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Coon Amplitudes and Their Generalizations

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ABSTRACT OF THE DISSERTATION

Coon Amplitudes and Their Generalizations

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In this dissertation we describe several recent advancements in the study of Coon amplitudes.

In the first chapter, we detail the properties of the Veneziano, Virasoro, and Coon amplitudes. These tree-level four-point scattering amplitudes may be written as infinite products with an infinite sequence of simple poles. Our approach for the Coon amplitude uses the mathematical theory of q-analysis. We interpret the Coon amplitude as a q-deformation of the Veneziano amplitude for all $q \geq 0$ and discover a new transcendental structure in its low-energy expansion. We show that there is no analogous q-deformation of the Virasoro amplitude.

In the second chapter, we analyze so-called generalized Veneziano and generalized Virasoro amplitudes. Under some physical assumptions, we find that their spectra must satisfy an over-determined set of non-linear recursion relations. The recursion relation for the generalized Veneziano amplitudes can be solved analytically and yields a two-parameter family which includes the Veneziano amplitude, the one-parameter family of Coon amplitudes, and a larger two-parameter family of amplitudes with an infinite tower of spins at each mass level. In the generalized Virasoro case, the only consistent solution is the string spectrum.

The dissertation of Nicholas Parker Geiser is approved.

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To my family, friends, and comrades.

TABLE OF CONTENTS

LIST OF FIGURES

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Chapter 1 of this dissertation is based on [GL22] with Lukas Lindwasser. Chapter 2 is based on [GL23] with Lukas Lindwasser.

CHAPTER 1

Infinite product amplitudes: Veneziano, Virasoro, and Coon

1.1 Introduction

This chapter is based on [GL22]. In this chapter, we shall detail the properties of the Veneziano [Ven68], Virasoro [Vir69], and Coon [Coo69] amplitudes with zero Regge intercept.[1] These amplitudes describe the tree-level scattering of four massless particles and may be written as infinite products with an infinite sequence of simple poles. For each amplitude, we shall discuss its unitarity, high-energy behavior, low-energy expansion, and number theoretic properties. We shall synthesize these properties in a unified manner to facilitate comparison between the amplitudes.

Two of these amplitudes are well-known. The Veneziano amplitude describes the scattering of four open strings. The Virasoro amplitude describes the scattering of four closed strings. The lesser-known Coon amplitude is a one-parameter deformation of the Veneziano amplitude with a real deformation parameter $q \geq 0$. At $q = 0$, the Coon amplitude reduces to a field theory amplitude. At $q = 1$, the Coon amplitude is equal to the Veneziano amplitude.

Our approach for the Coon amplitude uses the mathematical theory of q-deformations,

[1]For simplicity we only consider the scattering of four massless bosonic states. In this case, the tree-level open and closed superstring amplitudes respectively reduce to the Veneziano and Virasoro amplitudes with zero intercept.

or q-analysis. Using a well-established q-deformation of the gamma function, we write a new single formula (1.34) for the Coon amplitude valid for all $q \geq 0$. Previously, the Coon amplitude with $q < 1$ and the Coon amplitude with $q > 1$ were considered as distinct [BC76]. Our calculations confirm and extend the recent analysis of the Coon amplitude with $q < 1$ [FT22] to all $q \geq 0$. Moreover, we compute a compact formula (1.46) for the low-energy expansion of the Coon amplitude and discover a novel transcendental structure analogous to the number theoretic structure of the low-energy expansions of the Veneziano and Virasoro amplitudes.

As a function of q, the Coon amplitude demonstrates a subtle interplay between the properties of unitarity and meromorphicity (in the Mandelstam variables). For $0 < q < 1$, the Coon amplitude is unitary and non-meromorphic with an accumulation point spectrum.[2] For $q > 1$, the Coon amplitude is non-unitary and meromorphic with no accumulation point. Only the Veneziano amplitude at $q = 1$ is unitary and meromorphic with no accumulation point.

In any case, the Coon amplitude is a fruitful example for the study of general scattering amplitudes [CKS17, HLR21, Mai22, FT22, HR22]. While there is yet no definitive field theory or string worldsheet realization of the Coon amplitude, accumulation point spectra like those exhibited by the Coon amplitude with $q < 1$ have been recently found in a stringy setup involving open strings ending on a D-brane [MR22]. Similar accumulation point spectra have also appeared in recent amplitude studies [CKS17, Rid21, HLR21, BKZ21, HR22, MR22]. Famously, the hydrogen atom has energy levels $E_n = -13.6\ \text{eV}/n^2$ with an accumulation point at $E_\infty = 0$.

In both this chapter and in chapter 2 (based on [GL23]), we show that there is no naive q-deformation of the Virasoro amplitude or Virasoro-Coon amplitude analogous to our interpretation of the Coon amplitude as a q-deformed Veneziano amplitude. In this chapter, we attempt and fail to construct a q-deformed Virasoro amplitude using functions from q-

[2] A theory has an accumulation point spectrum if for some finite $M^2 > 0$, the number of particles with mass $m^2 < M^2$ is infinite.

analysis. Our assumptions include crossing symmetry and polynomial residues on the same sequence of poles as the Coon amplitude.

In chapter 2, we revisit this question by analyzing so-called generalized Veneziano and generalized Virasoro amplitudes, which are defined by generalizing the infinite product representations of the Veneziano and Virasoro amplitudes, respectively. Our procedure is an extension and clarification of Coon's original argument [Coo69] and related work [FN95] in which we search for tree-level amplitudes with an infinite product form. We again assume crossing symmetry, but we now demand physical residues on an a priori unspecified sequence of poles λ_n. In other words, we do not assume the mass spectrum (while we do assume the mass spectrum in our search for a q-deformed Virasoro amplitude here). Under these assumptions, we find that the poles λ_n must satisfy an over-determined set of non-linear recursion relations. The recursion relations for the generalized Veneziano amplitudes can be solved analytically. In the generalized Virasoro case, we numerically demonstrate that the only consistent solution to these recursion relations is the string spectrum. The Veneziano, Virasoro, and Coon amplitudes detailed here are in fact the healthiest amplitudes we find in [GL23]. This conclusion strengthens our present findings.

There may also be a simple physical argument for the non-existence of a Virasoro analog of the Coon amplitude. We recall that accumulation point spectra were recently found in a stringy setup involving open strings ending on a D-brane [MR22]. If accumulation point spectra in string theory generically require open strings, then there may be no consistent q-deformed Virasoro amplitude with an accumulation point for $q < 1$ since the Virasoro amplitude at $q = 1$ describes the scattering of closed strings. These ideas are speculative and worthy of future research.

Our approach is part of the modern S-matrix bootstrap program, which attempts to construct general amplitudes which satisfy various physical properties such as unitarity, crossing, and analyticity without relying on an underlying dynamical theory [CSZ21]. The modern S-matrix bootstrap is a revival of an old approach [ELO66] which predates modern quantum field theory (QFT) and attempts to constrain the space of physical theories, including those which may not be describable by QFT.

1.1.1 Conventions

We only consider tree-level scattering amplitudes for four massless particles in $d \geq 3$ spacetime dimensions. Such amplitudes have simple poles only. In a unitary theory, the residues of these poles equal finite sums of Gegenbauer polynomials,

$$C_j^{(\frac{d-3}{2})}(\cos\theta) \tag{1.1}$$

with positive coefficients. The positivity of these coefficients encodes the unitarity of the theory. The Mandelstam variables for this process are,

$$\begin{aligned} s &= -(p_1+p_2)^2 = \quad 4E^2 &&\geq 0 \\ t &= -(p_1+p_4)^2 = -2E^2(1-\cos\theta) &&\leq 0 \\ u &= -(p_1+p_3)^2 = -2E^2(1+\cos\theta) &&\leq 0 \end{aligned} \tag{1.2}$$

and satisfy the mass-shell relation $s+t+u=0$. Here E and θ are the center-of-mass energy and scattering angle, respectively, and the inequalities refer to the physical scattering regime. Since s-channel and t-channel Feynman diagrams correspond to the same cyclic ordering, color-ordered amplitudes (e.g. gluon amplitudes) will have only s-channel and t-channel poles and shall be denoted by $\mathcal{A}(s,t)$. Amplitudes with poles in all three channels (e.g. graviton amplitudes) shall be denoted by $\mathcal{A}(s,t,u)$. We use units in which the lowest massive state of any particular theory has mass $m^2 = 1$. In open (closed) string theory, this choice corresponds to $\alpha' = 1$ ($\alpha' = 4$).

1.1.2 Outline

In section 1.2, section 1.3, and section 1.4 we discuss the properties of the Veneziano, Virasoro, and Coon amplitudes, respectively. In section 1.5, we attempt and fail to construct a Virasoro-Coon amplitude by q-deforming the Virasoro amplitude. Three additional sections contain various technical details. In section 1.6, we review some properties of the gamma function. In section 1.7, we review some properties of the Gegenbauer polynomials. In section 1.8, we derive the low-energy expansion of the Coon amplitude.

1.2 The Veneziano amplitude

The Veneziano amplitude was discovered in 1968 [Ven68] and describes the scattering of four open strings. More recently, the Veneziano amplitude has been revisited in the context of the modern S-matrix bootstrap program [CKS17, GW19, HLR21, AHH21, Mai22, AEH22].

In $d \leq 10$ dimensions, the Veneziano amplitude is a physically-admissible UV-completion of the tree-level four-point amplitude of maximally supersymmetric Yang-Mills field theory. The color-stripped tree-level field theory amplitude which describes the scattering of any four massless particles in the Yang–Mills supermultiplet is given by,

$$\mathcal{A}_{\text{SYM}} = P_4 \frac{1}{st} \tag{1.3}$$

where P_4 is a kinematic prefactor which is determined by maximal supersymmetry and which contains the information about the particular states being scattered. For the four-gluon amplitude, schematically $P_4 = F^4$ where F is the linearized field strength. The second factor $\frac{1}{st}$ is symmetric in (s, t) and is a meromorphic function with simple poles from massless particle exchange in the s-channel and t-channel. In the high-energy Regge limit $s \to \infty$ with fixed polarizations and fixed $t < 0$, the prefactor $P_4 \propto s^2$, and the amplitude diverges as $\mathcal{A}_{\text{SYM}} \propto s$.

In tree-level open superstring theory, the color-stripped amplitude which describes the scattering process (1.3) is given by,

$$\mathcal{A}_{\text{open}} = P_4 \, \mathcal{A}_{\text{Ven}}(s, t) \tag{1.4}$$

where $\mathcal{A}_{\text{Ven}}$ is the Veneziano amplitude,

$$\mathcal{A}_{\text{Ven}}(s,t) = \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1-s-t)} \tag{1.5}$$

Like the corresponding field theory factor, the Veneziano amplitude is symmetric in (s,t) and is a meromorphic function with simple poles only. We may explicitly exhibit these poles using the infinite product representation (1.66) of the gamma function to write,

$$\mathcal{A}_{\text{Ven}}(s,t) = \frac{1}{st}\prod_{n\geq 1}\frac{n^2 - n(s+t)}{(s-n)(t-n)} \tag{1.6}$$

The infinite sequence of massive poles at $s = 1, 2, \ldots$ correspond to excited stringy states.

1.2.1 Unitarity

The kinematic prefactor P_4 which appears in both the field theory amplitude (1.3) and the open superstring amplitude (1.4) has a positive expansion on the Gegenbauer polynomials in $d \leq 10$ dimensions [AEH22, HR22]. It remains then to check the unitarity of the Veneziano amplitude itself. On the massless s-channel pole, $P_4 \propto t^2$, and the residue of the Veneziano amplitude agrees with field theory,

$$\operatorname*{Res}_{s=0} \mathcal{A}_{\text{Ven}}(s,t) = \operatorname*{Res}_{s=0} \frac{1}{st} = \frac{1}{t} \qquad \Longrightarrow \qquad \operatorname*{Res}_{s=0} \mathcal{A}_{\text{open}} \propto t \tag{1.7}$$

indicating the exchange of a massless spin-1 state, the gluon. The residue of each massive pole at $s = N \geq 1$ is a degree-$(N-1)$ polynomial in t,

$$\operatorname*{Res}_{s=N} \mathcal{A}_{\text{Ven}}(s,t) = \frac{1}{N!}(t+1)(t+2)\cdots(t+N-1) \tag{1.8}$$

indicating the exchange of states with mass $m^2 = N$ and spins $j \leq N+1$. These residues may be expanded in terms of Gegenbauer polynomials using the identities in section 1.7,

$$\operatorname*{Res}_{s=N} \mathcal{A}_{\text{Ven}}(s,t) = \sum_{j=0}^{N-1} c_{N,j}\, C_j^{\left(\frac{d-3}{2}\right)}\left(1 + \tfrac{2t}{N}\right) \tag{1.9}$$

with the first few coefficients given by,

$$\begin{aligned} c_{1,0} &= 1 & c_{2,0} &= 0 & c_{3,0} &= \tfrac{10-d}{24(d-1)} \\ & & c_{2,1} &= \tfrac{1}{2(d-3)} & c_{3,1} &= 0 \\ & & & & c_{3,2} &= \tfrac{3}{4(d-1)(d-3)} \end{aligned} \tag{1.10}$$

The coefficient $c_{3,0}$ is negative for $d > 10$, indicating the non-unitarity of the superstring above its critical dimension $d = 10$. These coefficients were recently studied in $d = 4$ [Mai22] and were recently shown to be positive for all $d \leq 6$ [AEH22]. The unitarity of superstring theory in $d \leq 10$ (and thus the positivity of the $c_{N,j}$) is known from the no-ghost theorem [Bro72, GT72, Tho85], but there is yet no direct proof that $c_{N,j} > 0$ for all $d \leq 10$.

1.2.2 High-energy

The high-energy behavior of the Veneziano amplitude may be calculated using Stirling's formula (1.68). In the Regge limit of large $|s| \gg 1$ with phase $0 < \arg(s) < 2\pi$ (to avoid the poles of the gamma function) and fixed $t < 0$, we find,

$$\mathcal{A}_{\text{Ven}}(s,t) \overset{|s|\to\infty}{\sim} (-s)^{t-1}\,\Gamma(-t)\left(1 + \mathcal{O}(s^{-1})\right) \tag{1.11}$$

Compared to field theory, the extra exponent t softens the UV behavior. For any scattered states with fixed polarizations, there is a range of fixed $t < 0$ such that $\lim_{|s|\to\infty} \mathcal{A}_{\text{open}} = 0$ while this limit diverges in the corresponding field theory amplitude.

1.2.3 Low-energy

At leading order in the low-energy expansion $|s|, |t| \ll 1$, the Veneziano amplitude reproduces field theory. At higher order, stringy corrections to field theory are given in terms of Riemann zeta-values. Using the Taylor expansion for the gamma function (1.67), we find,

$$\begin{aligned}\mathcal{A}_{\text{Ven}}(s,t) &= \frac{1}{st}\exp\sum_{k\geq 2}\frac{\zeta(k)}{k}\left[s^k+t^k-(s+t)^k\right]\\ &= \frac{1}{st}-\zeta(2)-\zeta(3)\,(s+t)-\zeta(4)\,(s^2+\tfrac{1}{4}st+t^2)+\cdots\end{aligned} \tag{1.12}$$

The Veneziano amplitude exhibits a remarkable property called uniform transcendentality, meaning each term in its low-energy expansion may be assigned the same transcendental weight. If we assign weight k to the zeta-value $\zeta(k)$ (the standard number theoretic assignment) and weight -1 to the Mandelstam variables, then each term in (1.12) has transcendental weight two. Uniform transcendentality is in fact a general property of tree-level superstring amplitudes [SS13], and the transcendental structure of one-loop superstring amplitudes is under active study [DG19, DG22]. In comparison, non-trivial transcendental structure in field theory only arises from loop integrals [KL03, BF09, ABC12, BDD19].

1.3 The Virasoro amplitude

The Virasoro amplitude was discovered in 1969 [Vir69] and describes the scattering of four closed strings. Like the Veneziano amplitude, the Virasoro amplitude has also been recently revisited in the context of the modern S-matrix bootstrap program [GW19, AHH21, AEH22].

In $d \leq 10$ dimensions, the Virasoro amplitude is a physically-admissible UV-completion of the tree-level four-point amplitude of maximal supergravity. The tree-level field theory amplitude which describes the scattering of any four massless particles in the supergravity multiplet is given by,

$$\mathcal{A}_{\text{SG}} = P_8\Big(-\frac{1}{stu}\Big) \tag{1.13}$$

where P_8 is a kinematic prefactor which is determined by maximal supersymmetry and which contains the information about the particular states being scattered. For the four-graviton amplitude, schematically $P_8 = R^4$ where R is the linearized Riemann curvature. The second factor $-\frac{1}{stu}$ is symmetric in (s,t,u) and contains poles from massless particle exchange in the s-channel, t-channel, and u-channel. In the high-energy Regge limit $s \to \infty$ with fixed polarizations and fixed $t < 0$, the prefactor $P_8 \propto s^4$, and the amplitude diverges as $\mathcal{A}_{\rm SG} \propto s^2$.

In tree-level closed superstring theory, the amplitude which describes the scattering process (1.13) is given by,

$$\mathcal{A}_{\rm closed} = P_8 \, \mathcal{A}_{\rm Vir}(s,t,u) \tag{1.14}$$

where $\mathcal{A}_{\rm Vir}$ is the Virasoro amplitude,

$$\mathcal{A}_{\rm Vir}(s,t,u) = \frac{\Gamma(-s)\Gamma(-t)\Gamma(-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} \tag{1.15}$$

Like the corresponding field theory factor, the Virasoro amplitude is symmetric in (s,t,u) and is a meromorphic function with simple poles only. We may explicitly exhibit these poles using the infinite product representation (1.66) of the gamma function to write,

$$\mathcal{A}_{\rm Vir}(s,t,u) = -\frac{1}{stu} \prod_{n\geq 1} \frac{-n^3 - n(st+tu+us) - stu}{(s-n)(t-n)(u-n)} \tag{1.16}$$

The infinite sequence of massive poles at $s = 1, 2, \dots$ correspond to excited stringy states.

