways of reasoning about the IMT, and there are multiple occasions during which he reasons about two or more ideas as conceptually equivalent. A particularly rich instance of this occurred during the second interview, when Abraham explained why he placed six different concept statement cards in the same pile when asked to group them according to ideas that "go together" for him (see Figure 5.1). After Abraham had discussed each of the other piles, the interviewer asked him, "What about the last pile, you have six cards, what was the common factor in those six cards?" For organizational reasons, I refer to each of the six cards within Abraham's pile by labeling them as Cards 1-6:

Card 1: For every **b** in R^n, there exists a solution **x** to A**x** = **b**
Card 2: For every **b** in R^n, there is a way to write **b** as a linear combination of the columns of A
Card 3: The column space of A is all of R^n
Card 4: The columns of A span all of R^n
Card 5: The transformation $T: R^n \rightarrow R^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ maps R^n onto R^n
Card 6: The determinant of A is not zero

In response to the interviewer's question, Abraham smiled as he replied, "Span. So I think of all of these as span." His initial response was a collection of partial phrases as he read the cards to himself, ending with the statement, "So then for every **b**, I can get to that. I think of that for a lot of them." Within this ending phrase of "I think of that for a lot of them," we see hints of the notion of conceptual equivalence existing for Abraham between at least some subset of the six concepts. The Toulmin scheme for this explanation, labeled "Interview 2 Q3 Argument 6," is given in Figure 5.4.

	Interview 2 Q3 Argument 6 [39:49]
Claim	The six cards in Pile 2 go together.
Data	*I think of all of these as span.*
Warrant	*So there exists a solution. So for every output vector, I can find. I think of it as for every output vector, I can find an input vector. So for every vector in this space, I can find a solution. So then for every* **b***, I can get to that. I think of that for a lot of them.*

Figure 5.4. Abraham's initial explanation for why he placed the six different concept statement cards in Pile 2 together.

Next, without prompting, Abraham explained why each particular card was in the pile. He began by relating Card 1 (For every **b** in R^n, there exists a solution **x** to A**x** = **b**) with Card 2 (For every **b** in R^n, there is a way to write **b** as a linear combination of the columns of A). He stated that the ideas were the "same thing" except that one (Card 1) was more transformation-oriented, whereas the other (Card 2) was vector-oriented.

Abraham: This [points to Card 1 and Card 2] is the same thing except we're not...This first one [Card 1] is matrix-oriented. This [Card 2] is vector-oriented. So it's really saying the same thing but it is saying, there's different, definitely some different things going on. I like to put them all together. But this [Card 1] is matrix-transformation-oriented. And this one [Card 2] is, let's think of the vectors within the matrix. So we know that these columns represent the vectors, so then if it's a linear combination of the columns, you get to everywhere. This one [Card 1] is dealing with a transformation. Can I send, can I find an input vector that sends to every output vector? So it's a little different, but they both concern spanning a space, whether it's through a transformation or by vectors being added together to get there.

To support his claim that he saw the concept statements on these two cards as the same thing, Abraham explained each of the cards separately. This is presented in his argumentation structure as two linked sub-arguments within the data (see Figure 5.5). A more explicit discussion of the use of the expanded Toulmin's Model to analyze the structure of Abraham's arguments is the subject of the subsequent section. What is noteworthy in the present section, however, is the nature of the justification provided by Abraham for the equivalence between Cards 1 and 2. While he did say they were "the same," he also said they were "a little different," and was able to clearly articulate that the distinction was contextual. In the first sub-argument within the data, he provided a justification for why Card 2 was vector-oriented (it described "the linear combinations of the columns of A," and the columns can be thought of as vectors). In the second sub-argument of the data he explained that, to him, the concept statement on Card 1, "For every **b** in R^n, there exists a solution **x** to A**x** = **b**" dealt with sending input vectors to output vectors. Finally, he provided a warrant, "it's a little different, but they both concern spanning a space," which connected the data to the main claim.

Abraham, despite his claim that the two concept statements were the same, provided an explanation in which he was able to distinguish notable differences between them. Thus, in this given argument, Abraham did *not* treat “for every **b** in R^n, there exists a solution **x** to A**x** = **b**” as conceptually equivalent to “for every **b** in R^n, there is a way to write **b** as a linear combination of the columns of A.”

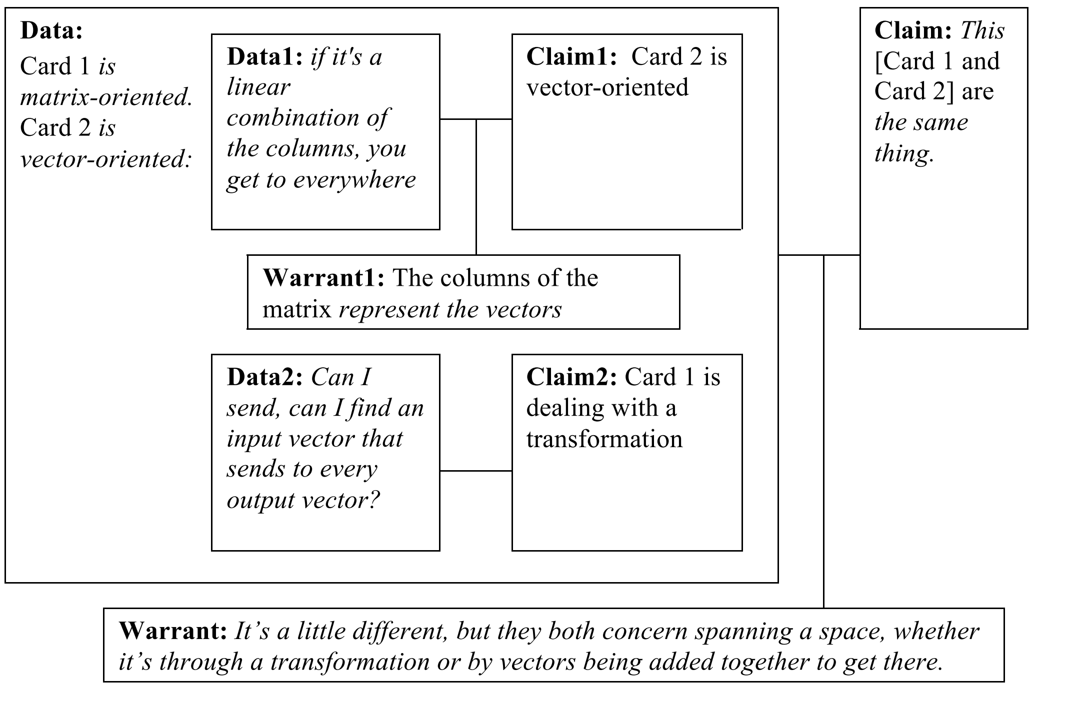


Figure 5.5. Interview 2, Q3, Argument 7: Abraham explains how, for him, the concept statement on Card 1 is the same thing as the one on Card 2.

Abraham’s explanation regarding the connection between Cards 3 and 4, however, is markedly different. As he discussed how Card 3 (The column space of A is all of R^n) and Card 4 (The columns of A span all of R^n) were connected for him, there is a change in the way in which he discussed their equivalency.

Abraham: Then this one [reads Card 3], ‘Column space of A is $\mathbf{R}^n$.’ Which is, if all the columns of A span all of $\mathbf{R}^n$ [points to Card 4]. So this one [points to both Cards 3 and 4] I don't know how to decipher, really. The column space. Because when I think of the column space, I really literally think of the space that the columns can get to. And this is talking about the columns of A spanning. So I can't really

> discern the difference of these ones [points to Cards 3 and 4]. Because like I said, this is to me where the columns can get to [points to Card 4] in this fashion. And this one [points to Card 3] is the space that the columns can get to. Which for me is like the same thing. I'm sure others might see that as two different totally.

In this argument (see Figure 5.6), Abraham claimed that he was unable to "really discern the difference" between "the column space of A is all of R^n" and "the columns of A span all of R^n." He also used the words, "can't decipher," but he did not verbally complete that thought. As the data for his claim, Abraham explained how he thought of the concept statement on each card individually. Finally, in his warrant, he summarized the way he understood each concept statement (his data) and used the phrase "where the columns can get to" in both explanations. He concluded by connecting the interpretations with the phrase, "Which for me is like the same thing." This echoes a statement Abraham made on Day 31 in class, when he told the members of his small group, "To me, this is just what I think, the column space of A is all of $\mathbf{R}^n$, and the column space of A spans all of $\mathbf{R}^n$, to me are exactly the same. Like those two to me are the most equivalent, the same" (see Day 31 SG Arg 5 in Appendix 5.1).

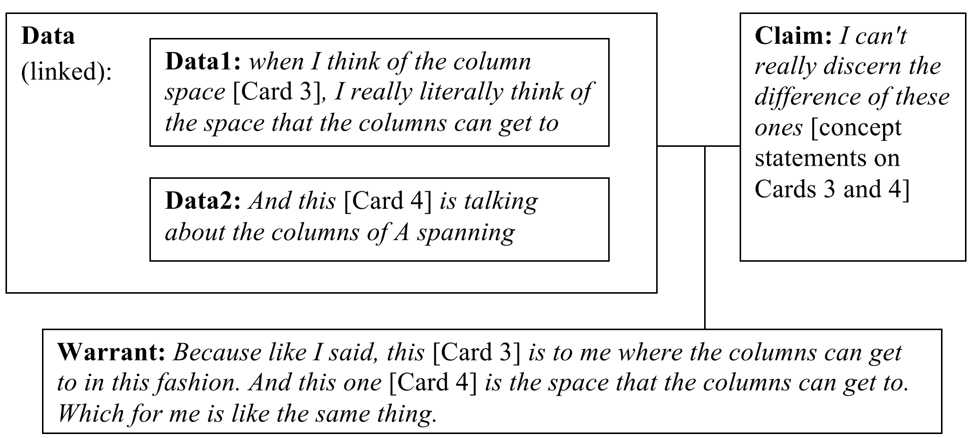


Figure 5.6. Interview 2, Q3, Argument 8: Abraham explains how he cannot discern between the concept statements on Card 3 and Card 4.

How does Toulmin's Model aid in illuminating conceptual equivalence within this argument? Again, within the data or warrant (here, the warrant), Abraham indicated that, for him, there was no difference in meaning between the two statements. Interesting to this argument, even in the claim there is an indication of conceptual equivalence because his entire explanation is given to support his claim that he cannot "really discern the difference" between the concept statements.

There is evidence of the six concept statements within this pile being conceptually equivalent for Abraham at other points during the semester as well. I conclude this section by reporting on one particularly interesting example, which occurred during the first individual interview. In a series of True/False questions, Abraham was asked to read the stem, "If A is a 3 x 3 matrix whose columns span $\mathbf{R}^3$," and decide if a series of subsequent statements were true or false based on assuming the stem. One of the prompts asked him to consider, "any vector **b** in $\mathbf{R}^3$ can be written as a linear combination of the columns of A" in conjunction with the stem. In other words, he was answering whether or not "Card 4 implies Card 1" (to use the language used alongside the previous arguments from Interview 2) was a true statement or not, particular to $\mathbf{R}^3$ rather than any $\mathbf{R}^n$. In Figure 5.7, the Toulmin scheme for this explanation is given.

This explanation is noteworthy because he states that when he reads the statement any vector **b** in $\mathbf{R}^3$ can be written as a linear combination of the columns of A," he "automatically think[s] of the definition of span" (the data in his argument). Furthermore, in his warrant, he stated, "If I was linear algebra, if I created it, that would be my definition of span." What creative language! In his warrant—which served to explain why the data is connected to the claim—Abraham creatively emphasized that he saw no difference between the two concept statements. He concluded by qualifying his explanation, stating he might be wrong but what he expressed was how he personally thought of the connection between the two concept statements. This qualifier is further evidence that Abraham saw the notions of Card 1 and Card 4 as conceptually equivalent. Furthermore, utilizing Toulmin's Model as an analytical tool helped to illuminate this equivalence. First, both his data and warrant served to confirm that, for Abraham, there was an absence of differences in meaning between the two written statements, which satisfies the definition of conceptually equivalent provided at the start of this subsection. Second, his qualifier indicated that he was aware that some other person may see those ideas differently.

	Interview 1 Q6b Argument 1 [59:42]
Claim	**'**If A is a 3x3 matrix whose columns span $\mathbf{R}^3$, then any vector **b** in $\mathbf{R}^3$ can be written as a linear combination of the columns of A' is a true statement.
Data	*When I read that, 'any vector* **b** *in* $\mathbf{R}^3$ *can be written as a linear combination of the columns of A,' I automatically think of the definition of span.*
Warrant	*And I think of the definition of span as any vector* **b** *being able to be written as a linear combination of the columns of A. If I was linear algebra, if I created it, that would be my definition of span.*
Qualifier	*I don't know, I might be wrong, but that's what I, what I think of, you know!*

Figure 5.7. Int 1 Q6b Arg 1: Abraham explains why "if A is a 3x3 matrix whose columns span $\mathbf{R}^3$, then any vector **b** in $\mathbf{R}^3$ can be written as a linear combination of the columns of A" is a true statement.

In conclusion, within the data presented here, we find evidence that the pair of concept statements "The column space of A is all of $\mathbf{R}^n$" and "The columns of A span $\mathbf{R}^n$" and "For every **b** in $\mathbf{R}^n$, there is a way to write **b** as a linear combination of the columns of A" and "The columns of A span $\mathbf{R}^n$" were conceptually equivalent for Abraham. The majority of the data presented came from Interview 2, when Abraham explained a pile of six cards all related to span for him. Within those arguments, he used the phrase, "can get to" or "get everywhere" six different times. While this is not directly related to the notion of conceptual equivalency, it is noteworthy with respect to Abraham's prevalent concept images of span, as well as the metaphorical saliency of the Magic Carpet Ride problem. Focusing on the "get everywhere" phrasing that occurred here, as well as other times during the semester, occurs in Ontogenetic Analysis via Adjacency Matrices section.

Justifications that Abraham provided concerning other pairs of concept statements from the IMT indicated that, for him, those pairs did not function as conceptually equivalent. Abraham, a relatively sophisticated mathematics student, *knew* that two given concept statements were in fact equivalent (if, for no other reason, because of their inclusion in the IMT), but he sometimes had to "work harder" to justify the equivalence, compared to his ways of reasoning for conceptually equivalent statements. The second category of ways of reasoning about equivalence that I highlight is referred to as logical equivalence.

5.1.1.1.2 ***Logical equivalence.*** For the purpose of the present research, I define *logical equivalence* as the existence of logical deductions between two or more concept statements. As was true for conceptual equivalence, this categorization is unique to each individual and the ways in which he or she structures a justification for the equivalence of two or more concepts. My purpose in utilizing the term logical equivalence in this study is towards the end of capturing various ways in which Abraham interacted with the notion of equivalence when reasoning about the concepts involved in the IMT.

Utilizing Toulmin's Model of Argumentation lends both specificity and an operational ability to this definition of logical equivalence. In Abraham's argumentation, I say that two concepts statements are *logically equivalent* for Abraham if, for a given claim of the form "concept statement A is equivalent to concept statement B," his data or warrant involved at least one other concept statement via logical deductions. As ubiquitous as this version of justification may seem, consider the examples of conceptual equivalence from the previous section and the absence of logical deductions involving a third concept statement. For Abraham, it was as if the conceptually equivalent ideas, loosely speaking, they were "defined" in nearly indistinguishable ways. To clarify, if the concept statements are airports, then logically equivalent concept statements, for Abraham, would be if there existed no "direct flight" between the two airports, but rather relied on at least one layover to arrive from the originating to the target destination.

For this section I present two examples. In both examples, Abraham only explained one direction of an implication (i.e., he only explains either '*p* implies *q'* or '*q* implies *p*,' although both are required to justify equivalence). Furthermore, these examples were chosen because they also help to exemplify the results in the subsequent section regarding for the expanded Toulmin's structure in Abraham's argumentation, and other examples in that section also complement the lend credence to this notion of logical equivalence.

The two examples come from the first interview, from the same set of true/false questions that was encountered in the previous section (see Figure 5.7). Here, consider Abraham's response to the prompt: "True or False: If *A* is a 3x3 matrix whose columns span $\mathbf{R}^3$, then the system $A\mathbf{x} = \mathbf{b}$ has no free variables." Abraham began by referencing a result he discovered in homework, namely that "# of free variables = # of unknowns - # pivots" (see Int 1 Q6e Arg 1 in Appendix 5.1). Abraham then lamented, "Now that's something I know, but *why*?" Abraham spent the next seven minutes working through why the stated implication in the True/False question was true, unsatisfied in using the "fact" as his sole justification. Here, I pick up on the fourth and fifth argumentation schemes from Abraham's work on that interview question, the first three of which involved him considering the interaction between free variables, infinitely many solutions, and pivots (see Int 1 Q63 Arguments 1-3 in Appendix 5.1).

Abraham: Basically I just, when I think of it as, if there is a free variable, then it can't have a pivot position; if it doesn't have a pivot position, it doesn't span. But we're assuming it does span, so the fact that it has no free variables, or the fact that it doesn't have a pivot position and doesn't span, then it's counter-example. I guess that would be a counter-example, if it's free. Let's assume, if I wrote out a proof, if it's free, then there's infinitely many solutions; if there's infinitely many solutions, then the bottom row is this. If the bottom row is this, then you know it only has two pivot positions. And if it only has two pivot positions, you keep going with proof, then it doesn't span all of $\mathbf{R}^3$. Then you write a symbol like this and you go, 'contradiction,' and your proof is done. Something like that.

This monologue is comprised of two arguments, the second of which begins with the phrase, "let's assume, if I wrote out a proof." In the first argument (its Toulmin scheme is given in Figure 5. 8), Abraham presented a sort of "proof by contrapositive" that if a system did have free variables, then the three columns could not span $\mathbf{R}^3$. Within his justification, he stated two explicit deductions: “if there is a free variable, then it can't have a pivot position;” and, “if it doesn't have a pivot position, it doesn't span.”

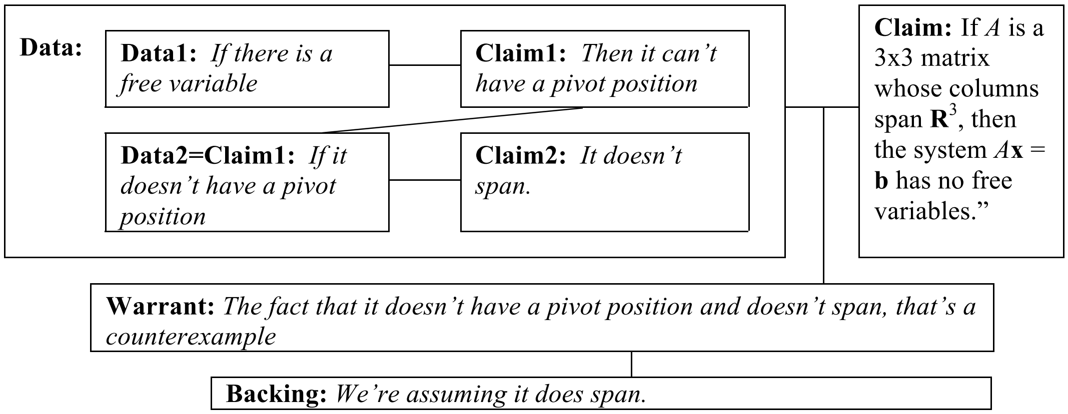


Figure 5.8. Example of logical equivalence, Int 1 Q6e Arg 4: Abraham explains why “if the columns of a 3x3 matrix A span $\mathbf{R}^3$,” then “there are no free variables.”

While Abraham’s first explanation seemed deliberate and thoughtful, during the subsequent one, which began, “let’s assume, if I wrote out a proof,” he spoke quickly and confidently, summarizing the major points of his four previous argumentation schemes. This argument, diagrammed in Figure 5.9, seemed to function like the culmination of the four previous ways of reasoning about how the concept statements regarding span and free variables were connected. Within his justification, the data was comprised of a sequence of four deductive statements. Abbreviating the chain of reasoning for this justification yields, “free variables → infinitely many solutions → less than n pivots → A does not span.” Abraham concluded with the statement, “But that would be a contradiction to what you're assuming in the very first sentence, so you write a symbol like this and you go, 'contradiction,' and your proof is done.”

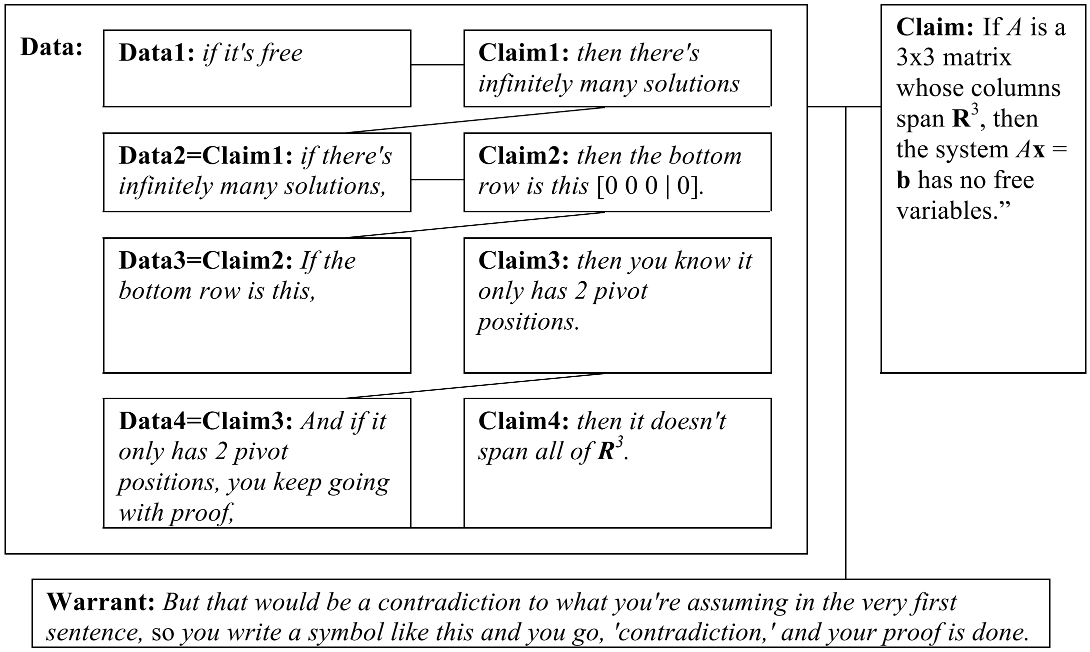


Figure 5.9. Example of logical equivalence, Int 1 Q6e Arg 5: Abraham explains why if the columns of a 3x3 matrix *A* span $\mathbf{R}^3$, then there are no free variables.

The two arguments (Figures 5.8 and 5.9) that exemplify logical equivalence demonstrate that Abraham was able to provide an explanation as to why, for a 3x3 matrix *A* whose columns span $\mathbf{R}^3$, any related system of equations would have no free variables. His justifications linked those two concept statements through a chain of logical deductions that involved more than one other concept statement from the IMT, and this was illuminated through Toulmin analysis via the use of these additional concept statements within the data or warrant. Finally, one may notice that the structure of the Toulmin scheme for both of the arguments is a Sequential structure, which was one of the expanded Toulmin schemes introduced in Chapter 4. While this coincidence is not surprising per se, it is not required. Nor is it repetitive to note that a student, through the Sequential structure of his justification, related two concepts together via a notion of logical equivalence. The former emphasis the structure of the argumentation, whereas the latter focuses on the mathematical content. To that end I now turn.

5.1.1.2 Expanded Toulmin Structures. In Chapter 4, I discussed the need to expand the Toulmin's model beyond the 6-part scheme in order to adequately and sufficiently capture the structure of the argumentation being presented during whole class discussion. Instances of the necessity of the expanded structures (embedded, linked, or sequential) also occurred when analyzing how Abraham reasoned about the IMT throughout the semester. Two of the twelve arguments he presented in whole class discussion were best captured with the expanded structure (Arg 20.16 WCD and Arg 20.26 WCD), as well as three of his 22 arguments from small group work (Day 20 SG Arg 7, Day 24 SG Arg 1, and Day 31 SG Arg 7). Finally, 21 of the 58 arguments from the two interviews were best captured with one of the expanded structures. One possible explanation for this distinction could be the nature of an individual interview: one student is given undivided attention for over ninety minutes and is asked to share his ways of reasoning. This is different than in, say, small group discussion, during which Abraham spoke so interactively with his group members that it was rare to have him speak without interruption for a long enough period of time (i.e., to be able to contribute an entire argument to only him) that the resulting argument necessitated the expanded Toulmin's model.

In this section I present two examples each of the Linked and the Sequential structures. I do not discuss the other two types, Embedded and Proof by Cases, in the present section. The Proof by Cases structure, in fact, did not surface as useful when analyzing Abraham's arguments. The Embedded structure, which I defined as a Toulmin scheme within which one or more of the data, warrant, or backing is itself composed of a Toulmin scheme, for the sake of brevity, does not gets its own treatment. Recall that the Linked and Sequential structures are specific versions of the Embedded structure, so by discussing the former two, the latter is evidenced as well. Finally, although the previous section on conceptual and logical equivalencies discussed arguments for which each of the expanded Toulmin structures had been utilized, the structures were not explicitly discussed, nor were they the focus point of those analyses; whereas previously the analysis focused on various instantiations of "equivalent," here the analysis focuses on the various structures of argumentation within which equivalency was discussed.

5.1.1.2.1 ***Linked Structure.*** Recall, from Chapter 4, I define a *linked structure* as a Toulmin scheme within which the data and/or warrant for the claim are composed of more than one embedded sub-argument that are linked by words such as "and" or "also." The first example of the Linked structure within Abraham's argumentation comes from small group work on Day 20, when the class investigated one-to-one and onto and how they related to other concepts from the semester, such as linear independence and span. During whole class discussion, Abraham and Mitchell had commented on various circumstances in which linear transformations were or were not one-to-one, and they mentioned linear independence. The instructor focused the classroom discussion around this suggestion.

Instructor: What in the world does linear dependency have to do with being one-to-one or not one-to-one? So I'll say that again, he said—actually, Abraham, can you say it, what did you say about connecting those two ideas?
Abraham: I just said that if it's one-to-one, the columns of *A* have to be linear independent.
...
Instructor: So take a minute in your groups, this is not a trivial thing, so take a minute to know why linear independence even has to do even has to do with being one-to-one, please.

Within his small group, Abraham shared how he saw the two concepts as related.

Abraham: So $T(\mathbf{x}) = \mathbf{0}$ to have only the trivial solution, right? And then because, one-to-one is unique solution or none at all, so it would have to, $T(\mathbf{x}) = \mathbf{b}$, I wrote it down, it would have to have unique solution or not at all. So if you simplify it down to $T(\mathbf{x}) = \mathbf{0}$, then we can see *T* of 0,0,0 must have the unique solution. Because it's not possible for it to have none at all. It has to have at least one, because it's 0,0,0.

First, Abraham stated the definition of linear independence (Data 1). He also provided, in Data 2, the definition of one-to-one by stating, "one-to-one is unique solution or none at all." Based on Data 2, he claimed that, in order for *T* to be one-to-one, the specific equation $T(\mathbf{x}) = \mathbf{b}$ would have to have a unique solution or none at all. His warrant started with, "So" (indicating that he was basing his conclusion on both Data 1 *and* the Data 2-Claim 2 pair), and he concluded that $T(\mathbf{x}) = \mathbf{0}$ must have $\mathbf{x} = \mathbf{0}$ as its unique solution. Here his data was comprised of the definitions of both linear independence and one-to-one. These two subparts of the data were not dependent on each other, yet both were necessary in Abraham's warrant in order to justify his main claim. It was through the warrant that Abraham linked the two data and stated why they were relevant to his main claim.

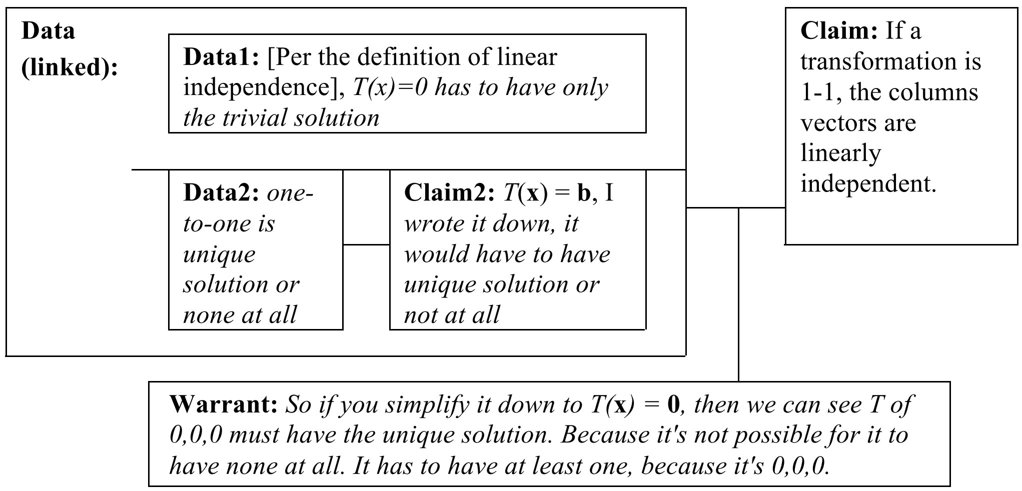


Figure 5.10. Example of the Linked Structure, Day 20 SG Arg 7: Abraham explains how linear independence and 1-1 are connected.

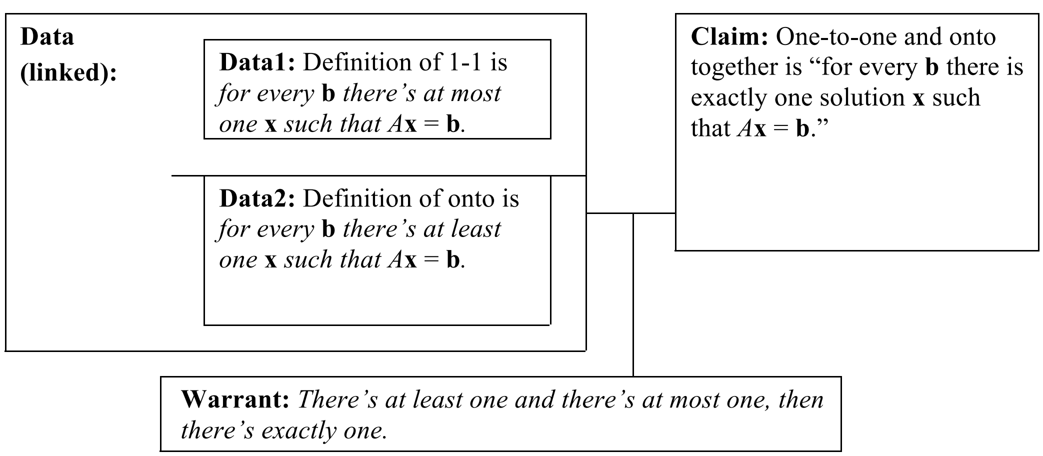


Figure 5.11. Example of the Linked Structure, Int 1 Q3b Arg 1a: Abraham explains how '1-1 and onto' together imply the existence of a unique solution.

The other example of the linked structure comes from Abraham's second interview. After he had explained his five piles of concept statements from the IMT that "went together" (see Figure 5.1), the interviewer asked him: "I was wondering if you can pull off [to the side] 'one-to-one,' 'onto' and '*A* is invertible.' And can you talk about, do you have any way to connect these three ideas together?" His initial response is given in Figure 5.11 (see Int 2 Q3b Arguments 1-9 in Appendix 5.1 for the entirety of his response to the interviewer's question).

He began connecting the three cards by discussing the implication for transformations that are both one-to-one and onto, and he concluded that, in that case, for every **b** there would be exactly one solution **x** such that $A\mathbf{x} = \mathbf{b}$. Within his data, he referenced both the definition of one-to-one and the definition of onto. The warrant brought together why, given *both* aspects of the data considered together, they had anything to do with the claim. Abraham argued for the claim twice and justified it in two slightly different ways. The first data (see Figure 5.11) linked the sub-data of "at most one" and "at least one" from the definitions of onto and one-to-one to conclude that any function with both properties must have "exactly one" solution. In the second argument, the data linked the subdata of "one or more" and "zero or one" to conclude that the intersection of those two "sets" is exactly one solution.

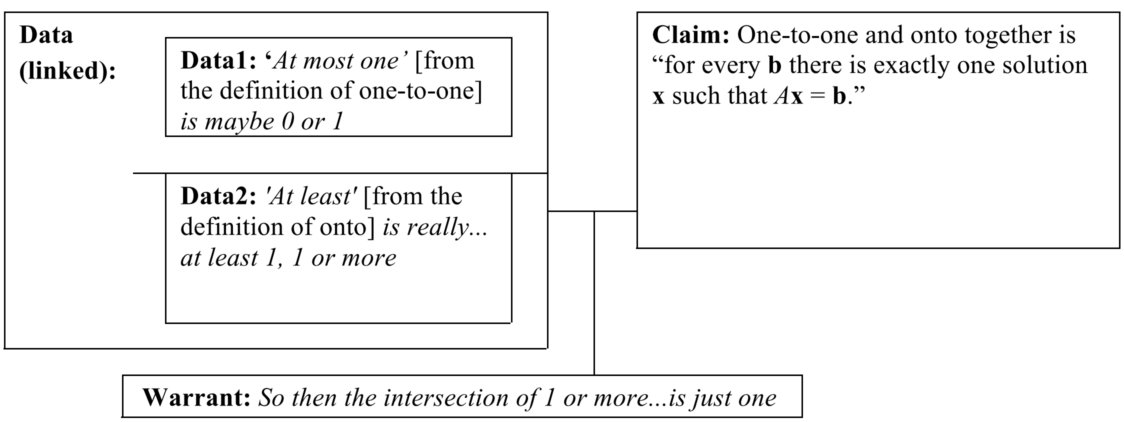


Figure 5.12. Example of the Linked Structure, Int 1 Q3b Arg 1b: Abraham explains how '1-1 and onto' together imply the existence of a unique solution.

5.1.1.2.2 *Sequential Structure*. Recall from Chapter 4 that I define a *sequential structure* as a Toulmin scheme within which the data for a specific claim contains an embedded string of data-claim pairs, such that claim C_k is data D_{k+1} for the next claim C_{k+1}. The first example of Abraham's argumentation being best captured via the Sequential structure is Argument 5 from Interview 1, Question 6e. This argument was presented in the previous section as an example of logical equivalence (see Figure 5.9, repeated here as Figure 5.13). In this argument, Abraham reasoned that if a 3x3 matrix has columns that span $\mathbf{R}^3$, then $A\mathbf{x} = \mathbf{b}$ has no free variables through a chain of logical deductions that included reasoning about the concept statements regarding infinitely many solutions, a row of zeroes in the RREF(A), and pivots. This same argument is considered here because of how well the Sequential structure of an expanded Toulmin scheme captured the essence of the argument's composition.

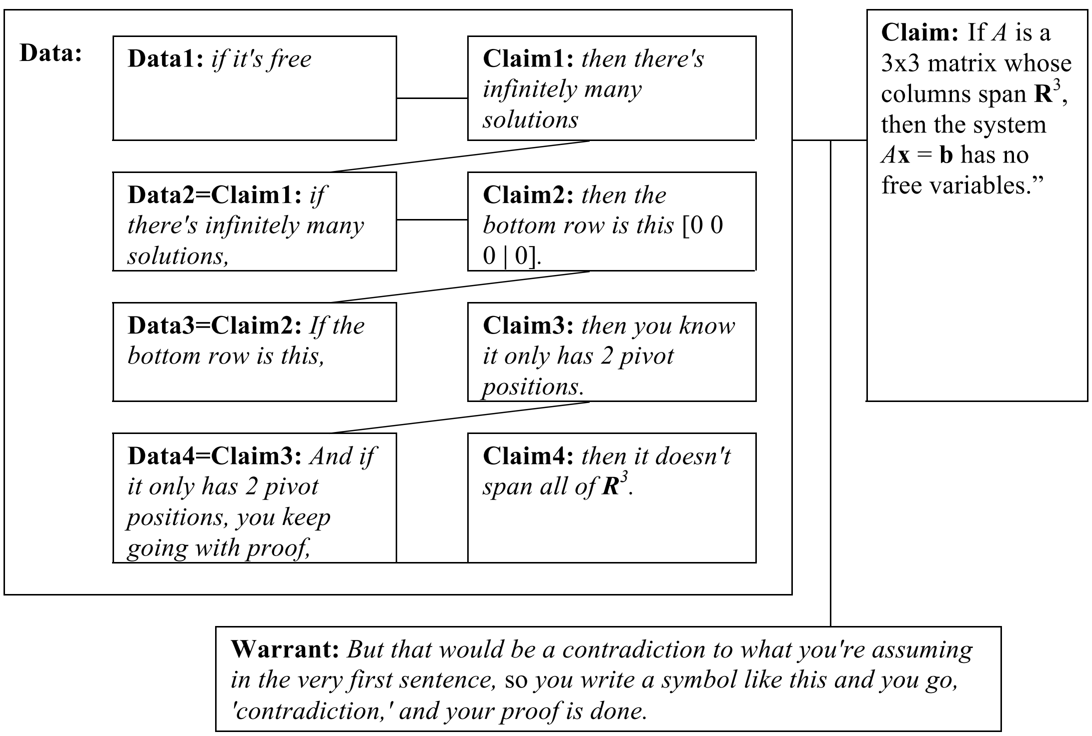


Figure 5.13. Example of Sequential Structure, Int 1 Q6e Arg 5: Abraham explains why if the columns of a 3x3 matrix A span $\mathbf{R}^3$, then there are no free variables.

Given the definition of a sequential structure, it is hard to imagine a cleaner example than that in Figure 5.13. As Abraham reflected on and summarized his immediately previous ways of reasoning about how span was connected to free variables, he very succinctly created a chain of data-claim pairs, starting with the data "assume there are free variables," in which each claim functioned as the data for the next data-claim pair. His warrant brought the proof to a close by explaining how the final sub-claim within the data contradicted the assumption in the original proof, thus leading to the conclusion that the system did not have free variables.

The last example of the Sequential structure comes from the second interview, as Abraham explained why he placed the cards "the number zero is not an eigenvalue of *A*" and "the null space of *A* contains only the zero vector" into a pile together (this was part of the card sorting task previously described, see Figure 5.1). The majority of his rather complex explanation is provided below.

Abraham: ... I'm thinking of an eigenvalue's definition something's, those are nonzero, $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. And so if the number zero, I always like to think, this [points to the 'zero is not an eigenvalue of *A*' card] says 'not.' But I always like to think of if it *is* [meaning 'zero *is* an eigenvalue of *A*']. So then if it *is*, if $\lambda = 0$, then $A\mathbf{x} = \mathbf{0}$. And then by the definition of an eigenvector, we can find a nonzero solution. Because if it's zero, it's not very interesting...But by the definition, we're trying to find a *non*-zero vector such that this eigenvalue stretches it. So then if $\lambda = 0$, then we can find a non-, this is saying by the definition, we can find a nonzero solution, such that $A\mathbf{x} = \mathbf{0}$...Therefore, I like that symbol [draws the three dots often used for "therefore"], if we can find a nonzero solution, therefore then it's, I'm going out of context. But this is, it's linear dependent, by definition, because it's a nontrivial or the only solutions aren't the trivial solution. There's a nonzero solution, so it's linear dependent if it's [the eigenvalue] zero. And how does this relate to null space for me? Because this is saying the number zero. Then if. Then I think of this because if there's a nonzero solution here, then the null space doesn't contain only the zero vector. So I think I think of them together, if I put a negation in front of them. Because then if the number zero is [an] eigenvalue, then the null space of *A* does not contain only the zero vector, the null space contains, the null space is part of the domain, so it contains all solutions to $A\mathbf{x}= \mathbf{0}$.

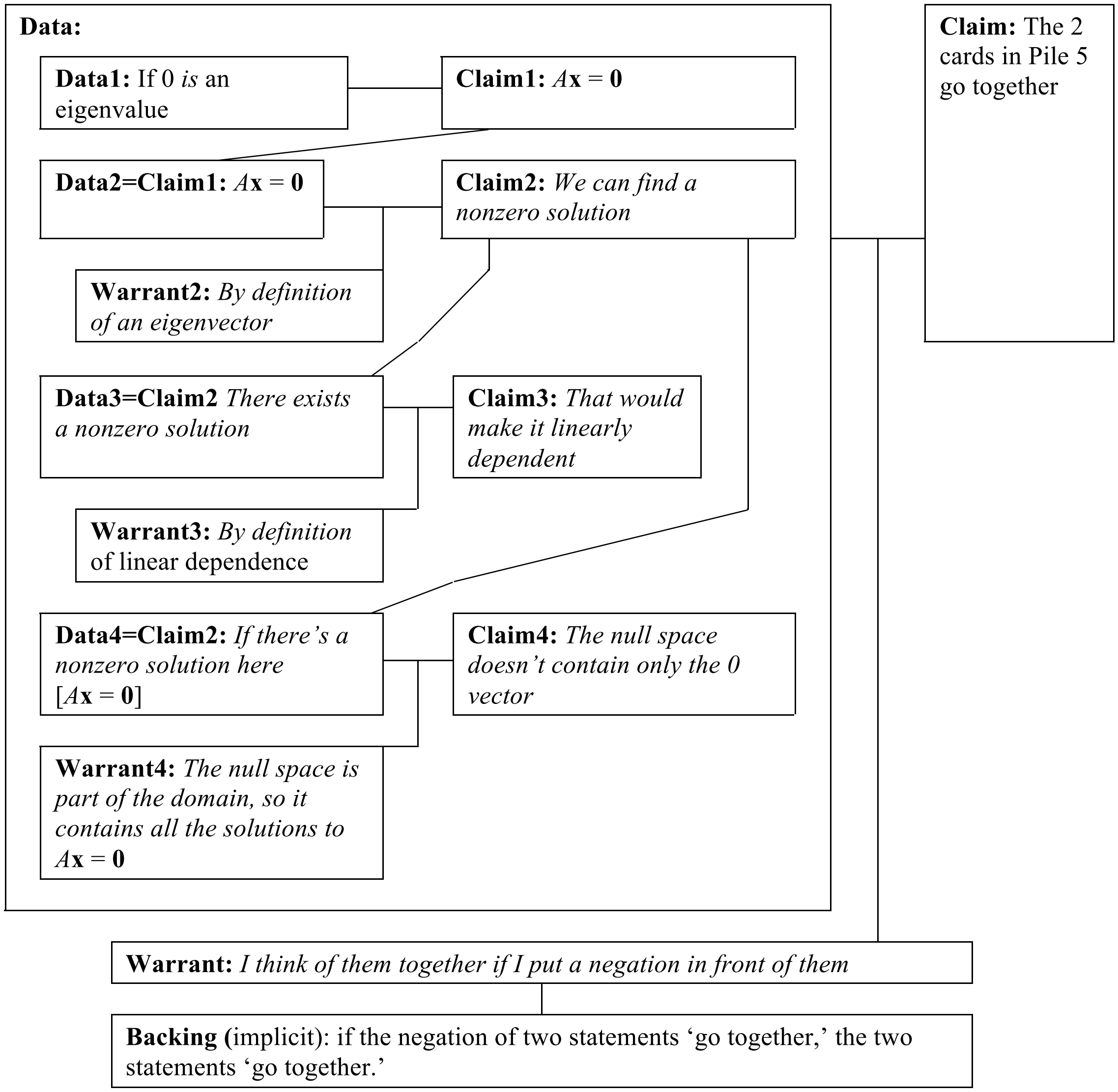


Figure 5.14. Example of the Sequential Structure, Int 2 Q3 Arg 3: Abraham explains why "if 0 is an eigenvalue, then the null space does not contain only the zero vector."

When asked to connect the cards, "zero is not an eigenvalue of *A*" and "the null space of *A* is only the zero vector," Abraham first stated he like to think of "zero is *not* an eigenvalue of *A*" instead. The remainder of his argument regarded him explaining what zero as an eigenvalue meant to him (the existence of nontrivial solutions to $A\mathbf{x} = \mathbf{0}$) and how that implies the null space of *A* was *not* only the zero vector. This main data for this argument consisted of four embedded data-claim pairs, for the later three of which he included a warrant. The sub-arguments, except for the first one, each used a claim from a previous argument explicitly as the data for a subsequent argument. Data 4, however, was identical to Claim 2 rather than Claim 3; Claim 3, which he never mentioned again, concluded that the columns of A were also linearly dependent. In sum, Abraham's argument analyzed in Figure 5.14 is a paradigmatic, albeit complex, example that demonstrated how the expanded Toulmin's model of argumentation is a worthwhile adaptation of the original 6-part Toulmin scheme when aiming to capture the complexity of mathematical justification.

5.1.1.3 Conclusion. Within this section of microgenetic analysis through the use of Toulmin's Model, I presented results concerning both content and structure for various discrete arguments from Abraham's ways of reasoning about the IMT throughout the semester. I first presented the distinctions of conceptual equivalence and logical equivalence as distinctions for Abraham's interaction with the mathematical content and the notion of equivalence. I then presented examples of how the expanded Toulmin scheme was necessary in order to capture the complex structure of Abraham's arguments. The expanded structures presented—embedded, linked, and sequential—were consistent with how they were used to capture argumentation at the collective level in Chapter 4.

5.1.2 Ontogenetic Analysis via Toulmin's Model

"Linear algebra connects with so many different points. I mean, just every concept is related somehow to the other. It's kind of interesting"

—Abraham, Interview 1

As both a teacher and researcher of linear algebra, I appreciate Abraham's sentiment and see both strengths and limitations in this statement. Indeed, linear algebra, as indicated by Abraham, is an interesting content area because of the many connections between ideas. As a researcher, it is a fascinating content area within which to investigate both learning and teaching. It is fascinating to investigate the ways of reasoning, logical structure of argumentation, and metaphorical problem settings that resonate with students—both at the collective and the individual level. On the other hand, however, Abraham's statement raises pragmatic limitations for research. There are 16 statements within what became formalized as the IMT for this particular class, producing numerous paths through which one could investigate how Abraham reasoned about the concepts within the IMT. For instance, one could take a pair of concepts statements and trace how Abraham discussed those in tandem; one could choose one concept and follow what other concept statements it is discussed in conjunction with over time; one could look for instances of the same claim being made multiple times at different points in time and compare the arguments that support them; and many more. Such complexity and intricacy, however captivating, requires narrowing. As such, this section of the present chapter carves out a portion of the possible results with the aim of telling a rich story of the ways in which Abraham reasoned about the IMT over time that highlights some of the aforementioned possibilities.

In particular, I focus on the development of Abraham's ways of reasoning about span and linear independence in conjunction with each other, as well as his prevalent use of reasoning about solutions to the matrix equations $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$ to make and support claims**.** Within the first broad strand of taking a pair of ideas and tracing how they are discussed in tandem, I present the analysis in two sections: (a) shifts in composition of arguments and placement of concepts within the arguments; and (b) comparison of arguments related to identical prompts at two different points in time during the semester. Regarding the second, I highlight his prevalence of reasoning about solutions by reporting on noteworthy examples from throughout the second half of the semester.

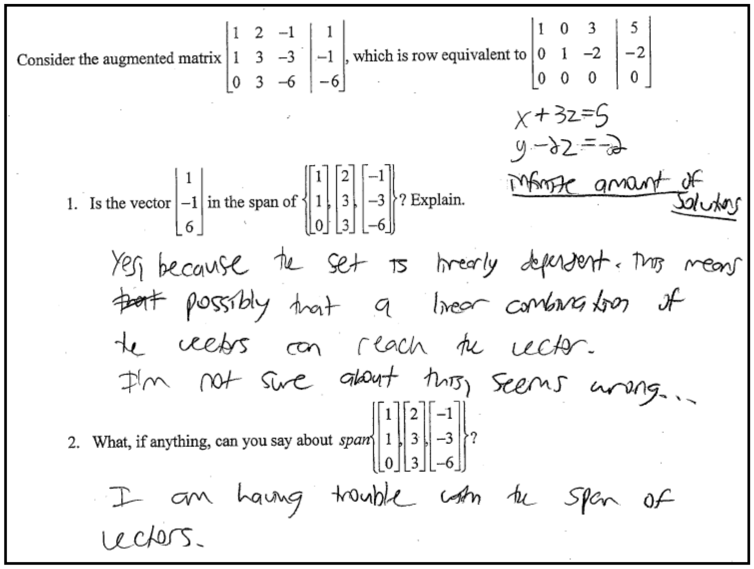
Recall that *ontogenesis* is defined as the shifts in patterns of thinking over the development of individuals and "is marked both by continuity in the individual's ways of understanding the experienced world and discontinuity as the individual structures new systems of understanding out of prior ones" (Saxe et al, 2009, p. 208). Furthermore, Roth and Ercikan (1996) contended that within genetic explanations, "a certain fact is not derived from antecedent conditions and laws (deduction) or observations and antecedents (induction) but rather is shown to be the endpoint of a longer development, the individual stages (phases) of which can be followed" (p. 19). My investigation into Abraham's ways of reasoning about the IMT throughout the semester is a compatible investigation. Considering the shifts in Abraham's reasoning as they occur throughout the semester is an integral part of genetic analysis; simply presenting conclusions in a manner such as, "Abraham used to think *X* but now he thinks *Y*," then, would be disingenuous with respect to the spirit of the aforementioned theoretical influences. Finally, even within the present ontogenetic section are aspects of microgenetic analyses because each data point within an ontogenetic progression constitutes a microgenetic construction (Saxe et al, 2009). Thus, although the present results focus on Abraham's ways of reasoning over time—an ontogenetic analysis which cannot be gained from discrete microgenetic analyses alone—additional information regarding the microgenetic development is necessarily gained within this section as well.

5.1.2.1 The development of ways of reasoning about span and linear independence in conjunction with each other. There are multiple reasons to pick this particular thread as one on which to focus. First, span and linear independence were two of the first ideas formalized within this class, so I have a wider window from which to draw data to analyze for shifts in ways of reasoning. Furthermore, many questions within a variety of data sources—interviews, written work, class conversation, etc.—specifically asked Abraham to discuss those two concepts in conjunction with each other. A third reason to follow this path is because of the classroom mathematics practices documented in Chapter 4. Recall that CMP 1 was entitled, “Reasoning and span and linear independence as equivalent ideas for square matrices.” To best set up a comparison for future research, it is useful to consider Abraham’s individual trajectory along a path with those same concepts in mind.

From the 105 Toulmin schemes created for Abraham’s verbal argumentation, over 30 of them explicitly mention both span and linear independence or dependence within the same argument. Some of these occurred within Abraham’s response to the same question, so removing for “repeats,” he discussed span and linear independence in conjunction with each other in 16 different instances. As already noted, span and linear independence were two of the first ideas formalized within this class, so it is not surprising to see their density and depth. This is only one reason to focus on span and linear independence. A second reason is to demonstrate the depth of argumentation from which to draw upon as I track the ways in which Abraham reasoned about these ideas in conjunction with each other over time.

5.1.2.1.1 ***Descriptive Overview.*** I first present a short chronological overview of Abraham’s reasoning about span and linear independence in conjunction with each other throughout the semester. I do so in order to allow the reader to establish some familiarity with the “terrain” of Abraham’s journey regarding these two ideas prior to analysis of this terrain. Substantive commentary on the data excerpts is not provided in the descriptive overview. In the section that follows the overview, however, I analysis in depth a selection of results from within the description. The Toulmin schemes that explicitly discuss span and linear independence in conjunction with each other are notated “(sp/LI)” in Appendix 5.1.

First, consider two occurrences from Day 7 of the semester. By that day, the class had developed the definitions of span and linear independence and was transitioning away from problems only couched in $\mathbf{R}^2$. The day’s reflection asked them to consider a particular augmented matrix in $\mathbf{R}^3$ and its row-reduced echelon form in order to answer questions about span. Abraham's response is given in Figure 5.15.


Consider the augmented matrix $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 1 & 3 & -3 & -1 \\ 0 & 3 & -6 & -6 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x+3z=5$
$y-2z=-2$
Infinite amount of solutions

1. Is the vector $\begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix}$ in the span of $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ -6 \end{bmatrix} \right\}$? Explain.

Yes, because the set is linearly dependent. This means ~~that~~ possibly that a linear combination of the vectors can reach the vector.
I'm not sure about this, seems wrong...

2. What, if anything, can you say about $span\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ -6 \end{bmatrix} \right\}$?

I am having trouble with the span of vectors.

Figure 5.15. Abraham's written response to the reflection on Day 7.

