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by

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MARKOV LEARNING MODELS FOR MULTIPERSON SITUATIONS, II.
METHODS OF ANALYSIS*/

by

Patrick Suppes and Richard C. Atkinson**/

§2.1. Introduction. The aim of this chapter is to present those methods of data analysis which are specific to the models we consider. Although some of these methods are standard, they may not be familiar to all readers. The emphasis is on maximum likelihood methods or variants thereof, which we shall term pseudo-maximum likelihood methods.

We first consider, in the next section, maximum likelihood estimates for the learning parameters in the case of noncontingent reinforcement and then for the two-person interaction situation described in §1.4. Necessarily, as we shall see, the methods described require the assumption of exactly one element in the stimulus set of each subject.

In the third section, the methods of parameter estimation are extended to models which assume more than one stimulus element. The methods developed formally resemble the maximum likelihood methods of the preceding section but, in fact, do not have the maximum likelihood property. In this context we also consider a method for estimating jointly the number N of stimuli and the value θ of the learning parameter.

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*/ This report assumes familiarity with Technical Report No. 21, which is labeled "Markov Learning Models for Multiperson Situations, I. The Theory." The authors are indebted to Mr. J. Merrill Carlsmith who assisted us in some of the computations reported here.

**/ On leave of absence from University of California, Los Angeles.

In the fourth section we consider various χ^2 tests for testing hypotheses that the transition probabilities of a first order chain are constant, that (in case the transition probabilities are constant) they are specific numbers, and that the process is a $\underline{u}^{th}$ order Markov chain against the alternative it is $\underline{r}^{th}$ but not $\underline{u}^{th}$ order. In this section no attempt is made to derive these tests, for which the reader is referred to the recent statistical literature on Markov chains.

In the fifth section we give some methods of estimation--peculiar to the models--for making what is "almost" a maximum likelihood estimate of the learning parameter. These methods are of perhaps more theoretical than practical issue. They hinge upon first making a least squares estimate of the not directly observed transition numbers m_{ij} from state i to state j in the transition matrix for the multi-element model.

Finally, in the sixth section a generalized conditioning model is developed in which the single parameter θ is replaced by several conditioning parameters. Simple maximum likelihood methods of estimation for these new parameters are derived.

§2.2. Maximum Likelihood Estimates. When the states of a Markov chain are observable and the transition probabilities depend on a real parameter, say θ, then it is often not difficult to obtain the maximum likelihood estimate $\hat{\theta}$ of θ.

Keeping in mind that we are mainly interested in Markov chains which consist of a sequence of response random variables, let $a_1, a_2, \ldots, a_n$

represent a finite sequence of values of the response random variables from trial 1 to n . Let s be the number of subjects. Then the maximum likelihood estimate of the learning parameter Θ is the number $\hat{\Theta}$ (if it exists) such that for all Θ'

$$(2.2.1)\qquad \prod_{\sigma=1}^{s} f^{(\sigma)}(a_1,a_2,\ldots,a_n;\hat{\Theta}) \geq \prod_{\sigma=1}^{s} f^{(\sigma)}(a_1,a_2,\ldots,a_n;\Theta') \quad .$$

In the inequality (2.2.1), $f^{(\sigma)}(a_1,a_2,\ldots,a_n;\hat{\Theta})$ is the probability of the sequence of responses $a_1,a_2,\ldots,a_n$ for subject σ when the learning parameter is $\hat{\Theta}$. It is important to understand that the notation $a_1,a_2,\ldots,a_n$ for each σ does not imply every subject has the same sequence of responses. It would be more explicit, but also more cumbersome to write: $a_1^{(\sigma)},a_2^{(\sigma)},\ldots,a_n^{(\sigma)}$. Use of the product notation in (2.2.1) expresses the assumption that the probabilities of response sequences of different subjects are statistically independent.

