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Supergrassmannian and large N limit of quantum field theory with bosons and fermions

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Abstract

We study a large N_c limit of a two-dimensional Yang-Mills theory coupled to bosons and fermions in the fundamental representation. Extending an approach due to Rajeev we show that the limiting theory can be described as a classical Hamiltonian system whose phase space is an infinite-dimensional supergrassmannian. The linear approximation to the equations of motion and the constraint yields the 't Hooft equations for the mesonic spectrum. Two other approximation schemes to the exact equations are discussed.

1 Introduction

To gain a better understanding of gauge theories, two dimensional models are often used as a testing ground. In a by now classic paper, 't Hooft has shown that the large- N_c limit allows us to obtain an equation describing the meson spectrum of two dimensional QCD [1]. The same model was analyzed using different approaches [2, 3, 4, 5] and they confirmed the results obtained by 't Hooft.

In this article we study the large- N_c limit of certain two dimensional theories following a general approach developed by S. Rajeev [6, 7] (see also [8] and [9] for similar approaches). In the large N_c limit of various quantum field theories (e.g., Quantum Chromodynamics

or QCD) the quantum fluctuations become small and the theories are well described by a classical limit. This classical limit however is different from the conventional one in that many of the essential non-perturbative features of the quantum theory survive the large N_c limit [10, 1, 11]. In the formulation of [7] the classical theory corresponding to large N_c limit of 2D QCD is described by a Hamiltonian system defined on an infinite dimensional Grassmannian. The points of this infinite dimensional manifold can be identified with subspaces in infinite-dimensional Hilbert space (see the main text for precise definitions). The Grassmannian is a topologically nontrivial manifold whose connected components are labeled by an integer which can be identified with the baryon number. The 't Hooft equation describing the meson mass spectrum can be obtained in the linear approximation to the equations of motion on the Grassmannian [7]. In addition to meson masses, this approach also allows to estimate the baryon mass via a variational ansatz [12, 6]. The overall scheme resembles the Skyrme model of baryons in four-dimensional QCD. However unlike the Skyrme model the Grassmannian system of [7] can be derived as a large N_c limit of an underlying gauge theory. The Grassmannian is a homogeneous manifold. It is equipped with an action of an infinite-dimensional group (which is unitary for the fermionic matter and pseudounitary in the case of bosonic matter). This fact is very important for the structure of the phase space. In particular, it can be used for quantization of the classical system which would allow one to get a handle on $1/N_c$ corrections (including nonperturbative ones). We believe that besides the possibility of describing baryons, not captured by the original 't Hooft approach [10], the present approach can be made mathematically more precise. We remark that when the matter fields are in the adjoint representation, the mathematical techniques required are also very elegant and interesting involving the Cuntz algebra in various forms. For this approach, we refer the reader to the papers of Halpern and Schwartz [13] and Rajeev and Lee [14].

The 2D QCD interacting with bosons in the fundamental representation was also worked out following 't Hooft, partly because bosonic theory resembles the four dimensional QCD in certain respects more than the fermionic one [15, 16, 17]. The approach of [7] was extended to the bosonic case in [18] (see also [19] for a similar approach to the problem).

In this paper we study the case when both bosonic and fermionic matter are present. One motivation for this is the fact that a dimensional reduction of four-dimensional QCD produces two dimensional fundamental fermions and bosons in the adjoint representation coupled to the fermions via gauge fields. We do not expect that the bosons in the fundamental representation capture completely the adjoint case, but it can be used again as a testing ground. We also explore a more general case that includes the Yukawa type interaction between bosons and fermions.

The model of fundamental bosons and fermions interacting via $SU(N_c)$ gauge field was studied, following the same ideas in the original paper of 't Hooft, by Aoki [20, 21]. The more general models in the large- N_c limit are presented in a paper by Cavicchi, where he uses a bilocal field approach in the path integral picture [22]. Some of the models discussed in [22] are more general containing more complicated interactions, some of which in fact require a coupling constant renormalization.

In [20, 21, 22] it is shown that there are 't Hooft like spectral equations for various types of mesons. In our case we have boson-boson, fermion-fermion, and boson-fermion type mesons, and they all satisfy essentially the same equation. In each case the meson spectrum is discrete

and these mesons are all stable in the large- N_c approximation. One cannot say much about the baryons using these methods.

In the present work we generalize the approach of [7] to QCD for the bosonic and fermionic matter fields coupled via gauge fields. We will see that the phase space of the theory corresponds to a certain superserversion of the infinite-dimensional Grassmanian. Although the original system does not have any supersymmetry the main objects describing the large N_c limit, such as the phase space, group action, symplectic form, can be described in supergeometric terms. (A similar phenomenon was observed in another two-dimensional model in [23], and indeed this is a general feature of bosons and fermions coupled via gauge fields). We obtain the equations describing the meson spectrum of the model within the linear approximation. These equations agree with the ones found by Aoki [20, 21]. The theory we will present is actually nonlinear and can accommodate solitonic solutions which should describe baryons. We identify the operator which gives us the baryon number. We also propose some approximations to the spectral equations going beyond the linear approximation and discuss some consequences.

The layout of the paper is as follows. In Section 2 we reformulate the model in terms of color invariant bilinears. We further derive the Poisson brackets and the constraints imposed on the bilinear variables in large N_c limit. In Section 3 we describe this Hamiltonian system in more precise terms using the language of supergeometry. The linear approximation to the equations of motion giving the meson mass spectrum is discussed in section 4. In Section 5 we propose two approximation schemes that incorporate some nonlinear corrections and give a qualitative discussion of their influence on the spectrum.

2 The algebra of color invariant operators

We start by writing the action functionals of the two theories that we are interested in. Both theories have a gauge field that can be completely eliminated in favor of static 2D Coulomb potential. We will use the light cone coordinates $x^+ = \frac{1}{\sqrt{2}}(t + x)$, $x^- = \frac{1}{\sqrt{2}}(t - x)$ and choose the $A_+ = 0$ gauge. We first look at the gauge-coupled complex bosons with a quartic self-interaction term and Dirac fermions both in the fundamental representation of $SU(N_c)$:

$$\begin{aligned}
S = & \int dx^+ dx^- \left[-\frac{1}{2} \text{Tr} F_{+-} F^{+-} + i\sqrt{2} \psi_L^{*\alpha} (\partial_- + igA_-)^\beta \psi_{L\beta} + i\sqrt{2} \psi_R^{*\alpha} \partial_+ \psi_{R\alpha} \right. \\
& - m_F (\psi_L^{*\alpha} \psi_{R\alpha} + \psi_R^{*\alpha} \psi_{L\alpha}) - 2\phi^{*\alpha} \partial_- \partial_+ \phi_\alpha + ig (\partial_+ \phi^{*\alpha} A_-^\beta \phi_\beta - \phi^{*\alpha} A_-^\beta \partial_+ \phi_\beta) \\
& \left. - m_B^2 \phi^{*\alpha} \phi_\alpha - \frac{\lambda^2}{4} \phi^{*\alpha} \phi_\alpha \phi^{*\beta} \phi_\beta \right] \tag{1}
\end{aligned}$$

In the other model we will look at parity broken and a Yukawa type interaction is added between fermions and bosons

$$\begin{aligned}
S_Y = & \int dx^+ dx^- \left[-\frac{1}{2} \text{Tr} F_{+-} F^{+-} + i\sqrt{2} \psi_L^{*\alpha} (\partial_- + igA_-)^\beta \psi_{L\beta} + i\sqrt{2} \psi_R^{*\alpha} \partial_+ \psi_{R\alpha} \right. \\
& \left. - 2\phi^{*\alpha} \partial_- \partial_+ \phi_\alpha + ig (\partial_+ \phi^{*\alpha} A_-^\beta \phi_\beta - \phi^{*\alpha} A_-^\beta \partial_+ \phi_\beta) - m_B^2 \phi^{*\alpha} \phi_\alpha \right]
\end{aligned}$$

$$-\frac{\lambda^2}{4}\phi^{*\alpha}\phi_\alpha\phi^{*\beta}\phi_\beta + \mu(\psi_R^*\psi_{L\alpha}\phi^{*\alpha} + \psi_L^{*\alpha}\psi_R\phi_\alpha)]. \quad (2)$$

In both cases we normalize the Lie algebra generators T^a as $\text{Tr}T^aT^b = \delta^{ab}$, and we choose them to be Hermitian. This second model is anomalous because it is not a chiral gauge theory. There exist some ideas in the literature to treat an anomalous two dimensional model [24], but we will not follow this path. Instead we will take the above model at the classical level and eliminate the gauge fields which are not dynamical, and subsequently quantize the effective theory. One can check that the resulting system has a global $SU(N_c)$ symmetry and relativistic invariance. We regard this as a toy model which is *inspired from gauge theory*.

We can further use the Gauss constraint to eliminate the gauge field A_- and the fermionic equations of motion to eliminate the right moving fermion ψ_R ($\psi_{R\alpha}$). We will do these reductions in the quantized model for the first case, and classically for the second one. The resulting action is first order in the ‘‘time direction’’ x^- so we can pass to Hamiltonian formalism in a straightforward way.

The Fourier mode expansions read,

$$\phi_\alpha(x^+) = \int a_\alpha(p)e^{-ipx^+} \frac{dp}{2\pi(2|p|)^{1/2}}, \quad \psi_{L\alpha}(x^+) = \int \chi_\alpha(p)e^{-ipx^+} \frac{dp}{2\pi 2^{1/4}},$$

(to simplify the notation instead of p_+ we write p). The normalization factors are chosen to give the correct classical limits. The commutation/anticommutation relations for the fields in the light cone gauge take the form [6],

$$[\chi_\alpha(p), \chi^{\dagger\beta}(q)]_+ = \delta_\alpha^\beta 2\pi\delta(p-q), \quad [a_\alpha(p), a^{\dagger\beta}(q)] = \text{sgn}(p)\delta_\alpha^\beta 2\pi\delta(p-q). \quad (3)$$

We define $\delta[p-q] = 2\pi\delta(p-q)$, and use $[dp] = \frac{dp}{2\pi}$ to keep track of factors of 2π .