1.3.1 Unitarity

The kinematic prefactor P_8 which appears in both the field theory amplitude (1.13) and the closed superstring amplitude (1.14) has a positive expansion on the Gegenbauer polynomials in $d \leq 10$ dimensions [AEH22, HR22]. It remains then to check the unitarity of the Virasoro amplitude itself. On the massless s-channel pole, $P_8 \propto t^4$, and the residue of the Virasoro amplitude agrees with field theory,

$$\operatorname*{Res}_{s=0} \mathcal{A}_{\rm Vir}(s,t,-s-t) = \operatorname*{Res}_{s=0} \frac{1}{st(s+t)} = \frac{1}{t^2} \qquad \Longrightarrow \qquad \operatorname*{Res}_{s=0} \mathcal{A}_{\rm closed} \propto t^2 \tag{1.17}$$

indicating the exchange of a massless spin-2 state, the graviton. The residue of each massive pole at $s = N \geq 1$ is a degree-$(2N-2)$ polynomial in t, indicating the exchange of states with mass $m^2 = N$ and spins $j \leq 2N+2$. In fact, the residues of the Virasoro amplitude equal the residues of the Veneziano amplitude (1.8) squared,

$$\begin{aligned} \operatorname*{Res}_{s=N} \mathcal{A}_{\text{Vir}}(s,t,-s-t) &= \left\{ \frac{1}{N!}(t+1)(t+2)\cdots(t+N-1) \right\}^2 \\ &= \left\{ \operatorname*{Res}_{s=N} \mathcal{A}_{\text{Ven}}(s,t) \right\}^2 \end{aligned} \tag{1.18}$$

These residues may be expanded in terms of Gegenbauer polynomials using the identities in section 1.7,

$$\operatorname*{Res}_{s=N} \mathcal{A}_{\text{Vir}}(s,t,-s-t) = \sum_{j=0}^{2N-2} c_{N,j}\, C_j^{(\frac{d-3}{2})}\big(1 + \tfrac{2t}{N}\big) \tag{1.19}$$

with the first few coefficients given by,

$$\begin{aligned}
&c_{1,0} = 1 \qquad && c_{2,0} = \tfrac{1}{4(d-1)} \qquad && c_{3,0} = \tfrac{224-18d+d^2}{576(d+1)(d-1)} \\
& && c_{2,1} = 0 && c_{3,1} = 0 \\
& && c_{2,2} = \tfrac{1}{2(d-1)(d-3)} && c_{3,2} = \tfrac{24-d}{16(d+3)(d-1)(d-3)} \\
& && && c_{3,3} = 0 \\
& && && c_{3,4} = \tfrac{27}{8(d+3)(d+1)(d-1)(d-3)}
\end{aligned} \tag{1.20}$$

The positivity of these coefficients below the critical dimension follows indirectly from the no-ghost theorem [Bro72, GT72, Tho85], but there is yet no direct proof that $c_{N,j} > 0$ for all $d \leq 10$.

1.3.2 High-energy

The high-energy behavior of the Virasoro amplitude may be calculated using Stirling's formula (1.68). In the Regge limit of large $|s| \gg 1$ with phase $0 < \arg(s) < \pi$ (to avoid the poles of the gamma function) and fixed $t < 0$, we find,

$$\mathcal{A}_{\text{Vir}}(s,t,-s-t) \overset{|s|\to\infty}{\sim} (-s)^{t-1}\, s^{t-1}\, \frac{\Gamma(-t)}{\Gamma(1+t)} \big(1 + \mathcal{O}(s^{-1})\big) \tag{1.21}$$

Compared to field theory, the extra exponent $2t$ softens the UV behavior. For any scattered states with fixed polarizations, there is a range of fixed $t < 0$ such that $\lim_{|s|\to\infty} \mathcal{A}_{\text{closed}} = 0$ while this limit diverges in the corresponding field theory amplitude.

1.3.3 Low-energy

At leading order in the low-energy expansion $|s|, |t|, |u| \ll 1$, the Virasoro amplitude reproduces field theory. At higher order, the stringy corrections to field theory are given in terms of Riemann zeta-values. Using the Taylor expansion for the gamma function (1.67), we find,

$$\begin{aligned}\mathcal{A}_{\text{Vir}}(s,t,u) &= -\frac{1}{stu} \exp \sum_{k\geq 1} \frac{2\,\zeta(2k+1)}{2k+1}(s^{2k+1} + t^{2k+1} + u^{2k+1}) \\ &= -\frac{1}{stu} - 2\,\zeta(3) - \zeta(5)\,(s^2 + t^2 + u^2) + \cdots \end{aligned} \tag{1.22}$$

Like the Veneziano amplitude (1.12), the low-energy expansion of the Virasoro amplitude exhibits uniform transcendentality. If we assign weight k to $\zeta(k)$ and weight -1 to the Mandelstam variables, then each term in (1.22) has transcendental weight three.

Furthermore, we note that only odd zeta-values occur in (1.22) while both even and odd zeta-values occurred in the low-energy expansion of the Veneziano amplitude (1.12). This discrepancy between the Veneziano and Virasoro amplitudes may be described by the so-called single-valued map, which maps the (motivic) zeta-values $\zeta(k)$ to the single-valued zeta-values $\zeta_{\text{sv}}(k)$, defined by,

$$\zeta_{\text{sv}}(2k) = 0 \qquad\qquad \zeta_{\text{sv}}(2k+1) = 2\,\zeta(2k+1) \tag{1.23}$$

The single-valued zeta-values are so-called because they descend from single-valued versions of the multi-valued polylogarithm functions $\text{Li}_k(z)$, which evaluate to the Riemann zeta function at $z = 1$,

$$\text{Li}_k(z) = \sum_{n\geq 1} \frac{z^n}{n^k} \quad \xrightarrow[z\to 1]{} \quad \zeta(k) = \sum_{n\geq 1} \frac{1}{n^k} \tag{1.24}$$

Comparing (1.12) and (1.22), we see that the Veneziano amplitude becomes the Virasoro amplitude under the single-valued map acting term-by-term on the low-energy expansion,

$$(st)\,\mathcal{A}_{\text{Ven}} \underset{\text{sv}}{\longrightarrow} (-stu)\,\mathcal{A}_{\text{Vir}} \tag{1.25}$$

General tree-level open and closed superstring amplitudes are in fact related by the single-valued map [SS13, Bro14, Sti14, ST14, BD21a, SS19, VZ18, BD21b], encoding a deep number theoretical relationship between the open and closed superstrings, and thus between gauge theories and theories of gravity.

Both the single-valued map (1.25) and the residue relation (1.18) are manifestations of another relationship between open and closed superstring amplitudes. The Kawai-Lewellen-Tye (KLT) relations [KLT86] express tree-level closed superstring amplitudes as bilinears of tree-level open superstring amplitudes. Informally, the closed superstring is equal to the open superstring squared. The four-point KLT relation is,

$$\mathcal{A}_{\text{Vir}}(s,t,u) = \mathcal{A}_{\text{Ven}}(s,t)\,S_{\text{KLT}}\,\mathcal{A}_{\text{Ven}}(s,t) \tag{1.26}$$

where S_{KLT} is the KLT kernel,

$$S_{\text{KLT}} = \frac{\sin(\pi s)\sin(\pi t)}{\pi \sin\big(\pi(s+t)\big)} \tag{1.27}$$

This expression for S_{KLT} follows from the definition of the Veneziano amplitude (1.5), the definition of the Virasoro amplitude (1.15), and the reflection formula for the gamma function (1.65). In the field theory (low-energy) limit, the KLT relations are known as the double copy between gauge theory and gravity [BCJ10, BCC19].

1.4 The Coon amplitude

The Coon amplitude was discovered in 1969 as a generalization of the Veneziano amplitude with non-linear Regge trajectories [Coo69]. The subsequent studies of the Coon amplitude were phenomenologically motivated. A concise survey of this early literature is given in a (quite difficult to find) 1989 review [Rom89] (which cites an expanded but unpublished pre-print [Rom88] which we could not locate). Around this time, the Coon amplitude was revisited in the broader context of string theory [Rom89, FN95]. Most recently, the Coon amplitude has reappeared in the modern S-matrix bootstrap program [CKS17, Rid21, HLR21, Mai22, FT22, HR22, MR22].

The Coon amplitude $\mathcal{A}_q$ is a one-parameter deformation of the Veneziano amplitude with a real deformation parameter $q \geq 0$. To construct a full four-point scattering amplitude, we replace $\mathcal{A}_{\text{Ven}}$ in the open superstring amplitude (1.4) with $\mathcal{A}_q$ to describe the scattering of four massless states in a putative q-deformed string theory,

$$\mathcal{A}_{q\text{-strings}} = P_4\, \mathcal{A}_q(s,t) \tag{1.28}$$

This deformation may be understood using the mathematical theory of q-deformations or q-analogs, also known as q-analysis.

1.4.1 q-analysis

In mathematics, a q-analog of a theorem, function, identity, or expression is a generalization involving a deformation parameter q that returns the original mathematical object in the limit $q \to 1$. Many special functions and differential equations have well-studied q-analogs dating back to the nineteenth century [GR04]. For our purposes, we shall only need a few q-ingredients. We first define the q-integers $[n]_q$ by,

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1} \quad \xrightarrow[q\to 1]{} \quad n \tag{1.29}$$

where the second equality holds for $n \geq 1$.

In passing from the Veneziano amplitude (1.5) to the Coon amplitude, we shall replace the linear Regge trajectory $\alpha(s) = s$ that appears in the arguments of the gamma functions with a non-linear deformation $\alpha_q(s)$ that satisfies $\alpha_q([n]_q) = n$. The q-deformed Regge trajectory is thus,

$$\alpha_q(s) = \frac{\ln\big(1+(q-1)s\big)}{\ln q} \tag{1.30}$$

This Regge trajectory becomes linear as $\lim_{q\to 1}\alpha_q(s) = s$.

The gamma functions in the Veneziano amplitude (1.5) are themselves replaced by the so-called q-gamma function, which is defined for complex q by [GR04],

$$\Gamma_q(z) = \begin{cases} (1-q)^{1-z}\displaystyle\prod_{n\geq 0}\frac{1-q^{+n+1}}{1-q^{+n+z}} & |q|<1 \\ q^{\frac{z(z-1)}{2}}(q-1)^{1-z}\displaystyle\prod_{n\geq 0}\frac{1-q^{-n-1}}{1-q^{-n-z}} & |q|>1 \end{cases} \tag{1.31}$$

The q-gamma function obeys a functional equation analogous to $\Gamma(z+1) = z\,\Gamma(z)$,

$$\Gamma_q(z+1) = \frac{1-q^z}{1-q}\,\Gamma_q(z) \tag{1.32}$$

and becomes the ordinary gamma function as $\lim_{q\to 1^\pm}\Gamma_q(z) = \Gamma(z)$. Many properties of the gamma function have precise q-analogs. For instance, the asymptotic behavior of the q-gamma function is given by a q-analog of Stirling's formula [Moa84],

$$\ln\Gamma_q(z) \overset{|q^z|\to 0}{\sim} (z-\tfrac{1}{2})\ln\frac{1-q^z}{1-q} + \frac{\mathrm{Li}_2(1-q^z)}{\ln q} + \tfrac{1}{2}\ln(2\pi) + C_q + \mathcal{O}\big(q^z\big) \tag{1.33}$$

which is valid for small $|q^z| \ll 1$. Here $\mathrm{Li}_2(z)$ is the dilogarithm and C_q is a known q-dependent constant.

1.4.2 q-deformed Veneziano

In terms of these q-ingredients, the Coon amplitude for all $q \geq 0$ is given by,

$$\mathcal{A}_q(s,t) = q^{\alpha_q(s)\alpha_q(t)-\alpha_q(s)-\alpha_q(t)}\,\frac{\Gamma_q\big(-\alpha_q(s)\big)\Gamma_q\big(-\alpha_q(t)\big)}{\Gamma_q\big(1-\alpha_q(s)-\alpha_q(t)\big)} \tag{1.34}$$

Our conventions for the Coon amplitude differ from the older literature by an overall normalization but are chosen so that its leading low-energy behavior is $\frac{1}{st}(1+\mathcal{O}(s,t))$ to facilitate comparison to the Veneziano amplitude. Clearly, $\lim_{q\to 1}\mathcal{A}_q = \mathcal{A}_{\text{Ven}}$. Moreover, our single formula contains both the Coon amplitude with $q < 1$ and the Coon amplitude with $q > 1$, which were previously considered as distinct [BC76]. However, many properties of the Coon amplitude, including its meromorphicity as a function of the Mandelstam variables, are obscured in the form (1.34). The prefactor $q^{\alpha_q(s)\alpha_q(t)}$ is explicitly non-meromorphic, but we shall soon see that it is perfectly natural.

Using the definition of q-gamma function, we may write the Coon amplitude in terms of one convergent infinite product for $q < 1$ and another for $q > 1$,

$$\begin{aligned}\mathcal{A}_q(s,t) = q^{\alpha_q(s)\alpha_q(t)}\,\Theta(1-q)\,\frac{1}{st}\prod_{n\geq 1}\frac{(1-q^{-\alpha_q(s)-\alpha_q(t)+n})(1-q^{+n})}{(1-q^{-\alpha_q(s)+n})(1-q^{-\alpha_q(t)+n})}\\ +\,\Theta(q-1)\,\frac{1}{st}\prod_{n\geq 1}\frac{(1-q^{+\alpha_q(s)+\alpha_q(t)-n})(1-q^{-n})}{(1-q^{+\alpha_q(s)-n})(1-q^{+\alpha_q(t)-n})}\end{aligned} \tag{1.35}$$

where the step function is defined by $\Theta(x \geq 0) = 1$ and $\Theta(x < 0) = 0$. This form is nice because the infinite product of each of the four factors in either infinite product separately converges. Moreover, for $q > 1$, the non-meromorphic prefactor $q^{\alpha_q(s)\alpha_q(t)}$ has canceled against similar non-meromorphic factors in the q-gamma functions.

We may further massage (1.35) into a form with just one infinite product for all $q \geq 0$ times a piecewise q-dependent prefactor,

$$\begin{aligned}\mathcal{A}_q(s,t) = \left\{q^{\frac{\ln(1+(q-1)s)}{\ln q}\frac{\ln(1+(q-1)t)}{\ln q}}\,\Theta(1-q)+\Theta(q-1)\right\}\\ \times\frac{1}{st}\prod_{n\geq 1}\frac{\left(\frac{1-q^n}{1-q}\right)^2-\left(\frac{1-q^n}{1-q}\right)(s+t)+(1-q^n)st}{\left(s-\frac{1-q^n}{1-q}\right)\left(t-\frac{1-q^n}{1-q}\right)}\end{aligned} \tag{1.36}$$

We must take care with this expression because the infinite products of the numerator and denominator do not separately converge. In this form, we see that the Coon amplitude with $q > 0$ has an infinite sequence of simple poles in both the s-channel and t-channel. These poles occur at the q-integers,

$$[n]_q = \frac{1-q^n}{1-q} \tag{1.37}$$

for integer $n \geq 0$. For $q = 0$, the infinite product is just $\prod_{n\geq 1} 1 = 1$. For $0 < q < 1$, the poles tend to an accumulation point at $\frac{1}{1-q}$. For $q \geq 1$, the poles tend to infinity, and $q = 1$ reproduces the string spectrum.

Like the Veneziano amplitude, the Coon amplitude is symmetric in (s, t) and has simple poles only. For $q \geq 1$, the Coon amplitude is meromorphic, but for $q < 1$, the Coon amplitude as written in (1.36) has an explicit non-meromorphic factor $q^{\alpha_q(s)\alpha_q(t)}$ with branch cuts in the complex s-plane and t-plane starting at the accumulation point $\frac{1}{q-1}$. In the limits $q \to 0$ and $q \to 1^-$, this non-meromorphic prefactor becomes $q^{\alpha_q(s)\alpha_q(t)} \to 1$, and the Coon amplitude reproduces the field theory and Veneziano amplitudes, respectively,

$$\begin{aligned} \mathcal{A}_q(s,t) &\xrightarrow[q\to 0]{} \frac{1}{st} \\ \mathcal{A}_q(s,t) &\xrightarrow[q\to 1]{} \mathcal{A}_{\text{Ven}}(s,t) \end{aligned} \tag{1.38}$$

1.4.3 Unitarity

To compute the residues of the Coon amplitude, we shall manipulate the form (1.35) with manifestly convergent infinite products. The algebra is tedious but straightforward. The residue of the massless pole agrees with field theory for all $q \geq 0$,

$$\operatorname*{Res}_{s=0} \mathcal{A}_q(s,t) = \operatorname*{Res}_{s=0} \frac{1}{st} = \frac{1}{t} \tag{1.39}$$

For $q > 0$, the residue of the massive pole at $s = [N]_q$ with $N \geq 1$ is a degree-$(N-1)$ polynomial in t, indicating the exchange of states with mass $m^2 = [N]_q$ and spins $j \leq N+1$,

$$\operatorname*{Res}_{s=[N]_q} \mathcal{A}_q(s,t) = q^N \prod_{n=1}^{N} \frac{1}{\left(\frac{1-q^n}{1-q}\right)} \prod_{n=1}^{N-1} \left(q^n\, t + \tfrac{1-q^n}{1-q}\right) \tag{1.40}$$

On the poles, the non-meromorphic factor $q^{\alpha_q([N]_q)\alpha_q(t)} = (1 + (q-1)t)^N$ ensures that the residues are polynomials in t for $q < 1$. In any case, these residues may be expanded in terms of Gegenbauer polynomials using the identities in section 1.7,

$$\operatorname*{Res}_{s=[N]_q} \mathcal{A}_q(s,t) = \sum_{j=0}^{N-1} c_{N,j}\, C_j^{\left(\frac{d-3}{2}\right)}\left(1 + \tfrac{2t}{[N]_q}\right) \tag{1.41}$$

with the first few coefficients given by,

$$
\begin{aligned}
c_{1,0} &= q \\
c_{2,0} &= \tfrac{q^2(1-q)(2+q)}{2(1+q)} \\
c_{2,1} &= \tfrac{q^3}{2(d-3)} \\
c_{3,0} &= \tfrac{q^3[4(d-1)+2q(d-1)-6q^2(d-1)-q^3(5d-6)-2q^4(d-2)+3dq^5+2dq^6+dq^7]}{4(d-1)(1+q)(1+q+q^2)} \\
c_{3,1} &= \tfrac{q^4(1-q)(1+3q+2q^2+q^3)}{2(d-3)(1+q)} \\
c_{3,2} &= \tfrac{q^6(1+q+q^2)}{2(d-1)(d-3)(1+q)}
\end{aligned}
\tag{1.42}
$$

For $q > 1$, the coefficient $c_{2,0}$ is negative in any number of dimensions, indicating non-unitarity. The non-unitarity of the Coon amplitude with $q > 1$ has been known since the early 1970s [BC71]. The unitarity of the Coon amplitude with $q < 1$ is more subtle. This case was studied in the 1990s [FN95] and again more recently [FT22]. The most recent numerical studies indicate that the Coon amplitude with $q < 1$ is unitary below some q-dependent critical dimension [FT22]. This critical dimension is $d = 10$ in the limit $q \to 1$ and $d = \infty$ in the limit $q \to 0$.

