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Author
Simons, Stephen

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Gossez's skew linear map and its pathological maximally monotone multifunctions

Stephen Simons *

Abstract

In this note, we give a generalization of Gossez's example of a maximally monotone multifunction such that the closure of its range is not convex, using more elementary techniques than in Gossez's original papers. We also discuss some new properties of Gossez's skew linear operator and its adjoint.

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1 Introduction

In [4] and [5], Gossez gives an example of a skew linear map $G\colon\ \ell_1 \to \ell_\infty = \ell_1^*$, and proves that there exist arbitrarily small values of $\lambda > 0$ such that $\overline{R(G + \lambda J)}$ is not convex. $\big($If E is a Banach space, $J\colon\ E \rightrightarrows E^*$ is the *duality map*, defined by $x^* \in Jx$ *exactly when* $\|x^*\| = \|x\|$ *and* $\langle x, x^*\rangle = \|x\|^2$. See eqn. (2.4).$\big)$ In Theorem 3.5, we shall prove the stronger result that $\overline{R(G + \lambda J)}$ is not convex whenever $0 < \lambda < 4$. In particular, $\overline{R(G + J)}$ is not convex. $\big(R(\cdot)$ stands for "range of".$\big)$

Gossez's analysis goes by way of the *monotone extension to the dual* introduced in [3]. This was critical to his definition of operators of *dense type*, which have been so important in the modern theory of monotone multifunctions. In addition to the use of the monotone extension to the dual, [4] and [5] use measure theory on the Stone-Čech compactification of the positive integers. In this paper, we use mainly elementary functional analysis, but we will make some comments about the measure theoretic approach in Remarks 4.4 and 5.3.

In Section 2, we define a *skew* linear operator, A, from a Banach space, E into its dual and, in Theorem 2.2, we establish an upper bound for the quadratic form $-\langle A^*x^{**}, x^{**}\rangle$ on $E^{**} \times E^{**}$ under certain circumstances. See eqn. (2.3).

*Department of Mathematics, University of California, Santa Barbara, CA 93106-3080, U.S.A. Email: stesim38@gmail.com.

In eqn.(3.1), we give the exact formula for G. Our presentation exploits the fact that G can be "factorized through c". In Lemma 3.2, we discuss a particular element $x_0^{**} \in \ell_1^{**} = \ell_\infty^*$ and give formulae for $G^* x_0^{**}$ and $\langle G^* x_0^{**}, x_0^{**} \rangle$. Lemma 3.3 appears in [5, Proposition, p. 360], but with a very different proof. Lemma 3.3 leads rapidly to our main result, Theorem 3.5.

In Section 4, we give some technical results on ℓ_1, ℓ_1^*, ℓ_1^{**} and ℓ_1^{***} and, in Theorem 4.2, define a particular element w^{***} of ℓ_1^{***} that will be used in Section 5 to obtain formulae for $G^* x^{**}$ and $\langle G^* x^{**}, x^{**} \rangle$ for *general* $x^{**} \in \ell_1^{**}$. It was proved in [4, Example, p. 89] and [1, Example 14.2.2, pp. 161–162] that, for all $x^{**} \in \ell_1^{**}$, $\langle G^* x^{**}, x^{**} \rangle \le 0$. In (5.2), we strengthen these results by showing that $\langle G^* x^{**}, x^{**} \rangle = -\langle x^{**}, w^{***} \rangle^2$.

All Banach spaces considered in this note are *real*.

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2 On skew linear operators on general Banach spaces

Definition 2.1. Let E be a nonzero Banach space and $A\colon\ E \to E^*$ be linear. We say that A is *skew* if,

$$\text{for all } w, x \in E,\ \langle w, Ax \rangle = -\langle x, Aw \rangle, \tag{2.1}$$

or, equivalently,

$$\text{for all } x \in E,\ \langle x, Ax \rangle = 0. \tag{2.2}$$

If $x \in E$, we write $\widehat{x}$ for the canonical image of x in E^{**}, that is to say $x \in E$ and $x^* \in E^* \Longrightarrow \langle x^*, \widehat{x} \rangle = \langle x, x^* \rangle$.