After class, Justin and Abraham stayed after class (on their own accord) to discuss a few mathematical questions with the instructor. Abraham's questions focused on trying to establish a way to determine the span of a set of vectors, as well as on trying to think of span and linear independence in a general way for the same set of vectors.

Abraham: I have a different question. My thing is like when I can't figure out this span stuff, because when say there's just three vectors in three dimensions, I don't know what I'm supposed to do to figure out the span of it. If it's, I'm assuming there's no multiples in it. Is it always going to span $\mathbf{R}^3$? I'm assuming.

Instructor: So the no-multiples thing is one, and then the other is you can write one as a combination of the other two.

Abraham: Right, you can write one, when it's a combo, it's linear dependent, right?

Instructor: Right, so then if you have three independent vectors in $\mathbf{R}^3$, then the span is all of $\mathbf{R}^3$.

Abraham: Right, okay, so if it's linearly independent, does it always span, or is there?

Instructor: If you have as many vectors, yes.

Abraham: I was trying to get a generalization going, but it wasn't happening.

Instructor: So let's just say we have these two [writes <1, 2, 0>, <2, 4, 1>], those are linearly independent. But I only have two, so I don't have enough.

Abraham: Oh, there's a case, when there's less entries than vectors, less vectors.

Instructor: But if these were all independent, yeah.

Abraham: Then if they're all independent, then they span all of. See, that's what I was thinking! I don't know why...Maybe it's just like in two dimensions, then. Because when they're linearly independent in two dimensions then it spans all of $\mathbf{R}^2$.

Justin: Then you have the same number of vectors as dimensions.

Abraham: Yeah, so that case generalizes all the way to what, same entries same dimensions.

Instructor: Yeah, and this Gaussian elimination that we have will help us actually prove that...So the reflection that's a little hard right now, we will be doing next relates to how can I get augmented matrix and see what the span is, just off the top of my head. That's what we do next.

Abraham: So then if it's linearly dependent, does that mean, is the opposite true, that it can never span?

Instructor: It depends on how many vectors you have.

Abraham: Because it could span, right, if it's linear dependent somehow, it could still span $\mathbf{R}^2$ even though it's.

Instructor: Like if you have more vectors than you need, if I throw some away, you still, you don't need them, but they're there.

Abraham: Like way more. Got you.

Justin: But if you have three vectors in $\mathbf{R}^3$ that are linearly dependent, it doesn't span of $\mathbf{R}^3$.

Abraham: Yeah, it can't, because it would make a plane, because two would be a linear combination, and then the other one would be able to go back, so that whole thing would create a plane.

On Day 9 (two class days later), the class had dedicated most of the class to investigate how pivots in the row-reduced echelon form of augmented matrices provide information regarding both linear independence and span. Abraham and Justin again stayed after class and approached the instructor to discuss issues from class. The below transcript begins with them examining matrices *A* and *B*. The exchange is significant because it was the first instance in which anyone from class expressed the beginnings of a generalization concerning both linear independence and span for *n* x *n* matrices.

$$A=\begin{bmatrix}1&0&2\\0&1&3\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}\qquad B=\begin{bmatrix}1&0&0&2&5\\0&1&0&3&6\\0&0&1&4&7\end{bmatrix}$$

Abraham: So that one [matrix *A*] doesn't span.

Instructor: Well what *does* it span then?

Justin: It spans a two-dimensional—

Abraham: It spans a 2-dimensional space in $\mathbf{R}^5$.

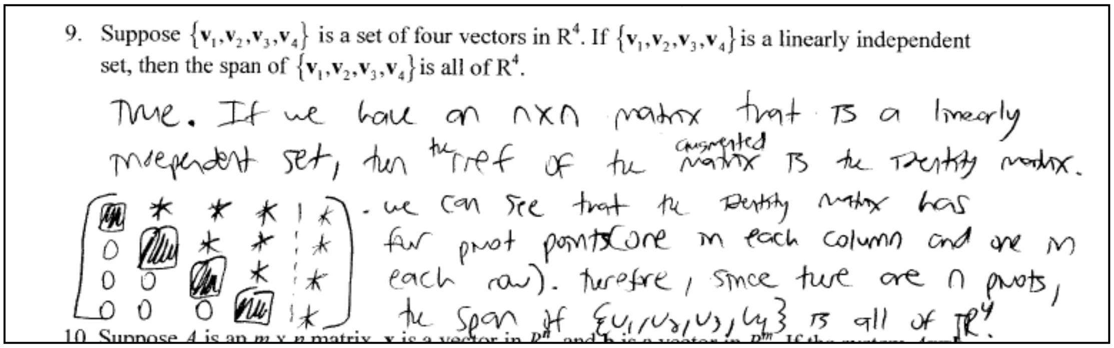
Instructor: Yeah.

Abraham: And it seems like when you have it like that, that the only time that it doesn't span the whole dimension when you have free variables.
Abraham: If you, if you have more, if it's the other way around, uh, if it's 5x3 [sic, 3x5], then you would, then it would definitely span, right? I mean, so I'm just trying to think about it as a generalization.
Instructor: [Makes up the example *B* on the board]...
Abraham: [There are] two free variables. And so now it's spanning. So you see what I'm saying? Like...[pauses, Justin and Gabe leave, Abraham remains, looking at the board in silence.]
Abraham: Yeah, and another thing that I think about is like when *m* is, uh, say *m* is *n*, I mean, then I start to think about if it like, to me, if it's linearly independent then, um, then it, then it spans all of the whole dimension. [Instructor: Hmmm.] That's another one. So I mean, when it's, when *m* and *n* are the same, then I think of it that way, and then if it's not, I start thinking about the free variable, does it matter, and then if it's more then free variable plays a role.
Instructor: Yeah, [inaud] we didn't do this case, right [writes "$m = n$" the board]. That's one thing that hasn't come out yet.
Abraham: I don't know...it just seems like it would span.

Also on Day 9, the class developed a theorem about four equivalent statements for any m x n matrix (see Figure 4.21), and on Day 10 they spent time considering how that theorem would change if $m = n$. Within their group, Justin suggested that the set of vectors would not be linearly dependent, and Abraham commented on why he agreed.

Justin: Okay. So I would say the set of vectors in *n* are not linear dependent.
Giovanni: The set of vectors in *n*? Are what?
Justin: If this is *n* [points to where he wrote "rows" in the empty matrix].
Giovanni: Are not linear dependent?
Justin: Are not required to be linearly—
Giovanni: Dependent?
Justin: Dependent.
Giovanni: Yeah, they can be, but they don't have to be.
Justin: They don't have to be. I feel like that's kind of a pointless thing to say.
Abraham: Well, I mean, um...if it has the same amount of vectors as entries, though... no matter, I mean it has to be linearly independent, right, for it to span?

On Day 12, the students took their first exam. Abraham's response to a true/false exam question regarding span and linear independence of four vectors in $\mathbf{R}^4$. The question and his response is given in Figure 5.16.



9. Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a set of four vectors in R^4. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a linearly independent set, then the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is all of R^4.

True. If we have an nxn matrix that is a linearly independent set, then the rref of the augmented matrix is the identity matrix. - we can see that the identity matrix has four pivot points (one in each column and one in each row). therefore, since there are n pivots, the span of $\{v_1, v_2, v_3, v_4\}$ is all of $\mathbb{R}^4$.

Figure 5.16. Abraham’s response to a true/false exam question regarding linear independence and span for four general vectors in $\mathbf{R}^4$.

Also on Day 12, students turned in portfolios, comprised of three problems from the course that documented their progress in understanding concepts and methods for solving problems. For each selected problem, they developed a rationale statement that explained why they chose that problem and what mathematics they learned through their work. For one of his portfolio responses, Abraham chose to discuss the reflection from Day 7 (see Figure 5.15). Although the portfolio response itself does not mention linear independence or dependence, it does provide a rare glimpse into a student himself reflecting on how his own ways of reasoning regarding span changed throughout the semester (see Figure 5.17).

Pivot Points and Span

I chose this problem because upon doing this problem the second time around, I was able to see how one can figure out the span by using the pivot points of the row reduced matrix. During the beginning of the semester, I was having trouble determining the span of a set of vectors. I knew the definition of span: the span of a set of vectors $\{v_1, v_2,\ldots,v_p\}$ in R^m is the set of all vectors in R^m that can be written as a linear combination of the vectors $v_1, v_2,\ldots,v_p$. In terms of the magic carpet situation, the span is all the places we can reach by using the various modes of transportation or vectors for a certain amount of time which corresponded to the scalar multiples. But how could I look at a set of vectors and know its span? What I used to try to do is set up a vector equation and by looking it, attempt to come to a conclusion as if you can manipulate the left side somehow to reach any vector on the right side. However, this didn't seem like the most efficient way. I mean, how could I check to see if this is true for every possible vector? When we introduced the concept of pivot points, it all began to make sense. If we have a m x n matrix and there are m pivots, then the columns of A span R^m. This makes sense to me because if there is a pivot in every row, this corresponds to being able to go in every direction the dimension offers. If we can go in every direction, then we can reach any point in the plane and thus span all of R^m.

Original Response #1

For the reflection question #2 on February 10, 2010, I wrote that I was having trouble determining the span of a set of vectors. The problem asked, "What can you say about span $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}\begin{bmatrix}2\\3\\3\end{bmatrix}\begin{bmatrix}-1\\-3\\-6\end{bmatrix}\right\}$." I stared at this matrix for a long time and had no idea how one could know its span. I asked myself, is there a c_1, c_2, and c_3 such that $c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}2\\3\\3\end{bmatrix} + c_3\begin{bmatrix}-1\\-3\\-6\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}$. In other words, can every x, y, z vector combination in the set of real numbers be written as a linear combination of the three vectors on the left side. You can see that this process would be inefficient. But since this process is closely related to the definition of span, it seemed most logical at first. However, looking at the problem again, I have noticed that you can use pivot points to discover the span of a set a vectors. If you row reduce the matrix stated above, it becomes $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}3\\-2\\0\end{bmatrix}\right\}$. You can see that there is a pivot point in the first two rows but not the third. This means by the theorem, the columns of A do not span all of R^m or R^3. This can be seen because the bottom row is all zeroes. This means that we cannot move in the z direction. Since we cannot move in the z direction, then we cannot get to every point in R^3 and instead the span is a two-dimensional space in R^3.

Figure 5.17. Abraham's portfolio entry concerning the reflection from Day 7.

The next set of arguments regarding Abraham reasoning about linear independence and span in conjunction with each other occurred during the first individual interview, which took place between the Day 17 and Day 18 of the semester. Within a set of true/false questions that began with the stem: “Suppose you have a 3 by 3 matrix A and you know that the columns of A span $\mathbf{R}^3$. Decide if the following statement is true or false, and explain your answer,” Abraham answered it was false that (a) the columns of A were linearly dependent, and (b) there was a nontrivial solution $\mathbf{x}$ to the equation $A\mathbf{x} = \mathbf{0}$ (questions 6a and 6d, respectively). Question 8 asked him to consider both concepts for non-square matrices, and he discussed why a set of vectors could both span $\mathbf{R}^n$ and be linearly independent only if there were n vectors in $\mathbf{R}^n$.

On Day 20, the class investigated ways of reasoning about one-to-one and onto transformations, and Abraham offered ways to think about how only transformations from $\mathbf{R}^n$ to $\mathbf{R}^n$ could be both one-to-one and onto, and he did so by reasoning about how those concepts were related to span and linear independence. On Day 31, when reflecting on the IMT, Abraham discussed the equivalence of the two concepts via both considering pivots, as well as the existence of unique solutions for matrix equations. Finally, during the second interview, Abraham was asked to respond to three questions that paralleled those from the first interview, as mentioned above. He also referenced span and linear independence when explaining why a one-to-one and onto transformation implied that the associated matrix was invertible (Question 3b), as well as explaining the equivalence between “the RREF(A) has n pivots” and “A is invertible” (Question 4a) and between “the null space of A contains only the zero vector” and “the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution” (Question 4c).

Whereas the previous description served to give a brief overview of what occurred, the present section offers selected results concerning what Toulmin analysis illuminated concerning how Abraham’s ways of reasoning about span and linear independence in combination shifted throughout the semester. I organize the results into two sections:

1. Shifts in composition of arguments and placement of concepts within the arguments
2. Comparison of arguments related to identical prompts at two different points in time during the semester

5.1.2.1.2 ***Shifts in composition of arguments and placement of concepts within the arguments.*** **The** subsequent analysis illuminates not only shifts in the composition of Abraham's arguments (e.g., when were qualifiers utilized) but also in the placement of concepts within the arguments (e.g., when claims later served as data or warrants for new claims). I begin by considering the after-class conversation between Abraham, Justin, and the instructor that occurred on Day 7. This particular exchange was not coded with Toulmin scheme because both the instructor and Justin had too prominent a role to be able to confidently assign any one argument as "Abraham's argument" alone. Furthermore, his utterances were intertwined with the instructor's, who supplied confirmation or clarification in response to Abraham's inquiries. Thus, although the after class data from Day 7 was not analyzed with Toulmin's Model, it serves to provide a rich background of Abraham's initial ways of reasoning about span and linear independence. A few highlights of this after-class transcript, given in the previous section, are the following questions and conjectures from Abraham:

- Asked if having no multiples in the columns implied they are always going to span
- Asked "if it's linearly independent, does it always span?"
- Claimed when "they're linearly independent in two dimensions then it spans all of $\mathbf{R}^2$." Justin responded, "then you have the same number of vectors as dimensions."
- Asked "if it's linearly dependent, does that mean...that it can never span?"
- Hypothesized that you can have a case that spans and is linearly dependent.

These questions and conjectures provide some evidence that Abraham is developing ways of reasoning about span and linear (in)dependence as a pair of related ideas. Specifically, we see him offer conjectures about if linearly independent vectors (in $\mathbf{R}^3$) necessarily span $\mathbf{R}^3$ and vice versa. He also began to wonder about how the number of vectors compared to the number of dimensions makes a difference.

In contrast to the after-class discussion on Day 7, the ways of reasoning in the after-class discussion on Day 9 were more readily attributed to Abraham, hence I was able code the discussion via a series of five Toulmin schemes. In these five arguments, there were two for which Abraham provided data, two that were only claim-qualifier, and one that was only a claim. The lack of support for his claims, coupled with his expressions of uncertainty, provided evidence that his reasoning about span and linear independence in conjunction with each other was still in its early stages.

	Day 9 AC Arg 1
Claim	Matrix *A* does not span $\mathbf{R}^5$.
Data	It spans a 2-D space in $\mathbf{R}^5$

	Day 9 AC Arg 2
Claim	*The only time it doesn't span the whole dimension is when there are free variables.*

	Day 9 AC Arg 3
Claim	*A 5x3 matrix* [sic, he pointed to a 3 x 5 matrix] *definitely spans, right?*
Qualifier	*I'm just trying to think about it as a generalization.*

	Day 9 AC Arg 4
Claim	*Now it's [matrix B] spanning*
Data	*It has 2 free variables*

	Day 9 AC Arg 5
Claim	*When m=n, if it's linearly independent, then it spans the whole dimension*
Qualifier	*I don't know...it just seems like it would span.*

Figure 5.18. Toulmin schemes for Arguments 1-5 from Day 9 after class.

Furthermore, his claims were not entirely mathematically correct: the columns of a 3 x 5 matrix do not always span $\mathbf{R}^3$ (Argument 3), and the existence of free variables does not always imply that the column vectors do not span (Argument 2). Also noteworthy is Argument 5, when Abraham stated that if $m = n$ and the columns are linearly independent, "it just seems like it would span." This was the first time a student had offered this conjecture to the instructor. It is interesting that Abraham thought it was true but was unable to provide a reason as of yet—it just *seemed* like it would be so.

	Day 12, Exam 1 Question
Claim	If {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$} is a linearly independent set of 4 vectors in $\mathbf{R}^4$, then the span of {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$} is all of $\mathbf{R}^4$.
Data	*The RREF of the augmented matrix is the identity matrix*
Warrant	The identity matrix has four pivots (one in each column and one in each row)
Backing	Having four pivots means the span is all of $\mathbf{R}^4$

Figure 5.19. Toulmin scheme for Abraham's exam question about linear independence and span.

Between Day 9 and Day 12, there was a clear shift in Abraham's certainty and ability to reason about linear independence and span when $m = n$. The concepts of pivots and row-reduced echelon form continued to be developed and connected to linear independence and span on Day 10, which likely influenced his response on the exam that occurred on Day 12. In the Toulmin scheme for Abraham's exam response, the true/false question is symbolized in terms of general vectors {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$} in $\mathbf{R}^4$, and he was asked to prove whether linear independence implies the vectors span $\mathbf{R}^4$. He first stated," if we have an n x n matrix that is a linearly independent set, then the RREF of the augmented matrix is the identity matrix" (see Figure 5.17). From his sketch, however, it is unclear where the fifth vector for his augmented matrix came from, and he stated, "We can see that the identity matrix has four pivots," but the matrix in his sketch was not the identity matrix. Indeed, his sketch is a valid reduced echelon form for a matrix with linearly independent columns, but it is not identically consistent with his written explanation. Thus, although the claims from Abraham's after-class conversation on Day 9 (see Day 9 AC Arg 5 in Figure 5.18) and from the exam question on Day 12 are quite similar, the surrounding justification had developed substantially.

Next I consider Interview 1. Question 6a is the focus of the subsequent section on comparative results, so here I focus on Abraham's responses to Question 8, within which he discussed why a set of vectors could both span $\mathbf{R}^n$ and be linearly independent only if there were n vectors in $\mathbf{R}^n$. Question 8a asked Abraham to construct, if possible, an example of a 3 x 5 matrix such that the columns spanned $\mathbf{R}^3$ and were linearly independent. His response was coded via a series of five argumentation schemes (see Int 1 Q8a Arg 1-5 in Appendix 5.1). Within the first two arguments, he supported his claim that he could create such an example that spanned all of $\mathbf{R}^3$, sketched the generic matrix in Figure 5.20, and stated,

> I believe those asterisks that I put there could be any numbers...these numbers don't really matter, you still have three pivot positions...If I have three pivot positions, then I'm spanning all of $\mathbf{R}^3$. These two vectors [points to the last two columns comprised of asterisks] don't even need to be used to span that, the $\mathbf{R}^3$.

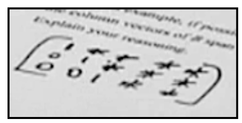


Figure 5.20. A 3x5 matrix whose columns spanned $\mathbf{R}^3$ but are linearly dependent.

Considering the possibility of linear independence, however, was more problematic for Abraham. After the statement above, he said, “You know what, though? If you have, hmm. This is a tricky question, I just actually thought of something...if you have more vectors than dimensions, that automatically makes it linearly dependent” (see Int 1 Q8a Arg 3). He thoughtfully re-read the prompt for a few seconds, crossed through his example matrix and wrote, “not possible” (see Figure 5.21). After a few more seconds of silence, he stated:

Abraham: I'm right to say they do span all of $\mathbf{R}^3$. If I just use these, right? [Covers the last two vectors in the matrix shown in Figure 5.21.] They do in fact span all of $\mathbf{R}^3$. But now we're talking about linearly independent. I can only construct one that spans all of $\mathbf{R}^3$ and is linearly *de*pendent. Which goes against my intuition, because I like to think of square matrices, and if they span all of $\mathbf{R}^3$, it's linearly independent.

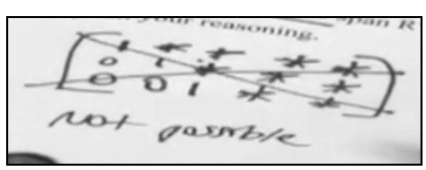


Figure 5.21. Abraham’s indication that it was not possible to create an example of a 3 x 5 matrix with linearly dependent column vectors.

Finally, the next two subsequent arguments concluded Abraham attempting to formulate a justification for his claim that "if you have more vectors than dimensions, that automatically makes it linearly dependent" (see Figure 5.22). In his data and warrant in both arguments, he used phrases such as, "get to any point," "get back to the origin," etc. These phrases stem from (a) the class's work with the Magic Carpet Ride problem, and (b) Justin's explanation (on Day 6) in class to support that very same claim's truth. Abraham made use of the travel analogy for the definition of linear dependence, but he was unable to convince himself why there was a way to "get back home" from any point in $\mathbf{R}^3$ with a set of five vectors.

	Interview 1 Q8a, Argument 4
Claim	*If you have more vectors than dimensions, that automatically makes it linearly dependent.*
Data	*I get to any point I want in* $\mathbf{R}^3$*. [draws an empty x,y,z coordinate axis] Let's just say I was here [puts a point in Quadrant 1]. Now I have two, that was after using three vectors, now I have two more I could use.*
Warrant	*So I should be able to make one that is linearly dependent and get back there, because I can get anywhere, so I can get, if I'm using the origin yet, I can get back to the origin*
Qualifier	*But how could I make it so that I don't get back to the origin? I don't think it's possible.*

	Interview 1, Q8a Argument 5
Claim	If you have more vectors than dimensions, that automatically makes it linearly dependent.
Qualifier	*This is weird to explain.*
Data	*I got here with three vectors, now since I can span all of* $\mathbf{R}^3$*, then we know I can get to this point, because this spans all of* $\mathbf{R}^3$*, so I know I can get to this point.*
Warrant	*If I can get there, then I can go back this way. So if I can get three vectors there, then I'm going to be able to get back there, using one of the other two vectors that are here.*
Qualifier	*But I'm still kind of, I want to be convinced that that's true, but at the same time, I don't know if I thought about it enough,*
Rebuttal	*But it seems like I should be able to make two vectors that don't get back here, but see, that's a problem. But I know by the definition that if they have more vectors than entries, it has to be dependent, so that convinces me. But it almost seems like maybe I should work on a computer program to try to make it so that I don't, I can't get back to here, and I probably would fail. But in my mind, it seems like you should be able to.*

Figure 5.22. Toulmin schemes for Interview 1, Question 8a, Arguments 4-5.

These two arguments are noteworthy because Abraham expressed his intuitive disbelief in his claim in conjunction with his conviction of its truth. In his rebuttal in Argument 5, he stated, "but I know by the definition...so that convinces me." It is unclear how Abraham understood the claim under debate as a definition. One explanation is because of its previous discussion in class. Recall from Chapter 4 that it was first conjectured as a generalization on Day 5 and then debated on Day 6. The reflection from Day 5 asked, "What one generalization from class today are you least confident about? Why?" Abraham wrote:

> If you have a set of vectors where you have more vectors than dimensions, then the set has to be linear dependent. I've only seen a couple of examples and examples cannot be used to prove anything. Therefore, it seems like it might be right but it has not been completely justified. I'm not completely sold on this generalization.

Thus, although Abraham's reflection response occurred on Day 5 and his interview after Day 17, and despite him being an active participant in a classroom that was using that generalization and him using it himself, he still questioned its validity.

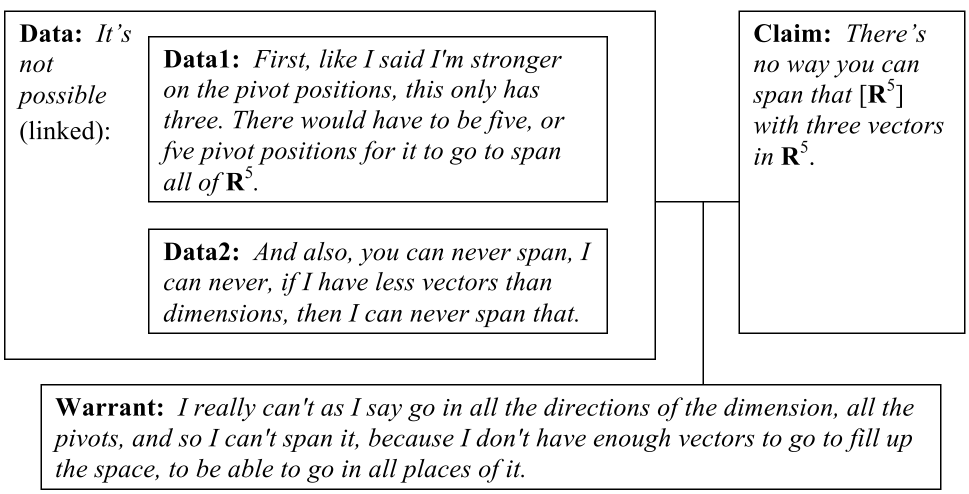


Figure 5.23. Interview 1, Q8b, Argument 1: Abraham explains why it is not possible to span $\mathbf{R}^5$ with three vectors in $\mathbf{R}^5$.

For the final data from Interview 1, I consider Question 8b, which asked Abraham to create an example, if possible, of a 5 x 3 matrix C such that the column vectors of C span all of

R^5 and are linearly independent. Within his response, he immediately said he was "stronger on the pivot positions," and explained that he needed five pivot positions to be able to span all of **R**^{5} because otherwise he would not "be able to go in all places of it" (see the Toulmin scheme for this argument in Figure 5.23).

Abraham was then asked how he would change the prompt so that he could create an example, and he replied he would change it to a 5 x 5 matrix, "just another square matrix." The Toulmin scheme for his justification is given in Figure 5.24.

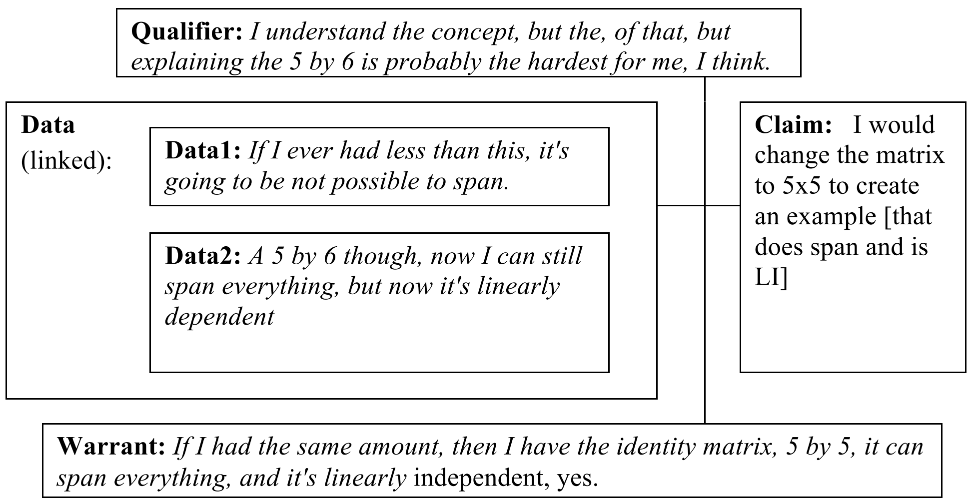


Figure 5.24. Interview 1, Q8b, Argument 2: Abraham explains why a 5x5 example could possibly have linearly independent vectors that span **R**^{5}.

In this argument, Abraham claimed that a matrix must be square for its column vectors to hold both desired properties. His data was composed of two linked aspects: if he had less than five columns, it wouldn't span; and the columns of a 5 x 6 would "span everything, but now it's linearly dependent." He concluded with the qualifier, "I understand the concept...but explaining the 5 x 6 is probably the hardest for me, I think."

The time in the semester between the first and second individual interviews, rather than being dedicated to developing the ideas of linear independence and span, used those ideas as ways to reason about new concepts in the course. As such, Abraham's arguments that explicitly related linear independence and span within the same argument occurred less frequently, and most often for the purpose of supporting a new conjecture. In particular, Abraham reasoned that: If a matrix is square*,* being linearly independent is the same as being onto (Arg 20.16 WCD); if the column vectors both span and are linearly independent, then the matrix would be invertible (Arg 20.24 WCD); having column vectors both span and be linearly independent is equivalent to "for every **b** in $\mathbf{R}^n$, there exists a unique solution to $A\mathbf{x} = \mathbf{b}$ (Arg 31.5-31.6 WCD).

During the second interview, rather than only talking about them as explicit points of conversation (which with I conclude this section), he used them in conjunction with each other to reason about one-to-one and onto transformations, invertible matrices, and basis. In each of these cases, the ideas of linear independence and span were within the data and/or warrant of the given claims. This is significant because this was a criterion used within Chapter 4 in order to document when mathematical ways of reasoning were functioning as-if shared for the collective. While the change in unit of analysis has methodological implications, it is worth noting, for the purposes of future work within the realm of coordination between individual and collective analyses, when possible instances of analytical compatibility arise. For instance, as part of an argument within which he attempted to explain why a transformation that was both one-to-one and onto would imply the associated matrix was invertible, Abraham had a sub-argument regarding unique solutions (See Figure 5.25).

	Interview 2, Q3b Argument 3
Claim	*It's spanning everywhere, and it's a unique solution.*
Data1	*This* [points to his written definition for 'onto'] *is span*
Data2	*This* [points to his written definition for '1-1'] *is linear independence*
Warrant	*If this is span, and I want to add linear independence to it, then I would say this definition of span [points to that same card again], but just adding the word 'unique. '.*

Figure 5.25. Int 2 Q3b Arg 3: Abraham reasons that the column vectors of a matrix for a given one-to-one and onto transformation span and are linearly independent.

His claim in this argument dealt with unique solutions, and his 2-part data to support it referenced how onto was equivalent to span and one-to-one was equivalent to linear independence, and then his warrant connected the two data to the claim. Here he was no longer justifying, for instance, why onto and span were related, why one-to-one and onto were related, or why linear independence and span implied a unique solution. Rather, he was using these ideas as part of a proof that eventually led him to a conclusion regarding invertibility (see Int 2 Q3b Arg 1-10 in Appendix 5.1 for the entire response).

As the last item to consider in this section regarding shifts in composition and content of arguments, I consider the following question regarding the role of $m = n$. In interview 2, Abraham was asked about the importance of the phrase, "Let A be an n x n matrix." He responded that "they [the concept statements] wouldn't, they'd be, they wouldn't go both ways, if the matrix wasn't n x n...if it can go both ways then they're equivalent. But if it can only go one way then they're not equivalent" (see Int 2 Q2c Arg 1 in Appendix 5.1). Here, "go one way" referred to logical implications of $p \Leftrightarrow q$; if it only "went one way," then either $p \rightarrow q$ or $q \rightarrow p$, but not both. When asked to create an example using specific concept statements from the IMT, Abraham chose, on his own accord, to discuss span and linear independence. He created two non-square examples, $\begin{bmatrix} 1 & 0 & 0 & 75 \\ 0 & 1 & 0 & 99 \\ 0 & 0 & 1 & 85 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, and stated that the former spanned $\mathbf{R}^3$ but was linearly dependent, and that the latter was linearly independent but did not span $\mathbf{R}^3$. His justifications were brief (see Int 2 Q2c Arg 2-5); twice he supported his claims about span by mentioning pivots, but he never supported his claims about linear independence.

From just this transcript, it is unclear whether he was unsure of how to provide justification, or if he thought is was unnecessary to do so. It is likely that it was the latter, given that in the immediately previous question during the interview, he explained why the column vectors of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ created a linearly dependent set. He did so by offering two justifications: (a) by showing two different solutions that "get to the same point": $1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $1\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$; and (b), by showing there was a solution that wasn't the trivial solution to the equation $A\mathbf{x} = \mathbf{0}$—namely, $1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (see Int 2 Q1b Arg 2-3 in Appendix 5.1). From his thorough and sophisticated response to Question 2b and his omission of an explanation in Question 8a in Interview 2, Abraham seemed more comfortable with the conjecture "if there are more vectors than dimensions, the vectors are linearly dependent" than he was during Interview 1.

5.1.2.1.3 *Comparison of arguments related to identical prompts at two different points in time during the semester.* In this section, I compare and contrast the Toulmin schemes for arguments in which Abraham was reasoning about the same question at two different points during the semester. In particular, I report on two specific comparisons: (a) Abraham's reflection and portfolio response concerning his reflection from Day 7; and (b) His responses to the interview prompt, "True or False**:** If the columns of a 3x3 matrix A span R3, then the columns vectors of A are linearly dependent."

The first comparison comes from a unique data source: a student reflecting on his own mathematical development. From the previous section, I demonstrated that Abraham's way of reasoning about span shifted throughout the semester. Analyzing his own reflection upon his growth, however, provides insight into how exactly his reasoning changed and what affected it. As discussed in the above description, the reflection from Day 7 asked students to consider $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 1 & 3 & -3 & -1 \\ 0 & 3 & -6 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ and respond to questions about the span of the three column vectors (see Figure 5.16). Here I focus on his answer to the second question, in which he said, "I'm having trouble with the span of vectors." The Toulmin scheme for his initial response is given in Figure 5.26.

	Day 7 Reflection, Question 2
Claim	Not sure what the span of $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}2\\3\\3\end{bmatrix},\begin{bmatrix}-1\\-3\\-6\end{bmatrix}\right\}$ is.
Qualifier	*I'm having trouble with the span of vectors.*

Figure 5.26. Toulmin scheme for Abraham's original response to Day 7 Reflection, Question 2.

From what was written on his paper, it is not possible to know more details about how he was "having trouble with the span of vectors," but we do see that he was unable at the time to make a definite claim about span. This is in contrast to his portfolio response from Day 12, which he turned in nearly three weeks after his original reflection response (see Figure 5.16). In that response, Abraham reflected that although he had a solid grasp on the definition of span (as well as a translation of it into terms compatible with the Magic Carpet Ride problem) on Day 7, he was not certain at that moment how to conclude anything about the span of *A*. He made an insightful remark considering the inefficiency of attempting to find a linear combination of the column vectors for *every* possible vector **b** in order to determine the span of the column vectors.

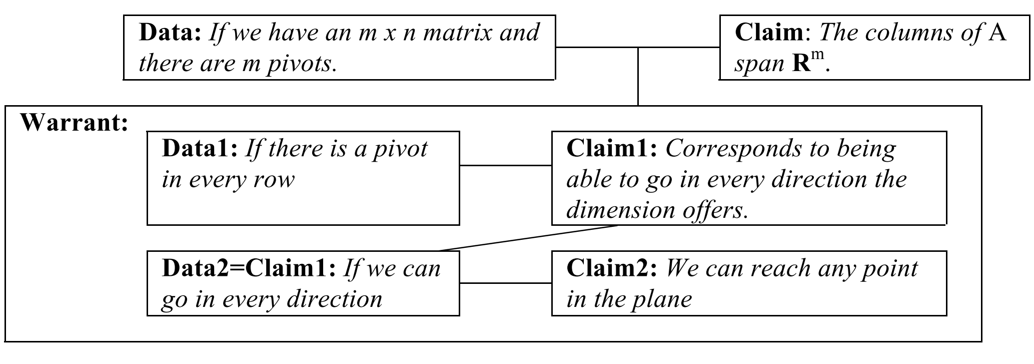


Figure 5.27. Toulmin scheme for Abraham's rationale for including a portfolio entry about Day 7, Reflection, Question 2.

He then transitioned out of his past ways of reasoning with the sentence, "When we introduced the concept of pivot points, it all began to make sense." The remainder of his response, in which he justified why having *m* pivots in the row-reduced form of an *m* x *n* matrix implies the span is all of $\mathbf{R}^m$, is coded via Toulmin's model (see Figure 5.27).

	Portfolio 1, description of his original response to Day 7 Reflection, Question 2
Claim	I *had no clue how one could know the* span of $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ -6 \end{bmatrix} \right\}$.
Data	It would be inefficient to determine if every *x, y, z vector combination in the set of real numbers can be written as a linear combination of the three vectors*
Qualifier	*Since this process is closely related to the definition of span, it seemed logical at first.*

Figure 5.28. Toulmin scheme for Abraham's description of his original response to Day 7 Reflection, Question 2, included in Portfolio 1.

The guidelines for a portfolio entry dictate that students should include their original response, as well as the re-worked problem, when applicable. The next page in his portfolio included that information (see Figure 5.16). Within that page, he explained how he had been thinking as he answered the original reflection question and then re-worked the problem by reasoning about pivots. The Toulmin schemes for those two portions are given in Figures 5.28 and 5.29, respectively.

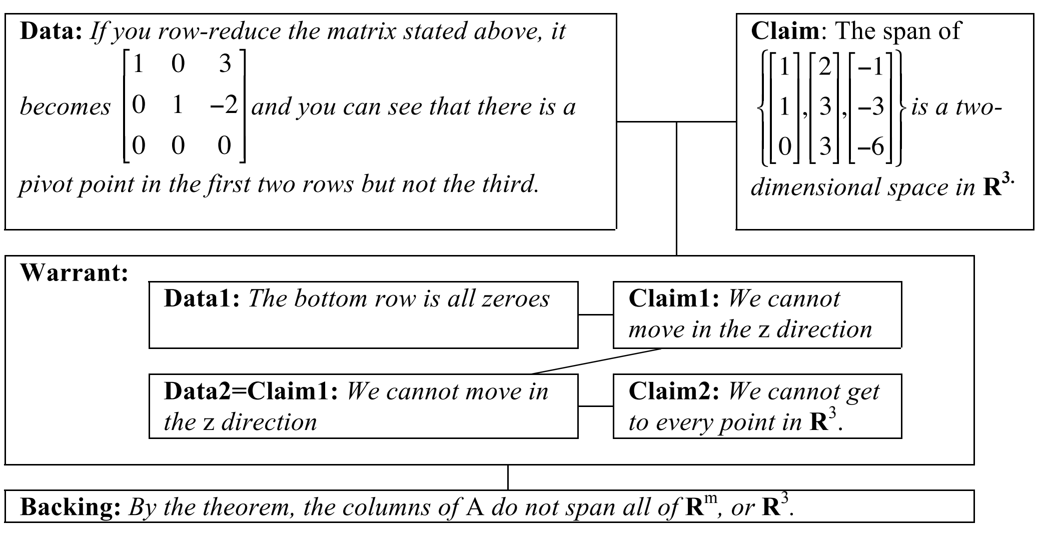


Figure 5.29. Toulmin scheme for Abraham's new solution to Day 7 Reflection, Question 2, included in Portfolio 1.

It is a rare opportunity to analyze when a student so clearly reflects upon his or her own thinking. Here Abraham, although he had only written, "I am having trouble with the span of vectors" in his original response, shared further information regarding what he recalled having thought at that moment. Twice (in his rationale statement and in his restatement of his original response), he shared, because of the inefficiency of trying to determine if there is a linear combination of the columns of a matrix for every possible output vectors, his preference for reasoning about pivots in order to make claims regarding span. He then, in his updated response, used as data that the RREF(A) only had two pivots and explained in the warrant why having two pivots had anything to do with the claim that the span was only a 2-dimensional space in $\mathbf{R}^3$. Overall, Abraham transitioned from not being able to really even formulate a claim about the span of the three given vectors to being able to, quite articulately, not only determine what the span was but also explain how pivots helped him to reason about span. Thus, we see a shift in Abraham's way of reasoning about the span of these particular vectors. We know from looking at all the arguments involved in the aforementioned section that Abraham's way of reasoning about span in general shifted over the course of the semester. By considering this portfolio and reflection response, however, we were able to gain a sharper glimpse at how exactly his reasoning changed and what affected it.

For the second example of direct comparison, I examine Abraham's responses to the interview prompt, "True or False**:** If the columns of a 3 x 3 matrix A span $\mathbf{R}^3$, then the columns vectors of A are linearly dependent," which he was asked during both individual interviews. During both interviews, he immediately replied 'false' to the prompt. For ease of comparison between the two responses, Table 5.2 has, in the leftmost column, the Toulmin schemes of his response from the first interview, and the rightmost column has the Toulmin schemes of his response from the second interview. In Table 5.2 I include only Abraham's *initial* responses to the question (barring small clarification questions from the interviewer). This is done to better align the responses for comparison. The Toulmin schemes for the full responses to the prompts are given in Appendix 5.1.

Table 5.2. Toulmin schemes for Abraham's two interview responses to the prompt regarding an implication between span and linear dependence.

<table>
<tr><td colspan="2">Suppose you have a 3 by 3 matrix A and you know that the columns of A span $\mathbf{R}^3$.
Decide if the following statement is true or false, and explain your answer:

The column vectors of A are linearly dependent.</td></tr>
<tr><th>Interview One (March 19)</th><th>Interview Two (May 19)</th></tr>
<tr><td><u>Int 1 Q6a Arg 1</u>

Claim: If the columns of a 3 x 3 matrix A span $\mathbf{R}^3$, then it is FALSE that the columns vectors of A are linearly dependent.
Data: The column vectors of A would have to be linearly independent
Qualifier (to the Data): This is hard to explain
Warrant: The only solution is the trivial solution
(Interviewer asks for clarification of the warrant)
Backing: And so if that holds for zero then that means that it should hold for, um, every point. So that every, um, every point would have a unique solution.

(Abraham reads "explain your answer" prompt and responds again)

<u>Int 1 Q6a Arg 2</u>

Claim: If the columns of a 3 x 3 matrix A span $\mathbf{R}^3$, then it is FALSE that the columns vectors of A are linearly dependent.
Data: I know it's linearly independent
Qualifier: It's so weird, it's like sometimes something makes like sense to you, and then you just know it's right, but sometimes you don't know how to really, like expl—you know what I mean? Like really explain, explain why, why it is true though.
Warrant: like automatically my mind just jumps to, 'they're linearly independent, they can, they span everywhere.'</td><td><u>Int 2 Q1a Arg 1</u>

Claim: If the columns of a 3 x 3 matrix A span $\mathbf{R}^3$, then it is FALSE that the columns vectors of A are linearly dependent.
Data: Matrices whose columns span $\mathbf{R}^3$ have three pivot positions
Warrant: And for like a square matrix, I just think like if this is three pivots in each row, then it's also going to be, automatically going to be three pivots in each column.
Backing: And that way you're always going to have a linearly independent set of...of, um, like x equals something, y equals something, z equals something, because of that.
Backing2: And then that's, so that's going to be, basically a unique solution for every output...There's a unique <x ,y, z> vector such that A$\mathbf{x} = \mathbf{b}$ in the output, so that'd be linearly independent, but not dependent.</td></tr>
</table>

Although Abraham correctly answered "false" in both cases, the Toulmin schemes for his two responses reveal significant differences between them. In the first interview, to support his claim that it was false that the columns vectors in the described matrix were linearly dependent, Abraham's data were "the column vectors of A would *have* to be linearly independent" and "I *know* it's linearly independent," respectively. He also qualified his data in both arguments. In the first, he stated it was hard to explain, and in the second, he commented on how "weird" it is to know something is true but not know how to explain why. In his response during the second interviewer, however, he uttered no qualifiers regarding the difficulty in explaining his claim. Furthermore, his data for the claim was that "matrices whose columns span $\mathbf{R}^3$ have three pivots positions," and he continued to explain (through a warrant and two backings) why having three pivot positions had anything to do with linear independence. He did provide warrants (in both) and backing (in the first) explanations from Interview 1; however, they are less developed than that of Interview 2. First, the prompt asked him to reason about "if span, then linear independence," but his justifications from Interview 1 both were more consistent with an "if linear independence, then span" form. Whether this was an oversight or a confusion with the logical order of a viable justification is unclear from the video and transcript data. In any case, this was not an issue in Interview 2.

Regarding the mathematical content of Abraham's justifications in Interview 1, he made use of the definition of linear independence and tried to conclude that if $A\mathbf{x} = \mathbf{0}$ had only the trivial solution, then there would be a unique solution to $A\mathbf{x} = \mathbf{b}$ for every **b.** Again, without a bit more work why this would possibly be true, the converse would have been more readily acceptable. In Interview 2, he articulated a justification that first concluded that A had three pivots, which implied there was a pivot in each column. From that, he stated, "so that's going to be, basically a unique solution for every output...so that'd be linearly independent, but not dependent." In the latter portion, there appeared to be an implicit assumption that a unique solution for $A\mathbf{x} = \mathbf{b}$ for every $\mathbf{b}$ implies linear independence (this would be valid, for the zero vector would be such a vector $\mathbf{b}$ and $A\mathbf{0} = \mathbf{0}$ for any matrix A because it defines a linear transformation). Elsewhere in Abraham's argumentation schemes (he was able to reason in a manner compatible with this. An alternative explanation is that Abraham was reasoning about, "there exists a unique solution to $A\mathbf{x} = \mathbf{b}$ for every **b**," as an alternative definition of linear independence. In any case, the overall structure of his response from Interview 2 was more confident, cohesive, and mathematically correct than his response from Interview 1.

5.1.2.2 The development of ways of reasoning about solutions to $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$. The second aspect of Abraham's ways of reasoning about the Invertible Matrix Theorem over time that I report on revolves around the common theme of solutions. When I first coded Abraham's argumentation with Toulmin's Model, I noticed that reasoning about solutions to either $A\mathbf{x} = \mathbf{0}$ or $A\mathbf{x} = \mathbf{b}$ was prominent in what he said throughout the semester. Indeed, a search through the compilation of all Toulmin schemes of his argumentation (given in Appendix 5.1) revealed that the word "solution" appeared 103 times within the Toulmin schemes. Furthermore, these occurrences were distributed across the claim, data, warrant, and backing of the different schemes within which it appeared.

There are two statements within the IMT that explicitly use the word "solution" in their phrasing: (a) The only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution, and (b) For every **b** in $\mathbf{R}^n$, there exists a solution **x** to $A\mathbf{x} = \mathbf{b}$. To an expert, the most obvious equivalence between these two concept statements and other aspects of the IMT are (c) The columns of A are linearly independent, and (d) The columns of A span $\mathbf{R}^n$, respectively. What is noteworthy with Abraham was his effort to reason about how (c) and (d) are related based on how (a) and (b) are related. During Interview 1, he inquired into what characteristics of $A\mathbf{x} = \mathbf{0}$ were applicable to $A\mathbf{x} = \mathbf{b}$. For instance, he reasoned about what the zero vector being the only solution to $A\mathbf{x} = \mathbf{0}$ implied when reasoning about the solutions to $A\mathbf{x} = \mathbf{b}$ for all possible **b**. Furthermore, this line of inquiry also led to Abraham reasoning about unique solutions to matrix equations. Given that this line of inquiry was not one that surfaced cleanly during whole class discussion, it is intriguing to have the opportunity to analyze it here. Thus, first section of analysis below is dedicated to this argumentation thread.

Additionally, Abraham reasoned about the notion of solution in another capacity, namely, to make claims about nearly every other concept captured within the IMT: one-to-one and onto transformations, invertibility, determinants, null space, and eigenvalues. I provide a short treatment of this category, listing a few arguments within which this occurs and present an example relating one-to-one and linear independence.

5.1.2.2.1 ***Reasoning about solutions to A*****x = 0** ***and A*****x = b** ***to form connections between span and linear independence.*** The first episode I present occurred during Interview 1, which took place approximately halfway through the semester. Abraham was asked to respond to the prompt, "True or False: If the columns of a 3x3 matrix span $\mathbf{R}^3$, then the column vectors are linearly dependent," to which he replied false. The first portion of Abraham's response to this prompt was analyzed in detail in the previous section (see Table 5.2). What was not presented in Table 5.2 was the remainder of his conversation with the interviewer, whose follow-up questions resulted in Abraham explaining why having a unique solution to $A\mathbf{x} = \mathbf{0}$ would imply there is a unique solution for every $\mathbf{b}$ to $A\mathbf{x} = \mathbf{b}$ (see Int 1 Q61 Arg 3 in Appendix 5.1). In Figure 5.30 I present this particular argument because it is the first time he clearly articulated that he was thinking about a relationship between solutions to $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$.

In his claim, Abraham stated "a lot of the same characteristics" apply to both $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$. While this claim is somewhat vague, his data and warrant serve to provide support for the claim by arguing that one similar characteristic is that of unique solutions. Within his data, he stated that when both systems are converted to augmented matrices, the coefficient matrix is identical in both. Thus, if you follow the same row-reducing steps, the final row-reduced echelon form of A in $A\mathbf{x} = \mathbf{b}$ will be the same as that in $A\mathbf{x} = \mathbf{0}$. Within his warrant, he stated a unique solution to $A\mathbf{x} = \mathbf{0}$ would imply that "most of the time, there should be a unique solution for any **b**” in his $A\mathbf{x} = \mathbf{b}$ system. The only elaboration provided for that somewhat unclear warrant was his qualifier, wherein he stated he remembered that when the bottom row of a row-reduced augmented matrix looks like [0 0 0 | 1], that there would be no solution to the system $A\mathbf{x} = \mathbf{b}$.

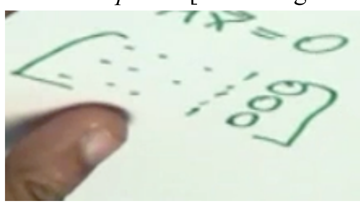
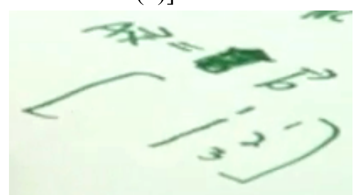
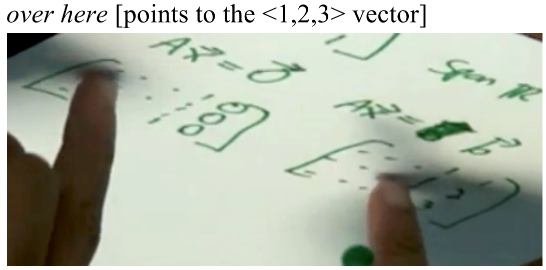
	Interview 1, Question 6a, Argument 3
Claim	*I started to think about this, like when you have* $A\mathbf{x} = \mathbf{0}$*, that a lot of the characteristics apply to* $A\mathbf{x}$ *= some vector* **b**.
Data	*Because, like, I have all these points here, and I have this augmented matrix here, and the 0's* [draws augmented matrix shown in (a)]. *And then let's say I was, um, I had some other points* [draws augmented matrix shown in (b)]. (a)  (b)  *Whatever I do over here with row reducing* this [points to augmented matrix in (a)], *it's really only going to affect this* [partitions off coefficient matrix in both (a) and (b) with his hands].
Warrant	*If there's a unique solution,* $c_1 = 0$, $c_2 = 0$ and $c_3 = 0$ for 0 [points to the zero vector in (a)] *then whatever, since, I mean, this* [points back and forth between coefficient matrices, shown in (e)] when *you row reduce it, it ends up being like the same thing on this side, then... then most of the time, there should be a unique solution for any* **b** *over here* [points to the <1,2,3> vector] (e) 
Qualifier	*But there are cases where it doesn't, and I remember thinking of one, where it doesn't work like that. And it's actually when the bottom row ends up being like that* [writes “0 0 0 | 1”]. That's when there's no solution.

Figure 5.30. Int 1 Q6a Arg 3: Abraham reasons that some characteristics of $A\mathbf{x}=\mathbf{0}$ apply to $A\mathbf{x}=\mathbf{b}$.

The previous example of Abraham reasoning about $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$ together stemmed from a question which asked him to reason about if a spanning set of three vectors in $\mathbf{R}^3$ implied the vectors were linearly dependent. While he knew that this implication was false (as seen in his initial response to the prompt, see Table 5.2), how to reason about that implication by reasoning about solutions, or even unique solutions, was a developing idea for him.

The next argument under consideration occurred during whole class discussion on Day 31, when Abraham again mentioned unique solutions. The class was presenting work about which concept statements were "most obviously equivalent" for their small group, and Justin had just presented an explanation for why his group thought the three statements are most readily seen as equivalent: (a) The columns of A span all of $\mathbf{R}^n$, (b) The column space of A is all of $\mathbf{R}^n$, and (c) For every $\mathbf{b}$ in $\mathbf{R}^n$, there's a way to write $\mathbf{b}$ as a linear combination of the columns of A. Abraham was preparing to explain why a fourth card, "'For every $\mathbf{b}$ in $\mathbf{R}^n$, there exists a solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$," could join that pile when Justin tried to add in a fifth, "the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution." Abraham, however, was quick to respond that, for him, that fifth card did not automatically belong in the pile with the other four. The Toulmin scheme for his response is given in Figure 5.31.

	Argument 31.6, from Whole Class Discussion
Claim	*If we added a word 'unique,' then I would put it* [the fifth card] *in there*
Data	*Because 'unique' makes it linear independence*
Qualifier	*But without the word, I just think of we can get to every* **b**.

Figure 5.31. Abraham explains why the 'trivial solution' card does not belong a pile with four other cards.

