The perpetual American put option for jump-diffusions with applications

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Abstract

In this paper we solve an optimal stopping problem with an infinite time horizon, when the state variable follows a jump-diffusion. Under certain conditions our solution can be interpreted as the price of an American perpetual put option, when the underlying asset follows this type of process.

We present several examples demonstrating when the solution can be interpreted as a perpetual put price. This takes us into a study of how to risk adjust jump-diffusions. One key observation is that the probability distribution under the risk adjusted measure depends on the equity premium, which is not the case for the standard, continuous version. This difference may be utilized to find intertemporal, equilibrium equity premiums, for example.

Our basic solution is exact only when jump sizes can not be negative. We investigate when our solution is an approximation also for negative jumps.

Various market models are studied at an increasing level of complexity, ending with the incomplete model in the last part of the paper.

KEYWORDS: Optimal exercise policy, American put option, perpetual option, optimal stopping, incomplete markets, equity premiums, CCAPM.

1 Introduction.

We consider the perpetual American put option when the underlying asset pays no dividends. This is known to be the same mathematical problem as pricing an infinite-lived American call option, when the underlying asset pays a continuous, proportional dividend rate, as shown by Samuelson (1965).

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The market value of the corresponding European perpetual put option is known to be zero, but as shown by Merton (1973a), the American counterpart converges to a strictly positive value. This demonstrates at least one situation where there is a difference between these two products in the situation with no dividend payments from the underlying asset.

We analyze this contingent claim when the underlying asset has jumps in its paths. We start by solving the relevant optimal stopping problem for a general jump-diffusion, and illustrate the obtained result by several examples. Our method does not provide a general solution when jumps can be negative. Here we consider some special cases, and demonstrate that our basic solution can still be used as an approximation for negative jumps. In many types of scientific applications the accuracy obtained this way is shown to be adequate.

In the pure jump model the probability distribution under the risk adjusted measure depends on the equity premium, which is not the case for the standard, continuous version. We briefly demonstrate how this difference may be utilized to find equilibrium equity premiums.

Various market models are studied from the rather simple to the more complex. As is usually the case, it is from the simple models we gain the most transparent insights.

The paper is organized as follows: Section 2 presents the model, Section 3 the American perpetual option pricing problem, Section 4 the solution to this problem in general, Section 5 treats adjustments to risk, Section 6 compares the solutions for the standard, continuous geometric Brownian model to the geometric Poisson model. In this section we see how equity premiums may be calibrated the the US data of the previous century. Section 7 presents solutions for a combined jump diffusion, Section 8 discusses a model where there are different possible jump sizes, Section 9 combines the latter case with a continuous component, and Section 10 treats the incomplete model, where jump sizes are continuously distributed. Section 11 concludes.

2 The Model

We start by establishing the dynamics of the assets in the model: There is an underlying probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ satisfying the usual conditions, where Ω is the set of states, $\mathcal{F}$ is the set of events, $\mathcal{F}_t$ is the set of events observable by time t, for any $t \geq 0$, and P is the given probability measure, governing the probabilities of events related to the stochastic price processes in the market. On this space is defined one locally riskless asset, thought as the evolution of a bank account with dynamics

$$d\beta_t = r\beta_t dt, \qquad \beta_0 = 1,$$

and one risky asset satisfying the following stochastic differential equation

$$dS_t = S_{t-}[\mu dt + \sigma dB_t + \alpha \int_R \eta(z)\tilde{N}(dt, dz)], \qquad S_0 = x > 0. \tag{1}$$

Here B is a standard Brownian motion, $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$ is the compensated Poisson random measure, $\nu(dz)$ is the Lévy measure, and $N(t, U)$ is the number of jumps which occur before or at time t with sizes in the set U

of real numbers. The process $N(t,U)$ is called the Poisson random measure of the underlying Lévy process. The function $\alpha\eta(z) \geq -1$ for all values of z. We will usually choose $\eta(z) = z$ for all z, which implies that the integral is over the set $(-1/\alpha, \infty)$. The Lévy measure $\nu(U) = E[N(1,U)]$ is in general a set function, where E is the expectation operator corresponding to the probability measure P. In our examples we will by and large assume that this measure can be decomposed into $\nu(dz) = \lambda F(dz)$ where λ is the frequency of the jumps and $F(dz)$ is the probability distribution function of the jump sizes. This gives us a finite Lévy measure, and the jump part becomes a compound Poisson process.

This latter simplification is not required to deal with the optimal stopping problem, which can in principle be solved for any Lévy measure ν for which the relevant equations are well defined, subject to certain technical conditions which we return to later. The processes B and N are assumed independent. Later we introduce more risky assets in some of the examples as need arises.

The stochastic differential equation (1) can be solved using Itô's lemma, and the solution is

$$\begin{aligned} S(t) = S(0) \exp\big\{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t \\ - \alpha \int_0^t \int_R \eta(z)\nu(dz)ds + \int_0^t \int_R \ln(1+\alpha\eta(z))N(ds,dz)\big\}. \end{aligned} \tag{2}$$

From this expression we immediately see why we have required the inequality $\alpha\eta(z) \geq -1$ for all z; otherwise the natural logarithm is not well defined. This solution is sometimes labeled a "stochastic" exponential, in contrast to only an exponential process which would result if the price Y was instead given by $Y(t) = Y(0)\exp(Z_t)$, where $Z_t = (X_t - \frac{1}{2}\sigma^2 t)$, and the accumulated return process X_t is given by the arithmetic process

$$X_t := \mu t + \sigma B_t + \alpha \int_0^t \int_R \eta(z)\tilde{N}(ds,dz). \tag{3}$$

Clearly the process Y can never reach zero in a finite amount of time if the jump term is reasonably well behaved [1], so there would be no particular lower bound for the term $\alpha\eta(z)$ in this case. We have chosen to work with stochastic exponential processes in this paper. There are several reasons why this is a more natural model in finance. On the practical side, bankruptcy can be modeled using S, so credit risk issues are more readily captured by this model. Also the instantaneous return $\frac{dS(t)}{S(t-)} = dX_t$, which equals ($\mu dt$ + "noise"), where μ is the rate of return, whereas for the price model Y we have that

$$\frac{dY(t)}{Y(t-)} = \big(\mu + \int_R (e^{\alpha\eta(z)} - 1 - \alpha\eta(z))\nu(dz)\big)\,dt + \sigma dB_t + \int_R (e^{\alpha\eta(z)} - 1)\tilde{N}(dt,dz),$$

which is in general different from dX_t, and as a consequence we do not have a simple interpretation of the rate of return in this model. [2]

[1] i.e., if it does not explode. The Brownian motion is known not to explode.

[2] If the exponential function inside the two different integrals can be approximated by the two first terms in its Taylor series expansion, which could be reasonable if the Lévy measure ν has short and light tails, then we have $\frac{dY(t)}{Y(t-)} \approx dX_t$.

3 The optimal stopping problem

We want to solve the following problem:

$$\psi(x) = \sup_{\tau \geq 0} E^x \Big\{ e^{-r\tau}(K - S_\tau)^+ \Big\}, \tag{4}$$

where $K > 0$ is a fixed constant, the exercise price of the put option, when the dynamics of the stock follows the jump-diffusion process explained above. By E^x we mean the conditional expectation operator given that $S(0) = x$, under the given probability measure P.

For this kind of dynamics the financial model is in general not complete, so in our framework the option pricing problem may not have a unique solution, or any solution at all. There will normally be many risk adjusted measures Q, and if it is not even clear that the pricing rule must be linear, none of these may be appropriate for pricing the option at hand. If one is, however, the pricing problem may in some cases be a variation of the solution to the above problem, since under any appropriate Q the price S follows a dynamic equation of the type (1), with r replacing the drift parameter μ, and possibly with a different Lévy measure $\nu^Q(dz)$, absolutely continuous with respect to $\nu(dz)$. Thus we first focus our attention on the problem (4).

There are special cases where the financial problem has a unique solution; in particular there are situations including jumps where the model either is, or can be made complete, in the latter case by simply adding a finite number of risky assets. We return to the the different situations in the examples.

The stopping problem (4) has been considered by other authors from different perspectives. Mordecki (2002) finds formulas based on extending the theory of optimal stopping of random walks. It hinges upon one's ability to compute the quantity $E(e^I)$, where $I = \inf_{0 \leq t \leq \tau(r)} Z(t)$, and $\tau(r)$ is an exponential random variable with parameter $r > 0$, independent of Z, and $\tau(0) = \infty$. No adjustments to risk was considered. See also Boyarchenko and Levendroskiĭ (2002).

In contrast, we base our development on the theory of integro-variational inequalities for optimal stopping. Although we do not obtain exact solutions in all situations considered, our procedure is well suited to many applications of option pricing theory.

4 The solution of the optimal stopping problem

In this section we present the solution to the optimal stopping problem (4) for jump-diffusions. Let $\mathcal{C}$ denote the continuation region, and let τ be the exercise time defined by $\tau = \inf\{t > 0; S(t) \notin \mathcal{C}\}$. We make the assumption that

$$S(\tau) \in \bar{\mathcal{C}} \qquad (\bar{\mathcal{C}} \text{ is the closure of } \mathcal{C}). \tag{5}$$

We then have the following result:

Theorem 1 *The solution $\psi(x) := \psi(x; c)$ of the optimal stopping problem is, under the assumptions (5), given by*

$$\psi(x) = \begin{cases} (K - c)\left(\frac{c}{x}\right)^\gamma, & \text{if } x \geq c; \\ (K - x), & \text{if } x < c, \end{cases} \tag{6}$$

where the continuation region $\mathcal{C}$ is given by

$$\mathcal{C} = \big\{(x,t) : x > c\big\},$$

and the trigger price c is a constant. This constant is given by

$$c = \frac{\gamma K}{\gamma + 1}, \tag{7}$$

where the constant γ solves the following equation

$$-r - \mu\gamma + \frac{1}{2}\sigma^2\gamma(\gamma+1) + \int_R \{(1+\alpha\eta(z))^{-\gamma} - 1 + \alpha\gamma\eta(z)\}\nu(dz) = 0. \tag{8}$$

Proof. As with continuous processes, there is an associated optimal stopping theory also for discontinuous processes. For an exposition, see e.g., Øksendal and Sulem (2004). In order to employ this, we need the characteristic operator, or generator $\bar{\mathcal{A}}$ of the process S. For any smooth function $f : R \to R$ not depending upon time, it is defined as

$$\bar{\mathcal{A}}f(x) = \lim_{t\to 0^+} \frac{1}{t}\{E^x[f(S_t)] - f(x)\} \qquad \text{(if the limit exists)},$$

where $E^x[f(S_t)] = E[f(S_t^x)], S_0^x = x$. Thus $\bar{\mathcal{A}}$ represents the expected rate of return of continuing at $t = 0$. For a time-homogeneous problem this is the expected rate of continuing at any time $t > 0$ as well. For our price process and with this kind of time-homogeneous function f, the generator for a jump-diffusion takes the following form:

$$\bar{\mathcal{A}}f(x)) = x\mu\frac{df(x)}{dx} + \frac{1}{2}x^2\sigma^2\frac{d^2f(x)}{dx^2} + \int_R \{f(x+\alpha x\eta(z)) - f(x) - \alpha\frac{df(x)}{dx}x\eta(z)\}\nu(dz),$$

where the last term stems from the jumps of the price process S. Since the objective function depends upon time via the discount factor, our problem can be classified as a time-inhomogeneous one. The standard theory of optimal stopping, and in particular the verification theorem, is formulated for the time-homogeneous case, but augmenting the state space of S by one more state, namely time itself,

$$Z_t = \begin{pmatrix} s+t \\ S_t \end{pmatrix}; \qquad t \geq 0$$

transforms the problem into a time-homogeneous one in the variable Z. (When $t = 0$, the process $Z(0) = (s,x)$.) Is is now convenient to reformulate our problem as follows: We seek the discounted value function $\phi(s,x)$ defined by

$$\phi(s,x) := \sup_{\tau\geq 0} E^{(s,x)}\Big\{e^{-r(s+\tau)}(K - S_\tau)^+\Big\}. \tag{9}$$

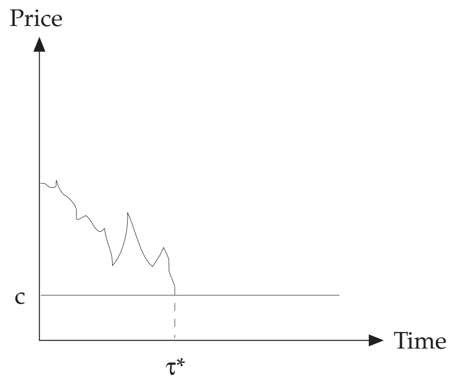
The generator $\mathcal{A}$ of the process Z is given in terms of the generator $\bar{\mathcal{A}}$ of the process S by $\mathcal{A}f(s,x) = \bar{\mathcal{A}}f(s,x) + \frac{\partial}{\partial s}f(s,x)$, where $\bar{\mathcal{A}}$ works on the x-variable.