We recall that if X and Y are Banach spaces and $A\colon\ X \to Y$ is linear then the adjoint $A^*\colon\ Y^* \to X^*$ is defined by $\langle x, A^* y^* \rangle = \langle Ax, y^* \rangle \quad (x \in X,\ y^* \in Y^*)$.

Theorem 2.2. *Let $A\colon\ E \to E^*$ be bounded, skew and linear. Suppose that $x^{**} \in E^{**}$, $\lambda > 0$ and $A^* x^{**} \in \overline{R(A + \lambda J)}$. Then*

$$-\langle A^* x^{**}, x^{**} \rangle \le \tfrac{1}{4} \lambda \|x^{**}\|^2. \tag{2.3}$$

Proof. Let $\varepsilon > 0$. By hypothesis, there exist $x \in E$, $x^* \in E^*$ with

$$\|x^*\| = \|x\|,\ \langle x, x^* \rangle = \|x\|^2, \tag{2.4}$$

and $z^* \in E^*$ such that $\|z^*\| < \varepsilon$ and $A^* x^{**} = Ax + \lambda x^* + z^*$. Then

$$A^* x^{**} - Ax = \lambda x^* + z^*, \tag{2.5}$$

and, using (2.4),

$$-\langle x^*, x^{**}\rangle \le \|x^*\|\|x^{**}\| = \|x^{**}\|\|x\|. \tag{2.6}$$

Let $Z_\varepsilon := \|x^{**}\| + \varepsilon/\lambda$. From the definition of A^*, (2.2), (2.4)–(2.6), and the inequalities $\|z^*\| < \varepsilon$ and $-\|x\|^2 + Z_\varepsilon\|x\| \le \frac{1}{4}Z_\varepsilon^2$,

$$\begin{aligned}-\langle A^*x^{**}, x^{**}\rangle &= -\langle x, A^*x^{**} - Ax\rangle - \langle A^*x^{**} - Ax, x^{**}\rangle \\ &= -\langle x, \lambda x^* + z^*\rangle - \langle \lambda x^* + z^*, x^{**}\rangle \\ &\le -\lambda\|x\|^2 + \varepsilon\|x\| + \lambda\|x^{**}\|\|x\| + \varepsilon\|x^{**}\| \\ &= -\lambda\|x\|^2 + \lambda Z_\varepsilon\|x\| + \varepsilon\|x^{**}\| \le \tfrac{1}{4}\lambda Z_\varepsilon^2 + \varepsilon\|x^{**}\|.\end{aligned}$$

Since $Z_\varepsilon \to \|x^{**}\|$ as $\varepsilon \to 0$, (2.3) now follows by letting $\varepsilon \to 0$. □

3 Gossez's skew linear operator

Definition 3.1. In the interest of precision, we shall use three different notations for the three duality pairings that appear in the rest of this paper. Then (noting that $\ell_1^* = \ell_\infty$), the bilinear form $\langle\cdot,\cdot\rangle_0\colon\ \ell_1 \times \ell_1^* \to \mathbb{R}$ is defined in the usual way. Then $\ell_1^{**} = \ell_\infty^*$, but this space does not have a convenient sequential representation. In this connection, see Remark 4.4. Also, $\langle\cdot,\cdot\rangle_1\colon\ \ell_1^* \times \ell_1^{**} \to \mathbb{R}$ and $\langle\cdot,\cdot\rangle_2\colon\ \ell_1^{**} \times \ell_1^{***} \to \mathbb{R}$. We write $\|\cdot\|_1$ and $\|\cdot\|_\infty$ for the usual norms on ℓ_1 and ℓ_∞. Let c be the subspace of ℓ_∞ consisting of all *convergent* sequences. Finally, let $e^* := (1, 1, \dots) \in c$. In what follows, all sequences are indexed by the set $\{1, 2, 3, \dots\}$.