Abraham claimed that, for him, the fifth card only belonged in the pile if he were to add the word "unique" to the statement, "'For every $\mathbf{b}$ in $\mathbf{R}^n$, there exists a solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$." His data was that with he added the word "unique" so that it read, "For every $\mathbf{b}$ in $\mathbf{R}^n$, there exists a *unique* solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$," then the column vectors of A would be linearly independent. He finished by saying, "but without the word [unique], I just think of 'we can get to every $\mathbf{b}$." He did not provide a rationale for why adding the word "unique" to the statement would imply the column vectors were linearly independent, and he was not asked for justification. Thus, although no claims can be made regarding *how* Abraham was reasoning about this distinction between linear independence, existence and uniqueness of solutions, and "getting to every **b**," Argument 31.6 demonstrates a shift in Abraham's comfort level and confidence in reasoning about these concepts in conjunction with one another.

Along the same lines as the above argument from Day 31, consider the following explanation from Interview 2, which occurred about a week after Day 31. This explanation occurred immediately after those presented in Figure 5.11 and 5.12, in which Abraham was responding to the interviewer's prompt to discuss connections between the concepts statements regarding one-to-one, onto, and invertibility. His response to this prompt was also discussed in section 5.1.2.1 regarding the shift in Abraham's reasoning about span and linear independence in combination (see Figure 5.25). I present the entire argument's Toulmin scheme in Figure 5.32 but only comment here on the warrant.

	Interview 2, Question 3b, Argument 3
Claim	*It's spanning everywhere, and it's a unique solution.*
Data1	*This* [points to his definition for 'onto'] *is span*
Data2	*This* [points to his definition for '1-1'] *is linear independence*
Warrant	*If this is span, and I want to add linear independence to it, then I would say this definition of* span [points to the "for every **b** there exists a solution **x** to $A\mathbf{x} = \mathbf{b}$" card], *but just adding the word 'unique.'*

Figure 5.32. Int 2 Q3b Arg 3: Abraham's warrant explains that span and linear independence are equivalent if there is a unique solution to $A\mathbf{x} = \mathbf{b}$ for every **b**.

While the specifics of the surrounding argument are not relevant to the present analysis, the warrant in Figure 5.23 cleanly states that Abraham thinks of, "for every **b** there exists a solution **x** to $A\mathbf{x} = \mathbf{b}$," as a definition of span, and that if you wanted a set of vectors that both spanned a given space *and* were linearly independent, then you only need to require that the solution mentioned in this aforementioned definition be unique. Again, he did not explain why adding the word 'unique' would imply linear independence, but this example does affirm a shift in his ability to confidently reason about the notions of span, linear independence, and existence and uniqueness of solutions for every **b**.

The final example of Abraham reasoning about solutions to $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$ to form connections between span and linear independence is Argument 1 from Interview 2, Question 1a. This argument was considered in the previous section to compare Abraham's response to the prompt, "True or False: If the columns of a 3x3 matrix span $\mathbf{R}^3$, then the column vectors are linearly dependent" from both interviews (see Table 5.2). The Toulmin scheme for Abraham's response from Interview 2 is shown in Figure 5.33.

	Interview 2, Question 1a, Argument 1
Claim	If the columns of a 3x3 matrix A span $\mathbf{R}^3$, then it is FALSE that the columns vectors of A are linearly dependent.
Data	Matrices whose columns span $\mathbf{R}^3$ have *three pivot positions*
Warrant	*And for like a square matrix, I just think like if this is 3 pivots in each row, then it's also going to be, automatically going to be 3 pivots in each column.*
Backing	*And that way you're always going to have a linearly independent set of...of, um, like x =something, y = something, z = something, because of that*
Backing 2:	*And then that's, so that's going to be, basically a unique solution for every output*...There's a unique x,y,z vector such that $A\mathbf{x} = \mathbf{b}$ in the output, so that'd be linearly independent, but not dependent.

Figure 5.33. Int 2 Q1a Arg 1: Abraham's explanation that a spanning set of three vectors in $\mathbf{R}^3$ is linearly independent

Here I draw attention to the two levels of backing within this argument. Recall from Table 5.2 that, during Interview 1, Abraham had difficulty explaining why the correct answer to the prompt's question was "false," with his first two arguments basically conveying that he "just knew" it was so. His third argument within that Interview 1 response, as discussed previously in the present section, indicated that Abraham beginning to think in terms of unique solutions (see Figure 5.30). In Figure 5.33, however, he used this very thing that he was struggling with in Interview 1 as backing for the original claim. In other words, he justified the claim that a spanning set of vectors were linearly independent by reasoning about unique solutions to $A\mathbf{x} = \mathbf{b}$. By comparing his work in Interview 1 to that of Interview 2, a shift in Abraham's ways of reasoning is illuminated.

5.1.2.2.2 ***Reasoning about solutions to support claims about other concepts.*** Abraham also used solutions to reason about other concept statements within the IMT. Examples for each relevant concept are: free variables (Int 1 Q63 Arg 2, 3, and 5); one-to-one (Argument 20.5, Int 2 Q3b Arg 1); onto (Int 2 Q3 Args 1 and 9); determinants (Int 2 Q4b Arg 2); null space (Day 31 SG Args 2, 6, and 7]; and eigenvalues (Int 2 Q3 Arg 3). The Toulmin schemes for each of these arguments are provided in Appendix 5.1.

I conclude with an example. In Interview 2, Abraham was asked to explain how one-to-one and linear independence were related for him. Within this one Toulmin scheme, he said the word "solution" 12 times (see Figure 5.34). Looking closer at the details of how he uses solution is also revealing.

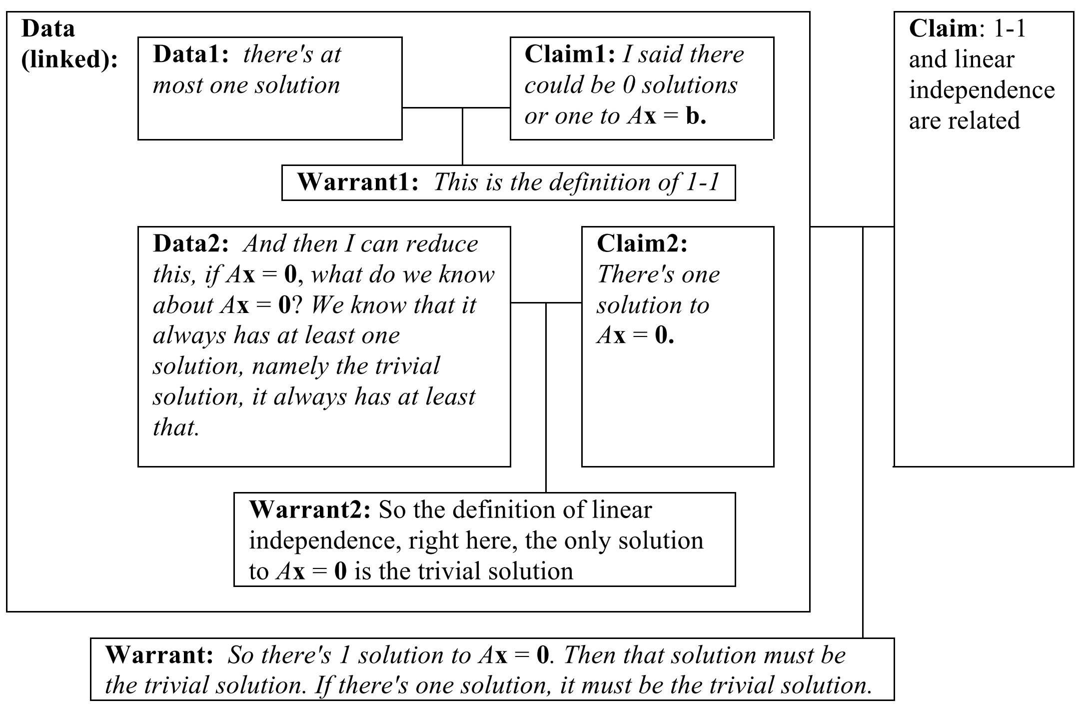


Figure 5.34. Int 2 Q3b Arg 10: Abraham explains how one-to-one and linear independence are related by reasoning about solutions to $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$.

The first sub-argument stated that based on the definition of one-to-one, there could be "zero solutions or one to $A\mathbf{x} = \mathbf{b}$." Within the second sub-argument, Abraham explained $A\mathbf{x} = \mathbf{0}$

always had to have at least one solution, and if there were only one (the trivial solution), the column vectors of A would be linearly independent. The warrant for the entire argumentation then concluded that if the only choices were one or zero solutions, that the only conclusion was that there was one solution to $A\mathbf{x} = \mathbf{0}$. In essence, Abraham's entire justification relied on reasoning about solutions to the matrix equation $A\mathbf{x} = \mathbf{0}$. He presented the definitions of both one-to-one and linear independence, as well as the zero property of linear transformations, to reason that if a given transformation is one-to-one, the column vectors of the associated matrix are linearly independent.

One may contend that Abraham's way of reasoning about the equivalence of one-to-one and linear independence is not surprising, given the definitions of those two terms; however, this does not rule out that other prevalent imageries could have been possible instead. For instance, within Chapter 4 we saw another student, Edgar, present an argument related to span and onto (which also have definitions that are solution-oriented). In Argument 20.19, Edgar claimed it was not possible to have a transformation from $\mathbf{R}^n \rightarrow \mathbf{R}^m$, where $m > n$, be onto. The data he provided was, "There's only two vectors, so you can't possibly span $\mathbf{R}^3$. You simply don't have enough vectors to get anywhere you have in $\mathbf{R}^3$" (see Figure 4.36). Edgar's justification relied more on the imagery of "getting everywhere" with linear combinations of column vectors than on solutions to a given matrix equation. Thus, it is noteworthy that Abraham has such a sophisticated ways of reasoning about solutions to $A\mathbf{x} = \mathbf{0}$ or $A\mathbf{x} = \mathbf{b}$ in a variety of situations.

Abraham's repeated use of reasoning about solutions to either $A\mathbf{x} = \mathbf{0}$ or $A\mathbf{x} = \mathbf{b}$ to support various new claims is also noteworthy because of its similarity to Criterion 3 from Chapter 4 regarding documenting normative ways of reasoning: When a particular idea is repeatedly used as either data or warrant for different claims across multiple days. Although a rigorous analysis of a possible adaptation of that criterion is beyond the scope of the present study, it provides a fruitful avenue of pursuit for future work in the coordination of individual and collective analyses.

5.1.3 Conclusion. Within the previous two sections, Microgenetic Analysis and Ontogenetic Analysis via Toulmin's Model, I presented results about how an individual student, Abraham, reasoned about the Invertible Matrix Theorem over the course of the semester. The results were achieved through the use of Toulmin's Model, coordinated with the microgenetic and ontogenetic strands of genetic analysis. In the microgenetic analysis section, I provided examples of how the expanded Toulmin structures were used to best capture the structure of Abraham's argumentation. In particular, I demonstrated the linked structure with an argument regarding how linear independence and one-to-one are connected, as well as through Abraham's explanation how 'one-to-one and onto' together imply the existence of a unique solution. I demonstrate the sequential structure through an argument regarding Abraham explains why if the columns of a 3x3 matrix A span $\mathbf{R}^3$, then there are no free variables, as well as through Abraham's explanation why "if 0 is an eigenvalue, then the null space does not contain only the zero vector."

I also presented results concerning the various ways in which Abraham interacted with the notion of equivalence when reasoning about the concepts involved in the IMT. I presented two categorizations to describe the nature of equivalence between different pairs of concept statements from the IMT that occurred within Abraham's argumentation. I defined the term conceptual equivalence and provided two examples of concept statement pairs about which Abraham reasoned as they were conceptually equivalent: (a) "The columns of A span Rn," and "The column space of A is all of Rn," and (b) "For every **b** in $\mathbf{R}^n$, there is a way to write **b** as a linear combination of the columns of A" and "The columns of A span $\mathbf{R}^n$." Second, I found that Abraham also reasoned about concept statements as logically equivalent. I provided two examples of this, both of which regarded Abraham's reasoning about "the columns of A span $\mathbf{R}^3$" and "the system $A\mathbf{x} = \mathbf{b}$ has no free variables."

In the ontogenetic analysis section, I first provided a descriptive overview of Abraham's ways of reasoning about span and linear independence in combination. I then presented selected analyses regarding these two ideas that focused on (a) shifts in composition of arguments and placement of concepts within the arguments, and (b) comparison of arguments that stemmed from identical prompts at two different points in time during the semester. I concluded by presenting results concerning the analysis of Abraham's prevalent use of reasoning about solutions to the matrix equations $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$ to make and support claims.

5.2 Adjacency Matrix Analysis

The remainder of this chapter continues to address results concerning the second research question: How did an individual student, Abraham, reason about the Invertible Matrix Theorem (IMT) over time? Whereas the first half of this chapter presented results utilizing Toulmin's Model to investigate Abraham's ways of reasoning, the remainder of the chapter presents results utilizing adjacency matrices as an analytical tool to address the same overarching research question. The microgenetic analysis revealed Abraham's tendency to reason about the negation of statements regarding determinants and eigenvalues from the IMT, as well as, through analyzing the $m < n$ sub-digraph, Abraham's difficulty with a particular generalization about linear dependence. Within the ontogenetic analysis section, I report results concerning the centrality measure for the concept of span, the prevalence of codes within the theme of "getting everywhere," and a comparison of arguments from identical prompts. I conclude the chapter with a discussion of the results and a reflection on and comparison of the separate analytical methods.

I first present the summary information regarding the adjacency matrix coding, as well as the associated total adjacency matrix $A(\mathrm{B})_{tot},$ for Abraham's ways of reasoning across the entire semester. I do so in order to frame the subsequent microgenetic and ontogenetic analysis, as it provides data that inform and drive the analyses on both levels. Furthermore, as stated in Chapter 4, conducting an ontogenetic analysis necessarily involves analysis at the microgenetic level. Per the methods described in Chapter 3, I conducted analysis over the same data set using both Toulmin's model and adjacency matrices as analytic tools. The analyzed portions of video and associated transcript come from portions of whole class discussion, small group work, and individual interviews. From this reduced data set, 105 arguments were analyzed using Toulmin's model. A selected portion of these results was presented in the first half of this chapter. The video and transcript from this same data set was also analyzed using adjacency matrices, and 98 of the 105 arguments were coded; the content of the other seven was inappropriate for the codes developed for the present adjacency matrix analysis. The complete list of adjacency matrix codes for each argument, as well as which arguments were not analyzed, can be found in Appendix 5.2.

E. Column vectors of *A* are linearly independent
- E1. *Trivial:* Only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution
- E2. *Unique:* There is a unique soln to matrix eqn/system of eqns
- E3. *Travel*: Can't get back (home)/get to origin with column vectors of *A*
- E4. *Geometric*: Vectors are not parallel/on the same line or plane
- E5. *Proportional*: No vector is a scalar multiple of another
- E6. *Linear combination*: No vector is a linear combo of another
- E7. *Placement*: No vector is in the span of the other vectors
- E8. *Extra*: Do not have an extra vector needed in order to return home

F. Column vectors of *A* are linearly dependent
- F1. *Trivial*: Is more than one solution to $A\mathbf{x} = \mathbf{0}$.
- F2. *Unique:* No unique/multiple/infinitely many solns to system/matrix eq
- F3. *Travel:* Can get back home/back to a point with column vectors of *A*
- F4. *Geometric*: Vectors are parallel/on the same line or plane
- F5. *Proportional*: One vector is a scalar multiple of another
- F6. *Linear combination*: One vector is a linear combo of others
- F7. *Placement*–A vector is in the span of the other vectors
- F8. *Zeroes*: the matrix *A* has a row or column of zeroes
- F9. *Extra:* Have an extra vector needed in order to return home

G. Column vectors of *A* span $\mathbf{R}^n$
- G1. *Size:* Are enough vectors to span the entire space
- G2. *Geometric*: Can use vectors to get to every pt/go everywhere
- G3. *Algebraic:* Is a linear combination of vectors for all pts in $\mathbf{R}^n$
- G4. *Direction:* Can use each vector to go in a certain direction
- G5. *Solution*: There is a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$

H. Column vectors of *A* do not span $\mathbf{R}^n$
- H1. *Size:* Are not enough vectors to span the entire space
- H2. *Geometric*: Can't use vectors to get to every pt/go everywhere in dim
- H3. *Direction:* Cannot go in all directions with the vectors
- H4. *Clarify*: The vectors of *A* span a *k*-dim subspace of $\mathbf{R}^n$
- H5. *Partial:* Span is only a point/line/plane
- H6: *Solution*: There is not a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$

I. Row-reduced echelon form of *A* has *n* pivots
- I1. *Diagonal*: RREF(*A*) has all ones on the main diagonal
- I2. *Identity:* Can row-reduce / is row equivalent to the identity
- I3. *Pivot-R:* Is a pivot in each row
- I4. *Pivot-C:* Is a pivot in each column
- I5. *Free Variable:* Each variable is defined in system/matrix eqn
- I6. *Zeroes:* RREF(*A*) has no rows of zeroes

J. Row-reduced echelon form of *A* has less than *n* pivots
- J1. *Diagonal:* RREF(*A*) does not have all ones on the main diagonal
- J2. *Identity:* Cannot row-reduce /not row equivalent to the identity
- J3. *Pivot-R:* Is not a pivot in each row
- J4. *Pivot-C:* Is not a pivot in each column
- J5. *Free Variable:* Not every variable is defined in system/matrix equation
- J6. *Zeroes:* RREF(*A*) has at least one row of zeroes

K. *A* is invertible
- K1. *Augment with the identity:* $[A \mid I] \sim [I \mid A^{-1}]$ is possible
- K2. *Formula:* Can calculat, no "divide by 0 errors"
- K3. *Undo transformation:* Can undo/get back from the transformation
- K4. *Inverse matrix:* Exists a *C* s.t. $AC=I$ and/or $CA=I$
- K5. *Transform:* Exists sequence of elementary row ops that turns *A* into *I*
- K6. *Necessary output*: Something gets sent to $\mathbf{e}_1$, $\mathbf{e}_2$, ...

L. *A* is not invertible
- L1. *Augment with the identity:* $[A \mid I] \sim [I \mid A^{-1}]$ is not possible
- L2. *Formula:* Can't calculate, get "divide by 0 errors"
- L3. *Undo transformation:* Can't undo/get back from the transformation
- L4. *Inverse matrix:* Does not exist a *C* s.t. $AC=I$ and/or $CA=I$
- L5. *Transform:* Does not exist sequence of elem row ops to turn *A* into *I*
- L6. *Necessary output*: Nothing gets sent to $\mathbf{e}_1$, $\mathbf{e}_2$, ...

M. The transformation defined by *A* is onto
- M1. *Definition:* For every $\mathbf{b}$ there is at least one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$
- M2. *Range:* Range is all of the codomain
- M3. *Images:* All values in codomain get used/mapped to as outputs/images

N. The transformation defined by *A* is not onto
- N1. *Definition:* For every $\mathbf{b}$ there is not at least one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$
- N2. *Range:* Range is not all of the codomain
- N3. *Images*: Not all values in codomain get used/mapped to as outputs/images
- N4. *Collapse:* Transformation collapses everything to a point/line/plane
- N5. *Transform:* The transformation goes up/adds a dimension

O. The transformation defined by *A* is 1-1
- O1. *Definition:* For every $\mathbf{b}$ there is at most one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$
- O2. *Input/Output:* Each output has at most one input
- O3. *Reachable:* There is only one way to "get to" the output/vector

P. The transformation defined by *A* is not 1-1
- P1. *Definition:* For every $\mathbf{b}$ there is more than one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$
- P2. *Input/Output:* Each output has more than one input
- P3. *Reachable:* There is more than one way to "get to" the output/vector
- P4. *Multiplicity:* At least 2 inputs give the same output
- P5: *Transform:* The transformation goes down/exludes a dimension

Q. $\det(A) \neq 0$: Determinant of *A* is nonzero
- Q1. *Area:* Unit square/cube has area/volume $\neq 0$ after transformation
- Q2. *Calculation*: formula to calculate determinant yields nonzero result

R. $\det(A) = 0$: Determinant of *A* is equal to 0
- R1. *Area:* Unit square/cube has area/volume = 0 after transformation
- R2. *Calculation*: formula to calculate determinant yields zero
- R3. *Measure:* No area/volume to a line/plane

S. Miscellaneous
- S1. *Calculation:* get inconsistent solutions (e.g., $0x + 0y = 1$)
- S2. *Column Space:* The Col *A* is all of $\boldsymbol{R}^n$
- S3. *Column Space:* The Col *A* is all not of $\boldsymbol{R}^n$
- S4. *Null Space:* The Nul *A* contains only the zero vector
- S5. *Null Space:* The Nul *A* contains more than the zero vector
- S6. *Eigen Value:* The number zero is not an eigenvalue of A
- S7. *Eigen Value:* The number zero is an eigenvalue of A
- S8: $m < n$: *A* has more vectors than dimensions
- S9: $m = n$: *A* has less vectors than dimensions
- S10: $m = n$: *A* has the same number of rows/columns
- S11: $m \neq n$: *A* does not have the same number of rows/columns
- S12: *Miscellaneous*

Figure 5.35. The 100 codes used as the rows and columns for the adjacency matrices.

Furthermore, the 100 codes used as the rows and columns of the adjacency matrix are the same as those used in Chapter 4 (see Figure 4.48). They are repeated here (see Figure 5.35). Within Abraham's coding, however, there were two uses for the "S12: Miscellaneous" code. The concept statement "Only the zero vector gets sent to the zero vector" is labeled S12a, and "The columns of A form a basis for $\mathbf{R}^n$" is labeled S12b. Consistent with the methods explained in Chapter 4, the transcript from Abraham's argumentation was coded based on whether he was reasoning about the case of $m < n$, $m = n$, $m > n$, or *any* m, n. The associated adjacency matrices for these four sub-digraphs are notated $A(\mathrm{B})_{m<n}$, $A(\mathrm{B})_{m=n}$, and $A(\mathrm{B})_{m>n}$, and $A(\mathrm{B})_{any}$. The sizes of the sub-digraphs, where size is the total number of arc between vertices, are $A(\mathrm{B})_{m<n} = 33$, $A(\mathrm{B})_{m=n} = 318$, $A(\mathrm{B})_{m>n} = 17$ $A(\mathrm{B})_{any} = 20$, making the size of the digraph $A(\mathrm{B})_{tot} = 33 + 318 + 17 + 20 = 388$. Furthermore, the order of the digraph was 71. In other words, 71 various interpretations of the concept statements (or concept statements themselves) associated with the IMT were used within the analyzed data set, and 388 edges exists in such a way as to match the implication offered by Abraham. Although not each of the three sub-digraphs utilized each of the 71 vertices, I used 71 as their order in my calculations of centrality for consistency purposes. This is consistent with my approach in Chapter 4.

The adjacency matrix $A(\mathrm{B})_{tot}$, which was partitioned into four figures because of its size, is provided in Figures 5.37-5.40. Figure 5.37 is the upper-left quadrant of $A(\mathrm{B})_{tot}$, displaying rows E-K3 and columns E-K3; Figure 5.38 is the lower-left quadrant, displaying rows K4-S12 and columns E-K3; Figure 5.39 is the upper-right quadrant of $A(\mathrm{B})_{tot}$, displaying rows E-K3 and columns K4-S12; and Figure 5.40 is the lower-right quadrant, displaying rows K4-S12 and columns K4-S12.

Because of the large number of arguments being analyzed, coding each argument in a different color was not possible for $A(\mathrm{B})_{tot}$ or its sub-digraph adjacency matrices. Thus, a different color was used for each of the class days and interview days coded. Furthermore, the colors used in $A(\mathrm{B})_{tot}$ correspond to those used for $A(\mathrm{T})_{tot}$ from Chapter 4 (see Figure 4.55), with an additional two colors (see Figure 5.36), light green and grey, to correspond to the two individual interviews in which Abraham participated (but were not data sources for the collective analysis of Chapter 4).

- Day 7: Pink
- Day 9: Red
- Day 10: Orange
- Interview 1: Light Green
- Day 17: Green
- Day 18: Dark Green
- Day 19: Light Blue
- Day 20: Dark Blue
- Day 24: Purple
- Day 31: Brown
- Interview 2: Grey

Figure 5.36. Key of colors used for entries in adjacency matrix $A(\mathrm{B})_{tot.}$

	E	E1	E2	E3	E4	E5	E6	E7	E8	F	F1	F2	F3	F4	F5	F6	F7	F8	F9	G	G1	G2	G3	G4	G5	H	H1	H2	H3	H4	H5	H6	I	I1	I2	I3	I4	I5	I6	J	J1	J2	J3	J4	J5	J6	K	K1	K2	K3
E. **Column vectors of A are linearly independent**		1 3 2																		1 1 2 1 1 2		1											2	1	1												1			
E1. *Trivial*: Only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution	1 5		2																			1											1																	
E2. *Unique:* There is a unique soln to matrix eqn/system of eqns	1 1 3	1 2																		1 1																														
E3. *Travel*: Can't get back (home)/get to origin with column vectors of A																																																		
E4. *Geometric*: Vectors are not parallel/on the same line or plane																																																		
E5. *Proportional*: No vector is a scalar multiple of another																												1																						
E6. *Linear combination*: No vector is a linear combo of another																																																		
E7. *Placement*: No vector is in the span of the other vectors																																																		
E8. *Extra*: Do not have an extra vector needed in order to return home																																																		
F. **Column vectors of A are linearly dependent**											1																				1															2				
F1. *Trivial*: Is more than one solution to $A\mathbf{x} = \mathbf{0}$.	1									2																																								
F2. *Unique*: No unique/multiple/infinitely many solns to system/matrix eqn																																							1	1				1		1				
F3. *Travel*: Can get back home/back to a point with column vectors of A										2 2						1			1																															
F4. *Geometric*: Vectors are parallel/on the same line or plane																																																		
F5. *Proportional*: One vector is a scalar multiple of another										1 1			1																																					
F6. *Linear combination*: One vector is a linear combo of others																																								1										
F7. *Placement*–A vector is in the span of the other vectors																																																		
F8. *Zeroes*: the matrix A has a row or column of zeroes										1																																								
F9. *Extra:* Have an extra vector needed in order to return home				1																																														
G. **Column vectors of A span $\mathbf{R}^n$**	7 1 4	3	2							3			1								2	4 1	2 1	2	2 1 2								4 2		1			4												
G1. *Size*: Are enough vectors to span the entire space																				1		2																												
G2. *Geometric*: Can use vectors to get to every pt/go everywhere			1										1 1						1	1			1																											
G3. *Algebraic*: Is a linear combination of vectors for all pts in $\mathbf{R}^n$																				3 2					1 1																									
G4. *Direction*: Can use each vector to go in a certain direction																						1																												
G5. *Solution*: There is a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$			1																	1 3		1 1	1 1																											
H. **Column vectors of A do not span $\mathbf{R}^n$**																															1																			
H1. *Size*: Are not enough vectors to span the entire space																										1		1																						
H2. *Geometric*: Can't use vectors to get to every pt/go everywhere in dim																										1																								
H3. *Direction*: Cannot go in all directions with the vectors																										1	1			1																				
H4. *Clarify*: The vectors of A span a k-dim subspace of $\mathbf{R}^n$																										1																								
H5. *Partial*: Span is only a point/line/plane																																																		
H6. *Solution*: There is not a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$																																																		
I. **Row-reduced echelon form of A has n pivots**	1 1 1	1																		2 1 1				2											1	1		2											1	
I1. *Diagonal*: RREF(A) has all ones on the main diagonal			3																	1 1																	1										1			
I2. *Identity*: Can row-reduce / is row equivalent to the identity		2																															2																	
I3. *Pivot-R*: Is a pivot in each row																				1																	1													
I4. *Pivot-C*: Is a pivot in each column	2		1																														1			1														
I5. *Free Variable*: Each variable is defined in system/matrix eqn																																																		
I6. *Zeroes*: RREF(A) has no rows of zeroes												1																												1										
J. **Row-reduced echelon form of A has less than n pivots**										2																5 1			3																					
J1. *Diagonal*: RREF(A) does not have all ones on the main diagonal																																																		
J2. *Identity*: Cannot row-reduce /not row equivalent to the identity																																																		
J3. *Pivot-R*: Is not a pivot in each row																													1																					
J4. *Pivot-C*: Is not a pivot in each column																																																		
J5. *Free Variable*: Not every variable is defined in system/matrix equation												3								1						1														1						1				
J6. *Zeroes*: RREF(A) has at least one row of zeroes																																1								2 1				1						
K. **A is invertible**	3																			1 1																														
K1. *Augment with the identity*: [A | I]~[I | A^{-1}] is possible																																																		
K2. *Formula*: Can calculat, no "divide by 0 errors"																																																		
K3. *Undo transformation*: Can undo/get back from the transformation																																																		

Figure 5.37. Upper left quadrant of $A(\mathrm{B})_{tot}$.

	K4	K5	K6	L	L1	L2	L3	L4	L5	L6	M	M1	M2	M3	N	N1	N2	N3	N4	N5	O	O1	O2	O3	P	P1	P2	P3	P4	P5	Q	Q1	Q2	R	R1	R	R3	S	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12
E. Column vectors of *A* are linearly independent											1										1 2	1																				2 1								
E1. *Trivial:* Only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution																																										1 2								1$_a$
E2. *Unique:* There is a unique soln to matrix eqn/system of eqns											1										1																													1$_b$
E3. *Travel:* Can't get back (home)/get to origin with column vectors of *A*																																																		
E4. *Geometric:* Vectors are not parallel/on the same line or plane																																																		
E5. *Proportional:* No vector is a scalar multiple of another																																																		
E6. *Linear combination:* No vector is a linear combo of another																																																		
E7. *Placement:* No vector is in the span of the other vectors																																																		
E8. *Extra:* Do not have an extra vector needed in order to return home																																																		
F. Column vectors of *A* are linearly dependent																									2									1																
F1. *Trivial:* Is more than one solution to $A\mathbf{x} = \mathbf{0}$.																													1														1							
F2. *Unique:* No unique/multiple/infinitely many solns to system/matrix eqn																																																		
F3. *Travel:* Can get back home/back to a point with column vectors of *A*																																																		
F4. *Geometric:* Vectors are parallel/on the same line or plane																																																		
F5. *Proportional:* One vector is a scalar multiple of another																		1																1																
F6. *Linear combination:* One vector is a linear combo of others																																																		
F7. *Placement*–A vector is in the span of the other vectors																																																		
F8. *Zeroes:* the matrix *A* has a row or column of zeroes																																																		
F9. *Extra:* Have an extra vector needed in order to return home																																																		
G. Column vectors of *A* span $\mathbf{R}^n$											1 1 2	1	1																											1 3										
G1. *Size:* Are enough vectors to span the entire space																																																		
G2. *Geometric:* Can use vectors to get to every pt/go everywhere																																								1										
G3. *Algebraic:* Is a linear combination of vectors for all pts in $\mathbf{R}^n$																																																		
G4. *Direction:* Can use each vector to go in a certain direction																																																		
G5. *Solution:* There is a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$											1	2		1												1														1										
H. Column vectors of *A* do not span $\mathbf{R}^n$															1																																			
H1. *Size:* Are not enough vectors to span the entire space																																																		
H2. *Geometric:* Can't use vectors to get to every pt/go everywhere in dim																																																		
H3. *Direction:* Cannot go in all directions with the vectors																																																		
H4. *Clarify:* The vectors of *A* span a *k*-dim subspace of $\mathbf{R}^n$																																																		
H5. *Partial:* Span is only a point/line/plane										2					1																																			
H6. *Solution:* There is not a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$																																																		
I. Row-reduced echelon form of *A* has *n* pivots	1																																																	
I1. *Diagonal:* RREF(*A*) has all ones on the main diagonal																																																		
I2. *Identity:* Can row-reduce / is row equivalent to the identity																																																		
I3. *Pivot-R:* Is a pivot in each row																																																		
I4. *Pivot-C:* Is a pivot in each column																																																		
I5. *Free Variable:* Each variable is defined in system/matrix eqn																																																		
I6. *Zeroes:* RREF(*A*) has no rows of zeroes																																																		
J. Row-reduced echelon form of *A* has less than *n* pivots																																									1									
J1. *Diagonal:* RREF(*A*) does not have all ones on the main diagonal																																																		
J2. *Identity:* Cannot row-reduce /not row equivalent to the identity																																																		
J3. *Pivot-R:* Is not a pivot in each row																																																		
J4. *Pivot-C:* Is not a pivot in each column																																																		
J5. *Free Variable:* Not every variable is defined in system/matrix equation																																																		
J6. *Zeroes:* RREF(*A*) has at least one row of zeroes																																																		
K. *A* is invertible	2																																																	
K1. *Augment with the identity:* $[A \mid I] \sim [I \mid A^{-1}]$ is possible																																																		
K2. *Formula:* Can calculat, no "divide by 0 errors"																																																		
K3. *Undo transformation:* Can undo/get back from the transformation																																																		

Figure 5.38. Upper right quadrant of $A(\mathrm{B})_{tot}$.

	K4	K5	K6	L	L1	L2	L3	L4	L5	L6	M	M1	M2	M3	N	N1	N2	N3	N4	N5	O	O1	O2	O3	P	P1	P2	P3	P4	P5	Q	Q1	Q2	R	R1	R	R3	S	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12
E. Column vectors of A are linearly independent											1										1 2	1																				2 1									
E1. *Trivial:* Only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution																																										1 2									1_a
E2. *Unique:* There is a unique soln to matrix eqn/system of eqns											1										1																														1_b
E3. *Travel:* Can't get back (home)/get to origin with column vectors of A																																																			
E4. *Geometric:* Vectors are not parallel/on the same line or plane																																																			
E5. *Proportional:* No vector is a scalar multiple of another																																																			
E6. *Linear combination:* No vector is a linear combo of another																																																			
E7. *Placement:* No vector is in the span of the other vectors																																																			
E8. *Extra:* Do not have an extra vector needed in order to return home																																																			
F. Column vectors of A are linearly dependent																									2									1																	
F1. *Trivial:* Is more than one solution to $A\mathbf{x} = \mathbf{0}$.																													1														1								
F2. *Unique:* No unique/multiple/infinitely many solns to system/matrix eqn																																																			
F3. *Travel:* Can get back home/back to a point with column vectors of A																																																			
F4. *Geometric:* Vectors are parallel/on the same line or plane																																																			
F5. *Proportional:* One vector is a scalar multiple of another																		1																1																	
F6. *Linear combination:* One vector is a linear combo of others																																																			
F7. *Placement*–A vector is in the span of the other vectors																																																			
F8. *Zeroes:* the matrix A has a row or column of zeroes																																																			
F9. *Extra:* Have an extra vector needed in order to return home																																																			
G. Column vectors of A span $\mathbf{R}^n$											1 1 2	1	1																											1 3											
G1. *Size:* Are enough vectors to span the entire space																																																			
G2. *Geometric:* Can use vectors to get to every pt/go everywhere																																								1											
G3. *Algebraic:* Is a linear combination of vectors for all pts in $\mathbf{R}^n$																																																			
G4. *Direction:* Can use each vector to go in a certain direction																																																			
G5. *Solution:* There is a solution to $A\mathbf{x}$=$\mathbf{b}$ for every $\mathbf{b}$											1	2		1												1														1											
H. Column vectors of A do not span $\mathbf{R}^n$															1																																				
H1. *Size:* Are not enough vectors to span the entire space																																																			
H2. *Geometric:* Can't use vectors to get to every pt/go everywhere in dim																																																			
H3. *Algebraic:* Is no linear combo of vectors for some pts in $\mathbf{R}^n$																																																			
H4. *Clarify:* The vectors of A span a k-dim subspace of $\mathbf{R}^n$																																																			
H5. *Partial:* Span is only a point/line/plane										2					1																																				
H6. *Solution:* There is not a solution to $A\mathbf{x}$=$\mathbf{b}$ for every $\mathbf{b}$																																																			
I. Row-reduced echelon form of A has n pivots	1																																																		
I1. *Diagonal:* RREF(A) has all ones on the main diagonal																																																			
I2. *Identity:* Can row-reduce / is row equivalent to the identity																																																			
I3. *Pivot-R:* Is a pivot in each row																																																			
I4. *Pivot-C:* Is a pivot in each column																																																			
I5. *Free Variable:* Each variable is defined in system/matrix eqn																																																			
I6. *Zeroes:* RREF(A) has no rows of zeroes																																																			
J. Row-reduced echelon form of A has less than n pivots																																									1										
J1. *Diagonal:* RREF(A) does not have all ones on the main diagonal																																																			
J2. *Identity:* Cannot row-reduce /not row equivalent to the identity																																																			
J3. *Pivot-R:* Is not a pivot in each row																																																			
J4. *Pivot-C:* Is not a pivot in each column																																																			
J5. *Free Variable:* Not every variable is defined in system/matrix equation																																																			
J6. *Zeroes:* RREF(A) has at least one row of zeroes																																																			
K. A is invertible	2																																																		
K1. *Augment with the identity:* $[A \mid I] \sim [I \mid A^{-1}]$ is possible																																																			
K2. *Formula:* Can calculat, no "divide by 0 errors"																																																			
K3. *Undo transformation:* Can undo/get back from the transformation																																																			

Figure 5.39. Lower left quadrant of $A(\mathrm{B})_{tot}$.

	K4	K5	K6	L	L1	L2	L3	L4	L5	L6	M	M1	M2	M3	N	N1	N2	N3	N4	N5	O	O1	O2	O3	P	P1	P2	P3	P4	P5	Q	Q1	Q2	R	R1	R	R3	S	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12
K4. *Inverse matrix:* Exists a C s.t. $AC=I$ and/or $CA=I$		1																																																
K5. *Transform:* Exists sequence of elementary row ops that turns A into I																																																		
K6. *Necessary output*: Something gets sent to $\mathbf{e}_1$, $\mathbf{e}_2$, ...																																																		
L. A is not invertible																																																		
L1. *Augment with the identity:* $[A \mid I] \sim [I \mid A^{-1}]$ is not possible																																																		
L2. *Formula:* Can't calculate, get "divide by 0 errors"																																																		
L3. *Undo transformation:* Can't undo/get back from the transformation																																																		
L4. *Inverse matrix:* Does not exist a C s.t. $AC=I$ and/or $CA=I$																																																		
L5. *Transform:* Does not exist sequence of elem row ops to turn A into I																																																		
L6. *Necessary output*: Nothing gets sent to $\mathbf{e}_1$, $\mathbf{e}_2$, ...				2																																														
M. The transformation defined by A is onto	1											2									1 1				1																									
M1. *Definition:* For every $\mathbf{b}$ there is at least one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$											2										1	1																												
M2. *Range:* Range is all of the codomain											1																																							
M3. *Images:* All values in codomain get used/mapped to as outputs/images																																																		
N. The transformation defined by A is not onto																																																		
N1. *Definition:* For every $\mathbf{b}$ there is not at least one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																																		
N2. *Range:* Range is not all of the codomain																																																		
N3. *Images*: Not all values in codomain get used/mapped to as outputs/images																																																		
N4. *Collapse:* Transformation collapses everything to a point/line/plane										1																																								
N5. *Transform:* The transformation goes up/adds a dimension																																																		
O. The transformation defined by A is 1-1											1 1											2 2																												
O1. *Definition:* For every $\mathbf{b}$ there is at most one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$											1	1									1 2																													
O2. *Input/Output:* Each output has at most one input																																																		
O3. *Reachable:* There is only one way to "get to" the output/vector																																																		
P. The transformation defined by A is not 1-1											1																																							
P1. *Definition:* For every $\mathbf{b}$ there is more than one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																													1																					
P2. *Input/Output:* Each output has more than one input																																																		
P3. *Reachable:* There is more than one way to "get to" the output/vector																																																		
P4. *Multiplicity:* At least 2 inputs give the same output																																																		
P5: *Transform:* The transformation goes down/exludes a dimension																																																		
Q. $\det(A) \neq 0$: Determinant of A is nonzero																																																		
Q1. *Area:* Unit square/cube has area/volume $\neq 0$ after transformation																																																		
Q2. *Calculation:* formula to calculate determinant yields nonzero result																																																		
R. $\det(A) = 0$: Determinant of A is equal to 0																										1		1							2		1													
R1. *Area:* Unit square/cube has area/volume = 0 after transformation																			1																															
R2. *Calculation:* formula to calculate determinant yields zero																																																		
R3. *Measure:* No area/volume to a line/plane																																																		
S. Miscellaneous																																																		
S1. *Calculation:* get inconsistent solutions (e.g., $0x + 0y = 1$)																																																		
S2. *Column Space:* The Col A is all of R^n																																																		
S3. *Column Space:* The Col A is all not of R^n																																																		
S4. *Null Space:* The Nul A contains only the zero vector																																												1					3_a
S5. *Null Space:* The Nul A contains more than the zero vector																																																		1_a
S6. *Eigen Value:* The number zero is not an eigenvalue of A																																										1								
S7. *Eigen Value:* The number zero is an eigenvalue of A																																											1							
S8: $m < n$: A has more vectors than dimensions																																																		
S9: $m = n$: A has less vectors than dimensions															1																																			
S10: $m = n$: A has the same number of rows/columns																																																		
S11: $m \neq n$: A does not have the same number of rows/columns																																																		
S12: *Miscellaneous*	2_b										1_b										1_b																					1_a 1_a								

Figure 5.40. Lower right quadrant of $A(\mathrm{B})_{tot}$.

Examination of $A(\mathrm{B})_{tot}$, reveals that the majority of argumentation occurred during the two interviews (noted with the light green and grey entries in Figures 5.37-5.40); this is not surprising given that the two interviews were designed to specifically reveal Abraham regarding his ways of reasoning about the IMT. This is not seen as a limitation, but rather an advantage in that the two interviews provided a more focused, personalized, and in-depth window of Abraham's ways of reasoning than could be gleaned from classroom data alone. As was done in Chapter 4, the measures of out-degree, out-connection, in-degree, and in-connection (see Figure 4.47 for the definitions of these terms), and centrality were computed for $A(\mathrm{B})_{tot}$; this information is given in Figure 5.41.

Both the adjacency matrix in Figures 5.37-40 and the summary information in 5.41 informed how I chose to conduct the microgenetic and ontogenetic analyses within the remainder of the chapter. I was also influenced by what analyses I conducted via adjacency matrices at the collective level in Chapter 4, as well as those conducted via Toulmin's model in the first half of this chapter. With those under consideration, the results I present at the microgenetic level involve (a) instances of reasoning about the IMT via the negation of statements; and (b) the structure of argumentation by considering the type of sub-digraph in which the argumentation occurs. At the ontogenetic level, my results concern (a) the centrality measure and various implications (b) the prevalence of codes within a common theme; and (c) comparison of arguments related to identical prompts at two different points in time during the semester.

	total		total		
	OD	OC	ID	IC	centrality
E	28	11	46	13	0.1690
E1	14	6	24	8	0.0986
E2	13	6	14	10	0.1127
E3			1	1	0.0070
E4					
E5	1	1			0.0070
E6					
E7					
E8					
F	7	5	27	11	0.1127
F1	5	4	3	2	0.0423
F2	4	4	4	2	0.0423
F3	6	3	4	3	0.0423
F4					
F5	5	4			0.0282
F6	1	1	1	1	0.0141
F7					
F8	1	1			0.0070
F9	1	1	2	2	0.0211
G	59	17	47	19	0.2535
G1	3	2	2	1	0.0211
G2	7	6	13	7	0.0915
G3	7	2	6	3	0.0352
G4	1	1	4	2	0.0211
G5	15	9	13	6	0.1056
H	2	2	16	7	0.0634
H1	2	2	1	1	0.0211
H2	1	1	3	3	0.0282
H3	3	3	4	2	0.0352
H4	1	1	2	1	0.0141
H5	3	2	2	2	0.0282
H6			1	1	0.0070
I	16	9	12	5	0.0986
I1	7	4	2	2	0.0423
I2	4	2	3	3	0.0352
I3	2	2	2	2	0.0282
I4	5	4	2	2	0.0423
I5			6	2	0.0141
I6	2	2	1	1	0.0211
J	12	4	8	6	0.0704
J1					
J2					
J3	1	1			0.0070
J4			2	2	0.0141
J5	7	5			0.0352
J6	5	3	4	3	0.0423
K	7	3	10	6	0.0634
K1					
K2			1	1	0.0070
K3					

	total		total		
	OD	OC	ID	IC	centrality
K4	6	2	6	4	0.0423
K5	1	1	1	1	0.0141
K6					
L			2	1	0.0070
L1					
L2					
L3					
L4					
L5					
L6	3	2	3	2	0.0282
M	13	9	15	10	0.1338
M1	9	6	6	4	0.0704
M2	2	2	1	1	0.0211
M3	2	2	1	1	0.0211
N			3	3	0.0211
N1					
N2					
N3			1	1	0.0070
N4	1	1	1	1	0.0141
N5					
O	15	6	11	6	0.0845
O1	8	6	6	3	0.0634
O2					
O3					
P	1	1	3	2	0.0211
P1	2	2	2	2	0.0282
P2					
P3	3	1	1	1	0.0141
P4			2	2	0.0141
P5					
Q	2	2			0.0141
Q1					
Q2					
R	6	5	2	2	0.0493
R1	1	1	2	1	0.0141
R					
R3			1	1	0.0070
S					
S1					
S2	7	3	6	3	0.0423
S3	1	1	1	1	0.0141
S4	12	4	9	4	0.0563
S5	1	1	2	2	0.0211
S6	1	1	1	1	0.0141
S7	4	3			0.0211
S8	8	2			0.0141
S9	7	3			0.0211
S10					
S11					
S12	14	7	6	4	0.0775

Figure 5.41. Out-degree, out-connection, in-degree, in-connection, and centrality information for adjacency matrix $A(\mathrm{B})_{tot}$.

5.2.1 Microgenetic Analysis via Adjacency Matrices

Within this section I present two categories of results: (a) instances of Abraham reasoning about the IMT via the negation of concept statements; and (b) Reasoning about the concepts within the IMT through various sub-digraphs. Within the first category, the examples I discuss concern Abraham reasoning about "the number zero is not an eigenvalue of *A*" and "the determinant of *A* is nonzero." The second category of results was influenced by those presented in the parallel section in Chapter 4, where I discussed ways in which investigating the various sub-digraphs of $m < n$, $m = n$, $m > n$, and any m, n illuminating rich aspects of the collective's ways of reasoning about the IMT. I focus on an argument within the $m < n$ sub-digraph that illuminates a mathematical difficulty within Abraham's reasoning.

5.2.1.1 Reasoning about the Invertible Matrix Theorem via the negation of statements. I present two examples, both from the $m = n$ sub-digraph, of Abraham reasoning about the negations of concept statements from the IMT, rather than as they appear in the IMT. Namely, he was much more likely to reason about "the number zero is an eigenvalue of *A*" and "the determinant of *A* is zero" than their respective negations. Adjacency matrix coding was crucial in determining this aspect of Abraham's ways of reasoning in that adjacency matrix $A(\text{B})_{tot}$ and the summary data in Figure 5.41 allowed me to compare the rows/columns of any given statement and its negation and consider the relative frequencies with which Abraham verbalized them in his reasoning.

5.2.1.1.1 *The number zero is not an eigenvalue of A*. The rows and columns relevant to the concept statement "the number zero is not an eigenvalue of *A*" corresponds to vertex S6, and its negation, "the number zero is an eigenvalue of *A*" corresponds to vertex S7. As stated above, this pair was chosen for analysis based on the low occurrence with which vertex S6 was needed to capture Abraham's reasoning, with a comparatively higher frequency of vertex S7's use (see Figure 5.42).

	OD	OC		ID	IC
S6. The number zero is not an eigenvalue of A	1	1		1	1
S7. The number zero is an eigenvalue of A	4	3			

Figure 5.42. The out-degree, out-connection, in-degree, and in-connection for vertices S6 and S7.

All of the adjacencies that appear within Figure 5.42 occurred as Abraham was responding to one particular question prompt during Interview 2. As mentioned in the previous Toulmin analysis section, Abraham was asked to place cards that had the 16 different concept statements from the IMT written on them into piles based on, for him, which ones went together. One of his piles (labeled Pile 5 in Figure 5.1) contained two cards: "The number zero is not an eigenvalue of *A*" and "The null space of *A* contains only the zero vector." Part of this explanation was seen in the previous section on Microgenetic Analysis via Toulmin's Model as an example of the expanded Toulmin's scheme (see Figure 5.14). In the present section I highlight how, within this argument, he reasons about the equivalence of these two concepts statements through their negations and how he was also very aware of this tendency.

The majority of Abraham's explanation for placing the two cards "The number zero is not an eigenvalue of *A*" and "The null space of *A* contains only the zero vector" together is below. The adjacency matrix coding is given within the transcript as well.

Abraham: These [the two cards in Pile 5] **(S4→S6, S6→S4)**, I'd have to write, it's only because I'm thinking of an eigenvalue's definition something's, those are nonzero, $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. And so if the number zero, I always like to think, this [points to card] says "not." But I always like to think of if it *is* [meaning 'is' an eigenvalue instead of 'is not']. So then if it *is*, if $\lambda = 0$, then $A\mathbf{x} = \mathbf{0}$. And then by the definition of an eigenvector, we can find a nonzero solution **(S7→F1)**. Because if it's zero, it's not very interesting.

Interviewer: What is the 'it,' if $\mathbf{x}$ is zero or? You said 'if it's zero, it's not very interesting,' what's the 'it'?

Abraham: Oh, sorry, it's the vector $\mathbf{x}$, because then *A* times zero is always going to be zero. So that would make it linear independent. **(E1→E)** But by the definition, we're trying to find a *nonzero* vector such that this eigenvalue stretches it. So then if $\lambda = 0$, then we can find a non-, this is saying by the definition, we can find a nonzero solution, such that $A\mathbf{x} = \mathbf{0}$ **(S7→F1)**. A non-0 solution. Therefore, I like that symbol, if we can find a nonzero solution, therefore then it's, I'm going out of context. But this is, it's linearly dependent, **(F1→F)** by definition, because it's a non-trivial or the only solutions are the trivial solution. There's a nonzero solution, so it's linearly dependent if it's zero **(S7→F)**.

First, note Abraham's reflection on his own reasoning practice. He stated early in his response, "I always like to think, this [points to the eigenvalue card] says 'not.' But I always like to think of if it *is* [meaning 'is' an eigenvalue instead of 'is not']." He, in essence, told the interviewer, in the language of the adjacency matrix coding, that he preferred to reason about S7 more than about S6. The code S6, which corresponds to the way the concept statement actually appears in the IMT, only appeared implicitly in Abraham's argumentation, by way of the fact that he put "the number zero is not an eigenvalue of *A*" and "the null space of *A* contains only the zero vector" in a pile together (coded S6→S4, S4→ S6). It does not appear again within Abraham's argumentation.

He then stated if zero was an eigenvalue for *A*, then the matrix equation *A*x = **0** would have nonzero solutions: this adjacency, (S7, F1), appeared twice. Having nonzero solutions to *A***x** = **0** led him to conclude that the columns of *A* were linearly dependent (F1→F), and finally that if zero was an eigenvalue of *A* then its columns had to be linearly dependent, shown in the (S7, F) adjacency above. Next in his explanation, Abraham shared how what he had just said connected to null space for him.

Abraham: And how does this relate to null space for me? Because this is saying the number zero. Then I think of this because if there's a nonzero solution here, then the null space doesn't contain only the zero vector **(F1→S5)**. So I think I think of them together, if I put a negation in front of them. Because then if the number zero is eigenvalue, then the null space of *A* does not contain only the zero vector, **(S7→S5)** the null space contains, the null space is part of the domain, so it contains all the solutions sent to zero **(→S12a)**...In terms of **x**, usually, right? In terms of the **x** vector, not, yeah, in terms of **x**, I don't know if that's enough, if that makes?

Interviewer: ...What is it that you mean by 'in terms of **x**?'

Abraham: Just when you write it out, you're really thinking of all the vectors sent to zero. So that's what I'm saying in terms of **x**, the vectors that are sent to zero. Because the null space is really the collection of vectors that are sent to zero. And if it's only the zero vector, then only the zero vector, the **x** only, **x** bar equals zero is sent to zero **(S4→S12a, S12a→S4)**.

In this second portion of transcript, Abraham connected “the null space of *A* contains only the zero vector” concept statement from the IMT to his previous explanation by discussing its negation. He began, “And how does this relate to null space for me? Because if there's a nonzero solution here, then the null space doesn't contain only the zero vector.” This was coded (F1, S5). Next, note Abraham’s reflective statement, “So I think I think of them together, if I put a negation in front of them.” He was explicit and aware that he was explaining the two concepts’ equivalence in terms of their negations; in other words, by discussing S7→S5 rather than S6→S4. Finally, Abraham concluded his explanation by interpreting the concept of null space in terms of the vectors that get sent to zero. He explained that if the null space of *A* only contained the zero vector, that was the same thing as saying only the **x** vector got sent to zero; this was coded (S4→S12a, S12a, S4). Thus, Abraham reasoned about the negation of “the number zero is not an eigenvalue of *A*” and “the null space contains only the zero vector” in order to justify their equivalence.