With a view towards the verification theorem - a version for jump-diffusion processes exists along the lines of the one for continuous processes - we now conjecture that the continuation region $\mathcal{C}$ has the following form

$$\mathcal{C} = \{(x,t) : x > c\},$$

where the trigger price c is some constant. The motivation for this is that for any time t the problem appears just the same, from a prospective perspective, implying that the trigger price $c(t)$ should not depend upon time. See Figure 1. In the

Figure 1: Continuation Region



Continuation region of the perpetual American put option.

continuation region $\mathcal{C}$, the principle of optimal stopping requires $\mathcal{A}\varphi = 0$, or

$$\frac{\partial \varphi}{\partial s} + \mu x \frac{\partial \varphi}{\partial x} + \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 \varphi}{\partial x^2}$$
$$+ \int_R \{\varphi(s, x + \alpha x \eta(z)) - \varphi(s,x) - \alpha \frac{\partial \varphi}{\partial x} x \eta(z)\} \nu(dz) = 0.$$

This is a typical dynamic optimization criterion saying that it is not optimal to exercise so long as the expected rate of change of the value function is not strictly negative.

Furthermore we conjecture that the function $\varphi(s,x) = e^{-rs}\psi(x)$. Substituting this form into the above equation allows us to cancel the common term e^{-rs}, and we are left with the equation

$$\begin{aligned} & -r\psi(x) + \mu x \frac{\partial \psi(x)}{\partial x} + \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 \psi(x)}{\partial x^2} + \\ & \int_R \{\psi(x + \alpha x \eta(z)) - \psi(x) - \alpha \frac{\partial \psi(x)}{\partial x} x \eta(z)\} \nu(dz) = 0 \end{aligned} \tag{10}$$

for the unknown function ψ.

Thus we were successful in removing time from the PDE, and reducing the equation to an ordinary integro-differential-difference equation.

The equation is valid for $c \leq x < \infty$. Given the trigger price c, let us denote the market value $\psi(x) := \psi(x; c)$. The relevant boundary conditions are then

$$\psi(\infty; c) = 0 \quad \forall c > 0 \tag{11}$$

$$\psi(c; c) = K - c \quad \text{(exercise)} \tag{12}$$

We finally conjecture a solution of the form $\psi(x) = a_1 x + a_2 x^{-\gamma}$ for some constants a_1, a_2 and γ. The boundary condition (11) implies that $a_1 = 0$, and the boundary condition (12) implies that $a_2 = (K - c)c^{\gamma}$. Thus the conjectured form of the market value of the American put option is the following

$$\psi(x; c) = \begin{cases} (K-c)\big(\frac{c}{x}\big)^{\gamma}, & \text{if } x \geq c; \\ (K-x), & \text{if } x < c. \end{cases}$$

In order to determine the unknown constant γ, we insert this function in the equation (10). This allows us to cancel the common term $x^{-\gamma}$, and we are left with the following nonlinear, algebraic equation for the determination of the constant γ:

$$-r - \mu\gamma + \frac{1}{2}\sigma^2\gamma(\gamma+1) + \int_R \{(1+\alpha\eta(z))^{-\gamma} - 1 + \alpha\gamma\eta(z)\}\nu(dz) = 0. \tag{13}$$

This is a well defined equation in γ, and the fact that we have successfully been able to cancel out the variables x and s, is a strong indication that we actually have found the solution to our problem.

If this is correct, it only remains to find the trigger price c, and this we do by employing the "high contact" or "smooth pasting" condition (e.g., McKean (1965)).

$$\frac{\partial \psi(c; c)}{\partial x}\Big|_{x=c} = -1.$$

This leads to the equation

$$(k - c)c^{\gamma}(-\gamma c^{-\gamma-1}) = -1,$$

which determines the trigger price c as

$$c = \frac{\gamma K}{\gamma + 1},$$

where γ solves the equation (13) (or (8)). See Figure 2.

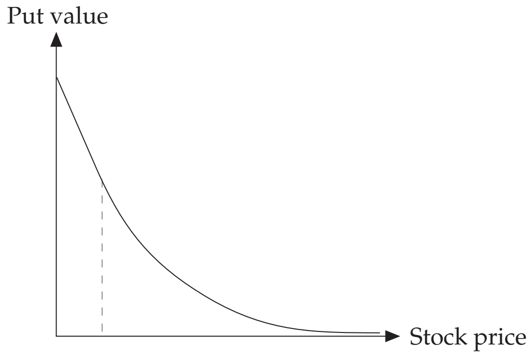
We can now finally use the verification theorem of optimal stopping for jump-diffusions (see e.g. Øksendal and Sulem (2004)) to prove that this is *the* solution to our problem. The main component of the verification theorem is the Dynkin formula, which states that

$$E^x\{\psi(S(\tau))\} = \psi(x) + E^x\{\int_0^{\tau} \mathcal{A}\psi(S(t))\, dt\}. \tag{14}$$

Here the requirement that $S(\tau) \in \bar{\mathcal{C}}$ comes into play, a sufficient condition for the theorem to hold. □

<u>Remarks:</u>

Figure 2: Perpetual American put option value



Market value of a perpetual American put option as a function of stock price.

1. If we use the exponential pricing model Y defined in Section 2 instead of the stochastic exponential, where $Y(t) = Y(0)\exp(Z_t)$, $Z_t = (X_t - \frac{1}{2}\sigma^2 t)$ and the the accumulated return process X_t is given by the arithmetic process in equation (3), this problem also has a solution, the above method works, and the corresponding equation for γ is given by

$$\begin{aligned} &-r - \gamma\big(\mu + \int_R (e^{\alpha\eta(z)} - 1 - \alpha\eta(z))\nu(dz)\big) \\ &+ \frac{1}{2}\sigma^2\gamma(\gamma+1) + \int_R \big(e^{-\gamma\alpha\eta(z)} - 1 + \gamma(e^{\alpha\eta(z)} - 1)\big)\nu(dz) = 0. \end{aligned} \tag{15}$$

2. By the verification theorem we get smooth pasting for free.
3. We may interpret the term $\left(\frac{x}{c}\right)^{-\gamma} I_{\{\tau(\omega)\in[t,t+dt)\}}(\omega)$ as the "state price" when $x \geq c$, where I indicates if exercise happens at time t or not: If exercise takes place at time t, then $(K-c)$ units are paid out at a price $(x/c)^{\gamma}$ per unit when $x \geq c$, and $(K-x)$ units are paid at price 1 per unit if $x < c$. Hence the term $(x/c)^{\gamma}$ can be interpreted as an "average state price" when $x \geq c$.
4. The assumption (5) may seem restrictive at this point, as it basically rules out jump processes having negative jumps. The problem arises if exercise occurs at a jump time of S. When this jump is negative it may carry $S(\tau)$ inside the exercise region where the value function $\psi(\cdot)$ is linear according to equation (6), in which case Dynkin's formula does not apply, since the value function has another form inside the integral in (14), also illustrated in Figure 2.

In the examples we indicate a direct method to solve the problem when jumps are negative, based on Dynkin's formula. We also demonstrate that the solution provided by Theorem 1 may still be a good approximation in this situation, especially when the current stock price x is away from the exercise boundary c. For the calibrations we consider, it turns out that the solution of Theorem 1 works well for price processes containing also negative jumps.

5 Risk adjustments

While the concept of an equivalent martingale measure is well known in the case of diffusion price processes with a finite time horizon $T < \infty$, the corresponding concept for jump price processes is less known. In addition we have an infinite time horizon, in which case it is not true that the "risk neutral" probability measure Q is equivalent to the given probability measure P.

Suppose P and Q are two probability measures, and let $P_t := P|_{\mathcal{F}_t}$ and $Q_t := Q|_{\mathcal{F}_t}$ denote their restrictions to the information set $\mathcal{F}_t$. Then P_t and Q_t are equivalent for all t if and only if $\sigma^P = \sigma^Q$ and the Lévy measures ν^P and ν^Q are equivalent.

We now restrict attention to the pure jump case, where the diffusion matrix $\sigma = 0$. Let $\theta(s,z) \leq 1$ be a process such that

$$\xi(t) := \exp\big\{ \int_0^t \int_R \ln(1-\theta(s,z))N(ds,dz) + \int_0^t \int_R \theta(s,z)\nu(dz)ds \big\} \tag{16}$$

exists for all t. Define Q_t by

$$dQ_t(\omega) = \xi(t)dP_t(\omega)$$

and assume that $E(\xi(t)) = 1$ for all t. Then there is a probability measure Q on $(\Omega, \mathcal{F})$ with the property that if we define the random measure $\tilde{N}^Q$ by

$$\tilde{N}^Q(dt,dz) := N(dt,dz) - (1-\theta(t,z))\nu(dz)dt,$$

then

$$\int_0^t \int_R \tilde{N}^Q(ds,dz) = \int_0^t \int_R N(ds,dz) - \int_0^t \int_R (1-\theta(s,z))\nu(dz)ds$$

is a Q local martingale. Notice that $\tilde{N}^Q(dt,dz) = \tilde{N}(dt,dz) + \theta(t,z)\nu(dz)dt$.

This result can be used to prove the following version of Girsanov's theorm for jump processes:

Theorem 2 *Let S_t be a 1-dimensional price process of the form*

$$dS_t = S_{t-}[\mu dt + \alpha \int_R \eta(z)\tilde{N}(dt,dz)].$$

Assume there exists a function $\theta(z) \leq 1$ such that

$$\alpha \int_R \eta(z)\theta(z)\nu(dz) = \mu \qquad a.s. \tag{17}$$

and such that the corresponding process ξ_t given in (16) (with $\theta(s,z) \equiv \theta(z)$ for all s) exists, with $E(\xi_t) = 1$ for all t. Consider a measure Q such that $dQ_t = \xi(t)dP_t$ for all t. Then Q is a local martingale measure for S.

Proof. By the above cited result and the equality (17) we have that

$$\begin{aligned} dS_t &= S_{t-}[\mu dt + \alpha \int_R \eta(z)\tilde{N}(dt,dz)] \\ &= S_{t-}[\mu dt + \alpha \int_R \eta(z)\{\tilde{N}^Q(dt,dz) - \theta(z)\nu(dz)dt\}] \\ &= S_{t-}\big[\alpha \int_R \eta(z)\tilde{N}^Q(dt,dz)\big], \end{aligned}$$

which is a local Q-martingale. $\square$

We will call Q a risk adjusted probability measure, and θ the market price of risk (when we use the bank account as a numeraire). The above results can be extended to a system of n-dimensional price processes, see e.g., Øksendal and Sulem (2004) for results on a finite time horizon, Sato (1999), Chan (1999) and Jacod and Shiryaev (2002) for general results, and Huang and Pagès (1992) or Revuz and Yor (1991)) for results on the infinite time horizon.

Recall that the computation of the price of an American option must take place under a risk adjusted, local martingale measure Q in order to avoid arbitrage possibilities. Under any such measure Q all the assets in the model must have the same rate of return, equal to the short term interest rate r. Thus we should replace the term μ by r in equation (10). However, this may not be the only adjustment required when jumps are present. Typically another, but equivalent, Lévy measure $\nu^Q(dz)$ will appear instead of $\nu(dz)$ in equation (10). We return to the details in the following sections.

6 Comparing two different models of the same underlying price process.

In this section we illustrate the above solution for two particular models of a financial market. We start out by recalling the solution in the standard lognormal continuous model, used by Black and Scholes and Merton.

6.1 The standard continuous model: $\alpha = 0$.

The standard geometric Brownian motion is given by

$$\frac{dS_t^c}{S_t^c} = \mu dt + \sigma dB_t. \tag{18}$$

Here the accumulated return process is defined by $R_t^c := \mu t + \sigma B_t$. It has mean μt, and variance $\sigma^2 t$.

The equation (8) has to be solved under a risk adjusted, local martingale measure Q in order for the solution to be the price of an American put option, we know that this is achieved in this model by replacing the drift rate μ by the interest rate r, and this is the only adjustment for Q required in this equation. Recall that the process $B_t^Q := B_t + \theta_c t$ is a standard Brownian motion for all $t \geq 0$ under Q, where $\theta_c = (\mu - r)/\sigma$ is the market price of risk, the Sharp ratio. Thus we have that the dynamics of S^c under Q is given by

$$\frac{dS_t^c}{S_t^c} = r dt + \sigma dB_t^Q. \tag{19}$$

The equation for γ then reduces to

$$-r - r\gamma + \frac{1}{2}\sigma^2\gamma(\gamma + 1) = 0 \tag{20}$$

which is a quadratic equation. It has the two solutions $\gamma_1 = 2r/\sigma^2$ and $\gamma_2 = -1$. The solution γ_2 is not possible, since the boundary condition $\psi(\infty; c) = 0$ for all c

simply can not hold true in this case. Thus the solution is $\gamma = \frac{2r}{\sigma^2}$, as first obtained by Merton (1973a).