Consider now for a single subject $f(a_1,a_2,\ldots,a_n;\Theta)$, where we omit temporarily the superscript σ . By virtue of the fundamental Markov property of the process, we have:

$$(2.2.2)\ f(a_n|a_{n-1};\Theta)f(a_{n-1}|a_{n-2};\Theta)\cdots f(a_2|a_1;\Theta)f(a_1;\Theta) = f(a_n,a_{n-1},\ldots,a_1;\Theta) \quad .$$

Thus, summing over trials and subjects, we want to maximize

$$(2.2.3)\qquad \prod_{\sigma=1}^{s} \prod_{m=2}^{n} f^{(\sigma)}(a_m|a_{m-1};\Theta)f^{(\sigma)}(a_1;\Theta) \quad .$$

Let N be the number of states in the process; let $p_{ij}(\Theta)$ be the probability of going from state i to state j with parameter value Θ; let n_{ij} be the observed number of transitions from state i to j, aggregated over trials and subjects (thus the n_{ij} are tabulated from experimental data); let $p_i(\Theta)$ be the probability of being in state i on trial 1; and, finally, let n_i be the number of subjects in state i on trial 1 (i.e., here the number of subjects who make response i on trial 1). Substituting this notation in (2.2.3) we may then replace (2.2.1) by*/

$$(2.2.4) \qquad \prod_{i,j=1}^{N} p_i^{n_i}(\hat{\Theta}) p_{i,j}^{n_{ij}}(\hat{\Theta}) \geq \prod_{i,j=1}^{N} p_i^{n_i}(\Theta') p_{ij}^{n_{ij}}(\Theta') \quad .$$

That is, to find $\hat{\Theta}$ we want to maximize with respect to Θ

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*/ To prevent any confusion we may make completely explicit the use of this notation. Suppose, for instance, there are two subjects with the following sequences of responses for the first five trials: 12121, 11222, where 1 indicates response A_1 and 2 indicates response A_2. Then clearly $n_1 = 2$, $n_2 = 0$, because both subjects made the A_1 response on the first trial, and, as may easily be checked, $n_{11} = 1$, $n_{12} = 3$, $n_{21} = 2$ and $n_{22} = 2$. The use of the n_{ij} as exponents in (2.2.4) comes from the simple equation:

$$p_1^2 p_{11} p_{12}^3 p_{21}^2 p_{22}^2 = p_1 p_{12} p_{21} p_{12} p_{21} p_1 p_{11} p_{12} p_{22} p_{22}$$

$$\prod_{i,j=1}^{N} p_i^{n_i}(\theta) p_{ij}^{n_{ij}}(\theta) ,$$

but the value of θ which maximizes this expression also maximizes the log of it, and in many cases the latter is easier to work with. Consequently we shall usually seek to maximize

$$(2.2.5) \qquad L(\theta) = \sum_i [n_i \log p_i(\theta) + \sum_j n_{ij} \log p_{ij}(\theta)] .$$

Ordinarily the function $L(\theta)$ will have a local maximum, and so we can find θ as an appropriate solution of

$$(2.2.6) \qquad \frac{dL(\theta)}{d\theta} = \sum_i [\frac{n_i p_i'(\theta)}{p_i(\theta)} + \sum_j \frac{n_{ij} p_{ij}'(\theta)}{p_{ij}(\theta)}] = 0 ,$$

where p' is the derivative with respect to θ of p.

We now apply the general results (2.2.5) and (2.2.6) to the one-element model for the noncontingent case, discussed in Chapter 1. The transition matrix (p_{ij}) is given by (1.3.7); moreover, if we begin with trial 1, $p_i(\theta)$ is independent of θ, and so we obtain from (2.2.6)

$$(2.2.7) \qquad \frac{dL(\theta)}{d\theta} = -\frac{n_{11}(1-\pi)}{1-\theta(1-\pi)} + \frac{n_{12}}{\theta} + \frac{n_{21}}{\theta} - \frac{n_{22}\pi}{1-\theta\pi} = 0 ,$$

and this simplifies to the quadratic equation given in the following theorem.

<u>Theorem</u>. <u>The maximum likelihood estimate of</u> Θ <u>for the one-element model in the noncontingent case is a root of the equation</u>:

$$(2.2.8) \quad (n_{11}+n_{12}+n_{21}+n_{22})\pi(1-\pi)\Theta^2 - [n_{11}(1-\pi)+n_{12}+n_{21}+n_{22}\pi]\Theta + (n_{12}+n_{21}) = 0 \ .$$

Numerical solutions of this equation for various experimental data are presented in subsequent chapters. From Descartes' rule of signs, it follows that there are no negative roots of (2.2.8), but it is sometimes true that a root greater than one, which is a maximum, will be found. The question naturally arises of what interpretation to place on $\hat{\Theta} > 1$. Clearly such a $\hat{\Theta}$ cannot be interpreted as a probability, even though the stochastic character of the matrix (1.3.7) may not itself be disturbed by $\hat{\Theta} > 1$. It seems to us that the best compromise is probably to think of such an estimate as approximating a true value of Θ which, as a probability, is very close to one. However, it has been our experience that when $\hat{\Theta} > 1$ the overall fit of the model is usually rather poor. We shall return to this point in later chapters.