5.2.1.1.2 ***The determinant of A is nonzero***. The second example of Abraham preferring to reason about the negation of a concept statement rather than the way it is stated in the IMT regards determinants. Within the adjacency matrix $A(\mathrm{B})_{tot}$, one can see there are relatively few statements about determinants (see Figures 5.38, 5.40, amd 5.41). As was true with eigenvalues, determinants were developed late in the semester. Furthermore, Interview 1 did not contain any questions about determinants (it had not been discussed in class yet), and Interview 2 only had one main question dedicated to it. Thus, I am not claiming that because reasoning about determinants had a low frequency, it wasn’t powerful to Abraham. The nature of the course (it being nearly the last concept developed) and the pre-determined interview questions do not lend themselves to such claims being made. Rather, what is noteworthy is Abraham’s preference for reasoning about “the determinant of *A* is zero,” rather than the way the concept statement is wording in the Invertible Matrix Theorem, which is “the determinant of *A* is nonzero.” The former is labeled vertex R, and the latter is labeled vertex Q. Looking at $A(B)_{tot}$, we see the following information for vertices Q and R and their subcodes:

	OD	**OC**		**ID**	**IC**
Q. det(*A*) ≠ 0: Determinant of *A* is nonzero	2	2			
Q1. *Area:* Unit square/cube has area/volume not equal to zero after the transformation					
Q2. *Calculation*: formula to calculate the determinant yields a nonzero result					
R. det(*A*) = 0: Determinant of *A* is equal to 0	6	5		2	2
R1. *Area:* Unit square/cube has area/volume equal to zero after the transformation	1	1		2	1
R2. *Calculation*: formula to calculate the determinant yields a result of zero					
R3. *Measure:* No area/volume to a line/plane				1	1

Figure 5.43. The out-degree, out-connection, in-degree, and in-connection for vertices Q-Q2 and R-R3.

Note that nearly all statements related to the concept of determinant fall within the R category: “the determinant of *A* is zero” and its interpretations. On Day 24, vertex R was adjacent from three different vertices: “F5: One vector is a scalar multiple of another,” “F8: the matrix *A* has a row or column of zeroes,” and “F: the columns of *A* are linearly dependent.” These occurred during small group work during which Abraham and his group members discussed what they had done for homework to investigate what how the column vectors of a transformation matrix being linearly dependent would affect the determinant of the matrix.

While the question prompt on Day 24 set up the students for discussing “det *A* = 0” rather than “det *A* ≠ 0,” the question prompt during Interview 2 left that choice up to Abraham. The prompt was, “For the following pairs of statements from the Invertible Matrix Theorem, please explain how you understand how they are equivalent: det *A* ≠ 0,” and “The columns of *A* are linearly independent.” The transcript, written work, and adjacency matrix coding for

Abraham's response is given below (these are the arguments coded Int 2 Q4b Arguments 1-4 in Appendix 5.1).

Abraham: Oh, I think I know how to do this one, or how I think about it, I should say. Determinant of *A*. I think of the determinant as. Uh-oh, I want to think of the negation again...This is an equivalent statement with this [the 'linear independence' card]. So since it is, I'll use my negations. Negation of the determinant of *A* does not equal zero, then the determinant of *A* is zero [wrote "~ det $A \neq 0$ → det $A = 0$" on his paper]. Now if the determinant of *A* is zero, then the area, I think of it as the area of the transformation **(R→R1)**. So I'm going to try to make some kind of drawing...let's just say that's some object, right, in $\mathbf{R}^2$. It's a door in $\mathbf{R}^2$, here's a little knob [draws the leftmost image seen in Figure 5.44]. No, but I think of the determinant as when you transform this by *A*, then the determinant [sic, object] after the transformation will have zero area **(R→R1)**. So the door is unfortunately not a door any more, it has, it's just a line [draws the rightmost image in Figure 5.44] **(→N4)**. So the area, but if I extend it to more dimensions, it's not always going to be a line, just something with zero area. It could be, if I think of some plane with area or volume or however you think of it, depending on dimension, I'll say volume or something, right? Then it could be smushed together or something and have zero volume **(R→R3)**, or something like that, you know.

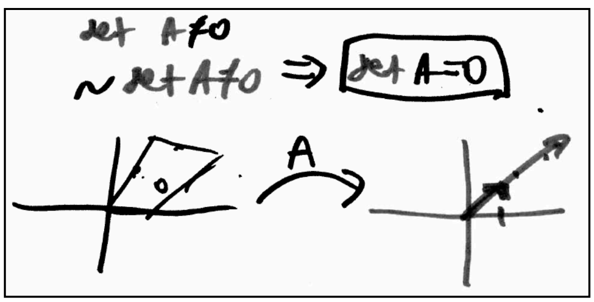


Figure 5.44. Abraham's written work while explaining the equivalence of "the determinant of *A* is zero" and "the columns of *A* are linearly dependent."

First, note his comment, "Uh-oh, I want to think of the negation again." As was true in the previous eigenvalue response, Abraham explicitly stated that he was aware he preferred to reason about the negation of the given concept statement. He stated he was permitted to do so because the statements, as said in the prompt, were equivalent. There may be a bit of circularity here, in that he is adopting a certain justification style based on his knowledge that the statements are in fact equivalent. Abraham's overall explanation is summarized with the codes: R→R1, R→R1→N4, and R→R3. Both the R1 and the R3 code relate to notions of area or volume, the former notating the area/volume of the unit square/cube after the transformation, and the latter noting a measurement-oriented idea in that lines/planes have no area/volume. Recall that the N4 code is “The transformation collapses everything to a point/line/plane.” Thus, Abraham’s explanation thus far focused on how he understood determinants in terms of geometric objects and area or volume (such as the door having area before the transformation but none after), but he had not yet related that to linear independence, as was requested in the prompt. The interviewer asked him to clarify whether his explanation was about a zero or nonzero determinant, which also led him to talk about linear dependence:

Interviewer: Is that what, having zero area, relates to which one of these, the equal to zero or not equal to zero?

Abraham: The determinant of *A* equals zero. So then the determinant of *A* equals zero, what am I try to say? Then [points to the prompt on the paper] the columns of *A* are linearly dependent **(R→F)**. Because, if, or 'show how they are equivalent,' I don't have to get to. I just know that from experience that if I plug in different points on this door or whatever, that I'm going to have multiple solutions to a given point. **(R→P1)** Let's think of vectors, to a given vector. Say that's <1, 1> or something; I might have two points on this door that transform to <1, 1> **(→P4)**. And if it was linearly independent, I would only have one point on this door that would go to one point over here. **(E→O1)** I guess that's one-to-one, so that'd be linearly independent. **(→O→E)** So if I start with determinant *A* equals zero, I'm going to have multiple points over here that are sent to the same point, so therefore it's linearly dependent **(R→P3→F)**. So then if the determinant of *A* does not equal zero, then the determinant, it's linearly independent, for the opposite reason. **(Q→E)**

Within this last section of transcript, Abraham first stated the implication (R, F): “if the determinant of *A* equals zero, then the columns of *A* are linearly dependent.” This is consistent with his comment that he wanted “want to think of the negation again.” However, his next statement stepped away from the notion and linear dependence in favor of revisiting his “door” explanation. This is noted through the coding R →P1→P4, where P1 is the definition of one-to-one, “For every **b** there is more than one **x** such that $T(\mathbf{x}) = \mathbf{b}$, “ and P4 is “At least two input vectors give the same output.” He transitioned to connecting one-to-one and linear independence with the statement, “I might have two points on this door that transform to <1, 1>.

And if it was linearly independent, I would only have one point on this door that would go to one point over here. I guess that's one-to-one, so that'd be linearly independent," the second sentence of which is captured by the string of codes E→O1→O→E. Finally, Abraham summarized his explanation, stating that a determinant of zero (code R) sends multiple points to the same point (code P3), which means the column vectors of the transformation matrix are linearly dependent (code F). He added on, "So then if the determinant of *A* does not equal zero...it's linearly independent, for the opposite reason." He was not asked to explain what he meant by "for the opposite reason," so it was not coded any further beyond (Q, E) as to avoid unnecessary inference.

5.2.1.1.3 *Conclusion*. Within this section, I presented two examples of concept statements for which Abraham was more prone to reason about the negation of the concept statement than how it is stated in the Invertible Matrix Theorem. Due to the lateness within the semester of when both ideas—determinants and eigenvalues—were presented, the frequency for all vertices within those two categories was low; however, what does exist demonstrates Abraham's preference for reasoning about the negations of the statements. Through this analysis, aspects of both structure and content of Abraham's ways of reasoning were investigated. That is, the adjacency matrix analysis aided me in determining which particular concepts Abraham reasoned about through the use of their negations. It then allowed me to locate those arguments and, through microgenetic analysis, determine the exact ways in which Abraham made use of negation within that argumentation. Furthermore, it simultaneously shed light on Abraham's understanding of the mathematical ideas about which he was reasoning. For instance, after Abraham concluded his explanation about determinants and linear independence (described above), the interviewer asked him if thinking of determinant in terms of area was the most salient for him, to which he replied yes.

Interviewer: Can you explain the way the determinant, do you always think about it in terms of area, is that what you said?

Abraham: I do, actually. Have I thought about it for anything else? I might have at one point, I don't think so though, I think I've always thought of it as the area after the transformation, like what is the area? If I start with some area, what's the area after I transform that?

As a final note, by examining adjacency matrix $A(\mathrm{B})_{tot}$ in Figures 5.37-40, one could determine how the use of other concept statements compared to that of their negations as well. In some instances, the opposite was true: Abraham reasoned about the concept statement's category more than that of the negation (e.g., there were more K codes than L, more I than J, and more M and N, which relate to invertibility, pivots, and onto transformations, respectively). One could also compare particular interpretations of concept statements. For instance, Abraham reasoned with "F3: Can get back home/back to a point with column vectors of *A*," and "G2: Can use vectors to get to every pt/go everywhere" much more frequently than their respective negations. I return to these two particular interpretations, F3 and G2, in greater detail in a subsequent section of the ontogenetic analysis regarding their commonality in phrasing in "get back home" and "get everywhere."

5.2.1.2 Reasoning about the concepts within the IMT through various sub-digraphs. Within the data set of the present study regarding Abraham's ways of reasoning about the IMT over the course of the semester, very few data points exist for the sub-digraphs of when $m < n$, $m > n$, or *any* m, n. (Recall the sizes of the sub-digraphs were $A(\mathrm{B})_{m<n} = 33$, $A(\mathrm{B})_{m=n} = 318$, $A(\mathrm{B})_{m>n} = 17$ $A(\mathrm{B})_{any} = 20$, out of 388 total arcs.) In Figure 5.45, I present a condensed version of $A(\mathrm{B})_{m<n}$. Upon inspection of $A(\mathrm{B})_{tot}$, I noticed only a few of his utterances were coded F9, "Have an extra vector needed in order to return home," an interpretation of linear independence that was rather prominent within whole class discussion (see the CMP 1 results section in Chapter 4). Furthermore, one such adjacency was (F9, E3), which is not mathematically correct. In other words, the adjacency matrix coding helped to illuminate not

only concept statements or interpretations that Abraham reasoned with infrequently, but it also helped to illuminate mathematically imprecise implications that he made.

	E	E1	E2	E3	E4	F	F1	F2	F3	F9	G	G1	G2	H	H5	H6	I	I1	J5	J6	M	M1	N	N1	P	P1	P2	P3	S8	S9
E																														
E1	1																													
E2																														
E3																														
E4																														
F																									2					
F1						2																								
F2																														
F3										1																				
F9				1																										
G						3							2																	
G1											1																			
G2									1	1																				
H															1															
H5																							1							
H6																														
I											1																			
I1											1																			
J5											1																			
J6																														
M																									1					
M1																														
N																														
N1																														
P																					1									
P1						1																								
P2																														
P3						2																								
S8						4 2 1					1																			
S9																														

Figure 5.45. A condensed version of adjacency matrix $A(\mathrm{B})_{m<n}$.

The green "1" within the (F9, E3) cell of $A(\mathrm{B})_{m<n}$ in Figure 5.45 corresponds to an argument from Interview 1. Upon inspection of the transcript, this occurred during Interview 1, Q8a Arg 5. In question 8a, Abraham was asked to create, if possible, a 3x5 matrix whose columns spanned $\mathbf{R}^3$ but were linearly independent. This particular question was investigated previously, in the section dedicated to exploring Abraham's ways of reasoning about span and linear independence in combination (see Figures 5.20-5.22). Below is the relevant portion of Abraham's explanation of his claim that "if you have more vectors than dimensions, that automatically makes it linearly dependent."

Abraham: This is weird to explain. I got here with three vectors, now since I can span all of $\mathbf{R}^3$, then we know I can get to this point, **(G→G2)** because this spans all of $\mathbf{R}^3$, so I know I can get to this point. **(G→G2)** So what makes me say I can't go from here to there and go back that way...So if I can get three vectors there,

> then I'm going to be able to get back there, using one of the other two vectors that are here **(G2→F9).** But I'm still kind of, I want to be convinced that that's true, but at the same time, I don't know if I thought about it enough, because for some reason it just seems that you'd be able to, it's probably because I can't picture 3-D so well, but it seems like I should be able to make two vectors that don't get back here, **(F9→E3)** but see, that's a problem. But I know by the definition that if they have more vectors than entries, it has to be dependent, **(S8→F)** so that convinces me. ...But in my mind, it seems like you should be able to.

Within this argumentation, the adjacency (F9, E3) occurred as Abraham was explaining to the interviewer that he was uncertain about the validity of the aforementioned claim. He referenced Justin's way of explaining it (that, if there were more vectors than dimensions and the first n spanned $\mathbf{R}^n$, then there would always exist a linear combination of those first n that would "let you return home" from the location of the remaining vectors, thus making the set linearly dependent). Here, however, Abraham stated that he thought it seemed like you should be able to have a vector for which this "getting back" was not possible. He concluded by stating he *knew* that a set with "more vectors than entries" had to be linearly dependent, coded (S8, F), but that it seemed like, to him, that it should not be true.

While the point of this microgenetic analysis is to investigate both Abraham's mathematical reasoning about the IMT at this point in time, as well as how the analytical tool of adjacency matrices aided in illuminating this, it is worth mentioning that, in general, reasoning about the notion of linear independence as an inability to "get back home" with the vectors within a given set seemed to not be a very powerful metaphor in general for Abraham. The adjacency matrix codes that describe this way of reasoning or its negation—E3, E8, F3, and F9—had a relatively low frequency for Abraham when considering the semester as a whole (see Figure 5.41). This is in stark opposition to Abraham's way of reasoning about span as the places you can get with a given set of vectors, both of which originated from the Magic Carpet Ride problem within the first two weeks of class. The prevalence of "getting everywhere" within

Abraham's ways of reasoning is the subject of a subsequent analysis within the forthcoming section on Ontogenetic Analysis.

5.2.2 Ontogenetic Analysis via Adjacency Matrices

In this final major section of the chapter, I present results regarding Abraham's ways of reasoning about the IMT throughout the semester in three main categories: (a) the centrality measure and various implications, (b) prevalence of codes within a common theme; and (c) comparison of arguments related to identical prompts at two different points in time during the semester. Rather than repeat the parallel results at the individual level that were presented in Chapter 4 at the collective level, the three results I present here shed a new layer of insight into the utility of adjacency matrices as an analytic tool.

5.2.2.1 The centrality measure and various implications. While the previous section discussed ways of reasoning that *were not* prevalent within Abraham's argumentation, the present section presents results for those ways of reasoning that *were* most prevalent within Abraham's argumentation. In Figure 5.41, I provided the summary data regarding the out-degree/connection, in-degree/connection, and centrality for every vertex. The most central vertex, highlighted with a red frame, was G; the second highest, indicated with an orange frame, was E; the third (shown with yellow) was M, and the fourth most central (shown with green) were E2 and F.

Taking a closer look, the most noticeable aspect was the overwhelming frequency of code G: the columns of A span $\mathbf{R}^n$. It by far had the highest measure of centrality, and (od G) = 59, (oc G) = 17, i(d G) = 47, and (ic G) = 19. This means that "the columns of A span $\mathbf{R}^n$" was adjacent to seventeen different vertices with 59 different arcs, which is approximately 15% of the total arcs. So one vertex, vertex G, out of 100 different possible vertices, was the "if" in the "if-then" structure of 15% of all such pairs. To be fair, some of the questions were worded "suppose the column vectors of matrix *A* span. T or F...." Questions like this undoubtedly boost the score in span's favor. In addition to having many questions specifically asked about span, it also was a concept developed early in the semester, thus providing a lengthy time frame within which to reason about the concept statement as well as use it to reason about new concepts. Despite these aspects, the overall prevalence of the notion of span within Abraham's argumentation is still noteworthy. Tiberghien and Malkoun (2009) discussed the notions of density (the number of key words or utterances in relation to the duration of a theme or sequence) and continuity (the distribution of key words or utterances that are most reused over the duration of a theme or sequence) as ways to investigate mathematical development. The adjacency matrix coding, coupled with the centrality measure, brought to light that reasoning about and with span was both a dense and continuous theme for Abraham throughout the semester.

5.2.2.2 Prevalence of codes within a common theme. The codes G2, G4, H2, H3, O3, and P3 (listed in Figure 5.46) are each worded with language consistent with the travelling metaphor introduced in the Magic Carpet Ride problem in that they each highlight some aspect of "getting to" a certain location. Whereas the previous section (regarding Abraham's reasoning in situations such that $m < n$) revealed that the linear (in)dependence thread of that travel metaphor was not prominent throughout Abraham's reasoning, the current section concludes that for the concept of span, the travel metaphor of the places you can get to with a given set of vectors did resonate with Abraham.

For the six codes listed in Figure 5.46, He used at least one of them throughout the semester, on Days 9, 10, 17, and 31, as well as during both Interview 1 and Interview 2. They also occurred within the $m < n$, $m = n$, and the $m > n$ sub-digraphs. Both of these results can be determined through examining the adjacency matrix $A(\mathrm{B})_{tot}$ and the adjacency matrices for the various sub-digraphs. This continuity (Tiberghien & Malkoun, 2009) is somewhat distinct from other prominent concept themes (such as reasoning about solutions, as discussed in the previous Toulmin analysis section) because none of the codes in Figure 5.46 are actually the way any of the concept statements are worded in the IMT. In other words, it speaks to the prominence of the given theme of interpretations—that of "getting places"—within Abraham's ways of reasoning throughout the semester and throughout the various *m*, *n* scenarios. Furthermore, the G2 code, which is itself not a concept statement within the IMT, had a rather high centrality score: the second highest for span (behind G5) and within the top ten percent of the most central vertices overall (see Figure 5.41).

	OD	**OC**		**ID**	**IC**
G2. *Geometric:* Can use vectors to get to every point/go everywhere	7	6		13	7
G4. *Direction:* Can use each vector to go in a certain direction	1	1		4	2
H2. Geometric: Can't use vectors to get to every point/go everywhere in dim	1	1		2	2
H3. *Direction*: Cannot go in all directions with the vectors	3	3		4	2
O3. *Reachable*: There is only one way to "get to" the output/vector	0	0		0	0
P3. *Reachable*: There is more than one way to "get to" the output/vector	3	1		1	1

Figure 5.46. Out-degree, out-connection, in-degree, and in-connection information for the "getting places" themed codes.

The in-degree for code G2 alone was 13, meaning it was adjacent from another vertex 13 times; furthermore, the in-connection of G2 in Figure 5.46 shows it was adjacent from seven different vertices. For instance, within Question 3 of Interview 2, Abraham relied on travel language as he explained his inability to discern the different between "the columns of *A* span $\mathbf{R}^n$" and "the column space of *A* is all of $\mathbf{R}^n$":

Abraham: Then this one, column space of *A* is $\mathbf{R}^n$. Which is, if all the columns of *A* span all of $\mathbf{R}^n$. **(S2→G, G→S2)** So this one I don't know how to decipher, really. Because when I think of the column space, I really literally think of the space that the columns can get to **(S2→G2, G2→S2)**. And this is talking about the columns of *A* spanning. So I can't really discern the difference of these ones **(S2→G, G→S2).** Because like I said, this is to me where the columns can get to [points to the 'span' card] in this fashion. **(G2→G, G→G2)** And this one [points to the 'column space' card] is the space that the columns can get to. Which for me is like the same thing.

5.2.2.3 Comparison of arguments related to identical prompts at two different points in time during the semester. In the previous Ontogenetic Analysis via Toulmin's Model section, I reported on an interview questions that Abraham was asked at two different points in the semester, one midway through and one after the semester ended. The question prompt was: "True or False**:** If the columns of a 3x3 matrix *A* span $\mathbf{R}^3$, then the columns vectors of *A* are linearly dependent."

	E	E1	E2	E3	E4	E5	E6	E7	E8	G	G1	G2	G3	G4	G5	I	I1	I2	I3	I4	I5	I6
E. Column vectors of *A* are linearly independent		2																				
E1. *Trivial:* Only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution			1																			
E2. *Unique:* There is a unique soln to matrix eqn/system of eqns	2																					
E3. *Travel*: Can't get back (home)/get to origin with column vectors of *A*																						
E4. *Geometric*: Vectors are not parallel/on the same line or plane																						
E5. *Proportional*: No vector is a scalar multiple of another																						
E6. *Linear combination*: No vector is a linear combo of another																						
E7. *Placement*: No vector is in the span of the other vectors																						
E8. *Extra*: Do not have an extra vector needed in order to return home																						
G. Column vectors of *A* span $\mathbf{R}^n$	3 1															1 1						
G1. *Size:* Are enough vectors to span the entire space																						
G2. *Geometric*: Can use vectors to get to every pt/go everywhere																						
G3. *Algebraic:* Is a linear combination of vectors for all pts in $\mathbf{R}^n$																						
G4. *Direction:* Can use each vector to go in a certain direction																						
G5. *Solution*: There is a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$																						
I. Row-reduced echelon form of *A* has *n* pivots																			1			
I1. *Diagonal*: RREF(*A*) has all ones on the main diagonal																						
I2. *Identity:* Can row-reduce / is row equivalent to the identity																						
I3. *Pivot-R:* Is a pivot in each row																				1		
I4. *Pivot-C:* Is a pivot in each column			1																			
I5. *Free Variable:* Each variable is defined in system/matrix eqn																						
I6. *Zeroes:* RREF(*A*) has no rows of zeroes																						

Red: Interview 1, Q61, Argument 1-2 *Blue:* Interview 2, Q1a Argument

Figure 5.47. Adjacency matrix coding for Abraham's two responses to the interview question: "True or False**:** If the columns of a 3x3 matrix A span R3, then the columns vectors of A are linearly dependent."

In Figure 5.47, the red entries mark Abraham's response during Interview 1. The adjacency (G, E) occurred three times, presumably once when he stated the statement was false (a set of vectors not being linearly dependent was coded as them being linearly independent). The only other vertex adjacent from G was vertex I: the row-reduced echelon form of *A* has *n* pivots. However, he did not continue that chain of implication, which can be seen by noting that the out-degree of row I is zero. Abraham's response also included (E, E1) twice and (E1, E2) once. Thus, the red entries in Figure 5.47 illustrate that Abraham's explanation for span → linearly independence (a) repeated the claim three times, said one implication (G to I) but did not continue from I to other vertices, and never completed a discernible chain of reasoning from the given concept statement G to the implied (concept statement E).

In the second interview, however, his explanation was structured differently. The blue entries in Figure 5.45 mark his response during Interview 2. He stated the implication (G, E) only once, presumably as he stated the question prompt was a false statement. Next, examining the other entry in the G row, one sees it is adjacent to I. Indeed, an entire chain of reasoning can be followed by tracing that string of blue entries within Figure 5.45, namely: G→I→I3→I4→E2→E. From only reading the concept statements or interpretations associated with this codes, this chain would read: "the columns of A span all of $\mathbf{R}^n$→the RREF(A) has n pivots→there is a pivot in each row→there is a pivot in each column→there is a unique solution to $A\mathbf{x} = \mathbf{b}$ for every →the columns of A are linearly independent." If one considers the transcript of this argument, provided in the Toulmin analysis of this argument (see Table 5.2 or Figure 5.33), it can be seen that the above translation from this chain of adjacency matrix codes very nearly matches Abraham's given explanation.

The results from investigating and comparing the adjacency matrix codes for these two arguments are twofold. First, it revealed a marked shift in Abraham's ways of reasoning about why three spanning vectors in $\mathbf{R}^3$ must also be linearly independent. Within the first interview, he had no discernible chain of reasoning between the two concept statements, but rather repeated the claim a few times. He also only stated one other concept statement, which he did not pursue. Within the second argument, Abraham provided a mathematically valid chain of reasoning that utilized the concept of pivot in a relatively sophisticated way in order to justify the claim. Second, comparing the adjacency matrix coding of these two arguments demonstrates the power and validity of adjacency matrices as an analytical tool for investigating ways in which students reason about various mathematical concepts and relationships over time. Finally, a comparison of the strengths and affordances of the two analytical tools (Toulmin's Model and adjacency matrices) to analyze this particular pair of arguments is presented in the chapter's conclusion.

5.3 Conclusion

Utilizing adjacency matrices as an analytic tool on Abraham's ways of reasoning throughout the semester revealed a variety of noteworthy results. Within the microgenetic analysis section, I reported on Abraham's tendency to reason about the negation of concept statements from the IMT in order to justify their equivalence. I determined this result by examining the out-degree/connection and in-degree/connection information for every vertex in the adjacency matrix associated with his set of argumentations. The concept statements of "the determinant of A is zero" and "the number zero is an eigenvalue of A," which are the negation of how the ideas are stated in the IMT, each scored higher in the four aforementioned measures then their respective counterpoints. I also presented one example of how the adjacency matrix for the $m < n$ sub-digraph, by revealing a mathematically incorrect adjacent pair of vertices, indicated a noteworthy argument within which Abraham struggled to develop a way of reasoning about a generalization he knew to be true but that "went against his intuition."

When compared to a potential Toulmin analysis, reasoning about the negations would not appear so cleanly. However, Abraham's reflective statements regarding his own ways of understanding the equivalence is lost when using adjacency matrix analysis alone. For instance, there was no way to code that something "went against his intuition" within the adjacency matrix analysis.

For ontogenetic analysis, I focused on three themes. First, the centrality measure showed that code G: the columns of A span $\mathbf{R}^n$," was by far the most central concept statement within Abraham's ways of reasoning. Despite having many questions specifically asked about span and also being a concept developed early in the semester, the overall prevalence of the notion of span within Abraham's argumentation is still noteworthy. Tiberghien and Malkoun (2009) discussed the notions of density and continuity as ways to investigate mathematical development. The adjacency matrix coding, coupled with the centrality measure, brought to light that reasoning about and with span was both a dense and continuous theme for Abraham throughout the semester.

Second, I utilized the total adjacency matrix $A(B)_{tot}$ to investigate vertices bound together by a common theme. In particular, I took note of all concept statements or interpretations worded with language consistent with "getting to" a certain location. This revealed that the interpretation of span as the places you can get to resonated with Abraham. In particular, code G2 ("you can get everywhere with the columns of A") was in the top ten percent of the most central vertices overall. This is noteworthy because this was not actually a concept statement within the IMT, yet it was very prominent within Abraham's ways of reasoning.

Finally, I compared the adjacency matrix coding for two arguments from an identical interview question. This revealed a marked shift in Abraham's ways of reasoning about why three spanning vectors in $\mathbf{R}^3$ must also be linearly independent. Within the first interview, he had no discernible chain of reasoning between the two concept statements, whereas within the second he was able to provide a mathematically valid chain of reasoning to justify the claim. This comparison added support to the power and validity of adjacency matrices as an analytical tool for investigating ways in which students reason about various mathematical concepts and relationships over time.

This chapter presented a variety of results concerning Abraham's ways of reasoning throughout the semester about the Invertible Matrix Theorem. Each analytical tool contributed a glimpse into understanding Abraham that the other did not. For instance, Toulmin's model allowed me a way to capture Abraham's reflection on his own learning in a way that adjacency matrices did not. The example captured in Figures 5.28, when within his portfolio response he explained how he had thought about span in the past and what he had learned (e.g., "I had no clue," "it seemed logical," etc.), had no way of being coded within adjacency matrix analysis. On the other hand, adjacency matrix analysis had powerful implication by providing easy to find holistic, macrogenetic information regarding with ideas were the most central and for Abraham throughout the entire semester.

CHAPTER SIX: CONCLUSION

In this study, I considered the development of mathematical meaning for both the classroom community and an individual by analyzing student reasoning about the Invertible Matrix Theorem over the course of the semester. To do so, I coordinated results from two tools in discourse analysis, Toulmin's model and adjacency matrices, and utilized the cultural change notions of microgenesis and ontogenesis as strands through which to analyze change over time at both the individual and collective levels. This work addressed the following two overarching research questions:

1. How did the collective classroom community reason about the Invertible Matrix Theorem over time?
2. How did an individual student, Abraham, reason about the Invertible Matrix Theorem over time?

This work drew on the perspective that mathematics is a human activity (Freudenthal, 1991) and was undergirded theoretically by the emergent perspective (Cobb & Yackel, 1996). This perspective on learning coordinates psychological constructivism and interactionism. From the position that learning is both an individual and a social process, within neither given primacy over the other, investigating mathematical development involves considering the individual's development as well as the collective activity and progression of the community in which the individual learner participates. The data for this study came from the third iteration of a classroom teaching experiment (Cobb, 2000) in an inquiry-oriented introductory linear algebra class (Rasmussen & Kwon, 2007).

My work was influenced methodologically by other research that investigates the emergence, development, and spread of ideas in a classroom community over time (e.g., Cobb, Stephan, McClain, & Gravemeijer, 2001; Cole et al., 2011; Rasmussen & Stephan, 2008; Saxe, 2002; Saxe et al., 2009; Tiberghien & Malkoun, 2009). The emergence and development of ways of reasoning about the Invertible Matrix Theorem occurred not only for the classroom as a collective entity but also for each individual student. These two forms of knowledge genesis—on an individual and on a collective level—are inextricably bound together in their respective development. This dissertation lays a substantial foundation towards building theory and understanding that individual and collective-level development with respect to the Invertible Matrix Theorem, as well as towards laying a foundation methodologically for the coordination of those developments.

In Chapter 3, I described a 6-Phase analysis through which I pursued my research questions. Table 6.1 illustrates how Phases 2-6 were actualized within Chapters 4, 5, and 6. After the first phase of data reduction, during Phases 2 and 3 I conducted microgenetic analysis through Toulmin's Model and adjacency matrices, respectively. Phases 4 and 5 compiled the results of that microgenetic analysis for analysis at the ontogenetic level though Toulmin's Model and adjacency matrices, respectively. The focus of the present chapter is Phase 6, a comparison of the results from the two analytical tools across both levels of analysis.

Table 6.1. Coordination between phases of analysis and organization of Chapters 4-6.

	Chapter Four: Collective	**Chapter Five: Individual**	**Chapter Six: Conclusion**
Phase 2: Microgenetic Analysis via Toulmin's Model	Section 4.1.1	Section 5.1.1	
Phase 4: Ontogenetic Analysis via Toulmin's Model	Section 4.1.2	Section 5.1.2	
Phase 3: Microgenetic Analysis via Adjacency Matrices	Section 4.2.1	Section 5.2.1	
Phase 5: Ontogenetic Analysis via Adjacency Matrices	Section 4.2.2	Section 5.2.2	
Phase 6: Comparison of results from the two analytical tools across both levels of analysis	Section 4.3	Section 5.3	Section 6.2

After presenting a summary of the results from Chapters 4 and 5, I compare results across the two analytical tools and discuss various aspects of their strengths and limitations. The chapter concludes with implications for teaching and potential avenues for future research.

6.1 Summary of Results

6.1.1 Chapter Four: Analysis at the Classroom Level

Within Chapter 4, I presented results concerning the classroom community's ways of reasoning about the Invertible Matrix Theorem over the course of the semester. Insights into both mathematical development via normative ways of reasoning and structure of argumentation that were revealed from both analytical tools are subsequently summarized.

6.1.1.1 Results from Toulmin's Model of Argumentation. I defined four types of argumentation schemes that are an expanded version of Toulmin's model. These expanded structures—Embedded, Proof by Cases, Linked, and Sequential (see Figure 6.1)—developed out of necessity when the original 6-part Toulmin's scheme proved inadequate while analyzing argumentation of the classroom community. Aspects of these four expanded structures were adapted from and are compatible with those presented by Aberdein (2006, 2009).

Expanded Toulmin Scheme Structures

1. *Embedded structure*: When data or warrants for a specific claim so complex, they had minor embedded arguments within them
2. *Proof by Cases structure*: When claims were justified using cases within the data and/or warrants
3. *Linked structure:* When data or warrants for a specific claim had more than one aspect that were linked by words such as "and" or "also"
4. *Sequential structure*: When data for a specific claim contained an embedded string of if-then statements, where a claim became data for the next claim

Figure 6.1. Definitions of the expanded Toulmin scheme structures developed in Chapter 4.

In the ontogenetic analysis via Toulmin's Model section, I presented two classroom mathematics practices: (a) Reasoning about span and linear independence as equivalent ideas for square matrices, and (b) Determining whether or not a transformation T: $\mathrm{R}^n \rightarrow \mathrm{R}^m$ is onto R^m

by considering the span of the column vectors of the associated matrix (see Figure 6.2). The first was a collection of four normative ways of reasoning that focused on span and linear independence for non-square cases (normative ways of reasoning 1.1 and 1.2), as well as, in the $m = n$ case, reasoning about linear independence implying span and vice versa (1.3 and 1.4). The second classroom mathematics practice was constituted by a collection of three normative ways of reasoning that made use of the travel language to consider onto as "getting everywhere" (2.1), established the equivalence of " the span is all $\mathbf{R}^n$" and "the transformation is onto $\mathbf{R}^n$" (2.2), and reasoned about onto transformations when $m < n$ (2.3).

Classroom Mathematics Practice #1:
Reasoning about Span and Linear Independence as Equivalent Ideas for Square Matrices

Associated Normative Ways of Reasoning:

1.5 For a given set of n vectors in $\mathbf{R}^m$, if $m < n$, the set must be linearly dependent.
1.6 For a given set of n vectors in $\mathbf{R}^m$, if $m > n$, the set cannot span all of $\mathbf{R}^m$.
1.7 If A is n x n and the columns of A span $\mathbf{R}^n$, then the columns of A are linearly independent.
1.8 If A is n x n and the columns of A are linearly independent, then the columns of A span $\mathbf{R}^n$.

Classroom Mathematics Practice #2:
Determining whether or not a transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ is onto $\mathbf{R}^m$ by considering the span of the column vectors of the associated matrix A.

Associated Normative Ways of Reasoning:

2.1 A transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ is onto $\mathbf{R}^m$ when you can "get everything" or "get everywhere" and not onto $\mathbf{R}^m$ when you cannot.
2.2 "A transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ is onto $\mathbf{R}^m$" is interchangeable with "the column vectors of the associated matrix A span all of $\mathbf{R}^m$."
2.3 For a given transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$, if $m > n$, the transformation is not onto $\mathbf{R}^m$.

Figure 6.2. The two classroom mathematics practices and associate normative ways of reasoning presented in Chapter 4.

6.1.1.2 Results from Adjacency Matrix Analysis. Within the microgenetic analysis via adjacency matrices section, I presented two categories of results regarding (a) where within a category an argumentation occurred, and (b) the type of sub-digraph in which the argumentation occurred. For (a), I reported on two main structures that existed in the data. First, when the argumentation served towards developing a way of reasoning about a new concept or connection between two concepts, the argument involved multiple uses of the "interpretation" subcodes; and (b) when the argumentation made use of relatively well-established concepts or connections between concepts, the argument involved mainly the concept statement codes.

The results of ontogenetic analysis via adjacency matrices were presented in two sections: those arguments involved in documenting the first classroom mathematics practice, and the compilation of all coded arguments. In the first section, analysis of the $m < n$ and $m = n$ sub-digraphs revealed high levels of compatibility with the Toulmin analysis for CMP 1. The results from analyzing the adjacency matrices for the compilation of all coded arguments were presented according to three categories: (a) centrality measures and implications for all *m, n* types; (b) frequencies and implications for individual cells; and (c) shifts in argumentation over the course of the semester. For (a), I reported that vertex G and vertex F had the highest and second highest centrality, respectively, in $A(\mathrm{T})_{tot}$. Furthermore, for both G and F, neither their in-connections nor out-connections were mutually exclusive across *m*, *n* types. These two analyses indicate that a high level of participation across numerous adjacent vertices and across the different *m*, *n* sub-digraphs.

The second category revealed that the adjacency matrix $A(\mathrm{T})_{tot}$ was also an informative tool regarding the participation of individual cells in the classroom discourse. It revealed which adjacency pair occurred the most often throughout the semester and which occurred on the highest number of days. These two analyses lend themselves to alignment with the notions of density and continuity of ideas, respectively (Tiberghien & Malkoun, 2009). Finally, the third category of results focused on shifts in argumentation patterns over the course of the semester. By considering the entries in the total adjacency matrix according to the color that corresponds to the day during the semester from which the entries originated, a summary of the emergence and shift of the collective's mathematical ways of reasoning were revealed.

6.1.2 Chapter Five: Analysis at the Individual Level

In Chapter 5, I presented results concerning Abraham's ways of reasoning about the Invertible Matrix Theorem over the course of the semester. Insights into both Abraham's mathematical development and broader aspects regarding structure of argumentation and methodological results are summarized here.

6.1.2.1 Results from Toulmin's Model of Argumentation. In the microgenetic analysis section, I reported that, as was the case in the collective-level analysis, the expanded Toulmin schemes introduced in Chapter 4 were sometimes necessary to best capture the structure of Abraham's argumentation. I provided examples of how three of the expanded structures—Embedded, Linked, and Sequential—were used to analyze Abraham's ways of reasoning. Within the second half of the microgenetic section, I presented results concerning how Abraham interacted with the notion of equivalence when reasoning about the IMT. I defined two terms, conceptual equivalence and logical equivalence, to describe the nature by which Abraham's argumentation described the equivalence between different pairs of concept statements from the IMT. I defined *conceptual equivalence* as the absence of differences in meaning between two or more concept statements and *logical equivalence* as the existence of logical deductions between two or more concept statements; I provided examples of both.

In the ontogenetic analysis section, I chose to focus on the shifts in form and function regarding Abraham's ways of reasoning about (a) span and linear independence in combination, and (b) solutions to matrix equations $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$. Regarding (a), I provided a descriptive overview of Abraham's ways of reasoning and two categories of selected analyses. First, I reported on shifts in composition of arguments and placement of concepts within the arguments. Specifically, the composition of Abraham's arguments shifted as he first made claims with little justification (shown through a low frequency of data and warrants) and/or expressed sentiments of uncertainty (shown in his particular qualifiers). He eventually presented well-developed justifications for these original claims. Additionally, concept placement with the arguments shifted throughout the semester, as those that Abraham first had trouble justifying later functioned as the mathematical ideas through which he justified new claims. The second category of analysis involved comparing Abraham's responses to identical question prompts at different instances during the semester in order to determine shifts in form and function. Regarding (b), I concluded the ontogenetic analysis section by presenting results concerning Abraham's prevalent use of reasoning about solutions to the matrix equations $A\mathbf{x} = \mathbf{0}$ as well as $A\mathbf{x} = \mathbf{b}$ to make and support claims**.**

6.1.2.2 Results from Adjacency Matrix Analysis. Within the microgenetic analysis section, I reported on Abraham's tendency to reason about the negation of some concept statements from the IMT in order to justify their equivalence. I determined this result by examining the out-degree/connection and in-degree/connection information for every vertex in the adjacency matrix associated with his set of argumentations. The concept statements of "the determinant of A is zero" and "the number zero is an eigenvalue of A," which are the negation of how the ideas are stated in the IMT, each scored higher in the four aforementioned measures then their respective counterpoints. I also presented one example of how the adjacency matrix for the $m < n$ sub-digraph, by revealing a mathematically incorrect adjacent pair of vertices, indicated a noteworthy argument within which Abraham struggled to develop a way of reasoning about a generalization he knew to be true but that "went against his intuition."

Within the strand of ontogenetic analysis, I chose to focus on three themes. First, the centrality measure showed that code G: "the columns of A span $\mathbf{R}^n$," was by far the most central concept statement within Abraham's ways of reasoning. Despite having many questions specifically asked about span and also being a concept developed early in the semester, the overall prevalence of the notion of span within Abraham's argumentation is still noteworthy. Tiberghien and Malkoun (2009) discussed the notions of density (the number of key words or utterances in relation to the duration of a theme or sequence) and continuity (the distribution of key words or utterances that are most reused over the duration of a theme or sequence) as ways to investigate mathematical development. The adjacency matrix coding, coupled with the centrality measure, brought to light that reasoning about and with span was both a dense and continuous theme for Abraham throughout the semester.

Second, I utilized the total adjacency matrix $A(\mathrm{B})_{tot}$ to investigate vertices bound together by a common theme. In particular, I took note of all concept statements or interpretations worded with language consistent with "getting to" a certain location; this language grew out of the class's work with the Magic Carpet Ride problem and its underlying travel metaphor. The analysis revealed that the interpretation of span as the places you can get to resonated with Abraham. In particular, code G2 ("you can get everywhere with the columns of *A*") was in the top ten percent of the most central vertices overall. This is noteworthy because this was not actually a concept statement within the IMT, yet it was very prominent within Abraham's ways of reasoning. Finally, I compared the adjacency matrix coding for two arguments from an identical interview question. This revealed a marked shift in Abraham's ways of reasoning about why three spanning vectors in $\mathbf{R}^3$ must also be linearly independent. Within the first interview, he had no discernible chain of reasoning between the two concept statements, whereas within the second he was able to provide a mathematically valid chain of reasoning to justify the claim. This comparison added support to the power and validity of adjacency matrices as an analytical tool for investigating ways in which students reason about various mathematical concepts and relationships over time.

6.1.3 Summary of Methodological Contribution and Significance

Through completing the analysis in pursuit of my two research questions, three main methodological contributions for the field of mathematics education came to the fore. First, the introduction and use of the expanded Toulmin scheme is a significant contribution to the field in that it advances the way Toulmin's Model of Argumentation is used in mathematics education research. While Toulmin's Model has been used in this field since Krummheuer (1995), the majority of research has only utilized the "core" of the argument—data, claim, and warrant. The work of Inglis, Mejia-Ramos, and Simpson (2007) argued for the use of the full 6-part Toulmin scheme, placing special emphasis on modal qualifiers, and Weber, Maher, Powell, and Lee (2008) made use of the 6-part scheme to analyze classroom-level debate. The need for the expanded Toulmin scheme in the present study may have been a function of the nature of the mathematical content at hand—that of linear algebra. An introductory linear algebra course often serves as a transitional point for students as they progress from more computationally based courses to more abstract courses that feature reasoning with formal definitions and proof construction. Thus, the complexity of proof-oriented argumentation involving formal, abstract concepts may have played a large role in why the expanded structures were needed. How may the expanded Toulmin scheme be helpful or necessary in analyzing other linear algebra data sets, or even other content domains? As such, the expanded structures may prove to be a powerful tool in developing theory and analyzing student development with regards to the discipline-specific practice of proving.

Second, the further adaptation and refinement of adjacency matrices as an analytic tool is a worthwhile pursuit for the field of mathematics education research. The work of Selinski, Rasmussen, Zandieh, and Wawro (2011) laid a strong foundation for the methodological guidelines in analyzing student thinking with this tool. That work differed from the present study in two main ways. First, Selinski et al. refined their methodology through an in-depth analysis of post-semester individual interviews with nine students. My research adapted and extended that work by using adjacency matrices on a data set that spanned the entire semester, rather than at one moment in time. Second, my current study also used adjacency matrices on the collective level of analysis. As such, it opens a door towards the possibility of gaining new insight into collective ways of reasoning and refining a new tool for this work.

Finally, the present study lays a strong foundation for a 2-fold analytical coordination. The first, coordinating results from adjacency matrix analysis with those from Toulmin's Model demonstrates that the results from both tools were often compatible, however the tools vary in their respective strengths and limitations; this coordination is discussed further in the subsequent section. Second, as is elaborated within the Avenues for Future Research section in this chapter, the results of the present study lay a foundation towards coordination between the individual and collective levels.

6.2 Discussion of the Two Analytical Tools

Within this section I summarize some of the points of compatibility and points of distinction that were revealed through using both Toulmin's model and adjacency matrices on the same data set. In what ways was adjacency matrix analysis compatible with the methodology for documenting normative ways of reasoning and classroom mathematics practices (Cole et al., 2011; Rasmussen & Stephan, 2008)? What was illuminated through Toulmin analysis but not in adjacency matrices, and vice versa? I conclude with an extended treatment regarding limitations of the adjacency matrix coding.

6.2.1 Points of compatibility

Within Chapter 4, I first documented a classroom mathematics practice for reasoning about span and linear independence as equivalent ideas in square matrices by using Toulmin scheme. I then analyzed the same set of arguments with adjacency matrices to investigate what similar and distinct results were revealed. Many aspects of the classroom's normative ways of reasoning that were established using Toulmin's model were also apparent through adjacency matrix analysis. By computing centrality for the various concept statements, those with a high degree of participation in the overall argumentation were revealed. In the case of $A(\mathrm{P}_1)_{m=n}$, concept statement G ("the columns of A span $\mathbf{R}^n$") had the highest measure of centrality. I then constructed a digraph of these connections in order to investigate how the use of "the columns of A span $\mathbf{R}^n$" shifted over the time. Second, within the $m = n$ sub-digraph for CMP 1, the adjacency (E4, G) was utilized in order to support a new claim. The vertex E4 was "vectors are not parallel/on the same line," thus this use of (E4, G) coincides with the second normative way of reasoning from Toulmin analysis, "The span of two non-parallel vectors in $\mathbf{R}^2$ is all of $\mathbf{R}^2$."

On a more general note, other aspects between the two analyses have potential to be complimentary as well. For instance, a vertex with a high out-connection means that the vertex was adjacent to a variety of other vertices. Given that the adjacency (u, v) for vertices u and v could be read as "if u then v," a vertex with a high out-connection means that the concept statement or interpretation represented by that vertex served as data for a variety of claims. This is precisely what was presented as Criterion 3 for documenting normative ways of reasoning (Cole et al., 2011). High diversity within a particular cell (such as the (G, E) cell), rather than over a particular row, indicates that the given implication (rather than a given concept) served a role in multiple arguments. While the specifics of the argumentation that these adjacent pairs belong to is not provided in the adjacency matrix, knowing on which days the pairs occurred provides easily accessible information regarding what concepts during the semester the pairs are used to reason about. This allows an analysis compatible with Criterion 2 (Rasmussen & Stephan, 2008).

6.2.2 Points of distinction

Each analytical tool often provides information that is difficult to glean from the other. For instance, by separating the argumentation into sub-digraphs according to whether students were reasoning about $m < n$, $m = n$, $m > n$, or any m, n pair for adjacency matrix analysis illuminated aspects of student reasoning that were not so readily apparent within Toulmin's model. Even though the Invertible Matrix Theorem is a set of equivalent statements of square matrices, this analysis helped to illuminate the role that reasoning about non-square matrices played in argumentation throughout the semester. Adjacency matrix analysis also provides a different sense of the entire data set at a glance than is possible with Toulmin's model. It provides an efficient, quantitative-oriented way to see, for instance, that Abraham hardly reasoned about "the determinant of A is nonzero" but rather preferred its negation. Toulmin's model has no readily efficient way to determine results such as this at a broad ontogenetic level.

Toulmin's model also revealed characteristics of argumentation that adjacency matrices did not. For instance, adjacency matrices—as created for the study reported here—had no way to code for aspects of uncertainty. That loosely correlates to losing the ability to code for qualifying statements, such as, "it seems like," or, "I just know it's true." Furthermore, Toulmin analysis was not restricted by the set of 100 codes that were used to code argumentation within adjacency matrices. Thus, Toulmin's model allowed me to stay closer to the data by using actual quotes from the transcript as the constituent parts of any given Toulmin scheme.

6.2.3 Limitations of adjacency matrices

Given that using adjacency matrices as analytic tools is a rather novel approach within mathematics education research, I conclude this section with a few limitations of this novel approach. First, the structure of an adjacency matrix can be limiting in two ways: (a) even though the various vertices are developed through grounded theory, there is still an inferential risk when coding utterances with those codes; and (b) the "if-then" structure is restricting on explanations that do not follow that in a clean way. Regarding (a), for instance, if Abraham reasoned about a specific 3x3 matrix or a general n x n matrix, there is no way to distinguish that within the adjacency matrix coding. Thus, one loses some degree of specificity when developing one set of codes that work for all data. Regarding (b), there was no clean way to code statements of the form " (p and q) → r" without straying too far from sense of the original utterance.

Second, as mentioned in the previous comparison with Toulmin's model, adjacency matrix coding does not account for "qualitative" aspects such as a student's inability to explain, as well as any reflective utterances. For instance, Abraham said more than once that he liked to reason about negations; that information is not captured with this coding scheme. Finally, I made the choice to try to keep the set of vertices as close as possible to only be about concept statements within the IMT or interpretations of those statements. Thus, reasoning about other notions, such as what translating between matrix equation and vector equation notation implies for what it means to be a solution, were not captured in the present study.

6.3 Implications for Teaching

What is learned through investigating the learning and teaching of linear algebra via design research inevitably has important implications for teaching. Indeed, the "intimate relationship between the development of theory and the improvement of instructional design for bringing about new forms of learning" (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) is a significant benefit of conducting a study that was part of a semester-long classroom teaching experiment (Cobb, 2000). In this section I present three implications for teaching that arise from the results of my research into students' ways of reasoning about the IMT over time.

6.3.1 Implications of the Magic Carpet Ride Problem on students' reasoning about the Invertible Matrix Theorem

In the Magic Carpet Ride problem students worked with an experientially real problem setting of travel in two and three dimensions as they developed more formal ways of reasoning about properties of vectors and vector spaces (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2011). Within that task sequence, the class first reasoned about span by considering "all the places you could get" with a linear combination of a set of given vectors. Furthermore, linear dependence was first explored as a property of a certain type of modes of transportation that allowed you to complete a journey that began and ended at home (find a nontrivial solution to the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$). Implications for teaching that arise from the current study of students' ways of reasoning about the IMT are (a) to further investigate how this problem setting aided students in developing sophisticated ways of reasoning about formal definitions, and (b) to adjust the instructional sequence when necessary.

As illustrated in the following excerpt from Interview 1 with a student, Jerry, the Magic Carpet Ride scenario offered students rich imagery to make sense of formal definitions.

Interviewer: Different people think about concepts in mathematics differently. I want to know, how you think about linear independence?

Jerry: Linear independence. Really, it goes way back to the first Magic Carpet problem. I think it's actually the first thing that enabled me to grasp it. Because I did take the class once before, and it was just definitions. I never would have thought of it as being, you have a set of vectors, can you get back to the same point? That never occurred. It's all abstract if you don't have that analogy.

While Jerry's statement provides evidence that the Magic Carpet Ride problem instructional sequence was beneficial for him, this dissertation contributes data that adds additional support of this benefit at both the collective and the individual level. One example from the collective level is the establishment of the first normative way of reasoning within the second classroom mathematics practice. This normative way of reasoning was, "A transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ is onto $\mathbf{R}^m$ when you can 'get everything' or 'get everywhere' and not onto $\mathbf{R}^m$ when you cannot." This idea that functioned as-if shared within the classroom was integral within the classroom mathematics practice of determining whether transformations were onto by reasoning about the span of a set of vectors. The Magic Carpet Ride instructional sequence provided a situational activity through which students developed a way of reasoning about span that they leveraged later in the semester to reason about new mathematical concepts. The fact that the language of "getting everywhere" was prominent throughout the semester provides evidence that the travel metaphor, instantiated through the Magic Carpet Ride sequence, not only offered students rich imagery to make sense of formal definitions, but also served to support their ways of reasoning about new mathematical concepts.

At the individual level of analysis, Chapter 5 discussed the prominence of the "getting everywhere" within Abraham's ways of reasoning, even within the last interview after the semester ended. Recall his explanation for why "the column space of A is all of $\mathbf{R}^n$" and "the columns of A span $\mathbf{R}^n$" are equivalent statements:

Abraham: So this one I don't know how to decipher, really. Because when I think of the column space, I really literally think of the space that the columns can *get to*. And this is talking about the columns of A spanning. So I can't really discern the difference of these ones. Because like I said, this is to me where the columns can *get to* [points to the 'span' card] in this fashion. And this one [points to the 'column space' card] is the space that the columns can *get to*. Which for me is like the same thing (emphasis added).