Comparative statics can be derived from the expression for the market value in (6). The results are directly comparable to the results for the finite-lived European put option: The put price ψ increases with K, ceteris paribus, and the put price decreases as the stock price x increases. Changes in the volatility parameter have the following effects: Let $v = \sigma^2$, then

$$\frac{\partial \psi}{\partial v} = \begin{cases} \frac{c}{v}\left(\frac{c}{x}\right)^{\gamma} \ln\left(\frac{x}{c}\right), & \text{if } x \geq c; \\ 0, & \text{if } x < c. \end{cases} \tag{21}$$

Clearly this partial derivative is positive as we would expect.

Similarly, but with opposite sign, for the interest rate r:

$$\frac{\partial \psi}{\partial r} = \begin{cases} -\frac{2c}{\gamma v}\left(\frac{c}{x}\right)^{\gamma} \ln\left(\frac{x}{c}\right), & \text{if } x \geq c; \\ 0, & \text{if } x < c. \end{cases} \tag{22}$$

The effect of the interest rate on the perpetual put is the one we would expect, i.e., a marginal increase in the interest rate has, ceteris paribus, a negative effect on the perpetual put value.

Notice that we have used above that

$$\frac{\partial \psi}{\partial \gamma} = \begin{cases} -\frac{c}{\gamma}\left(\frac{c}{x}\right)^{\gamma} \ln\left(\frac{x}{c}\right), & \text{if } x \geq c; \\ 0, & \text{if } x < c, \end{cases} \tag{23}$$

in other words the price is a decreasing function of γ when $x := S_t \geq c$, a result we will make use of below.

From the derivation in Section 4 we notice that the relationship (23) is true also in the jump-diffusion model, and because of this property, one can loosely think of the parameter γ as being inversely related to the "volatility" of the pricing process, properly interpreted.

6.2 A discontinuous model: The jump component is proportional to a Poisson process.

In this section we assume that $\nu(dz)$ is the frequency λ times the Dirac delta function at z_0, i.e., $\nu(dz) = \lambda\delta_{\{z_0\}}(z)dz$, $z_0 \in R\backslash\{0\}$ so that all the jump sizes are identical and equal to z_0 (which means that N is a Poisson process, of frequency λ, times z_0). We consider the pure jump case ($\sigma^2 = 0$), and choose the function $\eta(z) \equiv z$ and $\alpha = 1$. The dynamic equation for the discontinuous risky asset is then

$$\frac{dS_t^d}{S_{t-}^d} = \mu dt + z_0 d\tilde{N}_t \tag{24}$$

where $\tilde{N}_t = (N_t - \lambda dt)$ is a compensated, mean zero Poisson processes, and N_t is a Poisson process having frequency λ. This process is to be compared to the standard geometric Brownian motion given in equation (18).

We see that the accumulated return process is here $R_t^d := \mu t + z_0\tilde{N}_t$ to be compared to R_t^c. Both have means μt, and their respective variances are $z_0^2(\lambda t)$ and

$\sigma^2 t$. Since the Poisson process is known to be infinite divisible, meaning that N_t can be "divided" into an arbitrary number of i.i.d. Poisson random variables, the Central Limit Theorem comes into play stating that for sufficiently large λt, the probability distribution of the return processes R_t^d will be approximately normal.

We now use the results of Section 5, where we employ the risk free asset as a numeraire. By Theorem 2 the market price of risk $\theta(z)$ in equation (17) must satisfy the equation

$$\int_{-1/\alpha}^{\infty} z\theta(z)\nu(dz) = \mu - r. \tag{25}$$

Due to the form of the Lévy measure $\nu(dz)$, this equation reduces to

$$\theta_0 := \theta(z_0) = \frac{\mu - r}{z_0 \lambda}. \tag{26}$$

The constant θ_0 (or perhaps $\theta_0 \lambda$) could be compared to the familiar Sharpe ratio $\frac{\mu - r}{\sigma}$ in the standard lognormal case. Here the term $z_0^2 \lambda$ is the variance rate corresponding to the term σ^2 in the geometric Brownian motion model. This model is complete, and there is only one solution to the above equation (25).

Consider the risk adjusted probability measure Q. If we derive the dynamics of the discounted process $\tilde{S}_t := e^{-rt} S_t$, this process has drift zero under the measure Q, corresponding to the market price of risk in (25), or equivalently, S has drift r under Q. Turning to the American perpetual put option, we must e.g., replace μ by r in the equation (8). It turns out that this is *not* the only adjustment to Q we have to perform in this regard. Let us first turn to the frequency λ.

Recall that $\nu(dz) = \lambda \delta_{\{z_0\}}(z)dz$ so we have that $\tilde{N}^Q(dt, dz) := \tilde{N}(dt, dz) + \theta_0 \lambda \delta_{\{z_0\}}(z) dz dt$ or

$$d\tilde{N}_t^Q := dN_t - (1 - \theta_0)\lambda dt. \tag{27}$$

The term $\lambda(1 - \theta_0)$ can be interpreted as the frequency of the jumps under the risk adjusted measure Q, since θ_0 does not depend upon the jump size z, or

$$\lambda^Q := \lambda(1 - \theta_0) = \lambda + \frac{r - \mu}{z_0}. \tag{28}$$

Thus

$$\frac{dS_t^d}{S_{t-}^d} = r dt + z_0 d\tilde{N}_t^Q, \tag{29}$$

which is the analogue of (19) in the standard model.

Turning to the jump size parameter z_0, consider two Poisson processes with frequencies λ and λ^Q and jump sizes z_0 and z_0^Q corresponding to two measures P and Q respectively. Only if $z_0 = z_0^Q$ can the corresponding P_t and Q_t be equivalent. This means that changing the frequency of jumps amounts to "reweighting" the probabilities on paths, and no new paths are generated by simply shifting the intensity. However, changing the jump sizes generates a different kind of paths. The frequency of a Poisson process can be modified without changing the "support" of the process, but changing the sizes of jumps generates a new measure which assigns nonzero probability to some events which were impossible under the old one. Thus $z_0 = z_0^Q$ is the only possibility here.

By the stochastic exponential solution formula (2), it follows from (27), (28) and (29) that S^d has the following representation under Q:

$$S_t^d = S_0^d \exp\{rt + (\ln(1+z_0) - z_0)\lambda^Q t + (\ln(1+z_0))\tilde{N}_t^Q\}. \tag{30}$$

By properly approximating the logarithmic function, this can be written

$$S_t^d \approx S_0^d \exp\{rt - \frac{1}{2} z_0^2 \lambda^Q t + z_0 \tilde{N}_t^Q\}, \tag{31}$$

which can be compared to the representation for the standard price process S^c under Q:

$$S_t^c = S_0^c \exp\{rt - \frac{1}{2}\sigma^2 t + \sigma B_t^Q\}. \tag{32}$$

An completely analogous comparison can be made under the probability measure P.

Here we notice an important difference between the standard continuous model and the model containing jumps. While it is a celebrated fact that the probability distribution under Q in the standard model does not depend on the drift parameter μ, a direct consequence of (32), in the jump model we see from (30) that it does. The risk adjusted frequency λ^Q enters both the drift term, and is part of the variance of $\tilde{N}_t^Q$, and it follows from (28) that λ^Q depends upon the drift term μ. This will have as a consequence that values of options must also depend on μ in the latter type of models. For the American perpetual put option we see this directly from the equation for γ

$$\lambda^Q (1+z_0)^{-\gamma} = (r - \lambda^Q z_0)\gamma + \lambda^Q + r. \tag{33}$$

Since this equation depends on the drift parameter μ through the term λ^Q, given in equation (28), the solution $\gamma = \gamma_d$ also depends upon μ, and finally so does the option value given in equation (6).

Let us briefly recall the argument why the drift parameter can not enter into the pricing formula for any contingent claim in the standard model: If two underlying assets existed with different drift terms μ_1 and μ_2 but with the same volatility parameter σ, there would simply be arbitrage. In the jump model different drift terms lead to different frequencies λ_1^Q and λ_2^Q through the equation (28), but this also leads to different volatilities of the two risky assets, since the volatility depends upon the jump frequency (under Q). Thus no inconsistency arises when the drift term enters the probability distribution under Q in the jump model.

We may solve the equations (28) and (33) in terms of the equity premium e_p, which we here define as $e_p := r - \mu$. This results in a linear equation for e_p with solution

$$e_p = z_0 \left\{ \frac{r(\gamma+1)}{(1+z_0)^{-\gamma} - (1 - z_0\gamma)} - \lambda \right\}. \tag{34}$$

Although this formula indicates a very simple connection between the equity premium and the parameters of the model, it is in some sense circular, since the parameter γ on the right hand side is not exogenous, but depends on all the parameters of the model.

Let us focus on the equation (33) for γ. This equation is seen to have a positive root γ_d where the power function to the left in equation (33) crosses the straight line to the right in (33). If $z_0 > 0$, there exists exactly one solution if $r < \lambda^Q \alpha z_0$ for positive interest rate $r > 0$. If $r \geq \lambda^Q \alpha z_0 > 0$ there is no solution.

Example 1. Here we illustrate different solutions to the equations for γ, first without risk adjustments, but where we calibrate the variance rates of the two noise terms. Recall that the variance of a compound Poisson process X_t is $\text{var}(X_t) = \lambda t E(Z^2)$, where Z is the random variable representing the jump sizes. We can accomplish this by choosing $\lambda = \sigma = 1$, when the jump size parameter $z_0 = 1$, noticing that z_0^2 here corresponds to $E(Z^2)$. Fixing the short term interest rate $r = .06$, we get the solution $\gamma_d = .20$ of equation (33), while the corresponding solution to the equation (20) is $\gamma_c = .12$. Suppose the exercise price $K = 1$. Then we can compute the trigger price $c_c = .11$ in the continuous model, while $c_d = .17$ in the discontinuous model. This means that without any risk adjustments of the discontinuous model, it is optimal to exercise earlier using this model than using the continuous model, at least for this particular set of parameter values. $\square$

Using the respective formulas for the prices of the American put option in Example 1, by the formula for the price $\psi(x, c)$ in equation (6) of Section 4 it is seen that the price ψ^c based on the continuous model is larger then the price ψ^{du} based on the discontinuous model with no risk adjustments, or $\psi^c(x; c_c) > \psi^{du}(x; c_d)$ for all values $x > c_c$ of the underlying risky asset, $\psi^c(x; c_c) = \psi^{du}(x; c_d)$ for $x \leq c_c$. According to option pricing theory, this ought to mean that there is "less volatility" in the jump model without risk adjustment than in the continuous risk adjusted counterpart. Thus risk adjustments of the frequency λ^Q must mean that $\lambda^Q > \lambda$, when $z_0 > 0$.

6.3 Solution when jumps are negative.

In this section we demonstrate a direct method based on Dynkin's formula to deal with the case of negative jump sizes. to this end, consider the pure jump model with negative jumps only

$$dS_t = S_{t-}[\mu dt + \int_{-1}^{\infty} z\tilde{N}(dt, dz)] \tag{35}$$

where the Lévy measure $\nu(dz) = \lambda \delta_{\{z_0\}}(z)dz$, and $z_0 < 0$ $(\alpha = 1)$. Define the operator $\mathcal{A}$ by

$$\mathcal{A}\psi(x) = -r\psi(x) + \mu x \psi'(x) + \int_{-1}^{\infty} (\psi(x + zx) - \psi(x) - \psi'(x) z_0 x)\nu(dz).$$

We want to find a constant $c \in (0, K)$ and a function ψ on $(0, \infty)$ such that ψ is continuous in $(0, \infty)$ and (i) $\psi(x) = K - x$ for $0 < x \leq c$ and (ii) $\mathcal{A}\psi(x) = 0$ for $x > c$. We construct ψ on (c, ∞) by induction:

Case 1: $x \in \mathcal{C}_1 := (c, c/(1 + z_0))$. Then $x(1 + z_0) < c$ and therefore S jumps from $\mathcal{C}_1$ down to $(0, c)$ if it jumps, where ψ is given by (i). Thus condition (ii) becomes

$$\mathcal{A}\psi(x) = -r\psi(x) + \mu x \psi'(x) + [K - x(1 + z_0) - \psi(x) - \psi'(x) z_0 x]\lambda = 0$$

for $x \in \mathcal{C}_1$. This leads to the following standard first order in-homogeneous ODE

$$\psi'(x) + G(x)\psi(x) = H_1(x)$$

where $G(x) = -\frac{r+\lambda}{(\mu - z_0\lambda)x}$ and $H_1(x) = -\frac{[K - x(1+z_0)]\lambda}{(\mu - z_0\lambda)x}$. The solution, denoted $\psi_1(x)$ in $\mathcal{C}_1$, is

$$\psi_1(x) = e^{-\int_c^x G(v)dv}\Big[\int_c^x e^{\int_c^v G(u)du} H_1(v)dv + k_1\Big]. \tag{36}$$

By continuity of the value function we determine the integrating constant k_1 by $\psi_1(c) = K - c$, implying that $k_1 = K - c$.