Define the linear operator $G\colon\ \ell_1 \to \ell_1^* = \ell_\infty$ by

$$\text{for all } x \in \ell_1,\ (Gx)_m = \textstyle\sum_{n>m} x_n - \sum_{n<m} x_n. \tag{3.1}$$

G is the "Gossez operator". It is well known that G is skew and maximally monotone. See [4, Example, p. 89]. Clearly, for all $x \in \ell_1$,

$$\lim_{m\to\infty}(Gx)_m = -\textstyle\sum_{n=1}^{\infty} x_n = -\langle x, e^*\rangle_0. \tag{3.2}$$

Lemma 3.2. *There exists $x_0^{**} \in \ell_1^{**} = \ell_\infty^*$ such that*

$$\|x_0^{**}\| = 1, \tag{3.3}$$

$$G^*x_0^{**} = -e^* \in \ell_1^* = \ell_\infty \tag{3.4}$$

and

$$\langle G^*x_0^{**}, x_0^{**}\rangle_1 = -1. \tag{3.5}$$

Proof. The map from c into $\mathbb{R}$ defined by $x^* \mapsto \lim_{m\to\infty} x_m^*$ is bounded and linear and has norm 1 on the vector subspace c of ℓ_∞. So, from the extension form of the Hahn–Banach theorem, there exists $x_0^{**} \in \ell_\infty^*$ such that (3.3) is satisfied and,

$$\text{for all } x^* \in c,\ \langle x^*, x_0^{**}\rangle_1 = \lim_{m\to\infty} x_m^*. \tag{3.6}$$

For all $x \in \ell_1$, $Gx \in c$. Thus, from (3.2),

$$\langle x, G^* x_0^{**} \rangle_0 = \langle Gx, x_0^{**} \rangle_1 = \lim_{m\to\infty} (Gx)_m = -\langle x, e^* \rangle_0. \tag{3.7}$$

This completes the proof of (3.4). From this, $\langle G^* x_0^{**}, x_0^{**} \rangle_1 = \langle -e^*, x_0^{**} \rangle_1$ and (3.5) is immediate from (3.6). $\square$

The proof of Lemma 3.3 below is based on that of [5, Proposition, p. 360]. However, instead of using measure theory on $\beta\mathbb{N}$, we use the fact that a linear subspace is closed under differences (in (3.12) and (3.13)) and sums (in (3.14)). There is another way of establishing Lemma 3.3, using *Rugged Banach spaces*. See [1, Proposition 15.3.8, p. 176].

Lemma 3.3. *Let $\lambda > 0$. Suppose that*

$$\overline{R(G + \lambda J)} \text{ is convex.} \tag{3.8}$$

Then

$$\overline{R(G + \lambda J)} = \ell_\infty. \tag{3.9}$$

Proof. Let $k \geq 1$. As observed in [5, Proposition, p. 360], if, for all $m \notin \{1, 2\}$, $|u_m^*| \leq 2\lambda k$, then

$$(-k + 2\lambda k, -k - 2\lambda k, u_3^*, u_4^*, u_5^*, \dots) \in (G + \lambda J)(ke_1 - ke_2) \in \overline{R(G + \lambda J)}. \tag{3.10}$$

In particular,

$$(-k + 2\lambda k, -k - 2\lambda k, 0, 0, 0, \dots) \in \overline{R(G + \lambda J)}. \tag{3.11}$$

As observed in [5, Proposition, p. 360], (3.8) implies that $\overline{R(G + \lambda J)}$ is a linear subspace of ℓ_∞. So, by subtracting (3.11) from (3.10),

$$(0, 0, u_3^*, u_4^*, u_5^*, \dots) \in \overline{R(G + \lambda J)}. \tag{3.12}$$

Similarly, if, for all $m \notin \{3, 4\}$, $|v_m^*| \leq 2\lambda k$, then

$$(v_1^*, v_2^*, 0, 0, v_5^*, \dots) \in \overline{R(G + \lambda J)}. \tag{3.13}$$

Taking the Minkowski sum of (3.12) and (3.13),

$$(v_1^*, v_2^*, u_3^*, u_4^*, u_5^* + v_5^*, u_6^* + v_6^*, \dots) \in \overline{R(G + \lambda J)}. \tag{3.14}$$

(3.9) now follows easily by letting $k \to \infty$. $\square$

Lemma 3.4. *Suppose that $\lambda > 0$ and $\overline{R(G + \lambda J)}$ is convex. Then $\lambda \geq 4$.*