For Abraham, the travel language that the class developed regarding the mathematical concepts related to span was extremely salient. The travel language associated with linear independence, however, was less so. The adjacency matrix analysis presented in Chapter 5 revealed that, for Abraham, the vertices associated with the ability (or inability) to "get back home" had a low frequency of use within his argumentation throughout the semester. Just because Abraham did not make extensive use of travel imagery is not necessarily a deficiency. On the contrary, it is possible that it shows a true sophistication on Abraham's part in that he moved away from situational activity towards more general and formal activities, which, by definition, do not rely on the original problem setting (Gravemeijer, 1999). The issue, however, is whether the situational activity was pivotal for Abraham as he developed more formal ways of reasoning with and about linear independence.

How does this inform teaching? In broad terms, it shows that the Magic Carpet Ride sequence is fruitful for assisting students in developing sophisticated ways of reasoning about formal mathematical definitions, and it is worth refining and implementing in future instruction. Although we see that the span travel language was ubiquitous at both the individual and the collective level in the present study, Jerry's quote above and the evidence from analyzing Abraham are somewhat contradictory regarding the power of the task sequence in grounding students' ways of reasoning about linear (in)dependence. The implication is two-fold: First, further investigation, accomplished through analyzing more individual students' ways of reasoning, is necessary in order to see how, if it all, the travel language from the Magic Carpet Ride problem was prevalent or powerful for other individuals. Second, continue to refine the instructional sequence according to the implications of this further research. It could be the case that a different situational activity related to linear (in)dependence would provide a more fruitful starting point for students. For instance, reasoning about unique solutions to any equation $A\mathbf{x} = \mathbf{b}$ was very powerful for Abraham. Possibly shifting the original situational activity towards something that would lend itself to reasoning about unique solutions may prove to be more powerful for more students than the original setting of "trying to get back home." Additional research is needed to address this open question.

6.3.2 Weaknesses in the ways of reasoning about the transformation-oriented concept statements in the Invertible Matrix Theorem

Recall from Chapter 4 that the classroom mathematics practice analysis revealed a distinction, if not a weakness, in the ways of reasoning about some of the transformation-oriented concept statements within the IMT. The first classroom mathematics practice focused on ways of reasoning about span and linear independence as equivalent ideas for square matrices; I was surprised that I was unable to find data to support the existence of an analogous classroom mathematics practice regarding one-to-one and onto transformations. Rather, I found a collection of normative ways of reasoning around the notion of determining if a transformation is onto by reasoning about span. What is a possible explanation for the distinction between the two classroom mathematics practices? Why did the second not develop the same as the first? Within the collective argumentation analyzed in Chapter 4, I found that the justification of the equivalence of one-to-one and onto was firmly tied to the equivalence of span and linear independence in the case of *m* equals *n* (see Figure 4.46). Other arguments within the collective-level data set, although not included within a particular classroom mathematics practice presented here, indicated a similar lack of transformational reasoning (e.g., concluding that matrices are invertible because of calculational issues regarding finding a C such that $AC = I$ rather than because of inability to "undo" a transformation).

Similar issues seemed to exist for Abraham as well. Abraham, as reported in Chapter 5, developed consistent and accurate ways of reasoning about one-to-one and onto. For instance, he reasoned about transformations that were not one-to-one in order to establish equivalence between matrices with determinant of zero and linearly dependent column vectors. He was not as strong, however, when reasoning about invertibility. During the second interview, he struggled to explain a connection between one-to-one, onto, and invertibility. One possible explanation for this may be the phrasing of the concept statements in the Invertible Matrix Theorem itself. The statements about invertibility were matrix-oriented: "matrix A is invertible," and "there exists a matrix C such that $AC = I$ and $CA = I$." Consider the equivalent concepts worded in terms of linear transformations: "linear transformation T is invertible," and "there exists a linear transformation S such that $T(S(\mathbf{x})) = \mathbf{x}$ and $T(S(\mathbf{x})) = \mathbf{x}$." Although the matrix-oriented version may be helpful in some scenarios (such as relating why a matrix A is not invertible if its determinant is zero through exploring the row-reduced echelon form of A and seeing that $[\ A \mid I\ ] \sim [\ I \mid A^{-1}\ ]$ is not possible), it was *not* helpful, at least for Abraham, when trying to see that a one-to-one and onto function is an invertible function, and vice versa.

This “context shift” between matrix-oriented and transformation-oriented concept statements in the IMT is not unlike switching between various modes of representation (Hillel, 2000). Teachers are often not aware when they ask students to move between orientations, and, as Hillel stated, this is one aspect that makes linear algebra a difficult content area to learn and to teach. The fact that the IMT is a collective of equivalent statements that all “say the same thing but in different ways” is not a hindrance but a strength. Making this explicit with students may help them see the power of the IMT. It may also illuminate subtleties that help students see when it is the right situation to use the various concept statements (i.e., linear independence is a property of a set of vectors and to caution against saying “the matrix is linearly independent”).

6.3.3 Varieties of Equivalence within the Invertible Matrix Theorem

In Chapter 5, I reported that Abraham had two main ways of explaining why concept statements were equivalent: they were indistinguishable and/or defined in terms of each other (conceptual equivalence), or they could be shown to be connected through a sequence of if-then deductions (logical equivalence). These two types of equivalency might have arisen because the notion of equivalence is not consistent within the IMT. For instance, consider the concept statements “the columns of A are linearly independent” and “the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution.” These are equivalent by definition. That is different than, say, “the columns of A span $\mathbf{R}^n$” and “the number zero is not an eigenvalue of A,” which would take most anyone a substantial effort to demonstrate their equivalence. The notion of equivalence in general is most likely new for many students, and, given their limited experience in proof, students would benefit from conversations that explicitly bring this distinction to light.

6.4 Avenues for Future Research

The first avenue for future research involves conducting in-depth analyses of the other four students who were in Abraham’s small group during the linear algebra course. How did *those* individual students reason about the Invertible Matrix Theorem during the semester? The in-depth individual case study I conducted with Abraham provides a solid foundation from which to begin investigating how a wider range of individuals reasoned. I have shown the usefulness of the two analytical tools in illuminating how one individual reasoned, and I made some comparisons about how that reasoning is similar to or different from the collective growth of the classroom. For a future study, I want to look at variation in individual reasoning. That question would investigate major milestones and characteristics of similarity and diversity of individuals reasoning about the IMT. Of course this one case study of Abraham does not provide that breadth, but it does provide a depth from which to build a strong foundation for that future study.

A natural next step from researching other individual students' ways of reasoning about the Invertible Matrix Theorem would be to consider Abraham's small group as a unit of analysis—as a "mini-collective." Students in this inquiry-oriented linear algebra class spent a sizable portion of the semester working in small groups solving problems, developing and proving conjectures, and explaining their thinking to one another. The group members would often finish each other's sentences or communicate nonverbally, and through doing so, they often developed creative and sophisticated ways of reasoning. I think Abraham captured this sentiment well. During the last day of class, while working with his group members on a task about the Invertible Matrix Theorem (without the instructor nearby), he said to them: "It's crazy how we all think so differently. But then when we work as a group, is seems that we think a lot the same." Thus, conducting a analysis much like that of Chapter 4 on Abraham's small group as the collective unit of analysis would not only be a fascinating study but would also contribute to what is known about Abraham's ways of reasoning about the IMT.

The most immediate plan for future research is to coordinate the results concerning that mathematical development from both the individual and the collective units of analysis. Within the first chapter I discussed two of my theoretical and methodological influences as the emergent perspective and Saxe's view of genetic analysis. Both of these perspectives uphold the importance of documenting the mathematical growth of the individual and the collective and their mutual constitution. Consider quotes from the aforementioned influences, respectively:

> A basic assumption of the emergent perspective is, therefore, that neither individual students' activities nor classroom mathematical practices can be accounted for adequately except in relation to the other. (Cobb, 2000, p. 310)
>
> In collective practices, joint tasks are accomplished...through the interrelated activities of individuals. In such joint accomplishments, individual and collective activities are reciprocally related. Individual activities are constitutive of collective practices. At the same time, the joint activity of the collective gives shape and purpose to individuals' goal-directed activities. (Saxe, 2002, pp. 276-277)

As the above quotes imply, knowledge genesis on individual and collective levels are inextricably bound together in their development. Therefore, in order to gain the most fully developed understanding of the emergence, development, and spread of ideas in a particular classroom, analysis along both levels is warranted. However, the coordination of these analyses is still a relevant and open question in mathematics education research.

Works towards the coordination between individual and collective levels could take on various forms. Within this dissertation, I made use of the same analytical tools at both the individual and the collective levels in anticipation of a subsequent coordination. Thus, the most natural place to begin will be to coordinate the mathematical development of Abraham with that of the classroom community. For instance, a "double timeline" would map out and allow me to compare when certain ways of reasoning occurred for Abraham against when they occurred in whole class discussion. Comparing the development of ways of reasoning for both levels may be revealing. Were there ways of reasoning that were normative in the classroom but seemingly not meaningful to Abraham? Furthermore, how did the use of Toulmin's Model and adjacency matrices at both levels of analysis help or hinder this coordination?

Finally, an avenue of interest for future research would be ways in which to adopt the three criteria for documenting normative ways of reasoning at the classroom level in order to draw conclusions about an individual students' mathematical development. For example, consider the results concerning Abraham's ways of reasoning about span and linear independence in Section 5.1.2.1.2. That section highlighted shifts in the composition of Abraham's arguments as well as in the placement of concepts within the arguments. Early in the semester, he made claims with little justification (shown through a low frequency of data and warrants initially) and/or expressed sentiments of uncertainty (shown in his particular qualifiers). He eventually presented well-developed justifications for these original claims. Furthermore, the concepts that Abraham first had trouble justifying then became ways in which he justified new claims. That is reminiscent of the second criterion from Chapter 4 in that a way of reasoning was functioning as-if shared when any of the four parts of an argument (data, claim, warrant, or backing) shift position within subsequent arguments. Inquiring into possible points of compatibility such as this is a fruitful and complex domain for future work.

APPENDICES

Appendix 3.1. Interview questions and planned follow-ups from both Interview 1 and Interview 2 that were analyzed for this study.

Interview 1 Question 6

"For this next set of questions, suppose you have a 3 x 3 matrix *A*, and you know that the columns of *A* span R^3. One at a time, decide if each of the following statements is true or false, and explain your answer."

Interview 1 Question 6a

a. The column vectors of *A* are linearly dependent.

Follow-ups. Skip if redundant:

- "How do you think about what it means for vectors to be linearly dependent?
- "Do you have a way of thinking about this geometrically?"
- "How do you know that is the case?"
- "Do you want to give an example?"
- "How does linear dependence relate to span?"

Interview 1 Question 6b

b. Any vector **b** in R^3 can be written as a linear combination of the columns of *A*.

Follow-ups. Skip if redundant:

- "How do you think about what it means to be a linear combination?
- "Do you have a way of thinking about this geometrically?"
- "How do you know that is the case?"
- "Do you want to give an example?"
- "How does linear combinations relate to span?"

Interview 1 Question 6c

c. The row-reduced echelon form of *A* has three pivots.

Follow-ups. Skip if redundant:

- "How do you think about what a pivot is?
- "Do you have a way of thinking about this geometrically?"
- "How do you know that is the case?"
- "Do you want to give an example?"
- "How do pivots relate to span?"

Interview 1 Question 6d

d. There is a nontrivial solution to the equation $A\mathbf{x}=\mathbf{0}$.

Follow-ups. Skip if redundant:

- "What does it mean to be a solution to the equation $A\mathbf{x}=\mathbf{0}$?
- "What is meant by 'the trivial solution'?"
- "Do you have a way of thinking about this geometrically?"
- "How do you know that is the case?"
- "Do you want to give an example?"
- "How does this relate to span?"

Interview 1 Question 6e

e. The system $A\mathbf{x}=\mathbf{b}$ has no free variables (*make sure they attend to the equals* ***b***)

Follow-ups. Skip if redundant:

- "How do you think about what it means to be a free variable?"
- "Do you have a way of thinking about this geometrically?"
- "How do you know that is the case?"
- "Do you want to give an example?"
- "How do free variables relate to span?"

Example generation for non-square matrices

"Not all matrices in linear algebra are square. I am curious about how you think about some of the previous concepts when dealing with non-square matrices. Please read the following prompt and think out loud as you think through this problem."

Interview 1 Question 8a

Create an example of a 3 x 5 matrix *B* such that the column vectors of *B* span $\mathbf{R}^3$ and are linearly independent. If it is not possible, explain your reasoning why.

Follow-ups. Skip if redundant:

- *If they have an example:* "What were you thinking as you created your example?
- *If they have an example:* "Is that the only example that will work? (If so, why? If not, what else would work as an example?)"
- *If they have an example:* "You stated how this example works for (span). How is it working for (linear independence)?"
- *If they say not possible:* "How did you know this wasn't possible?
- *If they say not possible:* "Which part of the prompt is failing, to make creating an example impossible?"
- What would you change in the example requirements so that it is possible to create an example?

Interview 1 Question 8b

Create an example of a 5 x 3 matrix *C* such that the column vectors of *C* span $\mathbf{R}^5$ and are linearly independent. If it is not possible, explain your reasoning why.

Follow-ups. Skip if redundant:

- *If they have an example:* "What were you thinking as you created your example?
- *If they have an example:* "Is that the only example that will work? (If so, why? If not, what else would work as an example?)"
- *If they have an example:* "You stated how this example works for (span). How is it working for (linear independence)?"
- *If they say not possible:* "How did you know this wasn't possible?
- *If they say not possible:* "Which part of the prompt is failing, to make creating an example impossible?"
- What would you change in the example requirements so that it is possible to create an example?

Interview 2 Question 1

"Suppose you have a 3 x 3 matrix *A*, and you know that the columns of *A* span R^3. Decide if the following statements are true or false, and explain your answer:"

Interview 2 Question 1a

The column vectors of *A* are linearly <u>de</u>pendent.

Follow-ups. Skip if redundant:

- "How do you think about span?"
- "How do you think about what it means for vectors to be linearly <u>de</u>pendent?
- "How does linear <u>de</u>pendence relate to span of a set of vectors?

Interview 2 Question 1b

The row-reduced echelon form of A has three pivots.

Follow-ups. Skip if redundant:

- "How do you think about what a pivot is?
- "How do pivots of a matrix relate to span of a set of vectors?"

Interview 2 Question 2

General Questions about the IMT: "On this sheet of paper is what we developed in class over the course of the semester known as the Invertible Matrix Theorem."

Interview 2 Question 2a

What do you understand it to mean to even be a theorem? What is another example of a theorem from another math course? Why is that an example of a theorem?"

Interview 2 Question 2b

This phrase here in the theorem statement says 'the following are equivalent.' What does it mean, to you, to have equivalent statements?

Interview 2 Question 2c

What is the importance of the phrase, '*A* is *n* x *n*'? [Skip if redundant]: What if *A* is not *n* x *n*? [If it makes sense from what they say]: "Name two statements that are not equivalent if *A* is not *n* x *n*." (What about the "if" and "then" of their example?)

Interview 2 Question 3

Here is a set of 16 cards, and each card has written on it one of the statements from the IMT. Please sort these into different piles based on which concepts, *for you,* 'go together' and explain your choices out loud as you go.

- *Follow-ups dependent on student's pile choices. Skip if redundant:*
  - What were your criteria for how you chose what 'goes together?
  - [Pick two ideas from the same pile]: Why did you put all of these cards in the same pile? What makes them 'go together' for you?
  - [Skip if redundant]: I'm curious, why are these cards (see possible choices below) in the same pile?
    1. [span] —and— [Col *A* = Rn]
    2. [n pivots] —and— [row equivalent to the identity]
    3. [*T* is 1-1] —and— [*T* is onto]
  - [Pick two ideas from two different piles]: "So you put these two cards (pick two) in different piles. I'm curious: do you have any way that *these* two ideas connect? Explain.
    1. [*A* is invertible] —and— {[*T* is 1-1] –with- [*T* is onto]}
    2. [The det of *A* is nonzero] —and— [row equiv. to *I*]
    3. [Zero is not an eigenvalue] —and— [*anything*!]

Interview 2 Question 4

For the following pairs of statements from the Invertible Matrix Theorem, please explain how you understand how they are equivalent.

Interview 2 Question 4a

A has *n* pivots // *A* is invertible

- [If they do one direction]: "What about the other direction of "if-then"?
- [Skip if redundant]: "How do you think about what it means to be invertible?

Interview 2 Question 4b

det $A \neq 0$ // The columns of *A* are linearly independent

- [If they do one direction]: "What about the other direction of "if-then"?
- [Skip if redundant]: "How do you think about determinants?"

Interview 2 Question 4c

a. Nul A={0} // The only solution to Ax=0 is the trivial solution
   - [If they do one direction]: "How do you think about null space?"
   - [Skip if redundant]: "How do you think about A**x** = **0**?"
   - [Skip if redundant]: "How do you think about what it means to have only the trivial solution?"

Appendix 4.1. Argumentation Logs of Toulmin schemes from Whole Class Discussion

Day 5 Argumentation Log

Arg 5.1
C- D(C1-D1-W1 U C2-D2-W2-B2)-W proof by cases
Claim: When looking for an example of 3 vectors in R2 that form a linearly independent set, there is no solution
Data: Let's just start with any random vector, let's call it that one. Now after we have 1 vector down, there's only basically 2 situations we could have.
Claim1: *they're dependent*
Data1: *if we have 2 vectors that are parallel*
Warrant1: *we already said* [it]
Claim2: *no matter where we throw in our 3rd vector, we can get there with a combo of these 2 and make it back on that 3rd one*
Data2: If *we had 2 that weren't parallel*
Warrant2: the span of any 2 that aren't parallel is all of r2
Backing2: We did this when *we did our magic carpet hoverboard*
Warrant: *So* [$D_1 \vee D_2$] *there can't be any solution, so there's no, as long as we have 3 vectors in r2, it has to be linearly dependent.*

Day 6 Argumentation Log

Arg 6.1
C-D
Claim: Generalization #4 in my own words is *if you have more vectors than dimensions, you'll always be able to return to your original position* [Justin, 04:54]
Data: *There's no other way of putting it ... I don't know any other way to say it* [Nate, 05:26]

Arg 6.2
C-D-Q
Claim: If you have more vectors than dimensions, the set of vectors is linearly dependent [Instructor, 05:37]
Data: None
Qualifier [to the *lack* of warrant]: *We were confused on how to prove it, we didn't know where to start, where to go with a proof* [Saul, 06:04]

Arg 6.3
C-D-W
Claim: *I should always be able to get back home* [Instructor, 6:46, Lawson, 8:46]
Data: *If we had 3 nonparallel vectors in R2* [Instructor, 6:46, Lawson, 8:46]
Warrant [given after SGW]**:** *Graphically think about it. You say you have a vector over here, and then plus a vector that's not parallel to it [draws on board]. And we have just any other random vector, say it goes this way. If you add any 2 nonparallel vectors that at some point these vectors are going to come across this other vector, and hit it at a point. So if that happens,*

then you can just take the last vector and multiply it by some scalar to get you back to the origin. That's how graphically we got it. [Lawson, 08:58]

<u>Arg 6.4</u>
C-D-W--B**
Claim: It's linearly dependent [Lawson, 09:57, prompted by the instructor]
Data: *If you have the set of vectors <1,2>, <2,4>, <-9, 7> (*which has two 'parallel' vectors in it*)* [Lawson, prompted by the instructor at 09:44]
Warrant: *You can just ignore this last vector, take 1, like <1,2>, which would be, and just take the 2,4 back, it's a multiple of 1,2, and you can just not use this one. So it's linear dependent.* [Lawson, 09:57]
**Instructor asks if it makes sense
Backing: *It makes sense that pretty much if you have 3 vectors in r2, even in the 1st example, you could technically ignore the last one, because if you take 1,2 out, then you can still use any combination of the 1,2 and the 3,5 to get back* [Cayla, 10:28]

<u>Arg 6.5</u>
C--D (C1-D1 U C2-D2) proof by cases**
Claim1: *If a vector is contained in the span of 2 other vectors, then you should be able to reach the origin, get back to the origin***.** [Aziz, 11:40]
**Instructor asks for explanation
Data: *2 vectors span r2 or they don't*
Data1: *if they're linearly dependent already,*
Claim1: *then it makes the set linearly dependent.*
Data2: *If 2 aren't, then they span r2,*
Claim2: *the 3rd one has to be contained in [inaud] to be able to reach back to the origin.* [Aziz, 11:50]

<u>Arg 6.6</u>
C-D-W
Claim: The set with three vectors is linearly dependent [Matthew, 15:10]
Data: *It should be, if they're all existing within the x,y plane, they're all in the same plane. So once you have 3 of them, that is, you don't have 2 of them that lie along the same vector, on the same line,*
Warrant: *then we should be able to get back to the origin and it should be dependent.* [Matthew, 15:10]

<u>Arg 6.7</u>
C-D-R-W-B
Claim: The set with two vectors is linearly independent [Juan, 15:47]
Data: *Because it spans R2* [Juan, 15:51]
Rebuttal: *That's not a good reason* [Nate, 16:01]
Warrant: *You can go anywhere with the 1st 2 vectors in r2, but your way home, if you took away -9,7, there's no way back. So it's independent because you can go anywhere, but there's no home.* [Randall, 16:30]
Backing: *Yeah, we took away that 3rd vector, so that way home that we had no longer exists there.* [Instructor, 16:47]

Arg 6.8
C-D-W-B-Q
Claim: A set of vectors (in Rn) is linearly dependent [Justin, 17:19]
Data: You have more vectors than dimensions [Justin, 17:19]
Warrant: *So if you start in any Rn, and you just start with 1 vector and keep more. So let's do R3, just for an example. So we start with 1 vector. So either, we have 2 choices: The next vector we add can either be on the same line, which means it's already linearly dependent, so we don't want that, sowe're going to put it off somewhere else. Now the span of that is a plane in 3 dimensions. So now we're going to add another vector in. Our 3rd vector, now it can either be in that span or out of that span. And we want it to be linearly independent, so we're going to put it out of that span. But now that we have that going off of that plane, we just extended our span to all of r3. So our 4th vector, when we put it in, no matter where we put it, it's going to get us back home. Because just like in this case, we have to have the last one to get back home, we can get anywhere with those 1st 3 that we put in, but we have to have to have that 4th one to come back.* [Justin, 17:19]
Backing: *And so it works like that in any dimension, because the more you, if you keep adding, eventually you're going to get the span of your dimensions, and then you're going to have that extra one bringing you back.* [Justin, 17:19]
Qualifier: *Unless you have 2 vectors that are lying on the same line, then you won't have the span of all of your dimension, but it's negligible because those 2 will give you a linearly dependent set. Does that make sense?* [Justin, 17:19]
Backing 2: [think he's trying to support Justin's explanation] *I'd like to add on to this, if you're going from the plane to the volume of space, that's with 3 vectors in 3 dimensions. With the 4th one, you can cross all those dimensions back to the origin, that's what makes the 4th dimension, the 4th vector, it makes you able to go back to the beginning. Because each vector takes up a different point in space. You're going from the plane to the volume, gives you those 3 dimensions but you can't get back to origin, unless you have a 4th one that can cross all 3.* [Aziz, 19:46]

Day 9 Argumentation Log

Arg 9.1
C-D-W
Claim: The set $\left\{\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}2\\7\end{bmatrix},\begin{bmatrix}10\\5\end{bmatrix}\right\}$ is linearly dependent. [Aziz, 28:50]

Data: *If there are more vectors than there are columns, then it's linearly dependent.* [Aziz, 28:50]
Warrant: *So Aziz is reminding us of the thing we decided a while back, if we have more vectors than dimensions, it has to be linearly dependent* [Instructor, 28:56]

Arg 9.2
C-D-Q-W-B1-B2
Claim: The set $\left\{\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}2\\7\end{bmatrix},\begin{bmatrix}10\\5\end{bmatrix}\right\}$ is linearly dependent. [Instr, 29:07]

Data: *Do the row reduction and see, we get the identity matrix, I guess, c x=0, y=0. Actually it's on 2, so x=0, y=0 as well.* [Abe, 29:26]
Qualifier: *Yeah, so it's kind of weird out here, we have 3 vectors* [Instr, 28:56]
Warrant: *So the definition of linear independence, we've got, does there exist a non-0 solution to c1 of the 1st vector plus c2 of the 2nd plus c3 of the 3^{rd}* equals 0,0 [Instr, Justin, Abe, 29:50, 30:15]
Backing1: *Because that's what the definition said* [Justin, 30:20]
Baacking2: *Yeah, pretty much by definition, exactly. So set up to say, If this has a non-0 solution, then it's a linear dependent set.* [Instr, 30:24]

Arg 9.3
C-D-W-B
Claim: The set $\left\{\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}2\\7\end{bmatrix},\begin{bmatrix}10\\5\end{bmatrix}\right\}$ is linearly dependent. [Instr, 34:21]
Data: *c1-60c3 is 0 and c2 + 35c3 is 0.* [Instr, 33:51]
Warrant: *We don't have any way to define or nail down what c3 has to be. [board] So c3 here would be our free variable* [Instructor, 34:21]
Backing: *I can pick anything for that third one, then the first two depend on that third one, and there's more than one solution.* [Instr, 34:21]

Arg 9.4
C-D-W
Claim: The set $\left\{\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}2\\7\end{bmatrix},\begin{bmatrix}10\\5\end{bmatrix}\right\}$ is linearly dependent. [Instr, 34:53]
Data: *So if we look at the columns, if has the c1 in it. We've got a leading 1 here for the c1. If we look at the c2 column, we've got a leading 1 for the c2. If we look at c3, we don't have any leading 1...it has no pivot in its column* [Instr, 34:53]
Warrant: *c3 is a free variable* [Instr, 34:53]

Arg 9.5
C-D-W
Claim: The set $\left\{\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}2\\7\end{bmatrix},\begin{bmatrix}10\\5\end{bmatrix}\right\}$ is linearly dependent. [Instr, 35:38]
Data: As a system, it is row-equivalent to $\left[\begin{array}{ccc|c}1 & 0 & -60 & 0\\0 & 1 & 35 & 0\end{array}\right]$ [In, 35:38]
Warrant: Row equivalent systems have the same solution set and the *same type of linear dependence relationships* [Instr, 34:53]

Arg 9.6
C--D-W-B**
Claim: The span of $\left\{\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}2\\7\end{bmatrix},\begin{bmatrix}10\\5\end{bmatrix}\right\}$ is *all of R2* [Justin, 36:32]
****Teacher calls for data**
Data: *There's so many vectors, and they're not multiples of each other* [Justin, 36:38]
Warrant: *Yeah, so we only have 2 dimensions, they're not multiplies of each other, so we see that it span all of R2* [Instr, 36:43]

Backing: *Because there's multiple solutions* Justin, 36:49]

<u>**Arg 9.7**</u>
C-D-W
Claim: The span of $\left\{\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}2\\7\end{bmatrix},\begin{bmatrix}10\\5\end{bmatrix}\right\}$ is *all of R2* [Instructor, 38:03]
Data: From the row-reduced echelon form, $\begin{bmatrix}1 & 0 & -60\\0 & 1 & 35\end{bmatrix}$ [Instr, 38:03]
Warrant: *With the 1st 2 solutions, systems, it just stayed the set equals 0, and since the 3rd variable is free, it will just won't move* [Jorge, 38:32]

<u>**Arg 9.8**</u>
C-D-W--B1-B2**
Claim: The span of $\left\{\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}2\\7\end{bmatrix},\begin{bmatrix}10\\5\end{bmatrix}\right\}$ is *all of R2* [Instructor, 38:03]
Data: From the row-reduced echelon form, $\begin{bmatrix}1 & 0 & -60\\0 & 1 & 35\end{bmatrix}$ [Instr, 38:03]
Warrant: *We said that because the 1st 2, because if you've got, in 2 dimensions it can span everywhere, the 3rd one is a free variable, it would cause it to go wherever, in spanning.* [Giovanni, 38:49]
****Instr calls for backing**
Backing1: 1,0 and 0,1 tell you that it spans *because they are perpendicular* [Justin, 39:13]
Backing2: *it's easy to see that these are going in 2 different directions. If you row reduce it, we know since they have the same solution set, that if it spans a 2-dimensional space, the row reduce also spans a 2-dim space, etc. That this is really easy to see that we're going everywhere in r2. This gives us 1 way, this gives us another way, Justin's saying perpendicular.* [I, 39:14]

<u>**Arg 9.9**</u>
C-D-W-B
Claim: The span of $\left\{\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}2\\7\end{bmatrix},\begin{bmatrix}10\\5\end{bmatrix}\right\}$ is *all of R2* [Instructor, 38:03]
Data: From the row-reduced echelon form, $\begin{bmatrix}1 & 0 & -60\\0 & 1 & 35\end{bmatrix}$ [Instr, 38:03]
Warrant*: I have 2 pivots in these 2 rows* [Inst, 39:51]
Backing: *it's almost like this 0 and that 1, that tells us to go in x direction. Now I can see that this 1 takes us up the y direction, if you want to think in our standard x-y plane. So if I look at the pivot here and the pivot here, and the row that can tell us the columns span 2-dim space* [Instr, 39:51]

<u>**Arg 9.10**</u>
C--D-**-W**
Claim: *If you have 1 pivot position, is it a line, 2 pivot positions a plane, 3 volume and 4, I don't even know what to call it, like that* [Justin, 40:26]
****Instr calls for data**
Data: *That's the way it's been looking this whole time* [40:49]
****Inst calls for warrant**

Warrant*:* *Because so right now we're in r2, and if we have 2 pivot positions, I don't even really know what pivot position means, but just the way it seems to work is, if you have 1 pivot position, that's 1 position where you have a solution set? I don't know if that's right. And then you have 2, and then, I don't know, it just seems to go like that.* [Justin, 40:56]

Arg 9.11
C-D1-D2-W
Claim: The vectors [1,2,0,3,1; 2,4,0,6,2; 3,6,0,9,3] are linearly dependent [Instr, 44:22]
Data1: *The way we picked them* [Instructor, 44:22]
Data2: *the c1 has a pivot, you would say that these other 2 that don't have pivots, are the free variables.* [Instr, 45:25]
Warrant*:* *So as soon as we have 1 linear dependence relationship, i.e. a free variable, then we know it's a dependent set, so check that we've got linear dependence.* [Instr, 45:50]

Arg 9.12
C-D-W-Q
Claim: The span of [1,2,0,3,1; 2,4,0,6,2; 3,6,0,9,3] is a one-dim space in R5 [Instr, 47:33]
Data: *Once we start off on this 1,2,3,0,1, however that may travel in r5, the 2,0,4,0, blah is just twice that, in the same direction, but twice the magnitude. This one is in the same direction, but 3 times the magnitude.* We can't *get anywhere off that into other aspects of R5* [Instructor, 47:33]
Warrant: *We have one pivot position, so they span is a 1-dim space in R5.* [Instructor, 47:33] *then we know it's a dependent set, so check that we've got linear dependence.* [Instr, 45:50]
Qualifier: *I know it's hard to talk about directions and magnitudes and the dimensions higher than 3, but we need to be able to start to train our brains to think that way as well. And I think the imagery that we've built up so far allows us to do that. We can still try to generalize what it would mean to talk about traveling in a direction. It just might not make sense to try to picture that.* [Instructor, 48:26]

Arg 9.13
C-Q-D
Claim: The most you could span with 3 vectors in R5 is R3 [Abraham, 52:13]
Qualifier: *A 3-dimensional part in R5, yeah, so we have a 3D space in R5* [Instructor, 52:15]
Data: *if I have 5 dimensions in r5, I'm trying to figure out if there's any place that is a nook or cranny I can hide out in. If I have 3 vectors, we don't even want to say necessarily what they are. if I take 1 vector, it gets me 1 place. And the 2nd one gets me another place. And the 3rd one gets me another place*. [Instructor, 52:15]

Arg 9.14
C-D1-D2--W**
Claim: [1,0,0,0,0; 0,1,0,0,0; 0,0,1,0,0] spans 3 dimensions in R5 [Jerry, 59:59]
Data1: *We got one vector going in each of the 3 directions.* [Jerry, 59:59]
****Randall wanted to add**
Data2: *I want to say the amount of rows with values after reduction / pivots* [Randall, 1:00:28]
****Inst asks for warrant**
Warrant: *We were thinking about it, let's start with the 1st vector by itself. I know I'm still thinking x,y,z's and whatnot. So we have a number that's not 0 that we can multiply our x component by. So that gives us an infinite line like that. And so that's the 1st dimension. And then the next one, the leading 1 is in the 2nd row, so it's the y component, and we can multiply*

that one by infinity, and that gives us the 2nd dimension. And then 3 is z and gives us the 3rd dimension. And I don't know what the next variables are supposed to be, but that gives us the next one and the next one, however far we go. [Justin, 1:00:57]

<u>Arg 9.15</u>
C-D-W-B
Claim: if *m* is less than *n,* the columns are linearly *dependent* [Randall, 1:10:14]
Data: *If n is greater than m, that means there's going to be, say, 3 vectors in 2 space, and 2 of those are going to span, the last one makes it dependent.* [Randall, 1:10:25]
Warrant: *Yeah, so we could do that example from before, if we had 3 vectors in* $\boldsymbol{R}^2$*, we try to row reduce it, we could have a pivot here, a pivot here, and can we get a 3rd pivot?* [Randall: No.] [1:10:37]
Backing: *So this guy [the third column] has no pivot, so it has to be a linearly dependent set.* [Instr, 1:10:37]

<u>Arg 9.16</u>
C-Q-D-W-B
Claim: If $m > n$, *you won't be able to span the entire dimension* [Abraham and Justin, 1:11:42]
Qualifier: *I don't know, it just makes sense* [Abraham, 1:11:58]
Data: *You don't have enough vectors for the dimensions.* [Abraham, 1:11:58]
Warrant: *So it seems you wouldn't be able to go to all the directions in those dimensions.* [Abraham, 1:11:58]
Backing: *So you get one that's not enough vectors to get me to all the dimensions. Someone else would give an argument, I can't get enough pivots.* [Instructor, 1:12:15]

<u>Arg 9.17</u>
C--D (D1-C1 U D2-C2-W2)-Q1-Q2**
Claim: *You could always have a linearly dependent set, in any dimension* [Gabe, 1:12:38]
****Instructor calls for data**
Data: (cases)
D1: *If you have anything from 1 or greater vectors, if 1 of those vectors happens to be 0,*
C1: *it's dependent*
D2: *If you have 2 or more vectors, if 2 of those are the same vector, it doesn't matter what the other vectors are,*
W2: *so you can use one vector to go out and one vector to come back,*
C2: *so linearly dependent that way* [Gabe, 1:12:51]
Qualifier1: *So by 'same' we mean along the same path or they have the same span* [Instructor, 1:13:50]
Qualifier2: *same vector, different magnitude or scalar* [Gabe, 1:14:07]

Day 10 Argumentation Log

<u>Arg 10.1</u>
C--D-W**
Claim: If we have an nxn matrix that spans $\mathbf{R}^n$, then the set of vectors in *A* is linear independent [Justin 21:28]

Qualifier: *that's required,* [copying #1-4] *because if we don't require there to be a pivot in every row, this doesn't* have *to be true* [Justin, 22:09]
Data: *we have as many pivots as we have dimensions* [Justin, 22:09]
Warrant: *The simplest form...1's along that diagonal, which gives us the span.* [Justin, 22:09]

Arg 10.2
C-D-W
Claim: *We got the same thing* [as Justin's claim] [Nigel, 22:55]]
Data: *you're under the assumption that there's a pivot in every row, you can only use 1 vector once* [Nigel, 22:55]
Warrant: *you use each vector to go in a certain dimension, but you can never get back*. [Nigel, 22:55]

Arg 10.3
C-D
Claim: *Any system that spans Rn would be linear independent* [Matthew, 24:50] *(**must mean 'square system' b/c of data in #4 and esp in #5)*
Data: *the number of variables correspond to the number of rows, so it also corresponds to the number of dimensions.* [Matthew, 24:50]

Arg 10.4
C-D-W-B
Claim: *A system that spans all of Rn would be linear independent* [Matthew, 25:15]
Data: *it has the same number of rows and columns* [Matthew, 25:15]
Warrant: *the number of variables corresponds to the number of rows, and also corresponds to the number of dimensions in the system* [Matthew, 25:15]
Backing: *this n here is the number of equations. If you want to think of it and this as the # of variables, that sort of thing* [Instructor, 25:29]

Arg 10.5
C-*-D-*-W-R
Claim: If we have an nxn matrix that spans Rn, then *we took Rule 4 from the original one, but added that it'd be a unique solution.* [Mitchell, 26:04]
Data: *you have a pivot in every row, and it's square, then you have no free variables.* [Mitchell, 26:43]
Instructor asks for warrant
Warrant: *row reduced echelon form...which you can achieve if you have a pivot in every row... if you make it into augmented matrix...for any b you put in there, there's only 1 solution because pivot 1x whatever equals this y, equals this z, equals this and so forth*. [Mitchell, 26:54]
Rebuttal: It *seems like there's more solutions to me* [Justin, 27:55]

Arg 10.6
C-R
Claim: *If we span all of R2,* I can have all of these solutions [the different paths he drew] [Justin, 29:05]
Rebuttal: *But you're only given 2 vectors in* R^2 *because you have a square matrix* [Nate, 29:31]

__Arg 10.7__
C-D-W
Claim: You don't go this way around [Nigel, 29:44]
Data: *Because you only have 2 vectors* [Nigel, 29:44]
Warrant: *Those are more than 6 vectors you drew* [Nigel, 29:44]

__Arg 10.8__
C-D-W-Q
Claim: Another one to add on is that *none of the rows can be all zeros* [Cayla, 34:57]
Data: *There is a pivot in every row* [Cayla, 34:57]
Warrant: *There are no free variables* [Cayla, 34:57]
Qualifier: *we can say that the row reduced echelon form of A can't have a row of 0's* [Instructor, 35:44]

__Arg 10.9__
C-D-Q-W
Claim: *if a matrix can be in row reduced echelon form, then it's independent* Lawson, 37:39]
Data: *The only solution is the trivial solution* [Lawson, 37:39]
Qualifier: *It depends on what system you're talking about. So for here, we say the only solution to the system Ax=0, is one way we care about the answer, is the trivial solution* [Instructor, 37:48]
Warrant: It's actually by definition that 2 and 9 have to be the same...how we define what a linearly independent set was [Instructor, 38:27]

Day 17 Argumentation Log

__Arg 17.1__
D-C-W
Claim: *If you're* [a matrix] *completely full of zeroes* [it is not invertible] [Gabe, 30:49]
Data: *if you transform them using this matrix, you cannot see the original matrix back* [Gabe, 30:49]
Warrant: *there's no going back to that, it's over, because you'll end up with a matrix full of 0's, and then you can't multiply 0 by anything to get any another number* [Gabe, 30:49]

__Arg 17.2__
D-C-W
Claim: *having rows of 0's in your matrix actually accomplishes the same thing* [is not invertible] [Gabe, 30:49]
Data: *because if you multiply a certain row by 0, then again you can't get back to that* [Gabe, 30:49]
Warrant: *if you took out one of your rows, that's going to eliminate one of the dimensions that you're traveling* [Gabe, 32:32]

__Arg 17.3__
D-C-R-D-W-B
Claim: *if you can't span everything, there's no way it can have an inverse* [Kaleb, 33:40, attributed to Gabe]
Data*: You can still use the 0's to transverse it if you take out a column. But if you take out a*

row, you can't move in one direction, so you're done. [Gabe, 33:44]
Rebuttal: *I'm not convinced. The span thing, I don't get. I agree if there's a row of 0's, it doesn't work* [the matrix isn't invertible]. *But I don't know, I don't see how it relates to span, just because. So if I have a different matrix, that doesn't have a row of 0's but doesn't span, does that mean that there's no inverse for it?* [Justin, 34:03]
Data: So the rows of 0's, if you have a row of 0's anywhere, right here for example, when we try to multiply it by anything, by this row, we can never get this 1 over here [Nate, 35:16]
Warrant: *So we can't get the identity matrix. So if you have any row of 0, it messes that pivot point up* (his gestures imply any general row of zeroes will mess up the pivot in that associated row) [Nate, 35:33]
Backing: *if you had 0's = 0 = 0 f equals 1, so there's no way to figure out how to make that true. So I think at least for these square matrices we've looked at, we can see that this one* [having a row of zeroes makes the matrix not invertible]*, algebraically we can see it actually true* [Instructor, 36:07]

Arg 17.4
D-C
Claim: *we just said that if the matrix is dependent, then it doesn't have an inverse* [Randall, 36:44]
Data: *We basically just opened it up, instead of just saying, "yeah, the rows of 0's thing doesn't work, the column of 0's doesn't work," we also found that when vectors are multiples of each other, it doesn't work* [Randall, 36:44]

Arg 17.5
D-C-W
Claim: *the matrix with a column of 0's is not invertible* [Randall at 36:44, Instructor at 37:05]
Data: if you multiply it out, there's going to be someplace that there's a 1 in multiplication, addition doesn't work out [Instructor, 37:05]
Warrant: if you went through the algebra, it would be a similar argument to the one before [Instructor, 37:05

Arg 17.6
D-C-W-R-Q
Claim: *dependent matrices are not invertible* [Randall, 37:30]
Data: *We did multiples of each other, so we had 1,2;2,4 as a matrix...And we found that one of the rows just simplifies to a free variable* [Randall, 37:30]
Warrant: *that would make it 0's in a row, which you can't have* [Randall, 37:30]
Rebuttal (to claim): *But what if you have like 4 vectors in r3 that, those will span and they're dependent, what happens then?* [Justin, 38:03]
Qualifier (to claim): Randall only talked about square matrices in his claim [Instructor, Randall, 38:24]

Arg 17.7
D-C-Q
Claim: *A has to be square in order to have a chance to be invertible* [Instructor, 48:13, wrapping up from Lawson]
Data: Lawson showed it for one direction of the multiplication, the other will be in homework [Instructor, 48:54]
Qualifier: it's not exactly saying that all square matrices are invertible, because we've already

seen a couple that aren't. So not necessarily all square. But if you even want to have a chance of being right and left invertible, that CA and AC would have to be square. [Instructor, 49:13]

Arg 17.8
D-C--D**
Claim: The matrix [1,2; 2,4] is not invertible [Instructor and Edgar]
Data: *We pretty much just got inequivalencies* [Edgar, 50:15]
**Instructor pushes for explanation
Data: *Solve for a and then put that back in...Yeah, we solved for a, plugged it back in and then it just didn't match up...2=0*. [Edgar, Randall, and Lawson, 50:55-7]

Arg 17.9
D-C-W
Claim: The matrix [1,2; 2,4] is not invertible [Abraham, 51:23]
Data: *that span is only the line, the y=2x* [Abraham, 51:23]
Warrant: *And so it never even hits 1,0 or 0,1, because it has to go over 1, up 2, over 1, up 2.* [Abraham, 51:23]

Arg 17.10
C-Q-D1-D2-W-B1-B2
Claim: The matrix [1,2; 2,4] is not invertible [Abraham, 51:51]
Qualifier: *It doesn't seem like the matrix would be invertible* [Abraham, 51:51]
Data1: *We're trying to get* 1,2 to be 1,0 and 2,4 to be 0,1 *and these 2* [1,0 and 0,1] *don't really fall on that line* [where 1,2 and 2,4 are] [Abraham, 51:51]
**Instructor asks if she can help out, Abraham says yes
Data2: *if we do 1,2;2,4 is one on the outside and the a,b,c,d is the thing on the inside...think about this as a transformation, and let's just see what happens to some vector, what happens if you plug in 1,1 or whatever? ... it's actually going to 3,6. So if I had an input here, it's getting mapped to 3,6. What would happen to 0,1? That would get mapped to 1,2, which is along that line...So this is actually collapsing everything to that line* [Instructor, 52:38]
Warrant: *if it all collapses to a line, how it could, um, how it could be 1,0;0,1. It seems like it would have to be able to span to 1,0. It seems like it would have to be able to reach that point.* [Abraham, 54:03]
Backing1: If this is sending everything to that line, then what in the world would be sent to 1,0? Nothing is getting sent to 1,0. What would be getting sent to 0,1? Nothing is getting sent to 0,1. So there's no way for values of the matrix to be able to get sent to the 1,0;0,1 in order to even try to get back. [Instructor, 54:28]
Backing 2: *So we can only transform to multiples of 1,2. If any vectors are multiples of 1,2, you can transform to them, nothing else, though*. [Abraham, 54:51]

Arg 17.11
D-C
Claim: "If A is invertible, then the columns of A have to be linear independent" is logically equivalent to "If the columns of A are linearly dependent, then A is not invertible." [Abraham, 55:31]
Data: So if you think about this as the p statement, and we're saying now this is q. Then a logically equivalent statement would be to say not q to not p. [Abraham, 55:31]

Arguments 12-17 are considered one proof in count

Arg 17.12

Claim: To prove **"**If A is invertible, then the columns of A have to be linear independent," *we're trying to show that the only solution to Ax=0 is the trivial solution.* [Instructor, 56:06]
Data: Assuming A is invertible that would mean there *exists some C such that CA is I and AC is I* and *we're trying to prove that the columns of A are linear independent* [Instructor, 56:06]
Warrant: *what does it mean to be linear independent? That would mean the only solution, it's been a while, to Ax=0 is if x is 0* [Instructor, 56:06]

Arg 17.13

Claim: To get to x=0, we should consider CAx=C(Ax) [Instructor, 57:25]
Data: We know CA=I by assumption [Instructor, 57:25]

Arg 17.14

Claim: CAx=0 [Instructor, 58:06]
Data: Ax=0 by assumption and *we actually can say that C of 0 has to be 0* [M, 57:25]
Warrant: *because C is a linear transformation* [Instructor, 57:25

Arg 17.15

Claim: *we have i of x is equal to 0* [Instructor, 58:06]
Data: We know CA=I so CAx=Ix, and CAx=0 from before [Instructor, 57:25]

Arg 17.16

Claim: The vector x *is the trivial solution* [Abraham, 58:56]
Data: *We can see that x1=0, x2=0, x3=0* [Abraham, 59:00]
Warrant: *Yeah, you can see it, we know that the identity matrix is linear independent. We can see that the only answer is the trivial solution. So x has to be 0.* [Instructor, 59:04]

Arg 17.17

Claim: *A is a matrix with linear independent column vectors*. [Instructor, 59:04]
Data: *x has to be 0* [Instructor, 59:04]
Warrant: *if x has to be 0, then the only solution to A of x equaling 0 is if x=0.* [Instructor, 59:04]

Arg 17.18

C-D-W
Claim: *It* [the matrix A] *has to span* [Abraham, 1:00:45]
Data: *it has to be a square matrix, and you're also saying it has to be linear independent* [Abraham, 1:00:45]
Warrant: *based on our n by n theorem* [it says they relate] [Abraham, 1:00:45]

Arg 17.19

C-D-*-C-*-C-Q
Claim: *It has to span* [Abraham, 1:01:01]
Data: *based on having those 2, if those 2 are true* [A is square and the columns are LI] [Abraham, 1:01:01]
**Instructor asks for clarification of the claim [1:01:15]
Claim: *In order for A to be invertible, then A has to span* Rn [Abraham, 1:01:24]
**Instructor asks for assistance [1:01:39]
Claim: *If A is invertible, then the vectors within A span all or Rn, I guess.* [Kyle, 1:01:56 and Abraham, 1:02:15]
Qualifier: *Let's go ahead and say A is n x n.* [Instructor, 1:02:38]

<u>Arg 17.20</u>
C-Q1-D-Q2--W-R**
Claim: *The matrix you have will row reduce to the identity* [Justin, 1:02:50]
Qualifier1: *I feel like it's kind of redundant* [Justin, 1:02:50]
Data: *Any square matrix you have, that is independent is going to be reducible to the identity matrix* [Justin, 1:03:06]
Qualifier2: *I feel that's a waste, really, it's redundant* [Justin, 1:03:06]
**Instructor calls for a warrant [1:03:24]
Warrant: *If all of these things we came up with to say these have to be true for it to be invertible. So if we have, because what I do when I get a matrix, I put it in my calculator and I row-reduce it, since it's a hell of a lot easier to look at.* [Justin, 1:03:32]
Rebuttal: *I think there's a fine little line here with the implications between if then, which is what I was trying to get people to say well, that connects to what you guys are all saying. So let's think about this for a second, we've got one of them that starts off saying, If A is linearly dependent, then I can say it's not invertible. Then the other 2, the other one I added on here was If A is invertible, then the columns are linearly independent. If A is invertible, then the column vectors span. If A is invertible, it's reducible to the identity matrix. Great. What about if we wanted to go back this way, is it also true? Switching ifs and then?* [Instructor, 1:04:27-1:05:17]

<u>Arg 17.21</u>
C-D-W
Claim: *knowing that if A is invertible then the columns of A are linear independent is not the same as saying If A is linear independent, then A is invertible.* [Instructor, 1:05:55]
Data: there is a *subtle yet really major* difference between those two statements [Instructor, 1:05:46]
Warrant: So if we think about the ifs and then of what we can assume, because we're trying to really delve into this theorem we have, the New Theorem we called it from before, which is great, but we have to make sure we jump into it the right way [Instructor, 1:06:07]

<u>Arg 17.22</u>
C-D-W-Q
Claim: *If A is linearly independent, we're going to try to get that A has to be invertible* [Instructor, 1:06:31]
Data: *A can be row-reduced down to the identity matrix* [Instructor, 1:06:31, attributes to Abraham]
Warrant: We know this [the DATA] from the New Theorem [Instructor, 1:06:31, attributed to Abraham]
Qualifier: *A is row-equivalent to the identity. To me that does not speak anything about A being invertible yet* [Instructor, 1:06:54]

Day 18 Argumentation Log

<u>BIG ARG #1</u>
Claim: If A is square and has linearly independent column vectors, then A is invertible.
<u>Arg 18.1</u>
Claim: A is row-equivalent to the identity
Data: A is square and has linearly independent column vectors [49:10]
Warrant: The New Theorem (as an authority) [Instructor, 49:10]

Arg 18.2
Claim: *there exists a sequence of elementary row operations that turns A into i and vice versa [Nate, attributed by the Instructor, 49:34]*
Data*:* That's what it means to row-reduce to the identity [Nate, attributed by the instructor, 49:34]
Arg 18.3
Claim: there's something acting on A and changing it [Instructor, 50:24]
Data: There are *elementary row operations that turn A into I* [Instructor, 50:24]
Arg 18.4
Claim: *each elementary row operation can be thought of as a linear transformation* [Instructor, 50:24]
Data: *there's something that's acting on A and changing it* [Instructor, 50:24]
Arg 18.5
Claim: *eventually there's some last step, once I act on A by e1 and then e2, all the way down to Ek, then I end up getting the identity* [$E_k ... E_2 E_1 A=I$] [Instructor, 55:09]
Data: *We started with A, and we transformed it with some matrix that stood for an elementary row operation, so we started with A, and we transformed it with E1.* A second elem matrix would change that a little bit, and so on. [Instructor, 55:09-55:44]
Arg 18.6
Claim: The product of elementary matrices is a matrix, which can be called C. [Instr, 56:23]
Data: They [elementary matrices] *were matrices that are square and the same size* [Instructor, 56:23].
Arg 18.7
Claim: CA=I. [Instructor, 57:06]
Data: $E_k ... E_2 E_1 A=I$ and $(E_k ... E_2 E_1)=C$ [Instructor, 57:06]
Arg 18.8
Claim: *Each elementary matrix is invertible* [Instructor, 57:26]
Data: Each of the elementary row operations that takes A towards I is reversible [Instr, 57:26]
Arg 18.9
Claim: C is invertible [Instructor, 58:19]
Data: *The product of invertible matrices is invertible* [Instructor, 57:06]
Warrant: We saw it through Pat and Jamie
Qualifier: *It wasn't a super formal*
Arg 18.10
Claim: *AC=I and CA=I* [Instructor, 58:19]
Data: *C is invertible* [Instructor, 58:19]
Arg 18.11
Claim: *A is C's inverse* [Instructor, 58:52]
Data: *C is invertible and CA=I* [Instructor, 58:52]
Arg 18.12
Claim: *A is invertible* [Instructor, 59:08]
Data: *if you have the inverse of something, it better be true that you're invertible* [Instr, 59:08]
End of big proof

