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# CONSUMER'S SURPLUS WITHOUT APOLOGY: ANOTHER COMMENT

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# CONSUMER'S SURPLUS WITHOUT APOLOGY: ANOTHER COMMENT†

In his 1973 and 1976 papers, Robert Willig provided an ingenious analysis of the relationship between the Marshallian measure of consumer's surplus and the concepts of compensating and equivalent variation which has found widespread application. In this comment I shall show that his analysis can produce anomalous results in certain cases. An investigation of these anomalies reveals that Willig's methodology is valid only over certain subsets of price-income space, a restriction which has not previously been emphasized. George McKenzie recently criticized the validity of Willig's approximation procedures. In order to stress that this is not an issue here, I focus on cases where Willig's methodology yields exact results rather than upper and lower bounds or approximations.

## I. The Anomalies

Suppose that a consumer behaves as though he were choosing his consumption bundle, $x = (x_1, \ldots, x_N)$, so as to maximize a strictly quasi-concave ordinal utility function, $u(x)$, subject to a budget constraint. Since I shall focus on changes in the price of good 1, I write the price vector as $p = (p_1, \bar{p})$ where $\bar{p} = (p_2, \ldots, p_N)$. I denote the ordinary demand functions by $h^i(p_1, \bar{p}, m)$ where m is the consumer's income, the income elasticities of demand by $\eta_i$, the indirect utility function by $v(p_1, \bar{p}, m)$, and the compensated demand functions by $g^i(p_1, \bar{p}, u)$. Suppose that the price of good 1 changes from $p_1^0$ to $p_1^1$ while other prices and income stay constant at $(\bar{p}, m^0)$. Accordingly, the consumption of good 1 changes from $x_1^0 = h^1(p_1^0, \bar{p}, m^0)$ to $x_1^1 = h^1(p_1^1, \bar{p}, m^0)$. The effect of this change on the consumer's welfare can be measured in terms of the compensating or equivalent variations; for convenience, I focus on the former defined by

(1) $$C = \int_{p_1^0}^{p_1^1} g^1(p_1, \bar{p}, u^0)\, dp_1$$

where $u^0 = v(p_1^0, \bar{p}, m^0)$. The Marshallian consumer's surplus measure is

$$A = \int_{p_1^0}^{p_1^1} h^1(p_1, \bar{p}, m^0)\, dp_1.$$

Let $c = C/m^0$ and $a = A/m^0$. In his 1976 paper Willig proves that, if $\eta_1$ is constant over the relevant range,

(2a), (2b) $$c = \begin{cases} [1 + (1 - \eta_1)\, a]^{\frac{1}{1-\eta_1}} - 1, & \text{if } \eta_1 \neq 1 \\ e^a - 1, & \text{if } \eta_1 = 1. \end{cases}$$

One might wish to apply this result to the ordinary demand function

(3) $$h^1(p_1, \bar{p}, m) = \alpha p_1^{-\beta} m^{\eta} \quad \alpha, \beta, \eta > 0$$

where $\alpha = \alpha(\bar{p})$, say. Suppose, alternatively, that the ordinary demand function took the form

(4) $$h^1(p_1, \bar{p}, m) = \alpha - \beta p_1 + \gamma m \quad \beta, \gamma > 0.$$

In that case one could apply Theorem 3, Corollary 2, of Willig's 1973 paper to obtain

(5) $$C = \frac{e^{\gamma(p_1^1 - p_1^0)}}{\gamma}\left(x_1^0 - \frac{\beta}{\gamma}\right) - \frac{1}{\gamma}\left(x_1^1 - \frac{\beta}{\gamma}\right).$$

As another alternative, suppose that the demand function took the semilog form

(6) $$h^1(p_1, \bar{p}, m) = \alpha e^{-\beta p_1 + \gamma m} \qquad \alpha, \beta, \gamma > 0.$$

By imitating Willig's approach and solving the relevant ordinary differential equation for the income compensation function, one obtains

(7) $$C = -\frac{1}{\gamma}\ln\left[\frac{\gamma}{\beta}(x_1^1 - x_1^0) + 1\right] = -\frac{1}{\gamma}\ln[1 - \gamma A].$$