Proof. Let x_0^{**} be as in Lemma 3.2. From Lemma 3.3, $G^* x_0^{**} \in \overline{R(G + \lambda J)}$. From Theorem 2.2, (3.5) and (3.3), $1 = -\langle G^* x_0^{**}, x_0^{**} \rangle \leq \frac{1}{4}\lambda \|x_0^{**}\|^2 = \frac{1}{4}\lambda$. This gives the desired result. $\square$

Theorem 3.5. *If $0 < \lambda < 4$ then $\overline{R(G + \lambda J)}$ is not convex. In particular, $\overline{R(G + J)}$ is not convex.*

Proof. This is immediate from Lemma 3.4. $\square$

Problem 3.6. Is $\overline{R(G + 4J)}$ convex?

4 On the dual, bidual and tridual of ℓ_1

This section is devoted to the technical results that will be needed for our discussion of G^* in Section 5. We point, in particular, to Lemma 4.1(c), in which p^* is moved from being the first variable in $\langle\cdot,\cdot\rangle_1$ to being the second variable in $\langle\cdot,\cdot\rangle_0$, *i.e.*, from being a primal variable to being a dual variable. Lemma 4.1(c) will be critical in the proof of (4.3), which will be used in Theorem 5.1.

Let c_0 be the Banach space of sequences that converge to 0. For all $m \ge 1$, let e_m^* be the element $(0,\ldots,0,1,0,0,\ldots)$ of ℓ_1^*, with the 1 in the mth place. Define the linear map $W\colon\ \ell_1^{**} \to \ell_1$ by $Wx^{**} := \big(\langle e_m^*, x^{**}\rangle_1\big)_{m\ge1}$.

Lemma 4.1. (a) *Let $x^{**} \in \ell_1^{**}$. Then $\sum_{m=1}^\infty |\langle e_m^*, x^{**}\rangle_1| \le \|x^{**}\| < \infty$.*
(b) $\|W\| = 1$ *and, for all $x \in \ell_1$, $W\widehat{x} = x$.*
(c) *Let $p^* \in c_0 \subset \ell_1^*$ and $x^{**} \in \ell_1^{**}$. Then $\langle p^*, x^{**}\rangle_1 = \langle Wx^{**}, p^*\rangle_0$.*

Proof. For all $m \ge 1$, find δ_m such that $|\delta_m| = 1$ and $\delta_m\langle e_m^*, x^{**}\rangle_1 = |\langle e_m^*, x^{**}\rangle_1|$. Let $n \ge 1$. Then

$$\begin{aligned}\textstyle\sum_{m=1}^n |\langle e_m^*, x^{**}\rangle_1| &= \textstyle\sum_{m=1}^n \delta_m\langle e_m^*, x^{**}\rangle_1 = \big\langle \sum_{m=1}^n \delta_m e_m^*, x^{**}\big\rangle_1\\ &\le \big\| \textstyle\sum_{m=1}^n \delta_m e_m^*\big\|_\infty \|x^{**}\| = \sup_{m=1}^n |\delta_m|\|x^{**}\| = \|x^{**}\|.\end{aligned}$$

(a) now follows by letting $n \to \infty$. It also follows that $\|Wx^{**}\|_1 \le \|x^{**}\|$. Since this holds for all $x^{**} \in \ell_1^{**}$, $\|W\| \le 1$. Now let $x \in \ell_1$. Then, for all $m \ge 1$, $(W\widehat{x})_m = \big\langle e_m^*, \widehat{x}\big\rangle_1 = \langle x, e_m^*\rangle_0 = x_m$, and so $W\widehat{x} = x$, as required. It follows from this that $\|W\| = 1$, which completes the proof of (b).

Let $p^* = (p_m)_{m\ge1} \in c_0$. Since $p^* = \lim_{n\to\infty} \sum_{m=1}^n p_m e_m^*$ in ℓ_∞,

$$\begin{aligned}\langle p^*, x^{**}\rangle_1 &= \textstyle\lim_{n\to\infty} \sum_{m=1}^n p_m\langle e_m^*, x^{**}\rangle_1 = \sum_{m=1}^\infty p_m\langle e_m^*, x^{**}\rangle_1\\ &= \textstyle\sum_{m=1}^\infty \langle e_m^*, x^{**}\rangle_1 p_m = \langle Wx^{**}, p^*\rangle_0.\end{aligned}$$

This completes the proof of (c). □

In what follows, we define $w^{***} := \widehat{e^*} - W^*e^* \in \ell_1^{***}$.