Case 2: $x \in \mathcal{C}_2 := (c/(1+z_0), c/(1+z_0)^2)$. Then $x(1+z_0) < c/(1+z_0)$ and therefore S jumps from $\mathcal{C}_2$ down to $\mathcal{C}_1$ if it jumps, where ψ is given by $\psi_1(\cdot)$ just determined. Thus condition (ii) becomes

$$\mathcal{A}\psi(x) = -r\psi(x) + \mu x\psi'(x) + [\psi_1(x(1+z_0)) - \psi(x) - \psi'(x)z_0 x]\lambda = 0$$

for $x \in \mathcal{C}_2$. This leads to the same kind of ODE as above

$$\psi'(x) + G(x)\psi(x) = H_2(x)$$

where $G(x) = -\frac{r+\lambda}{(\mu - z_0\lambda)x}$ and $H_2(x) = -\frac{\psi_1(x(1+z_0))\lambda}{(\mu - z_0\lambda)x}$. The solution, denoted $\psi_2(x)$ in $\mathcal{C}_2$, is

$$\psi_2(x) = e^{-\int_{c/(1+z_0)}^x G(v)dv}\Big[\int_{c/(1+z_0)}^x e^{\int_{c/(1+z_0)}^v G(u)du} H_2(v)dv + k_2\Big]. \tag{37}$$

By continuity of the value function we determine the integrating constant k_2 by $\psi_1(c/(1+z_0)) = \psi_2(c/(1+z_0))$. Thus $k_2 = \psi_1(c/(1+z_0))$, where $\psi_1(\cdot)$ is given above. This determines the value function ψ on $\mathcal{C}_2$.

Next we define $\mathcal{C}_3 = (c/(1+z_0)^2, c/(1+z-0)^3)$ and proceed as above to determine ψ on $\mathcal{C}_3$ etc. We summarize as

Theorem 3 *The solution of the optimal stopping problem*

$$\phi(s,x) = \sup_{\tau \geq 0} E^{s,x}\Big\{e^{-r(s+\tau)}(K - S_\tau)^+\Big\},$$

with S_t given by (35) has the form $\phi(s,x) = e^{-rs}\psi(x)$ where $\psi(x)$ is given inductively by the above procedure. In particular we have that

$$\psi(x) = \begin{cases} K - x, & \text{for } 0 < x \leq c; \\ \psi_1(x) & \text{for } x \in \mathcal{C}_1; \\ \psi_2(x) & \text{for } x \in \mathcal{C}_2; \end{cases}$$

and $\psi(x) = \psi_n(x)$ for $x \in \mathcal{C}_n$, $n = 3, 4, \cdots$, where $\psi_1(x)$ is given in equation (36), $\psi_2(x)$ is given in equation (37), etc.

Since we here have a first order ODE, it is not a natural requirement that the first derivative of the value function $\psi'(x)$ is continuous in the patching point $x = c$. It is true that the function itself is continuous there, a requirement we have already used. Thus we may seem to be lacking a criterion to determine the trigger price c.

The solution $\psi(x)$ above is the value of an American perpetual put option if we adjust for risk, i.e., when $\mu = r$ and λ is interpreted as the risk adjusted frequency under Q. If we consider the requirement $\psi'(c) = -1$, we only get a solution for c if $\mu \neq r$, and hence this trigger value does not correspond to the solution of the American put problem. One could perhaps conjecture that requiring the function $\psi(x)$ to be C^1 in the point $c/(1+z_0)$ would provide the "missing" equation, but this turns out to yield a tautology, i.e., $\psi_2'(c/(1-z_0)) = \psi_1'(c/(1+z_0)$ is automatically satisfied by the solution provided above and thus does not give anything new. The value of c must in fact be determined in the other end, namely by requiring that $\psi(x)$ approaches zero as $x \to \infty$.

Let us turn to an approximate solution when jumps are allowed to be negative, and focus on the equation (33) for γ. If $-1 < z_0 < 0$ this equation has exactly one solution for $r > 0$ provided that $\alpha z_0 > -1$, and is accordingly well defined. Reexamining the exact procedure above, notice that if we approximate the linear function $(K - x(1 + z_0))$ by the curved one $\psi(x(1 + z_0))$ in the term dictating the inhomogeneous part of the first order ODE, we would obtain the solution given in Theorem 1. The effect of this perturbation will be more and more diluted as x increases. This we can see by comparing ψ_1 to ψ_2, where the linear term in the numerator of H_1 has already been replaced by a curved one in the numerator of H_2. Thus we conjecture that for reasonably large values of the spot price x of the underlying asset, the solution obtained using Theorem 1 is a good approximation.

Example 2. Consider now the case where $z_0 < 0$, and let us pick $z_0 = -.5$. Now $\gamma_d = .29$ for the same set of parameter values as above. In order to properly calibrate the variance rates of the two models, we compare to the continuous model having $\sigma^2 = \lambda z_0^2 = .25$ or $\sigma = .50$. This yields $\gamma_c = .48$, which means that the situation is reversed from the situation in Example 1. The price commanded by the continuous model has decreased more than the corresponding price derived using discontinuous dynamics, without risk adjustments. Thus risk adjustments of the frequency λ^Q must now $(z_0 < 0)$ mean that $\lambda^Q < \lambda$. For $K = 1$ we find a trigger value $c_d = .23$, while the corresponding one in the standard model is $c_c = .32$, meaning that exercise tends to come earlier in the continuous model than in the discontinuous one. □

The results in this example does not seem unreasonable. From these numerical examples it seems like we have the following picture, at least at the moment: When the jumps are all positive (and identical) and we do not adjust for risk, the jump model produces put option values reflecting less risk than the continuous one. When the jump sizes are all negative (and identical), and we continue to consider the risk neutral case, the situation is reversed. These conclusions seem natural for a put option, since price increases in the underlying tend to lower the value of this insurance product. In Example 1 only upward, sudden price changes are possible for the underlying asset, whereas the downward movement stemming from the compensated term in the price path is slower and predictable. Thus a put option that is not adjusted for risk ought to have less value under such dynamics, than in a situation where only negative, sudden price changes can take place.

In the next section we consider numerical results after risk adjustments of the jump model. This leads to some interesting, economic results.

6.4 A calibration exercise.

We now use the two different models of sections 6.1 and 6.2 for the same phenomenon to infer about equity premiums in equilibrium. In order to do this, we calibrate the two models, which we propose to do in two steps. First we ensure that the martingale terms have the same variances in both models, just as in Example 1. Second, both models ought to yield the same option values. Let us present the argument for the latter. Recall the accumulated return processes for the two models. For the continuous standard model it is

$$R_t^c = \mu t + \sigma B_t$$

and for the geometric Poisson processes it is ($\alpha = 1$)

$$R_t^d = \mu t + z_0 \tilde{N}_t.$$

In both cases $E(R_t^c) = E(R_t^d) = \mu t$ and the variances are $\sigma^2 t$ and $z_0^2 \lambda t$ respectively. Furthermore the quantity $\tilde{N}_t/\sqrt{\lambda t}$ converges in distribution to the standard normal $\mathcal{N}(0,1)$-distribution as λt increases, and of course, $B_t/\sqrt{t}$ is $\mathcal{N}(0,1)$-distributed for any value of t. As a consequence, when we calibrate the variances, these two models come across as almost identical, at least for large enough values of λt. When a reasonably large value of λ is multiplied by the average time a typical investor would choose to hold this option, the normal approximation should be very appropriate for the Poisson process. Since the Poisson random variable is infinitely divisible, the normal approximation is particularly adequate. Note that this argument does not depend upon the size of the jumps parameter z_0.

Also consider the solutions to the option valuation problem in these two cases. The value functions are in both cases given by

$$\psi(x) = \begin{cases} (K-c)\left(\frac{c}{x}\right)^{\gamma}, & \text{if } x \geq c; \\ (K-x), & \text{if } x < c, \end{cases} \tag{38}$$

where the trigger price c is

$$c = \frac{\gamma K}{\gamma + 1}. \tag{39}$$

If investors are convinced that the probability distributions are approximately the same, they would typically equate the average state prices in the two situations. These are both being given by $(x/c)^{\gamma}$ when $x \geq c$. Clearly for the same contracts both the prices (x) of the underlying at initiation of the contract, and the exercise prices (K) are the same, which means that it suffices to equate the two γ_i -parameters and the trigger prices c_i, $i = \{c, d\}$. From the equations (38) and (39) we see that it is enough to equate the γ-values, and this leads in turn to the same values for the American put options in these two situations. Consider the following example:

Example 3. Choose $\sigma = .165$ and $r = .01$. (The significance of these particular values will be explained below.) Our calibration consists in the following two steps: (i) First, we match the volatilities. This gives the equation $z_0^2 \lambda = \sigma^2 = .027225$. We start with $z_0 = .01$, i.e., each jump size is positive and of size one per cent. The compensated part of the noise term consists of a negative drift, precisely "compensating" for the situation that all the jumps are positive. Then means that $\lambda = 272.27$ which is a roughly one jump each trading day on the average, where the time unit is one year.

(ii) Second, we calibrate the average state prices. From the discussion above, it follows from the equations (38) and (39) that this is equivalent to equating the values of γ. Thus we find the value of λ^Q that yields γ_d as a solution of equation (33) equal to the value γ_c resulting from solving the equation (20) for the standard, continuous model. For the volatility $\sigma = .165$, the latter value is $\gamma_c = .73462$. By trying different values of λ^Q in equation (33), we find that the equality in prices is obtained when $\lambda^Q = 274.73$.

Finally, from the equation (28) for the risk adjusted frequency λ^Q we can solve for the equity premium $e_p = (r - \mu)$, which is found to be .0248, or about 2.5 per cent.

Equivalent to the above is to use the formula for the equity premium e_p in equation (34) directly, using $\gamma = \gamma_c$ in this equation and the above parameters values of z_0, r and λ.

Is this value dependent of our choice for the jump size z_0? Let us instead choose $z_0 = .1$. This choice gives the value of the frequency $\lambda = 2.7225$ in step (i), the risk adjusted frequency $\lambda^Q = 2.9700$ in step (ii) and the value for the equity premium is consequently $(r - \mu) = .0248$, or about 2.5 per cent again. Choosing the more extreme value $z_0 = 1.0$, i.e., the upward jump sizes are all 100 per cent of the current price, gives the values $\lambda = .0272$ in step (i), the risk adjusted frequency $\lambda^Q = .0515$ follows from step (ii), and finally the equity premium $(r - \mu) = .0248$, again exactly the same value for this quantity. □.

In the above example the value of $\sigma = .165$ originates from an estimate of the volatility for the Standard and Poor's composite stock price index during parts of the last century. Thus the value of the equity premium around 2.5 per cent has independent interest in financial and macro economics.

Notice that $\mu < r$ in equilibrium. This is a consequence of the fact that we are analyzing a perpetual put option, which can be thought of as an insurance product. The equilibrium price of a put is larger than the expected pay-out, because of risk aversion in the market. For a call option we have just the opposite, i.e., $\mu > r$, but the perpetual call option is of no use for us here, since its market value equals x, the initial stock price.

Notice that in the above example we have essentially two free parameters to choose, namely z_0 and λ. The question remains how robust this procedure is regarding the choice of these parameters. The example indicates that our method is rather insensitive to the choice of these two parameters as long as the volatility $z_0^2\lambda$ stays constant.

Example 4. Set $\sigma = .165$ and $r = .01$, and consider the case when $z_0 < 0$. Using Theorem 1 to approximate the put price also for negative jumps, first choose $z_0 = -.01$. i.e., each jump size is negative and of size one per cent all the time. The compensated part of the noise term will now consist of a positive drift, again "compensating" for the situation that all the jumps are negative. Now we get $\lambda = 272.27$ in step (i) and the risk adjusted frequency is $\lambda^Q = 269.78$ in step (ii). This gives for the equity premium $(r - \mu) = .0247$, or again close to 2.5 per cent.