It is to be noticed that we cannot, in terms of observable quantities, make a similar maximum likelihood estimate of Θ for the two-element model of the noncontingent case, whose transition matrix is given by (1.3.17). The difficulty is that the states of the Markov chain are unobservable, and the sequence of response random variables, which is a sequence of observables, is not a Markov chain, but a chain of infinite order, that is, the probability of response, on a given trial depends on all preceding responses.

The quadratic equation (2.2.8) suggests that the maximum likelihood estimate of Θ would be the root of a polynomial of higher degree in the zero-sum, two-person situation described by matrix (1.4.5). (For the present, we shall assume $\Theta_A = \Theta_B$ in (1.4.5) and thus consider estimation of a single parameter.) Surprisingly enough this is not the case. In fact, the estimate has a much simpler form.

<u>Theorem</u>. <u>If</u> $0 < a_i < 1$, <u>then the maximum likelihood estimate of</u> Θ <u>for the one-element model in the zero-sum, two-person case is</u>:

$$(2.2.9)\quad \hat{\Theta} = \frac{n_{12}+n_{13}+n_{21}+n_{24}+n_{31}+n_{34}+n_{42}+n_{43}}{n_{11}+n_{22}+n_{33}+n_{44}+n_{12}+n_{13}+n_{21}+n_{23}+n_{31}+n_{34}+n_{42}+n_{43}} \quad .$$

<u>Proof</u>: As in the noncontingent case, $p_i(\Theta)$ is independent of Θ , and we have from (1.4.5)

$$\frac{dL(\Theta)}{d\Theta} = -\frac{n_{11}}{1-\Theta} + \frac{n_{12}a_1}{a_1\Theta} + \frac{n_{13}(1-a_1)}{(1-a_1)\Theta} + \frac{n_{21}a_2}{a_2\Theta} - \frac{n_{22}}{1-\Theta} + \frac{n_{24}(1-a_2)}{(1-a_2)\Theta} + \frac{n_{31}(1-a_3)}{(1-a_3)\Theta}$$

$$- \frac{n_{33}}{1-\Theta} + \frac{n_{34}a_3}{a_3\Theta} + \frac{n_{42}(1-a_4)}{(1-a_4)\Theta} + \frac{n_{43}a_4}{a_4\Theta} - \frac{n_{44}}{1-\Theta} = 0 \quad .$$

Simplifying, we obtain:

$$\frac{n_{12}+n_{13}+n_{21}+n_{24}+n_{31}+n_{34}+n_{42}+n_{43}}{\Theta} - \frac{n_{11}+n_{22}+n_{33}+n_{44}}{1-\Theta} = 0 \quad .$$

We solve this linear equation to obtain the desired result. If a_i is 0 or 1 this estimate is changed in the obvious way.

The simultaneous estimate of Θ_A and Θ_B in the zero-sum, two-person situation is more complicated. Specifically

$$(2.2.10)\quad \frac{\partial L(\Theta_A,\Theta_B)}{\partial \Theta_A} = \frac{n_{11}(a_1-1)}{a_1(\Theta_A-\Theta_B)+1-\Theta_A} + \frac{n_{13}}{\Theta_A} + \frac{n_{22}(a_2-1)}{a_2(\Theta_A-\Theta_B)+1-\Theta_A} + \frac{n_{24}}{\Theta_A} + \frac{n_{31}}{\Theta_A}$$

$$+ \frac{n_{33}(a_3-1)}{a_3(\Theta_A-\Theta_B)+1-\Theta_A} + \frac{n_{42}}{\Theta_A} + \frac{n_{44}(a_4-1)}{a_4(\Theta_A-\Theta_B)+1-\Theta_A} = 0$$

$$\frac{\partial L(\Theta_A,\Theta_B)}{\partial \Theta_B} = -\frac{n_{11}a_1}{a_1(\Theta_A-\Theta_B)+1-\Theta_A} + \frac{n_{12}}{\Theta_B} + \frac{n_{21}}{\Theta_B} - \frac{n_{22}a_2}{a_2(\Theta_A-\Theta_B)+1-\Theta_A}$$

$$-\frac{n_{33}a_3}{a_3(\Theta_A-\Theta_B)+1-\Theta_A} + \frac{n_{34}}{\Theta_B} + \frac{n_{43}}{\Theta_B} - \frac{n_{44}a_4}{a_4(\Theta_A-\Theta_B)+1-\Theta_A} = 0 \quad .$$