Theorem 4.2. *We have*

$$\|w^{***}\| = 1, \tag{4.1}$$

for all $x \in \ell_1$,

$$\langle \widehat{x}, w^{***}\rangle_2 = 0, \tag{4.2}$$

and, for all $x^* = (x_m^*)_{m\ge1} \in c$ and $x^{**} \in \ell_1^{**}$,

$$\langle x^*, x^{**}\rangle_1 = \langle Wx^{**}, x^*\rangle_0 + \langle x^{**}, w^{***}\rangle_2 \lim_{n\to\infty} x_n^*. \tag{4.3}$$

Proof. Let $n \ge 1$. Then

$$\begin{aligned}\langle e^*, x^{**}\rangle_1 - \textstyle\sum_{m=1}^n \langle e_m^*, x^{**}\rangle_1 &= \big\langle e^* - \textstyle\sum_{m=1}^n e_m^*, x^{**}\big\rangle_1\\ &\le \big\|e^* - \textstyle\sum_{m=1}^n e_m^*\big\|_\infty \|x^{**}\| = \|x^{**}\|.\end{aligned}$$

Letting $n \to \infty$, $\langle e^*, x^{**}\rangle_1 - \sum_{m=1}^{\infty}\langle e^*_m, x^{**}\rangle_1 \le \|x^{**}\|$. Thus

$$\begin{aligned}\langle x^{**}, w^{***}\rangle_2 &= \left\langle x^{**}, \widehat{e^*}\right\rangle_2 - \left\langle x^{**}, W^*e^*\right\rangle_2 = \langle e^*, x^{**}\rangle_1 - \langle Wx^{**}, e^*\rangle_0 \\ &= \langle e^*, x^{**}\rangle_1 - \textstyle\sum_{m=1}^{\infty}\langle e^*_m, x^{**}\rangle_1 \le \|x^{**}\|.\end{aligned} \tag{4.4}$$

Since this holds for all $x^{**} \in \ell_1^{**}$, $\|w^{***}\| \le 1$. On the other hand, if x_0^{**} is as in Lemma 3.2, then (3.3) gives $\|x_0^{**}\| = 1$ and, from (3.6) and the above,

$$\langle x_0^{**}, w^{***}\rangle_2 = \langle e^*, x_0^{**}\rangle_1 - \textstyle\sum_{m=1}^{\infty}\langle e^*_m, x_0^{**}\rangle_1 = 1 - \sum_{m=1}^{\infty} 0 = 1,$$

which gives (4.1). Now let $x \in \ell_1$. Then, from Lemma 4.1(b),

$$\begin{aligned}\langle \widehat{x}, w^{***}\rangle_2 &= \left\langle \widehat{x}, \widehat{e^*}\right\rangle_2 - \left\langle \widehat{x}, W^*e^*\right\rangle_2 = \langle e^*, \widehat{x}\rangle_1 - \langle W\widehat{x}, e^*\rangle_0 \\ &= \langle x, e^*\rangle_0 - \langle x, e^*\rangle_0 = 0,\end{aligned}$$

which gives (4.2). Finally, let $x^* = (x^*_m)_{m\ge1} \in c$, $x^{**} \in \ell_1^{**}$, and write $\Lambda := \lim_{n\to\infty} x^*_n$. From Lemma 4.1(c), with $p^* := x^* - \Lambda e^* \in c_0$,

$$\langle x^* - \Lambda e^*, x^{**}\rangle_1 = \langle Wx^{**}, x^* - \Lambda e^*\rangle_0.$$