One defines a Fock vacuum state $|0\rangle$ by the conditions,

$$a_\alpha(p)|0\rangle = \chi_\alpha(p)|0\rangle = 0 \text{ for } p > 0 \quad a^{\dagger\alpha}(p)|0\rangle = \chi^{\dagger\alpha}(p)|0\rangle = 0 \text{ for } p < 0. \quad (4)$$

The corresponding normal orderings are defined as

$$:\chi^{\dagger\alpha}(p)\chi_\beta(q): := \begin{cases} -\chi_\beta(q)\chi^{\dagger\alpha}(p) & \text{if } p < 0, q < 0, \\ \chi^{\dagger\alpha}(p)\chi_\beta(q) & \text{otherwise,} \end{cases} \quad (5)$$

$$:a^{\dagger\alpha}(p)a_\beta(q): := \begin{cases} a_\beta(q)a^{\dagger\alpha}(p) & \text{if } p < 0, q < 0, \\ a^{\dagger\alpha}(p)a_\beta(q) & \text{otherwise.} \end{cases} \quad (6)$$

(Later on we also use the extension of normal ordering to product of four operators, and it is the standard one). For our purposes it is most convenient to remember the normal orderings of bilinears in the following form:

$$\begin{aligned} :\chi^{\alpha\dagger}(p)\chi_\beta(q): &= \chi^{\alpha\dagger}(p)\chi_\beta(q) - \frac{\delta_\beta^\alpha}{2}[1 - \text{sgn}(p)]\delta[p-q] \\ :a^{\alpha\dagger}(p)a_\beta(q): &= a^{\alpha\dagger}(p)a_\beta(q) - \frac{\delta_\beta^\alpha}{2}[1 - \text{sgn}(p)]\delta[p-q]. \end{aligned}$$

Written as quantum operators, we have in the first model,

$$\psi_{R\alpha} = \frac{m_F}{\sqrt{2i\partial_+}} \psi_{L\alpha} \quad (7)$$

and its hermitian conjugate, and

$$\psi_R = -\frac{\mu}{\sqrt{2i\partial_+}} \phi^{*\alpha} \psi_{L\alpha} \quad (8)$$

and its hermitian conjugate for the second model. In the first case, A_- is given in terms of the other fields as,

$$A_-^a = -\frac{g}{\partial_+^2} : (\sqrt{2}\psi_L^{\dagger\alpha}(T^a)_\alpha^\beta \psi_{L\beta} + i[\phi^{\dagger\alpha}(T^a)_\alpha^\beta \partial_+ \phi_\beta - \partial_+ \phi^{\dagger\alpha}(T^a)_\alpha^\beta \phi_\beta]) : \quad (9)$$

In the second model we are using the same equation to eliminate A_- at the classical level (which means without the normal ordering).

By eliminating the redundant degrees of freedom we can express the action in terms of the bilinears of the fields $\psi_{L\alpha}$ and ϕ_α only. We introduce,

$$\begin{aligned} \hat{M}(p, q) &= \frac{2}{N_c} : \chi^{\dagger\alpha}(p) \chi_\alpha(q) : \\ \hat{N}(p, q) &= \frac{2}{N_c} : a^{\dagger\alpha}(p) a_\alpha(q) : \end{aligned} \quad (10)$$

and their odd counterparts,

$$\hat{Q}(p, q) = \frac{2}{N_c} \chi^{\dagger\alpha}(p) a_\alpha(q), \quad \hat{Q}(r, s) = \frac{2}{N_c} a^{\dagger\alpha}(r) \chi_\alpha(s) \quad (11)$$

Once the redundancies are removed the resulting action is already first order in the “time” variable hence we can read off the Hamiltonian, and the resulting commutation relations are consistent with the ones obtained from the conventional canonical quantization. The reduction is straightforward in principle but requires a long and careful computation. Since the details are explained in Rajeev’s lecture notes [6] we only give the result:

$$H = H_0 + H_I, \quad (12)$$

$$H_0 = \frac{1}{4} M_B^2 \int \frac{[dp]}{|p|} N(p, p) + \frac{1}{4} M_F^2 \int \frac{[dp]}{p} M(p, p), \quad (13)$$

where for the first model we use,

$$M_F^2 = m_F^2 - \frac{g^2}{\pi}, \quad M_B^2 = m_{RB}^2 - \frac{g^2}{\pi}. \quad (14)$$

We employ a logarithmic renormalization on the bare mass of the bosonic field [19], and denote the renormalized mass as m_{RB} . For the second model, since we reduce it at the classical level there are no corrections coming to the boson mass term,

$$M_F^2 = 0, \quad M_B^2 = \frac{m_B^2}{4}. \quad (15)$$

The interaction parts are given by

$$H_I = \int [dpdqdsdt] G_1(p, q; s, t) M(p, q) M(s, t) + \int [dpdqdsdt] G_2(p, q; s, t) N(p, q) N(s, t) \\ + \int [dpdqdsdt] G_3(p, q; s, t) Q(p, q) \bar{Q}(s, t),$$

where both for the first and second models,

$$G_1(p, q; s, t) = -\frac{g^2}{16} \left(\frac{1}{(p-t)^2} + \frac{1}{(q-s)^2} \right) \delta[p+s-t-q] \quad (16)$$

$$G_2(p, q; s, t) = \frac{g^2}{64} \left(\frac{1}{(p-t)^2} + \frac{1}{(q-s)^2} \right) \delta[p+s-t-q] \frac{qt+ps+st+pq}{\sqrt{|pqst|}} + \frac{\lambda^2}{64} \frac{\delta[p+s-t-q]}{\sqrt{|pqst|}}. \quad (17)$$

In the first model we use,

$$G_3(p, q; s, t) = \frac{g^2}{8} \frac{q+s}{(q-s)^2} \frac{\delta[p+s-t-q]}{\sqrt{|qs|}}, \quad (18)$$

and for the second model we only have an additional term,

$$G_3(p, q; s, t) = \frac{\mu^2}{16} \frac{1}{(p-q)} \frac{1}{\sqrt{|qs|}} \delta[p-t-q+s] + \frac{g^2}{8} \frac{q+s}{(q-s)^2} \frac{\delta[p+s-t-q]}{\sqrt{|qs|}}. \quad (19)$$

Above we rescaled our coupling constants by a factor of N_c and keep the same symbols for the couplings (so $g^2 N_c \mapsto g^2$, $\mu^2 N_c \mapsto \mu^2$ and $\lambda^2 N_c \mapsto \lambda^2$) to simplify notation. For the precise meaning of these singular integral kernels we refer to the lecture notes of Rajeev[6]: we should interpret them as Hadamard principal value. We will continue to write the ordinary integrals but keep in mind that the integrals are evaluated with this prescription.

The theory we obtained still possesses a global $SU(N_c)$ invariance. The corresponding generator of symmetry is

$$\hat{Q}_\beta^\alpha = \int [dp] \left(: \chi^{\dagger\alpha}(p) \chi_\beta(p) : - \frac{1}{N_c} \delta_\beta^\alpha : \chi^{\dagger\gamma}(p) \chi_\gamma(p) : \right) + \\ + \int [dp] \text{sgn}(p) \left(: a^{\dagger\alpha}(p) a_\beta(p) : - \frac{1}{N_c} \delta_\beta^\alpha : a^{\dagger\gamma}(p) a_\gamma(p) : \right). \quad (20)$$

It is known (at least for the purely spinor and purely scalar QCD₂) that in the light-like axial gauge only the color singlet sector of the model can be quantized in a way that preserves Lorentz invariance ([5, 4]). In this paper we will therefore consider only the restrictions of our models to this sector. In general for a gauge theory it is expected that in the large N_c limit any gauge invariant correlator factorizes, i.e. $\langle AB \rangle = \langle A \rangle \langle B \rangle + O(1/N_c)$. So when the two dimensional theory restricted to the color invariant subspace in the large N_c limit any color invariant correlator should be expressible in terms of correlators of color invariant bilinear operators, \hat{M} , \hat{N} and $\hat{Q}, \hat{\bar{Q}}$ given in (10) and (11). We compute the (anti)commutation relations between these bilinears:

$$\begin{aligned}
[\hat{M}(p, q), \hat{M}(r, s)] &= \frac{2}{N_c} [\hat{M}(p, s)\delta[q - r] - \hat{M}(r, q)\delta[p - s] \\
&\quad - \delta[p - s]\delta[q - r](\text{sgn}(p) - \text{sgn}(q))] \\
[\hat{N}(p, q), \hat{N}(r, s)] &= \frac{2}{N_c} [\hat{N}(p, s)\text{sgn}(q)\delta[q - r] - \hat{N}(r, q)\text{sgn}(p)\delta[p - s] \\
&\quad + \delta[q - r]\delta[p - s](\text{sgn}(p) - \text{sgn}(q))] \\
[\hat{Q}(p, q), \hat{Q}(r, s)]_+ &= \frac{2}{N_c} [\hat{M}(p, s)\text{sgn}(q)\delta[q - r] + \hat{N}(r, q)\delta[p - s] \\
&\quad + \delta[p - s]\delta[q - r](1 - \text{sgn}(p)\text{sgn}(q))] \\
[\hat{M}(p, q), \hat{Q}(r, s)] &= \frac{2}{N_c} \delta[q - r]\hat{Q}(p, s) \\
[\hat{N}(p, q), \hat{Q}(r, s)] &= -\frac{2}{N_c} \delta[p - s]\text{sgn}(p)\hat{Q}(r, q) \\
[\hat{M}(p, q), \hat{\hat{Q}}(r, s)] &= -\frac{2}{N_c} \delta[p - s]\hat{\hat{Q}}(r, q) \\
[\hat{N}(p, q), \hat{\hat{Q}}(r, s)] &= \frac{2}{N_c} \delta[q - r]\text{sgn}(q)\hat{\hat{Q}}(p, s)
\end{aligned} \tag{21}$$