The maximum likelihood estimates of Θ_A and Θ_B are determined by the simultaneous solution of the above equations.

Maximum likelihood estimates for conditioning parameters in other two and three-person interaction situations are presented in later chapters.

§2.3. Pseudo-Maximum Likelihood Estimates. When more than one stimulus element is assumed, the problems of estimating the parameter Θ are considerably more difficult. As has already been remarked in Chapter 1 the states of the Markov chain are the possible states of conditioning of the stimuli, and these states are unobservable. The observable sequence of response random variables $< \underline{A}_1, \underline{A}_2, \ldots, \underline{A}_n, \ldots >$ is itself a chain of infinite order, that is, the probability distribution on trial n of $\underline{A}_n$ depends, not just on the response on trial $n-1$, but on that of all preceding trials. As in the case of the one-element model, (2.2.1) defines the maximum likelihood estimate $\hat{\theta}$, but (2.2.2) no longer holds. The approach of the pseudo-maximum likelihood method of estimation is directly to use (2.2.3) as the expression to maximize even though the resulting estimate is not the maximum likelihood one. The justification of this approach is that it represents a crude but computationally practical approximation of the maximum likelihood estimate. If the transition numbers n_{ijk} and the accompanying theoretical expressions $f(a_m | a_{m-1} a_{m-2}; \theta)$ are used, a still better estimate is obtained at the cost of considerably more work. Similarly, use of the transition numbers n_{ijkl} yield a still better approximation. Let us call (2.2.3) the first order pseudo-maximum likelihood estimate θ^*, and the estimate which uses two preceding trials the second-order estimate, and so on. It is not difficult to show that for all the models considered in this book the n-order pseudo-maximum likelihood estimates rapidly converge to the maximum likelihood estimate $\hat{\theta}$. This result follows directly from showing that the chains of infinite order constituted by the response random variables satisfy the conditions for convergence given in Lamperti and Suppes [1959].

As a simple example, we compute the (first order) pseudo-maximum likelihood estimate θ^* for the two-element noncontingent model discussed in §1.3. To begin with, we need the probabilities

(2.3.1) $$\lim_{n \to \infty} P(\underline{A}_{n+1} = i \mid \underline{A}_n = j)$$

for $i, j = 1, 2$. To abbreviate notation in derivations we replace the random variable notation '$\underline{A}_{n+1} = i$' by the event notation '$A_{i,n+1}$', etc., that is, $A_{i,n+1}$ is the event of response i on trial $n+1$. Similarly, $E_{k,n}$ is the event of reinforcing event E_k on trial n .

Moreover, for purposes of subsequent generalization and to illustrate some technical simplifications which are often useful, we shall consider the two-element noncontingent model in terms of a Markov chain whose states are the number of stimulus elements conditioned to the A_1 response. Thus there will be three states, 0, 1, 2, rather than the four states $0, \{s_1\}, \{s_2\}, \{s_1, s_2\}$ of the process defined by (1.3.17). From the discussion in §1.3 it should be obvious how to construct the trees of the Markov chain in these three states, and we simply give here the transition matrix.

(2.3.2)

	0	1	2
0	$1-\theta\pi$	$\theta\pi$	0
1	$\frac{1}{2}\theta(1-\pi)$	$1-\frac{1}{2}\theta$	$\frac{1}{2}\theta\pi$
2	0	$\theta(1-\pi)$	$1-\theta(1-\pi)$.