Arg 18.13
C-D-W
Claim: *The columns of A are linearly independent* [Instructor, 59:53]
Data: *A is invertible* [Instructor, 59:53]
Warrant: *From that paper that's still up there, we had it before* [Instructor, 59:53]

<u>Arg 18.14</u>
C-D-W
Claim: *A is invertible* [Instructor, 1:00:09]
Data: *The columns of A are linearly independent* [Instructor, 1:00:09]
Warrant: *we now have* [from the previous long chain of arguments [Instructor, 1:00:09]

<u>Arg 18.15</u>
C-D-W
Claim: *We can add that* [A is invertible] *to the theorem* [the New Theorem] [Instr, 1:00:28]
Data: *now that we have this thing here, and we actually have 2 arrows, we know that this direction is true, and we actually now have that this direction is true* [Instructor, 1:00:28]
Warrant: *in order to be a part of that* [the New Theorem], *you have to have double equivalent.* [1:00:09]

<u>Arg 18.16</u>
C-D-W
Claim: We can add there exists a *C that's n x n such that AC=I. And we can also say there exists a C that's n x n such that CA = I* to the theorem [Instructor, 1:01:17]
Data: *11 and 12* [those two statements] *are the definition of 10* [A is invertible] [Instr, 1:01:17]
Warrant: *they were equivalent because that's what the definition is.*
Backing: *And we said that if one is true, the other one has to be true for 11 and 12, what we had up here that Anthony's question was* [Instructor, 1:02:09]
ta: *for all your x values, you're going to have y values, but you're not going to have these y values at all* [Nigel, 16:35]
Warrant: For instance, the y-value of -5 is impossible *because there's no x value that would get you to that.* [Nigel & Instructor, 17:03-17:11]
Backing: *from the graph you can see, the graphical representation of the function never crosses at y equals -5* [Instructor, 17:14]

Day 19 Argumentation Log

<u>Arg 19.1</u>
C-D-W-B
Claim: the function $y=x^2$ is not onto [Nigel, 16:35]
Data: *for all your x values, you're going to have y values, but you're not going to have these y values at all* [Nigel, 16:35]
Warrant: For instance, the y-value of -5 is impossible *because there's no x value that would get you to that.* [Nigel & Instructor, 17:03-17:11]
Backing: *from the graph you can see, the graphical representation of the function never crosses at y equals -5* [Instructor, 17:14]

<u>Arg 19.2</u>
C-D-W
Claim: the function y=sqrt x is not onto [Nate, 17:40]
Data: [draws graph] *you cannot, there's not, any point on this side* [Nigel,]
Warrant: *Yeah, so altitude up and down, the y values, there's no way to put in something, take the square root of it and get a negative number, that would be a real value solution, right?.* [Instructor, 18:22]

Arg 19.3

C-D-W

Claim: The transformation T: R2→ R3 given by [4,0,0; 0,0,2] *can't be onto* [all of R3] [Justin, 22:32]

Data: *you have a row of 0's, so it's going to give you a column of 0's. And so you will have 2 vectors with numbers in it, or 2 columns with numbers in it, and a column with zeroes in it* [Justin, 22:32]

Warrant: *And that breaks it spanning all of r3* [Justin, 22:32]

Arg 19.4

C-D-W-Q

Claim: the span of [4,0,0; 0,0,2] is not R3 (says *you're not spanning, no matter what, you're not going to span*) [Justin, 23:17]

Data: *put in just a 2 by 3 full of whatever. And the output would give you a 3 by 3 square matrix. And the middle row, the middle column, excuse me, will always be 0's* [Justin, 23:17]

Warrant: I *saw the algebra. Since it's 4,0;0,0;0,2, there's no values in the middle, so you're really convinced you're not spanning that dimension where the x's and y's are can't span where the 0's are, that dimension.* [Nigel, 24:08]

Qualifier: *There are some good ideas, and they all involve a little more thought* [Instr, 24:39]

Arg 19.5

C-D-W-Q--B1-B2**

Claim: The range of the transformation T: R2→ R3 given by [4,0,0; 0,0,2] is the x,z point. [Kaleb, 27:31]

Data: *the 2nd row that vector is all 0's,* [Cayla, 27:40]

Warrant: *essentially you can't* ***get to any y*** *value.* [Cayla, 27:40]

Qualifier: You can't get any y-value *other than 0, because 0 is a value* [Instructor, 27:57]

*******Instructor asks for a reinterpretation (I'll call a backing)

Backing: *No matter where your input vector is, when you multiply it out with that transformation function is going to give you the middle row of the 3 by 3 as 0's.* [Male, 28:33]

Backing2*: So let's actually multiply by this input vector. So we said no matter what your input vector is, if we do x times 4,0,0 + y times 0,0,2. If we add it together, we get 4x + 0,0x + 0y, 0x + 2y. So that I could probably make any number if I want. If I want it to be a million, I could figure out what x would be. If I wanted this to be 47, I could figure out what y would be. But is there any way to get that to not be 0? Not really* [Instructor, 28:41]

Arg 19.6

C-D-Q

Claim: The transformation T[x1,x2,x3]=[x1,x2] is given by the matrix [0,1,0; 0,0,1] [Saul, 40:35]

Data: *We knew that this t want to get x2, x3, those 2 rows. So you multiply this out, you have 1 for the x2, and another 1 for the x3.* [Saul, 40:02]

Qualifier: *Made sense in my head* [Saul, 40:35]

Arg 19.7

C-D-W

Claim: The transformation T[x1,x2,x3]=[x1,x2] is onto R2 [Giovanni, 41:45]

Data*: it has 2 separate vectors, the 2 pivot points right there. Minus, you don't even necessarily need that 1st vector.* [Giovanni, 41:45]

Warrant: *So with those 2 pivot points, that's how it* [the vectors in the matrix] *can span all of* $\mathbf{R}^2$. [Giovanni, 41:45, 42:05]

Arg 19.8

C-D-W1--D2-**-W2**

Claim: Since the columns vectors span R2, the transformation is onto. [Giovanni, restated by Instructor 42:37]

Data1*: If you can't choose any value, you're not going to be able to get output*. [For instance] *If those were 0's, you're dictated what your values, your output is going to be.* [Justin, 42:52]

Warrant1: *But since you can go anywhere, you can use anywhere as an, do you know what I'm trying to say?* [Justin, 42:52]

****Instructor asks for help [43:17]**

Data2*: Having the required amount of pivot points in the correct positions can span that number of dimension*. [Josh, 43:25]

****Instructor asks for a warrant [43:34]**

Warrant2: *Because it has 2 pivots, and it can span r2. And then since those 2 pivots are located in the y and z column, you're trying to have it in the y and z, it just, it works out.* [Josh, 43:40]

Arg 19.9

C-D-Q-R-Q

Claim: *We can use span and onto interchangeably* [Jerry, Abraham, Justin, 44:44-45:23]

Data*: If it spans, the range equals the codomain* [Abraham, 45:01]

Qualifier: *If you're spanning all of your dimensions, then you can* [use span & onto interchangeably] [Justin, 45:09]

Rebuttal: *The span of that one* [an example] *is a line, but it's not onto* [Justin, 45:15]

Qualifier: *We use span to describe a set of vectors, and now we're saying onto describes a transformation* [Instructor, 45:27-48]

Arg 19.10

C-D-W

Claim: the function y = sine x is not 1-1 [Nigel, 59:56]

Data: *The same way that we know that x2 is not 1 to 1. The horizontal line test, I guess* [Nigel 1:00:10]

Warrant: *If you draw as Suzanna did, for any given input...you had the same output... sign of pi over 2 is 1...the sign of 5 pi over 2 is also 1. And that would happen infinitely many times, every multiple of 2pi.* [Instructor, 1:00:16]

Arg 19.11

C-D-W

Claim: This matrix here, 1,1; 2,2, it's not 1 to 1 [Kaleb, 1:12:08]

Data: *you multiply it out by 2 different things* (<0,1> and <2,0>) *to be 2,2* [Kaleb, 1:12:08-1:12:33]

Warrant: *you have 2 inputs giving you that same output, so it's not 1 to 1* [1:12:33]

Day 20 Argumentation Log

Arg 20.1
C-D
Claim: Onto is the same as saying *the range is the same as the entire codomain* [Instr, 02:48]
Data: *We established* [it] *last Wednesday* [Instructor, 02:48]

Arg 20.2
C-D
Claim: *Onto is the same as saying the column vectors of the matrix span all of Rm* [Instr, 02:48]
Data: *A lot of you had also said* [it] [Instructor, 02:48]
Arg 20.3
C-D-W--D-Q-W2-B2**
Claim: The transformation defined by the matrix [1, 0; 0, 1] is 1-1 [Kaleb, 34:40]
Data1: *identity is perfect, it transfers from a 2-dimension to a 2-dimension matrix, and it's going to cover each dimension.* [Kaleb, 34:52]
Warrant1: *If it's 1-1, then you'll looking for a 1 dimensional going each way, it doesn't cover everything* [Kaleb, 34:52]
****Teacher calls for more explanation [35:06]**
Data2: *If you try to transform any matrix with the identity, it's going to be its own matrix* [Jerry, 35:12]
Qualifier (to Data2): *I'm going to change 1 word he's been saying...I'm going to say 'vector' instead.* [Instructor, 35:38]
Warrant2: *you have the identity and then 2,5, well 2,5 gets sent to 2,5; 7,9 gets sent to 7,9; 14,-7 gets sent to 14,-7, etc.* [Instructor, 35:38]
Backing2: *each unique input is only going to itself, so it has 1 output, and there's only 1 of those, so it's only going to map to it once.* [Instructor, 35:38]

Arg 20.4
C-D1--D2-W**
Claim: *You can't be more than 1-1 if m < n* [Mitchell, 36:25]
Data1: *It's just not possible* [Mitchell, 36:25]
****Call for another data**
Data2: having *more vectors than dimensions* means *it has to be linear dependent* [Abraham, 36:42]
Warrant: *In order for it to be 1-1, it has to be linear independent* [Abraham, 36:42]

Arg 20.5
C-D-W-B
Claim: *if it's 1 to 1, the columns of A have to be linear independent.* [Abraham, 37:34]
****Call for data [37:34]**
Data: *1 to 1 is at most 1 solution* [Abraham, 37:43]
Warrant: But we know the trivial, or the Tx=0 always has to have 1 solution. So then it has to have the trivial solution. But if their columns are linear dependent, then T of x is 0 has more than 1 solution. So there would be 1 going to 2 different things [Abraham, 37:43]
Backing (implicit): logical equivalence of proof by contrapositive.

Arg 20.6
C-D-W-B

Claim: *The columns of A are linearly independent is another way to say that the transformation T is 1-1* [Instructor, 41:26]
Data: *When you reduce matrices linear dependent, you're going to have a free variable* [Brad, 40:48]
Warrant: *When you have that free variable, it has to be more than 1 input to get the same output* [Brad, 40:48]
Backing: A free variable gives *you some leniency in how you put down a solution for what's in the span on the vectors, and that variability is giving you more than one way to get a certain output* [Instructor, 40:59]

Arg 20.7
C-D-W-B
Claim: The transformation defined by the matrix [1,0,5; 0,1,7] is 1-1 [Instructor, 43:08, attributed to Donna's table]
Data: *Those two column vectors are linearly independent* [Instructor, 43:08]
Warrant: *That fits our qualification* [Instructor, 43:08]
Backing: *So if you are in the span, there's only one way to get there* [Instructor, 43:08]

Arg 20.8
C-D1-D2
Claim: The transformation defined by the matrix [1,2; 1,2]] is not 1-1 [Jerry, 44:13]
Data1: *I just saw it as row reducing gets you 0's on the bottom* [Jerry, 44:13]
Data2: *And it is linearly dependent because it's a multiple* [Jerry, 44:13]

Arg 20.9
C-D-W
Claim: The transformation defined by the matrix [1,2;1,2;e,f] is not 1-1 [Jerry, 44:13]
Data: *You have the 1,2,1,2 on the left side of that, and e and f can be whatever it wants* [Jerry, 44:54]
Warrant: *Because you have 2 multiples of each other* [Jerry, 44:54]

Arg 20.10
C-D--W**
Claim: For any values a-f, the transformation defined by the matrix [a,c;b,d;e,f] is not 1-1 [Donna, 45:35]
Data: *you could use any number, because it'll still [inaud], no matter what matrix you try to transform it by.* [Donna, 45:35]
****Call for warrant [Instructor, 45:48]**
Warrant: *it could be any number* [the entries in the matrix]*, that means that, it meant that it could have the possibility of having more than 1 solution.* [Donna, 45:53]

Arg 20.11
C-D
Claim: *if we have more column vectors than we have dimensions, this right here always has to be linear dependent* [Instructor, 46:19]
Data: *We heard a long time ago...That was one of the first things we had about linear dependency* [Instructor, 46:19]

Arg 20.12
C-C
Claim1: *It's not going to be 1 to 1 if they are linear dependent,* [Giovanni, 47:36]
Claim2: *and it is 1 to 1 if they are linear independent* [Giovanni, 47:36]

Arg 20.13
C-D-W-B
Claim: The transformation defined by the matrix [3,4; 5,6] is onto [Cayla, 48:21]
Data: *columns are linear independent.* [Cayla 48:21]
Warrant: *if you row reduce it, you would be able to get a pivot position in every row*
Backing: *so it would span all of r2*. [Cayla, 48:21]

Arg 20.14
C-D-W-B-R
Claim: a *transformation being onto* and *the column vectors spanning all of Rn*, *they're actually similar ideas* [Instructor, 49:15, Matthew, 50:10]
Data: *If you're excluding a dimension, you're excluding all the possible answers that would lead to that dimension* [Matthew, 50:10]
**** Instructor calls for warrant [50:16]**
Warrant: *I have an example up there. It's linear dependent vector in r2 times a vector input x in r2. It cannot equal its input. Xy represents a valid b in r2, and the combination of the transformation and input cannot equal that* [Lawson, 50:43]
Backing: *because it's not linear independent.* [Lawson, 50:43]
Rebuttal: *Is there some other way you could have gotten <[x,y>?* [Instructor, 50:43]

Arg 20.15
C-D-W-B
Claim: If the span of a set of vectors is a plane in R3, that transformation can't be onto [Instructor, 51:28, attributed to Matthew]
Data: *All the things that aren't on that plane we're excluding as outputs* [Instructor 51:28]
Warrant: *Those aren't the image of anything* [Instructor, 51:28]
Backing: *The transformation can't be onto* [Instructor, 51:28]

Arg 20.16
C--Q-D (D1-C1- C2/D2)-W (W1 U W2) iff, chain**
Claim: *If the matrix is square,* being linearly independent is the same as being onto [Abraham, 52:21]
****Instructor calls for data [52:26]**
Qualifier: *I just remember*
Data: *if it's square, we had the nxn theorem* [Abraham, 52:52]
Data1: *If a square matrix is linear independent,*
Claim1: *it also spans* [Abraham, 52:52]
Claim2/Data2: *And vice versa.*
Warrant: *As so that means* (union of those two D-C)
Warrant1: *if it's 1 to 1, it has to be onto*
Warrant2: *if it's onto, it has to be 1 to 1*

<u>**Arg 20.17**</u>
C-D-W1-W2-B
Claim: If a set of vectors is linearly dependent, we cannot *automatically conclude that the transformation is not onto* [Instructor & Male, 54:27]
Data: The 3 vectors [1,0; 0,1; 0,0] are linearly dependent but the transformation is onto [Instructor & Male, 54:27]
Warrant1: *There's 2 pivot positions* [Male, 45:53]
Warrant 2: *It spans R2* [Kaleb, *54:53*]
Backing: *So the matrix has 2 pivot points, so the vectors span all of r2. And since those span all of r2, we can get anywhere in r2. So we say that T is onto* [Instructor, 55:05]

<u>**Arg 20.18**</u>
C-D-W-B1-B2
Claim: It is not possible to have a transformation from Rn→ Rm, m > n, be onto [Nate, 55:45]
Data: *you'll always go up a dimension* [Nate, 55:45]
Warrant: *that means we're going to have, more or less the way I think of it as dots in a diagram, we have more dots in our results in our codomain than in our domain* [Nate, 55:45]
Backing: *so there's no way for every dot in R3 to be mapped to*. [Nate, 55:45]
Backing: *I think that was really cool ... He's saying if you start out in r2 but we have more outputs vectors in r3 in a certain way, well, they can't all get mapped to*. [Instructor, 56:13]

<u>**Arg 20.19**</u>
C-D-W-B
Claim: It is not possible to have a transformation from Rn→ Rm, m > n, be onto [Edgar, 56:48]
Data: *There's only 2 vectors, so you can't possibly span r3. You simply don't have enough vectors to get anywhere you have in r3.* [Edgar, 56:48]
Warrant: *If we only have 2 vectors then all combinations of those 2 vectors are only going to give us a plane in r3.* [Instructor, 57:02]
Backing: *You don't have any way to get to all of r3. So there's no way it could be onto.* [Instructor, 57:02]

<u>**Arg 20.20**</u>
C-D
Claim: It is *possible to have a transformation that is both 1 to 1 and onto* [Jerry, 58:59, Instructor, 59:11]
Data: [from example] *I think if we look at the board, we have m = n, we have a 1 to 1 and onto.* [Instructor, 59:11]

<u>**Arg 20.21**</u>
C-D-W-B
Claim: It is possible to have a transformation that is both 1 to 1 and onto [Cayla, 59:40]
Data: *It has to be square, and the columns have to be independent in order to get pivot points in each row to make it span all of the dimension.* [Cayla, 59:40]
Warrant: *we need the column vectors are linear independent, and we need the column vectors to be able to span Rn.* [Instructor, 1:00:05]
Backing: *it's 1 to 1 if the column vectors are linear independent, and it spans if the column vectors span Rm, so we need both of those to be true.* [Instructor, 1:00:05]

<u>Arg 20.22</u>
C-D (D1-C1 U D2-C2)-W
Claim: *the only time that we have them both being true* [1-1 and onto] *is if they're square* [Instructor, 1:00:44, attr to Cayla]
Data1: *Because if it's wider than tall,*
Claim1: *then 1-1 isn't possible.*
Data2: *And if it's taller than wide,*
Claim2: *then onto isn't possible.* [Instructor, 1:00:44]
Warrant: *we can look at the boards around the room. If we go down the columns, it seems to be the case that the only time that we have them both being true* is when m = n [Instr, 1:00:44]

<u>Arg 20.23</u>
C-D
Claim: *we said then if it's row reducible to the identity matrix then it's both 1 to 1 and onto?* [Jerry, 1:01:22]
**Instructor calls for data
Data: *None.*
Qualifer: *I don't know* [Jerry, 1:01:50]

<u>Arg 20.24</u>
C-Q--D-W**
Claim: *It* [a matrix] *would be invertible if all those things are true* [Abraham, 1:02:33]
Qualifier: *It seems like it* [Abraham, 1:02:33]
****Teacher calls for data**
Data: *I remember linear independent, span and row reduces to identity matrix and square all being in the invertible matrix theorem.* [Abraham, 1:02:54]
Warrant: *those are all the same things we had up there* [Abraham, 1:02:54]

<u>Arg 20.25</u>
C-D (D1 U D2) W (W1 (C3-D3-W3, C4-D4) hard one
Claim: *if it's square, if it's 1-1, it is always also onto* [Edgar, 1:05:18, Abraham/Kaleb, 1:05:23]
Data is D1 U D2
Data1: *If it's, to be 1 to 1, it seems like it's really into the dependence and independence of the vectors involved* [Gabe, 1:06:18]
Data2: *onto seems to be related to span* [Gabe, 1:06:18]
Warrant: *So* [the union of D1 and D2] (now we have cases)
Warrant1:
D3: *If it's both independent and spanning*
C3: *That means it's both onto and 1-1.*
W3: *It has to be able to reduce to the identity matrix*
D4: *If it's both dependent and non-spanning*
C4: *That means it's neither 1-1 or onto.*

<u>Arg 20.26</u>
C-D-W
Claim: *if it's either 1 to 1 or onto but not both, then it has to be non-square* [Abraham, 1:09:55]
Data: *Because of the chart* [Abraham, 1:09:55]
Warrant: *if you're one and not the other, then you can't be square.*
Data1: *If you're square*

Claim1: *you're either both or neither.*
Data2: *And if you're one but not the other, meaning 1 to 1 or onto*
Claim2*: then you must be either wider or taller but you can't be square.*

Day 24 Argumentation Log

Arg 24.1
C-D-W-B-Q
Claim: For the transformation matrix [1,1; 2,2] *the area would be 0* [Cayla, attrib to Brad or Jerry, 09:59]
Data: *Basically that's a line, and there is no way to transform that into a parallelogram* Cayla, attributed to Brad or Jerry, 09:59]
Warrant: *So if we try to transform the first unit vector for the unit square that goes to 1,1. And if we try to transform the other one, 0,1, that goes to 2,2.* [Instructor, 10:26]
Backing: *So that's a line, and there's basically no area to it.* [Cayla, 10:52]
Qualifier: *We didn't know if there was no area to the line* [Suzanna, 11:11]

Arg 24.2
C-D-W
Claim: Linearly dependent matrices have determinant zero [David, attributed to Donna, 12:23]
Data*: if you a linear dependent matrix you multiple all the vectors by, it'll just put them on a line* [David, attributed to Donna, 12:23]
Warrant: *The line doesn't have any area* [David, attributed to Donna, 12:23]

Arg 24.3
C-D (D1U D2)-Wim
Claim: In the 2D case**,** *if the column vectors are linear dependent, then the determinant of that matrix is 0* [Instructor, 13:02]
Data*:* **D1 U D2** (linked)
Data1: *we can use this area argument.* [in arg #1 and #2] [Instructor, 13:28]
Data2: *If we would try to compute the determinant of this thing, see if it would actually get, 1 times 2 minus 2 times 1, so it would get to be 0 anyway.* [Instructor, 13:28]
Warrant (implicit): D1 U D2 is valid data

Arg 24.4
C-D1-Q-D2-**W-Q-D3-W**
Claim: *If the determinant is 0, then the column vectors have to be linear dependent* [Abraham, 13:55, after Instructor asked if it was true]
Data1*: I have no idea* [Abraham, 13:55]
Qualifier: *It just seems [Abraham, 13:58]*
Data2: *it's a square matrix, and it's independent so it's going to span.* [Randall, 14:10]
****Instructor calls for a warrant [14:16]**
Warrant: *Well, if it spans, then it's invertible...* [Randall, 14:19]
Qualifier: *I don't really know where to go with that.* [Randall, 14:19]
Data3: *Even with this geometric approach, it seems pretty true* [Instructor, 14:28], *we would say it* [the claim] *is true* [Instructor, 15:09]
Warrant: You will go through it in homework [Instructor, 14:54]

<u>**Arg 24.5**</u>
C-D (C1-D1-W1 U D2) –W (em1 U D2)
Claim: *For the 2x2 case...* A is invertible, the columns of A are linear independent, and the determinant of A is not 0 are equivalent statements [Instructor, 16:43]
Embedded1:
Claim1: *Since it's linearly dependent, it's not invertible, just the opposite* [Edgar, 16:02
Data1: *if it was linear independent, it's invertible.* [Edgar, 15:43]
Warrant1: *That's one of those from your exams, so it's what we had before, so for the 2x2 case, we can say this* [claim1] [Instructor, 16:10]
Data 2: *We just had up here if the columns are linear dependent then the determinant is 0.* [Instructor, 16:10]
Warrant: (implicit Em1 U D2]*So we can add that one down here as well, add it to the mix of, at least for 2 by 2 cases, equivalent statements.* [Instructor, 16:10]

<u>**Arg 24.6**</u>
C-D-Wim-Q
Claim: *if A is not invertible, that happens when the columns are dependent, and we can say that's connected to when the determinant is 0* [Instructor, 17:00]
Data: *We can say the opposite of all these [the claim from Arg #5]* [Instructor, 17:00]
Warrant1: *(implicit)* p ⇔ q ⇔ r ➔ not p ⇔ not q ⇔ not r is logically valid
Qualifier (for both Arg #5 and Arg #6): *we've only been talking about 2-dimensional space, so we have 2-vector matrices. And linear dependence is kind of the odd case in 2 by 2's, because the only way you could have it is with scalar multiples. So let's see if this holds true in r3.* [Instructor, 17:29]

<u>**Arg 24.7**</u>
C-D-W-B
Claim: *The determinant of [a, c, 0; b, d, 0; 0, 0, k] is (ad-bc)*k* [Adam, 30:47]
Data: *the original ad - bc and multiply it by k to get the volume.* [Adam, 30:47]
Warrant: *Since we had area of parallelogram [points board], we thought that you take the volume, you multiply it by k, because it's just the 3 sides.* [Adam, 30:47]
Backing: *the ad - bc is the base, and we learned in high school that volume is based times height for certain prisms, then we get this k as the height* [Instructor, 31:42]

<u>**Arg 24.8**</u>
C-D
Claim: *The determinant of [a, c, 0; b, d, 0; 0, 0, k] is (ad-bc)*k* [Justin, 31:32]
Data: *I just remembered it from a previous class* [Justin, 31:32]

<u>**Arg 24.9**</u>
C--D-W-B**
Claim: *The volume is zero when all the columns of the transformations are dependent, all the different equations of that* [Gabe, 32:18]
****Call for data (what are the possible ways?)**
Data: *If k is zero, we're down to a plane or if column 1 and column 2 are scalar multiples of each other* [Instructor, 32:59]
Warrant: *Does a plane have any volume? Not really* [Instructor, 32:59]

Backing: *So* [there are] *a couple different ways that scrunches down to a point or a line or a plane, there are a couple different ways that could come in* [Instructor, 33:38]

<u>**Arg 24.10**</u>
C-D (C1-D1-R-D2=C1-C2-Q(chain) (or is the Q a W)
Claim: *It makes sense to add det A not equal to 0 to the IMT* [Instructor, 35:06]
Data (Chain):
Data1: *If the determinant is zero* [Lawson, 37:17]
Claim1: *Then A is linearly independent* [Lawson, 37:17]
Rebuttal (to claim1): *Dependent* [Student, 37:30]
Data2=Claim1 (implicit): *If A is linearly dependent* [Lawson, 37:17]
Claim2: *Therefore not invertible* [Lawson, 37:17]
Qualifier: *I'm letting Lawson just say that without reasons why for right now, that sort of thing you can do on your homework this week* [37:44]

<u>**Arg 24.11**</u>
C-D (C1-D1)-W
Claim: *If ad-bc is 0, you can't row-reduce to I.* [Instructor, 39:59]
Data: (embedded): When row-reducing A, *ad-bc* shows up *at the bottom left corner*.
Claim1:*It's linearly dependent* [Student, 39:55]]
Data1: *If that [the bottom right corner] is zero* [Instructor, 39:31]
Warrant: *that was one of the exam questions we had, it's some matrix that had a row of 0's on the bottom, it was a 2 by 2. We know that's not invertible, it can't be row reduced to the identity.* [Instructor, 39:59]

<u>**Arg 24.12**</u>
C-D
Claim: The inverse of a 2x2 matrix A is [d/(ad-bc), -c/(ad-bc); -b/(ad-bc), a/(ad-bc)] [Instructor, 42:56]
Data: That is the matrix that shows up *on the left* [sic, right] *side of the row reduced form* [of [A|I]~[I|A-1] [Instructor, 42:56]

<u>**Arg 24.13**</u>
C-D-W
Claim: If det A = 0, A is not invertible [Instructor, 42:56]
Data: *Inverse is undefined* is you had 1 over 0 [Gabe, 44:00]
Warrant: *Yeah, that one doesn't make sense, right?* [to divide by zero] [Instructor, 44:01]

<u>**Arg 24.14**</u>
C-D-W-B
Claim: *It makes sense that* the determinant of C = [0,0,0; 2,5,1; 3,1,-2] is 0 [Nate, 49:54]
Data: *There's a whole column full of zeroes* [Nate, 49:54]
Warrant: *That shows the matrix is dependent* [Nate, 49:54]
Backing: *We learned before, if it's linearly dependent, it collapses the volume, so you can't have volume, it's collapsing* [Nate, 49:54]

Day 31 Argumentation Log

Arg 31.1
C-D-W
Claim: *We had an arrow between 'Invertible matrix' or 'matrix B such that AB=I.' And 'there exists an invertible matrix C such that CA=I.* [Randall, 30:56]
Data: These three concepts *are almost exactly the same thing* [Randall, 30:56]
Warrant: *It shows commutativity, they can be multiplied either way, and you're supposed to get I, but they're both the same.*[Randall, 30:56]

Arg 31.2
C-D-W-B
Claim: We put det A not being zero at the top of our board [i.e, it's an idea we are most comfortable with] [Nila's group, 35:32]
Data: *We think of determinant as area or matrices, when a transformation...*[Nila, 35:44]
Warrant: *The determinant as the area of the transformation, so if it's 0, then it's not changing it, so it can't be 0* [Carter, 36:05]
Backing: *If the determinant is a way to measure area, then if it does equal 0, then there's no area in between and we see that those 2 lines are linearly dependent, it can't be invertible.* [Instructor, 36:19]

Arg 31.3
C-D (C1-D1 and C2-D2-W2)-Q (linked data)
Claim: The most obviously equivalent set for us is *'the columns of A span all of Rn, the column space of A is all of Rn, and for every b in Rn, there's a way to write b as a linear combination of the columns of A* [Justin, 45:37]
Data: *we think they go together very well because:*
Claim1: *It's* [span and column space] *like the same thing*
Data1: *because, if your columns of A do span all of Rn, then your column space is going to be all of Rn*
AND (linked data)
Claim 2: The "for every b" card is also connected (implicit claim)
Data2: *When we're trying to get to Gauss' cabin or whatever, you know, we had to be able to make a linear combination*
Warrant2: *our columns can make a linear combo for everywhere and span everywhere*
Qualifier; *I think I'm rambling*

Arg 31.4
C-D-W
Claim: you'll be able to make a linear combination of them to get to where you want. [Jorge, 46:41]
Data: *using Gauss' cabin, since you can go anywhere because of the 1st 2,* **[**Jorge, 46:41]
Warrant: *since b would be a linear combination of all the vectors there* [Jorge, 46:41]

Arg 31.5
C-D-W-B
Claim: "'For every b in Rn, there exists a solution x to Ax=b **"** is also equiv to the previous 3 (span, column space, lin combo [Abraham, 47:03]
Data: *There's solution to every b,.* [Abraham, 47:11]

Warrant: *so we can get to that output vector, and that's span* [Abraham, 47:11]
Backing: *So they're kind of equivalent ideas for you, because they by definition are the same* [Instructor, 47:42, Abraham, 48:26]

<u>Arg 31.6</u>
C-D-Q
Claim: *If we added a word 'unique,' then I would put it* [the columns are linearly independent] *in there* [Abraham, 47:11]
Data: *Because 'unique' makes it linear independence.* [Abraham, 47:11]
Qualifier: *but without the word, I just think of we can get to every b.* [Abraham, 47:11]

<u>Arg 31.7</u>
C-D
Claim: *the only solution to Ax=0 being the trivial solution and being linear independent are obviously equivalent* [Edgar's table's board, stated by Instructor, 48:32]
Data: *By definition* [they] *have to be the same, that's how we define both of them* [Instructor, 48:32]

<u>Arg 31.8</u>
C-D (D1-C1=D2-C2) chained data
Claim: *'the columns of A span Rn,' and 'the columns are linear independent'* are *an obviously equivalent pair.* [Nate's table's board, stated by Instructor, 48:57]
Data:
Claim1: *It has n amount of pivot points*
Data1: *If the columns of A are linearly independent,*
Data 2-Claim1: *If it has n amount of pivot points*
Qual*: And it's an nxn matrix, assuming that*
Claim2: *Then it spans all of Rn.*

<u>Arg 31.9</u>
C-D-W-B
Claim: *The main 'hub' for our table was A is invertible* [Randall, 1:08:30]
Data: *because all of these, they're all pointing to that, that's the reason we have these.*
Warrant: *I just felt that each one of these was a different way of saying that.*
Backing: *we didn't have 'A is invertible' on the list, these would all be equivalent to each other, but not really get anywhere* [Randall, 1:08:05]

Appendix 4.2. Transcript for the section of Day 6 that was analyzed

(0:01:53.7) Instructor: And upon reading your Reflections, the last one was the one that a lot of you said you weren't sure about. Sorry, I don't have your Reflections today. I had asked you, 'What generalization are you confident about and why, and which one are you not confident about and why?' And I caught quite a few people saying, 'I'm not really sure about #4 yet.' So let's go ahead and start off class by thinking about #4 together. So just take a couple minutes in your group to think about #4. Let's take about 3 minutes and let's talk about why #4 might be true and how you'd be convinced of that. We'll reconvene and talk about it for a couple more minutes.

—SMALL GROUP WORK: (0:02:31.4-0:04:26)—

(0:04:26.9) Instructor: As I was walking around, I saw a little more puzzled faces than I expected, it is Monday and I actually heard someone said, 'How are we supposed to know how to make a generalization if we don't know what it's about, or we don't understand what it's about?' I think that's a great question. So how about, can I have someone who understands #4 restate #4 in their own words, what do we mean by this generalization?

(0:04:54.3) Justin: If you have more vectors than dimensions, you'll always be able to return to your original position.

(0:05:03.9) Instructor: Could you say that louder? If you have more vectors than dimensions?

(0:05:10.2) Justin: Then you can always return to your original position.

(0:05:17.5) Instructor: [writes board] Does that resonate with anyone else's way of thinking about this problem? Nate, can you say anything about the way you understand what #4 is about?

(0:05:26.2) Nate: It's saying if you have, there's no other way of putting it, if there's more vectors than. I don't know another way to say it.

(0:05:37.8) Instructor: That's fine. So that's the problem statement, now the question would be, how can we explain why that is true? Jerry, did your table get to talk about a reason why #4 is going to make sense to your table?

(0:05:51.8) Jerry: Not really.

(0:05:56.8) Instructor: Not really. Saul, how did your table talk about #4?

(0:06:04.0)Saul: We were confused on how to prove it, we didn't know where to start, where to go with a proof. [inaud]

(0:06:12.4) Instructor: Okay, why don't we start with an example? I think we had some from last class that, where let's just say we have 3 vectors in r2. So we're just going to have any 3 vectors in r2. Part of the generalization says, it doesn't matter what the vectors are, since I have more than I have dimensions, I should be able to get back home or return back to my starting point. [pause]

(0:06:46.7) Instructor: So why don't you take another minute in your group to think about this particular example that I just made up. If you don't like my numbers, you can change them, that's fine, too. Let's think about why if we had 3 vectors in r2, that would mean I should always be able to get back home?

(0:07:00.1) Instructor: Edgar, can you tell me again what I'm asking you to work on in your groups? I want to make sure everyone, when you go in your groups, I want to make sure you know what I'm asking you to work on. I don't want to give you 5 minutes of dead time, that's not the point.

(0:07:12.7)Edgar: You want us to basically find a way to prove whether or not it's true.

(0:07:18.0) Instructor: Okay, take 1 more minute, please.

—SMALL GROUP WORK: (0:07:23.9-0:08:33.9)—

Instructor: Okay, Lawson is going to come up and share his explanation for #4 with us, please? I'm sorry, are you guys okay? I don't want to cut off.

(0:08:46.0) Justin: No, go ahead, it's alright.

(0:08:58.9)Lawson: If you have 3 nonparallel vectors, if you graphically think about it. You say you have a vector over here, and then plus a vector that's not parallel to it. And we have just any other random vector, say it goes this way. If you add any 2 nonparallel vectors that at some point these vectors are going to come across this other vector, and hit it at a point. So if that happens, then you can just take the last vector and multiply it by some scalar to get you back to the origin. That's how graphically we got it.

(0:09:44.4) Instructor: I totally agree basically. I heard him say something that I'm curious about. What if the set of vectors were 1,2; 2,4; -9,7?

(0:09:57.9)Lawson: You can just ignore this last vector, take 1, like 1,2 which would be, and just take the 2,4 back, it's a multiple of 1,2, and you can just not use this one. So it's linear dependent.

(0:10:21.7) Instructor: Cayla, I see you nodding, can you explain how this makes sense to how you're thinking?

(0:10:28.7)Cayla: I was nodding because it makes sense that pretty much if you have 3 vectors in r2, even in the 1st example, you could technically ignore the last one, because if you take 1,2 out, then you can still use any combination of the 1,2 and the 3,5 to get back.

(0:11:20.5) Instructor: So Aziz, I may come back to you. I heard her say something about taking this vector out, right? [-9,7] And still getting everywhere. Aziz, tell me what you want to say?

(0:11:40.0)Aziz: If a vector is contained in the span of 2 other vectors, then you should be able to reach the origin, get back to the origin.

Instructor: Say more about that?

(0:11:50.8)Aziz: 2 vectors span r2 or they don't, if they're linearly dependent already, then it makes the set linearly dependent. If 2 aren't, then they span r2, and the 3rd one has to be contained in R2 to be able to reach back to the origin.

(0:12:09.3) Instructor: David, did you hear what Aziz said?

(0:12:13.7) David: I didn't really hear what he said, no.

(0:12:15.7) Instructor: Could you say it again? I want everyone to listen, because I think he said quite a few generalizations in there, and some important things I want to hit on.

(0:12:22.4)Aziz: If 2 vectors span r2, if the 3rd one is contained in r2, then you should be able to reach the origin back. But if the 2 vectors are multiples of each other, then it already makes the set linearly dependent. Them 2 making the span makes the 3rd one be able to reach back to the origin.

(0:12:46.3) Instructor: Yeah, so what I heard was, we're debating whether linear dependence or not, he had 2 cases, he had the case of if this and this are multiples of each other. Then it sounds like this set is already linearly dependent, so the 3rd one [inaud]. Now the other case, where none of them are multiples of each other, I know that if the 1st 2 are linearly independent, that their span is all of r2, so the 3rd one, if it's in r2, has to be able to take you back to the origin. I hope the notetaker is getting some of these down, I know we're not writing them all down, but these are the kind of things you should be trying to write down, please, as you're taking notes.

(0:13:54.1) Instructor: So I want to go back to something else that she said in here about taking a vector out. If you think about the definitions on your page, you have linear independence and you have span. They're slightly different but they're definitely related. If we look at this set, the span of this set is all of r2. Now Cayla was saying that actually if we took out 1 of those vectors, the span is still all of r2. Now the question would be, how does this relate to linear independence or dependence, are they both linearly independent, are they both linearly dependent, one of each?

(0:14:59.7) Instructor: Matthew, your table back there, can you tell me about the 1st one up here, how are you thinking about linear independence or dependence?

(0:15:10.0) Matthew: It should be, if they're all existing within the x,y plane, they're all in the same plane. So once you have 3 of them, that is, you don't have 2 of them that lie along the same vector, on the Same line, then we should be able to get back to the origin and it should be dependent.

(0:15:32.1) Instructor: So you're saying dependent for this 1st one? Okay. And Juan, what about the 2nd set?

(0:15:47.9) Juan: Linear independent.

(0:15:49.7) Instructor: Can you gve us a reason?

(0:15:51.3) Juan: Because it spans r2.

(0:15:55.0) Instructor: He says the 2nd set is linearly independent because it spans r2. What do you guys say?

(0:16:01.4) Nate: That's not a good reason.

(0:16:07.4) Instructor: I would say we need a little bit more. Caleb, can you add on some more? Are you the one that whispered something under your breath? Student: [inaud]

(0:16:18.0) Instructor: Okay, yeah, it is a linearly independent set, and it does span r2, but so does this one, right? So what's the difference? Randall: [inaud]

(0:16:28.8) Instructor: The x [inaud], say more?

(0:16:30.3)Randall: I was just saying you can go anywhere with the 1st 2 vectors in r2, but your way home, if you took away -9,7, there's no way back. So it's independent because you can go anywhere, but there's no home.

(0:16:47.2) Instructor: Yeah, we took away that 3rd vector, so that way home that we had no longer exists there. The original question I was talking about, the generalization #4. So I don't want to beat this thing to death, but let's go back to this. We did a case of 3 vectors in r2, but what about 4 vectors in r3? I think that was also in the chart. So I think table 4, you guys were the ones who came up with this generalization, can you guys say a little bit more in general how this makes sense to you, not just in the r2 case?

(0:17:19.3) Justin: So if you start in any Rn, and you just start with 1 vector and keep adding more. So let's do R3, just for an example. So we start with 1 vector. So either, we have 2 choices: The next vector we add can either be on the Same line, which means it's already linearly dependent, so we don't want that, so we're going to put it off somewhere else. Now the span of that is a plane in 3 dimensions. So now we're going to add another vector in. Our 3rd vector, now it can either be in that span or out of that span. And we want it to be linearly independent, so we're going to put it out of that span. But now that we have that going off of that plane, we just extended our span to all of r3.

(0:18:09.9) Justin: So our 4th vector, when we put it in, no matter where we put it, it's going to get us back home. Because just like in this case, we have to have the last one to get back home, we can get anywhere with those 1st 3 that we put in, but we have to have to have that 4th one to come back. And so it works like that in any dimension, because the more you, if you keep adding, eventually you're going to get the span of your dimensions, and then you're going to have that extra one bringing you back. Unless you have 2 vectors that are lying on the Same line, then you won't have the span of all of your dimension, but it's negligible because those 2 will give you a linearly dependent set. Does that make sense?

(0:19:00.3) Instructor: I think Nate has a question for you?

(0:19:02.2) Nate: The only, the big question that I have is, how can you make generalization about let's say 10th dimension, if you don't know what it really is?

(0:19:10.0) Instructor: That a good question. A lot of the ways we've been trying to think of this is in r2 and r3 because we can visualize it physically, right? But after a while we don't know what r10 means. [Said under his breath, C2: (0:19:24.1)Greg: Why should we care?] But could we still have something along those lines that makes sense? I think as we keep going in the semester, even today, we're going to start some more like computational techniques, but lose a little bit of this context, and it will maybe help us see how we could get to r10. Any other ways that you can think about this stuff, for you guys, can we reach all of r2 and r3

(0:19:46.7)Aziz: I'd like to add on to this, if you're going from the plane to the volume of space, that's with 3 vectors in 3 dimensions. With the 4th one, you can cross all those dimensions back to the origin, that's what makes the 4th dimension, the 4th vector, it makes you able to go back to the beginning. Because each vector takes up a different point in space. You're going from the plane to the volume, gives you those 3 dimensions but you can't get back to origin, unless you have a 4th one that can cross all 3.

(0:20:24.9) Instructor: I think Ali and Justin said a lot and it's all accurate, but I think it might have been a lot to take in, so take a minute with the person next to you, and try to restate what you heard from them, try to solidify it while you're thinking of it, then we'll move on to old Reflection.

—SMALL GROUP WORK: (0:20:45.2-0:21:52.8)—

(Instructor: Allen, can you tell me what you and your table talked about, a short summary?

(0:22:00.0)Allen: We were just summarizing what he was saying, whatever dimensions we have, like say r3, whatever, more vectors that we have more than 3, like 4 or 5, all those vectors are always going to come back to the origin. Whenever it's the same amount, it's just going to span.

(0:22:22.1) Instructor: Same amount is going to span. Any more comments?

Appendix 4.3. Compilation of Adjacency Matrix codes for Whole Class Discussion

Argument	*Used for* ***m < n*** *in general or a specific example*	*Used for* ***m = n*** *in general or a specific example*	*Used for* ***m > n*** *in general or a specific example*	*Used for* ***all m, n*** *or was vague*
Arg 5.1: CMP 1	G3→F9→ S8→F	F4→F E4→G E4→G2		
Arg 6.1: CMP 1	S8→F3 S8→F3			
Arg 6.2	*Content not appropriate for coding scheme*			
Arg 6.3: CMP 1	E4→G2→F9			
Arg 6.4: CMP 1	F4→F3→ F5→F S8→G4→F4			
Arg 6.5: CMP 1	G→F9 G→F9 F5→F G→F9 F5→F G→F9	F→F E→G		F7→F3
Arg 6.6:CMP 1	F4→E4→ F9→F			
Arg 6.7: CMP 1		G**→E G**→E G2→E8 G→E3**→E		
Arg 6.8: CMP 1	S10→F9 G2→F9 G1→F9 F5→H1 F5→F G1→F9 G4→G1 →F9→F4	E7**→E →G4→G	F4→F E5**→E→H5	
Arg 9.1: CMP 1	S10→F S10→F			
Arg 9.2	E→E1			

	F1→F			
Arg 9.3	J5→F2**→F			
Arg 9.4	J1→J4**→J5			
Arg 9.5	*Content not appropriate for coding scheme*			
Arg 9.6	E5→G E5→G F3→G			
Arg 9.7	J4→J5**→G			
Arg 9.8	I3→G→J5→ G1**→G I3→G4→G I→G→G2			
Arg 9.9				I3→G4→G
Arg 9.10	*Content not appropriate for coding scheme*			
Arg 9.11	J4→J5 J5→F			
Arg 9.12	J3→H4 F4→H2 J3→H4			
<u>Arg 9.13</u>	H1→H H1→H4			
<u>Arg 9.14</u>	J→H4 G4→H4 J3→H4, I3→G I→G4→G			
Arg 9.15	S10→F S10 →G→F S10→J4→F			
Arg 9.16			S9→H H1→H2 S9→H1 S9→J3→H	
Arg 9.17				F8→F F5→F4→F F5→F3→F
Arg 10.1: CMP 1		I3**→G→E J3→F I3→I1→G E4→E		
Arg 10.2: CMP 1		I3→I4→E3** →G→E		
Arg 10.3: CMP 1		S10→**G→E		
Arg 10.4: CMP 1		S10→**G→E		
Arg 10.5		G3 →E3 I3 →I5 I3→I1→E2 G→F2		
Arg 10.6		G→F2		

Arg 10.6		G→F2 S10→E2		
Arg 10.7	***Content not appropriate for coding*** *scheme*			
Arg 10.8		I3→I5→I6 G→I5→I6		
Arg 10.9		I1→E2**→E E→E2, E2→E		
Arg 17.1		F8→**S1→L3		
Arg 17.2		F8→L3 F8→H2		
Arg 17.3		H→L F9→L F8→L4 F8→S1**→L4 S1→L4 F8→J3 S1→L S1→L F8→L		
Arg 17.4		F5→L F→L		
Arg 17.5		S1**→F8→L		
Arg 17.6	G→F	F5→J5→ J6→L→ F→L→F→H F→H		
Arg 17.7	F→L G→F→L	K→S10	E→L F→L K→K4→S10	
Arg 17.8		F→S1→ L4→L		
Arg 17.9		F→H5→ L6→L		
Arg 17.10		H5→L6→L H5→N4 N4→L6→H2 N4→L6→L3 F5→N3		
Arg 17.11:		F→L K→E		
Arg 17.12-17		K→K4 E→E1 E1→E F→L		

Arg 17.18		E → G K→G		
Arg 17.19		E→G E→G2 K→G K→G		
Arg 17.20		E → I2 K→I2 F→L K→E K→G K→I2		
Arg 17.21	*Content not appropriate for coding scheme*			
Arg 17.22		E→I2		
Arg 18.1-12:		E→I2 I2→K5, K5→I2 K5→K4→K E→K I2→K		
Arg 18.13		K→E		
Arg 18.14		E→K		
Arg 18.15		E→K, K→E		
Arg 18.16		I5→K K→K4 K4→K		
Arg 19.1				N3→N N1→N
Arg 19.2	*Content not appropriate for coding scheme*			
Arg 19.3: CMP 2			F8→N F8→H	
Arg 19.4: CMP 2			F8→N3→ **H→N F8→H→N3	
Arg 19.5: CMP 2			F8→N3** →N2 F8→N3 F8→N3	
Arg 19.6	*Content not appropriate for coding scheme*			
Arg 19.7: CMP 2			I3→**M I3→G	
Arg 19.8: CMP 2	H5→N4		G→M J3→N3 G2→M3 I→G I→G	

	H→N3 H5→N4			
Arg 19.9: CMP 2				G→M2→ G→M, M→G H5→N→ H→N G→M
Arg 19.10				P4→P
Arg 19.11				P4→P
Arg 20.1: CMP 2				M→M1→M3 M→M2, M2→M
Arg 20.2: CMP 2				M→G, G→M
Arg 20.3		S10→M2→O O2→O		
Arg 20.4: CMP 1	S8 →P S8→F, E**→O			
Arg 20.5				S8→F O→E O1→O, O→O1**→E1 F→F1→P4
Arg 20.6				F2→P F→J5→P4 J5→F2→P3→ P1 E→O, O→E G→M, M→G E→O, O→M
Arg 20.7			E→N2 E→O3	
Arg 20.8-9	F5→P	F5→J6→P		
Arg 20.10	P5→P F2→**P			
Arg 20.11	S8→F S8→F F→P			
Arg 20.12	F→P E→O			
Arg 20.13: CMP 2		E→I3→G→** M		M→M2 M→M1
Arg 20.14: CMP 2		F→N3		H4→N3
Arg 20.15:				H5→N3→N

CMP 2				
Arg 20.16: CMP 1&2		E→M, M→E G→M E→G G→E→O →M, M→ O O→G G→O F→H H→N		
Arg 20.17: CMP 2			I3→G I3→G→ G2→M	
Arg 20.18: CMP 2			N5→$S12_1$→ N3→ N	
Arg 20.19: CMP 2			S9**→H1→H S9→H5→H2 →N	
Arg 20.20: CMP 2	*Content not appropriate for coding scheme*			
Arg 20.21: CMP 2		E→I3→G E→G E→O, G→M O→M, M→O		
Arg 20.22: CMP 2		M→O, O→M S8→P S9→N O→M, M→O		
Arg 20.23: CMP 2		I2→O, I2→M		
Arg 20.24: CMP 2		O→K, M→K		
Arg 20.25: CMP 2		O→E, E→O, M→G, G→M E→I3→O G→I3→M F→P H→N		O→ M
Arg 20.26: CMP 2	M→P, P→M M→P, P→M	M→O, O→M, N→P P→N	N→O, O→N N→O, O→N	
Arg 24.1		F→H5→R1		

Arg 24.1		F→H5→R1 N4→R3 N4→R3→R1		
Arg 24.2		E→N4→R3		
Arg 24.3		R1**→F→R F→R2→R		
Arg 24.4		R→F E→G G→K F→R R→F F→R, R→F		
Arg 24.5-6		E→K F→L F→L F→R L→R, R→L K→E, E→K E→Q, Q→E K→Q, Q→K F→L, L→F F→R, R→F		
Arg 24.7-8	*Content not appropriate for coding scheme*			
Arg 24.9		F→R F6→N4→R1		
Arg 24.10		R→F→K Q→E→K R→E→L R→F→L		
Arg 24.11		J6→F J2→L R2→J2 R→L		
Arg 24.12		K→K1		
Arg 24.13		R→L2 L1→J6→L K→I6		
Arg 24.14		F8→F→N4→ R1→R		
Arg 31.1		K→K4, K4→K		
Arg 31.2		R→R1→F→L		
Arg 31.3		G→S2, S2→G G→G3, G3→G S2→G3, G3→S2 G→S2		

Arg 31.4		G→G2, S2→G2 G3→G2		
Arg 31.5-31.6		G5→G G→ G5 G5→S2 S2→ G5 G5→G3 G3→ G5 E2→E G5→G2 G5→M3→G		
Arg 31.7		E→E1, E1→E		
Arg 31.8		G →E, E→G E→I→G E→I I→G		
Arg 31.9	*Content not appropriate for coding scheme*			

Appendix 4.4. The adjacency matrix $A(\mathrm{P}_1)_{m<n}$, condensed to show only used rows and columns.