Thus

$$\begin{aligned}\langle x^*, x^{**}\rangle_1 &= \langle Wx^{**}, x^* - \Lambda e^*\rangle_0 + \langle \Lambda e^*, x^{**}\rangle_1 \\ &= \langle Wx^{**}, x^*\rangle_0 - \langle Wx^{**}, \Lambda e^*\rangle_0 + \langle \Lambda e^*, x^{**}\rangle_1,\end{aligned}$$

which gives (4.3). This completes the proof of Theorem 4.2. □

Remark 4.3. For all $x^{**} \in \ell_1^{**} = \ell_\infty^*$, $\langle x^{**}, W^*e^*\rangle_2 = \langle Wx^{**}, e^*\rangle_0 = \sum_{m=1}^{\infty}\langle e^*_m, x^{**}\rangle_1 = \sum_{m=1}^{\infty}\langle x^{**}, \widehat{e^*_m}\rangle_2$, so we could write $W^*e^* = \sum_{m=1}^{\infty}\widehat{e^*_m}$, with the understanding that the convergence is in the $w(\ell_1^{***}, \ell_1^{**})$ (weak*) sense. Thus, with this understanding, $w^{***} = \widehat{e^*} - \sum_{m=1}^{\infty}\widehat{e^*_m}$. Note from (4.1) that this does not imply that $w^{***} = 0$.

Since $\widehat{W\widehat{x}} = \widehat{x}$, the map $x^{**} \mapsto \widehat{Wx^{**}}$ is a *linear retraction* from ℓ_1^{**} onto $\widehat{\ell_1}$.

Remark 4.4. It is known from standard results in point–set topology that the set $\mathbb{N}$ of positive integers (considered as a discrete topological space) can be embedded as a dense open subspace of a compact Hausdorff space, $\beta\mathbb{N}$ (the *Stone-Čech compactification* of $\mathbb{N}$), such that, for all $x^* \in \ell_1^* = \ell_\infty$, there exists a unique element βx^* of $C(\beta\mathbb{N})$ (the set of continuous functions on $\beta\mathbb{N}$) extending x^*. (The fact that $\mathbb{N}$ is open in $\beta\mathbb{N}$ is a consequence of the result proved in [2, XI.8.3, pp. 245–246] that any locally compact completely regular space is open in any compactification.) Obviously $\beta e^* = 1$.

For all $m \ge 1$, $\{m\}$ is open (in $\mathbb{N}$ and hence) in $\beta\mathbb{N}$, and so, if $f_m\colon \beta\mathbb{N} \to \mathbb{R}$ is defined by $f_m(m) := 1$ and $f_m := 0$ on $\beta\mathbb{N} \setminus \{m\}$ then $f_m = \beta e^*_m \in C(\beta\mathbb{N})$. It follows from the Riesz representation theorem (see, for instance, [6, Theorem 6.19, pp. 130–132] for a considerably more general result) that $\ell_1^{**} = \ell_\infty^*$ can be identified with the set $\mathcal{M}(\beta\mathbb{N})$ of (signed) Radon measures on $\beta\mathbb{N}$. If $x^{**} \in$

$\ell_1^{**} = \ell_\infty^*$ and $\mu \in \mathcal{M}(\beta\mathbb{N})$ represents x^{**} then $\langle e_m^*, x^{**}\rangle_1 = \int f_m d\mu = \mu(\{m\})$, and so $Wx^{**} = \big(\mu(\{m\})\big)_{m\geq 1} \in \ell_1$. Furthermore, $\langle e^*, x^{**}\rangle_1 = \int 1 d\mu = \mu(\beta\mathbb{N})$. Thus, from (4.4) and standard measure-theoretic arguments,

$$\langle x^{**}, w^{***}\rangle_2 = \mu(\beta\mathbb{N}) - \textstyle\sum_{m=1}^\infty \mu(\{m\}) = \mu(\beta\mathbb{N} \setminus \mathbb{N}). \tag{4.5}$$

This discussion will be continued in Remark 5.3.

5 G^*

Theorem 5.1 below extends the results proved in (3.4) and (3.5) for a particular element x_0^{**} of ℓ_1^{**} to a general element x^{**} of ℓ_1^{**}.