We now use (2.3.2) to compute the quantities (2.3.1). Again, for notational purposes, let $C_{i,n}$ be the event of i stimulus elements conditioned to A_1 on trial n . We first note that

$$(2.3.3) \qquad P(A_{1,n+1}|A_{1,n}) = \sum_{i=0}^{2} P(A_{1,n+1}C_{i,n+1}|A_{1,n})$$

$$= P(A_{1,n+1}|C_{1,n+1})P(C_{1,n+1}|A_{1,n}) +$$

$$P(A_{1,n+1}|C_{2,n+1})P(C_{2,n+1}|A_{1,n}) ,$$

because

$$P(A_{1,n+1}|C_{i,n+1}A_{1,n}) = P(A_{1,n+1}|C_{i,n+1}) ,$$

which is a consequence of the "independence of path" axioms, and

$$P(A_{1,n+1}|C_{0,n+1}) = 0 .$$

Also, from the sampling and response axioms, we have that

$$(2.3.4) \qquad \begin{cases} P(A_{1,n+1}|C_{1,n+1}) = \frac{1}{2} \\ P(A_{1,n+1}|C_{2,n+1}) = 1 , \end{cases}$$

and thus (2.3.3) simplifies to:

$$(2.3.5) \qquad P(A_{1,n+1}|A_{1,n}) = \frac{1}{2} P(C_{1,n+1}|A_{1,n}) + P(C_{2,n+1}|A_{1,n}) .$$

Our problem now is to compute the two quantities on the right of (2.3.5). We first observe that

$$(2.3.6)\quad P(C_{1,n+1}|A_{1,n}) = \sum_{i=0}^{2} P(C_{1,n+1}A_{1,n}C_{i,n})/P(A_{1,n})$$

$$= P(C_{1,n+1}|A_{1,n}C_{1,n})P(A_{1,n}|C_{1,n})P(C_{1,n})/P(A_{1,n})$$

$$+ P(C_{1,n+1}|A_{1,n}C_{2,n})P(A_{1,n}|C_{2,n})P(C_{2,n})/P(A_{1,n}) ,$$

where again we have eliminated the state C_0 because $P(A_{1,n}|C_{0,n}) = 0$. Now

$$(2.3.7)\quad P(C_{1,n+1}|A_{1,n}C_{1,n}) = P(C_{1,n+1}|E_{1,n}A_{1,n}C_{1,n})P(E_{1,n}) +$$

$$P(C_{1,n+1}|E_{2,n}A_{1,n}C_{1,n})P(E_{2,n})$$

$$= \pi + (1-\theta)(1-\pi) ,$$

for in order for $A_{1,n}$ to occur, the sampled element must be conditioned to A_1 . Whence if E_1 occurs, regardless of the effectiveness of conditioning, the new conditioning state is the same. However, if E_2 occurs the conditioning stays the same only with probability $(1-\theta)$. Now for the second term on the right of (2.3.6), we can pass from $C_{2,n}$ to $C_{1,n+1}$ only if an E_2 reinforcement occurs to reduce the number of stimulus elements conditioned to A_1 . Thus

$$(2.3.8)\quad P(C_{1,n+1}|A_{1,n}C_{2,n}) = \theta(1-\pi) .$$

It easily follows from the asymptotic results in §1.3 that

$$(2.3.9)\qquad \begin{cases} \lim C_{0,n} = (1-\pi)^2 \\ \lim C_{1,n} = 2\,\pi\,(1-\pi) \\ \lim C_{2,n} = \pi^2 \end{cases}$$

and of course $\lim P(A_{1,n}) = \pi$. We thus infer from (2.3.4), (2.3.6) - (2.3.9) that

$$(2.3.10)\ \lim P(C_{1,n+1}|A_{1,n}) = [\pi + (1-\Theta)(1-\pi)]\cdot\frac{1}{2}\cdot 2\pi(1-\pi)/\pi + \Theta(1-\pi)\pi^2/\pi$$

$$= \pi(1-\pi) + (1-\Theta)(1-\pi)^2 + \Theta\,\pi\,(1-\pi)\quad .$$

We next compute $\lim P(C_{2,n+1}|A_{1,n})$. It is clear that given the A_1 response on trial n, the sampled stimulus element on trial n must be conditioned to A_1 and thus to have event $C_{2,n+1}$ it is necessary to have $C_{2,n}$. That is, given $A_{1,n}$, conditioning state $C_{2,n+1}$ only can be reached from $C_{2,n}$. Thus

$$(2.3.11)\ \lim P(C_{2,n+1}|A_{1,n}) = \lim P(C_{2,n+1}|A_{1,n}C_{2,n})P(A_{1,n}|C_{2,n})P(C_{2,n})/P(A_{1,n})$$