Although the Coon amplitude with $q < 1$ may be unitary, it is non-meromorphic due to the factor $q^{\alpha_q(s)\alpha_q(t)}$. As we discussed above, this explicit non-meromorphic factor is necessary for the Coon amplitude to have polynomial residues. The Coon amplitude with $q < 1$ also has an accumulation point of poles at $\frac{1}{1-q}$. By definition, meromorphic functions can only have isolated poles. Thus, the infinite product itself is non-meromorphic even without the explicit non-meromorphic factor.

For $q > 1$ the situation is reversed. There the Coon amplitude is meromorphic with no accumulation point, but it is non-unitary. Only the Veneziano amplitude at $q = 1$ is both meromorphic and unitary.

1.4.4 High-energy

The high-energy behavior of the Coon amplitude may be calculated using the q-analog of Stirling's formula (1.33). In the Regge limit with fixed $t < 0$ and large $|s| \gg 1$ with phase $0 < \arg(s) < 2\pi$ (to avoid the poles of the q-gamma function as well as the branch cut for $0 < q < 1$), we find,

$$\mathcal{A}_q(s,t) \overset{|s|\to\infty}{\sim} (-s)^{\alpha_q(t)-1} \frac{\Gamma_q\big(-\alpha_q(t)\big)}{(q-1)t+1} \left[1 + \mathcal{O}\big((q-1)^{-1}s^{-1}\big)\right] \tag{1.43}$$

which agrees with the Regge limit (1.11) of the Veneziano amplitude as $q \to 1$ (ignoring the subtlety that the small parameter blows up at $q = 1$). For both $0 < q < 1$ and $q > 1$, the exponent $\alpha_q(t) = \ln(1 + (q-1)t)/\ln q$ can be made arbitrarily large and negative as a function of $t < 0$. For any scattered states with fixed polarizations, there is thus a range of fixed $t < 0$ such that $\lim_{|s|\to\infty} \mathcal{A}_{q\text{-strings}} = 0$ while this limit diverges in the corresponding field theory amplitude.

1.4.5 Low-energy

The low-energy expansion of the Coon amplitude with $q < 1$ was recently studied [FT22]. Here we extend that result to all $q \geq 0$. The details of our calculation are given in section 1.8.

Like the Veneziano amplitude, the Coon amplitude reproduces field theory at leading order. At higher order and for all $q > 0$, corrections to field theory are given in terms of the q-deformation $\mathrm{Li}_k(z;q)$ of the polylogarithm $\mathrm{Li}_k(z)$, which evaluates to the Riemann zeta function $\zeta(k)$ at $q = z = 1$ [Sch01],

$$\mathrm{Li}_k(z;q) = \sum_{n\geq 1} \frac{z^n}{\left(\frac{1-q^n}{1-q}\right)^k} \quad \xrightarrow[q\to 1]{} \quad \mathrm{Li}_k(z) = \sum_{n\geq 1} \frac{z^n}{n^k} \quad \xrightarrow[z\to 1]{} \quad \zeta(k) = \sum_{n\geq 1} \frac{1}{n^k} \tag{1.44}$$

The low-energy expansion includes in particular the q-deformed polylogarithms $\mathrm{Li}_k(q^j;q)$ with integers $k > j \geq 1$. For all $q \geq 0$, the defining sums for these special functions are absolutely convergent and finite,

$$\text{Li}_k(q^j;q) = \sum_{n\geq 1} \frac{q^{nj}}{\left(\frac{1-q^n}{1-q}\right)^k} \leq \begin{cases} \dfrac{q^j}{1-q^j} & q<1 \\ \zeta(k) & q=1 \\ \dfrac{q^j}{1-q^{j-k}} & q>1 \end{cases} \tag{1.45}$$

For $q < 1$, there is also a contribution from the non-meromorphic factor $q^{\alpha_q(s)\alpha_q(t)}$ which appears in (1.35) and (1.36). In total we find,

$$\begin{aligned} \mathcal{A}_q(s,t) &= \frac{1}{st} \exp \sum_{\ell_1,\ell_2\geq 1} \left\{ \Theta(1-q) \frac{(1-q)^{\ell_1+\ell_2}}{\ell_1\ell_2 \ln q} - \sum_{j=1}^{\ell_{\min}} d_{j;\ell_1,\ell_2} \, \text{Li}_{\ell_1+\ell_2}(q^j;q) \right\} s^{\ell_1} t^{\ell_2} \\ &= \frac{1}{st} - \left[\text{Li}_2(q;q) - \Theta(1-q) \tfrac{(1-q)^2}{\ln q} \right] - \left[\text{Li}_3(q;q) - \Theta(1-q) \tfrac{(1-q)^3}{2 \ln q} \right] (s+t) + \cdots \end{aligned} \tag{1.46}$$

where $\ell_{\min} = \min(\ell_1, \ell_2)$ and $d_{j;\ell_1,\ell_2}$ is the following rational number,

$$d_{j;\ell_1,\ell_2} = \sum_{i=j}^{\ell_{\min}} \frac{(\ell_1+\ell_2-i-1)!}{(\ell_1-i)!(\ell_2-i)!(i-j)!j!} (-)^{i-j} \tag{1.47}$$

The limit $q \to 1$ reproduces the low-energy expansion of the Veneziano amplitude (1.12) since $\text{Li}_{\ell_1+\ell_2}(1;1) = \zeta(\ell_1+\ell_2)$ and,

$$\sum_{j=1}^{\ell_{\min}} d_{j;\ell_1,\ell_2} = \frac{1}{\ell_1+\ell_2} \binom{\ell_1+\ell_2}{\ell_1} \tag{1.48}$$

for positive integers ℓ_1, ℓ_2.

The q-deformed polylogarithms, like the usual polylogarithms and the Riemann zeta-values, may be assigned a transcendental weight. If we assign weight k to $\text{Li}_k(q^j;q)$, then we must assign weight one to the factor $(1-q)$ since,

$$(1-q)\,\text{Li}_k(q^j;q) = \text{Li}_{k+1}(q^j;q) - \text{Li}_{k+1}(q^{j+1};q) \tag{1.49}$$

Under these assignments, each side of this equation has weight $k+1$. If we assign weight -1 to the Mandelstam variables as we did in the low-energy expansions of the Veneziano and

Virasoro amplitudes, then each term in the low-energy expansion of the Coon amplitude with $q \geq 1$ has uniform transcendental weight two, just like the Veneziano amplitude (1.12).

For $q < 1$, the transcendental structure is not as clear. In this case, the argument of the exponential (which should have transcendental weight zero) includes the terms,

$$\frac{1}{\ell_1 \ell_2 \ln q} (1-q)^{\ell_1+\ell_2}\, s^{\ell_1} t^{\ell_2} \tag{1.50}$$

The factor $(1-q)^{\ell_1+\ell_2}\, s^{\ell_1} t^{\ell_2}$ has weight zero under our previous assignments, but it is customary to assign weight one to logarithms. After all, the logarithm is just the weight-one polylogarithm,

$$\mathrm{Li}_1(z) = -\ln(1-z) \tag{1.51}$$

so that (1.50) naively has transcendental weight -1 rather than weight zero.

We are not, however, out of luck. We may write the reciprocal $1/\ell_1\ell_2$ in terms of finite harmonic sums,

$$H_1(k) = \sum_{n=1}^{k-1} \frac{1}{n} \qquad \Longrightarrow \qquad \frac{1}{\ell_1 \ell_2} = H_1(\ell_1\ell_2 + 1) - H_1(\ell_1 \ell_2) \tag{1.52}$$

We then assign transcendental weight one to the finite harmonic sums so that (1.50) has weight zero. This assignment is delicate. One should think of $H_1(k)$ not as its value for a single k (which is a rational number whose natural transcendental weight assignment is zero) but instead as a function of k to be inserted into an infinite series in k. For instance, $H_1(k)$ occurs in this manner in the double zeta-value $\zeta(\ell, 1)$,

$$\zeta(\ell, 1) = \sum_{n_1 > n_2 \geq 1} \frac{1}{n_1^{\ell} n_2} = \sum_{n \geq 2} \frac{H_1(n)}{n^{\ell}} \tag{1.53}$$

The standard weight assignments of $\zeta(\ell)$ and $\zeta(\ell, 1)$ are ℓ and $\ell + 1$, respectively, which justifies assigning weight one to the function $H_1(k)$. This assignment of non-zero transcendental weight to finite harmonic sums is familiar to the low-energy expansion of one-loop superstring amplitudes [DG19, DG22] and to loop amplitudes in $\mathcal{N} = 4$ supersymmetric quantum field theory [KL03, BF09].

Under these assignments, each term in the low-energy expansion of the Coon amplitude for all $q \geq 0$ has uniform transcendental weight two, in perfect analogy with the low-energy expansion of the Veneziano amplitude (1.12). For $q \geq 1$, the subtleties involving $\ln q$ and finite harmonic sums can be ignored.

1.5 The Virasoro-Coon amplitude?

In this section, we shall attempt to construct a q-deformed Virasoro or Virasoro-Coon amplitude in analogy with our interpretation of the Coon amplitude as a q-deformed Veneziano amplitude. Specifically, we shall try to construct an amplitude $\mathcal{A}_{q\text{-Vir}}(s,t,u)$ with the following properties:

- (s,t,u) crossing symmetry
- simple poles in each channel only at the q-integers $[n]_q$ with $n \geq 0$
- polynomial residues on the massive poles
- the field theory amplitude at $q = 0$ and the Virasoro amplitude at $q = 1$,

$$\begin{aligned} \mathcal{A}_{q\text{-Vir}}(s,t,u) &\quad \xrightarrow[q\to 0]{} \quad -\frac{1}{stu} \\ \mathcal{A}_{q\text{-Vir}}(s,t,u) &\quad \xrightarrow[q\to 1]{} \quad \mathcal{A}_{\text{Vir}}(s,t,u) \end{aligned} \tag{1.54}$$

- the low-energy expansion $-\frac{1}{stu}\big(1 + \mathcal{O}(s,t,u)\big)$

We shall first consider the location of the poles. For $q \neq 1$, a convergent infinite product which contains our desired sequence of poles in each channel is,

$$\prod_{n\geq 0} \frac{1}{\big(1-\hat{q}^{\,n-\alpha_q(s)}\big)\big(1-\hat{q}^{\,n-\alpha_q(t)}\big)\big(1-\hat{q}^{\,n-\alpha_q(u)}\big)} \tag{1.55}$$

where $\hat{q} = \min(q, q^{-1})$. This infinite product, and thus $\mathcal{A}_{q\text{-Vir}}$, is proportional to the following product of three q-gamma functions,

$$\Gamma_q\big(-\alpha_q(s)\big)\Gamma_q\big(-\alpha_q(t)\big)\Gamma_q\big(-\alpha_q(u)\big) \tag{1.56}$$

If we are to have a polynomial residue on each massive s-channel pole, then the infinite product of t-channel poles from $\Gamma_q\big(-\alpha_q(t)\big)$ must be canceled by an infinite product of zeroes. This cancellation can only be achieved by a function proportional to the ratio,

$$\frac{\Gamma_q\big(-\alpha_q(t)\big)}{\Gamma_q\big(1-\alpha_q(s)-\alpha_q(t)\big)} \tag{1.57}$$

The Coon amplitude achieves polynomial residues through the same cancellation. The further requirement that $\mathcal{A}_{q\text{-Vir}} = -\frac{1}{stu}$ at low-energy implies that all but t^{-1} should be canceled from $\Gamma_q\big(-\alpha_q(t)\big)$ on the massless pole at $s = 0$ for all $q \geq 0$. We may satisfy this condition by multiplying (1.57) by $q^{-\alpha_q(t)}$. Demanding (s,t,u) symmetry, we find that $\mathcal{A}_{q\text{-Vir}}$ must be proportional to,

$$q^{-\delta_q(s,t,u)}\,\frac{\Gamma_q\big(-\alpha_q(s)\big)\,\Gamma_q\big(-\alpha_q(t)\big)\,\Gamma_q\big(-\alpha_q(u)\big)}{\Gamma_q\big(1-\alpha_q(t)-\alpha_q(u)\big)\,\Gamma_q\big(1-\alpha_q(u)-\alpha_q(s)\big)\,\Gamma_q\big(1-\alpha_q(s)-\alpha_q(t)\big)} \tag{1.58}$$

where $\delta_q(s,t,u) = \alpha_q(s) + \alpha_q(t) + \alpha_q(u)$. At $q = 1$, this expression reproduces the Virasoro amplitude as desired.

Now on each massive s-channel pole, the factor $1/\Gamma_q\big(1-\alpha_q(t)-\alpha_q(u)\big)$ contributes an infinite product of zeroes in t, spoiling the polynomial residue. These zeroes do not appear if $q = 0$ or $q = 1$ because $\alpha_q(s) + \alpha_q(t) + \alpha_q(u) = 0$ when $q = 0$ or $q = 1$. For general q, the infinite product of zeroes must be canceled by an infinite product of poles, and this cancellation can only be achieved by a function proportional to the ratio,

$$\frac{\Gamma_q\big(\ell-\delta_q(s,t,u)\big)}{\Gamma_q\big(1-\alpha_q(t)-\alpha_q(u)\big)} \tag{1.59}$$

for some integer $\ell \geq 1$. While the factor $\Gamma_q\big(\ell-\delta_q(s,t,u)\big)$ cancels the infinite product of zeroes, it also introduces an infinite number of new poles, spoiling our initial assumption.

Despite this complication of additional poles, we shall proceed with $\ell = 1$. We now have the following ansatz for $\mathcal{A}_{q\text{-Vir}}$,

$$q^{-\delta_q(s,t,u)} \frac{\Gamma_q\big(-\alpha_q(s)\big)\,\Gamma_q\big(-\alpha_q(t)\big)\,\Gamma_q\big(-\alpha_q(u)\big)\,\Gamma_q\big(1-\delta_q(s,t,u)\big)}{\Gamma_q\big(1-\alpha_q(t)-\alpha_q(u)\big)\,\Gamma_q\big(1-\alpha_q(u)-\alpha_q(s)\big)\,\Gamma_q\big(1-\alpha_q(s)-\alpha_q(t)\big)} \tag{1.60}$$

which has the following convergent infinite product form for all $q \geq 0$,

$$-\frac{1}{stu}\prod_{n\geq 1} \frac{\big(1-\hat{q}^{\,n-\alpha_q(t)-\alpha_q(u)}\big)\big(1-\hat{q}^{\,n-\alpha_q(u)-\alpha_q(s)}\big)\big(1-\hat{q}^{\,n-\alpha_q(s)-\alpha_q(t)}\big)\big(1-\hat{q}^{\,n}\big)}{\big(1-\hat{q}^{\,n-\alpha_q(s)}\big)\big(1-\hat{q}^{\,n-\alpha_q(t)}\big)\big(1-\hat{q}^{\,n-\alpha_q(u)}\big)\big(1-\hat{q}^{\,n-\delta_q(s,t,u)}\big)} \tag{1.61}$$

where again $\hat{q} = \min(q, q^{-1})$. This ansatz reproduces the field theory amplitude at $q = 0$ and the Virasoro amplitude at $q = 1$. The residues of this ansatz, however, are not polynomials. Near the massive pole at $s = [N]_q$, (1.61) becomes,

$$-\frac{1}{[N]_q}\frac{1}{tu}\frac{1}{\big(1-\hat{q}^{\,N-\alpha_q(s)}\big)}\prod_{n=1}^{N-1}\frac{1}{\big(1-\hat{q}^{\,n-N}\big)}\prod_{n=1}^{N}\frac{\big(1-\hat{q}^{\,n-N-\alpha_q(t)}\big)\big(1-\hat{q}^{\,n-N-\alpha_q(u)}\big)}{\big(1-\hat{q}^{\,n-N-\alpha_q(t)-\alpha_q(u)}\big)} \tag{1.62}$$

After some straightforward algebra, we see that the residue at $s = [N]_q$ is a non-polynomial rational function of t unless $q = 0$ or $q = 1$.

We have thus failed to construct a q-deformed Virasoro amplitude under our stated assumptions. Therefore, we conclude that there is no amplitude with (s, t, u) symmetry, simple poles at the q-integers, and polynomial residues. Only the field theory amplitude at $q = 0$ (with no massive poles) and the Virasoro amplitude at $q = 1$ (with poles at the integers) satisfy our constraints. It seems then that there is no q-deformed Virasoro or Virasoro-Coon amplitude.

In our companion work [GL23], we revisit this question by analyzing so-called generalized Virasoro amplitudes, defined by a generalization of the infinite product representation of the Virasoro amplitude (1.16). In this analysis, we assume (s, t, u) symmetry and demand physical residues on an a priori unspecified sequence of poles λ_n. In other words, we do not assume a given mass spectrum as we have done in our search for a q-deformed Virasoro amplitude here. We find that the poles λ_n must satisfy an over-determined set of non-linear recursion relations. We then numerically demonstrate that the only consistent solution to these recursion relations is the string spectrum with $\lambda_n = n$.

1.6 Gamma function

In this section, we shall collect some well-known properties of the gamma function $\Gamma(z)$. The gamma function is a meromorphic function with poles at the non-positive integers, defined by the following integral,

$$\Gamma(z) = \int_0^\infty dx\, x^{z-1} e^{-x} \tag{1.63}$$

The gamma function obeys the functional equation,

$$\Gamma(z+1) = z\,\Gamma(z) \tag{1.64}$$

and the reflection formula,

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{1.65}$$

A useful infinite product representation of the gamma function is,

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^\infty \frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}} \tag{1.66}$$

A Taylor expansion for $\Gamma(1+z)$ with $|z|<1$ is given by,

$$\ln \Gamma(1+z) = -\gamma_E\, z + \sum_{k=2}^\infty \frac{\zeta(k)}{k} (-z)^k \tag{1.67}$$

where γ_E is the Euler-Mascheroni constant and $\zeta(k) = \sum_{n=1}^\infty n^{-k}$ are Riemann zeta-values. The asymptotic behavior of the gamma function is given by Stirling's formula,

$$\ln \Gamma(z) \overset{|z|\to\infty}{\sim} \left(z-\tfrac{1}{2}\right) \ln z - z + \tfrac{1}{2} \ln(2\pi) + \mathcal{O}\big(z^{-1}\big) \tag{1.68}$$

which is valid for large $|z| \gg 1$ with phase $|\arg(z)| < \pi$.