All the other (anti)commutators vanish. The first two relations were used before in [7] and [18] respectively. These (anti)commutation relations define an infinite dimensional Lie superalgebra. Its even part is isomorphic to a direct sum of central extensions of infinite-dimensional unitary and pseudo unitary groups each one generated by operators $\hat{M}(p, q)$ and $\hat{N}(p, q)$ respectively (see [18] for details). We will talk more about this Lie superalgebra and the corresponding supergroup in the next section. As the right hand sides of (21) all contain a factor of $1/N_c$ in the large N_c limit all of the bilinears commute (or anticommute respectively) and can be thought of as coordinates on a classical phase space. We denote the classical variables corresponding to \hat{M} , \hat{N} , \hat{Q} , $\hat{\hat{Q}}$ by the same letters with hats removed. This classical phase space is an infinite dimensional supermanifold endowed with a super Poisson structure inherited from the (anti)commutation relations (21). The corresponding Poisson superbrackets are obtained from the (anti)commutators in (21) by substituting $-i$ instead of $1/N_c$ factors (note that this brings an extra factor of 2). There is no simple way to decide which multiple of $1/N_c$ should be the quantum parameter. If one attempts a geometric quantization of this model, the symplectic form should be an integer multiple of the Chern character of the line bundle, the symplectic form we have in the next section is in fact the basic two form. The other possibility is to write the symplectic form in the action and use single valuedness of the path integral as is done in [7]. (We note in passing that there is a factor of 2 missing in the reference [18], due to an error in the conventions, but we scale the Hamiltonian with the same parameter so the large- N_c results are the same. The geometric quantization parameter instead should have been $1/N_c$).

However the super-Poisson structure corresponding to (21) only gives a local structure of the classical phase space of the theory. In addition to that there are some global constraints on the classical variables assigned to the color invariant bilinears. The constraints emerge in

the large N_c limit as consequences of the color invariance condition $\hat{Q}_\beta^\alpha = 0$.

To write down these constraints it is convenient to introduce the following operator product convention

$$(AB)(p, q) = \int [dr] A(p, r)B(r, q)$$

where A, B stand for any of the above (classical) bilinears. We also introduce operators M and ϵ as the operators with the kernels $\delta[p - q]$ and $-\text{sgn}(p)\delta[p - q]$, respectively. In this notation the constraints read as follows

$$\begin{aligned} (M + \epsilon)^2 + Q\epsilon Q^\dagger &= 1 \\ \epsilon Q^\dagger M + \epsilon Q^\dagger \epsilon + \epsilon N \epsilon Q^\dagger + Q^\dagger &= 0 \\ MQ + \epsilon Q + Q\epsilon N + Q\epsilon &= 0 \\ (\epsilon N + \epsilon)^2 + \epsilon Q^\dagger Q &= 1. \end{aligned} \quad (22)$$

For brevity we will present here a derivation only of the first constraint in (22). The derivations of all the others are very similar. We will restrict ourselves to the zero subspace of the operator \hat{Q}_β^α and we define the number operators

$$\hat{F} \equiv \frac{1}{N_c} \int [dp] : \chi^{\dagger\alpha}(p) \chi_\alpha(p) :, \quad \hat{B} \equiv \frac{1}{N_c} \int [dp] \text{sgn}(p) : a^{\dagger\alpha}(p) a_\alpha(p) : \quad . \quad (23)$$

(Note that these operators are scaled by a factor of $\frac{1}{N_c}$ so taking the limit $N_c \rightarrow \infty$ gives us zero when these operators are acting on mesonic states. They are nonzero when we look at the baryonic states as we will see shortly.)

By writing out the product of operators at hand in terms of the variables a, a^\dagger, χ and χ^\dagger and moving the suitable combinations to the right one can prove the identity (that holds on the whole Fock space)

$$((\hat{M} + \epsilon)^2 + \hat{Q}\epsilon\hat{Q}^\dagger)(r, s) = \delta[r - s] + \frac{2}{N_c^2} \chi^{\dagger\alpha}(r) \chi_\beta(s) (\hat{Q}_\alpha^\beta + \delta_\alpha^\beta (\hat{F} + \hat{B}))$$

On the subspace $\hat{Q}_\alpha^\beta = 0$, the operator $\hat{B} + \hat{F}$ will be equal to the baryon number operator $\hat{\mathbf{B}}$. Thus when we restrict ourselves to a fixed baryon number B , we get,

$$((\hat{M} + \epsilon)^2 + \hat{Q}\epsilon\hat{Q}^\dagger)(r, s) = \delta[r - s] + (\hat{M} + 1 - \epsilon)(r, s) \frac{B}{N_c},$$

this in the large N_c limit produces the first constraint in (22).

When we look at a possible exotic baryon state:

$$\int \epsilon_{\alpha_1 \dots \alpha_s \alpha_{s+1} \dots \alpha_{N_c}} Z(p_1, \dots, p_s; p_{s+1}, \dots, p_{N_c}) \chi^{\dagger\alpha_1}(p_1) \dots \chi^{\dagger\alpha_s}(p_s) a^{\dagger\alpha_{s+1}}(p_{s+1}) \dots a^{\dagger\alpha_{N_c}}(p_{N_c}) |0 \rangle, \quad (24)$$

where p_1, p_2, \dots, p_{N_c} are all positive, and $Z(p_1, \dots, p_s, p_{s+1}, \dots, p_{N_c})$ is symmetric in p_1, \dots, p_s and antisymmetric in p_{s+1}, \dots, p_{N_c} . The operator $\hat{\mathbf{B}}$ gives 1 acting on such a state. On mesonic states this operator has vanishing matrix elements in the large- N_c limit. One can prove more generally therefore that this operator is the baryon number operator. If we act by this

operator on a product of such exotic baryons and finite number of mesons in the large- N_c limit we get the number of baryons, B . In this discussion we see the possibility of having exotic baryons, and we will come back to the geometric meaning of this in the next section (and show that it is indeed an integer in our model). We will also show that just as in the purely bosonic and purely fermionic cases the constraints (22) have an elegant geometric interpretation in terms of infinite dimensional disc and Grassmannian.

3 Phase Space of the Theory: Super-Grassmannian

In this section we present the geometry of the phase space without going into the mathematical intricacies. We believe the most proper treatment requires an infinite dimensional extension of Berezin's \mathbf{Z} -graded version of super-geometry. We do not give such a complete presentation, and develop a more formal approach (in many cases we provide paranthetical remarks on the general case). We plan to provide a more detailed discussion in a later publication when we discuss geometric quantization of this system. The proper treatment of second quantization with bosons and fermions, which fits to our point of view, can be found in [25, 26], and also in [27]. In order to understand the geometry of the phase space, we define an operator in super-matrix form;

$$\Phi = \begin{pmatrix} \epsilon N + \epsilon & \epsilon Q^\dagger \\ Q & M + \epsilon \end{pmatrix}, \quad (25)$$

where $\Phi : \mathcal{H}^e | \mathcal{H}^o \rightarrow \mathcal{H}^e | \mathcal{H}^o$. We think of the direct sum $\mathcal{H}^e \oplus \mathcal{H}^o$ of one-particle Hilbert spaces of bosons and fermions respectively as even and odd graded and the notation $\mathcal{H}^e | \mathcal{H}^o$ is used to emphasize this grading. We use $\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ in both of these spaces. This matrix realization corresponds to the decomposition of the Hilbert spaces into positive and negative energy subspaces as $\mathcal{H}_+^e \oplus \mathcal{H}_-^e$ for bosons and $\mathcal{H}_+^o \oplus \mathcal{H}_-^o$ for fermions.

The constraints and the conditions that we found in the previous section on the basic variables of our theory in terms of Φ become

$$\Phi^2 = 1 \quad E \Phi^\dagger E = \Phi, \quad (26)$$

where $E = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$.

If we introduce a super-group of operators acting on $\mathcal{H}^e | \mathcal{H}^o$, obeying the relations

$$g E g^\dagger = E, \quad g^\dagger E g = E, \quad (27)$$

we see that the action of this group on the variable Φ , $(g, \Phi) \mapsto g \Phi g^{-1}$ preserves the above stated conditions on Φ . The orbit of $\hat{\epsilon} = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$ under the action of this super-unitary group, can be parametrized by Φ .

The condition that the bilinears, originally defined on the Fock space of the quantum theory, create finite norm vectors implies that the off-diagonal components of M and N are Hilbert-Schmidt operators (see [7, 6, 18] and for the ideals in the non-super case see [28, 29]).