Theorem 5.1. *Let $x^{**} \in \ell_1^{**}$. Then*

$$G^* x^{**} = -GWx^{**} - \big\langle x^{**}, w^{***}\big\rangle_2 e^* \in \ell_1^* \tag{5.1}$$

and

$$\langle G^* x^{**}, x^{**}\rangle_1 = -\langle x^{**}, w^{***}\rangle_2^2. \tag{5.2}$$

Proof. Let $x \in \ell_1$. Setting $x^* = Gx$ in (4.3), writing α for $\langle x^{**}, w^{***}\rangle_2$ to simplify the expressions, and using (2.1) (with $w = Wx^{**}$) and (3.2),

$$\begin{aligned}\langle x, G^* x^{**}\rangle_0 = \langle Gx, x^{**}\rangle_1 &= \langle Wx^{**}, Gx\rangle_0 + \alpha \lim\nolimits_{n\to\infty}(Gx)_n \\ &= -\langle x, GWx^{**}\rangle_0 - \alpha\langle x, e^*\rangle_0.\end{aligned}$$

Since this holds for all $x \in \ell_1$, this completes the proof of (5.1). From (5.1), the definition of G^*, (5.1) again, (2.2) (with $x = Wx^{**}$), the definition of W^*, and the definition of w^{***} (in sequence),

$$\begin{aligned}\langle G^* x^{**}, x^{**}\rangle_1 &= -\langle GWx^{**}, x^{**}\rangle_1 - \alpha\langle e^*, x^{**}\rangle_1 \\ &= -\langle Wx^{**}, G^* x^{**}\rangle_0 - \alpha\langle e^*, x^{**}\rangle_1 \\ &= \langle Wx^{**}, GWx^{**}\rangle_0 + \alpha\langle Wx^{**}, e^*\rangle_0 - \alpha\langle e^*, x^{**}\rangle_1 \\ &= \alpha\langle Wx^{**}, e^*\rangle_0 - \alpha\langle e^*, x^{**}\rangle_1 \\ &= \alpha\big[\langle x^{**}, W^* e^*\rangle_2 - \langle x^{**}, \widehat{e^*}\rangle_2\big] = -\alpha^2.\end{aligned}$$

This gives (5.2), and completes the proof of Theorem 5.1. ∎

Remark 5.2. There is an analysis of G^* and $\langle G^* x^{**}, x^{**}\rangle_1$ in [1, Example 14.2.2, pp. 161–162] that is, on the surface, different from the one presented in Theorem 5.1 above. A linear operator $T\colon\ \ell_1 \to \ell_\infty$ is defined by

$$\text{for all } x \in \ell_1,\ (Tx)_m = x_m + 2\textstyle\sum_{n>m} x_n.$$

If n is odd, we define $y^{(n)} \in \ell_1$ by $y^{(n)} := (2, -2, \ldots, 2, -2, 1, 0, 0, \ldots)$, where the "1" is in the nth place. Then

$$T\big(y^{(n)}\big) = (2, -2, \ldots, 2, -2, 1, 0, 0, \ldots) + 2\big(-1, 1, \ldots, -1, 1, 0, 0, 0, \ldots\big) = e_n^*.$$

Similarly, if n is even, we define $y^{(n)} \in \ell_1$ by $y^{(n)} := (-2, 2, \ldots, 2, -2, 1, 0, 0, \ldots)$. Again, $T\big(y^{(n)}\big) = e_n^*$. The analysis in [1] rests on the assumption (see [1, Eqn. (1), pp. 159]) that, for all $x^{**} \in \ell_1^{**}$,

$$\text{there exists } x \in \ell_1 \text{ such that, for all } x^* \in R(T), \langle x^*, \widehat{x}\rangle_1 = \langle x^*, x^{**}\rangle_1.$$

In particular, the argument above implies that, for all $n \geq 1$, $\langle e_n^*, \widehat{x}\rangle_1 = \langle e_n^*, x^{**}\rangle_1$. Thus, for all $n \geq 1$, $x_n = \langle e_n^*, x^{**}\rangle_1$. Consequently, $x = Wx^{**}$, and the formulae for G^* and $\langle G^*x^{**}, x^{**}\rangle_1$ given in [1] reduce to the more explicit ones given in Theorem 5.1 above.

Remark 5.3. This is a continuation of Remark 4.4. A comparison of (4.5) and (5.2) leads to the conclusion that $\langle G^*x^{**}, x^{**}\rangle_1 = -\mu(\beta\mathbb{N} \setminus \mathbb{N})^2$. This is exactly the formula obtained in Gossez, [4, Example, p. 89].

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