1.7 Gegenbauer polynomials

In this section, we shall review some properties of the Gegenbauer polynomials [Vil68, GR07]. The Gegenbauer polynomials may be defined by a generating function,

$$\frac{1}{(1-2xt+t^2)^\lambda} = \sum_{j=0}^\infty C_j^{(\lambda)}(x)\, t^j \tag{1.69}$$

or in terms of the hypergeometric function,

$$C_j^{(\lambda)}(x) = \tfrac{\Gamma(j+2\lambda)}{\Gamma(j+1)\Gamma(2\lambda)}\, {}_2F_1\big(-j,\, j+2\lambda;\, \lambda+\tfrac{1}{2};\, \tfrac{1}{2}(1-x)\big) \tag{1.70}$$

The first few Gegenbauer polynomials are,

$$\begin{aligned} C_0^{(\lambda)}(x) &= 1 \\ C_1^{(\lambda)}(x) &= 2\lambda\, x \\ C_2^{(\lambda)}(x) &= -\lambda + 2\lambda(1+\lambda)\, x^2 \\ C_3^{(\lambda)}(x) &= -2\lambda(1+\lambda)\, x + \tfrac{4}{3}\lambda(1+\lambda)(2+\lambda)\, x^3 \end{aligned} \tag{1.71}$$

In $d \geq 3$ spacetime dimensions, the polynomials $C_j^{(\frac{d-3}{2})}(\cos\theta)$ diagonalize the Lorentz group Casimir operator. In $d=3$ we must omit the normalization factor $\frac{\Gamma(j+2\lambda)}{\Gamma(j+1)\Gamma(2\lambda)}$ which vanishes. The case $d=4$ reduces to the familiar Legendre polynomials. The Gegenbauer polynomials obey an orthogonality relationship,

$$\int_{-1}^{1} dx\, (1-x^2)^{\lambda-\frac{1}{2}}\, C_j^{(\lambda)}(x)\, C_\ell^{(\lambda)}(x) = \begin{cases} \frac{\pi}{2^{2\lambda-1}(j+\lambda)} \frac{\Gamma(j+2\lambda)}{\Gamma(j+1)\Gamma(\lambda)^2} & j=\ell \\ 0 & j\neq \ell \end{cases} \tag{1.72}$$

and the following integration identity,

$$\int_{-1}^{1} dx\, (1-x^2)^{\lambda-\frac{1}{2}}\, C_j^{(\lambda)}(x)\, x^\ell = \begin{cases} \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(j+2\lambda)\Gamma(\ell+1)\Gamma(\frac{\ell-j+1}{2})}{2^j\,\Gamma(2\lambda)\Gamma(j+1)\Gamma(\ell-j+1)\Gamma(\frac{\ell+j}{2}+\lambda+1)} & j+\ell \text{ even} \\ 0 & j+\ell \text{ odd} \end{cases} \tag{1.73}$$

for integers $j, \ell \geq 0$. These two integrals may be used to write the residues of any tree-level four-point amplitude in terms of Gegenbauer polynomials. The product of two Gegenbauer polynomials may be expanded as,

$$C_{j_1}^{(\lambda)}(x)\, C_{j_2}^{(\lambda)}(x) = \sum_{\ell=|j_1-j_2|}^{j_1+j_2} c_{j_1,j_2;\ell}^{(\lambda)}\, C_\ell^{(\lambda)}(x) \tag{1.74}$$

for integers $j_1, j_2 \geq 0$, where,

$$c_{j_1,j_2;\ell}^{(\lambda)} = \begin{cases} \frac{(\ell+\lambda)\,\Gamma(\ell+1)\Gamma(g+2\lambda)}{\Gamma(\lambda)^2\Gamma(\ell+2\lambda)\Gamma(g+\lambda+1)} \frac{\Gamma(g-\ell+\lambda)}{\Gamma(g-\ell+1)} \frac{\Gamma(g-j_1+\lambda)}{\Gamma(g-j_1+1)} \frac{\Gamma(g-j_2+\lambda)}{\Gamma(g-j_2+1)} & j_1+j_2+\ell \text{ even} \\ 0 & j_1+j_2+\ell \text{ odd} \end{cases} \tag{1.75}$$

with $g = \frac{1}{2}(j_1+j_2+\ell)$. For $d \geq 4$, the coefficients $c_{j_1,j_2;\ell}^{(\frac{d-3}{2})} \geq 0$ are non-negative.

1.8 Deriving the Coon amplitude low-energy expansion

In this section, we shall derive the low-energy expansion (1.46) of the Coon amplitude for all $q \geq 0$. Our starting point is (1.36). The low-energy expansion of the factor $q^{\alpha_q(s)\alpha_q(t)}$ may be computed using the Taylor expansion for $\ln(1-z)$,

$$q^{\frac{\ln(1+(q-1)s)}{\ln q}\frac{\ln(1+(q-1)t)}{\ln q}} = \exp \sum_{\ell_1,\ell_2 \geq 1} \frac{(1-q)^{\ell_1+\ell_2}}{\ell_1 \ell_2 \ln q} s^{\ell_1} t^{\ell_2} \tag{1.76}$$

The low-energy expansion of the infinite product is similarly given by,

$$\begin{aligned}&\prod_{n\geq 1} \frac{\left(\frac{1-q^n}{1-q}\right)^2 - \left(\frac{1-q^n}{1-q}\right)(s+t) + (1-q^n)st}{\left(s - \frac{1-q^n}{1-q}\right)\left(t - \frac{1-q^n}{1-q}\right)} \\ &= \exp \sum_{n\geq 1}\sum_{k\geq 1} \frac{1}{k\left(\frac{1-q^n}{1-q}\right)^k}\left[s^k + t^k - \left(s+t+(q-1)st\right)^k\right]\end{aligned} \tag{1.77}$$

At this point we cannot interchange the sums over n and k and perform the sum over n because the resultant q-deformed polylogarithms $\mathrm{Li}_k(1;q)$ diverge for $q < 1$. Instead, we expand the summand using the multinomial theorem and collect powers of s and t to find,

$$\exp \sum_{n\geq 1}\sum_{\ell_1,\ell_2\geq 1}\sum_{i=0}^{\ell_{\min}} \frac{(\ell_1+\ell_2-i-1)!}{(\ell_1-i)!(\ell_2-i)!i!} \frac{(-)(q^n-1)^i}{\left(\frac{1-q^n}{1-q}\right)^{\ell_1+\ell_2}} s^{\ell_1} t^{\ell_2} \tag{1.78}$$

where $\ell_{\min} = \min(\ell_1, \ell_2)$. We now expand the factor $(q^n-1)^i$ and find,

$$\exp \sum_{n\geq 1}\sum_{\ell_1,\ell_2\geq 1}\sum_{j=1}^{\ell_{\min}} d_{j;\ell_1,\ell_2} \frac{(-)\,q^{nj}}{\left(\frac{1-q^n}{1-q}\right)^{\ell_1+\ell_2}} s^{\ell_1} t^{\ell_2} \tag{1.79}$$

with the rational numbers $d_{j;\ell_1,\ell_2}$ defined in (1.47). The $j=0$ terms vanish because,

$$d_{0;\ell_1,\ell_2} = \sum_{i=0}^{\ell_{\min}} \frac{(\ell_1+\ell_2-i-1)!}{(\ell_1-i)!(\ell_2-i)!i!}(-)^i = 0 \tag{1.80}$$

We may now interchange the order of the infinite sums and perform the sum over n because the resultant q-deformed polylogarithms $\mathrm{Li}_k(q^j;q)$ are absolutely convergent for all $q \geq 0$. Combining our results, we arrive at (1.46).

CHAPTER 2

Generalized Veneziano and Virasoro amplitudes

2.1 Introduction

This chapter is based on [GL23]. In this chapter, we shall search for consistent generalizations of the Veneziano amplitude [Ven68] and the Virasoro amplitude [Vir69] with zero Regge intercept. For simplicity we shall only consider the scattering of four massless bosonic states, in which case the tree-level open and closed superstring amplitudes respectively reduce to the Veneziano and Virasoro amplitudes with zero intercept. Both amplitudes may be written as infinite products with an infinite number of simple poles.

So-called *generalized Veneziano amplitudes* and *generalized Virasoro amplitudes* are defined by modifying these infinite products subject to some general physical constraints. The name generalized Veneziano amplitude originates in [FN95], and the Coon amplitude [Coo69] is one well-studied example. In chapter 1 (based on [GL22]), we detail the properties of the Veneziano, Virasoro, and Coon amplitudes, including their unitarity, high-energy behavior, low-energy expansion, and number theoretic properties.

Our present procedure is an extension and clarification of Coon's original argument [Coo69] and related work [FN95]. These previous studies only considered generalized Veneziano amplitudes, but we shall also consider the generalized Virasoro case. In either case, we simply assume crossing symmetry in the Mandelstam variables and demand physical residues on an a priori unspecified sequence of poles λ_n. In other words, we do not assume the mass spectrum of the theory. Under our assumptions, we find that the poles λ_n must satisfy an over-determined set of non-linear recursion relations. These recursion relations fix all the subsequent poles in terms of the first three poles and highly constrain the space of generalized Veneziano and generalized Virasoro amplitudes.

In the generalized Veneziano case, the recursion relations can be solved analytically. The solutions correspond to the Veneziano amplitude, the one-parameter family of Coon amplitudes, and a larger two-parameter family of amplitudes with an infinite tower of spins at each mass level. This two-parameter family of solutions has been previously identified [Coo69, FN95] but never systematically studied. Only the one-parameter sub-family of Coon amplitudes has been studied in detail [FT22, GL22]. In this chapter, we shall systematically analyze the entire two-parameter space of generalized Veneziano amplitudes. We also begin an initial study of the unitarity properties of this space

In the generalized Virasoro case, we numerically demonstrate that the only consistent solution to the aforementioned recursion relations is the string spectrum. That is, we do not find any consistent generalized Virasoro amplitudes beyond the Virasoro amplitude itself. We reached a similar, though less general, conclusion in [GL22] by failing to construct a generalization of the Virasoro amplitude with the same poles as the Coon amplitude (a so-called Virasoro-Coon amplitude).

The authors of [CR23] approach this same problem under a different set of assumptions and reach many of the same conclusions that we reach here, such as the uniqueness of the Virasoro amplitude. Our work is complementary.

Our approach is part of the modern S-matrix bootstrap program [CSZ21], a revival of an old approach [ELO66] which attempts to construct general amplitudes which satisfy various physical properties without relying on an underlying dynamical theory.

2.1.1 Conventions

In this chapter, we shall only consider crossing-symmetric tree-level scattering amplitudes for four massless external particles in weakly-coupled theories in $d \geq 3$ spacetime dimensions. We use units in which the lowest massive state of any particular theory has mass $m^2 = 1$. In open (closed) string theory, this choice corresponds to $\alpha' = 1$ ($\alpha' = 4$).

2.1.1.1 Kinematics

We shall primarily consider amplitudes stripped of their dependence on the polarizations or colors of the scattered states, leaving functions $\mathcal{A}(s_{ij})$ which depend only on the Mandelstam variables $s_{ij} = -(p_i + p_j)^2$,

$$\begin{aligned} s = s_{12} = s_{34} = & \quad 4E^2 & \geq 0 \\ t = s_{14} = s_{23} = & -2E^2(1-\cos\theta) & \leq 0 \\ u = s_{13} = s_{24} = & -2E^2(1+\cos\theta) & \leq 0 \end{aligned} \tag{2.1}$$

which satisfy the mass-shell relation $s + t + u = 0$. Here E and θ are the center-of-mass energy and scattering angle, respectively. The inequalities refer to the physical scattering regime with real s_{ij}.

2.1.1.2 Crossing

Crossing symmetry refers to permutation symmetry in (s, t) or (s, t, u). The Veneziano, Virasoro, and Coon amplitudes were discovered under the assumption of crossing symmetry, and we are searching for their generalizations.

Since s-channel and t-channel Feynman diagrams correspond to the same cyclic ordering, color-ordered amplitudes (e.g. gluon amplitudes) will have only s-channel and t-channel poles and shall be denoted by $\mathcal{A}(s, t)$ to emphasize that they are analytic functions of two complex variables. For these amplitudes, crossing symmetry is the requirement that $\mathcal{A}(s, t) = \mathcal{A}(t, s)$.

Amplitudes with poles in all three channels (e.g. graviton amplitudes) shall be denoted by $\mathcal{A}(s,t,u)$. We shall regard these amplitudes as analytic functions of three complex variables restricted to the algebraic variety defined by $s+t+u=0$. For these amplitudes, crossing symmetry is the requirement that $\mathcal{A}(s,t,u)=\mathcal{A}(\sigma(s),\sigma(t),\sigma(u))$ for any permutation σ of the variables (s,t,u).

2.1.1.3 Analytic structure

The amplitude $\mathcal{A}(s_{ij})$ is an analytic function of the complexified s_{ij} with simple poles and branch cuts dictated by unitarity. At high-energy, we demand that $\mathcal{A}(s_{ij})\to 0$ vanishes as $|s|\to\infty$ with physical t, in analogy with the high-energy behavior of the Veneziano, Virasoro, and Coon amplitudes [GL22].

Tree-level amplitudes have simple poles at $s_{ij}=m_n^2$ for each state n which couples to the external states through the s_{ij}-channel. It is often assumed that physical tree-level amplitudes are meromorphic, i.e. that $\mathcal{A}(s_{ij})$ is analytic outside its simple poles with no branch cuts or other singularities. However, the Coon amplitude with $q<1$ provides a counterexample of a seemingly healthy non-meromorphic tree-level amplitude [FT22, GL22].

In a physical four-point amplitude, the t-channel and u-channel poles should cancel on each s-channel pole (and vice versa). Typically, the residue of each s-channel pole is then a polynomial in t (after using the mass-shell relation to eliminate any u-dependence). The highest power of t in this residue corresponds to the highest-spin state exchanged on that pole. Non-polynomial residues can in principle result and may be Taylor expanded, corresponding to the exchange of an infinite tower of spinning states. In any case, the residues of these poles can be written as a sum of Gegenbauer polynomials and the amplitude may be written as follows (under the assumption that $\mathcal{A}(s_{ij})$ vanishes at high-energy [CKS17]),

$$\mathcal{A}(s_{ij})=\sum_n \frac{1}{s-m_n^2}\sum_j c_{n,j}\, C_j^{(\frac{d-3}{2})}(\cos\theta) \tag{2.2}$$

In a unitary theory, the partial wave coefficients $c_{n,j}>0$ will be positive.

We shall restrict our discussion to amplitudes with an infinite number of simple poles (à la Veneziano, Virasoro, and Coon) because the assumptions of crossing symmetry, physical residues, and that $\mathcal{A}(s_{ij})$ vanishes at high-energy imply that there must be an infinite number of poles in each channel [CEM16, CKS17]. The argument may be summarized as follows. A crossing-symmetric tree-level amplitude $\mathcal{A}(s,t) = \mathcal{A}(t,s)$ may be expanded on either its s-channel or t-channel poles, leading to the following equality,

$$\sum_n \frac{f_n(t)}{s - m_n^2} = \sum_n \frac{f_n(s)}{t - m_n^2} \tag{2.3}$$

The functions $f_n(z)$ must be finite at each $z = m_n^2$ because the t-channel poles should cancel on each s-channel pole. However, the left-hand side of (2.3) can then only produce the t-channel poles which appear on the right-hand side if the sum over n is infinite.

2.1.1.4 Accumulations points

In this chapter, we shall encounter two distinct notions of accumulation point spectra:

- infinite tower of masses $m_n^2 < \lambda_\infty$ with *finite spin exchange* at each mass level
- infinite tower of masses $m_n^2 < \lambda_\infty$ with *infinite spin exchange* at each mass level

for some finite accumulation point of masses $0 < \lambda_\infty < \infty$.

Finite spin exchange results from a polynomial residue on a given mass pole and corresponds to a finite tower of states at that mass level. The Coon amplitude with $q < 1$ exhibits this type of accumulation point spectrum with $\lambda_\infty = \frac{1}{1-q}$. While there is yet no definitive physical realization of the Coon amplitude, similar accumulation point spectra have been found in a stringy setup involving open strings ending on a D-brane [MR22]. Most famously, the hydrogen atom has a spectrum of this type with energy levels $E_n = -13.6\text{ eV}/n^2$ and an accumulation point at $E_\infty = 0$.

Infinite spin exchange results from a non-polynomial residue on a given mass pole and is generally considered unphysical. Indeed, sensible quantum field theories are typically assumed to have a finite number of particle types below any finite mass. In the case of finite spin exchange, this assumption only fails at masses $m^2 \geq \lambda_\infty$. In the case of infinite spin exchange, this assumption fails at all masses above the mass gap. Nevertheless, amplitudes with infinite spin exchange were recently considered in [HR22]. Moreover, amplitudes with this type of accumulation point were recently found to have interesting extremal properties in the context of the EFT-hedron [CV21, AHH21, HLR21, BKZ21].

In any case, amplitudes with either type of accumulation point spectra are not well understood and are fruitful examples for the study of general scattering amplitudes.

2.1.2 Outline

In section 2.2, we shall briefly review our conventions for the Veneziano, Virasoro, and Coon amplitudes. In section 2.3, we review some complex analysis and motivate our infinite product ansatz for the generalized Veneziano and generalized Virasoro amplitudes. In section 2.4 and section 2.5, we respectively analyze the generalized Veneziano and generalized Virasoro amplitudes by solving an infinite set of non-linear constraints on their poles λ_n. Finally, in section 2.6, we discuss our results and present some questions for future research.

2.2 Veneziano, Virasoro, and Coon amplitudes

The Veneziano, Virasoro, and Coon amplitudes are each tree-level four point amplitudes with an infinite sequence of simple poles and polynomial residues. A detailed review of their properties may be found in [GL22]. The Coon amplitude was also recently discussed in [FT22, CMM22, BDS22]. Here we shall briefly review our conventions and give each amplitude's infinite product representation.

2.2.1 Veneziano

The Veneziano amplitude $\mathcal{A}_{\text{Ven}}$ describes the scattering of four open strings and is a UV-completion of maximally supersymmetric Yang-Mills field theory. The color-stripped tree-level field theory amplitude which describes the scattering of any four massless particles in the Yang–Mills supermultiplet is,

$$\mathcal{A}_{\text{SYM}} = P_4 \, \frac{1}{st} \tag{2.4}$$

where $P_4 = \mathcal{O}(s,t)^2$ is a kinematic pre-factor. For the four-gluon amplitude, $P_4 = F^4$ where F is the linearized field strength. In tree-level open superstring theory, the color-stripped amplitude which describes the same process is,

$$\mathcal{A}_{\text{open}} = P_4 \, \mathcal{A}_{\text{Ven}} \tag{2.5}$$

where,

$$\mathcal{A}_{\text{Ven}}(s,t) = \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1-s-t)} = \frac{1}{st} \prod_{n\geq 1} \frac{\left(1 - \frac{s+t}{n}\right)}{\left(1 - \frac{s}{n}\right)\left(1 - \frac{t}{n}\right)} \tag{2.6}$$

Like the field theory factor $\frac{1}{st}$, the Veneziano amplitude is symmetric in (s,t) and is a meromorphic function with simple poles only.