The value $z_0 = -.1$ gives $\lambda = .0273$ in step (i), the risk adjusted frequency $\lambda^Q = 2.473$ in step (ii) and an estimate for the equity premium is $(r - \mu) = .0251$. The more extreme value of $z_0 = -.5$, i.e., each jump results in cutting the price in half, provides us with the values $\lambda = .1089$ in step (i), $\lambda^Q = .0585$ in step (ii) and $(r - \mu) = .0252$ follows. This indicates a form of robustness regarding the choice of the jump size parameter and frequency. □

Note how well the results of this example match the results in the previous one. This we take as an indication that the procedure of approximating the value of the American perpetual put option in Example 3 by the one obtained in Theorem 1, is fairly accurate, at least for our purposes.

Comparative statics using numerical solutions of the equation for γ show that when $z_0 > 0$ and increases, the values of γ decreases, so by (23) the put price increases. When $z_0 < 0$ and decreases to -1, the parameter γ decreases, and consequently the option value increases. When λ^Q increases, the parameter γ decreases, and the option price increases.

This can be used to show that when the drift rate μ decreases, the put option gets more valuable. Similarly, when the equity premium e_p increases, the put option gets more valuable. An increase in the risk premium typically go along with an increase in the risk aversion in the market, in which case it is natural for for the put price to increase, since this product can be interpreted as an insurance product.

Notice that this kind of economic reasoning does not apply to the standard model, which must be considered as a weakness of that model, compared to the equally simple, pure jump Poisson model.

In turns out that the results of the examples 3 and 4 are robust to the choice of the parameter values z_0 and λ (as long at they produce the same volatility). We have tried ranges of z_0 from $z_0 = -0.9$ to $z_0 = 100$, and the variations in the corresponding equity premiums are insignificant. When we change r to 0.04, the corresponding values for the equity premium is about 0.045. The full explanation behind this really requires a jump version of the Consumption based Capital Asset Pricing Model (CCAPM), and is pursued elsewhere (see Aase (2005)).

7 A combination of the standard model and a Poisson process.

7.1 A complete model.

We now introduce diffusion uncertainty in the pure jump model of the previous section, and choose the standard Black and Scholes model as before for the diffusion part. Taking a look at the equation (8) for γ, at first sight this seems like an easy extension of the last section, including one more term in this equation. But is is more to it than that. First we should determine the market price of risk. We have now two sources of uncertainty, and by "Girsanov type" theorems this would lead to an equation of the form

$$\sigma\theta_1 + \alpha z_0 \lambda \theta_2(z_0) = \mu - r,$$

where θ_1 is the market price of diffusion risk and $\theta_2(z)$ is the market price of jump size risk for any z. This constitutes only one equation in two variables, and has consequently infinitely many solutions, so this model is not complete. The problem is that there is too much uncertainty compared to the number of assets. In the present situation we can overcome this difficulty by introducing one more risky asset in the model. Hence we assume that the market consists of one riskless asset as before, and two risky assets with price processes S_1 and S_2 given by

$$dS_1(t) = S_1(t-)[\mu_1 dt + \sigma_1 dB(t) + \alpha_1 \int_A z_1 \tilde{N}(dt, dz)], \tag{40}$$

where $S_1(0) = x_1 > 0$ and

$$dS_2(t) = S_2(t-)[\mu_2 dt + \sigma_2 dB(t) + \alpha_2 \int_A z_2 \tilde{N}(dt, dz)], \tag{41}$$

where $S_2(0) = x_2 > 0$. Here the set of integration $A = (-1/\alpha_1, \infty) \times (-1/\alpha_2, \infty)$, and $z = (z_1, z_2)$ is two-dimensional. We now choose the following Lévy measure:

$$\nu(dz) = \lambda \delta_{z_{1,0}}(z_1)\delta_{z_{2,0}}(z_2) dz_1 dz_2$$

meaning that at each time τ of jump, the relative size jump in S_1 is $z_{1,0}$ units multiplied by α_1, and similarly the percentage jump in S_2 is $z_{2,0}$ units times α_2. (One could perhaps say that the jump sizes are independent, but since there is just one alternative jump size for each "probability distribution", we get the above interpretation.)

These joint jumps take place with frequency λ. These returns have a covariance rate equal to $\sigma_1\sigma_2$ from the diffusion part and $\lambda\alpha_1\alpha_2 z_{1,0} z_{2,0}$ from the jump part, so the risky assets display a natural correlation structure stemming from both sources of uncertainty. This gives an appropriate generalization of the model of the previous section.

In order to determine the market price of risk for this model, we are led to solving the following two equations:

$$\sigma_1\theta_1 + \alpha_1 \int_A z_1 \theta_2(z) \nu(dz) = \mu_1 - r,$$

and

$$\sigma_2\theta_1 + \alpha_2 \int_A z_2 \theta_2(dz) \nu(dz) = \mu_2 - r.$$

Using the form of the Lévy mesure indicated above, the market price of jump size risk $\theta_2(z) = \theta_2$, a constant. The above two functional equations then reduce to the following set of linear equations

$$\sigma_1\theta_1 + \lambda\alpha_1 z_{1,0}\theta_2 = \mu_1 - r,$$

and

$$\sigma_2\theta_1 + \lambda\alpha_2 z_{2,0}\theta_2 = \mu_2 - r,$$

which leads to the solution

$$\theta_1 = \frac{(\mu_1 - r)\alpha_2 z_{2,0} - (\mu_2 - r)\alpha_1 z_{1,0}}{\sigma_1\alpha_2 z_{2,0} - \sigma_2\alpha_1 z_{1,0}}$$

for the market price of diffusion risk, and

$$\theta_2 = \frac{(\mu_1 - r)\sigma_2 - (\mu_2 - r)\sigma_1}{\lambda(\sigma_2\alpha_1 z_{1,0} - \sigma_1\alpha_2 z_{2,0})} \tag{42}$$

for the market price of jump size risk. Here $(\sigma_2\alpha_1 z_{1,0} - \sigma_1\alpha_2 z_{2,0}\sigma_2) \neq 0$, and the constant $\theta_2 \leq 1$. This solution is unique, so the model is complete provided the parameters satisfy the required constraints.

Consider the risk adjusted probability measure Q determined by the pair (θ_1, θ_2) via the localized, standard density process for the infinite horizon situation of Section 5. If we define $\tilde{N}^Q(dt, dz) := N(dt, dz) - (1 - \theta_2(z))\nu(dz)dt$, and $B^Q(t) := \theta_1 t + B(t)$, then $\int_0^t \int_A \tilde{N}^Q(dt, dz)$ is a local Q-martingale and B^Q is a Q-Brownian motion. The first risky asset can be written under Q,

$$dS_1(t) = S_1(t-)[rdt + \sigma_1 dB^Q(t) + \alpha_1 \int_A z_1 \tilde{N}^Q(dt, dz)], \tag{43}$$

and thus $\bar{S}_1(t) := S_1(t)e^{-rt}$ is a local Q-martingale. A similar result holds for the second risky asset.

We are now in the position to find the solution to the American put problem. Consider the option written on the first risky asset. It follows from the above that the equation for γ can be written

$$\lambda^Q(1 + \alpha_1 z_{1,0})^{-\gamma} = (r - \lambda^Q\alpha_1 z_{1,0})\gamma - \frac{1}{2}\sigma_1^2\gamma(\gamma + 1) + \lambda^Q + r, \tag{44}$$

where $\lambda^Q := \lambda(1 - \theta_2)$, and θ_2 is given by the expression in (42). Again we have dependence from the drift term(s) μ on the risk adjusted probability distribution. Here both of the parameters of the second risky asset enter into the expression for the risk adjusted frequency λ^Q, which means that the market price of jump risk must be determined in this model from equation (42) in order to price the American perpetual put option.

In the case when $z_{1,0} > 0$ (and $\alpha_1 > 0$), this equation can be seen to have one positive solution for $r > 0$. (When $r \leq 0$ there is a range of parameter values where the equation has two positive solutions, then one solution, and finally no solutions.)

Example 5. In order to compare this situation to the two pure models considered in examples 1 and 2, let us again choose the parameter values such that the variance rates of all three models are equal, and first we do not risk adjust the pure jump model, neither do we risk adjust the jump part of the model of this section. This means that we have set $\theta_1 = 0$ and $\theta_2 = 0$. This is accomplished, for example, by choosing $\alpha = 1$ and $\lambda = .7$, $\sigma = .55$ and $z_0 = 1$. For $r = .06$, we get the solution $\gamma_{d,c} = .17$ to the equation (44), while the solution to the equation (33) is $\gamma_d = .20$, and the corresponding solution to the equation (20) is $\gamma_c = .12$. Thus the present solution lies between the two first numerical cases considered in Example 1.

Considering the situation when $z_0 < 0$, we now calibrate to the situation of Example 2. When $-1 < z_{1,0} < 0$, there is exactly one solution of equation (44) when $r > 0$ (and no positive solutions when $r \leq 0$). Using the approximation in Theorem 1, we can choose $\alpha = \lambda = 1$ and $\sigma^2 + \lambda\alpha^2 z_0^2 = .25$, which is accomplished, for example, by choosing $\sigma^2 = .125$ and $z_0 = -.35$. This gives the solution $\gamma_{d,c}$

to the equation (44) equal to $\gamma_{d,c} = .40$, while the solution to the equation (33) is $\gamma_d = .29$, and the corresponding solution to the equation (20) is $\gamma_c = .48$, still using the same value for the short interest rate. Thus the present solution also lies between the two pure cases in Example 2. □

As a preliminary conclusion to this example we may be led to consider a combined jump-diffusion model as a compromise between the two pure counterparts.

Turning to calibration, with two risky assets we quickly get many parameters, and it is not obvious that we can proceed as before. We choose to equate both the drift rates and the variance rates, but use different characteristics for the latter. This way the market price of risk parameters will be well defined. According to the CCAPM for jump-diffusions we may then get different equity premiums, but the discrepancies may be small if the jump sizes are small, so we shall ignore them here.

Example 6. We choose $\alpha_1 = \alpha_2 = 1$, and consider the two equations $\sigma_c^2 = \sigma_1^2 + \lambda z_{1,0}^2 = .027225$ and $\sigma_c^2 = \sigma_2^2 + \lambda z_{2,0}^2 = .027225$, where we choose $\sigma_1 = .01$ ($\sigma_1 \leq .165$ is the obvious constraint here), and $z_{1,0} = .1$, $z_{2,0} = .01$. This leads to $\lambda = 2.7125$ and $\sigma_2 = .1642$. Then we calibrate the solution $\gamma_{c,d}$ to the equation (44) to the value for the standard continuous model $\gamma_c = .73462$ for the US data, and find that this corresponds to the risk adjusted frequency $\lambda^Q = 2.9593$. Assuming that $(\mu_1 - r) \approx (\mu_2 - r) := (\mu - r)$, the relationship $\lambda^Q = \lambda(1 - \theta_2)$ can now be written, using the expression (42) for the market price of jump risk θ_2:

$$\lambda^Q \approx \lambda + (r - \mu)\left(\frac{\sigma_2 - \sigma_1}{\sigma_2 \alpha_1 z_{1,0} - \sigma_1 \alpha_2 z_{2,0}}\right).$$

The only unknown quantity in this equation is the equity premium, which leads to the estimate $(r - \mu) = .0261$ when $r = .01$. The market price of diffusion risk $\theta_1 = -.14$, and the market price of jump risk is $\theta_2 = -.09$.