$$= [\pi + (1-\Theta)(1-\pi)]\pi^2/\pi$$

$$= \pi - \Theta\,\pi\,(1-\pi)\quad .$$

Going back now to (2.3.3) and applying (2.3.4), (2.3.10) and (2.3.11), we have:

$$(2.3.12)\quad \lim P(A_{1,n+1}|A_{1,n}) = \tfrac{1}{2}[\pi(1-\pi)+(1-\theta)(1-\pi)^2+\theta\,\pi(1-\pi)]+\pi-\theta\,\pi(1-\pi)$$

$$= \pi + \frac{(1-\theta)(1-\pi)}{2}\,,$$

which is the desired result. From considerations of symmetry, or by direct computation, it is easily established that

$$(2.3.13)\qquad \lim P(A_{2,n+1}|A_{2,n}) = 1-\pi+\frac{(1-\theta)\pi}{2}\,,$$

and thus also

$$(2.3.14)\qquad \begin{cases} \lim P(A_{1,n+1}|A_{2,n}) = \pi-\dfrac{(1-\theta)\pi}{2} \\ \lim P(A_{2,n+1}|A_{1,n}) = 1-\pi-\dfrac{(1-\theta)(1-\pi)}{2} \end{cases} .$$

Following then (2.2.7) of the preceding section, the first order pseudo-maximum likelihood estimate θ^* is a solution of the equation

$$(2.3.15)\quad \frac{dL^*(\theta)}{d\theta} = -\frac{n_{11}(1-\pi)}{1+\pi-\theta(1-\pi)}+\frac{n_{12}}{(1+\theta)}+\frac{n_{21}}{(1+\theta)}-\frac{n_{22}\pi}{2-(1+\theta)\pi} = 0\,,$$

which simplifies to the quadratic equation of the following theorem.

Theorem. The first order pseudo-maximum likelihood estimate θ^* for the two-element model in the noncontingent case is a root of the equation:

$$(2.3.16)\quad (n_{11}+n_{12}+n_{21}+n_{22})\pi(1-\pi)\theta^2 - 2[n_{11}(1-\pi)^2+(n_{12}+n_{21})(\pi^2-\pi+1)$$

$$+ n_{22}\pi^2]\theta - n_{11}(1-\pi)(2-\pi) + (n_{12}+n_{21})(1+\pi)(2-\pi) - n_{22}\pi(1+\pi) = 0 \ .$$

The statistical properties of this method of estimation need investigation. However, it can be shown that the method is consistent for all the models considered in this book.

Equations (2.3.12) and (2.3.13) are special cases for $N = 2$ of more general results for arbitrary N. In order to use pseudo-maximum likelihood methods to estimate jointly N and θ, it is necessary to derive the general result.

Theorem. For the N-element model (with equiprobable sampling of stimulus elements), in the noncontingent case

$$(2.3.17)\qquad \lim P(A_{1,n+1} | A_{1,n}) = \pi + \frac{(1-\theta)(1-\pi)}{N}$$

and

$$(2.3.18)\qquad \lim P(A_{2,n+1} | A_{2,n}) = 1 - \pi + \frac{(1-\theta)\pi}{N} \ .$$

Proof: We simply generalize the method of attack used for $N = 2$. Corresponding to (2.3.3), we have

$$(1)\quad P(A_{1,n+1}|A_{1,n}) = \sum_{i=1}^{N}\sum_{j=1}^{N} P(A_{1,n+1}C_{i,n+1}A_{1,n}C_{j,n})/P(A_{1,n})$$

$$= \sum_i \sum_j P(A_{1,n+1}|C_{i,n+1})P(C_{i,n+1}|A_{1,n}C_{j,n})\cdot$$

$$P(A_{1,n+1}|C_{j,n})P(C_{j,n})/P(A_{1,n}) \quad .$$

Analogous to (2.3.4), we know that

$$(2)\qquad P(A_{1,n}|C_{j,n}) = \frac{j}{N} ,$$

and corresponding to (2.3.7), (2.3.8) and (2.3.11)

$$(3)\qquad P(C_{i,n+1}|A_{1,n}C_{j,n}) = \begin{cases} 0 , & i > j \\ \pi + (1-\Theta)(1-\pi) , & i = j \\ \Theta(1-\pi) , & i = j-1 \\ 0 , & i < j-1 \quad . \end{cases}$$