	E	E1	E2	E3	E4	E5	E6	E7	E8	F	F1	F2	F3	F4	F5	F6	F7	F8	F9	G	G1	G2	G3	G4	G5	H	H1	H2	J3	J4	J5	J6	O	O1	O2	O3	P	P1	P2	S7	S8	S9	S10	S11	S12
E. Column vectors of *A* are linearly independent																																	1												
E1. *Trivial:* Only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution																																													
E2. *Unique:* There is a unique soln to matrix eqn/system of eqns																																													
E3. *Travel:* Can't get back (home)/get to origin with column vectors of *A*																																													
E4. *Geometric:* Vectors are not parallel/on the same line or plane																			1			1																							
E5. *Proportional:* No vector is a scalar multiple of another																																													
E6. *Linear combination:* No vector is a linear combo of another																																													
E7. *Placement:* No vector is in the span of the other vectors																																													
E8. *Extra:* Do not have an extra vector needed in order to return home																																													
F. Column vectors of *A* are linearly dependent																																													
F1. *Trivial:* Is more than one solution to $A\mathbf{x} = \mathbf{0}$.																																													
F2. *Unique:* No unique/multiple/infinitely many solns to system/matrix eqn																																													
F3. *Travel:* Can get back home/back to a point with column vectors of *A*															1																														
F4. *Geometric:* Vectors are parallel/on the same line or plane					1								1																																
F5. *Proportional:* One vector is a scalar multiple of another										1 1 1																	1																		
F6. *Linear combination:* One vector is a linear combo of others																																													
F7. *Placement*–A vector is in the span of the other vectors																																													
F8. *Zeroes:* the matrix *A* has a row or column of zeroes																																													
F9. *Extra:* Have an extra vector needed in order to return home										1				1																											1				
G. Column vectors of *A* span $\mathbf{R}^n$										1									3																										
G1. *Size:* Are enough vectors to span the entire space																			3																										
G2. *Geometric:* Can use vectors to get to every pt/go everywhere																			1 1																										
G3. *Algebraic:* Is a linear combination of vectors for all pts in $\mathbf{R}^n$																			1																										
G4. *Direction:* Can use each vector to go in a certain direction											1			1																															
G5. *Solution:* There is a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$																																													
H. Column vectors of *A* do not span $\mathbf{R}^n$																																													
H1. *Size:* Are not enough vectors to span the entire space																																													
H2. *Geometric:* Can't use vectors to get to every pt/go everywhere in dim																																													
J3. *Pivot-R:* Is not a pivot in each row																																													
J4. *Pivot-C:* Is not a pivot in each column										1																																			
J5. *Free Variable:* Not every variable is defined in system/matrix equation																																													
J6. *Zeroes:* RREF(*A*) has at least one row of zeroes																																													
O. The transformation defined by *A* is 1-1																																													
O1. *Definition:* For every $\mathbf{b}$ there is at most one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																													
O2. *Input/Output:* Each output has at most one input																																													
O3. *Reachable:* There is only one way to "get to" the output/vector																																													
P. The transformation defined by *A* is not 1-1																																													
P1. *Definition:* For every $\mathbf{b}$ there is more than one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																													
P2. *Input/Output:* Each output has more than one input																																													
S7. *Eigen Value:* The number zero is an eigenvalue of A																																													
S8: $m < n$: *A* has more vectors than dimensions										1 1			2											1													1								
S9: $m = n$: *A* has less vectors than dimensions																																													
S10: $m = n$: *A* has the same number of rows/columns										2 1									1	1										1															
S11: $m \neq n$: *A* does not have the same number of rows/columns																																													
S12: *Miscellaneous*																																													

Arg 5.1	Gray
Arg 6.1	Black
Arg 6.2	
Arg 6.3	Red
Arg 6.4	Orange
6.5	Yellow
6.6	light green
6.7	green
6.8	dark green
9.1	light teal
9.15	teal
10.1	light blue
10.2	blue
10.3	dark blue
10.4	light purple
20.4	purple
20.16	brown
31.8	light pink

Appendix 4.5. Upper left quadrant of the adjacency matrix $A(P_1)_{m=n}$

	E	E1	E2	E3	E4	E5	E6	E7	E8	F	F1	F2	F3	F4	F5	F6	F7	F8	F9	G	G1	G2	G3	G4	G5	H	H1	H2	H3	H4	H5	H6	I	I1	I2	I3	I4	I5	I6	J	J1	J2	J3	J4	J5	J6	K	K1	K2	K3
E. Column vectors of *A* are linearly independent																				1 1 1				1									2																	
E1. *Trivial:* Only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution																																																		
E2. *Unique:* There is a unique soln to matrix eqn/system of eqns																																																		
E3. *Travel:* Can't get back (home)/get to origin with column vectors of *A*	1																			1																														
E4. *Geometric:* Vectors are not parallel/on the same line or plane	1																			1		1																												
E5. *Proportional:* No vector is a scalar multiple of another																																																		
E6. *Linear combination:* No vector is a linear combo of another																																																		
E7. *Placement:* No vector is in the span of the other vectors	1																																																	
E8. *Extra:* Do not have an extra vector needed in order to return home																																																		
F. Column vectors of *A* are linearly dependent										1																1																								
F1. *Trivial:* Is more than one solution to $A\mathbf{x} = \mathbf{0}$.																																																		
F2. *Unique:* No unique/multiple/infinitely many solns to system/matrix eqn																																																		
F3. *Travel:* Can get back home/back to a point with column vectors of *A*																																																		
F4. *Geometric:* Vectors are parallel/on the same line or plane										1																																								
F5. *Proportional:* One vector is a scalar multiple of another																																																		
F6. *Linear combination:* One vector is a linear combo of others																																																		
F7. *Placement*–A vector is in the span of the other vectors													1																																					
F8. *Zeroes:* the matrix *A* has a row or column of zeroes																																																		
F9. *Extra:* Have an extra vector needed in order to return home																																																		
G. Column vectors of *A* span $\mathbf{R}^n$	2 1 1 1 1 1 1			1																																														
G1. *Size:* Are enough vectors to span the entire space																																																		
G2. *Geometric:* Can use vectors to get to every pt/go everywhere									1																																									
G3. *Algebraic:* Is a linear combination of vectors for all pts in $\mathbf{R}^n$				1																																														
G4. *Direction:* Can use each vector to go in a certain direction																				1																														
G5. *Solution:* There is a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$																																																		
H. Column vectors of *A* do not span $\mathbf{R}^n$																																																		
H1. *Size:* Are not enough vectors to span the entire space																																																		
H2. *Geometric:* Can't use vectors to get to every pt/go everywhere in dim																																																		
H3. *Direction:* Cannot go in all directions with the vectors																																																		
H4. *Clarify:* The vectors of *A* span a *k*-dim subspace of $\mathbf{R}^n$																																																		
H5. *Partial:* Span is only a point/line/plane																																																		
H6: *Solution:* There is not a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$																																																		
I. Row-reduced echelon form of *A* has *n* pivots																				2																														
I1. *Diagonal:* RREF(*A*) has all ones on the main diagonal																				1																														
I2. *Identity:* Can row-reduce / is row equivalent to the identity																																																		
I3. *Pivot-R:* Is a pivot in each row																				1													1				1													
I4. *Pivot-C:* Is a pivot in each column				1																																														
I5. *Free Variable:* Each variable is defined in system/matrix eqn																																																		
I6. *Zeroes:* RREF(*A*) has no rows of zeroes																																																		
J. Row-reduced echelon form of *A* has less than *n* pivots																																																		
J1. *Diagonal:* RREF(*A*) does not have all ones on the main diagonal																																																		
J2. *Identity:* Cannot row-reduce /not row equivalent to the identity																																																		
J3. *Pivot-R:* Is not a pivot in each row										1																																								
J4. *Pivot-C:* Is not a pivot in each column																																																		
J5. *Free Variable:* Not every variable is defined in system/matrix equation																																																		
J6. *Zeroes:* RREF(*A*) has at least one row of zeroes																																																		
K. *A* is invertible																																																		
K1. *Augment with the identity:* $[A \mid I] \sim [I \mid A^{-1}]$ is possible																																																		
K2. *Formula:* Can calculat, no "divide by 0 errors"																																																		
K3. *Undo transformation:* Can undo/get back from the transformation																																																		

Arg 5.1	Gray
Arg 6.1	**Black**
Arg 6.2	
Arg 6.3	Red
Arg 6.4	Orange
6.5	Yellow
6.6	light green
6.7	green
6.8	dark green
9.1	light teal
9.15	teal
10.1	light blue
10.2	blue
10.3	dark blue
10.4	light purple
20.4	purple
20.16	brown
31.8	light pink

Appendix 4.5. Upper right quadrant of the adjacency matrix $A(\mathrm{P}_1)_{m=n}$

	K4	K5	K6	L	L1	L2	L3	L4	L5	L6	M	M1	M2	M3	N	N1	N2	N3	N4	N5	O	O1	O2	O3	P	P1	P2	P3	P4	P5	Q	Q1	Q2	R	R1	R	R3	S	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12
E. Column vectors of *A* are linearly independent											1																																							
E1. *Trivial:* Only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution																																																		
E2. *Unique:* There is a unique soln to matrix eqn/system of eqns																																																		
E3. *Travel:* Can't get back (home)/get to origin with column vectors of *A*																																																		
E4. *Geometric:* Vectors are not parallel/on the same line or plane																																																		
E5. *Proportional:* No vector is a scalar multiple of another																																																		
E6. *Linear combination:* No vector is a linear combo of another																																																		
E7. *Placement:* No vector is in the span of the other vectors																																																		
E8. *Extra:* Do not have an extra vector needed in order to return home																																																		
F. Column vectors of *A* are linearly dependent																																																		
F1. *Trivial:* Is more than one solution to $A\mathbf{x} = \mathbf{0}$.																																																		
F2. *Unique:* No unique/multiple/infinitely many solns to system/matrix eqn																																																		
F3. *Travel:* Can get back home/back to a point with column vectors of *A*																																																		
F4. *Geometric:* Vectors are parallel/on the same line or plane																																																		
F5. *Proportional:* One vector is a scalar multiple of another																																																		
F6. *Linear combination:* One vector is a linear combo of others																																																		
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K1. *Augment with the identity:* $[A \mid I] \sim [I \mid A^{-1}]$ is possible																																																		
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Appendix 4.5. Lower left quadrant of the adjacency matrix $A(P_1)_{m=n}$

	E	E1	E2	E3	E4	E5	E6	E7	E8	F	F1	F2	F3	F4	F5	F6	F7	F8	F9	G	G1	G2	G3	G4	G5	H	H1	H2	H3	H4	H5	H6	I	I1	I2	I3	I4	I5	I6	J	J1	J2	J3	J4	J5	J6	K	K1	K2	K3
K4. *Inverse matrix:* Exists a C s.t. $AC=I$ and/or $CA=I$																																																		
K5. *Transform:* Exists sequence of elementary row ops that turns A into I																																																		
K6. *Necessary output*: Something gets sent to $\mathbf{e}_1$, $\mathbf{e}_2$, ...																																																		
L. *A* is not invertible																																																		
L1. *Augment with the identity:* $[A \mid I] \sim [I \mid A^{-1}]$ is not possible																																																		
L2. *Formula:* Can't calculate, get "divide by 0 errors"																																																		
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M. The transformation defined by *A* is onto	1																																																	
M1. *Definition:* For every **b** there is at least one **x** s.t. $T(\mathbf{x})=\mathbf{b}$																																																		
M2. *Range:* Range is all of the codomain																																																		
M3. *Images:* All values in codomain get used/mapped to as outputs/images																																																		
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P3. *Reachable:* There is more than one way to "get to" the output/vector																																																		
P4. *Multiplicity:* At least 2 inputs give the same output																																																		
P5. *Transform:* The transformation goes down/exludes a dimension																																																		
Q. det(A) ≠ 0: Determinant of *A* is nonzero																																																		
Q1. *Area:* Unit square/cube has area/volume ≠ 0 after transformation																																																		
Q2. *Calculation*: formula to calculate determinant yields nonzero result																																																		
R. det(A) = 0: Determinant of *A* is equal to 0																																																		
R1. *Area:* Unit square/cube has area/volume = 0 after transformation																																																		
R2. *Calculation*: formula to calculate determinant yields zero																																																		
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S. Miscellaneous																																																		
S1. *Calculation:* get inconsistent solutions (e.g., $0x + 0y = 1$)																																																		
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S12: *Miscellaneous*																																																		

Appendix 4.5. Lower right quadrant of the adjacency matrix $A(P_1)_{m=n}$

	K4	K5	K6	L	L1	L2	L3	L4	L5	L6	M	M1	M2	M3	N	N1	N2	N3	N4	N5	O	O1	O2	O3	P	P1	P2	P3	P4	P5	Q	Q1	Q2	R	R1	R	R3	S	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12
K4. *Inverse matrix:* Exists a C s.t. $AC=I$ and/or $CA=I$																																																		
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Q. $\det(A) \neq 0$: Determinant of *A* is nonzero																																																		
Q1. *Area:* Unit square/cube has area/volume $\neq 0$ after transformation																																																		
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Appendix 4.6. The adjacency matrix $A(\mathrm{P}_1)_{m>n}$, condensed to show only used rows and columns.

	E	E1	E2	E3	E4	E5	E6	E7	E8	F	F1	F2	F3	F4	F5	F6	F7	F8	F9	G	G1	G2	G3	G4	G5	H	H1	H2	H3	H4	H5	H6
E. Column vectors of *A* are linearly independent																															1	
E1. *Trivial:* Only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution																																
E2. *Unique:* There is a unique soln to matrix eqn/system of eqns																																
E3. *Travel*: Can't get back (home)/get to origin with column vectors of *A*																																
E4. *Geometric*: Vectors are not parallel/on the same line or plane																																
E5. *Proportional*: No vector is a scalar multiple of another	1																															
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F. Column vectors of *A* are linearly dependent																																
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G. Column vectors of *A* span $\mathbf{R}^n$																																
G1. *Size:* Are enough vectors to span the entire space																																
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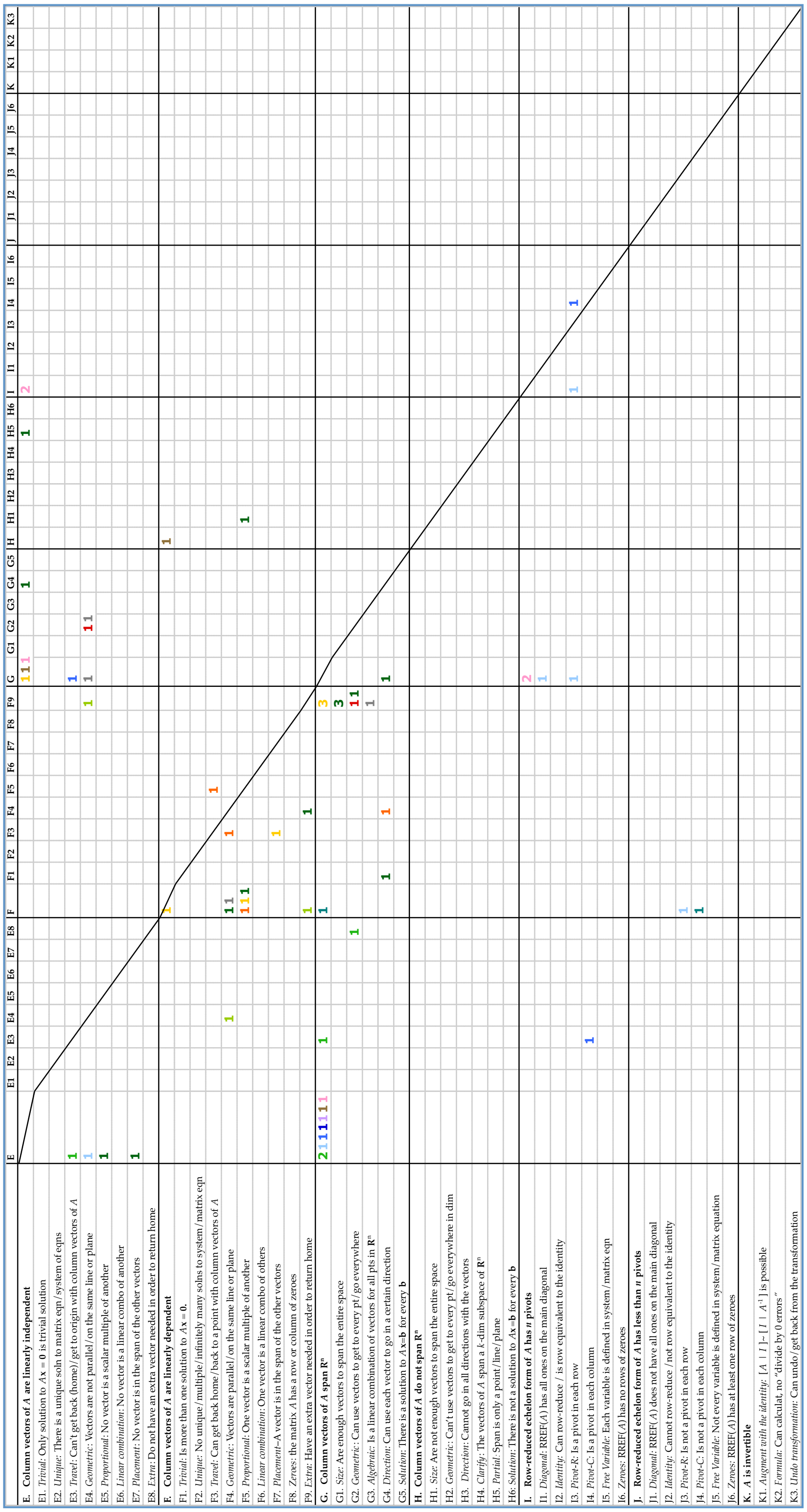
Arg 5.1	Gray
Arg 6.1	**Black**
Arg 6.2	
Arg 6.3	Red
Arg 6.4	Orange
6.5	Yellow
6.6	light green
6.7	green
6.8	dark green
9.1	light teal
9.15	teal
10.1	light blue
10.2	blue
10.3	dark blue
10.4	light purple
20.4	purple
20.16	brown
31.8	light pink

Appendix 4.7. The upper left quadrant of adjacency matrix $A(\text{P}_1)_{tot}$



Arg 5.1	Gray
Arg 6.1	Black
Arg 6.2	
Arg 6.3	Red
Arg 6.4	Orange
6.5	Yellow
6.6	light green
6.7	green
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9.1	light teal
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10.1	light blue
10.2	blue
10.3	dark blue
10.4	light purple
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Appendix 4.7. The upper right quadrant of adjacency matrix $A(\mathrm{P}_1)_{tot}$

	K4	K5	K6	L	L1	L2	L3	L4	L5	L6	M	M1	M2	M3	N	N1	N2	N3	N4	N5	O	O1	O2	O3	P	P1	P2	P3	P4	P5	Q	Q1	Q2	R	R1	R	R3	S	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12
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Appendix 4.7. The lower left quadrant of adjacency matrix $A(P_1)_{tot}$

	E	E1	E2	E3	E4	E5	E6	E7	E8	F	F1	F2	F3	F4	F5	F6	F7	F8	F9	G	G1	G2	G3	G4	G5	H	H1	H2	H3	H4	H5	H6	I	I1	I2	I3	I4	I5	I6	J	J1	J2	J3	J4	J5	J6	K	K1	K2	K3
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R. $\det(A) = 0$: Determinant of A is equal to 0																																																		
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S8: $m < n$: A has more vectors than dimensions										1 1			2											1																										
S9: $m = n$: A has less vectors than dimensions																																																		
S10: $m = n$: A has the same number of rows/columns										2 1									1	1 1 1																								1						
S11: $m \neq n$: A does not have the same number of rows/columns																																																		
S12: *Miscellaneous*																																																		

Appendix 4.7. The lower right quadrant of adjacency matrix $A(P_1)_{tot}$

	K4	K5	K6	L	L1	L2	L3	L4	L5	L6	M	M1	M2	M3	N	N1	N2	N3	N4	N5	O	O1	O2	O3	P	P1	P2	P3	P4	P5	Q	Q1	Q2	R	R1	R	R3	S	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12
K4. *Inverse matrix:* Exists a C s.t. $AC=I$ and/or $CA=I$																																																		
K5. *Transform:* Exists sequence of elementary row ops that turns A into I																																																		
K6. *Necessary output*: Something gets sent to $\mathbf{e}_1$, $\mathbf{e}_2$, ...																																																		
L. *A* is not invertible																																																		
L1. *Augment with the identity:* $[A \mid I] \sim [I \mid A^{-1}]$ is not possible																																																		
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Appendix 4.8. The upper left quadrant for the adjacency matrix $A(\mathrm{T})_{m<n}$

A(T) m<n UL	E	E1	E2	E3	E4	E5	E6	E7	E8	F	F1	F2	F3	F4	F5	F6	F7	F8	F9	G	G1	G2	G3	G4	G5	H	H1	H2	H3	H4	H5	H6	I	I1	I2	I3	I4	I5	I6	J	J1	J2	J3	J4	J5	J6	K	K1	K2	K3
E. Column vectors of *A* are linearly independent		1																																																
E1. *Trivial:* Only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution																																																		
E2. *Unique:* There is a unique soln to matrix eqn/system of eqns																																																		
E3. *Travel:* Can't get back (home)/get to origin with column vectors of *A*																																																		
E4. *Geometric:* Vectors are not parallel/on the same line or plane																			1			1																												
E5. *Proportional:* No vector is a scalar multiple of another																				2																														
E6. *Linear combination:* No vector is a linear combo of another																																																		
E7. *Placement:* No vector is in the span of the other vectors																																																		
E8. *Extra:* Do not have an extra vector needed in order to return home																																																		
F. Column vectors of *A* are linearly dependent																																																		
F1. *Trivial:* Is more than one solution to $A\mathbf{x} = \mathbf{0}$.										1																																								
F2. *Unique:* No unique/multiple/infinitely many solns to system/matrix eqn										1																																								
F3. *Travel:* Can get back home/back to a point with column vectors of *A*															1					1																														
F4. *Geometric:* Vectors are parallel/on the same line or plane					1								1															1																						
F5. *Proportional:* One vector is a scalar multiple of another										3																	1																							
F6. *Linear combination:* One vector is a linear combo of others																																																		
F7. *Placement*–A vector is in the span of the other vectors																																																		
F8. *Zeroes:* the matrix *A* has a row or column of zeroes																																																		
F9. *Extra:* Have an extra vector needed in order to return home										1				1																																				
G. Column vectors of *A* span $\mathbf{R}^n$										1 2									3			1																							1					
G1. *Size:* Are enough vectors to span the entire space																			3	1																														
G2. *Geometric:* Can use vectors to get to every pt/go everywhere																			2																															
G3. *Algebraic:* Is a linear combination of vectors for all pts in $\mathbf{R}^n$																			1																															
G4. *Direction:* Can use each vector to go in a certain direction														1						2	1									1																				
G5. *Solution:* There is a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$																																																		
H. Column vectors of *A* do not span $\mathbf{R}^n$																																																		
H1. *Size:* Are not enough vectors to span the entire space																										1				1																				
H2. *Geometric:* Can't use vectors to get to every pt/go everywhere in dim																																																		
H3. *Direction:* Cannot go in all directions with the vectors																																																		
H4. *Clarify:* The vectors of *A* span a *k*-dim subspace of $\mathbf{R}^n$																																																		
H5. *Partial:* Span is only a point/line/plane																																																		
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I. Row-reduced echelon form of *A* has *n* pivots																				1				1																										
I1. *Diagonal:* RREF(*A*) has all ones on the main diagonal																																																		
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Appendix 4.8. The upper right quadrant for the adjacency matrix $A(\mathrm{T})_{m<n}$

A(T) m<n UR	K4	K5	K6	L	L1	L2	L3	L4	L5	L6	M	M1	M2	M3	N	N1	N2	N3	N4	N5	O	O1	O2	O3	P	P1	P2	P3	P4	P5	Q	Q1	Q2	R	R1	R	R3	S	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12
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Appendix 4.8. The lower left quadrant for the adjacency matrix $A(\mathrm{T})_{m<n}$

A(T) m<n LL	E	E1	E2	E3	E4	E5	E6	E7	E8	F	F1	F2	F3	F4	F5	F6	F7	F8	F9	G	G1	G2	G3	G4	G5	H	H1	H2	H3	H4	H5	H6	I	I1	I2	I3	I4	I5	I6	J	J1	J2	J3	J4	J5	J6	K	K1	K2	K3
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Appendix 4.8. The lower right quadrant for the adjacency matrix $A(\mathrm{T})_{m<n}$

$A(T)$ $m<n$ LR	K4	K5	K6	L	L1	L2	L3	L4	L5	L6	M	M1	M2	M3	N	N1	N2	N3	N4	N5	O	O1	O2	O3	P	P1	P2	P3	P4	P5	Q	Q1	Q2	R	R1	R	R3	S	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12
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R. det(A) = 0: Determinant of *A* is equal to 0																																																		
R1. *Area:* Unit square/cube has area/volume = 0 after transformation																																																		
R2. *Calculation:* formula to calculate determinant yields zero																																																		
R3. *Measure:* No area/volume to a line/plane																																																		
S. Miscellaneous																																																		
S1. *Calculation:* get inconsistent solutions (e.g., $0x + 0y = 1$)																																																		
S2. *Column Space:* The Col A is all of $\mathbf{R}^n$																																																		
S3. *Column Space:* The Col A is all not of $\mathbf{R}^n$																																																		
S4. *Null Space:* The Nul A contains only the zero vector																																																		
S5. *Null Space:* The Nul A contains more than the zero vector																																																		
S6. *Eigen Value:* The number zero is not an eigenvalue of A																																																		
S7. *Eigen Value:* The number zero is an eigenvalue of A																																																		
S8: $m < n$: A has more vectors than dimensions																									1																									
S9: $m = n$: A has less vectors than dimensions																																																		
S10: $m = n$: A has the same number of rows/columns																																																		
S11: $m \neq n$: A does not have the same number of rows/columns																																																		
S12: *Miscellaneous*																																																		

Appendix 4.9. The upper left quadrant of the adjacency matrix $A(\mathrm{T})_{m=n}$

A(T) m=n UL	E	E1	E2	E3	E4	E5	E6	E7	E8	F	F1	F2	F3	F4	F5	F6	F7	F8	F9
E. **Column vectors of A are linearly independent**		11	1																
E1. *Trivial*: Only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution	11																		
E2. *Unique*: There is a unique soln to matrix eqn/system of eqns	21																		
E3. *Travel*: Can't get back (home)/get to origin with column vectors of A	1																		
E4. *Geometric*: Vectors are not parallel/on the same line or plane	1																		
E5. *Proportional*: No vector is a scalar multiple of another																			
E6. *Linear combination*: No vector is a linear combo of another																			
E7. *Placement*: No vector is in the span of the other vectors	1																		
E8. *Extra*: Do not have an extra vector needed in order to return home																			
F. **Column vectors of A are linearly dependent**										1									
F1. *Trivial*: Is more than one solution to $A\mathbf{x} = \mathbf{0}$.																			
F2. *Unique*: No unique/multiple/infinitely many solns to system/matrix eqn																			
F3. *Travel*: Can get back home/back to a point with column vectors of A																			
F4. *Geometric*: Vectors are parallel/on the same line or plane										1									
F5. *Proportional*: One vector is a scalar multiple of another																			
F6. *Linear combination*: One vector is a linear combo of others																			
F7. *Placement*–A vector is in the span of the other vectors													1						
F8. *Zeroes*: the matrix A has a row or column of zeroes										1									
F9. *Extra*: Have an extra vector needed in order to return home																			
G. **Column vectors of A span $\mathbf{R}^n$**	2411			1								2							
G1. *Size*: Are enough vectors to span the entire space																			
G2. *Geometric*: Can use vectors to get to every pt/go everywhere									1										
G3. *Algebraic*: Is a linear combination of vectors for all pts in $\mathbf{R}^n$				1															
G4. *Direction*: Can use each vector to go in a certain direction																			
G5. *Solution*: There is a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$																			
H. **Column vectors of A do not span $\mathbf{R}^n$**																			
H1. *Size*: Are not enough vectors to span the entire space																			
H2. *Geometric*: Can't use vectors to get to every pt/go everywhere in dim																			
H3. *Direction*: Cannot go in all directions with the vectors																			
H4. *Clarify*: The vectors of A span a k-dim subspace of $\mathbf{R}^n$																			
H5. *Partial*: Span is only a point/line/plane																			
H6: *Solution*: There is not a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$																			
I. **Row-reduced echelon form of A has n pivots**																			
I1. *Diagonal*: RREF(A) has all ones on the main diagonal			2																
I2. *Identity*: Can row-reduce / is row equivalent to the identity																			
I3. *Pivot-R*: Is a pivot in each row																			
I4. *Pivot-C*: Is a pivot in each column				1															
I5. *Free Variable*: Each variable is defined in system/matrix eqn																			
I6. *Zeroes*: RREF(A) has no rows of zeroes																			
J. **Row-reduced echelon form of A has less than n pivots**																			
J1. *Diagonal*: RREF(A) does not have all ones on the main diagonal																			
J2. *Identity*: Cannot row-reduce /not row equivalent to the identity																			
J3. *Pivot-R*: Is not a pivot in each row										1									
J4. *Pivot-C*: Is not a pivot in each column																			
J5. *Free Variable*: Not every variable is defined in system/matrix equation																			
J6. *Zeroes*: RREF(A) has at least one row of zeroes										1									
K. **A is invertible**	321																		
K1. *Augment with the identity*: $[A \mid I] \sim [I \mid A^{-1}]$ is possible																			
K2. *Formula*: Can calculat, no "divide by 0 errors"																			
K3. *Undo transformation*: Can undo/get back from the transformation																			

A(T) m=n UL	G	G1	G2	G3	G4	G5	H	H1	H2	H3	H4	H5	H6
E	12211		1		1								
E1													
E2													
E3	1												
E4	1		1										
E5													
E6													
E7													
E8													
F							21					11	
F1													
F2													
F3													
F4													
F5													
F6													
F7													
F8									1				
F9													
G			1	1		1							
G1													
G2	1												
G3	1		2			1							
G4	1												
G5	1		1	1									
H													
H1													
H2													
H3													
H4													
H5													
H6													
I	2												
I1	1												
I2													
I3	12												
I4													
I5													
I6													
J													
J1													
J2													
J3													
J4													
J5													
J6													
K	4												
K1													
K2													
K3													

A(T) m=n UL	I	I1	I2	I3	I4	I5	I6	J	J1	J2	J3	J4	J5	J6	K	K1	K2	K3
E	2		21	3											33			
E1																		
E2																		
E3																		
E4																		
E5																		
E6																		
E7																		
E8																		
F															1			
F1																		
F2																		
F3																		
F4																		
F5													1	1				
F6																		
F7																		
F8											1							
F9																		
G				1		1									1			
G1																		
G2																		
G3																		
G4																		
G5																		
H																		
H1																		
H2																		
H3																		
H4																		
H5																		
H6																		
I																		
I1																		
I2															1			
I3	1	1			1	2												
I4																		
I5							2								1			
I6																		
J																		
J1																		
J2																		
J3																		
J4																		
J5														1				
J6																		
K			2				1									1		
K1																		
K2																		
K3																		

Arg 5.1	Gray
Arg 6.1	Black
Arg 6.2	
Arg 6.3	Red
Arg 6.4	Orange
6.5	Yellow
6.6	light green
6.7	green
6.8	dark green
9.1	light teal
9.15	teal
10.1	light blue
10.2	blue
10.3	dark blue
10.4	light purple
20.4	purple
20.16	brown
31.8	light pink

Appendix 4.9. The upper right quadrant of the adjacency matrix $A(\mathrm{T})_{m=n}$

A(T) m=n UR	K4	K5	K6	L	L1	L2	L3	L4	L5	L6	M	M1	M2	M3	N	N1	N2	N3	N4	N5	O	O1	O2	O3	P	P1	P2	P3	P4	P5	Q	Q1	Q2	R	R1	R	R3	S	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12
E. Column vectors of A are linearly independent				1							1								1		2										1																			
E1. *Trivial:* Only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution																																																		
E2. *Unique:* There is a unique soln to matrix eqn/system of eqns																																																		
E3. *Travel*: Can't get back (home)/get to origin with column vectors of A																																																		
E4. *Geometric*: Vectors are not parallel/on the same line or plane																																																		
E5. *Proportional*: No vector is a scalar multiple of another																																																		
E6. *Linear combination*: No vector is a linear combo of another																																																		
E7. *Placement*: No vector is in the span of the other vectors																																																		
E8. *Extra*: Do not have an extra vector needed in order to return home																																																		
F. Column vectors of A are linearly dependent				541														1	1						1									6		1			1											
F1. *Trivial*: Is more than one solution to $A\mathbf{x} = \mathbf{0}$.																																																		
F2. *Unique:* No unique/multiple/infinitely many solns to system/matrix eqn																																																		
F3. *Travel:* Can get back home/back to a point with column vectors of A																																																		
F4. *Geometric*: Vectors are parallel/on the same line or plane																																																		
F5. *Proportional*: One vector is a scalar multiple of another				1														1																																
F6. *Linear combination*: One vector is a linear combo of others																			1																															
F7. *Placement*–A vector is in the span of the other vectors																																																		
F8. *Zeroes*: the matrix A has a row or column of zeroes				2			1	1																															2											
F9. *Extra:* Have an extra vector needed in order to return home				1																																														
G. Column vectors of A span $\mathbf{R}^n$											4										1																			2										
G1. *Size:* Are enough vectors to span the entire space																																																		
G2. *Geometric*: Can use vectors to get to every pt/go everywhere																																																		
G3. *Algebraic:* Is a linear combination of vectors for all pts in $\mathbf{R}^n$																																								1										
G4. *Direction:* Can use each vector to go in a certain direction																																																		
G5. *Solution*: There is a solution to $A\mathbf{x}$=$\mathbf{b}$ for every $\mathbf{b}$														1																										1										
H. Column vectors of A do not span $\mathbf{R}^n$				1											2																																			
H1. *Size:* Are not enough vectors to span the entire space																																																		
H2. *Geometric*: Can't use vectors to get to every pt/go everywhere in dim																																																		
H3. *Direction*: Cannot go in all directions with the vectors																																																		
H4. *Clarify*: The vectors of A span a k-dim subspace of $\mathbf{R}^n$																																																		
H5. *Partial:* Span is only a point/line/plane										2									1																1															
H6: *Solution*: There is not a solution to $A\mathbf{x}$=$\mathbf{b}$ for every $\mathbf{b}$																																																		
I. Row-reduced echelon form of A has n pivots																																																		
I1. *Diagonal*: RREF(A) has all ones on the main diagonal																																																		
I2. *Identity:* Can row-reduce / is row equivalent to the identity		1									1										1																													
I3. *Pivot-R:* Is a pivot in each row											1										1																													
I4. *Pivot-C:* Is a pivot in each column																																																		
I5. *Free Variable*: Each variable is defined in system/matrix eqn																																																		
I6. *Zeroes:* RREF(A) has no rows of zeroes																																																		
J. Row-reduced echelon form of A has less than n pivots																																																		
J1. *Diagonal*: RREF(A) does not have all ones on the main diagonal																																																		
J2. *Identity:* Cannot row-reduce /not row equivalent to the identity				1																																														
J3. *Pivot-R:* Is not a pivot in each row																																																		
J4. *Pivot-C:* Is not a pivot in each column																																																		
J5. *Free Variable:* Not every variable is defined in system/matrix equation																																																		
J6. *Zeroes:* RREF(A) has at least one row of zeroes				11																					1																									
K. A is invertible	111																														1																	1		
K1. *Augment with the identity:* $[A \mid I] \sim [I \mid A^{-1}]$ is possible																																																		
K2. *Formula*: Can calculat, no "divide by 0 errors"																																																		
K3. *Undo transformation:* Can undo/get back from the transformation																																																		

Appendix 4.9. The lower left quadrant of the adjacency matrix $A(\mathrm{T})_{m=n}$

A(T) m=n LL	E	E1	E2	E3	E4	E5	E6	E7	E8	F	F1	F2	F3	F4	F5	F6	F7	F8	F9	G	G1	G2	G3	G4	G5	H	H1	H2	H3	H4	H5	H6	I	I1	I2	I3	I4	I5	I6	J	J1	J2	J3	J4	J5	J6	K	K1	K2	K3
K4. *Inverse matrix:* Exists a C s.t. $AC=I$ and/or $CA=I$																																															2 1			
K5. *Transform:* Exists sequence of elementary row ops that turns A into I																																			1															
K6. *Necessary output:* Something gets sent to $\mathbf{e}_1$, $\mathbf{e}_2$, ...																																																		
L. A is not invertible										2 1																																								
L1. *Augment with the identity:* $[A \mid I] \sim [I \mid A^{-1}]$ is not possible																																														1				
L2. *Formula:* Can't calculate, get "divide by 0 errors"																																																		
L3. *Undo transformation:* Can't undo/get back from the transformation																																																		
L4. *Inverse matrix:* Does not exist a C s.t. $AC=I$ and/or $CA=I$																																																		
L5. *Transform:* Does not exist sequence of elem row ops to turn A into I																																																		
L6. *Necessary output:* Nothing gets sent to $\mathbf{e}_1$, $\mathbf{e}_2$, ...																												1																						
M. The transformation defined by A is onto	1																			1																											1			
M1. *Definition:* For every $\mathbf{b}$ there is at least one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																																		
M2. *Range:* Range is all of the codomain																																																		
M3. *Images:* All values in codomain get used/mapped to as outputs/images																				1																														
N. The transformation defined by A is not onto																																																		
N1. *Definition:* For every $\mathbf{b}$ there is not at least one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																																		
N2. *Range:* Range is not all of the codomain																																																		
N3. *Images:* Not all values in codomain get used/mapped to as outputs/images																																																		
N4. *Collapse:* Transformation collapses everything to a point/line/plane																																																		
N5. *Transform:* The transformation goes up/adds a dimension																																																		
O. The transformation defined by A is 1-1	1																			1																											1			
O1. *Definition:* For every $\mathbf{b}$ there is at most one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																																		
O2. *Input/Output:* Each output has at most one input																																																		
O3. *Reachable:* There is only one way to "get to" the output/vector																																																		
P. The transformation defined by A is not 1-1																																																		
P1. *Definition:* For every $\mathbf{b}$ there is more than one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																																		
P2. *Input/Output:* Each output has more than one input																																																		
P3. *Reachable:* There is more than one way to "get to" the output/vector																																																		
P4. *Multiplicity:* At least 2 inputs give the same output																																																		
P5: *Transform:* The transformation goes down/exludes a dimension																																																		
Q. det(A) ≠ 0: Determinant of A is nonzero	2																																														1			
Q1. *Area:* Unit square/cube has area/volume ≠ 0 after transformation																																																		
Q2. *Calculation:* formula to calculate determinant yields nonzero result																																																		
R. det(A) = 0: Determinant of A is equal to 0	1									6																																								
R1. *Area:* Unit square/cube has area/volume = 0 after transformation										1 1																																								
R2. *Calculation:* formula to calculate determinant yields zero																																										1								
R3. *Measure:* No area/volume to a line/plane																																																		
S. Miscellaneous																																																		
S1. *Calculation:* get inconsistent solutions (e.g., $0x + 0y = 1$)																		1																																
S2. *Column Space:* The Col A is all of R^n																				1		1	1		1																									
S3. *Column Space:* The Col A is all not of R^n																																																		
S4. *Null Space:* The Nul A contains only the zero vector																																																		
S5. *Null Space:* The Nul A contains more than the zero vector																																																		
S6. *Eigen Value:* The number zero is not an eigenvalue of A																																																		
S7. *Eigen Value:* The number zero is an eigenvalue of A																																																		
S8: $m < n$: A has more vectors than dimensions																																																		
S9: $m = n$: A has less vectors than dimensions																																																		
S10: $m = n$: A has the same number of rows/columns			1																	2																														
S11: $m \neq n$: A does not have the same number of rows/columns																																																		
S12: *Miscellaneous*																																																		

Appendix 4.9. The lower right quadrant of the adjacency matrix $A(\mathrm{T})_{m=n}$

A(T) m=n LR	K4	K5	K6	L	L1	L2	L3	L4	L5	L6	M	M1	M2	M3	N	N1	N2	N3	N4	N5	O	O1	O2	O3	P	P1	P2	P3	P4	P5	Q	Q1	Q2	R	R1	R	R3	S	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12
K4. *Inverse matrix:* Exists a C s.t. $AC=I$ and/or $CA=I$																																																		
K5. *Transform:* Exists sequence of elementary row ops that turns A into I	1																																																	
K6. *Necessary output*: Something gets sent to $\mathbf{e}_1$, $\mathbf{e}_2$, ...																																																		
L. A is not invertible																																		1																
L1. *Augment with the identity:* $[A \mid I] \sim [I \mid A^{-1}]$ is not possible																																																		
L2. *Formula:* Can't calculate, get "divide by 0 errors"																																																		
L3. *Undo transformation:* Can't undo/get back from the transformation																																																		
L4. *Inverse matrix:* Does not exist a C s.t. $AC=I$ and/or $CA=I$				1																																														
L5. *Transform:* Does not exist sequence of elem row ops to turn A into I																																																		
L6. *Necessary output*: Nothing gets sent to $\mathbf{e}_1$, $\mathbf{e}_2$, ...				2			1																																											
M. The transformation defined by A is onto																					5																													
M1. *Definition:* For every $\mathbf{b}$ there is at least one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																																		
M2. *Range:* Range is all of the codomain																					1																													
M3. *Images:* All values in codomain get used/mapped to as outputs/images																																																		
N. The transformation defined by A is not onto																									1																									
N1. *Definition:* For every $\mathbf{b}$ there is not at least one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																																		
N2. *Range:* Range is not all of the codomain																																																		
N3. *Images*: Not all values in codomain get used/mapped to as outputs/images																																																		
N4. *Collapse:* Transformation collapses everything to a point/line/plane										2																									2		3													
N5. *Transform:* The transformation goes up/adds a dimension																																																		
O. The transformation defined by A is 1-1											5																																							
O1. *Definition:* For every $\mathbf{b}$ there is at most one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																																		
O2. *Input/Output:* Each output has at most one input																					1																													
O3. *Reachable:* There is only one way to "get to" the output/vector																																																		
P. The transformation defined by A is not 1-1															1																																			
P1. *Definition:* For every $\mathbf{b}$ there is more than one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																																		
P2. *Input/Output:* Each output has more than one input																																																		
P3. *Reachable:* There is more than one way to "get to" the output/vector																																																		
P4. *Multiplicity:* At least 2 inputs give the same output																																																		
P5: *Transform:* The transformation goes down/exludes a dimension																																																		
Q. det(A) ≠ 0: Determinant of A is nonzero																																																		
Q1. *Area:* Unit square/cube has area/volume ≠ 0 after transformation																																																		
Q2. *Calculation:* formula to calculate determinant yields nonzero result																																																		
R. det(A) = 0: Determinant of A is equal to 0				2		1																													1															
R1. *Area:* Unit square/cube has area/volume = 0 after transformation																																		1																
R2. *Calculation:* formula to calculate determinant yields zero																																		1																
R3. *Measure:* No area/volume to a line/plane																																			1															
S. Miscellaneous																																																		
S1. *Calculation:* get inconsistent solutions (e.g., $0x + 0y = 1$)				2			1	3																																										
S2. *Column Space:* The Col A is all of R^n																																																		
S3. *Column Space:* The Col A is all not of R^n																																																		
S4. *Null Space:* The Nul A contains only the zero vector																																																		
S5. *Null Space:* The Nul A contains more than the zero vector																																																		
S6. *Eigen Value:* The number zero is not an eigenvalue of A																																																		
S7. *Eigen Value:* The number zero is an eigenvalue of A																																																		
S8: $m < n$: A has more vectors than dimensions																									1																									
S9: $m = n$: A has less vectors than dimensions															1																																			
S10: $m = n$: A has the same number of rows/columns													1																																					
S11: $m \neq n$: A does not have the same number of rows/columns																																																		
S12: *Miscellaneous*																																																		

Appendix 4.10. The adjacency matrix $A(\mathrm{T})_{m>n}$, condensed for used rows and columns

	E	E1	E2	E3	E4	E5	F	F1	F2	F3	F4	F5	F6	F7	F8	F9	G	G1	G2	G3	G4	G5	H	H1	H2	H3	H4	H5	H6	I	I1	I2	I3	J3	J4	J5	J6	K	K4	K5	K6	L	L1	M	M1	M2	M3	N	N1	N2	N3	N4	N5	O	O1	O2	O3	S8	S9	S10	S11	S12
E. **Column vectors of A are linearly independent**																												1														1								1							1					
E1. *Trivial*: Only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution																																																														
E2. *Unique*: There is a unique soln to matrix eqn/system of eqns																																																														
E3. *Travel*: Can't get back (home)/get to origin with column vectors of A																																																														
E4. *Geometric*: Vectors are not parallel/on the same line or plane																																																														
E5. *Proportional*: No vector is a scalar multiple of another	1																																																													
F. **Column vectors of A are linearly dependent**																																										1																				
F1. *Trivial*: Is more than one solution to $A\mathbf{x} = \mathbf{0}$.																																																														
F2. *Unique*: No unique/multiple/infinitely many solns to system/matrix eqn																																																														
F3. *Travel*: Can get back home/back to a point with column vectors of A																																																														
F4. *Geometric*: Vectors are parallel/on the same line or plane							1																																																							
F5. *Proportional*: One vector is a scalar multiple of another																																																														
F6. *Linear combination*: One vector is a linear combo of others																																																														
F7. *Placement*–A vector is in the span of the other vectors																																																														
F8. *Zeroes*: the matrix A has a row or column of zeroes																							2																									1			4											
F9. *Extra:* Have an extra vector needed in order to return home																																																													
G. **Column vectors of A span $\mathbf{R}^n$**																			1																									1																		
G1. *Size:* Are enough vectors to span the entire space																																																														
G2. *Geometric*: Can use vectors to get to every pt/go everywhere																																												1			1															
G3. *Algebraic:* Is a linear combination of vectors for all pts in $\mathbf{R}^n$																																																														
G4. *Direction:* Can use each vector to go in a certain direction																																																														
G5. *Solution*: There is a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$																																																														
H. **Column vectors of A do not span $\mathbf{R}^n$**																																																2														
H1. *Size:* Are not enough vectors to span the entire space																							1		1																																					
H2. *Geometric*: Can't use vectors to get to every pt/go everywhere in dim																																																1														
H3. *Direction*: Cannot go in all directions with the vectors																																																														
H4. *Clarify*: The vectors of A span a k-dim subspace of $\mathbf{R}^n$																																																														
H5. *Partial:* Span is only a point/line/plane																									1																																					
H6: *Solution*: There is not a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$																																																														
I. **Row-reduced echelon form of A has n pivots**																	2																																													
I1. *Diagonal*: RREF(A) has all ones on the main diagonal																																																														
I2. *Identity:* Can row-reduce / is row equivalent to the identity																																																														
I3. *Pivot-R:* Is a pivot in each row																	1 2																											1																		
J3. *Pivot-R:* Is not a pivot in each row																							1																												1											
J4. *Pivot-C:* Is not a pivot in each column																																																														
J5. *Free Variable:* Not every variable is defined in system/matrix equation																																																														
J6. *Zeroes:* RREF(A) has at least one row of zeroes																																																														
K. **A is invertible**																																							1																							
K4. *Inverse matrix*: Exists a C s.t. $AC=I$ and/or $CA=I$																																																											1		
K5. *Transform:* Exists sequence of elementary row ops that turns A into I																																																														
K6. *Necessary output*: Something gets sent to $\mathbf{e}_1$, $\mathbf{e}_2$, ...																																																														
L. **A is not invertible**																																																														
L1. *Augment with the identity*: $[A \mid I] \sim [I \mid A^{-1}]$ is not possible																																																														
M. **The transformation defined by A is onto**																																																														
M1. *Definition*: For every $\mathbf{b}$ there is at least one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																																														
M2. *Range*: Range is all of the codomain																																																														
M3. *Images*: All values in codomain get used/mapped to as outputs/images																																																														
N. **The transformation defined by A is not onto**																																																						2								
N1. *Definition*: For every $\mathbf{b}$ there is not at least one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																																														
N2. *Range*: Range is not all of the codomain																																																														
N3. *Images*: Not all values in codomain get used/mapped to as outputs/images																							1																									1		1												
N4. *Collapse:* Transformation collapses everything to a point/line/plane																																																														
N5. *Transform:* The transformation goes up/adds a dimension																																																													1
O. **The transformation defined by A is 1-1**																																																2														
O1. *Definition:* For every $\mathbf{b}$ there is at most one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																																													
O2. *Input/Output:* Each output has at most one input																																																														
O3. *Reachable:* There is only one way to "get to" the output/vector																																																														
S8: $m < n$: A has more vectors than dimensions																																																														
S9: $m = n$: A has less vectors than dimensions																							1	1 1				1						1																												
S10: $m = n$: A has the same number of rows/columns																																																														
S11: $m \neq n$: A does not have the same number of rows/columns																																																														
S12: *Miscellaneous*																																																			1											

Day 5/6 light pink
Day 9 red
Day 10 orange
Day 17 green
Day 18 dark green
Day 19 light blue
Day 20 dark blue
Day 24 light purple
Day 31 brown

Appendix 4.11. The adjacency matrix $A(\mathrm{T})_{m>n}$, condensed for used rows and columns

A(T) any m,n	E	E1	F	F1	F2	F3	F4	F5	F6	F7	F8	F9	G	G1	G2	G3	G4	G5	H	H1	H2	H3	H4	H5	H6	I	I1	I2	I3	J4	J5	J6	M	M1	M2	M3	N	N1	N2	N3	N4	N5	O	O1	O2	O3	P	P1	P2	P3	P4	P5	S8	S9
E. Column vectors of A are linearly independent																																											2											
E1. *Trivial*: Only solution to $A\mathbf{x} = \mathbf{0}$ is trivial solution																																																						
F. Column vectors of A are linearly dependent				1																											1																							
F1. *Trivial*: Is more than one solution to $A\mathbf{x} = \mathbf{0}$.																																																			1			
F2. *Unique*: No unique/multiple/infinitely many solns to system/matrix eqn																																															1			1				
F3. *Travel*: Can get back home/back to a point with column vectors of A			1																																																			
F4. *Geometric*: Vectors are parallel/on the same line or plane			1																																																			
F5. *Proportional*: One vector is a scalar multiple of another						1	1																																															
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F7. *Placement*–A vector is in the span of the other vectors																																																						
F8. *Zeroes*: the matrix A has a row or column of zeroes			1																																																			
F9. *Extra*: Have an extra vector needed in order to return home																																																						
G. Column vectors of A span $\mathbf{R}^n$																																	33		1																			
G1. *Size*: Are enough vectors to span the entire space																																																						
G2. *Geometric*: Can use vectors to get to every pt/go everywhere																																																						
G3. *Algebraic*: Is a linear combination of vectors for all pts in $\mathbf{R}^n$																																																						
G4. *Direction*: Can use each vector to go in a certain direction													1																																									
G5. *Solution*: There is a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$																																																						
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H5. *Partial*: Span is only a point/line/plane																																					1			1														
H6: *Solution*: There is not a solution to $A\mathbf{x}=\mathbf{b}$ for every $\mathbf{b}$																																																						
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I2. *Identity*: Can row-reduce / is row equivalent to the identity																																																						
I3. *Pivot-R*: Is a pivot in each row																	1																																					
J4. *Pivot-C*: Is not a pivot in each column																																																						
J5. *Free Variable*: Not every variable is defined in system/matrix equation					1																																														1			
J6. *Zeroes*: RREF(A) has at least one row of zeroes																																																						
M. The transformation defined by A is onto													13																					2	2																			
M1. *Definition*: For every $\mathbf{b}$ there is at least one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																				1																		
M2. *Range*: Range is all of the codomain													1																				1																					
M3. *Images*: All values in codomain get used/mapped to as outputs/images																																																						
N. The transformation defined by A is not onto																			1																																			
N1. *Definition*: For every $\mathbf{b}$ there is not at least one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																					1																	
N2. *Range*: Range is not all of the codomain																																																						
N3. *Images*: Not all values in codomain get used/mapped to as outputs/images																																	1				11																	
N4. *Collapse*: Transformation collapses everything to a point/line/plane																																																						
N5. *Transform*: The transformation goes up/adds a dimension																																																						
O. The transformation defined by A is 1-1	2																																1											1										
O1. *Definition*: For every $\mathbf{b}$ there is at most one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$	1																																										1											
O2. *Input/Output*: Each output has at most one input																																																						
O3. *Reachable*: There is only one way to "get to" the output/vector																																																						
P. The transformation defined by A is not 1-1																																																						
P1. *Definition*: For every $\mathbf{b}$ there is more than one $\mathbf{x}$ s.t. $T(\mathbf{x})=\mathbf{b}$																																																						
P2. *Input/Output*: Each output has more than one input																																																						
P3. *Reachable*: There is more than one way to "get to" the output/vector																																																1						
P4. *Multiplicity*: At least 2 inputs give the same output																																															1							
P5: *Transform*: The transformation goes down/exludes a dimension																																																						
S8: $m < n$: A has more vectors than dimensions			1																																																			
S9: $m = n$: A has less vectors than dimensions																																																						

Day 5/6 light pink
Day 9 red
Day 10 orange
Day 17 green
Day 18 dark green
Day 19 light blue
Day 20 dark blue
Day 24 light purple
Day 31 brown

Appendix 5.1. Compiled list of Toulmin schemes for Abraham

Day 7 Argumentation Log

Arg 7.2 WCD
Claim: *The set of vectors (??) is linearly dependent*
Data: *Because they're multiples of each other.*
Warrant: *I can take one in 1 direction and the opposite in the other, you get back home.*
** Instructor asks him to repeat
Data: *because it's essentially the same vector to the origin.*
Warrant: You can go on 1 vector and I can come back in the opposite direction

Day 9 Argumentation Log

Arg 9.16 WCD
Claim: If $m > n$, *you won't be able to span the entire dimension* [Abraham and Justin, 1:11:42]
Qualifier: *I don't know, it just makes sense* [Abraham, 1:11:58]
Data: *You don't have enough vectors for the dimensions.* [Abraham, 1:11:58]
Warrant: *So it seems you wouldn't be able to go to all the directions in those dimensions.* [Abraham, 1:11:58]

Day 9 AC Arg 1 (sp/LI)
Claim: Matrix A does not span R5. [Abe, 1:29:24]
Data: It spans a 2-D space in R5 [Abe]

Day 9 AC Arg 2 (sp/LI)
Claim: *The only time it doesn't span the whole dimension is when there are free variables.* [Abe]

Day 9 AC Arg 3 (sp/LI)
Claim: *A 5x3 matrix definitely spans, right? [Abe]*
Qualifier (to C): *I'm just trying to think about it as a generalization.*

Day 9 AC Arg 4 (sp/LI)
Claim: *Now it's [matrix B] spanning [Abe, 01:30:10]*
Data: *It has 2 free variables [Abe]*
Day 9 AC Arg 5 (sp/LI)
Claim: *When m=n, if it's linearly independent, then it spans the whole dimension [Abe, 1:31:15]*
Qualifier: *I don't know...it just seems like it would span.*

Day 10 Argumentation Log

Day 10 SG Arg 1 [12:47]
Claim: *You have a solution for everything*
Data: *It sill spans, the columns still span*

<u>Day 10 SG Arg 2 [14:02]</u> (sp/LI)
Claim: *It has to be linearly independent*
Data1: *it has the same amount of vectors as entries*
Data2: *It spans*

<u>Day 10 SG Arg 3 [16:20]</u>
Qualifier: *I might be thinking about this wrong*
Claim: If the columns of an nxn span Rn, *the solution can't be unique*
****Justin calls for data**
Data: *It spans everywhere, then you have a lot of different b's*
Warrant: *It couldn't be the same weights on them*
Backing: *The weights are really what the solution is.*

<u>Day 10 SG Arg 4 [19:45]</u>
Claim: *you can still go in all the paths*
Data: *If the span is everywhere, you can go everywhere to get anywhere*
Qualifier: *That was not really in linear algebra terms, though.*

Interview 1 Argumentation Log

<u>Int 1 Q6a Arg 1</u> (sp/LI)
Claim: If the columns of a 3x3 matrix *A* span R^3, then it is FALSE that the columns vectors of *A* are linearly dependent. [Abe, 0:45:59]
Data: The column vectors of A would *have* to be linearly independent [Abe, 0:46:09]
Qualifier (to the Data): this is hard to explain
Warrant: The only solution [to Ax=**0**] is the trivial solution
**Int calls for clarification of the warrant
Backing: And so if that holds for 0 then that means that it should hold for, um, every point. So that every, um, every point would have a unique solution.