Now for the anomaly. Although it has not been mentioned up to now, there is an upper bound on the magnitude of C which holds regardless of the form of the demand function. This bound is

(8) $$C \leq x_1^0(p_1^1 - p_1^0).$$

The quantity on the right-hand side of (8) is obviously the maximum compensation which would be required for a price increase since, with this additional income, the consumer could afford to purchase his original bundle of goods and thus enjoy the same level of welfare as before the price change. However, none of the three formulas for C given above--(2), (5), or (7)--necessarily satisfy this bound. In each case one can concoct numerical examples where the bound is violated. For the demand function (3), assume that $\alpha = m = 1$, $\beta = 1.5$, $p_1^0 = 0.25$, and $p_1^1 = 0.5$; for values of $\eta \geq 0.9$, $C > 2$, which is the value of the bound in this case. Similarly, for the demand function (4), assume that $\alpha = \beta = \gamma = m = 1$ and $p_1^0 = 0.5$; for all $p_1^1 > p_1^0$, the bound is exceeded. Finally, for the demand function (6), assume that $\alpha = \gamma = m = 1$, $\beta = 0.5$, $p_1^0 = 1$, and $p_1^1 = 1.5$; then $C = 1.307$ while the value of the bound is 0.824. Thus, for each of these common demand functions, one can produce examples where Willig's formulas yield absurd results.

## II. The Explanation

The key to Willig's methodology is the system of partial differential equations

$$\text{(9)} \qquad \frac{\partial \mu(p|p^0, m^0)}{\partial p_i} = h^i\,[p, \mu(p|p^0, m^0)] \qquad i = 1, \ldots, N$$

with the boundary condition

$$\mu(p^0|p^0, m^0) = m^0$$

where $\mu(p|p^0, m^0)$ is the income compensation function, and $C = \mu(p^1) - \mu(p^0)$. Let $\sigma = [\sigma_{ij}]$ be the Slutsky-Hicks substitution matrix where $\sigma_{ij} = h_j^i + x_j\, h_m^i$.[1] Hurwicz and Uzawa prove (their Lemma 1) that, if the functions $h^i(\cdot)$ are single valued, possess a differential, and satisfy a smoothness condition, and if

$$\text{(10)} \qquad \sigma_{ij} = \sigma_{ji} \quad \text{all } i, j,$$

then the system (9) is uniquely integrable. Hurwicz [p. 177] refers to this as "mathematical integrability," which he distinguishes from "economic integrability," by which he means that the utility function implied by the income compensation function is quasi-concave and has convex indifference surfaces. Hurwicz and Uzawa show (their Lemma 8 and Theorem 4) that if, in addition to the above conditions, the demand functions satisfy a budget equation

$$\Sigma p_i \, h^i \, (p, m) = m$$

and the substitution matrix is negative semidefinite, which implies that

(11) $$\sigma_{ii} = h^i_i + x_i \, h^i_m \leq 0 \quad i = 1, \ldots, N,$$

then a single-valued, monotone, and quasi-concave utility function can be constructed from the income compensation function.

In the present context, where only one price changes and (9) becomes an ordinary differential equation, the symmetry condition (10) is not an issue. However, the negative semidefiniteness condition (11) is still relevant. If it does not hold over the relevant range, the underlying utility function is not quasi-concave, and, since $\sigma_{ii} = g^i_i$, application of the integral mean value theorem to (1) shows that the bound in (8) cannot be satisfied. If (11) is violated over only part of the relevant range, the bound in (8) may or may not be satisfied. But, irrespective of whether (8) happens to hold, I would regard the estimate of C as invalid because indifference surfaces are not convex over the whole range. Thus, the key test is satisfaction of (11) rather than (8). For the ordinary demand function (3), (11) is satisfied if $(p_1 \, x_1/m^0) \leq \beta/\eta$ over the relevant range; for the demand functions (4) and (6), the condition is that $x_1 \leq \beta/\gamma$. All of the numerical examples given above violate these conditions--the first two over part of the range and the third over the entire range.

Footnotes

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[1]Subscripts denote partial derivatives; thus, $h^i_j = \partial h^i/\partial p_j$ and $h^i_m = \partial h^i/\partial m$.