A similar computation shows that the off-diagonal components of the super-operators Q and Q^\dagger also satisfy these conditions. These finite norm conditions in two dimensions can be written in an economical way as the Hilbert-Schmidt condition on the super-matrix Φ . To state these convergence conditions more properly we should decompose $\mathcal{H}^e|\mathcal{H}^o$ into negative and positive energy subspaces and think of Φ as an operator acting from $\mathcal{H}_+^e|\mathcal{H}_+^o \oplus \mathcal{H}_-^e|\mathcal{H}_-^o$ to the same space. Thus we have the convergence conditions,

$$[\hat{\epsilon}, \Phi] \in \mathcal{I}_2 \quad \hat{\epsilon} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (28)$$

where we write $\hat{\epsilon}$'s matrix realization with respect to this positive-negative energy decomposition. Here for the upper off-diagonal component the ideal of Hilbert-Schmidt operators \mathcal{I}_2 refers to the set of operators $B : \mathcal{H}_-^e|\mathcal{H}_-^o \rightarrow \mathcal{H}_+^e|\mathcal{H}_+^o$, such that $\text{Tr} B^\dagger B$ is convergent. (This definition in the \mathbf{Z}_2 graded case is the usual one, since the operators have ordinary numbers as their matrix entries, in a fully \mathbf{Z} graded case these questions are delicate and we have to give a precise meaning to the Hilbert-Schmidt condition. For this work we ignore this question but see [30] for further comments on it). The lower off-diagonal block will be a Hilbert-Schmidt operator $C : \mathcal{H}_+^e|\mathcal{H}_+^o \rightarrow \mathcal{H}_-^e|\mathcal{H}_-^o$ as well. (Since the even and odd Hilbert spaces are isomorphic, it is convenient to drop the superscript when there is no confusion). The above considerations suggest that we should use as our symmetry group the restricted super-unitary group:

$$U_1(\mathcal{H}_-, \mathcal{H}_+|\mathcal{H}) = \left\{ g \mid g E g^\dagger = E, \quad g^\dagger E g = E \quad [\hat{\epsilon}, g] \in \mathcal{I}_2 \right\}, \quad (29)$$

where \mathcal{I}_2 denotes the ideal of Hilbert-Schmidt operators as in the above positive-negative energy decomposition used for Φ 's convergence conditions.

We look at the orbit of $\hat{\epsilon} = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$, this time we write it with respect to the original decomposition $\mathcal{H}^e|\mathcal{H}^o$, under the restricted super-unitary group. We notice that this orbit is in fact a homogeneous super-symplectic manifold:

$$SGr_1 = \frac{U_1(\mathcal{H}_-, \mathcal{H}_+|\mathcal{H})}{U(\mathcal{H}_-|\mathcal{H}_-) \times U(\mathcal{H}_+|\mathcal{H}_+)}. \quad (30)$$

The stability subgroup has a natural embedding into the full group. This physically means that we allow mixing of the positive energy states of bosons and fermions as well as the negative ones.

Notice that a tangent vector V_u at any point on this super-Grassmannian is given by its effect on Φ , $V_u(\Phi) = i[u, \Phi]_s$, where we use the super-Lie bracket which is defined by

$$\begin{aligned} \left[\begin{pmatrix} a_1 & \beta_1 \\ \gamma_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & \beta_2 \\ \gamma_2 & d_2 \end{pmatrix} \right]_s &= \left[\begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ 0 & d_2 \end{pmatrix} \right] + \left[\begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix}, \begin{pmatrix} 0 & \beta_2 \\ \gamma_2 & 0 \end{pmatrix} \right] \\ &+ \left[\begin{pmatrix} 0 & \beta_1 \\ \gamma_1 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ 0 & d_2 \end{pmatrix} \right] + \left[\begin{pmatrix} 0 & \beta_1 \\ \gamma_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta_2 \\ \gamma_2 & 0 \end{pmatrix} \right]_+ \end{aligned}$$

for a decomposition of u into $\begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}$, with respect to $\mathcal{H}^e|\mathcal{H}^o$. In general the super-Lie algebra element u will depend on the position Φ .

Our homogeneous manifold carries a natural two-form, this turns it into a phase space. We formally define a two-form;

$$\Omega = \frac{i}{4} \text{Str} \Phi d\Phi \wedge d\Phi. \quad (31)$$

One can give the symplectic form explicitly via its action on vector fields, and this defines the above two form:

$$i_{V_u} i_{V_v} \Omega = \frac{i}{8} \text{Str} \Phi [[u, \Phi]_s, [v, \Phi]_s]_s. \quad (32)$$

Using exactly the same methods as in [7, 18], we can show that it is closed and non-degenerate.

In fact the above form is also a homogeneous two-form invariant under the group action, as can be verified in a simple manner. We note that the super-Poisson brackets which we introduced in the first section as a result of the large- N_c limit, are precisely the ones given by this symplectic form. Therefore we may introduce a classical dynamical system defined on this super-Grassmannian with this symplectic form which gives us the same set of super-Poisson brackets. This shows that the large- N_c limit of our theory has an independent geometric formulation: the phase space is an infinite dimensional homogeneous manifold with a natural symplectic structure on it.

The group action is generated by moment maps $F_u = -\frac{i}{2} \text{Str}_\epsilon u (\Phi - \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix})$, where we use the even-odd decomposition to write all the operators and conditional trace to be defined below. They satisfy the following super-Poisson realization of the super-unitary group:

$$\{F_u, F_v\} = F_{-i[u, v]_s} - \frac{i}{2} \text{Str}_\epsilon \left[\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}, u \right]_s v, \quad (33)$$

where $[\cdot, \cdot]_s$ again denotes the super-commutator(super-Lie bracket). To see this, one way is to compute both sides, the other is to use general principles and evaluate both sides at $\hat{\epsilon} = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$ (written with respect to the even-odd decomposition). The moment function on the right vanishes there and the central term is constant on the phase space, this gives us,

$$\begin{aligned} \Sigma_s(u, v) &= -\frac{i}{8} \text{Str} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \left[\left[\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}, u \right], \left[\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}, v \right] \right]_s \\ &= -\frac{i}{2} \left(\text{Tr}_\epsilon [\epsilon, a(u)] a(v) - \text{Tr}_\epsilon [\epsilon, \beta(u)] \gamma(v) + [\epsilon, \beta(v)] \gamma(u) - \text{Tr}_\epsilon [\epsilon, d(u)] d(v) \right) \\ &= -\frac{i}{2} \text{Str}_\epsilon \left[\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}, u \right]_s v. \end{aligned}$$

The conditional super-trace is defined by $\text{Str}_\epsilon \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{Tr}_\epsilon A - \text{Tr}_\epsilon D$, and $\text{Tr}_\epsilon A = \frac{1}{2} \text{Tr}(A + \epsilon A \epsilon)$. Notice that the convergence conditions on Φ guarantees that the conditional trace exists (in fact a better convergence is possible, see below). This can be seen most easily by using, $\Phi - \hat{\epsilon} = g \hat{\epsilon} g^{-1} - \hat{\epsilon} = -[\hat{\epsilon}, g] g^{-1}$. It is more natural to compute this with respect to the positive-negative energy decomposition, (we use the subscripts \pm to denote

the super-matrix elements acting between various subspaces),

$$[\hat{\epsilon}, g]g^{-1} = \begin{pmatrix} 0 & g_{+-} \\ g_{-+} & 0 \end{pmatrix} \begin{pmatrix} (g^{-1})_{++} & (g^{-1})_{+-} \\ (g^{-1})_{-+} & (g^{-1})_{--} \end{pmatrix} = \begin{pmatrix} \mathcal{I}_1 & \mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{I}_1 \end{pmatrix}, \quad (34)$$

where \mathcal{I}_1 denotes the ideal of trace class operators and \mathcal{I}_2 is the ideal of Hilbert-Schmidt operators. We used the fact that the off diagonal elements are Hilbert-Schmidt and the others are bounded, and the analog of the well-known conditions $\mathcal{I}_2\mathcal{I}_2 \in \mathcal{I}_1$ in the super-case. If we multiply this with an element of the Lie algebra we see that the conditional traces exist. It suggests a slightly better way to write the moment maps, $F_u = -\frac{i}{2}\text{Str}_{\hat{\epsilon}}u(\Phi - \hat{\epsilon})$, which shows that the conditional convergence could be actually defined with respect to the positive-negative energy decomposition.

The above discussion further implies that $\text{Str}_{\hat{\epsilon}}(\Phi - \hat{\epsilon})$ is convergent. This expression is in fact conserved by the equations of motion of a quadratic Hamiltonian. We may understand the meaning of this number, if we think of its action on color invariant states before we take the large- N_c limit. We can prove that in this case this operator gives us twice the baryon number. Recall that the baryons in this theory can be exotic, that is we may have color singlet combinations of the form,

$$\int \epsilon_{\alpha_1\alpha_2\dots\alpha_{N_c}} Z(q_1, \dots, q_s; q_{s+1}, \dots, q_{N_c}) \chi^{\dagger\alpha_1}(q_1) \dots \chi^{\dagger\alpha_k}(q_k) a^{\dagger\alpha_{k+1}}(q_{k+1}) \dots a^{\dagger\alpha_{N_c}}(q_{N_c}) |0\rangle, \quad (35)$$

where all the momenta are positive, and Z is symmetric in p_1, \dots, p_s and antisymmetric in p_{s+1}, \dots, p_{N_c} , as we have seen in the previous section. The negative momenta case,

$$\int \epsilon^{\alpha_1\alpha_2\dots\alpha_{N_c}} \bar{Z}(q_1, \dots, q_k; q_{k+1}, \dots, q_{N_c}) \chi_{\alpha_1}(q_1) \dots \chi_{\alpha_k}(q_k) a_{\alpha_{k+1}}(q_{k+1}) \dots a_{\alpha_{N_c}}(q_{N_c}) |0\rangle, \quad (36)$$

where all the momenta negative, and similar symmetry properties for \bar{Z} corresponds to an anti-baryon and $\hat{\mathbf{B}}$ acting on such a state gives -1 . So we identify the large- N_c limit of the baryon number operator as,

$$\mathbf{B} = -\frac{1}{2}\text{Str}_{\hat{\epsilon}}(\Phi - \hat{\epsilon}). \quad (37)$$

We show in the appendix A that the baryon number operator is indeed an integer using the geometry of our phase space. We will leave the discussion of the geometry of the phase space at this point and return to the dynamics.