2.2.2 Virasoro

The Virasoro amplitude $\mathcal{A}_{\text{Vir}}$ describes the scattering of four closed strings and is a UV-completion of maximal supergravity. The tree-level field theory amplitude which describes the scattering of any four massless particles in the supergravity multiplet is,

$$\mathcal{A}_{\text{SG}} = P_8 \Big(-\frac{1}{stu} \Big) \tag{2.7}$$

where $P_8 = \mathcal{O}(s,t,u)^4$ is a kinematic pre-factor. For the four-graviton amplitude, $P_8 = R^4$ where R is the linearized Riemann curvature. In tree-level closed superstring theory, the amplitude which describes the same process is,

$$\mathcal{A}_{\text{closed}} = P_8\,\mathcal{A}_{\text{Vir}} \tag{2.8}$$

where,

$$\mathcal{A}_{\text{Vir}}(s,t,u) = \frac{\Gamma(-s)\Gamma(-t)\Gamma(-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} = -\frac{1}{stu}\prod_{n\geq 1}\frac{\left(1+\frac{st+tu+us}{n^2}+\frac{stu}{n^3}\right)}{\left(1-\frac{s}{n}\right)\left(1-\frac{t}{n}\right)\left(1-\frac{u}{n}\right)} \tag{2.9}$$

Like the field theory factor $-\frac{1}{stu}$, the Virasoro amplitude is symmetric in (s,t,u) and is a meromorphic function with simple poles only.

2.2.3 Coon

The Coon amplitude $\mathcal{A}_q$ is a generalization of the Veneziano amplitude with a real-valued deformation parameter $q \geq 0$. This deformation moves the poles of the Veneziano amplitude from the integers to the q-integers,

$$[n]_q = \frac{1-q^n}{1-q} \quad \xrightarrow[q\to 1]{} \quad n \tag{2.10}$$

The Coon amplitude may be written as an infinite product with a q-dependent pre-factor,[1]

$$\begin{aligned}\mathcal{A}_q(s,t) = &\left\{ q^{\frac{\ln(1+(q-1)s)}{\ln q}\frac{\ln(1+(q-1)t)}{\ln q}}\,\Theta(1-q) + \Theta(q-1)\right\}\\ &\times \frac{1}{st}\prod_{n\geq 1}\frac{\left(1-\frac{s+t}{[n]_q}+(1-q)\frac{st}{[n]_q}\right)}{\left(1-\frac{s}{[n]_q}\right)\left(1-\frac{t}{[n]_q}\right)}\end{aligned} \tag{2.11}$$

where the step function is defined by $\Theta(x \geq 0) = 1$ and $\Theta(x < 0) = 0$. For $0 < q < 1$, the poles tend to an accumulation point at $\frac{1}{1-q}$. For $q \geq 1$, the poles tend to infinity. In the limits $q \to 0$ and $q \to 1$, the Coon amplitude reproduces the field theory factor and the Veneziano amplitude, respectively,

$$\begin{aligned}\mathcal{A}_q(s,t) \quad &\xrightarrow[q\to 0]{} \quad \frac{1}{st}\\ \mathcal{A}_q(s,t) \quad &\xrightarrow[q\to 1]{} \quad \mathcal{A}_{\text{Ven}}(s,t)\end{aligned} \tag{2.12}$$

[1] A more natural expression for the Coon amplitude may be given in terms of a special function called the q-deformed gamma function [GL22].

For all $q \geq 0$, the Coon amplitude is symmetric in (s, t) with simple poles only, but its meromorphicity is subtle. For $0 < q < 1$, the pre-factor in (2.11) is explicitly non-meromorphic with branch cuts at $s, t = \frac{1}{1-q}$. This pre-factor ensures that the Coon amplitude has polynomial residues. For $q \geq 1$, there is no pre-factor, and the Coon amplitude is meromorphic.

2.3 Infinite products and Weierstrass factorization

As we have seen in (2.6), (2.9), and (2.11), the Veneziano, Virasoro, and Coon amplitudes each have an infinite product form. Hence, we shall assume that more general tree-level scattering amplitudes with an infinite sequence of simple poles may be similarly written as infinite products. To motivate our ansatz for these generalized Veneziano and generalized Virasoro amplitudes, we shall first review some complex analysis.

2.3.1 Some complex analysis

Let $f : U \to \mathbb{C}$ be a function of one complex variable z on an open set $U \subset \mathbb{C}$. We first recall some standard definitions from single-variable complex analysis [FL12].

- $f(z)$ is *complex differentiable* at a point $z_0 \in U$ if $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists.
- $f(z)$ is *holomorphic* on U if it is complex differentiable on U.
- $f(z)$ is *meromorphic* on U if it is holomorphic on U except for a set of isolated points.
- $f(z)$ is *entire* if it is holomorphic on the full complex plane.
- $f(z)$ is *complex analytic* on U if for every $z_0 \in U$ it can be written as a convergent power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ with $a_n \in \mathbb{C}$.

It is a fundamental theorem of single-variable complex analysis that holomorphicity is equivalent to complex analyticity, but with tree-level scattering amplitudes in mind, we will be most interested in meromorphic functions.

A meromorphic function $f(z)$ may always be written as the ratio of two holomorphic functions and is characterized by its (possibly infinite) sequence of zeros ζ_n and poles λ_n (counted with multiplicity). If these sequences are finite, then $f(z)$ can be written as,

$$f(z) = z^m \, e^{g(z)} \, \frac{\prod_n (1 - z/\zeta_n)}{\prod_n (1 - z/\lambda_n)} \tag{2.13}$$

where $|m| \in \mathbb{N}$ is the order of the zero or pole at $z = 0$, $g(z)$ is an entire function (so that the factor $e^{g(z)}$ has no zeros or poles), and the two finite products run over the non-zero zeros and poles. We note that each numerator and denominator factor is separately linear in the variable z. This factorization is a consequence of the fundamental theorem of algebra.

If $f(z)$ is meromorphic but with an infinite number of zeros and poles, it will admit a similar factorization. In this case, we may formally combine the two finite products in (2.13) into one infinite product,

$$f(z) = z^m \, e^{g(z)} \, \prod_n \frac{(1 - z/\zeta_n)}{(1 - z/\lambda_n)} \tag{2.14}$$

which converges if the zeros ζ_n and poles λ_n obey,

$$\sum_n \left| \frac{1}{\zeta_n} - \frac{1}{\lambda_n} \right| < \infty \tag{2.15}$$

However, the formal product (2.14) need not converge.

A convergent product representation of any function $f(z)$ which is meromorphic on the full complex plane is given by the Weierstrass factorization theorem [FL12]. To ensure that this (possibly infinite) product converges, it is written in terms of the so-called elementary factors $E_\ell(z)$,

$$E_\ell(z) = \begin{cases} (1 - z) & \ell = 0 \\ (1 - z) \exp\left(\frac{z}{1} + \frac{z^2}{2} + \cdots + \frac{z^\ell}{\ell} \right) & \ell \geq 1 \end{cases} \tag{2.16}$$

Using the elementary factors, it is always possible to find sequences $N_n, D_n \in \mathbb{N}$ and an entire function $\tilde{g}(z)$ such that,

$$f(z) = z^m \, e^{\tilde{g}(z)} \, \prod_n \frac{E_{N_n}(z/\zeta_n)}{E_{D_n}(z/\lambda_n)} \tag{2.17}$$

where again $|m| \in \mathbb{N}$ is the order of the zero or pole at $z = 0$. Such a factorization always exists but is not unique. For any Weierstrass factorization, the pre-factor $e^{\tilde{g}(z)}$ will have neither zeros nor poles.

If, in fact, the formal product (2.14) converges, then the entire functions $g(z)$ and $\tilde{g}(z)$ which appear in (2.14) and the general Weierstrass factorization (2.17) are related by,

$$\exp\big(g(z) - \tilde{g}(z)\big) = \exp\left\{\sum_n \sum_{\ell=0}^{N_n} \frac{(z/\zeta_n)^\ell}{\ell} - \sum_n \sum_{\ell=0}^{D_n} \frac{(z/\lambda_n)^\ell}{\ell}\right\} \tag{2.18}$$

where the sums $\sum_n$ are over the non-zero zeros and poles.

For example, the gamma function $\Gamma(z)$ has the following infinite product representation in Weierstrass form,

$$\Gamma(z) = e^{-\gamma_E z}\,\frac{1}{z}\prod_{n\geq 1}\frac{1}{\big(1+z/n\big)e^{-z/n}} \tag{2.19}$$

where γ_E is the Euler-Mascheroni constant and the denominators are just the elementary factors $E_1(-z/n)$. This infinite product clearly demonstrates the simple poles of the gamma function at the non-positive integers, but without the factors $e^{-z/n}$ (from the elementary factors) the product would diverge.

The Weierstrass factorization theorem is no longer applicable if the zeros and poles are bounded and tend to a common limit point $\zeta_\infty = \lambda_\infty < \infty$ because then $f(z)$ is no longer meromorphic at $z = \lambda_\infty$. In this case, however, we may still factorize the function $f(z)$ in a form analogous to (2.14). Such a factorization will not need elementary factors because the elementary factors do not improve the convergence of the infinite product in the case that $\zeta_\infty = \lambda_\infty < \infty$. Moreover, the pre-factor $e^{g(z)}$ in this factorization will have no zeros or poles of finite order but may have essential singularities, branch points, etc. since in this case the function $f(z)$ is not meromorphic at $z = \lambda_\infty$.

2.3.2 Ansatz for infinite product amplitudes

We shall now use the infinite product factorization described above to motivate the ansatz for our generalized Veneziano and generalized Virasoro amplitudes. While a Weierstrass factorization (2.17) necessarily exists for all meromorphic functions of a single complex variable, there is no analogous theorem for functions of several complex variables. Moreover, while tree-level amplitudes are expected to be meromorphic functions of the Mandelstam variables, this need not be true. The Coon amplitude with $q < 1$ is non-meromorphic. Thus, we shall proceed without assuming meromorphicity. Instead, we shall simply write down and analyze an infinite product ansatz analogous to (2.14).

We begin with the generalized Veneziano case. Our crossing symmetric tree-level generalized Veneziano amplitude $\mathcal{A}(s,t)$ should have an infinite sequence of simple poles λ_n in both the s-channel and the t-channel. We shall assume that the leading poles are at $s = 0$ and $t = 0$ and that the amplitude reduces to field theory at low-energy,

$$\mathcal{A}(s,t) = \frac{1}{st}\big(1 + \mathcal{O}(s,t)\big) \tag{2.20}$$

Without loss of generality, we assume the poles are ordered $\lambda_n > \lambda_{n-1}$ and choose $\lambda_1 = 1$. These assumptions can always be made true by a relabeling of the poles and a choice of units. Beyond these assumptions, the poles are wholly unspecified. In addition to its poles, $\mathcal{A}(s,t)$ will have an infinite sequence of t-dependent s-zeros $\zeta_n(t)$ and an identical sequence of s-dependent t-zeros $\zeta_n(s)$.

Ignoring issues of convergence momentarily, we shall consider the following ansatz which satisfies these constraints and resembles the infinite product representation of the Veneziano amplitude (2.6),

$$\mathcal{A}(s,t) = \mathcal{W}(s,t)\,\frac{1}{st}\prod_{n\geq 1}\frac{1 - A_n(s+t) + B_n st}{(1 - s/\lambda_n)(1 - t/\lambda_n)} \tag{2.21}$$

where A_n and B_n are yet undetermined coefficients and the pre-factor $\mathcal{W}(s,t) = \mathcal{W}(t,s)$ has neither zeros nor poles below the largest mass pole, i.e. for $|s|, |t| < \lambda_\infty$ where λ_∞ may be finite or infinite. The pre-factor $\mathcal{W}(s,t)$ is analogous to the pre-factor $e^{g(z)}$ in (2.14) and has the low-energy behavior $\mathcal{W}(s,t) = 1+\mathcal{O}(s,t)$. We note that the numerator and denominator of the infinite product in (2.21) are both separately linear in s and t so that the zeros and poles in either channel can be written as,

$$\frac{1}{s}\prod_{n\geq 1}\frac{\big(1-s/\zeta_n(t)\big)}{(1-s/\lambda_n)} \qquad \text{or} \qquad \frac{1}{t}\prod_{n\geq 1}\frac{\big(1-t/\zeta_n(s)\big)}{(1-t/\lambda_n)} \tag{2.22}$$

with the zeros given by,

$$\zeta_n(x) = \frac{1\ \ - A_n x}{A_n - B_n x} \tag{2.23}$$

In this form, the amplitude resembles the Weierstrass factorization (2.17) but without the elementary factors. The formal product in (2.21) converges if the coefficients A_n and B_n and the poles λ_n obey,

$$\sum_{n\geq 1}\left|A_n - \frac{1}{\lambda_n}\right| < \infty \qquad\qquad \sum_{n\geq 1}\left|B_n - \frac{1}{\lambda_n^2}\right| < \infty \tag{2.24}$$

We shall return to this ansatz in section 2.4.

We now consider the generalized Virasoro case. Our crossing symmetric tree-level generalized Virasoro amplitude $\mathcal{A}(s,t,u)$ should have an infinite sequence of simple poles λ_n in the s-channel, t-channel, and u-channel. We shall again assume that the leading poles are at $s=0$, $t=0$, and $u=0$ and that the amplitude reduces to field theory at low-energy,

$$\mathcal{A}(s,t,u) = -\frac{1}{stu}\big(1+\mathcal{O}(s,t,u)\big) \tag{2.25}$$

Without loss of generality, we assume the poles are ordered $\lambda_n > \lambda_{n-1}$ and choose $\lambda_1 = 1$. Beyond these assumptions, the poles are again wholly unspecified.

Again momentarily ignoring issues of convergence, we shall consider the following ansatz which satisfies these constraints and resembles the infinite product representation of the Virasoro amplitude (2.9),

$$\mathcal{A}(s,t,u) = \mathcal{W}(s,t,u)\left(-\frac{1}{stu}\right)\prod_{n\geq 1}\frac{1 + A_n(st+tu+us) - B_n stu}{(1-s/\lambda_n)(1-t/\lambda_n)(1-u/\lambda_n)} \tag{2.26}$$

where A_n and B_n are yet undetermined coefficients and the (s,t,u)-symmetric pre-factor $\mathcal{W}(s,t,u)$ has neither zeros nor poles below the largest mass pole, i.e. for $|s|,|t|,|u| < \lambda_\infty$ where λ_∞ may again be finite or infinite. As before, the pre-factor $\mathcal{W}(s,t,u)$ is analogous to the pre-factor $e^{g(z)}$ in (2.14) and has the low-energy behavior $\mathcal{W}(s,t,u) = 1 + \mathcal{O}(s,t,u)$. We note that the numerator and denominator of the infinite product in (2.26) are both separately linear in s, t, and u. Moreover, there is no term proportional to $s+t+u$ in the numerator because this combination vanishes on-shell for massless external states. The formal product in (2.26) converges if the coefficients A_n and B_n and the poles λ_n obey,

$$\sum_{n\geq 1}\left|A_n - \frac{1}{\lambda_n^2}\right| < \infty \qquad\qquad \sum_{n\geq 1}\left|B_n - \frac{1}{\lambda_n^3}\right| < \infty \tag{2.27}$$

We shall return to this ansatz in section 2.5.

In both the generalized Veneziano and generalized Virasoro case, demanding that the t-channel poles cancel on each s-channel pole will enforce strong constraints on the undetermined coefficients A_n and B_n as well as the poles λ_n. In the following two sections, we shall analyze these constraints in detail.

2.4 Generalized Veneziano amplitudes

In this section, we shall systematically analyze our infinite product ansatz (2.21) for the generalized Veneziano amplitude.

2.4.1 Veneziano truncation

We first recall the infinite product form (2.6) of the Veneziano amplitude, which has simple poles at each non-negative integer. The residue of the massless s-channel pole is $1/t$, and the residue of each massive pole at $s = N$ is a polynomial of degree-$(N-1)$ in t. The Veneziano amplitude achieves these residues because on each s-pole, its zeros cancel the t-poles, leaving a finite polynomial in t. This cancellation can be described in terms of the numerator factors,

$$\mathcal{N}_n(s,t) = 1 - (s+t)/n \tag{2.28}$$

When $s = N$, each numerator factor $\mathcal{N}_{N+n}(N,t) = \frac{n}{N+n}(1 - t/n)$ cancels the t-channel pole from the factor $(1 - t/n)^{-1}$, and the infinite product truncates. In short, the condition,

$$\mathcal{N}_{N+n}(N,n) = 0 \tag{2.29}$$

ensures that the Veneziano amplitude has polynomial residues.

2.4.2 Generalized Veneziano truncation

We now return to our generalized Veneziano ansatz (2.21). We shall demand that the zeros and poles of this amplitude cancel in a similar fashion as those of the Veneziano amplitude. We first demand that the residue at $s = 0$ is $1/t$ so that the amplitude reproduces the massless spectrum of super Yang-Mills analogously to the Venziano amplitude,

$$\operatorname*{Res}_{s=0} \mathcal{A}(s,t) = \frac{1}{t} \qquad \Longrightarrow \qquad \mathcal{W}(0,t) \prod_{n\geq 1} \frac{1 - A_n t}{1 - t/\lambda_n} = 1 \tag{2.30}$$

which implies that $\mathcal{W}(0,t) = 1$ and $A_n = 1/\lambda_n$ since $\mathcal{W}(s,t)$ has neither zeros nor poles. In other words, the coefficients A_n are determined by the poles λ_n.