The same procedure when $r = .04$ leads to $\lambda^Q = 3.1658$, and the estimate $(r - \mu) = .048$. Now the market prices of risk are $\theta_1 = -.26$ and $\theta_2 = -.17$. □

7.2 An exact procedure when the jumps are negative.

Again we attempt to solve the problem by a direct method when jumps can be negative, using Dynkin's formula. To this end consider the above jump-diffusion

$$dS_t = S_{t-}[\mu dt + \sigma dB_t + \int_{-1}^{\infty} z\tilde{N}(dt, dz)], \tag{45}$$

where the Lévy measure $\nu(dz) = \lambda\delta_{\{z_0\}}(z)dz$, and μ, σ and $z_0 \in (-1, 0)$ are all constants ($\alpha = 1$). Define the operator $\mathcal{A}$ by

$$\mathcal{A}\psi(x) = -r\psi(x) + \mu x\psi'(x) + \frac{1}{2}\sigma^2 x^2 \psi''(x) + \int_{-1}^{\infty} (\psi(x + zx) - \psi(x) - \psi'(x)z_0 x)\nu(dz).$$

We want to find a constant $c \in (0, K)$ and a function ψ on $(0, \infty)$ such that $\psi \in C^1(0, \infty)$ (continuously differentiable) and (i) $\psi(x) = K - x$ for $0 < x \leq c$ and (ii) $\mathcal{A}\psi(x) = 0$ for $x > c$. Let us adjust for risk and set $\mu = r$ and interpret the frequency λ to be under the risk adjusted measure Q. We construct ψ on (c, ∞) by induction:

Case 1: $x \in \mathcal{C}_1 := (c, c/(1+z_0))$. Then $x(1+z_0) < c$ and therefore S jumps from $\mathcal{C}_1$ down to $(0, c)$ if it jumps, where ψ is given by (i). Thus condition (ii) becomes

$$\mathcal{A}\psi(x) = -r\psi(x) + rx\psi'(x) + \frac{1}{2}\sigma^2 x^2 \psi''(x) + [K - x(1+z_0) - \psi(x) - \psi'(x) z_0 x]\lambda = 0$$

for $x \in \mathcal{C}_1$. This leads to the following second order inhomogeneous ODE

$$\frac{1}{2}\sigma^2 x^2 \psi''(x) + (r - z_0\lambda)x\psi'(x) - (r+\lambda)\psi(x) = \lambda(1+z_0)x - K\lambda. \tag{46}$$

The general solution solution of (46) in $\mathcal{C}_1$ we denote by $\psi_1(x)$. It has the following form

$$\psi_1(x) = C_1 x^{\gamma_1} + C_2 x^{\gamma_2} - x + \frac{K\lambda}{r+\lambda}, \tag{47}$$

where C_1 and C_2 are arbitrary constants and $\gamma_1 < 0 < \gamma_2$ are the two solutions of

$$\frac{1}{2}\sigma^2 \gamma(\gamma - 1) + (r - \lambda z_0)\gamma - (\lambda + r) = 0. \tag{48}$$

By continuity of the value function and the assumption that $\psi(x)$ is continuously differentiable (high contact) we determine the integrating constant C_1 and C_2 from the following two equations:

$$(I) \qquad C_1 c^{\gamma_1)} + C_2 c^{\gamma_2} - c + \frac{K\lambda}{\lambda + r} = K - c$$

and

$$(II) \qquad C_1 \gamma_1 c^{(\gamma_1 - 1)} + C_2 \gamma_2 c^{(\gamma_2 - 1)} = 0.$$

The solution is

$$C_1 = \frac{Kr}{(\lambda + r)} \frac{\gamma_2}{(\gamma_2 - \gamma_1)} c^{(-\gamma_1)}. \tag{49}$$

and

$$C_2 = \frac{Kr}{(\lambda + r)} \frac{\gamma_1}{(\gamma_1 - \gamma_2)} c^{(-\gamma_2)}. \tag{50}$$

Case 2: $x \in \mathcal{C}_2 := (c/(1+z_0), c/(1+z_0)^2)$. Then $x(1+z_0) < c/(1+z_0)$ and therefore S jumps from $\mathcal{C}_2$ down to $\mathcal{C}_1$ if it jumps, where ψ is given by $\psi_1(\cdot)$ just determined. Thus condition (ii) becomes

$$\mathcal{A}\psi(x) = -r\psi(x) + \mu x\psi'(x) + \frac{1}{2}\sigma^2 x^2 \psi''(x) + [\psi_1(x(1+z_0)) - \psi(x) - \psi'(x) z_0 x]\lambda = 0$$

for $x \in \mathcal{C}_2$. This leads to the following second order ODE:

$$\frac{1}{2}\sigma^2 x^2 \psi''(x) + (r - z_0\lambda)x\psi'(x) - (r+\lambda)\psi(x) = \\ \lambda(1+z_0)x - \frac{K\lambda^2}{r+\lambda} - \lambda C_1((x(1+z_0))^{\gamma_1} - \lambda C_2((x(1+z_0))^{\gamma_2}. \tag{51}$$

Then we solve this equation in $\mathcal{C}_2$, denote the solution by $\psi_2(x)$, and proceed to $\mathcal{C}_3 = (c/(1+z_0)^2, c/(1+z-0)^3)$ etc.

In this situation we have required that the solution satisfies the high contact condition in $x = c$, and this led to the determination of the two constants C_1 and C_2 from equations (I) and (II) above. However, we still seem to lack a criterion to determine the trigger price c. One could perhaps conjecture that requiring the function $\psi(x)$ to be C^2 in the point $c/(1+z_0)$ would provide the "missing" equation, but again this turns out to be void, i.e., $\psi_2''(c/(1-z_0)) = \psi_1''(c/(1+z_0)$ is automatically satisfied by the solution provided above. The determination of c must, as before, be attempted in the other end as $x \to \infty$, where $\psi(x) \to 0$. We summarize as follows:

Theorem 4 *The solution of the optimal stopping problem*

$$\phi(s,x) = \sup_{\tau \geq 0} E_Q^{(s,x)} \Big\{ e^{-r(s+\tau)}(K - S_\tau)^+ \Big\},$$

with S_t given by (45) is, under adjustments for risk, equal to the value of an American perpetual put option, and has the form $\phi(s,x) = e^{-rs}\psi(x)$ where $\psi(x)$ is given inductively by the above procedure. In particular we have that

$$\psi(x) = \begin{cases} K - x, & \text{for } 0 < x \leq c; \\ \psi_1(x) & \text{for } x \in \mathcal{C}_1; \\ \psi_2(x) & \text{for } x \in \mathcal{C}_2; \end{cases}$$

where $\psi_1(x)$ is given in equation (47), the two constants C_1 and C_2 are provided in (49) and (50) respectively, and $\gamma_1 < 0 < \gamma_2$ are the two solutions of the equation (48); etc.

8 Different jump sizes.

We now turn to the situation where several different jump sizes can occur in the price evolution of the underlying asset. Suppose the Lévy measure ν is supported on n different points $a_1, a_2, \cdots, a_n$, where $-1 < a_1 < a_2 < \cdots < a_n < \infty$, $a_i \neq 0$ for all i. In our interpretation we may think of the jump size distribution function $F(dz)$ as having n simple discontinuities at each of the numbers $a_1, a_2, \cdots, a_n$ with sizes of the discontinuities equal to $p_1, p_2, \cdots, p_n$, p_i being of course the probability of the jump size a_i, $i = 1, 2, \cdots, n$.

A purely mechanical extenison of the model in section 6.2 leads to an incomplete model, since by proceeding this way we end up with one equation of the type

$$\alpha\lambda \sum_{i=1}^{n} a_i \theta(a_i) p_i = \mu - r,$$

containing n unknown market price of risk parameters $\theta(a_1), \theta(a_2), \cdots, \theta(a_n)$. Instead we consider the following market. A riskless asset exists as before, and n risky assets exist having price processes $S(t) = (S_1(t), S_2(t), \cdots, S_n(t))$ given by

$$\frac{dS_i(t)}{S_i(t-)} = \mu_i dt + \sum_{j=1}^{n} \alpha_{i,j} \int_{-1/\alpha_{i,j}}^{\infty} z\tilde{N}_j(dt, dz), \qquad i = 1, 2, \cdots, n. \tag{52}$$

Here N_j is a Poisson process, of frequency λ_j, times a_j, independent of N_i for $i \neq j$, and $\tilde{N}_j$ is the corresponding compensated process. If we define the return rate process R_i of asset i by $dR_i(t) = \frac{dS_i(t)}{S_i(t-)}$, this means that the jump distribution of R_i is $\alpha_{i,1}a_1$ with probability $p_1 := \frac{\lambda_1}{\sum_{j=1}^n \lambda_j}$, $\alpha_{i,2}a_2$ with probability $p_2 := \frac{\lambda_2}{\sum_{j=1}^n \lambda_j}, \cdots, \alpha_{i,n}a_n$ with probability $p_n := \frac{\lambda_n}{\sum_{j=1}^n \lambda_j}$. The covariance rate between returns R_i and R_j is given by $(\sum_{k=1}^n \alpha_{i,k}\alpha_{j,k}\lambda_k a_k^2)$, which can vary freely because of the relatively large freedom of choice of the parameters $\alpha_{i,j}$. Also note that jumps occur in any of the price processes with frequency $\lambda := \sum_{i=1}^n \lambda_i$.

This model gives us the following n equations to determine the market price of risk processes $\theta(z)$:

$$\sum_{k=1}^n \alpha_{i,k}\lambda_k \int_{-1/\alpha_{i,k}}^{\infty} z\theta_k(z)\delta_{\{a_k\}}(z)dz = \mu_i - r, \qquad i = 1, 2, \cdots, n. \tag{53}$$

Since $\delta_{\{a_k\}}(z)$ are the Dirac delta distributions at the points $\{a_k\}$, this system of equations reduce to the following system of n linear equations in n unknowns

$$\sum_{k=1}^n \alpha_{i,k}a_k\lambda_k\theta_k = \mu_i - r, \qquad i = 1, 2, \cdots, n, \tag{54}$$

where

$$\theta_i(z) = \theta_i(a_i) := \theta_i, \qquad \text{a constant for all } z. \tag{55}$$

This system of equations has a unique solution if the associated coefficient determinant is non-vanishing. The solution to the system (54) is

$$\theta = A^{-1}(\mu - r) \tag{56}$$

where θ is the vector of θ_i's, A^{-1} is the inverse of the matrix A with element $a_{i,j} = \alpha_{i,j}\lambda_j a_j$, $i, j = 1, 2, \cdots, n$, and $(\mu - r)$ is the vector with i-th element equal to $(\mu_i - r)$, $i = 1, 2, \cdots, n$. A unique solution exists when $\det(A) \neq 0$, which is equivalent to $\det(\tilde{\alpha}) \neq 0$, where $\tilde{\alpha}$ is the matrix with (i, j)-th element $\alpha_{i,j}$. As an illustration, if $n = 2$ this means that the requirement is $(\alpha_{1,1}\alpha_{2,2} - \alpha_{2,1}\alpha_{1,2}) \neq 0$.

We now turn to the density process associated with the change of probability measure from P to Q. It is given by

$$\xi(t) := \exp\Big\{\sum_{i=1}^n\sum_{j=1}^n \int_0^t \int_{-1/\alpha_{i,j}}^{\infty} [\ln(1-\theta_j)N_j(ds,dz) + \theta_j\lambda_j\delta_{\{a_j\}}(z)dzdt]\Big\}. \tag{57}$$

This means that the restriction Q_t of Q to $\mathcal{F}_t$ is given by $dQ_t(\omega) = \xi(t)dP_t(\omega)$ for any given time horizon t, where P_t is the restriction of P to $\mathcal{F}_t$, and $E\xi(t) = 1$ for all t, then P_t and Q_t assign zero probability to the same events in $\mathcal{F}_t$. As before we call Q the risk adjusted measure for the price system S in the infinite time horizon case. This means that if we define the processes $N_i^Q(dt,dz) := N_i(dt,dz) - (1-\theta_i)\nu_i(dz)dt$, $i = 1, 2, \cdots, n$, the processes $\int_0^t \int_{-1/\alpha_{i,j}}^{\infty} \tilde{N}_i^Q(ds,dz)$ are local Q-martingales for $j = 1, 2. \cdots, n$, where the risky assets have the following dynamics under Q:

$$\frac{dS_i(t)}{S_i(t-)} = rdt + \sum_{j=1}^n \alpha_{i,j}\int_{-1/\alpha_{i,j}}^{\infty} z\tilde{N}_j^Q(dt,dz), \qquad i = 1, 2, \cdots, n, \tag{58}$$

implying that $\bar{S}_i(t) := S_i(t)e^{-rt}$ is a zero drift local Q-martingale for all i.

This model is accordingly complete provided $\theta_i \leq 1$ for all $i = 1, 2, \cdots, n$, and pricing e.g., the perpetual American put option written on, say, the first asset, is a well defined problem with a unique solution. The equation for γ for this option can be written

$$-r - r\gamma + \sum_{j=1}^{n} \lambda_j^Q [(1 + \alpha_{1,j} a_j)^{-\gamma} - 1 + \alpha_{1,j} a_j \gamma] = 0, \tag{59}$$

where $\lambda_j^Q = \lambda_j(1 - \theta_j)$, $j = 1, 2, \cdots, n$. Note that this equation follows from equation (8) after the appropriate risk adjustments of the various frequecies, if we set $\alpha_{1,j} = \alpha$ for all j, and Lévy measure $\nu(dz) = \lambda^Q F(dz)$, where the probability distribution function F has discontinuities at the points $a_1, a_2, \cdots, a_n$ with probabilities $p_1^q, p_2^q, \cdots, p_n^q$, where $p_i^q = \frac{\lambda_i^Q}{\lambda^Q}$ and λ^Q is the frequency of the jumps under the probability Q ($\lambda^Q = \sum_i \lambda_i^Q$). Thus our model captures the general situation, under P, with a frequency of jumps equal to λ and a pdf of jump sizes F with support on n different points. Here is an example:

Example 7. We consider the case of $n = 2$, where we do not adjust for risk. The parameters $\alpha_{1,1} = \alpha_{1,2} = 1$, and $a_1 = -.5$, $a_2 = 1$ so that each price jump either cuts the price in half, or doubles the current price of the underlying asset. We let $\lambda_1 = \lambda_2 = 1$ so that the two different jump sizes are equally probable under P, and the total frequency λ of jumps equals two per time unit.