4 The Linear Approximation

In this section we discuss the linear approximation to the above theory. At present we do not have a simple physical interpretation of the full equations of motion. In principle they are straightforward to compute using the Hamiltonians we have and the defining Poisson brackets. Our phase space is defined by the Poisson brackets we get from the super-commutators for this system in the large- N_c limit and the constraints which define the global nature of the phase space. We note that part of the interactions of this theory are in these constraints.

We give the super Poisson brackets, that defines the kinematics of our theory:

$$\begin{aligned}
\{M(p, q), M(r, s)\} &= -2i[M(p, s)\delta[q - r] - M(r, q)\delta[p - s] \\
&\quad - \delta[p - s]\delta[q - r](\text{sgn}(p) - \text{sgn}(q))] \\
\{N(p, q), N(r, s)\} &= -2i[N(p, s)\text{sgn}(q)\delta[q - r] - N(r, q)\text{sgn}(p)\delta[p - s] \\
&\quad + \delta[q - r]\delta[p - s](\text{sgn}(p) - \text{sgn}(q))] \\
\{Q(p, q), \bar{Q}(r, s)\}_+ &= -2i[M(p, s)\text{sgn}(q)\delta[q - r] + N(r, q)\delta[p - s] \\
&\quad + \delta[p - s]\delta[q - r](1 - \text{sgn}(p)\text{sgn}(q))] \\
\{M(p, q), Q(r, s)\} &= -2i\delta[q - r]Q(p, s) \\
\{N(p, q), Q(r, s)\} &= 2i\delta[p - s]\text{sgn}(p)Q(r, q) \\
\{M(p, q), \bar{Q}(r, s)\} &= 2i\delta[p - s]\bar{Q}(r, q) \\
\{N(p, q), \bar{Q}(r, s)\} &= -2i\delta[q - r]\text{sgn}(q)\bar{Q}(p, s).
\end{aligned} \tag{38}$$

We have the constraints for the basic variables given in equation (21).

If we are given a Hamiltonian we can compute the equations of motion using the above super-Poisson brackets. This is a complete description of a classical system. Of course since the theory is infinite dimensional there are various delicate questions, such as, is it possible to define trajectories for a any given initial data, what is the dense domain on which the Hamiltonian is defined, etc. We will not attempt to answer these questions here. In the limit $N_c \rightarrow \infty$, we can rewrite the Hamiltonians of interest in terms of these classical variables, the answers are given in Section 2.

$$H = H_0 + H_I, \tag{39}$$

here $H_0 = \int [dp] h_F(p) M(p, p) + \int [dp] h_B(p) N(p, p)$, and we take $h_F(p) = \frac{M_F^2}{4} \frac{1}{p}$ and $h_B(p) = \frac{M_B^2}{4} \frac{1}{|p|}$ with the interpretation that these mass terms are given by the previous expressions. H_I , the interaction part, is given generally by

$$\begin{aligned}
H_I &= \int [dpdqdsdt] G_1(p, q; s, t) M(p, q) M(s, t) + \int [dpdqdsdt] G_2(p, q; s, t) N(p, q) N(s, t) \\
&\quad + \int [dpdqdsdt] G_3(p, q; s, t) Q(p, q) \bar{Q}(s, t).
\end{aligned}$$

In the next section, it will be useful to keep this general form of the Hamiltonian, but their explicit forms are given in the discussion of the models in Section 2 in (16),(17), (19), we will use them directly (in the calculations we keep μ^2 always, for the first model we must set $\mu^2 = 0$).

It is straightforward to find the resulting non-linear equations of motion simply by computing

$$\frac{\partial O(x^-)}{\partial x^-} = \{O(x^-), H\}_s, \tag{40}$$

for any observable O of the theory (we allow for an odd Hamiltonian in the above form, but in our cases, the Hamiltonians are even). However, it is simpler to first look at the linearization where everything decouples (equations for M and N were analyzed in this approximation in

previous publications [7, 18, 6]). We will see that we get the same equations for M, N as in [7, 18] in our linearized theory.

Let us ignore all the quadratic terms in the equations of motion and all the quadratic terms in the constraints. First let us write down the resulting constraints in this approximation:

$$\begin{aligned}\epsilon M + M\epsilon &= 0 \\ \epsilon N\epsilon + N &= 0 \\ \epsilon Q^\dagger\epsilon + Q^\dagger &= 0 \\ \epsilon Q + Q\epsilon &= 0.\end{aligned}$$

We note that the last two equations are identical and the constraints on these variables decouple hence they can be solved independently. The solutions are,

$$M(u, v) = 0, \quad N(u, v) = 0, \quad Q(u, v) = 0 \quad \text{for } uv > 0. \quad (41)$$

The other components, that is the ones which have opposite sign momenta, are not restricted. The equations of motion one gets for the variable M in the linear approximation is (for $u > 0, v < 0$):

$$\frac{\partial M(u, v; x^-)}{\partial x^-} = i\frac{M_F^2}{2}\left(\frac{1}{u} - \frac{1}{v}\right)M(u, v) - \frac{ig^2}{2\pi} \int_{-\frac{u-v}{2}}^{\frac{u-v}{2}} dp \frac{M(p - \frac{u-v}{2}, p + \frac{u-v}{2})}{(p - \frac{u+v}{2})^2}, \quad (42)$$

which is identical with the one in [7, 6]. If we make the ansatz (see [7]) $M(u, v) = \xi_M(x)e^{iP_-x^-}$, where $x = \frac{u}{u-v}$, and define the invariant mass $\Lambda_M^2 = 2P_-(u-v)$, (recall that $(u-v) = P_+$), we get,

$$\Lambda_M^2 \xi_M(x) = M_F^2 \left(\frac{1}{x} + \frac{1}{1-x}\right) \xi_M(x) - \frac{g^2}{\pi} \int_0^1 \frac{dy}{(y-x)^2} \xi_M(y). \quad (43)$$

This is the well-known 't Hooft equation [10]. Similarly for $N(u, v)$ using the same type of ansatz, $N(u, v) = \xi_N(x)e^{iP_-x^-}$, and the invariant mass $\Lambda_N^2 = 2P_-(u-v)$, we get,

$$\begin{aligned}\Lambda_N^2 \xi_N(x) &= M_B^2 \left(\frac{1}{x} + \frac{1}{1-x}\right) \xi_N(x) - \frac{g^2}{4\pi} \int_0^1 \frac{dy}{(y-x)^2} \frac{(x+y)(2-x-y)}{\sqrt{x(1-x)y(1-y)}} \xi_N(y) \\ &+ \frac{\lambda^2}{8\pi} \int_0^1 \frac{dy}{\sqrt{x(1-x)y(1-y)}} \xi_N(y).\end{aligned} \quad (44)$$

This is the bosonic analog of the 't Hooft equation [15, 16, 17, 19]. The equations for Q, \bar{Q} are given in the next section in a slightly more general context, so we will not repeat it here. If we again use an ansatz for the $Q(u, v)$ given by $Q(u, v; x^-) = c_Q(x)e^{iP_-x^-}$ and the same interpretation of the symbols, and an invariant mass, $\Lambda_Q^2 = 2P_-(u-v)$, we get

$$\begin{aligned}\Lambda_Q^2 c_Q(x) &= \left(\frac{M_F^2}{x} + \frac{M_B^2}{1-x}\right) c_Q(x) - \frac{g^2}{2\pi} \int_0^1 \frac{dy}{(y-x)^2} \frac{2-x-y}{\sqrt{(1-x)(1-y)}} c_Q(y) \\ &+ \frac{\mu^2}{4\pi} \int_0^1 \frac{dy}{\sqrt{(1-y)(1-x)}} c_Q(y).\end{aligned} \quad (45)$$

Setting $\mu^2 = 0$ we recover the equations found by Aoki[20, 21]. Similarly for the complex conjugate variable \bar{Q} , we get,

$$\begin{aligned} \Lambda_{\bar{Q}}^2 c_{\bar{Q}}(x) &= \left(\frac{M_B^2}{x} + \frac{M_F^2}{1-x} \right) c_{\bar{Q}}(x) - \frac{g^2}{2\pi} \int_0^1 \frac{dy}{(y-x)^2} \frac{x+y}{\sqrt{xy}} c_{\bar{Q}}(y) \\ &+ \frac{\mu^2}{4\pi} \int_0^1 \frac{dy}{\sqrt{yx}} c_{\bar{Q}}(y). \end{aligned} \quad (46)$$

We remark that the equation for $c_{\bar{Q}}$ can be obtained from the equation for c_Q if we make the change of variable $x \mapsto 1-x$, and interchange M_B and M_F and use the principal value prescription (this ultimately comes from the charge conjugation invariance).

The properties of these equation have been discussed in the literature. The two kernels above differ from the ones given in [10, 15, 20, 21] by a relatively compact perturbation so they behave in the same way. What is remarkable about them is that they only allow for discrete spectrum, they do not have scattering states. The corresponding eigenvectors form a basis.

We make a short digression and note an interesting limit: in the second model let us set $g^2 = 0$. There is no coupling to gauge fields thus there is no reason to assume that the observables of the theory are color invariant. We can study this case along the same lines assuming *it is a sort of mean-field approximation only* and we search for bound states of a fermion and a boson in the linear approximation. The Hamiltonian is quite simple,

$$H = \frac{1}{4} m_B^2 \int \frac{[dp]}{|p|} N(p, p) + \frac{\mu^2}{16} \int \frac{[dpdqdsdt]}{\sqrt{|qs|}} \frac{\delta[p-q+s-t]}{t-s} Q(p, q) \bar{Q}(s, t). \quad (47)$$

The linearization is the same as before, for the bound state solution we obtain equation (44) with $g^2 = 0$. Unfortunately this equation will not have a solution for the bound state energy. We need the opposite sign in the Hamiltonian for the coupling of $Q\bar{Q}$. It is an amusing exercise to check that the seemingly different interaction $i\mu(\phi^{*\alpha}\psi_R^*\psi_{L\alpha} - \psi_L^{*\alpha}\psi_R\phi_\alpha)$ produces the same Hamiltonian, so we will still not find a bound state for fermion-boson pair. We hope to come back to some of these issues in a separate work.