The results in §1.3 for $N = 2$ easily generalize to show that

$$(4)\qquad \lim P(C_{i,n}) = \binom{N}{i} \pi^i (1-\pi)^{N-i} ,$$

where $\binom{N}{i} = \frac{N!}{i!(N-i)!}$, the binomial coefficient. Substituting (2) and (3) in (1), we infer:

(5) $$P(A_{1,n+1}|A_{1,n}) = \frac{1}{P(A_{1,n})}\left\{\sum_{i=1}^{N}\frac{i^2}{N^2}[\pi+(1-\Theta)(1-\pi)]P(C_{i,n}) + \sum_{i=1}^{N}\frac{i(i-1)}{N^2}\Theta(1-\pi)P(C_{i,n})\right\}$$

and thus using (4) as $n \to \infty$

$$\lim P(A_{1,n+1}|A_{1,n}) = \frac{\pi+(1-\Theta)(1-\pi)}{\pi N^2}\sum_i i^2 \binom{N}{i}\pi^i(1-\pi)^{N-i} + \frac{\Theta(1-\pi)}{\pi N^2}\sum_i i(i-1)\binom{N}{i}\pi^i(1-\pi)^{N-i} \quad .$$

Now the first summation is just the second raw moment of the binomial distribution with parameter π, and the second summation is the second raw moment minus the mean $N\pi$, whence

$$\lim P(A_{1,n+1}|A_{1,n}) = \frac{\pi+(1-\Theta)(1-\pi)}{\pi N^2}[N^2\pi^2+N\pi(1-\pi)] + \frac{\Theta(1-\pi)}{\pi N^2}[N^2\pi^2+N\pi(1-\pi)-N\pi]$$

$$= \frac{1}{N}[\pi(N\pi+1-\pi)+(1-\pi)\left\{(1-\Theta)(N\pi+1-\pi)+\Theta\pi(N-1)\right\}]$$

$$= \frac{1}{N}[N\pi+1-\Theta-\pi+\Theta\pi]$$

$$= \pi + \frac{(1-\Theta)(1-\pi)}{N} ,$$

which establishes (2.3.17), and the argument for (2.3.18) is completely symmetrical with π and $1-\pi$ interchanged. Q.E.D.

It is perfectly straightforward to use (2.3.17) and (2.3.18) to make a joint pseudo-maximum likelihood estimate of Θ and N, exactly as was done for Θ with $N = 2$. However, the equation which arises from differentiating $L^*(\Theta,N)$ with respect to N is a polynomial of high degree, and it is desirable to have a simpler approach. Applying the methods just used, we may establish for the N-element model that

(2.3.19) $$\lim P(A_{1,n+1} | E_{1,n}) = \pi + \frac{\Theta(1-\pi)}{N}$$

and

(2.3.20) $$\lim P(A_{2,n+1} | E_{2,n}) = 1 - \pi + \frac{\Theta\pi}{N} .$$

Let

(2.3.21) $$\begin{cases} x = \dfrac{1-\Theta}{N} \\ y = \dfrac{\Theta}{N} . \end{cases}$$

Then

(2.3.22) $$\begin{cases} N = \dfrac{1}{x+y} \\ \Theta = \dfrac{y}{x+y} . \end{cases}$$

We may use (2.3.17) and (2.3.18) to make a pseudo-maximum likelihood estimate of x, and (2.3.19) and (2.3.20) to estimate y. For x, we have

$$
(2.3.23) \qquad \begin{array}{c|cc} & A_{1,n+1} & A_{2,n+1} \\ \hline A_{1,n} & \pi + x(1-\pi) & (1-x)(1-\pi) \\ A_{2,n} & (1-x)\pi & 1-\pi+x\pi \end{array} ,
$$

which is just the transition matrix (1.3.7) for the one-element model with θ replaced by $1-x$, whence the quadratic equation (2.2.8) yields the pseudo-maximum likelihood estimate of $1-x$, and thus of x. This is particularly convenient if, as we have done, maximum likelihood estimates of θ for the one-element model are computed, for $\hat{\theta} = 1-x^*$.