Fixing $r = .06$ as before, we get the solution $\gamma_{2d} = .12$ to the equation (59). In order to compare to the standard model and the model with only upward jumps of the same size (= 1) of Example 1, we find the corresponding γ-values adjusted so that the various variance rates are equal. They are $\gamma_c = .10$ for $\sigma^2 = 5/4$ and $\gamma_d = .16$ for $\lambda = 5/4, \alpha = 1$, and $z_0 = 1$ in the jump model. Since $\gamma_c < \gamma_{2d} < \gamma_d$, the corresponding American perpetual put option prices are ranked $\psi_c > \psi_{2d} > \psi_d$.

In this situation, when both upward and downward sudden jumps are possible in the price paths of the underlying asset, the corresponding put price is between the polar cases of only continuous movements or only upward jumps.

Comparing to the situation with only downward jumps of size $-.5$ of Example 2, this is calibrated to have the same variance by choosing $z_0 = -.5, \lambda = 5, \alpha = 1$ which gives $\gamma_d = .06$. Thus we get $\psi_d > \psi_c > \psi_{2d}$, so here the situation with two jumps reflects the "least risky" situation. □

We notice that also in the situation with several jumps prices of contingent claims depend on the drift rates μ_i of the basic risky assets. In addition to requiring a risk adjustment of the frequencies λ_i, the *probabilities* p_i of the different jump sizes must also be risk adjusted under Q. Thus the system (54) of n linear equations in n unknowns for the market prices of jump risk θ_i must be solved in order to correctly price options and other contingent claims in this model.

8.1 Calibration when $n = 2$

An attempt to calibrate this model to the data from the Standard and Poor's composite stock index during the time period 1889-1979 is not likely to succeed, since only estimates of the short time interest rate and the stock index volatility are not enough to determine all the parameters in this model. Consider for example the case of $n = 2$. For the model to be complete, we need one risky asset in addition to the index,

From the solution (56) of the system of equations (54) when $n = 2$, we get for the market prices of risk parameters θ_1 and θ_2 the following two expressions:

$$\theta_1 = \frac{\alpha_{2,2}(\mu_1 - r) - \alpha_{1,2}(\mu_2 - r)}{\lambda_1 a_1(\alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\alpha_{2,1})}, \tag{60}$$

and

$$\theta_2 = \frac{\alpha_{2,1}(\mu_1 - r) - \alpha_{1,1}(\mu_2 - r)}{\lambda_2 a_2(\alpha_{1,2}\alpha_{2,1} - \alpha_{1,1}\alpha_{2,2})}. \tag{61}$$

From the equations $\lambda_i^Q = \lambda_i(1 - \theta_i), i = 1, 2$, we find the risk adjusted frequencies,

$$\lambda_1^Q = \lambda_1 + \frac{\alpha_{2,2}(r - \mu_1) - \alpha_{1,2}(r - \mu_2)}{a_1(\alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\alpha_{2,1})}, \tag{62}$$

and

$$\lambda_2^Q = \lambda_2 + \frac{\alpha_{2,1}(r - \mu_1) - \alpha_{1,1}(r - \mu_2)}{a_2(\alpha_{1,2}\alpha_{2,1} - \alpha_{1,1}\alpha_{2,2})}. \tag{63}$$

We must choose the constants in the matrix $\tilde{\alpha}$ such that the determinant $(\alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\alpha_{2,1}) \neq 0$. Choosing the first risky asset similar to the composite stock index, its variance rate must satisfy

$$\alpha_{1,1}^2\lambda_1 a_1^2 + \alpha_{1,2}^2\lambda_2 a_2^2 = \sigma^2, \tag{64}$$

where $\sigma^2 = 0.027225$ as for the index. The variance rate of the second risky asset is given by

$$\alpha_{2,1}^2\lambda_1 a_1^2 + \alpha_{2,2}^2\lambda_2 a_2^2. \tag{65}$$

In equilibrium there is a connection between the equity premiums and the standard deviation rate, which we now wish to utilize. By the CCAPM for jump-diffusions (Aase (2004)), while a linear relationship is almost exact for the model of Section 6, for the present model this is no longer the case. By Schwartz's inequality this linear relationship is at the best approximately true when the jump sizes are small and different in absolute value. Assuming we can use this approximation here, we get the following:

$$(\mu_2 - r) \approx (\mu_1 - r)\sqrt{\frac{\alpha_{2,1}^2\lambda_1 a_1^2 + \alpha_{2,2}^2\lambda_2 a_2^2}{\alpha_{1,1}^2\lambda_1 a_1^2 + \alpha_{1,2}^2\lambda_2 a_2^2}}. \tag{66}$$

We are now in position to derive an approximate expression for the equity premium $e_p = (r - \mu_1)$. Using (66) in the expressions (62) and (63), we get $\lambda_1^Q = \lambda_1 + k_1 e$ and $\lambda_2^Q = \lambda_2 + k_2 e$, where

$$k_1 = \frac{\alpha_{2,2} - \alpha_{1,2}\sqrt{\frac{\alpha_{2,1}^2\lambda_1 a_1^2 + \alpha_{2,2}^2\lambda_2 a_2^2}{\alpha_{1,1}^2\lambda_1 a_1^2 + \alpha_{1,2}^2\lambda_2 a_2^2}}}{a_1(\alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\alpha_{2,1})},$$

and

$$k_2 = \frac{\alpha_{2,1} - \alpha_{1,1}\sqrt{\frac{\alpha_{2,1}^2\lambda_1 a_1^2 + \alpha_{2,2}^2\lambda_2 a_2^2}{\alpha_{1,1}^2\lambda_1 a_1^2 + \alpha_{1,2}^2\lambda_2 a_2^2}}}{a_2(\alpha_{1,2}\alpha_{2,1} - \alpha_{1,1}\alpha_{2,2})}.$$

Inserting these expressions in the equation (59) for γ when $n = 2$, we get a linear equation for e_p, which solution is

$$\begin{aligned} e_p =& \big[r(\gamma+1) + \lambda_1\big((1 - a_1\alpha_{1,1}\gamma) - (1 + a_1\alpha_{1,1})^{-\gamma}\big) \\ &+ \lambda_2\big((1 - a_2\alpha_{1,2}\gamma) - (1 + a_2\alpha_{1,2})^{-\gamma}\big)\big] / \big[k_1\big((1 + a_1\alpha_{1,1})^{-\gamma} - (1 - a_1\alpha_{1,1}\gamma)\big) \\ &+ k_2\big((1 + a_2\alpha_{1,2})^{-\gamma} - (1 - a_2\alpha_{1,2}\gamma)\big)\big]. \end{aligned} \tag{67}$$

A numerical example is the following.

Example 8. Choosing the parameters $\alpha_{1,1} = \alpha_{2,2} = \alpha_{1,2} = 1$ and $\alpha_{2,1} = 2$, the absolute value of the determinant $|\tilde{\alpha}|$ equals one, so the risk premiums are well defined. We choose $a_1 = 0.02$ and $a_2 = -0.01$, and $p_1 = 0.5$, and consider first the case where the short term interest rate $r = 0.01$. Since $p_1 = \lambda_1/(\lambda_1 + \lambda_2)$, we obtain that $\lambda_1 = \lambda_2 = 54.45$ from equation (64). From the relation (66) we find that $(r - \mu_2) = 1.84(r - \mu_1)$, and this enables us to compute the market price of risk parameters θ_1 and θ_2, and hence the risk adjusted frequencies, which are

$$\lambda_1^Q = 54.45 + 42.16(r - \mu_1), \qquad \lambda_2^Q = 54.45 - 15.69(r - \mu_1)$$

in terms of the equity premium $(r - \mu_1)$ of the index. By inserting these values in the equation (59) for γ_{2d}, we can find the value of the risk premium that satisfies $\gamma_{2d} = \gamma_c$, where γ_c is the corresponding solution for the standard model. For $r = 0.01$ this value is $\gamma_c = 0.73462$. This calibration gives the value $(r - \mu_1) = 0.0226$, or 2.26 per cent equity premium for the composite stock index. The forgoing can alternatively (and computationally less requiring) be accomplished by using $\gamma = \gamma_c$ in the expression for e given in (67), together with the other parameter values indicated.

A similar procedure for the spot rate $r = 0.04$ calibrates γ_{2d} to $\gamma_c = 2.93848$, and this gives $(r - \mu_1) = 0.041$, or an equity premium of 4.1 per cent for the stock index. Both these values are reasonably close to the values obtained in Section 6. □

Our results for the present model indicate that linear relationship implied by the CCAPM does not hold, so any calibration to the continuous, standard model becomes less interesting here than for the simpler model in Section 6. This is not to say that our results in Section 6 are not valuable, or correct, it only means that the present model is not as well suited to produce these results as the geometric Poisson.

For the present model one could instead proceed as follows: (a) Observe option prices in the market. (b) Estimate the parameters of the index from historical observations. From this one could find a market estimate of γ. Then the correct version of the CCAPM should be used to improve the approximation (66), and finally use the corresponding expression to (67) to compute e_p. This procedure would presumably need some consumption data when using the CCAPM.

9 A combination of the standard model and the jump model with different jump sizes.

By introducing also diffusion uncertainty in the model of the previous section, there will be "too much uncertainty" compared to the number of assets, but again we may use the method of Section 8 to enlarge the space of jumps and add one risky asset. This will lead to a complete model, and our valuation problem again becomes well defined. The equation for γ is now

$$-r - r\gamma + \frac{1}{2}\sigma_1^2\gamma(\gamma+1) + \sum_{j=1}^{n} \lambda_j^Q[(1+\alpha_{1,j}a_j)^{-\gamma} - 1 + \alpha_{1,j}a_j\gamma] = 0, \qquad (68)$$

where σ_1 is the volatility parameter of the continuous part of the first risky asset.

Example 9. We compare the combined model of this section, not adjusted for jump risk, with the purely discontinuous model of the previous section, also not risk adjusted, and the standard, continuous model. Using the same parameter values as in the first part of Example 7, we again obtain $\gamma_{2d} = .12$, and $\gamma_c = .10$ from the standard model having the same variance rate. Choosing $\alpha_{1,1} = \alpha_{1,2} = .9$ and $\sigma^2 = .2375$, the combined model of this section has the same variance as the other two models. This gives the solution to equation (68) equal to $\gamma_{2d,c} = .11$. Thus $\gamma_c < \gamma_{2d,c} < \gamma_{2d}$, or $\psi_{2d} > \psi_{2d,c} > \psi_c$ so that the combined model fits in between the two pure models, as we also saw in Example 5. □

10 The model with a continuous jump size distribution.

We round off this paper by considering the situation with a continuous distribution for the jump sizes in the jump part of the model. In this case the model is incomplete as long as there is a finite number of assets, since there is "too much uncertainty" compared to the number of assets.

The case with countably many jump sizes in the underlying asset could be approached along the lines of Section 8, by introducing more and more risky assets. In order for the market prices of risk $\theta_1, \theta_2, \cdots$ to be well defined, presumably only mild additional technical conditions need to be imposed. One line of attack is to weakly approximate any such distribution by a sequence of discrete distributions with finite supports. This would require more and more assets, and in the limit, an infinite number of primitive securities in order for the model to possibly be complete.

Here we will not elaborate further on this, but only make the assumption that the pricing rule is linear, which would be the case in a frictionless economy where it is possible to take any short or long position. This will ensure that there is *some* probability distribution and frequency for the jumps giving the appropriate value for γ, corresponding to a value for the perpetual American put option.

Below we limit ourselves to a discussion of the prices obtained this way for two particular choices of the jump distribution, where the risk adjustment is carried out mainly through the frequency of jumps.

The model is the same as in Section 2 with one risky security S and one locally riskless asset β. The risky asset has price process S satisfying

$$dS_t = S_{t-}[\mu dt + \alpha \int_{-1/\alpha}^{\infty} z\tilde{N}(dt, dz)], \tag{69}$$

where the density process of S is given by

$$\xi(t) = \exp\{\int_0^t \int_{-1/\alpha}^{\infty} \ln(1-\theta(z))N(ds, dz) + \int_0^t \int_{-1/\alpha}^{\infty} \theta(z)\lambda F(dz)ds\}, \tag{70}$$

Here $\theta(z)$ is the market price of risk process and $F(dz)$ is the distribution function of the jump sizes, assumed absolutely continuous with a probability density $f(z)$. According to our results in Section 5, if the market price of risk satisfies the following equation

$$\int_{-1/\alpha}^{\infty} z\theta(z)f(z)dz = \frac{\mu - r}{\lambda\alpha}, \tag{71}$$

then the risk adjusted compensated jump process can be written

$$\tilde{N}^Q(dt, dz) = N(dt, dz) - (1-\theta(z))\lambda f(z)dzdt. \tag{72}$$

This means that the term

$$\lambda^Q f^Q(z) := \lambda(1-\theta(z))f(z) \tag{73}$$

determines the product of the risk adjusted frequency λ^Q and the risk adjusted density $f^Q(z)$, when θ satisfies equation (71). If the market price of risk θ is a constant, there is no risk adjustment of the density $f(z)$. The densities $f(z)$ and $f^Q(z)$ are mutually absolutely continuous with respect to each other, which means in particular that the domains where they are both positive must coincide.