5 Beyond the linear approximation

In this section we will discuss the equations of motion of our theory in a semi-linear approximation. The exact equations of motion can of course be written, but it is hard to grasp their meaning at this point for the most general case. It will be interesting to look at other approximations to see what new information they contain.

Our first semi-linear approach is this: We will keep everything linear in the variables M and N and terms second order in Q and Q^\dagger only. We will drop terms of the form MQ , NQ and M^2, N^2 . Even though we have not found a justification for why this should be a good approximation, we expect that it may give us a better feeling for the system. We first show that this is a consistent approximation, that is, if the equations of motion are also kept to the same approximation, the truncated constraints are preserved.

The constraints in this new approximation become

$$\begin{aligned} M\epsilon + \epsilon M + Q\epsilon Q^\dagger &= 0 \\ Q\epsilon + \epsilon Q &= 0 \\ \epsilon N\epsilon + N + \epsilon Q^\dagger Q &= 0. \end{aligned}$$

We should also obtain semi-linearized equations of motion for these variables. We now show that the linearized constraints are left invariant by the semi-linearized equations of motion. We will present the proof for a general quadratic Hamiltonian. The solution of the constraint on Q is simple: $Q(u, v) = 0$ when u and v have the same sign. We notice that the first constraint does not impose anything on $M(u, v)$ for $u > 0, v < 0$ or $u < 0, v > 0$, and the constraint is consistent since for this case we have $\int Q(u, q)[- \text{sgn}(q)]\bar{Q}(q, v)[dq] = 0$. Thus we should look at $u > 0, v > 0$ or both negative case for M in the constraint:

$$-2M(u, v) + \int Q(u, q)[- \text{sgn}(q)]\bar{Q}(q, v)[dq] = -2M(u, v) + \int_{-\infty}^0 [dq]Q(u, q)\bar{Q}(q, v) = 0. \quad (48)$$

Let us check that it is preserved by the linearized equations of motion.

$$\begin{aligned} \frac{\partial M(u, v)}{\partial x^-} &= \{M(u, v), H\} \\ &= 2i(h_F(u) - h_F(v))M(u, v) + \int [dpdrdsdt]G_1(p, r, s, t)\{M(u, v), M(p, r)M(s, t)\} \\ &+ \int [dpdrdrdt]G_3(p, r, s, t)\{M(u, v), Q(p, r)\bar{Q}(s, t)\} \\ &= 2i(h_F(u) - h_F(v))M(u, v) \\ &+ 4i \int [dpdr]G_1(p, r; v, u)M(p, r)[\text{sgn}(u) - \text{sgn}(v)] \\ &- 2i \int [drdsdt]G_3(v, r, s, t)Q(u, r)\bar{Q}(s, t) + 2i \int [dpdrds]G_3(p, r, s, u)Q(p, r)\bar{Q}(s, v). \end{aligned}$$

The equations of motion for Q in this approximation becomes,

$$\frac{\partial Q(u, q)}{\partial x^-} = 2ih_F(u)Q(u, q) - 2i\text{sgn}(q)h_B(q)Q(u, q) + 2i \int G_3(p, r, q, u)Q(p, r)[1 - \text{sgn}(u)\text{sgn}(q)]. \quad (49)$$

Similarly for Q^\dagger ,

$$\frac{\partial \bar{Q}(q, v)}{\partial x^-} = -2ih_F(v)\bar{Q}(q, v) + 2i\text{sgn}(q)h_B(q)\bar{Q}(q, v) - 2i \int G_3(v, q; s, t)\bar{Q}(s, t)[1 - \text{sgn}(v)\text{sgn}(q)]. \quad (50)$$

Combining these equations and using the constraint again we obtain,

$$2\frac{\partial M(u, v)}{\partial x^-} - \int_{-\infty}^0 \left[\frac{\partial Q(u, q)}{\partial x^-}\bar{Q}(q, v) + Q(u, q)\frac{\partial \bar{Q}(q, v)}{\partial x^-} \right][dq] = 0. \quad (51)$$

Using the same equations, we can check that the condition $Q(u, v) = 0$ when u, v have the same sign, is also preserved by the equations of motion, hence also for $\bar{Q}(u, v)$.

We write down the equation of motion for $N(u, v)$;

$$\frac{\partial N(u, v)}{\partial x^-} = 2i[h_B(u)\text{sgn}(u) - h_B(v)\text{sgn}(v)]N(u, v)$$

$$\begin{aligned}
& - 4i \int [dpdq] G_2(p, q; v, u) [\text{sgn}(u) - \text{sgn}(v)] N(p, q) \\
& + 2i \int [dpdqdt] [G_3(p, u; q, t) Q(p, v) \bar{Q}(q, t) \text{sgn}(u) - G_3(p, q; v, t) Q(p, q) \bar{Q}(u, t) \text{sgn}(v)].
\end{aligned}$$

Using the above equations of motion we can check that the truncated constraint on N is preserved under time evolution:

$$(1 + \text{sgn}(u)\text{sgn}(v)) \frac{\partial N(u, v)}{\partial x^-} - \text{sgn}(u) \int_{-\infty}^0 [dq] \left[\frac{\partial \bar{Q}(u, q)}{\partial x^-} Q(q, v) + \bar{Q}(u, q) \frac{\partial Q(q, v)}{\partial x^-} \right] = 0. \quad (52)$$

Next we discuss the equations of motion for the unconstraint components. From the above equations we see that the equations for Q and Q^\dagger are independent of M and N , therefore they can be solved independently. Furthermore, the solution acts as a source term for the M and N equations. Let us write down the equation of motion for Q in the case of $u > 0$ and $v < 0$ for our model:

$$\begin{aligned}
\frac{\partial Q(u, v)}{\partial x^-} & = 2ih_F(u)Q(u, v) + 2ih_B(v)Q(u, v) + i \frac{\mu^2}{8(u-v)} \int_{-\frac{u-v}{2}}^{\frac{u-v}{2}} [dq] \frac{Q(q - \frac{u-v}{2}, q + \frac{u-v}{2})}{\sqrt{|q + \frac{u-v}{2}||v|}} \\
& - i \frac{g^2}{2} \int_{-\frac{u-v}{2}}^{\frac{u-v}{2}} [dp] \frac{p - \frac{u}{2} + \frac{3v}{2} Q(p + \frac{u-v}{2}, p - \frac{u-v}{2})}{(p - \frac{u+v}{2})^2 \sqrt{|p - \frac{u-v}{2}||v|}}.
\end{aligned}$$

A similar equation for $\bar{Q}(u, v)$ holds (which can also be found by complex conjugation of the $Q(v, u)$).

Notice that the equations of motion for $M(u, v)$ (for $u > 0, v < 0$) becomes,

$$\begin{aligned}
\frac{\partial M(u, v)}{\partial x^-} & = 2i(h_F(u) - h_F(v))M(u, v) - ig^2 \int [ds] \frac{M(s + (u-v)/2, s - (u-v)/2)}{[s - (u+v)/2]^2} \\
& + \frac{ig^2}{4} \int [dqds] \frac{q+s}{(q-s)^2} \frac{1}{\sqrt{|qs|}} [Q(q+u-s, q)\bar{Q}(s, v) - Q(u, q)\bar{Q}(s, v+s-q)] \\
& - \frac{i\mu^2}{8} \int \frac{[dqds]}{\sqrt{|qs|}} \left[\frac{Q(u, q)\bar{Q}(s, s-v+q)}{v-q} - \frac{Q(u+q-s, q)\bar{Q}(s, v)}{u-s} \right].
\end{aligned}$$

We note that in the above integral over M we should separate the constrained variables from the unconstrained ones. At the same time we do some shift of integration variables, and obtain,

$$\begin{aligned}
\frac{\partial M(u, v; x^-)}{\partial x^-} & = i \frac{M_F^2}{2} \left[\frac{1}{u} - \frac{1}{v} \right] M(u, v) - ig^2 \int_{-\frac{u-v}{2}}^{\frac{u-v}{2}} [ds] \frac{M(s + (u-v)/2, s - (u-v)/2)}{[s - (u+v)/2]^2} \\
& + f_+(u, v; x^-) + f_-(u, v; x^-) + g_+(u, v; x^-) + g_-(u, v; x^-) \\
& + Y_+(u, v; x^-) + Y_-(u, v; x^-),
\end{aligned}$$

where all the forcing terms are functions of Q, \bar{Q} and their explicit expressions are given in the appendix B. Note that once we know the solution for Q and Q^\dagger , f 's, g 's and Y 's just become time dependent sources for the M and N equations. Therefore we can think of this

as a forced linear equation. Let us also write down the resulting equation of motion for $N(u, v)$, for $u > 0, v < 0$.

$$\begin{aligned}
\frac{\partial N(u, v; x^-)}{\partial x^-} &= i \frac{M_B^2}{2} \left[\frac{1}{u} - \frac{1}{v} \right] N(u, v) \\
&- i \int_{-\frac{u-v}{2}}^{\frac{u-v}{2}} [ds] \frac{N(s + (u-v)/2, s - (u-v)/2)}{\sqrt{|s - \frac{u-v}{2}| |s + \frac{u-v}{2}| |uv|}} \left[\frac{g^2 (s + \frac{3u}{2} - \frac{v}{2})(s + \frac{3v}{2} - \frac{u}{2})}{4 [s - \frac{(u+v)}{2}]^2} - \frac{\lambda^2}{8} \right] \\
&+ \tilde{f}_+(u, v; x^-) + \tilde{f}_-(u, v; x^-) + \tilde{g}_+(u, v; x^-) + \tilde{g}_-(u, v; x^-) \\
&+ \tilde{Y}_+(u, v; x^-) + \tilde{Y}_-(u, v; x^-),
\end{aligned}$$

where we have again the forcing terms determined by the variables Q, \bar{Q} (the explicit formulae are given in the appendix B).