<u>Int 1 Q6a Arg 2</u> (sp/LI)
Claim: If the columns of a 3x3 matrix A span R3, then it is FALSE that the columns vectors of A are linearly dependent
Data: *I know it's linearly independent*
Qualifier: *It's so weird, it's like sometimes something makes like sense to you, and then you just know it's right, but sometimes you don't know how to really, like expl—you know what I mean? Like really explain, explain, you know, why, why it is true though.*
Warrant: *like automatically my mind just jumps to, 'they're linearly independent, they can, they span everywhere.'*

<u>Int 1 Q6a Arg 3</u> (sp/LI)
Claim: *I started to think about this, like when you have [* $A\vec{x} = 0$*], this is Ax=0, that a lot of the characteristics apply to Ax = some vector b*
Data: *because, like, I have all these points here [draws dots in an augmented matrix], and I have this augmented matrix here, and the 0's* [in [A|0]], *And then let's say I was, um, I had some other points [draws an empty augmented matrix with last column* <1,2,3>]. *whatever I do over here with row reducing this [original augmented matrix], it's really only going to affect this [the coefficient matrix].*

Warrant: *if there's a unique solution, c1=0, c2=0 and c3=0 for 0 [points to the <0,0,0> in the first augmented matrix] then whatever, since, I mean, this [points back and forth between the two coefficient matrices] when you row reduce it, it ends up being like the same thing on this side, then...then most of the time, there should be a unique solution for any b over here [points to the <1,2,3> in the second augmented matrix]*
Qualifier: *but there are cases where it doesn't, and I remember thinking of 1, where it doesn't work like that. And it's actually when the bottom row ends up being like that. [writes "0 0 0 | 1"] That's when there's no solution.*

Int 1 Q6b Arg 1
Claim: 'If A is a 3x3 matrix whose columns span R3, then any vector b in R3 can be written as a linear combination of the columns of A' is a TRUE statement.
Data1: *When I read that, 'any vector b in r3 can be written as a linear combination of the columns of A,' I automatically think of the definition of span.*
Data2: *And I think of the definition of span as any vector b being able to be written as a linear combination of the columns of A.*
Warrant: *if I was linear algebra, if I created it, that would be my definition of span.*
Qualifier: *I don't know, I might be wrong, but that's what I, what I think of, you know!*

Int 1 Q6b Arg 2
Claim: 'If A is a 3x3 matrix whose columns span R3, then any vector b in R3 can be written as a linear combination of the columns of A' is a TRUE statement.
Data1: The column vectors span
Claim1: *I can figure out the scalar multiples that would make these* [column] *vectors get to here* [b]
Warrant: *You could get anywhere, which just seems like the definition of span.*

Int 1 Q6c Arg 1
Claim: If the columns of a 3 by 3 matrix A span r3, then A has 3 pivots is a true statement
Data: Let's assume it doesn't have 3 pivots [writes [1,0,0; 0,1,0; 0,0,0]]
Qualifier: *I guess it would be a hard way to prove something actually, because them you would have to think about all the different things.*
(embedded D1-C1)
Data1: *pivot, you know, you pivot and you go a different position...when you kind of, you go in one, like I'm a basketball person, I pivot [turns his body] you know! And then I go somewhere else. So I'm able to go a different direction.*
Claim1: *I think that's why they call it 'pivot positions.'*
Warrant: *And so if I can't pivot [makes smaller version of basketball move], go in this direction, pivot [repeats basketball pivot], go in this direction, pivot and go in all 3 directions, I can't span everywhere, I can't span.*
Backing: *Since it does say that I can span, then I must be able to pivot in all 3 of these positions* [implicit proof by contrapositive]

Int 1 Q6c Arg 2
Claim: *I won't be able to span* with the columns of [1,0,0; 0,1,0; 0,0,0]
Data: *this is a 1 vector [1st column] say I can go in x direction; this [2nd column] is another vector say I can go in y direction. But how [3rd column] do I go in the z* direction?
Warrant: *I'm only able to pivot in 2 positions, I'm only able to go in 2 directions.*

<u>Int 1 Q6d Arg 1 [1:06:46]</u> (sp/LI)
Claim: If the columns of a 3 by 3 matrix A span r3, then there is a non-trivial solution to the equation Ax=0' is a false statement
Data: (embedded)
Qualifier: *Assuming that my answer to A was correct* [that span → LD was false]
Claim1: *I know they're linearly independent*
Data1: *the definition of linear independence says that there is only a trivial solution to the equation Ax=0*
Warrant: (embedded)
Claim1: *The only solution* [to Ax=0] *would have to be 0,0,0*
Data1: *that would also be a unique solution within the plane* [*I think he means the origin would have a unique solution to get there]
Warrant1: *And like I was talking about before, that trivial solution, there has to be a trivial solution because you're spanning everywhere, and I covered that whatever point you get to has a unique solution*

<u>Int 1 Q6d Arg 2</u>
Claim: If the columns of a 3x3 matrix span R3, x*=0 y=0 z=0 is the only solution to Ax=0*
Data: Ax=0 is [A | 0] in the augmented matrix, *and we're trying to find the x, x is pretty much what I think of as the solution.*
Warrant: From [A | 0], *the solution is x=0, y=0, z=0, there is no other solution.*

<u>Int 1 Q6d Arg 3</u>
Claim: If the columns of a 3x3 matrix span R3, x*=0 y=0 z=0 is the only solution that you're going to get*
Data: *We also proved there is 3 pivots*

<u>Int 1 Q6e Arg 1 [1:10:31]</u>
Claim: If the columns of a 3 by 3 matrix A span r3, then the system Ax=b has no free variables' is a true statement
Qualifier (to C): *I'm trying to think of a counterexample. I think, I'm pretty sure it's true.*
Data: *# of free variable*s = *# of unknowns - # of pivot positions*
Qualifier (to D): *I don't know if we talked about this in class but I'm pretty sure the homework, everybody figured it out from doing the homework*
Warrant: *I'm still thinking of this as a vector equations, so it's basically the number of unknowns corresponds to the number of columns.*
Backing: *So we have 3 columns, we have 3 pivot positions, so there's 0 free variables* [writes '3 – 3 = 0']

<u>Int 1 Q63 Arg 2</u>
Claim: *If there are free variables, then there infinitely many solutions to Ax=b*
Data: *In x1=5-x2, x2=7+x3, x3 is free, it's called free because it can be anything it wants.*
Warrant: *So if I plug anything I want in here, then x1 depends on x3, x2 depends on x3. And so x3 could be whatever it wants, and x1 and x2 can still be found, a solution back to Ax=b.*
Backing: *And now I end up with infinitely many solutions* [to Ax-b]

<u>Int 1 Q63 Arg 3</u>
Claim: *If there are infinitely many solutions, there are not n pivot positions*

Data: *infinitely many solutions to Ax=b relates back to this bottom thing [0 0 0 | 0] I was talking about*
Warrant: *that 0 in the third column means that it can't be a pivot position*

Int 1 Q63 Arg 4
Claim: If it spans, there are no free variables
Data1: *If there is a free variable*
Claim1: *Then it can't have a pivot position*
Data2=Claim1: *If it doesn't have a pivot position*
Claim2: *It doesn't span.*
Data3=Claim1=Claim2: *The fact that it doesn't have a pivot position and doesn't span*
Claim3: *that's a counterexample*
Warrant: *We're assuming it does span.*

Int 1 Q63 Arg 5
BIG CLAIM: If the columns of a 3x3 matrix A span R3, then there are no free variables.
Sequential Data:
Data1: [assume] *if it's free*
Claim1: *then there's infinitely many solutions;*
Data2=Claim1: *if there's infinitely many solutions,*
Claim2: *then the bottom row is this* [0 0 0 | 0].
Data3=Claim2: *If the bottom row is this,*
Claim3: *then you know it only has 2 pivot positions.*
Data4=Claim3: *And if it only has 2 pivot positions, you keep going with proof,*
Claim4: *then it doesn't span all of r3.*
Warrant: *but that would be a contradiction to what you're assuming in the very first sentence,* so *you write a symbol like this and you go, 'contradiction,' and your proof is done.*

Int 1 Q8a Arg 1
Claim: Those asterisks could be any numbers.
Data: I saw this notation (with the asterisks) in the book and it makes sense
Warrant: When you're trying to row-reduce, you're trying to get zeroes under the pivot positions. You stop once that happens.
Backing: So everything else, the other numbers, don't really matter.

Int 1 Q8a Arg 2 (sp/LI)
Claim: I can create a 3x5 matrix that spans all of R3
Data: *I have 3 pivot positions, these 2 vectors don't even need to be used to span that, the r3*
Warrant: *I can span r3 still, which is those 3 [covers 1st 3 columns]. So there's extra vectors in the problem it seems like, it's still 3 dimensions, but you have 5 vectors.*

Int 1 Q8a Arg 3 (sp/LI)
Claim: *if you have more vectors than dimensions, that automatically makes it [the columns of B] linearly dependent.*
Qualifier: *I just actually thought of something. This, if, I was talking about earlier*
Data: the first 3 vectors [in his example] *do span all of R3*
Warrant: *But now we're talking about linearly independent. I can only construct one that spans all of r3 and is linearly dependent.*

Int 1 Q8a Arg 4 (sp/LI)
Claim: *if you have more vectors than dimensions, that automatically makes it [the columns of B] linearly dependent.*
Data: *I get to any point I want in r3. [draws an empty x,y,z coordinate axis] Let's just say I was here [puts a point in Q1]. Now I have 2, that was after using 3 vectors, now I have 2 more I could use.*
Warrant: *So I should be able to make 1 that is linearly dependent and get back there, because I can get anywhere, so I can get, if I'm using the origin yet, I can get back to the origin.*
Qualifier: *But how could I make it so that I don't get back to the origin? I don't think it's possible.*

Int 1 Q8a Arg 5u (sp/LI)
Claim: if you have more vectors than dimensions, that automatically makes it linearly dependent.
Qualifier: *This is weird to explain.*
Data: *I got here with 3 vectors, now since I can span all of r3, then we know I can get to this point, because this spans all of r3, so I know I can get to this point.*
Warrant: *If I can get there, then I can go back this way. So if I can get 3 vectors there, then I'm going to be able to get back there, using one of the other 2 vectors that are here*
Qualifier: *But I'm still kind of, I want to be convinced that that's true, but at the same time, I don't know if I thought about it enough,*
Rebuttal: *but it seems like I should be able to make 2 vectors that don't get back here, but see, that's a problem. But I know by the definition that if they have more vectors than entries, it has to be dependent, so that convinces me. But it almost seems like maybe I should work on a computer program to try to make it so that I don't, I can't get back to here, and I probably would fail. But in my mind, it seems like you should be able to*

Int 1 Q8b Arg 1 (sp/LI)
Claim**:** There's no way you can span that [R5] with 3 vectors in R5
Data: *It's not possible***. (linked)**
Data1: *First, like I said I'm stronger on the pivot positions, this only has 3. There would have to be 5, or 5 pivot positions for it to go to span all of r5.*
Data2: *And also, you can never span, I can never, if I have less vectors than dimensions, then I can never span that.*
Warrant: *I really can't as I say go in all the directions of the dimension, all the pivots, and so I can't span it, because I don't have enough vectors to go to fill up the space, to be able to go in all places of it.*

Int 1 Q8b Arg 2 (sp/LI)
Claim: I would change the matrix to 5x5 to create an example [that does span and is LI]
Data1: *If I ever had less than this, it's going to be not possible to span.*
Data2: *A 5 by 6 though, now I can still span everything, but now it's linearly dependent*
Warrant: *If I had the same amount, then I have the identity matrix, 5 by 5, it can span everything, and it's linearly* independent, yes.
Qualifier: *I understand the concept, but the, of that, but explaining the 5 by 6 is probably the hardest for me, I think.*

Day 17 Argumentation Log

Arg 17.9
Claim: The matrix [1,2; 2,4] is not invertible [Abraham, 51:23]
Data: *that span is only the line, the y=2x* [Abraham, 51:23]
Warrant: *And so it never even hits 1,0 or 0,1, because it has to go over 1, up 2, over 1, up 2.* [Abraham, 51:23]

Arg 17.10 (partial)
Claim: The matrix [1,2; 2,4] is not invertible [Abraham, 51:51]
Qualifier: *It doesn't seem like the matrix would be invertible* [Abraham, 51:51]
Data1: *We're trying to get* 1,2 to be 1,0 and 2,4 to be 0,1 *and these 2* [1,0 and 0,1] *don't really fall on that line* [where 1,2 and 2,4 are] [Abraham, 51:51]
Warrant: *if it all collapses to a line, how it could, um, how it could be 1,0;0,1. It seems like it would have to be able to span to 1,0. It seems like it would have to be able to reach that point.* [Abraham, 54:03]
Backing 2: *So we can only transform to multiples of 1,2. If any vectors are multiples of 1,2, you can transform to them, nothing else, though*. [Abraham, 54:51]

Arg 17.19 WCD (partial)
Claim: *It has to span* [Abraham, 1:01:01]
Data: *based on having those 2, if those 2 are true* [A is square and the columns are LI] [Abraham, 1:01:01]
**Instructor asks for clarification of the claim [1:01:15]
Claim: *In order for A to be invertible, then A has to span* Rn [Abraham, 1:01:24]
**Instructor asks for assistance [1:01:39]

Day 18 Argumentation Log

Day 18 SG Arg 1
Claim: *A matrix is invertible*
Data: *I just look at it and go, "man that looks linearly independent, it's got to be invertible"*

Day 19 Argumentation Log

Day 19 SG Arg 1 [15:32]
Claim: $y = x^2$ is not onto
Data: *It hits 2 points*
Warrant: *it doesn't pass the horizontal line test*

Day 19 SG Arg 2 [16:10]
Claim: *If it's unique, then it's bijective*
Data1: *at least one is the definition of 1-1*
Warrant: *At most one is the definition of onto*

Arg 19.9 (partial)
Claim: *We can use span and onto interchangeably* [asked by Jerry, answered by Abe, 44:44]

Data*: If it spans, the range equals the codomain* [Abraham, 45:01]

Day 20 Argumentation Log

Day 20 SG Arg 1
Claim: [1,0; 0,1] is onto R2
Data: [the columns] *span all of R2, spans the whole codomain*

Day 20 SG Arg 2
Claim: [1,0; 1,0] is not onto R2
Data: *That one doesn't span*

Day 20 SG Arg 3 [11:29, 12:39]
Claim: *It's not possible for it to be 1-1 and be a 2x3 matrix*
Data: *1-1, really the columns have to be linearly independent*
Warrant: *If it has more vectors, it's linearly dependent*

Day 20 SG Arg 4 [13:50]
Claim: [11; 2,3; 3,3] is not onto R2
Data1: *That doesn't span R2*
Data2: *It just actually is a line.*

Day 20 SG Arg 5 [24:22]
Claim: If m<n, *it's not possible for the columns of A to be linearly independent*
Data1: *Since we have more vectors than dimensions, then it has to be linearly dependent.*

Day 20 SG Arg 6 [28:09]
Claim: *If there are less vectors than dimensions*, *it's not possible for the transformation to be onto*
Data: *If we have less, not enough pivot positions because there aren't enough vectors*

Arg 20.5
Claim: *if it's 1 to 1, the columns of A have to be linear independent.* [Abraham, 37:34]
****Call for data [37:34]**
Data: *1 to 1 is at most 1 solution* [Abraham, 37:43]
Warrant: But we know the trivial, or the Tx=0 always has to have 1 solution. So then it has to have the trivial solution. But if their columns are linear dependent, then T of x is 0 has more than 1 solution. So there would be 1 going to 2 different things [Abraham, 37:43]
Backing (implicit): logical equivalence of proof by contrapositive.

Day 20 SG Arg 7 [38:38]
Claim: Linear independence and 1-1 are connected
Data1: *T(x)=0 has to have only the trivial solution*
(AND)
Data2: *1-1 is unique solution or none at all*
Claim2: *Tx=b would have to have unique solution or none at all*

Warrant: *So if you simplify it down to T of x = 0, then we can see T of 0,0 must have the unique solution. Because it's not possible for it to have none at all. It has to have at least 1, because it's 0,0,0.*

Arg 20.16 (sp/LI)
Claim: *If the matrix is square,* being linearly independent is the same as being onto [Abraham, 52:21]
****Instructor calls for data [52:26]**
Qualifier: *I just remember*
Data: *if it's square, we had the nxn theorem* [Abraham, 52:52]
Data1: *If a square matrix is linear independent,*
Claim1: *it also spans* [Abraham, 52:52]
Claim2/Data2: *And vice versa.*
Warrant: *As so that means* (union of those two D-C)
Warrant1: *if it's 1 to 1, it has to be onto*
Warrant2: *if it's onto, it has to be 1 to 1*

Arg 20.24 (sp/LI)
Claim: *It* [a matrix] *would be invertible if all those things are true* [Abraham, 1:02:33]
Qualifier: *It seems like it* [Abraham, 1:02:33]
****Teacher calls for data**
Data: *I remember linear independent, span and row reduces to identity matrix and square all being in the invertible matrix theorem.* [Abraham, 1:02:54]
Warrant: *those are all the same things we had up there* [Abraham, 1:02:54]

Arg 20.26
Claim: *if it's either 1 to 1 or onto but not both, then it has to be non-square* [Abraham, 1:09:55]
Data: *Because of the chart* [Abraham, 1:09:55]
Warrant: *if you're one and not the other, then you can't be square.*
Data1: *If you're square*
Claim1: *you're either both or neither.*
Data2: *And if you're one but not the other, meaning 1 to 1 or onto*
Claim2: *then you must be either wider or taller but you can't be square.*

Day 24 Argumentation Log

Day 24 SG Arg 1 [02:07]
Claim: If the column vectors are linearly dependent, the determinant is 0
Data: *I just made like all the different examples of 2x2 matrices that would be linearly dependent, just to be sure*
Data1: *1 with multiples*
Data2: *1 with the same vector*
Data3: *1 with 0 as a column vector*
Data4: *1 with 0 as rows*
Warrant: *Every time, the determinant was zero.*

Day 31 Argumentation Log

Day 31 SG Arg 1 [23:19] (span/LI)
Claim: the 'n pivots' card goes with the 'span' and 'linear independence' cards
Data: *It has to be square, and it has to have n pivots, that means it's linear independent and spans. That's the first thing I think of, as far as n pivots*

Day 31 SG Arg 2 [26:54]
Claim: 'null space is only 0' and 'linear independence' *go together*
Data: *That's just the vector of x = 0, so that's just a solution. That's just a solution to the Ax=0, the null space*
Warrant: *I never thought about it like that, though*

Day 31 SG Arg 3 [27:06]
Claim: 'the number 0 is not an eigenvalue' also goes with 'null space is only 0' and 'linear independence'
Data: *if the Eigen value was 0, I'd just reduce down to Ax=0.*

Day 31 SG Arg 4 [33:40]
Claim: If there exists a C s.t. AC=I, then A is invertible
Data: *these* [points to C] *are the row reducing functions that take this from A to the identity matrix, which makes it invertible*
Qualifier: *That's the only way I see it, I don't see it both ways.*

Day 31 SG Arg 5 [43:15]
Claim: *the column space of A is all of Rn, and the columns of A spans all of Rn are exactly the same*
Data: *Like those 2 to me are the most equivalent, the same*

Arg 31.5 WCD (sp/LI)
Claim: "'For every b in Rn, there exists a solution x to Ax=b **"** is also equiv to the previous 3 (span, column space, lin combo [Abraham, 47:03]
Data: *There's solution to every b,.* [Abraham, 47:11]
Warrant: *so we can get to that output vector, and that's span* [Abraham, 47:11]

Arg 31.6 WCD (sp/LI)
Claim: *If we added a word 'unique,' then I would put it* [the columns are linearly independent] *in there* [Abraham, 47:11]
Data: *Because 'unique' makes it linear independence.* [Abraham, 47:11]
Qualifier: *but without the word, I just think of we can get to every b.* [Abraham, 47:11]

Day 31 SG Arg 6 [1:02:45]
Claim: *To me, the null space contains only the 0 vector, that goes with linear independence really well.*
Data: *Because that* [null space] *means that the only solution is 0*

Day 31 SG Arg 7 [1:04:11]
Claim: 'The null space contains only the zero vector' is the same as 'the only solution to Ax=0 is the trivial solution'

Data: *The null space is like, we talk about it as part of the domain.*
Data1: *If this is only the 0 vector*
Claim1: *then this is the same thing to me as 'the only solution to Ax=0 is the trivial solution'*
Warrant: *this* [null space] *is the domain and this is the input vector of the domain*
****Giovanni asked for clarification of the warrant**
Claim: If 'The null space contains only the zero vector' the 'the only solution to Ax=0 is the trivial solution'
Data: *The null space is part of the domain*
Warrant: *The only solution to Ax=0 is the vector x=0*

Interview 2 Argumentation Log

<u>Int 2 Q1a Arg 1</u> (sp/LI)
Claim: If the columns of a 3x3 matrix A span R3, then it is FALSE that the columns vectors of A are linearly dependent. [Abraham, 00:53]
Data: Matrices whose columns span R3 have *three pivot positions* [Abraham, 00:53]
Warrant: *And for like a square matrix, I just think like if this is 3 pivots in each row, then it's also going to be, automatically going to be 3 pivots in each column.*
Backing: *And that way you're always going to have a linearly independent set of...of, um, like x =something, y = something, z = something, because of that.*
Backing L2**: *And then that's, so that's going to be, basically a unique solution for every output*...There's a unique x,y,z vector such that Ax = b in the output, so that'd be linearly independent, but not dependent.

<u>Int 2 Q1a Arg 2</u> (sp/LI)
Claim: 3 vectors that span R3 relates to a 3x3 matrix with ones on the diagonal
Data: To span R3 the vectors need to go in every direction, and pivot positions allow it to do that.

<u>Int 2 Q1a Arg 3</u> (sp/LI)
Claim: Having 3 pivot positions allow you to go in three directions [Interviewer after Abraham, 0:03:43] **In response to Abe's previous explanation]
Data: *so I mean I go in this direction and then I can kind of pivot up, you know to go to this direction. And then I have another value in here, so I can come back and go, so I can kind of go any direction*
Warrant: *based on these 3 pivots, like pivoting in each direction, to get to any point.*

<u>Int 2 Q1a Arg 1</u> (sp/LI)
Claim: Square & span → RREF(A) has 3 pivots is TRUE
Data (embedded cases)
Data1: *Whenever I have that* [one vector can be written as a linear combination of the other vectors],
Claim1: *it doesn't row-reduce to 3 pivots.*
Warrant1: *Whenever it's a linearly dependent set, it row reduces to some kind of row of zeroes*
Qualifier1 [to the W]: *Usually, yeah, usually a row of zeroes I believe*
Data2: *But since I don't have these multiples, and I can't use these 2 vectors to get to there.*
Claim2: *it's going to row-reduce down to 3 pivot positions*

Warrant: *just imagine I went here, and I somehow arrived at, you know, 3,3,3. Let's say I got to that point and then I could take that point back to get, to get back to home. Which is still how I kind of think of it. Then that would make that linearly dependent because I can, I can kind of get there and take that vector back. And um, but if I don't have that, then it's going to row reduce down, without this.*
Backing (implicit): there are only two choices: it does or it does not row-reduce to 3 pivots.
Qualifier: *So...hmm, I don't know if that makes.*

Int 2 Q2b Arg 2 [09:14] (sp/LI)
Claim: [1, 2; 2, 4; 3, 6] is linearly dependent
Data: $1\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}3\\6\end{bmatrix}$ and $1\begin{bmatrix}3\\6\end{bmatrix}=\begin{bmatrix}3\\6\end{bmatrix}$

Warrant: *So now I have 2 solutions, getting to the same point*

Backing: *So that makes it, in a square*[sic, nonsquare] *matrix, linearly dependent, because I'm getting there 2 different ways.*

Int 2 Q2b Arg 3 [11:16] (sp/LI)
Claim: [1, 2; 2, 4; 3, 6] is linearly dependent
Data: *I can also get to that point, and then I can subtract, multiply by negative vector, I can get back to 0,0* $1\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}2\\4\end{bmatrix}-\begin{bmatrix}3\\6\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$
Warrant: *But you want the only solution to be the 0,0, for it to be linearly independent*
Backing: *But this is, you see that this [points to his new equation] is a solution is 0,0*

Int 2 Q2b Arg 4 [15:05] (sp/LI)
Claim: *If it doesn't have 3 pivots, this isn't going to span.*
Data1: *There's only one pivot position here* (RREF(A)) $\begin{bmatrix}1&2&3\\0&0&0\\0&0&0\end{bmatrix}$
Data2: *It kind of relates to column space, which would be anything times this vector for A* [writes *a*<1,1,1> in set notation]
Warrant: *Yeah, so then it's just, it's only spanning this 1-dimensional subset.*
Backing: *I still think of column space as being span*

Int 2 Q2c Arg 1 (sp/LI)
Claim: "Let A be n x n" is an important statement in the Invertible Matrix Theorem [Int, Abe, 0:21:49]
Data: *They wouldn't, they'd be, they wouldn't go both ways, if the matrix wasn't n by n.* [Abe, 0:22:01]
Warrant: *if it can go both ways then they're equivalent. But if it can only go 1 way then they're not equivalent*

Int 2 Q2c Arg 2 (sp/LI)
Claim: If m > n, then 'linear independence' and 'span' are not equivalent statements [implicit from before, and 0:22:36]
Data: The example matrix A (3x4: [1,0,0; 0,1,0; 0,0,1; 75, 99, 85]) spans but is not linearly

independent [Abe, 0:23:01]
Warrant: *But if this one spans here [points to the IMT statement sheet], so if it spans, then to say it's linear independent, that's false. Let's see, if it's linear independent it spans? Not true either.*

<u>Int 2 Q2c Arg 3</u> (sp/LI)
Claim: If m > n, then 'linear independence' and 'span' are not equivalent statements [implicit from before, and 0:24:34]
Data: If it has less vectors than dimensions it won't span but could be linearly independent [Abe, 0:24:34]

<u>Int 2 Q2c Arg 4</u> (sp/LI)
Claim: If m > n, span and linearly independent are not equivalent
Data: Example B: [1,0,0; 0,1,0]
Data1: (embedded):
Data1a: *So this only has the two pivot positions*
Claim1a: it *spans 2-dimensional subset of r3*
Warrant1a: *you can't go in all the different directions*
Data2: *it is linearly independent*
Warrant: (D1 and D2) *This is, 1 is false,* and *6 is true*

<u>Int 2 Q3 Arg 1 [29:43]</u>
Claim: The 3 cards in Pile 3 go together
Data: *For me those are together on linear independence*

<u>Int 2 Q3 Arg 2 [29:43]</u>
Claim: The 2 cards in Pile 1 go together
Data: *I always see those ones and I automatically think, if it's row equivalent to that, it has n pivots.*
Warrant: *And I see those ones in my head.*

<u>Int 2 Q3 Arg 3 [29:43]</u>
Claim: The 2 cards in Pile 5 go together
Data: (sequential of 4 args)
Claim1: Ax=0
Data1: If 0 *is* an eigenvalue
Claim2: *We can find a nonzero solution*
Data2=Claim1: Ax=0
Warrant2: *By definition of an Eigen vector*
Qualifier: The zero solution is *not very interesting*
****Teacher asks for small clarification of "it"**
Claim3: *That would make it linearly dependent*
Data3=Claim2 *There exists a nonzero solution*
Warrant3: *By definition* of linear dependence
Claim4: *The null space doesn't contain only the 0 vector*
Data4=Claim2: *If there's a nonzero solution here* [Ax=0]
Warrant4: *The null space is part of the domain, so it contains all the solutions to Ax=0.*
Backing: *I think of them together if I put a negation in front of them* (implicit: if the negation of two statements 'go together,' the two statements 'go together.')

Int 2 Q3 Arg 4
Claim: If the null space contains only the 0 vector, then zero is not an eigenvalue of A
Qualifier: *That's a hard one to explain, that's why I don't like to think of it this way.*
Data: *Only this* [x=0] *is being sent to zero*
Warrant: $A<0,0,0>=0=\lambda x=\lambda 0$
Rebuttal: *But then that also tells me that any lambda will satisfy this equation. But my only problem with this isn't a very good proof, because also 0 satisfies this. So I'm sorry about that.*
Backing: *it's not* [true] *in all cases, but I know it is in this case, because we're talking about equivalent statements. An equivalent statements I say are like this [writes '⇔']. With an equivalent statement, I don't remember when I learned this but, the negation of the statement will also go both ways, if it's an equivalent statement. So since I have an equivalent statement, I can prove these 2 by using the opposites of it. And it's way, maybe this is probably simple too, but I see this as a way of being way simpler.*

Int 2 Q3 Arg 5 [39:22]
Claim: The three cards in Pile 4 go together
Data: "AD=I" and "CA=I" are the *definition of invertibility...the core definition*.

Int 2 Q3 Arg 6 [39:49]
Claim: The six cards in Pile 2 go together
Data: *I think of all of these as span.* [39:55]
Warrant: *So there exists a solution. So for every output vector, I can find. I think of it as for every output vector, I can find an input vector. So for every vector in this space, I can find a solution. So then for every b, I can get to that. I think of that for a lot of them.*

Int 2 Q3 Arg 7
Claim: the cards 'exists a solution for every b' and 'a way to write b as a linear combo are *the same thing*.
Data (linked/embedded): Card 1 *is matrix-oriented.* Card 2 *is vector-oriented:*
Claim1: Card 2 is vector-oriented
Data1: The card talks about *it's the linear combination of the columns to get everywhere*
Warrant1: The columns of the matrix *represent the vectors.*
Claim2: Card 1 is dealing with a transformation
Data2: *Can I send, can I find an input vector that send to every output vector?*
Warrant: *It's a little different, but they both concern spanning a space, whether it's through a transformation or by vectors being added together to get there.*

Int 2 Q3 Arg 8 [41:50]
Claim: *I can't really discern the difference of these ones* **[**'column space of A is Rn' and 'the vectors span Rn']
Data (linked):
Data1: *when I think of the column space, I really literally think of the space that the columns can get to*
Data2: *And this* [span card] *is talking about the columns of A spanning*
Warrant: *Because like I said, this* [Col A] *is to me where the columns can get to in this fashion. And this one* [span] *is the space that the columns can get to. Which for me is like the same thing*

<u>Int 2 Q3 Arg 9 [0:42:27]</u>
Claim: 'Onto' and 'determinant' are *connected* [to the 'span' pile], *but not as connected*
Data: *when I think of onto, I think of span automatically. But not as fast as I think of span if I'm looking at this [points to (2)].*
Warrant [embedded]:
Claim1: *This one* ['onto'] *is just very related to span*
Data1: *For it to be onto, for every b there's at least 1 solution.*
Warrant1: *this doesn't say 'at least.' [points to a card in the span pile] But it says a solution. There exists a solution*
Backing1: *If there exists a solution, then there exists at least a solution, so that goes back to this one* [card 1].

BIG CLAIM: 1-1, onto, and invertible connect
His proof: Sequential. Does a "1-1/onto together"→Invertible (Arg 1-7 in total.)

<u>Int 2 Q3b Arg1:</u>
Claim1: 1-1/onto together is "for every b there is exactly one solution x s.t. Ax=b."
Data1 (linked by "and"):
D1a: Definition of 1-1 is *for every b there's at most 1 x s.t. Ax=b.*
Data1b: Definition of onto is *for every b there is at least one x s.t. Ax=b*]
Warrant1: *There's at least one and there's at most 1, then there's exactly one.*
Data2 (linked by "and"):
D2a: *'At least' is really... at least 1, 1 or more*
Data2b: *At most 1 is maybe 0 or 1*
Warrant2: *So then the intersection of 1 or more...is just one.*

<u>Int 2 Q3b Arg2</u>
Claim: There is a unique solution to Ax=b for every b
Data=Claim1: *There is exactly 1* [solution to Ax=b]
Warrant: *'Exactly one solution' is also a 'unique' solution.*

<u>Int 2 Q3b Arg3</u> (sp/LI)
Claim: *it's spanning everywhere, and it's an unique solution.*
Data3a; *this* [onto] *is span*
Data3b: *this* [1-1] *is linear independence*
Warrant: *if this is span, and I want to add linear independence to it, then I would say this definition of span [points to that same card again], but just adding the word 'unique. '.*
Backing (implicit)=Claim from Arg 2 [there is a unique solution]

<u>Int 2 Q3b Arg4</u> (sp/LI)
Claim: *I can think about the basis vectors*, which are *always 1's and 0's, 0's and 1's*
Data(=Claim from 3): *it's spanning everywhere, and it's an unique solution.*
Warrant: *that's the definition of a basis: one that spans the subspace and is a linearly independent set.*

<u>Int 2 Q3b Arg5</u>
Claim: *There exists a C s.t. CA=I*
Data: *because you have C, the definition of invertibility is such that C times the basis vectors =I*

<u>Int 2 Q3b Arg 6</u>
Claim: *No matter what C is, C is equal to the inverse*
Data(=Claim from 5): *There exists a C s.t. CA=I*

<u>Int 2 Q3b Arg7</u>
Claim: *A* is invertible
Data: *An inverse C can be found*

<u>Int 2 Q3b Arg 8</u> (sp/LI)
Claim*:* *It's an invertible* [matrix]
Data: *Combining the definitions of span and linear independence*
Warrant: *what else combines the 2 definitions of span and linear independence? Kind of a basis does. But then what arises from basis is our basis vectors. And then if it's the basis vectors, we see that the inverse c is always, itself, or however you want to think of it*

<u>Int 2 Q3b Arg 9</u>
Claim: Span and onto are related but a little bit different
Data: "at least one solution" [wording in onto definition] *is really close to this definition* [card of "there is a solution for every b..."]
Warrant: *And that's* [that card] *is the definition of span to me.*
Backing: *If there's a solution x for Ax=b, then there's at least a solution***.** this "at least" holds. But then it also means that things can repeat.

<u>Int 2 Q3b Arg10</u>
Claim: 1-1 and linear independence are related
Data: (linked)
Data1: (embedded)
Data1a*: there's at most 1 solution*
Claim1a: *I said there could be 0 solutions or 1 to Ax=b*
Warrant1: *This is the definition of 1-1*
Data2: (embedded)
Data2a*:* And then I can reduce this, if Ax=0, what do we know about Ax=0? We know that it always has at least 1 solution, namely the trivial solution, it always has at least that.
Claim2a: , there's 1 solution to Ax=0
Warrant2: So the definition of linear independence, right here, the only solution to Ax=0 is the trivial solution
Big Warrant: *So there's 1 solution to Ax=0. Then that solution must be the trivial solution. If there's 1 solution, it must be the trivial solution*

<u>Int 2 Q4a Arg 1</u>
Claim: If A has n pivots, then A is invertible
Data1: *A has n pivots*
Claim1: *It has n x n.*
Data2(=Claim1): *We always see that it goes to anything time that* [the 3x3 identity] *AC=I*
Claim2: *it would be invertible*
Warrant2: *You multiply, you can find the inverse easily if it has n pivots.*

<u>Int 2 Q4a Arg 2</u> (sp/LI)
Data3: *It has n pivots*

Claim3: *it spans and is linearly independent.*
Data4(=Claim3) *We connect these ideas of span, LI* [to] *onto and 1-1*
Claim4: *A would be invertible*
Warrant4: [We] *said that then you would get to this [the 3x3 identity]*
Backing4: *Then we used those to show the A invertible by using the basis vectors*

<u>Int 2 Q4a Arg 3</u>
Claim: If A is invertible, A has n pivots
Data: *I know it's true*
Qualifier: *I don't know why it is, I have never thought about it.*

<u>Int 2 Q4a Arg 4</u> (sp/LI)
Claim: If A is invertible, A has n pivots
Data:
Data1: *When I have A is invertible*
Claim1: *The columns of A are linearly independent*
Qualifier: *I don't remember how I used to think of that, but that's the connection I think about right away.*
Data2=Claim1: *The columns of A are linearly independent*
Claim2: *I'm going to have a pivot position here, here, here.*
Data3=Claim2: *I have a pivot position in these columns*
Claim2: *I have* pivots in the rows *as well*
Dat4-Claim3: *It follows naturally that those pivots there* [rows]
Claim4: *Then it spans*

<u>Int 2 Q4a Arg 5</u> (sp/LI)
Claim: If A is invertible, A has n pivots
Data:
Data1: *When I have A is invertible*
Claim1: *The columns of A are linearly independent*
Warrant: There is a pivot in every column [inferred b/c he didn't cover up the column arrows]
Qualifier1: *you don't need to talk about this [covers up his arrows over the rows], it's already has n pivots.*
Qualifier2; *But we could have talked about span as being the first connection to invertibility. But that's* [LI as first connection] *just what we chose to do in class!*

<u>Int 2 Q4b Arg 1</u>
<u>Claim:</u> *if the determinant of A is 0 then the area after the transformation is zero*
Data: *I think of it* [determinant] *as the area of the transformation.*
Warrant: *For some reason, let's just say that's some object, right, in r2. It's a door in r2, here's a little [knob]. No, but I think of the determinant as when you transform this by A, then the determinant after the transformation will have 0 area. So the door is unfortunately not a door any more, it has just the line*
Backing: *So the area, but if I extend it to more dimensions, it's not always going to be a line, just something with 0 area. It could be, if I think of some plane with area or volume or however you think of it, depending on dimension, I'll say volume or something, right? Then it could be smushed together or something and have 0 volume, or something like that.*

Int 2 Q4b Arg2
Claim: *if the determinant of A is zero then the columns are LD*
Data: *if I plug in different points on this door or whatever, that I'm going to have multiple solutions to a given point. Let's think of vectors, to a given vector. Say that's 1,1 or something; I might have 2 points on this door that transform to 1,1.*
Warrant: *And if it was linear independent, I would only have 1 point on this door that would go to 1 point over here*
Backing: *That's 1-1, so that'd be linearly independent.*

Int 2 Q4b Arg 3
Claim: *if the determinant of A is zero then the columns are LD*
Data: *if I start with determinant A equals 0, I'm going to have multiple points over here that are sent to the same point*

Int 2 Q4b Arg3
Claim: *if the determinant of A does not equal zero then* the columns are LD
Data: *For opposite reasons*

Int 2 Q4c Arg1
Claim: 'Null of A equal to 0,' 'The only solution to Ax=0 is the trivial solution' are the same thing.
Data [embedded]
Data1: *the definition of null space* [is] *the solutions to Ax = 0.*
Claim1: *that means right away to me that the 0 vector is being sent to the 0 vector. Or the only thing being sent to the 0 vector is the 0 vector.*
Warrant: *Which is the only solution to Ax=0 is the trivial solution*
Qualifier: *That's hard for me to explain*

Int 2 Q4c Arg2
Claim: 'Null of A equal to 0,' 'The only solution to Ax=0 is the trivial solution' are the same thing.
Data: *Because I automatically think if the null space is 0, then the set of vectors that satisfy Ax=0 is some form of all the multiples of A,0,0,*
Qualifier: *which is silly because there are no multiples*
Warrant: so *the only thing that is a solution to Ax=0 is the 0 vector, which is the trivial solution to this, to this last statement.*

Int 2 Q4c Arg3 (sp/LI)
Claim: Nul A = {a<0,0,0>} = 0 vector
Data: *I want to figure out what the null space is,* I solve for the x vector in this matrix equation (Ix=0)
Qualifier (to D): The matrix in the equation *could be anything, let's just say it's linear independent and it spans.*
Warrant: I can solve for the x in the matrix equation by *setting up an augmented matrix*
Qualifier (to W): *It's a simple case, but let's just say this is some matrix that's linear independent and spans.*
Backing: From row-reduction of the augmented matrix, *the solutions to Ax=- are x1=0, x2=0 and x3=0*

<u>Int 2 Q4c Arg4</u>
Claim: Nul A = {a<0,0,0>} = 0 vector
Data: *That's kind of where I got it* [the set notation] *from, is just going about solving for the vector in the domain.*
Qualifier (to D): *it doesn't really need the 'a'*
Warrant: *I went row reducing some matrix over here, and I can find out which vectors satisfy the equation Ax=0*
Backing: *And if it equals 0, then that's the null space.*

<u>INT 2 Q4c Arg5</u>
Claim: *The only solution is the trivial solution*
Data: *The null space is 0, only the zero vector*

Appendix 5.2. Compilation of Adjacency Matrix codes for Abraham

Argument	*Used for **m < n** in general or a specific example*	*Used for **m = n** in general or a specific example*	*Used for **m > n** in general or a specific example*	*Used for **all m, n** or was vague*
Arg 7.2 WCD		F5→F F3→F F3→F		
Arg 9.16 WCD			S9→H H1→H2	
Day 9 AC Arg 1 (sp/LI)			H4→H	
Day 9 AC Arg 2 (sp/LI)			J5→H	
Day 9 AC Arg 3 (sp/LI)	S8→G			
Day 9 AC Arg 4 (sp/LI)	J5→G			
Day 9 AC Arg 5 (sp/LI)		E→G		
Day 10 SG Arg 1		G→G5		
Day 10 SG Arg 2 (sp/LI)		E→G		
Day 10 SG Arg 3		G→G5→P1		
Day 10 SG Arg 4		G→G1 G→G1→G2 G1→G2		
Int 1 Q6a Arg 1 (sp/LI)		G→E G→E→E1 E→E1 E1→E2		
Int 1 Q6a Arg 2 (sp/LI)		G→I→E		
Int 1 Q6a Arg 3 (sp/LI)		I→E1→E2 I1→G E1→E I1→E2 I1→E2→G→E E2→E E1→E2 J6→H6		
Int 1 Q6b Arg 1 [59:42]		G3→G G→G3 G3→G, G→G3		
Int 1 Q6b Arg 2		G3→G→G2 G**→G2		
Int 1 Q6c Arg 1		G→I J→H3 J→H3→H1→H G→I		
Int 1 Q6c Arg 2		J→H3 J→H		

Int 1 Q6d Arg 1 (sp/LI)		G→E E→E1 G→E1→G2→E 2 E2**→E1 G→E1		
Int 1 Q6d Arg 2		E1→I1→E2→E 1		
Int 1 Q6d Arg 3		I→E1		
Int 1 Q6e Arg 1		G→I5 I4→I→I5		
Int 1 Q6e Arg 2		J5→F2		
Int 1 Q6e Arg 3		F2→I6 I6→F2 I6 → J F2→J J6→J4 F2→J4 G →I5 J5→F2 J5→J6→J→H G→I→I5		
Int 1 Q6e Arg 4		J5→J J→H G→I5		
Int 1 Q6e Arg 5		J5→F2 F2→J6 J6→J J→H G→I5		
Int 1 Q8a Arg 1	*Content not appropriate for coding scheme*			
Int 1 Q8a Arg 2 (sp/LI)	I→G G1→G S8→F	 E→G		
Int 1 Q8a Arg 3 (sp/LI)	S8→F G→F	 G→E		
Int 1 Q8a Arg 4 (sp/LI)	 G→F G2→F3→F9	G→E		
Int 1 Q8a Arg 5 (sp/LI)	G→G2 G→G2 G2→F9 F9→E3 S8→F			
Int 1 Q8b Arg 1 (sp/LI)			S9→H J→H S9→H H7→H	 I→G

			H2→H	
Int 1 Q8b Arg 2 (sp/LI)	S8→F G→F	G→E, E→G	S9→H	
Arg 17.9 WCD		F→H5→L6→L		
Arg 17.10 WCD		H5→L6→L N4→L6→H2 F5→N3		
Arg 17.19 WCD		E→G E→G2 K→G		
Day 18 SG Arg 1				E→K
Day 19 SG Arg 1	*Content not appropriate for coding scheme*			
Day 19 SG Arg 2				M1→O O1→M E2→M, E2→O
Arg 19.9 WCD				G→M2→G→M M→G
Day 20 SG Arg 1		G→M M2→M		
Day 20 SG Arg 2		H→N		
Day 20 SG Arg 3	S8→F→P			O→E
Day 20 SG Arg 4	H→H5→N			
Day 20 SG Arg 5	F→P S8→F			
Day 20 SG Arg 6			S9→J S9→N	
Arg 20.5 WCD				O→E O1→O O→O1**→E1 F→F1→P4
Day 20 SG Arg 7				E→E1 O→O1 E2→E1
Arg 20.16 WCD (sp/LI)		E→M, M→E E→G G→E→O→M M→O		
Arg 20.24 WCD (sp/LI)		O→K, M→K		
Arg 20.26	M→P, P→M			
Day 24 SG Arg 1		F5→F, F8→F F5→R F8→R F→R		
Day 31 SG Arg 1 (span/LI)		I→E I→G		
Day 31 SG Arg 2		S4→E, E→S4 S4→E1, E1→S4		
Day 31 SG Arg 3	*Too vague to code reliably*			

Day 31 SG Arg 4		K4→K5→K		
Day 31 SG Arg 5		S2→G, G→S2		
Arg 31.5&6 WCD (sp/LI)		G5→G, G→ G5 G5→S2, S2→ G5 G5→G3, G3→ G5 E2→E G5→G2 G5→M3→G		
Day 31 SG Arg 6		S4→E, E→S4 S4→E1 S4→E, E→S4		
Day 31 SG Arg 7		S12a→E1, E1→S12a→S4 S4→E1		
Int 2 Q1a Arg 1 (sp/LI)		G→E G→I→I3→I4→ E2→E E2→E		
Int 2 Q1a Arg 2 (sp/LI)		G→G4 S12b→G→G4 I→G4		
Int 2 Q1a Arg 3 (sp/LI)		I→G4→G2		
Int 2 Q1b Arg 1 (sp/LI)		G→I F5→F3→F6 F6→J→F→J6→ J E5→H2 G2→F3→F G→F3→F E→I G→E		
Int 2 Q1b Arg 2 (sp/LI)	P1→F P3→F			
Int 2 Q1b Arg 3 (sp/LI)	E1→E F1→F P3→F F1→F			
Int 2 Q1b Arg 4 (sp/LI)		J→F F→J6 J→H S2→G, G→S2 J→S3→H4		
Int 2 Q2c Arg 1 (sp/LI)	*Content not appropriate for coding scheme*			
Int 2 Q2c Arg 2 (sp/LI)	I1→G S8→F			
Int 2 Q2c Arg 3 (sp/LI)			S9→H I4→E	
Int 2 Q2c Arg 4 (sp/LI)			J3→H7→H4 I4→E	

Int 2 Q3 Arg 1		O→E, E→O E→E1, E1→E			
Int 2 Q3 Arg 2		I2→I I2→I, I2→I			
Int 2 Q3 Arg 3		S4→S6, S6→S4 S7→F1 E1→E S7→F1 F1→F S7→F F1→S5 S7→S5→S12a S4→S12a, S12a→S4			
Int 2 Q3 Arg 4	*Content not appropriate for coding scheme*				
Int 2 Q3 Arg 5	K4→K, K→K4				
Int 2 Q3 Arg 6		S2→G G5→G G3→G M→G Q→G M1→G5→G2			
Int 2 Q3 Arg 7		G3→G5, G5→G3 G2→G3 M3→G5			
Int 2 Q3 Arg 8		S2→G, G→S2 S2→G2, G2→S2 S2→G, G→S2 G2→G, G→G2			
Int 2 Q3 Arg 9		M→G G3→G, G→G3 M→M1 M1→M, M1→M M1→G5 G5→M1 M→G5, G5→M			
Int 2 Q3b Arg 1		O→O1 M→M1 M1→O1, O1→M1 O1→E2, M1→E2 M→O→E2, O→M→E2			
Int 2 Q3b Arg 2	*Content not appropriate for coding scheme*				
Int 2 Q3b Arg 3 (sp/LI)		G5→G, G→G5 G5→G→M O→E G→G5→E2→E G→E2, E2→G			

Int 2 Q3b Arg 4 (sp/LI)		G→E2→S12b S12b→G S12b→E		
Int 2 Q3b Arg 5		S12b→K4		
Int 2 Q3b Arg 6		K→K4, K4→K		
Int 2 Q3b Arg 7		S12b→O S12b→M→K4 →K		
Int 2 Q3b Arg 8 (sp/LI)		S12b→G S12b→E S12b→K4→K		
Int 2 Q3b Arg 9		M1→G, G→M1 M1→G5, G5→M1		
Int 2 Q3b Arg10		O→E O→O1, O1→O O1→E1→E E→E1, E1→E		
Int 2 Q4a Arg 1		I→K4→K I→K2		
Int 2 Q4a Arg 2 (sp/LI)		I→G, I→E G→M E→O→I1→K		
Int 2 Q4a Arg 3	*Content not appropriate for coding scheme*			
Int 2 Q4a Arg 4 (sp/LI)		K→E E→I1→I4 I4→I3 I3→G→E→G		
Int 2 Q4a Arg 5 (sp/LI)		K→E→I K→E K→G		
Int 2 Q4b Arg 1		R→R1 R→R1→N4 R→R3		
Int 2 Q4b Arg 2		R→F R→P1→P4 E→O1→O→E		
Int 2 Q4b Arg 3		R→P3→F		
Int 2 Q4b Arg 4		Q→E		
Int 2 Q4c Arg 1		S4→S12a→E1 S4→E1, E1→S4		
Int 2 Q4c Arg 2		S4→S12a→E1		
Int 2 Q4c Arg 3 (sp/LI)		E→G, G→E E→I2 G→I2→E1		
Int 2 Q4c Arg 4		I2→E1→S4		
Int 2 Q4c Arg 5		S4→E1		

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