Next, in analogy with the truncation condition for the Veneziano amplitude (2.29), we demand that the generalized numerator factor,

$$\mathcal{N}_n(s,t) = 1 - A_n(s+t) + B_n st \tag{2.31}$$

obeys the generalized truncation condition,

$$\mathcal{N}_{N+n}(\lambda_N, \lambda_n) = 0 \tag{2.32}$$

so that $\mathcal{N}_{N+n}(\lambda_N, t) \propto (1 - t/\lambda_n)$ and the infinite sequence of t-channel poles cancels on each s-channel pole.[2] This truncation condition determines the coefficients B_n in terms of the

[2] At this point, we are no longer considering the most general possible infinite product amplitude but are instead working in close analogy with the Veneziano amplitude. A more general truncation condition, $\mathcal{N}_{N+n+\alpha}(\lambda_N, \lambda_n) = 0$ for some positive integer α, is considered in [CR23].

poles λ_n,

$$B_n = \frac{\lambda_k + \lambda_{n-k} - \lambda_n}{\lambda_n \lambda_{n-k} \lambda_k} \qquad\qquad k = 1, 2, \ldots, n-1 \tag{2.33}$$

For fixed $n \geq 2$, both k and $k' = n - k$ yield the same equation for B_n so that there are $\lfloor \frac{n}{2} \rfloor$ independent equations for B_n, where $\lfloor x \rfloor$ is the floor function. The coefficient B_1 is left undetermined, the coefficients B_2 and B_3 are uniquely determined, and the coefficients B_n with $n \geq 4$ are all over-determined.

This over-determination of the B_n highly constrains the poles. Any sequence of poles λ_n must leave the following combination independent of k for all $n \geq 2$,

$$\Lambda_n(k) = \frac{\lambda_k + \lambda_{n-k} - \lambda_n}{\lambda_{n-k} \lambda_k} \tag{2.34}$$

We shall refer to these equations as the generalized Veneziano amplitude constraints. The Veneziano solution $\lambda_n = n$ (i.e. the string theory spectrum) solves these constraints with $\Lambda_n(k) = 0$ for all n and k. We shall search for other, more general sequences of poles λ_n which solve the generalized Veneziano amplitude constraints.

2.4.3 Generalized Veneziano amplitude constraints

Since $\Lambda_n(k)$ must be independent of k, we may fix $n \geq 2$ and choose two distinct values of (k, k') in the appropriate range to find,

$$\Lambda_n(k) = \Lambda_n(k') \qquad \Longrightarrow \qquad \frac{\lambda_k + \lambda_{n-k} - \lambda_n}{\lambda_{n-k} \lambda_k} = \frac{\lambda_{k'} + \lambda_{n-k'} - \lambda_n}{\lambda_{n-k'} \lambda_{k'}} \tag{2.35}$$

This equation is a non-linear recursion relation for the poles λ_n of order $\max(k, k')$ which determines all the λ_n with $n > \max(k, k')$ in terms of the lower λ_n (except for $n = k + k'$, in which case the equation is vacuous). Because we are free to choose (k, k') within the appropriate range, the poles λ_n must solve an infinite set of these non-linear recursion relations. This system is highly constrained, and there is no guarantee that a general solution (other than the Veneziano solution) exists!

It turns out, however, that from (2.35) we can derive a simple first-order recursion relation which determines all the poles λ_n with $n \geq 4$ in terms of λ_1, λ_2, and λ_3. We consider the following three equations for fixed $n \geq 4$,

$$\Lambda_n(1) = \Lambda_n(2) \qquad \Lambda_n(1) = \Lambda_n(3) \qquad \Lambda_{n-1}(1) = \Lambda_{n-1}(2) \tag{2.36}$$

These three equations include the poles λ_1, λ_2, λ_3, λ_{n-3}, λ_{n-2}, λ_{n-1}, and λ_n, but we may eliminate λ_{n-3} and λ_{n-2} to find the following first order recursion relation for λ_n in terms of only λ_1, λ_2, λ_3, and λ_{n-1},

$$\lambda_n = \frac{a\lambda_{n-1} + b}{c\lambda_{n-1} + d} \tag{2.37}$$

where the coefficients a, b, c, and d are given by,

$$\begin{aligned}
a &= \lambda_2(1 - 2\lambda_3 + \lambda_2\lambda_3) = (1+x)(x^2 + xy - y) \\
b &= \lambda_2(\lambda_3 - \lambda_2) = (1+x)y > 0 \\
c &= 1 + \lambda_2^2 - \lambda_2 - \lambda_3 = x^2 - y \\
d &= \lambda_2(\lambda_3 - \lambda_2) = (1+x)y > 0
\end{aligned} \tag{2.38}$$

Here we have defined the positive numbers $x = \lambda_2 - \lambda_1 = \lambda_2 - 1 > 0$ and $y = \lambda_3 - \lambda_2 > 0$, using the fact that the poles $\lambda_n > \lambda_{n-1}$ are ordered. The recursion relation (2.37) was derived for $n \geq 4$ but is in fact true for all $n \geq 1$ if we define $\lambda_0 = 0$. For $n = 1, 2, 3$, (2.37) is only vacuously true and does not determine λ_1, λ_2, or λ_3. The choice $\lambda_1 = 1$ simply sets our units, and the free parameters λ_2 and λ_3 (or equivalently x and y), define a two-parameter space of possible solutions in the region $x, y > 0$. The string spectrum $\lambda_n = n$ is at the point $x = y = 1$ of this two-parameter space.

2.4.4 Solving the Riccati relation

The recursion relation (2.37) is known as the Riccati recursion relation with constant coefficients, and its solutions are well known. An exhaustive study of non-linear recursion relations of this kind may be found in [KL93].

Although the Riccati recursion relation (2.37) is generally non-linear, there is a curve $c = x^2 - y = 0$ in parameter space where it becomes linear,

$$\lambda_n = x\lambda_{n-1} + 1 \tag{2.39}$$

and yields the Coon spectrum (relabeling $x \to q$),

$$\lambda_n = \frac{1 - q^n}{1 - q} \tag{2.40}$$

The Coon spectrum reproduces the string spectrum at $q = 1$ and accounts for all the spectra reviewed in section 2.2. For $q > 1$, the poles grow exponentially, and for $0 < q < 1$, they monotonically accumulate to the limit point $\lambda_\infty = \frac{1}{1-q}$.

Beyond these well-studied solutions, there is, however, a much larger space of solutions to (2.37) with $c = x^2 - y \neq 0$. In this case, the non-linear first-order recursion relation (2.37) can be reduced to a linear second-order recursion relation using the following change of variables,

$$c\lambda_n + d = (a + d)\frac{z_{n+1}}{z_n} \tag{2.41}$$

with the boundary condition $z_0 = 1$. Substituting this expression into (2.37), we find,

$$z_{n+2} - z_{n+1} + R\, z_n = 0 \tag{2.42}$$

with the positive coefficient R given by,

$$R = \frac{ad - bc}{(a + d)^2} = \frac{y}{(1 + x)(x + y)} > 0 \tag{2.43}$$

The solutions of this linear recursion relation are governed by the quadratic equation,

$$r^2 - r + R = 0 \tag{2.44}$$

whose roots are,

$$r_\pm = \frac{1 \pm \sqrt{1 - 4R}}{2} \tag{2.45}$$

We shall separately analyze the cases $R \neq \frac{1}{4}$ and $R = \frac{1}{4}$.

2.4.4.1 The case $R \neq \frac{1}{4}$

If $R \neq \frac{1}{4}$, then the roots $r_+ \neq r_-$ are distinct and z_n is given by,

$$z_n = \frac{z_1 - r_-}{r_+ - r_-} r_+^n + \frac{r_+ - z_1}{r_+ - r_-} r_-^n \tag{2.46}$$

Subsequently, λ_n is given by,

$$\lambda_n = \frac{(1+x)(1-p^n)}{(1-xp) - (1-x/p)p^n} \tag{2.47}$$

with $p = r_-/r_+$ so that $|p| \leq 1$. We shall refer to these solutions as p-type spectra. These spectra were first identified in [Coo69] and were later called Möbius trajectories in [FN95]. Our parametrization in terms of p is novel and can be clearly related to the first three mass levels through the parameters x and y since,

$$p = p(x,y) = \frac{1 - \sqrt{1 - 4y/(1+x)(x+y)}}{1 + \sqrt{1 - 4y/(1+x)(x+y)}} \tag{2.48}$$

When $p = x$ or $p = x^{-1}$, these spectra reduce to the Coon solution (2.40) with $q = p < 1$ or $q = p^{-1} > 1$, respectively.

The expression (2.47) solves the generalized Veneziano amplitude constraints (2.34) for all $x, y > 0$, but the resultant λ_n will not necessarily be monotonically ordered and positive. We shall now determine the values of x and y which yield a monotonically increasing sequence of poles λ_n.

We first note that $R > \frac{1}{4}$ implies that the parameter $p = r_-/r_+ = e^{i\phi}$ is a phase so that the λ_n are periodic as a function of n. These periodic solutions always produce negative (and thus unphysical) λ_n. The condition $R > \frac{1}{4}$ is equivalent to $y > \frac{x(1+x)}{3-x}$, so this region of parameter space is ruled out.

We now consider $0 < R < \frac{1}{4}$ which corresponds to $0 < y < \frac{x(1+x)}{3-x}$. In this case, the roots $r_\pm$ are real and positive, so the parameter p is in the range $0 < p < 1$. To determine whether the λ_n increase monotonically in this region, we shall momentarily treat n as a continuous variable so that $\lambda_n \to \lambda(n)$ is an analytic function of n with a discrete set of singularities at the points $n = n_*$ on the complex n-plane,

$$n_* = \frac{\ln\left(\frac{1-xp}{1-x/p}\right)}{\ln p} - \frac{2\pi i k}{\ln p} \tag{2.49}$$

with $k \in \mathbb{Z}$. Since $\frac{d}{dn}\lambda(n) > 0$ for all real $n \geq 0$, the function $\lambda(n)$ can only fail to be monotonic if there is a singularity $n_* > 0$ on the real n-axis such that $\lim_{n \to n_*^{\mp}} \lambda(n) = \pm\infty$. From (2.49), we see that there is at most one such singularity with $k = 0$, which occurs if and only if,

$$0 < \frac{1-xp}{1-x/p} < 1 \tag{2.50}$$

We first suppose that (2.50) is satisfied with $1 - xp > 0$ and $1 - x/p > 0$, which then implies $1 - xp < 1 - x/p$ and thus $p > 1$. Since $0 < p < 1$, we must instead have $1 - xp < 0$ and $1 - x/p < 0$. To proceed, we shall separately consider the cases $0 < x < 1$ and $x \geq 1$.

- For $0 < x < 1$, it is not possible to fulfill the condition $1 - xp < 0$, so the whole region corresponding to $0 < R < \frac{1}{4}$ and $0 < x < 1$ yields monotonically increasing and positive λ_n.

- For $x \geq 1$, the condition $1 - x/p < 0$ is always satisfied, but $1 - xp < 0$ implies,

$$x > \frac{1}{p(x,y)} \quad \Longrightarrow \quad \left(\frac{x-1}{x+1}\right)^2 > \frac{x(1+x) - (3-x)y}{(1+x)(x+y)} \quad \Longrightarrow \quad y > x^2 \tag{2.51}$$

  Therefore, when either $1 \leq x < 3$ and $x^2 < y < \frac{x(1+x)}{3-x}$ or when $x \geq 3$ and $y > x^2$, the function $\lambda(n)$ has a singularity at finite $n = n_* > 0$ and is not monotonic. Moreover, in this region, the limit point $\lambda_\infty = \frac{1+x}{1-xp} < 0$ is negative and thus non-physical.

2.4.4.2 The case $R = \frac{1}{4}$

If $R = \frac{1}{4}$, then the roots $r_+ = r_-$ are equal and z_n is given by,

$$z_n = \frac{1}{2^n}\left(1 + \frac{1-x}{2x}n\right) \tag{2.52}$$

Subsequently, λ_n is given by,

$$\lambda_n = \frac{(1+x)n}{2x+(1-x)n} \tag{2.53}$$

We shall refer to these solutions as r-type spectra (where the r is for rational). The expression (2.53) solves the generalized Veneziano amplitude constraints (2.34) for all $x > 0$, but the resultant λ_n will only be monotonically ordered and positive for $0 < x \leq 1$. When $x > 1$, the limit point $\lambda_\infty = \frac{1+x}{1-x} < 0$ is negative and thus non-physical. When $x = 1$, this solution reduces to the string spectrum $\lambda_n = n$. Notably, these r-type solutions were not identified in the previous literature [Coo69, FN95].

2.4.4.3 Summary

We have now fully classified all the monotonically ordered and positive solutions of the generalized Veneziano amplitude constraints (2.34). These solutions exist in the region of the xy-plane defined by,

$$\left\{0 < x < 1 \ , \ 0 < y \leq \tfrac{x(1+x)}{3-x}\right\} \cup \left\{1 \leq x \ , \ 0 < y \leq x^2\right\} \tag{2.54}$$

where again $x = \lambda_2 - \lambda_1 > 0$ and $y = \lambda_3 - \lambda_2 > 0$ are positive parameters which determine the second and third masses. This region and the various solutions are shown in Figure 2.1. Notably, all the non-monotonically-ordered solutions to the Riccati equation (2.37), i.e. the points within the excluded regions of parameter space, yield negative λ_n and are unphysical.

For completeness, we shall rewrite all the solutions and the ranges of their parameters. The Coon spectra have one free parameter and are given by,

$$\lambda_n = \frac{1-q^n}{1-q} \qquad\qquad 0 < q < \infty \tag{2.55}$$

where q is related to the parameters x and y by $x = q$ and $y = q^2$. The p-type spectra have two free parameters and are given by,

$$\lambda_n = \frac{(1+x)(1-p^n)}{(1-xp)-(1-x/p)p^n} \qquad 0 < x < \infty \qquad 0 < p < \min(1, x^{-1}) \qquad p \neq x, x^{-1} \tag{2.56}$$

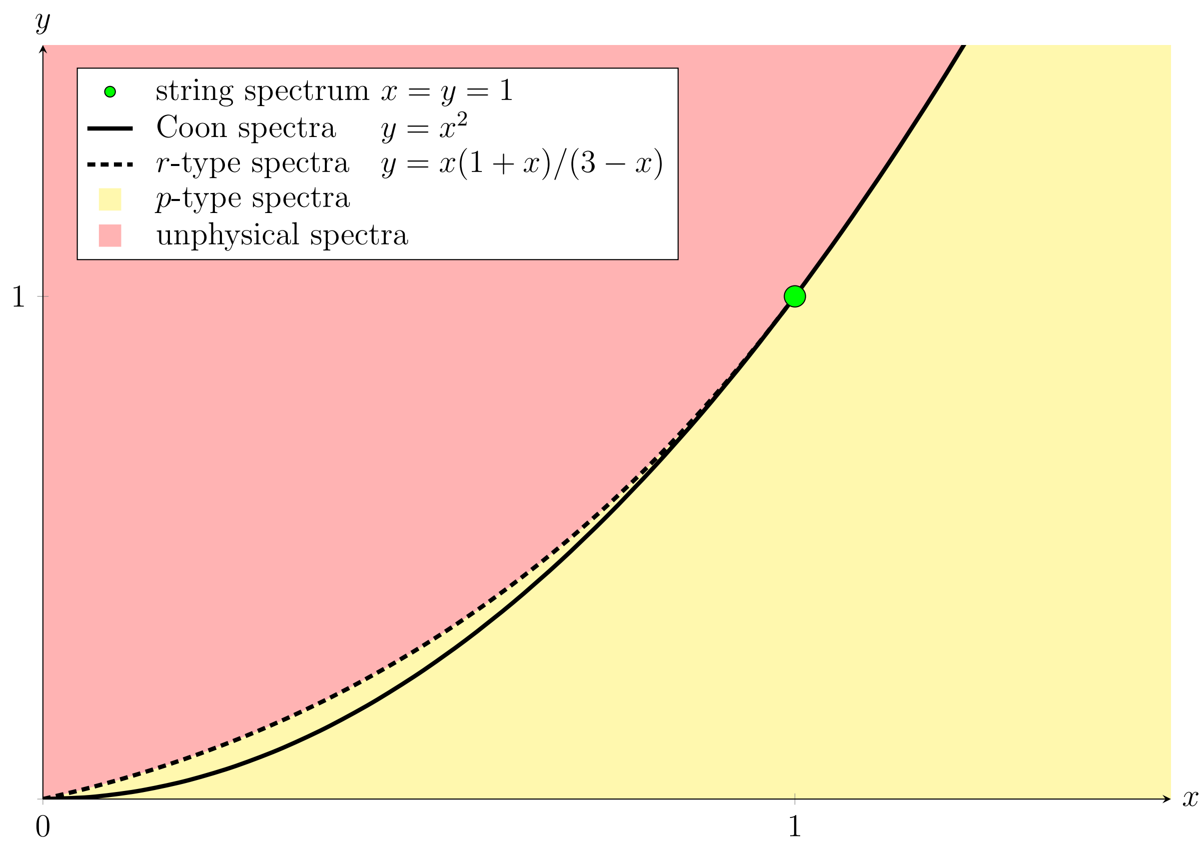

Figure 2.1: The two-parameter space of solutions to the generalized Veneziano amplitude constraints. The point $x = y = 1$ corresponds to the string spectrum. The solid black line corresponds to the one-parameter subspace of Coon spectra. The dashed black line corresponds to the one-parameter subspace of r-type spectra. The yellow region corresponds to the two-parameter subspace of p-type spectra. The red region corresponds to unphysical spectra with negative mass squared.

where we have excluded $p = x, x^{-1}$ to avoid double-counting the Coon spectra. Finally, the r-type spectra have one free parameter and are given by,

$$\lambda_n = \frac{(1+x)n}{2x + (1-x)n} \qquad\qquad 0 < x < 1 \tag{2.57}$$

The string spectrum $\lambda_n = n$ is located at $x = y = 1$ in parameter space and can be obtained by taking various limits of each of these solutions. All of these spectra have a finite accumulation point λ_∞, except for the Coon spectra (2.55) with $q \geq 1$ (which includes the string spectrum at $q = 1$).