We can give a rough argument how these equations behave. If we look at the formulae given in the appendix B, we notice that the singular looking kernels are actually harmless, since the integration regions are outside of the singular points. This means that once we have the solutions for the Q, \bar{Q} variables we can treat them as small perturbations to the equations. If we could find the Green's function for these linear operator equations given the sources we should be able to solve them. Let us assume that we have the linear equation $i \frac{\partial M}{\partial x^-} = \mathbf{L}M + S(x^-)$, where \mathbf{L} is a linear Hermitian operator. If we know the eigenvectors $\mathbf{L}M_\lambda = \lambda M_\lambda$ then we can use a general ansatz as $M = \sum_\lambda a_\lambda(x^-) M_\lambda(x^-)$, and get $a_\lambda(x^-) = -i \int_0^{x^-} dx^- \langle M_\lambda(x^-), S(x^-) \rangle$. (In our case the leading singular integral operators are hermitian and have only discrete spectrum, hence the expansion makes sense). This is the full solution and represents transition probabilities among the stationary states of the operator \mathbf{L} . Perhaps it is better to think of the ordinary forced harmonic oscillator problem. When we have a time dependent forcing, this causes transitions between the stationary levels of the oscillator. So, without actually solving the above equation we see that the forcing terms will cause transition between the stationary levels. That physically means that the energy levels of the mesons will have a broadening due to possible transitions.

There is another possible approximation, for which we drop all MM, NN , and $Q\bar{Q}$ terms and allow for the cross terms MQ, NQ etc, and neglect any higher orders. In some sense this is the complementary approximation to the previous one. This implies that we should write the constraint as;

$$\begin{aligned}
M\epsilon + \epsilon M &= 0 \\
MQ + Q\epsilon N + \epsilon Q + Q\epsilon &= 0 \\
\epsilon N\epsilon + N &= 0
\end{aligned}$$

The first and the last one are familiar conditions. The middle one has the following solution (in the given approximation): For $u, v > 0$ (recall that $\epsilon(p) = -\text{sgn}(p)$),

$$-2Q(u, v) + \int_{-\infty}^0 [dq] M(u, q) Q(q, v) + \int_{-\infty}^0 [dq] Q(u, q) (-\text{sgn}(q)) N(q, v) = 0. \quad (53)$$

For $u > 0, v < 0$ we have,

$$\int_{-\infty}^0 [dq] M(u, q) Q(q, v) - \int_0^{\infty} [dq] Q(u, q) N(q, v) = 0. \quad (54)$$

We satisfy the lower equation by noting that the same momenta case for Q is given by the first constraint and the integrands then become of lower order in this case. The consistency of these approximations could be checked. In fact if we write down the time derivative of the above constraint,

$$\begin{aligned} -2 \frac{\partial Q(u, v)}{\partial x^-} &+ \int_{-\infty}^0 [dq] \left(\frac{\partial M(u, q)}{\partial x^-} Q(q, v) + M(u, q) \frac{\partial Q(q, v)}{\partial x^-} \right) \\ &+ \int_{-\infty}^0 [dq] \left(\frac{\partial Q(u, q)}{\partial x^-} N(q, v) + Q(u, q) \frac{\partial N(q, v)}{\partial x^-} \right) = 0 \end{aligned}$$

To see this we use,

$$\begin{aligned} \frac{\partial Q(u, v; x^-)}{\partial x^-} &= 4i \int [dpdsdt] G_1(p, u; s, t) Q(p, v) M(s, t) \\ &+ 4i \int [dqdsdt] G_2(v, q; s, t) Q(u, q) N(s, t) \text{sgn}(v) \\ &+ 2i(h_F(u) - \text{sgn}(v)h_B(v))Q(u, v) + 2i \int [dpdqdt] G_3(p, q; v, t) Q(p, q) M(u, t) \text{sgn}(v) \\ &+ 2i \int [dpdqds] G_3(p, q; s, u) Q(p, q) N(s, v) \\ &+ 2i \int [dpdq] G_3(p, q; v, u) Q(p, q) [1 - \text{sgn}(u)\text{sgn}(v)]. \end{aligned}$$

For the first time derivative in the constraint we insert this expression, for the time derivatives of Q inside the integral we only retain the linear terms in Q , since other combinations are of lower order by assumption. We should also use the equations of motion of M and N for the opposite momenta case and only within the linear approximation as is given in the previous semi-linear case, we do not repeat them, higher order terms get multiplied by Q and become small. Then we see that the constraint is preserved within the given approximation.

This time we have decoupled linear equations for M and N for the opposite momenta case, since we ignore $Q\bar{Q}$ type terms, and in principle they can be solved independently. When we look at the equations for Q , we should again be careful. The opposite momenta case are to be treated as independent dynamical variables: if we use the constraint equation, we may express the same sign momenta in terms of the solutions of M and N and the opposite momenta terms of Q . When we look at the opposite sign momenta equation for Q we may separate the same sign momenta contributions in the integral operators. But these same momenta terms in the integral equation become of higher order, since all these terms are multiplied by other variables, and the central part vanishes in this case, hence they can be dropped. Let us denote the resulting integral equation which only acts on the opposite momenta terms by \mathbf{K} , this is the expression we have found before, and write the remaining parts as an abstract integral operator $\mathbf{F}(x^-)$. Notice that it has dependence on x^- via the solutions of M and N . The time dependence of M and N are rather simple for this case, since we have singular integral equations with discrete spectra. We can in principle substitute the solutions we picked into this equation. Hence we have an integral equation

$$\frac{\partial Q(u, v; x^-)}{\partial x^-} = [\mathbf{K}Q](u, v; x^-) + [\mathbf{F}(x^-)Q](u, v; x^-). \quad (55)$$

It is most natural to think of the last term as a time dependent perturbation. We can write this perturbation term $\mathbf{F}(x^-)$:

$$\begin{aligned}
[\mathbf{F}(x^-)Q](u, v; x^-) &= -\frac{ig^2}{2} \left[\int_0^u [ds] \int_0^{u-s} [dp] + \int_{-\infty}^0 [ds] \int_{u-s}^{\infty} [dp] \right] \frac{M(s, s+p-u; x^-)}{(s-u)^2} Q(p, v) \\
&+ i \left[\int_{-\infty}^v [dq] \int_0^{v-q} [dt] + \int_v^0 [dq] \int_{v-q}^0 [dt] \right] \left[\frac{g^2}{8} \frac{(t+q)^2 + (q-v)^2 - 2q^2}{(v-t)^2} - \frac{\lambda^2}{16} \right] \\
&\times \frac{N(t+q-v, t; x^-)}{\sqrt{|vq(t+q-v)|}} Q(u, q) \\
&- i \left[\int_v^0 [dt] \int_0^{t-v} [dp] + \int_{-\infty}^v [dt] \int_{t-v}^0 [dp] \right] \left(\frac{\mu^2}{8(t-v)} + \frac{g^2}{4} \frac{p-t+2v}{(p-t)^2} \right) \\
&\times \frac{M(u, t; x^-)}{\sqrt{|(p-t+v)v|}} Q(p, p-t+v) \\
&+ i \left[\int_0^u [ds] \int_0^{u-s} [dp] + \int_u^{\infty} [ds] \int_{u-s}^0 [dp] \right] \left(\frac{\mu^2}{8(u-s)} + \frac{g^2}{4} \frac{p+2s-u}{(p-u)^2} \right) \\
&\times \frac{N(s, v; x^-)}{\sqrt{|(p+s-u)s|}} Q(p, p+s-u)
\end{aligned}$$

The method of solving such equations is known in principle. We can treat the last term as a truly time dependent perturbation, but this time it involves the unknown itself and thus cannot be solved in closed form. However, we can solve it perturbatively. The kernels again do not become singular within the given ranges of the integrals except at the boundaries. The singularities are not as severe and we expect that the perturbations are small, so that one can obtain a reasonable answer from this approach. We will not go into further details, but the basic result is again the possibility of transitions between the different levels of the boson-fermion mesons due to the interactions.

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7 Appendix A: Baryon number

We define the Fredholm operators in a \mathbf{Z}_2 graded context as in the ordinary case (there is an extension to the \mathbf{Z} grading which should fit to our model better: The definition of the Fredholm operator shows that the body is an ordinary Fredholm operator and the rest is compact. So below use the body for all the formulae, and take the super-trace of $\Phi - \hat{\epsilon}$'s body part): a Fredholm operator is an invertible operator up to compact operators. This again implies the kernel and the cokernel are actually finite dimensional. Let us write down the kernel in a decomposition $V_e|V_o$, and define a super-dimension, which is the dimension of the even part of the kernel minus the dimension of the odd part. $\text{Sdim}(\text{Ker}(A)) = \dim(P_e \text{Ker}(A) P_e) - \dim(P_o \text{Ker}(A) P_o)$, where P_e and P_o denote projections onto the even and odd subspaces respectively. Then the index should be,

$$\text{SInd}(A) = \text{Sdim}(\text{Ker}(A)) - \text{Sdim}(\text{Coker}(A)). \quad (56)$$