Clearly the equation (71) has many solutions θ, so the model is incomplete.

In solving the American perpetual put problem for this model, it follows from our previous results that the equation for γ is given by

$$-r - r\gamma + \int_{-1/\alpha}^{\infty} \{(1+\alpha z)^{-\gamma} - 1 + \alpha\gamma z\}\lambda^Q f^Q(dz) = 0, \tag{74}$$

where we have carried out the relevant risk adjustments.

As before, the corresponding solution in Theorem 1 is only an approximation when jumps can be negative. The problem arises if exercise can happen at a time of jump of the underlying price process S. For a given stock price $S_t = x$ and jump size z, we are asking what is the probability that $x(1+\alpha z) < c$. In this case the term $\psi(x + \alpha xz)$ in equation (10) should be replaced by the linear function $(K - (x + \alpha xz))$. This probability is

$$\int_{(-\frac{1}{\alpha})}^{(\frac{c}{\alpha x} - \frac{1}{\alpha})} f(z)\, dz$$

which is seen to become small as x increases. Thus we conjecture that the error committed can not be large if we approximate the linear function by the curve ψ

in this situation, which is what Theorem 1 does. Below we consider two situations where the jumps can be both negative and positive, but still we use Theorem 1, so one may interpret our results as approximations, and relatively more accurate the larger the values of x.

We now turn to the illustrations by considering two special cases for the jump density $f(z)$.

10.1 The truncated normal case

Here we analyze normally distributed returns. In our model formulation, where we have chosen the stochastic exponential, we observe from the expression (2) for S that we can not allow jump sizes less than $-1/\alpha$, so the domain of F is the interval $[-1/\alpha, \infty)$. In this case we choose to consider a truncated normal distribution at $-1/\alpha$. By and large we restrict our attention to risk adjustments associated with a constant θ only. In the present case the most straightforward risk adjustments of the normal density $f(z)$ having mean m and standard deviation s would be another normal density having mean m^Q and standard deviation s^Q, with a similar adjustment for the truncated normal distribution. Here we only notice that a joint risk adjustment of the jump distribution f to another truncated normal with parameters m^Q and s^Q, and of the frequency λ to λ^Q, means that the equity premium can be written

$$e_p = \alpha\big(\lambda^Q E^Q\{Z|m^Q, S^Q\} - \lambda E\{Z|m, s\}\big), \tag{75}$$

where the expectations are taken of the truncated normal random variable Z with respect to the parameters indicated. The above formula then follows from (71) and (73). Notice that α does not change under the measure Q, since the supports of f and f^Q must coincide.

The exponential pricing model with normal jump sizes was considered by Merton (1976). In that case the probability density of the pricing model S_t is known explicitly. In contrast to Merton, who assumed that the jump size risk was not priced, or, he did not adjust for this type of risk, we will risk adjust precisely the jump risk, and our model is the stochastic exponential, not the exponential as he used.

Below we have calibrated this model to the standard continuous one using the same technique as outlined earlier. Since the equity premium is not proportional to the volatility of S in this model, we can not expect to confirm the simple results of Section 7. For Z a random variable with a truncated normal distribution at $-1/\alpha$, we first solve the equation $\lambda\alpha^2 E(Z^2) = \sigma_c^2 = .027225$, or

$$\lambda\alpha^2 \frac{\int_{-1/\alpha}^{\infty} z^2 \frac{1}{\sqrt{2\pi}s} e^{-\frac{1}{2}\left(\frac{z-m}{s}\right)^2} dz}{\int_{-1/\alpha}^{\infty} \frac{1}{\sqrt{2\pi}s} e^{-\frac{1}{2}\left(\frac{z-m}{s}\right)^2} dz} = \sigma_c^2$$

for various values of m and s, and find the frequency λ. Then we solve equation (74), using the relevant values for r and $\gamma = \gamma_c(r)$, to find the risk adjusted frequency λ^Q, and finally we use equation (71) to find the equity premium $e_p = (r - \mu)$, assuming θ is a constant, so that $\lambda^Q = \lambda(1-\theta)$ and $f = f^Q$. Some results are summarized in tables 1 and 2.

α	1	1	1	.01	.8	3
(m,s)	(.1, .1)	(.4, .7)	(.4, 2.0)	(10, 10)	(.01, .01)	(.01, .01)
λ	1.36	.042	0.0065	1.36	212.70	15.13
λ^Q	1.60	.079	.019	1.60	215.78	15.94
e_p	0.024	0.026	0.025	0.024	0.025	0.024

Table 1: The equity premium e_p when $r = 0.01$ and $\gamma = 0.73462$, for various values of the parameters. The jumps are truncated normally distributed.

By decreasing the parameter α we notice from the above equation that this has the effect of increasing the frequency of jumps λ. Alternatively this can be achieved by decreasing the values of m and s, as can be observed in Table 15, where the spot rate is equal to 4 per cent. A decrease in the standard deviation s, within certain limits, moves the present model closer to the one of Section 7.

α	1	1	1	.9	2	10
(m,s)	(.1, .1)	(-.01, .01)	(1.0, 0.1)	(.01, .01)	(.011, .01)	(.01, .01)
λ	1.36	136.13	0.027	136.06	30.80	1.36
λ^Q	1.79	131.62	.076	173.01	32.91	1.79
e_p	0.043	0.045	0.049	0.045	0.046	0.043

Table 2: The equity premium e_p when $r = 0.04$ and $\gamma = 2.93848$, for various values of the parameters. The jumps are truncated normally distributed.

10.2 Exponential tails

In this model the distribution of the jump sizes is an asymmetric exponential with density of the form

$$f(z) = pae^{-a|z|}I_{[-\infty,0]}(z)/(1-e^{-a/\alpha}) + (1-p)be^{-bz}I_{[0,\infty]}(z)$$

with $a > 0$ and $b > 0$ governing the decay of the tails for the distribution of negative and positive jump sizes and $p \in [0,1]$ representing the probability of a negative jump. Here $I_A(z)$ is the indicator function of the set A. The probability distribution of returns in this model has semi-heavy (exponential) tails. Notice that we have truncated the left tail at $-1/\alpha$. The exponential pricing version of this model, without truncation, has been considered by Kou (2002).

Below we calibrate this model along the lines of the previous section. Also here we restrict attention to risk adjusting the frequency only. We then have the following expression for the equity premium:

$$e_p = \alpha(\lambda^Q - \lambda)\Big(p\big(\frac{e^{-a/\alpha}}{\alpha(1-e^{-a/\alpha})} - \frac{1}{a}\big) + (1-p)\frac{1}{b}\Big), \qquad (76)$$

where the frequency is risk adjusted, but not f. A formula similar to (75) can be obtained if also the density f is to be adjusted for risk. The simplest way to accomplish this here is to consider another probability density f^Q of the same type as the above f with strictly positive parameters p^Q, a^Q and b^Q. This would constitute an absolutely continuous change of probability density, but there are of course very many other possible changes that are allowed. In finding the expression (76) we have first solved the equation (71) with a constant θ, and then substituted for the market price of risk using the equation $\lambda^Q = \lambda(1-\theta)$.

Proceeding as in the truncated normal case, we first solve the equation $\lambda\alpha^2 E(Z^2) = \sigma_c^2 = 0.027225$, which can be written

$$\lambda\alpha^2\bigg(p\big(1-e^{-a/\alpha}\big)^{-1}\big(\frac{2}{a^2} - e^{-a/\alpha}\big(\frac{1}{\alpha^2} + \frac{2}{a\alpha} + \frac{2}{a^2}\big)\big) + (1-p)\big(\frac{2}{b^2}\big)\bigg) = \sigma_c^2. \quad (77)$$

Then we determine reasonable parameters through the equation $\alpha E(Z) = R_e$ for various values of R_e. This equation can be written:

$$R_e := \alpha\bigg(p\bigg(\frac{e^{-a/\alpha}}{\alpha(1-e^{-a/\alpha})} - \frac{1}{a}\bigg) + (1-p)\frac{1}{b}\bigg). \quad (78)$$

In order to arrive at reasonable values for the various parameters, we solve the two equations (77) and (78) in a and b for various values of the parameters α, p and R, where we have fixed the value of $\lambda = 250$. Then for the spot rates $r = 0.01$ and $r = 0.04$ with corresponding values of $\gamma = \gamma_c(r)$ respectively, we solve the equation (74) to find the value of λ^Q. Finally we compute the value of the equity premium from the formula (76). Some results are the following:

(α, p)	(1, .45)	(1, .55)	(1, .60)	(.01, .40)	(.01, .45)	(.01, .60)
R_e	.004	-.004	-.004	.0045	.004	.0035
a	350.23	104.07	110.34	3.76	3.50	5.54
b	140.07	350.23	278.21	1.08	1.04	.87
λ^Q	255.92	243.92	244.55	255.85	252.71	257.41
e_p	0.024	0.024	0.022	0.026	0.024	0.026

Table 3: The equity premium e_p when $r = 0.01$ and $\gamma = .73462$, for various values of the parameters, where $\lambda = 250$. The jumps are truncated, asymmetric exponentials.

(α, p)	(1, .40)	(1, .45)	(1, .60)	(.01, .40)	(.01, .45)	(.01, .60)
R_e	- .0035	-.0035	-.0035	.0045	.004	.0035
a	87.29	93.62	113.58	3.76	3.50	5.54
b	554.30	420.88	224.41	1.08	1.04	.87
λ^Q	236.24	237.53	241.29	260.54	260.67	263.38
e_p	0.048	0.044	0.048	0.047	0.043	0.047

Table 4: The equity premium e_p when $r = 0.04$ and $\gamma = 2.93848$, for various values of the parameters, where $\lambda = 250$. The jumps are truncated, asymmetric exponentials.

Since the equity premium is not proportional to the volatility of S in this model, we can not expect to obtain the simple and unique results of Section 6. As in the case of several jumps in Section 8 and the truncated normal case of the previous section, we typically get a wide variety of equity premiums for a given standard deviation of the price process, as the parameters vary. There is simply too much freedom in these models to obtain the unique results of Section 6. The volatility of the stock is not a "sufficient statistic" for its risk premium in these models.

The tables 1-4 identify parameters that are consistent of the simple results obtained in Section 6, and are not meant to be representative of the variation one may obtain for e_p. Obviously there is a large amount of parameter values that satisfy this. These tables primarily illustrate numerical solutions of the basic equation (8) for γ, and how the calibration procedure works to infer about e_p in more complex models.

Notice that, as for the simple case of the geometric Poisson process, the probability distribution of returns under any risk adjusted probability measure Q depends on the equity premium here as well, as we have demonstrated in this section. This means that once we have estimated the various process parameters from, say, time series data, and observed option prices in the market, we may find implied equity premiums in much the same manner as implied volatility is found in various option pricing models. This method does not require a comparison to the standard model.

11 Conclusions

In this paper we have solve an optimal stopping problem with an infinite time horizon, when the state variable follows a jump-diffusion. Under certain conditions, explained in the paper, our solution can be interpreted as the price of an American perpetual put option, when the underlying asset follows this type of process.

We present several examples demonstrating when the solution can be interpreted as a perpetual put price. This takes us into a study of how to risk adjust jump-diffusions. One key observation is that the probability distribution under the risk adjusted measure depends on the equity premium, which is not the case for the standard, continuous version. This difference may be utilized to find intertemporal, equilibrium equity premiums, for example.

We applied this technique to the US equity data of the last century, and found an indication that the risk premium on equity was about two and a half per cent if the risk free short rate was around one per cent. On the other hand, if the latter rate was about four per cent, we similarly find that this corresponds to an equity premium of around four and a half per cent.

Our basic solution is exact only when jump sizes can not be negative. We investigate when our solution is an approximation also for negative jumps.

The advantage with our approach is that we needed only equity data and option pricing theory, no consumption data was necessary to arrive at these conclusions.

Various market models were studied at an increasing level of complexity, ending with the incomplete model in the last part of the paper. In these models the equity premiums are no longer proportional to the volatility of the assets. An econometric investigation, where option prices are observed in the market, would enable us to find implied equity premiums also for these more complex models, since the probability distribution under the risk adjusted measure still depends on the equity premium.

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