Although we derived these solutions from the Riccati relation (2.37), they do in fact satisfy the full generalized Veneziano amplitude constraints (2.34). For each case, we may compute $\Lambda_n(k)$ and verify that it is independent of k. Since any solution of the generalized Veneziano amplitude constraints (2.34) necessarily satisfies the Riccati relation (2.37), we have thus fully solved (2.34). Explicitly, the $\Lambda_n(k)$ are given by,

$$\begin{aligned}
\text{Coon}: &\qquad \Lambda_n(k) = 1 - q \\
p\text{-type}: &\qquad \Lambda_n(k) = \frac{1}{(1+x)} \cdot \frac{(1-xp)^2 - (1-x/p)^2 p^n}{(1-xp) \ - (1-x/p) \ p^n} \\
r\text{-type}: &\qquad \Lambda_n(k) = \frac{1-x}{1+x} \cdot \frac{4x + (1-x)n}{2x + (1-x)n}
\end{aligned} \tag{2.58}$$

These three expression can be written in a universal form,

$$\Lambda_n(k) = \frac{1}{\lambda_\infty} + \frac{1}{\lambda_{-\infty}} - \frac{\lambda_n}{\lambda_\infty \lambda_{-\infty}} \tag{2.59}$$

where we have defined the (possibly infinite) quantities $\lambda_{\pm\infty} = \lim_{n\to\pm\infty} \lambda_n$. We have,

$$\begin{aligned}
\text{Coon}\,(q<1): &\qquad \frac{1}{\lambda_\infty} = 1 - q &\qquad \frac{1}{\lambda_{-\infty}} = 0 \\
\text{Coon}\,(q\geq 1): &\qquad \frac{1}{\lambda_\infty} = 0 &\qquad \frac{1}{\lambda_{-\infty}} = 1 - q \\
p\text{-type}: &\qquad \frac{1}{\lambda_\infty} = \frac{1-xp}{1+x} &\qquad \frac{1}{\lambda_{-\infty}} = \frac{1-x/p}{1+x} \\
r\text{-type}: &\qquad \frac{1}{\lambda_\infty} = \frac{1-x}{1+x} &\qquad \frac{1}{\lambda_{-\infty}} = \frac{1-x}{1+x}
\end{aligned} \tag{2.60}$$

When λ_∞ is finite, it is of course the limit point of poles. The quantity $\lambda_{-\infty}$ does not have a clear physical interpretation but is still useful to define.

Finally, we note that with $A_n = 1/\lambda_n$ and $B_n = \Lambda_n/\lambda_n$, all of the solutions we find give convergent infinite product amplitudes (2.21) satisfying the convergence condition (2.24). As we noted above, the coefficient B_1 is undetermined by our constraints, but we shall choose $B_1 = \Lambda_1/\lambda_1$ to fit the pattern. This choice will not affect our subsequent analysis.

2.4.5 Polynomial residues?

We derived the spectra above from the generalized Veneziano amplitude constraints (2.34), which we in turn derived by demanding that the infinite sequence of t-channel poles cancels on each s-channel pole within our infinite product ansatz (2.21). However, this truncation condition will not necessarily imply that our generalized Veneziano amplitudes have polynomial residues. With our explicit expressions for the poles λ_n and the coefficients A_n and B_n, we can explicitly compute the residues of (2.21).

We shall denote the Coon amplitudes by $\mathcal{A}_q(s,t)$, the p-type amplitudes by $\mathcal{A}_p(s,t)$, and the r-type amplitudes by $\mathcal{A}_r(s,t)$. We may then manipulate (2.21) and write each of these amplitudes in a form such that each factor in its infinite product is manifestly convergent. For the Coon amplitudes, we have,

$$\begin{aligned}\mathcal{A}_q(s,t) &= \mathcal{W}_q(s,t)\,\frac{1}{st}\prod_{n\geq 1}\frac{\left(1-\hat{q}^{n-\alpha_q(s)-\alpha_q(t)}\right)(1-\hat{q}^n)}{\left(1-\hat{q}^{n-\alpha_q(s)}\right)\left(1-\hat{q}^{n-\alpha_q(t)}\right)}\\ \alpha_q(s) &= \frac{\ln\big(1+(q-1)s\big)}{\ln q}\end{aligned}\tag{2.61}$$

where $\hat{q} = \min(q, q^{-1})$. For the p-type amplitudes, we have,

$$\begin{aligned}\mathcal{A}_p(s,t) &= \mathcal{W}_p(s,t)\,\frac{1}{st}\prod_{n\geq 1}\frac{\left(1-p^{n-\alpha_p(s)-\alpha_p(t)}\right)(1-p^n)}{\left(1-p^{n-\alpha_p(s)}\right)\left(1-p^{n-\alpha_p(t)}\right)}\\ \alpha_p(s) &= \frac{\ln\left(\frac{(1+x)-(1-xp)s}{(1+x)-(1-x/p)s}\right)}{\ln p}\end{aligned}\tag{2.62}$$

Finally, for the r-type amplitudes, we have,

$$\mathcal{A}_r(s,t) = \mathcal{W}_r(s,t)\,\frac{1}{st}\prod_{n\geq 1}\frac{1-\big(\alpha_r(s)+\alpha_r(t)\big)/n}{\big(1-\alpha_r(s)/n\big)\big(1-\alpha_r(t)/n\big)}$$

$$\alpha_r(s) = \frac{2xs}{1+x-(1-x)s} \tag{2.63}$$

In each case, the functions $\alpha(s)$ are the respective amplitudes' leading Regge trajectory and obey $\alpha(\lambda_N) = N$.

For simplicity, we have omitted the exponential factors needed to make the infinite product of each factor in (2.63) convergent. As in the infinite product representation for the Veneziano amplitude (2.6), these factors cancel between the numerator and denominator.

From these expressions, we may simply compute the residues at $s = \lambda_N$ for $N \geq 1$. We recall that the residue of the massless pole at $s = 0$ is $1/t$ by construction. For the massive poles, we find the following. For the Coon amplitudes with $q \geq 1$, we have,

$$\operatorname*{Res}_{s=\lambda_N} \mathcal{A}_q(s,t) = \mathcal{W}_q(\lambda_N,t)\,\frac{q^N}{\lambda_N}\prod_{n=1}^{N-1}\left(\frac{q^n}{\lambda_n}\,t+1\right) \tag{2.64}$$

For the Coon amplitudes with $q < 1$, we have,

$$\operatorname*{Res}_{s=\lambda_N} \mathcal{A}_q(s,t) = \mathcal{W}_q(\lambda_N,t)\,\frac{q^N}{\lambda_N}\frac{1}{(1-t/\lambda_\infty)^N}\prod_{n=1}^{N-1}\left(\frac{q^n}{\lambda_n}\,t+1\right) \tag{2.65}$$

For the p-type amplitudes, we have,

$$\operatorname*{Res}_{s=\lambda_N} \mathcal{A}_p(s,t) = \mathcal{W}_p(\lambda_N,t)\,\frac{p^N}{\lambda_N}\frac{x^2(1-p^2)^2}{\big[p(1-xp)-(p-x)p^N\big]^2}$$

$$\times\,\frac{1}{(1-t/\lambda_\infty)^N}\prod_{n=1}^{N-1}\left(\frac{(1-xp)p^n-(1-x/p)}{(1+x)(1-p^n)}\,t+1\right) \tag{2.66}$$

Finally, for the r-type amplitudes, we have,

$$\operatorname*{Res}_{s=\lambda_N} \mathcal{A}_r(s,t) = \mathcal{W}_r(\lambda_N,t)\,\frac{1}{\lambda_N}\frac{4x^2}{\big(2x+(1-x)N\big)^2}$$

$$\times\,\frac{1}{(1-t/\lambda_\infty)^N}\prod_{n=1}^{N-1}\left(\frac{2x-(1-x)n}{(1+x)n}\,t+1\right) \tag{2.67}$$

Using the quantities $\lambda_{\pm\infty}$ defined above, we may write these expressions in the following universal form,

$$\begin{aligned}\operatorname*{Res}_{s=\lambda_N} \mathcal{A}(s,t) = \mathcal{W}(\lambda_N, t)\, \frac{1}{\lambda_N}\left(1-\frac{\lambda_N}{\lambda_\infty}\right)\left(1-\frac{\lambda_N}{\lambda_{-\infty}}\right) \\ \times \frac{1}{(1-t/\lambda_\infty)^N}\prod_{n=1}^{N-1}\left[\left(\frac{1}{\lambda_n}-\frac{1}{\lambda_\infty}-\frac{1}{\lambda_{-\infty}}\right)t+1\right]\end{aligned} \tag{2.68}$$

Ignoring for now the $\mathcal{W}(s,t)$ pre-factors, only the amplitudes $\mathcal{A}_q(s,t)$ with $q \geq 1$ have polynomial residues. In other words, the amplitudes with accumulation point spectra all have non-polynomial residues! In each case, though, the non-polynomial behavior is captured by the factor $(1-t/\lambda_\infty)^{-N}$ which multiplies a degree-$(N-1)$ polynomial in t.

This, however, is not the end of the story. It may be possible to find a pre-factor $\mathcal{W}(s,t)$ which cancels the non-polynomial factors $(1-t/\lambda_\infty)^{-N}$ on each pole. We recall that the pre-factor obeys $\mathcal{W}(s,t) = 1 + \mathcal{O}(s,t)$. We must then require $\mathcal{W}(\lambda_N, t) \propto (1-t/\lambda_\infty)^N$ for all $N \geq 1$ to cancel the non-polynomial factors in each residue. A natural guess is simply $\mathcal{W}(s,t) = (1-t/\lambda_\infty)^{\alpha(s)}$ for the appropriate Regge trajectory $\alpha(s)$.[3] In fact, any pre-factor $\mathcal{W}(s,t)$ which cancels the non-polynomial behavior on every residue must be proportional to this guess, but this guess is not generally crossing symmetric. Only for the Coon amplitude do we have,

$$(1-t/\lambda_\infty)^{\alpha_q(s)} = (1-s/\lambda_\infty)^{\alpha_q(t)} = q^{\alpha_q(s)\alpha_q(t)} \tag{2.69}$$

As described in section 2.2, this pre-factor is explicitly non-meromorphic and introduces branch cuts beginning at $s,t = \lambda_\infty = \frac{1}{1-q}$. We recall, however, that we explicitly allowed for such non-meromorphic behavior in the pre-factor of our ansatz (2.21) so long as $\mathcal{W}(s,t)$ had no zeros nor poles in the region $|s|,|t| < \lambda_\infty$. For the p-type and r-type amplitudes, the crossing-symmetric guess $\mathcal{W}(s,t) = (1-t/\lambda_\infty)^{\alpha(s)}(1-s/\lambda_\infty)^{\alpha(t)}$ adds further non-polynomial behavior to each residue which cannot be fixed by any other crossing symmetric factor.

[3] A more general pre-factor is considered in [CR23].

Hence, we conclude that we can only cancel the non-polynomial residues in the case of the Coon amplitude with $q < 1$. We thus take,

$$\mathcal{W}_{q<1}(s,t) = q^{\alpha_q(s)\alpha_q(t)} \qquad \text{and} \qquad \mathcal{W}_{q\geq 1}(s,t) = \mathcal{W}_p(s,t) = \mathcal{W}_r(s,t) = 1 \tag{2.70}$$

since there is no way to construct a crossing-symmetric pre-factor which cancels the non-polynomial behavior of each residue for the p-type and r-type amplitudes.

Of all the spectra which solve the generalized Veneziano amplitude constraints (2.34), only the Coon spectra (2.40) can be included in an infinite product amplitude with polynomial residues. Moreover, for $q < 1$ polynomial residues can only be achieved by introducing the non-meromorphic pre-factor $\mathcal{W}_q(s,t) = q^{\alpha_q(s)\alpha_q(t)}$.

The other generalized Veneziano amplitudes $\mathcal{A}_p(s,t)$ and $\mathcal{A}_r(s,t)$ do not have polynomial residues, but the non-polynomial behavior of their residues is captured by the universal factor $(1 - t/\lambda_\infty)^{-N}$. These residues can be expanded in t for all $|t| < \lambda_\infty$, resulting in infinite spin exchange on each massive pole as described in section 2.1. By construction, the massless poles have finite spin exchange with $\ell_{\max} = 1$ (from the residue $1/t$ multiplied by the kinematic pre-factor $P_4 = \mathcal{O}(t^2)$ described in section 2.2).

2.4.6 Unitarity?

Although the non-polynomial residues of $\mathcal{A}_p(s,t)$ and $\mathcal{A}_r(s,t)$ are novel, these amplitudes may still be interesting. Amplitudes with non-polynomial residues have appeared in the context of extremized EFT-hedron bounds [CV21, AHH21]. Moreover, it has been recently shown that amplitudes with non-polynomial residues may be unitary [HR22]. The unitarity properties of the Coon amplitudes were also recently studied in [GL22, FT22, CMM22, BDS22]. Here we shall begin a unitarity analysis of the generalized Veneziano amplitudes.

In a unitary theory, the residue of each pole of the four-point amplitude must have an expansion on the Gegenbauer polynomials with positive partial wave coefficients. This expansion is described in (2.2). Several useful properties of the Gegenbauer polynomials are listed in the appendix of [GL22]. One particularly useful property is that the product of two Gegenbauer polynomials has a positive expansion on the Gegenabauer polynomials.

The unitarity properties of a given theory may depend on the number of spacetime dimensions d. The Coon amplitudes exhibit a particularly rich dimension-dependence [FT22]. For $q > 1$, the Coon amplitude is non-unitary in any dimension. For $0 < q \leq \frac{2}{3}$, the Coon amplitude is unitary in any dimension. For $\frac{2}{3} < q \leq 1$, the Coon amplitude is unitary below a q-dependent critical dimension $d_c(q)$. At $q = 1$, this critical dimension $d_c(1) = 10$ reproduces the critical dimension of the superstring. We shall derive similar results for the larger space of generalized Veneziano amplitudes.

2.4.6.1 Analytic results

We begin with a dimension-agnostic analysis which will provide sufficient but not strictly necessary conditions for unitarity. We define $z = \cos\theta$, where θ is the scattering angle in the center-of-mass frame, so that $t = \frac{1}{2}s(z-1)$. In terms of z, the residue of the generalized Veneziano amplitude at the massive pole $s = \lambda_N$ is given by,

$$\begin{aligned}\operatorname*{Res}_{s=\lambda_N} \mathcal{A}(s,t) = \mathcal{W}\big(\lambda_N, \tfrac{1}{2}\lambda_N(z-1)\big)\, \frac{1}{\lambda_N}\left(1 - \frac{\lambda_N}{\lambda_\infty}\right)\left(1 - \frac{\lambda_N}{\lambda_{-\infty}}\right) \\ \times \frac{1}{\big(1 - \frac{1}{2}(z-1)\lambda_N/\lambda_\infty\big)^N} \prod_{n=1}^{N-1}\left[\left(\frac{1}{\lambda_n} - \frac{1}{\lambda_\infty} - \frac{1}{\lambda_{-\infty}}\right)\frac{\lambda_N}{2}(z-1) + 1\right]\end{aligned} \tag{2.71}$$

The z-independent factor,

$$\frac{1}{\lambda_N}\left(1 - \frac{\lambda_N}{\lambda_\infty}\right)\left(1 - \frac{\lambda_N}{\lambda_{-\infty}}\right) \tag{2.72}$$

is always a positive number. The z-dependent factor,

$$\frac{\mathcal{W}\big(\lambda_N, \frac{1}{2}\lambda_N(z-1)\big)}{\big(1 - \frac{1}{2}(z-1)\lambda_N/\lambda_\infty\big)^N} \tag{2.73}$$

has a positive expansion on the Gegenbauer polynomials since positive powers of z have a positive expansion on the Gegenbauer polynomials [GL22]. In the case of the Coon amplitude (for any q), this factor simply equals one. In all other cases, the factor $\mathcal{W}(s,t)=1$, and we may use the binomial theorem to write,

$$\frac{1}{\left(1-\frac{1}{2}(z-1)\lambda_N/\lambda_\infty\right)^N} = \left(1+\frac{\lambda_N}{2\lambda_\infty}\right)^{-N} \sum_{k=0}^{\infty} \binom{N+k-1}{k} \left(\frac{\lambda_N}{\lambda_N+2\lambda_\infty}\right)^k z^k \tag{2.74}$$

which is a sum of powers of z with manifestly positive coefficients. It remains to study the polynomial part of the residue,

$$\prod_{n=1}^{N-1} \left[\left(\frac{1}{\lambda_n}-\frac{1}{\lambda_\infty}-\frac{1}{\lambda_{-\infty}}\right)\frac{\lambda_N}{2}\, z - \left(\frac{1}{\lambda_n}-\frac{1}{\lambda_\infty}-\frac{1}{\lambda_{-\infty}}\right)\frac{\lambda_N}{2}+1\right] \tag{2.75}$$

This factor will be a sum of powers of z with manifestly positive coefficients if,

$$\frac{1}{\lambda_n}-\frac{1}{\lambda_\infty}-\frac{1}{\lambda_{-\infty}} \geq 0 \qquad \text{and} \qquad \left(\frac{1}{\lambda_n}-\frac{1}{\lambda_\infty}-\frac{1}{\lambda_{-\infty}}\right)\frac{\lambda_N}{2} \leq 1 \tag{2.76}$$

for $n=1,2,\ldots,N-1$ for each $N \geq 1$. Because the poles are ordered $\lambda_n > \lambda_{n-1}$, these conditions are satisfied for all n at fixed N if,

$$\frac{1}{\lambda_{N-1}}-\frac{1}{\lambda_\infty}-\frac{1}{\lambda_{-\infty}} \geq 0 \qquad \text{and} \qquad \left(1-\frac{1}{\lambda_\infty}-\frac{1}{\lambda_{-\infty}}\right)\frac{\lambda_N}{2} \leq 1 \tag{2.77}$$

where we have used $\lambda_1 = 1$. These conditions are in turn satisfied for all N if,

$$\frac{1}{\lambda_\infty}-\frac{1}{\lambda_\infty}-\frac{1}{\lambda_{-\infty}} \geq 0 \qquad \text{and} \qquad \left(1-\frac{1}{\lambda_\infty}-\frac{1}{\lambda_{-\infty}}\right)\frac{\lambda_\infty}{2} \leq 1 \tag{2.78}$$

Rearranging, we find,

$$\frac{1}{\lambda_{-\infty}} \leq 0 \qquad \text{and} \qquad \frac{3}{\lambda_\infty}+\frac{1}{\lambda_{-\infty}} \geq 1 \tag{2.79}$$

We have carefully written these conditions in terms of the reciprocals $1/\lambda_\infty$ and $1/\lambda_{-\infty}$ since λ_∞ or $\lambda_{-\infty}$ may be infinite.