We can extend the Calderon theorem to our case (see [31] for a good introduction and the original result): If we have an operator A which is Fredholm, and assume we have an operator B such that $(I - AB)^m$ and $(I - BA)^m$ are trace class for an integer m , then we can compute the super-Fredholm index as $\text{Str}(I - BA)^m - \text{Str}(I - AB)^m$. Let us now see that in our problem the supertrace of Φ is indeed this index. It will be more convenient to decompose our operator with respect to the positive and negative subspaces, thus we write everything with respect to $\mathcal{H}_+|\mathcal{H}_+ \oplus \mathcal{H}_-|\mathcal{H}_-$, we do not repeat odd and even superscripts, since the bar indicates this separation. In this decomposition our group conditions can be found from,

$$g^\dagger E g = E \quad g E g^\dagger = E \quad E = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}, \quad (57)$$

so E is the same as before in this matrix representation, it is interpreted differently. The orbit is with respect to this decomposition,

$$\Phi = g \hat{\epsilon} g^{-1} \quad \hat{\epsilon} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (58)$$

So if we write $g : \mathcal{H}_+|\mathcal{H}_+ \oplus \mathcal{H}_-|\mathcal{H}_- \rightarrow \mathcal{H}_+|\mathcal{H}_+ \oplus \mathcal{H}_-|\mathcal{H}_-$, explicitly, we have

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad B, C \in \mathcal{I}_2(\mathcal{H}_\mp|\mathcal{H}_\mp, \mathcal{H}_\pm|\mathcal{H}_\pm). \quad (59)$$

From the first group condition we get, $A^\dagger \epsilon A + C^\dagger C = \epsilon$ and $D^\dagger D + B^\dagger \epsilon B = 1$ and from the second one we get, $A \epsilon A^\dagger + B B^\dagger = \epsilon$ and $D D^\dagger + C \epsilon C^\dagger = 1$. Since B, C are Hilbert-Schmidt in the more generalized sense, we have A, D super Fredholm. Further we can use the above theorem to compute the index of A, D , for example

$$\text{SInd}(D) = \text{Str}(B^\dagger \epsilon B) - \text{Str}(C \epsilon C^\dagger). \quad (60)$$

Let us compute the conditional supertrace of $\Phi - \hat{\epsilon}$, (in fact we see that this is the correct way we should be computing it), first we write it explicitly with respect to the above decomposition,

$$g \hat{\epsilon} g^{-1} - \hat{\epsilon} = \begin{pmatrix} -A \epsilon A^\dagger \epsilon + B B^\dagger \epsilon + 1 & * \\ * & -C \epsilon C^\dagger + D \epsilon D^\dagger - 1 \end{pmatrix}. \quad (61)$$

If we use the above group properties, we get

$$\Phi - \hat{\epsilon} = \begin{pmatrix} 2BB^\dagger\epsilon & * \\ * & -2C\epsilon C^\dagger \end{pmatrix}. \quad (62)$$

The conditional supertrace of this operator gives us, $\text{Str}_\epsilon(\Phi - \hat{\epsilon}) = 2(\text{Str}(BB^\dagger\epsilon) - \text{Str}(C\epsilon C^\dagger))$, which is equal to $2\text{SInd}(D)$ (using $\text{Str}(BB^\dagger\epsilon) = \text{Str}(B^\dagger\epsilon B)$). Thus we prove using only the geometry of the super-Grassmanian that this is an integer.

8 Appendix B: Forcing terms

Here we present the forcing functions for the inhomogeneous equations of the previous section. The ones we got for M in the first semi-linear approximation is given by,

$$\begin{aligned} f_+(u, v; x^-) &= -i\frac{g^2}{2} \int_{\frac{u-v}{2}}^\infty [dp] \int_{-\infty}^0 [dq] \frac{Q(p - \frac{u-v}{2}, q) \bar{Q}(q, p + \frac{u-v}{2})}{[p - \frac{u+v}{2}]^2} \\ f_-(u, v; x^-) &= -i\frac{g^2}{2} \int_{-\infty}^{-\frac{u-v}{2}} [dp] \int_0^\infty [dq] \frac{Q(p - \frac{u-v}{2}, q) \bar{Q}(q, p + \frac{u-v}{2})}{[p - \frac{u+v}{2}]^2} \\ g_+(u, v; x^-) &= i\frac{g^2}{4} \left[\int_0^u [ds] \int_{-\frac{u-s}{2}}^{\frac{u-s}{2}} [dq] + \int_u^\infty [ds] \int_{-\frac{s-u}{2}}^{\frac{s-u}{2}} [dq] \right] \frac{q + \frac{3s}{2} - \frac{u}{2}}{[q - \frac{u+s}{2}]^2} \frac{Q(q + \frac{u-s}{2}, q - \frac{u-s}{2}) \bar{Q}(s, v)}{\sqrt{|q - \frac{u-s}{2}||s|}} \\ g_-(u, v; x^-) &= -i\frac{g^2}{4} \left[\int_v^0 [ds] \int_{-\frac{s-v}{2}}^{\frac{s-v}{2}} [dq] + \int_{-\infty}^v [ds] \int_{-\frac{v-s}{2}}^{\frac{v-s}{2}} [dq] \right] \frac{q + \frac{3s}{2} - \frac{v}{2}}{[q - \frac{v+s}{2}]^2} \frac{Q(u, s) \bar{Q}(q - \frac{v-s}{2}, q + \frac{v-s}{2})}{\sqrt{|q - \frac{v-s}{2}||s|}} \\ Y_-(u, v; x^-) &= +\frac{i\mu^2}{8} \left[\int_0^u [ds] \int_{-\frac{u-s}{2}}^{\frac{u-s}{2}} [dq] + \int_u^\infty [ds] \int_{-\frac{s-u}{2}}^{\frac{s-u}{2}} [dq] \right] \frac{Q(q + \frac{u-s}{2}, q - \frac{u-s}{2}) \bar{Q}(s, v)}{(u-s)\sqrt{|q - \frac{u-s}{2}||s|}} \\ Y_+(u, v; x^-) &= -\frac{i\mu^2}{8} \left[\int_v^0 [ds] \int_{-\frac{s-v}{2}}^{\frac{s-v}{2}} [dq] + \int_{-\infty}^v [ds] \int_{-\frac{v-s}{2}}^{\frac{v-s}{2}} [dq] \right] \frac{Q(u, s) \bar{Q}(q - \frac{v-s}{2}, q + \frac{v-s}{2})}{(v-s)\sqrt{|q - \frac{v-s}{2}||s|}}. \end{aligned}$$

The forcing terms for the first semi-linear approximation for the N variable,

$$\begin{aligned} \tilde{f}_-(u, v; x^-) &= -i \int_{\frac{u-v}{2}}^\infty [dp] \int_{-\infty}^0 [dq] \frac{\bar{Q}(p + \frac{u-v}{2}, q) Q(q, p - \frac{u-v}{2})}{\sqrt{|p - \frac{u-v}{2}||p + \frac{u-v}{2}||uv|}} \left[\frac{g^2}{8} \frac{(p + \frac{3u}{2} - \frac{v}{2})(p + \frac{3v}{2} - \frac{u}{2})}{[p - \frac{u+v}{2}]^2} - \frac{\lambda^2}{4} \right] \\ \tilde{f}_+(u, v; x^-) &= i \int_{-\infty}^{-\frac{u-v}{2}} [dp] \int_0^\infty [dq] \frac{\bar{Q}(p + \frac{u-v}{2}, q) Q(q, p - \frac{u-v}{2})}{\sqrt{|p - \frac{u-v}{2}||p + \frac{u-v}{2}||uv|}} \left[\frac{g^2}{8} \frac{(p + \frac{3u}{2} - \frac{v}{2})(p + \frac{3v}{2} - \frac{u}{2})}{[p - \frac{u+v}{2}]^2} - \frac{\lambda^2}{4} \right] \\ \tilde{g}_+(u, v; x^-) &= i\frac{g^2}{4} \left[\int_0^u [dp] \int_{-\frac{u-p}{2}}^{\frac{u-p}{2}} [ds] + \int_u^\infty [dp] \int_{-\frac{p-u}{2}}^{\frac{p-u}{2}} [ds] \right] \frac{s - \frac{p}{2} + \frac{3u}{2}}{[s - \frac{u+p}{2}]^2} \frac{Q(p, v) Q(s + \frac{u-p}{2}, s - \frac{u-p}{2})}{\sqrt{|s - \frac{p-u}{2}||u|}} \\ \tilde{g}_-(u, v; x^-) &= i\frac{g^2}{4} \left[\int_v^0 [dp] \int_{-\frac{p-v}{2}}^{\frac{p-v}{2}} [ds] + \int_{-\infty}^v [dp] \int_{-\frac{v-p}{2}}^{\frac{v-p}{2}} [ds] \right] \frac{q + \frac{3s}{2} - \frac{v}{2}}{[q - \frac{v+s}{2}]^2} \frac{Q(s - \frac{v-p}{2}, s + \frac{v-p}{2}) \bar{Q}(u, p)}{\sqrt{|s - \frac{p-v}{2}||v|}} \end{aligned}$$

$$\begin{aligned}\tilde{Y}_+(u, v; x^-) &= -\frac{i\mu^2}{8} \left[\int_0^u [dp] \int_{-\frac{u-p}{2}}^{\frac{u-p}{2}} [ds] + \int_u^\infty [dp] \int_{-\frac{p-u}{2}}^{\frac{p-u}{2}} [ds] \right] \frac{Q(p, v) \bar{Q}(s + \frac{u-p}{2}, s - \frac{u-p}{2})}{(p-u) \sqrt{|s + \frac{u-p}{2}| |u|}} \\ \tilde{Y}_-(u, v; x^-) &= \frac{i\mu^2}{8} \left[\int_v^0 [dp] \int_{-\frac{p-v}{2}}^{\frac{p-v}{2}} [ds] + \int_{-\infty}^v [dp] \int_{-\frac{v-p}{2}}^{\frac{v-p}{2}} [ds] \right] \frac{Q(s - \frac{v-p}{2}, s + \frac{v-p}{2}) \bar{Q}(u, p)}{(p-v) \sqrt{|s + \frac{v-p}{2}| |v|}}.\end{aligned}$$

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