The conditions (2.79) are satisfied as follows. For the Coon amplitudes with $q<1$, the first condition is trivially satisfied since $1/\lambda_{-\infty}=0$, leaving only the second condition,

$$\text{Coon}\,(q<1): \qquad\qquad 3(1-q) \geq 1 \qquad \Longrightarrow \qquad q \leq \tfrac{2}{3} \tag{2.80}$$

For the Coon amplitudes with $q \geq 1$, we have $1/\lambda_\infty = 0$, and the two conditions become,

$$\begin{aligned} \text{Coon}\,(q \geq 1): \qquad & 1 - q \leq 0 \qquad \Longrightarrow \qquad q \geq 1 \\ & 1 - q \geq 1 \qquad \Longrightarrow \qquad q \leq 0 \end{aligned} \tag{2.81}$$

which is never satisfied. For the p-type amplitudes, the two conditions become,

$$\begin{aligned} p\text{-type}: \qquad & \frac{1 - x/p}{1+x} \leq 0 \qquad \Longrightarrow \qquad x \geq p \\ & 3\,\frac{1 - xp}{1+x} + \frac{1 - x/p}{1+x} \geq 1 \qquad \Longrightarrow \qquad x \leq \frac{3p}{1 + p + 3p^2} \end{aligned} \tag{2.82}$$

Finally, for the r-type amplitudes, the two conditions become,

$$\begin{aligned} r\text{-type}: \qquad & \frac{1 - x}{1+x} \leq 0 \qquad \Longrightarrow \qquad x \geq 1 \\ & 4\,\frac{1 - x}{1+x} \geq 1 \qquad \Longrightarrow \qquad x \leq \frac{3}{5} \end{aligned} \tag{2.83}$$

which is never satisfied.

In summary, we have found that the Coon amplitudes with $0 < q \leq \frac{2}{3}$ are unitary in any dimension, in agreement with [FT22]. Moreover, we have analytically demonstrated that the p-type generalized Veneziano amplitudes with $p \leq x \leq 3p/(1 + p + 3p^2)$ are unitary in any dimension. These inequalities define a region of parameter space with infinite critical dimension. In terms of the parameters x and y, the first inequality $p \leq x$ becomes $y \leq x^2$ while the second inequality becomes $f_-(x) \leq y \leq f_+(x)$, where,

$$f_\pm(x) = \frac{x^2(6 + x - 3x^2 \pm \sqrt{9 - 6x - 11x^2})}{9 - 3x - 5x^2 + 3x^3} \tag{2.84}$$

This infinite critical dimension region is displayed in Figure 2.2.

While the conditions (2.79) are sufficient to prove unitarity in all dimensions, they are by no means necessary. In general, for a given finite dimension d, the unitary region in the xy-plane will be larger than the region of infinite critical dimension.

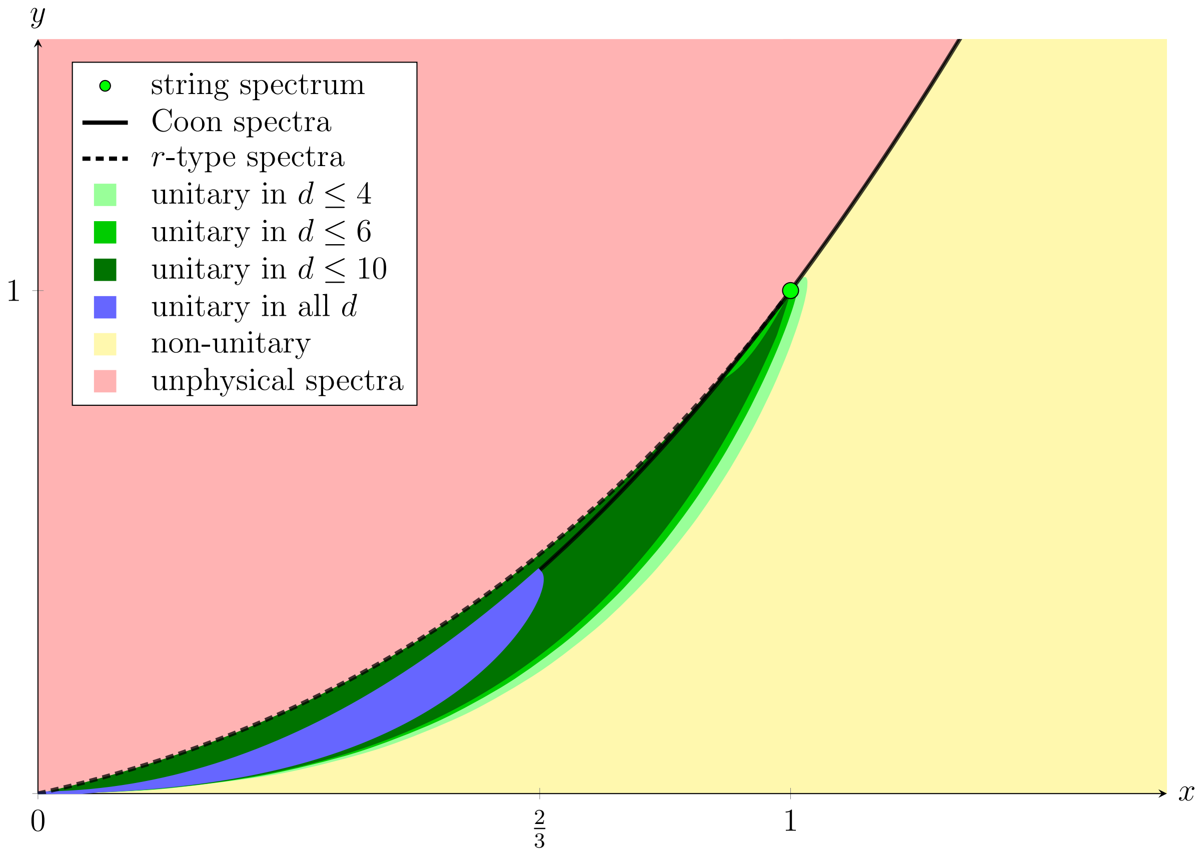

Figure 2.2: The two-parameter space of solutions to the generalized Veneziano amplitude constraints with unitary regions in various dimensions. The blue region is unitary in all dimensions and includes the Coon amplitudes with $0 < q \leq \frac{2}{3}$. The dark green region is unitary in $d \leq 10$ and includes the Veneziano amplitude at $x = y = 1$. The middle green region is unitary in $d \leq 6$. The light green region is unitary in $d \leq 4$. The blue region was computed analytically, and the green regions were computed by numerically analyzing the first few partial wave coefficients $c_{n,j}$.

2.4.6.2 Numerical results

To study the regions of parameter space with finite critical dimension, we shall employ numerical methods to analyze the first few partial wave coefficients. One cannot mathematically prove unitarity by examining a finite number of partial wave coefficients, but if any of those coefficients are negative, then the amplitude in question is non-unitary.

In this way, one can find evidence for the critical dimension of the superstring by computing one of the first partial wave coefficients of the Veneziano amplitude, $c_{3,0} \propto 10 - d$. Similarly, one can deduce that the Coon amplitudes with $q > 1$ are non-unitary by computing the coefficient $c_{2,0} \propto 1 - q$ [GL22]. In this spirit, we hope to provide some evidence on the qualitative structure of the unitary regions of parameter space for the generalized Veneziano amplitudes.

The analytic expressions for the partial wave coefficients $c_{n,j}$ are given by the following overlap integral against the Gegenbauer polynomials [GL22],

$$c_{n,j} = \mathcal{N}_j^{(\frac{d-3}{2})} \int_{-1}^{1} dz \, (1-z^2)^{\frac{d-4}{2}} \, C_j^{(\frac{d-3}{2})}(z) \times \operatorname*{Res}_{s=\lambda_n} \mathcal{A}\big(s, \tfrac{1}{2}s(z-1)\big) \tag{2.85}$$

where the residue is given by (2.68) and the normalization is,

$$\mathcal{N}_j^{(\frac{d-3}{2})} = 2^{d-5}(2j+d-3)\frac{\Gamma(j+1)\Gamma(\frac{d-3}{2})^2}{\pi\Gamma(j+d-3)} \tag{2.86}$$

The apparent poles in this formula at $d = 3$ are a remnant of the normalization of the Gegenbauer polynomials and can be trivially removed by a change in normalization.

For the p-type and r-type amplitudes which exhibit infinite spin exchange, the coefficients $c_{n,j}$ with $n \geq 1$ will generally be non-zero for all spins $j \geq 0$. Remarkably, Mathematica can explicitly compute these integrals in terms of generalized hypergeometric functions. The expressions are incredibly long, so we shall omit them here. Instead, we shall numerically examine the region of parameter space where $c_{n,j} \geq 0$ for $1 \leq n \leq 4$ and $0 \leq j \leq 3$ in several dimensions, namely $d = 4, 6, 10$. The unitary regions of parameter space are displayed in Figure 2.2. The qualitative structures of these regions do not appreciably change upon probing larger values of n or j.

As expected, the unitary region in d dimensions envelopes those in $d' > d$ dimensions, and they all contain the region of infinite critical dimension. For $d \leq 10$ the unitary region includes the Veneziano amplitude at $x = y = 1$. The unitary region also appears to contain some r-type amplitudes, albeit with finite critical dimension. It would be interesting to study the features of Figure 2.2 in more detail. Perhaps the methods of [GL22, FT22, CMM22, BDS22] which were used to study the Coon amplitudes could be adapted to study the unitary properties of generalized Veneziano amplitudes.

2.5 Generalized Virasoro amplitudes

In this section, we shall systematically analyze our infinite product ansatz (2.26) for the generalized Virasoro amplitude.

2.5.1 Virasoro truncation

We first recall the infinite product form (2.9) of the Virasoro amplitude, which has simple poles at each non-negative integer. After applying the mass-shell relation $s + t + u = 0$, the residue of the massless s-channel pole is $1/t^2$, and the residue of each massive pole at $s = N$ is a polynomial of degree-$(2N - 2)$ in t. The Virasoro amplitude achieves these residues because on each s-pole, its zeros cancel the t-poles and u-poles, leaving a finite polynomial in t. This cancellation can be described in terms of the numerator factors,

$$\mathcal{N}_n(s, t, u) = 1 + (st + tu + us)/n^2 + stu/n^3 \tag{2.87}$$

When $s = N$, each numerator factor $\mathcal{N}_{N+n}(N, t, -N - t) \propto (1 - t/n)(1 - u/n)$ cancels both the t-channel and u-channel pole factors $(1 - t/n)^{-1}(1 - u/n)^{-1}$, and the infinite product truncates. In short, the condition,

$$\mathcal{N}_{N+n}(N, n, -N-n) = 0 \tag{2.88}$$

ensures that the Virasoro amplitude has polynomial residues.

These features bare a striking resemblance to those of the Veneziano amplitude. Hence, our analysis of the generalized Virasoro amplitude (2.26) will mirror our analysis of the generalized Veneziano amplitudes in the previous section.

2.5.2 Generalized Virasoro truncation

We now return to our generalized Virasoro ansatz (2.26). We shall demand that the zeros and poles of this amplitude cancel in a similar fashion as those of the Virasoro amplitude. We first demand that the residue at $s = 0$ is $1/t^2$ so that the amplitude reproduces the massless spectrum of supergravity analogously to the Virasoro amplitude,

$$\operatorname*{Res}_{s=0} \mathcal{A}(s,t,u) = \frac{1}{t^2} \qquad \Longrightarrow \qquad \mathcal{W}(0,t,-t) \prod_{n\geq 1} \frac{1 - \ A_n t^2}{1 - t^2/\lambda_n^2} = 1 \tag{2.89}$$

which implies that $\mathcal{W}(0,t,-t) = 1$ and $A_n = 1/\lambda_n^2$ since $\mathcal{W}(s,t,u)$ has neither zeros nor poles. In other words, the coefficients A_n are again determined by the poles λ_n.

Next, in analogy with the truncation condition for the Virasoro amplitude (2.88), we demand that the generalized numerator factor,

$$\mathcal{N}_n(s,t,u) = 1 + A_n(st + tu + us) - B_n stu \tag{2.90}$$

obeys the generalized truncation condition,

$$\mathcal{N}_{N+n}(\lambda_N, \lambda_n, -\lambda_N - \lambda_n) = 0 \tag{2.91}$$

so that $\mathcal{N}_{N+n}(\lambda_N, t, -\lambda_N - t) \propto (1 - t/\lambda_n)(1 - u/\lambda_n)$ and the infinite sequence of t-poles and u-poles cancels on each s-channel pole. This truncation condition determines the coefficients B_n in terms of the poles λ_n,

$$B_n = \frac{\lambda_k^2 + \lambda_k\lambda_{n-k} + \lambda_{n-k}^2 - \lambda_n^2}{\lambda_n^2 \lambda_{n-k}\lambda_k(\lambda_k + \lambda_{n-k})} \qquad k = 1, 2, \ldots, n-1 \tag{2.92}$$

For fixed $n \geq 2$, both k and $k' = n - k$ yield the same equation for B_n so that there are again $\left\lfloor \frac{n}{2} \right\rfloor$ independent equations for B_n. Once more, the coefficient B_1 is left undetermined, the coefficients B_2 and B_3 are uniquely determined, and the coefficients B_n with $n \geq 4$ are all over-determined.

As in the previous section, this over-determination of the B_n highly constrains the poles. Any sequence of poles λ_n must leave the following combination independent of k for all $n \geq 2$,

$$\Lambda_n(k) = \frac{\lambda_k^2 + \lambda_k \lambda_{n-k} + \lambda_{n-k}^2 - \lambda_n^2}{\lambda_{n-k}\lambda_k(\lambda_k + \lambda_{n-k})} \tag{2.93}$$

We shall refer to these equations as the generalized Virasoro amplitude constraints. The Virasoro solution $\lambda_n = n$ (i.e. the string theory spectrum) solves these constraints with $\Lambda_n(k) = -1/n$ for all n and k. We shall now search for other, more general sequences of poles λ_n which solve the generalized Virasoro amplitude constraints.

2.5.3 Generalized Virasoro amplitude constraints

Since $\Lambda_n(k)$ must be independent of k, we may fix $n \geq 2$ and choose two distinct values of (k, ℓ) in the appropriate range to find,

$$\frac{\lambda_k^2 + \lambda_k \lambda_{n-k} + \lambda_{n-k}^2 - \lambda_n^2}{\lambda_{n-k}\lambda_k(\lambda_k + \lambda_{n-k})} = \frac{\lambda_\ell^2 + \lambda_\ell \lambda_{n-\ell} + \lambda_{n-\ell}^2 - \lambda_n^2}{\lambda_{n-\ell}\lambda_\ell(\lambda_\ell + \lambda_{n-\ell})} \tag{2.94}$$

We may then solve this equation for λ_n in terms of λ_k, λ_{n-k}, λ_ℓ, and $\lambda_{n-\ell}$,

$$\lambda_n = \sqrt{\frac{\lambda_k\lambda_{n-k}(\lambda_{n-k}+\lambda_k)(\lambda_{n-\ell}^2+\lambda_{n-\ell}\lambda_\ell+\lambda_\ell^2)-\lambda_\ell\lambda_{n-\ell}(\lambda_{n-\ell}+\lambda_\ell)(\lambda_{n-k}^2+\lambda_{n-k}\lambda_k+\lambda_k^2)}{\lambda_k\lambda_{n-k}(\lambda_{n-k}+\lambda_k)-\lambda_\ell\lambda_{n-\ell}(\lambda_{n-\ell}+\lambda_\ell)}} \tag{2.95}$$

As in the generalized Veneziano case, the first three poles are free parameters, and (2.95) determines all the subsequent poles in terms of λ_1, λ_2, and λ_3. We shall again define the positive numbers $x = \lambda_2 - \lambda_1 = \lambda_2 - 1 > 0$ and $y = \lambda_3 - \lambda_2 > 0$, using the fact that the poles $\lambda_n > \lambda_{n-1}$ are ordered. The choice $\lambda_1 = 1$ simply sets our units.

For $n = 4$ and $n = 5$, there is a unique choice of (k, ℓ) and thus a single equation determining $\lambda_4 = \lambda_4(x, y)$ and $\lambda_5 = \lambda_5(x, y)$. For $n = 6$, we can write two different equations for $\lambda_6 = \lambda_6(x, y)$. These equations are exceedingly large and include several nested radicals. Equating these two expressions implicitly defines a curve in the xy-plane. Any solution of the generalized Virasoro amplitude constraints must be on this curve. We have analyzed this curve numerically and verified that it passes through $x = y = 1$. Repeating this process at $n = 7$ yields a second curve in the xy-plane, and any solution of the generalized Virasoro amplitude constraints must again be on this curve. Through a straightforward numerical analysis, we find that the λ_6 and λ_7 curves only intersect at $x = y = 1$, corresponding to the string spectrum.

In other words, only the string spectrum $\lambda_n = n$ satisfies the generalized Virasoro amplitude constraints (2.93). Thus, the construction which led to several infinite families of generalized Veneziano amplitudes fails to yield any new generalizations of the Virasoro amplitude. The closed string is highly constrained.

2.6 Discussion

In this chapter, we have systematically analyzed generalizations of both the Veneziano and Virasoro amplitudes by considering the infinite product ansatz (2.21) and (2.26). Demanding that the poles cancel on each residue, we arrived at the generalized Veneziano and generalized Virasoro amplitude constraints, (2.34) and (2.93), respectively. These constraints are equivalent to an infinite set of non-linear recursion relations obeyed by the poles of each amplitude.

In the generalized Veneziano case, we solved the recursion relations analytically by reducing them to the Riccati recursion relation (2.37). The solutions corresponded to the Veneziano amplitude, the one-parameter family of Coon amplitudes, and a larger two-parameter family of amplitudes with an infinite tower of spins at each mass level. Of these generalized Veneziano amplitudes, only the Veneziano and Coon amplitudes have polynomial residues. We also began an initial study of the unitarity properties of these amplitudes and found that a subspace of them, including the Coon amplitudes with $0 < q \leq \frac{2}{3}$, are unitary in any dimension. A larger subspace is unitary with finite critical dimension.

In the generalized Virasoro case, we numerically demonstrated that the only consistent solution to the generalized Virasoro amplitude constraints is the string spectrum. These infinitely many constraints did not allow any deviation outside of closed string theory. Our results are consistent with those of [GL22, CR23].

In future work, it would be interesting to explore where the low-energy expansion coefficients of the generalized Veneziano amplitudes $\mathcal{A}_p(s,t)$ and $\mathcal{A}_r(s,t)$ lie in relation to the EFT-hedron [AHH21] and other positivity bounds [CV21]. The low-energy expansion coefficients of the Coon amplitudes were recently studied in this context in [FT22, GL22]. It would also be interesting to further study the unitarity properties of these amplitudes. Further generalizations of this work may study other truncation conditions leading to polynomial residues for our infinite product ansatz. Recent progress in this direction has been made in [CR23].

Finally, we hope to find a definitive field theory or string theory realization of the Coon amplitudes or their generalizations. Recently, accumulation point spectra like those exhibited by Coon amplitudes were found in a setup involving open strings ending on a D-brane [MR22]. Moreover, accumulation point spectra have appeared in various contexts in the modern S-matrix bootstrap program, so it is imperative to better understand the Coon amplitudes' physical origins.

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