For y, we have

$$
(2.3.24) \qquad \begin{array}{c|cc} & A_{1,n+1} & A_{2,n+1} \\ \hline E_{1,n} & \pi + y(1-\pi) & (1-y)(1-\pi) \\ E_{2,n} & (1-y)\pi & 1-\pi+y\pi \end{array} .
$$

Because the matrix (2.3.24) is just like (2.3.23) with y replacing x, the same formal remarks hold. Namely, the quadratic equation (2.2.8) may yield the estimate of $1-y$ and thus of y. Of course, in this case the transition numbers n_{ij} are different from those used in making the maximum likelihood estimate of θ for the one-element model. In fact, it should be noticed that (2.3.24) does not even approximate a chain of infinite order, for the "transitions" are from reinforcements to responses. Applications of this method for jointly estimating θ and N are to be found in Chapter 10.

§2.4. Some Statistical Tests for Markov Chains. There are four standard tests which we shall use in analyzing experimental data for Markov characteristics. The first tests the hypothesis that the Markov process is stationary, that is, that the transition probabilities are independent of n, the trial number. The second is concerned with the order of the Markov chain, for example, first order versus second order. The third test is one for goodness of fit. The fourth test deals with the hypothesis of whether or not response protocols for different subjects can be viewed as a collection of samples from the same r^{th} order Markov chain. The derivation of these tests and an analysis of their statistical properties are given in Anderson and Goodman [1957]; also relevant are Bartlett [1951] and Hoel [1954]. For our purposes it will be sufficient to present a brief description of the tests.

Stationarity. Let $p_{ij}(t)$ be the probability of transition from state i on trial $t-1$ to state j on trial t. The null hypothesis to be considered is that $p_{ij}(t) = p_{ij}$, that is, that the transition probabilities of the Markov process are independent of t. The χ^2-test of homogeneity is appropriate, and we calculate for each row i of the transition matrix

$$\chi_i^2 = \sum_{t,j} n_i(t-1)\left[\frac{n_{ij}(t)}{n_i(t-1)} - \frac{n_{ij}}{n_i}\right]^2 \Bigg/ \left(\frac{n_{ij}}{n_i}\right) \tag{2.4.1}$$

where $n_{ij}(t)$ denotes the observed number of cases in state i at $t-1$ and state j at t. Further, let

$$n_i(t-1) = \sum_j n_{ij}(t)$$

$$n_{ij} = \sum_t n_{ij}(t)$$

$$n_i = \sum_j n_{ij} \quad .$$

If the null hypothesis is true, χ_i^2 has the usual limiting distribution with $(m-1)(T-1)$ degrees of freedom, where m is the number of states and T is the number of trials. In the experiments reported in subsequent chapters the number of pairs of subjects, or triples of subjects, as the case may be, ranges between 20 and 40 , which is not sufficient to yield stable estimates of $\hat{p}_{ij}(t) = \frac{n_{ij}(t)}{n_i(t-1)}$ for individual trials t . Consequently, our procedure is to sum over blocks of trials, in which case t now represents a particular block of trials.

Finally, it may be shown that the set of χ_i^2 are asymptotically independent, whence the sum

$$\chi^2 = \sum_i \chi_i^2 \tag{2.4.2}$$

has the usual limiting distribution with $m(m-1)(T-1)$ degrees of freedom.

<u>Order</u>. The Markov character of the sequence of response random variables, or of other sequences of random variables, may be tested directly without recourse to details of stimulus sampling theory. Again a χ^2-test of

homogeneity is appropriate. To begin with, suppose our null hypothesis is that the outcomes of trials are statistically independent (zero order process) against the alternative hypothesis that the process is a first order Markov chain. A test of this hypothesis can be made by computing the sum

$$\chi^2 = \sum_{i,j} n_i \left(\frac{n_{ij}}{n_i} - \frac{n_j}{N} \right)^2 \Bigg/ \left(\frac{n_j}{N} \right) , \tag{2.4.3}$$

where n_{ij} and n_i are as defined above, and

$$n_j = \sum_i n_{ij} ,$$

$$N = \sum_{i,j} n_{ij} .$$

Again, χ^2 has the usual limiting distribution with $(m-1)^2$ degrees of freedom.

A second null hypothesis is that the process is a first order Markov chain against the hypothesis that it is a second order chain. Rejection of the null hypothesis in this case would mean that a better prediction of the response probabilities can be made by observing the two immediately preceding responses rather than simply the single immediately preceding response. Results of this test are of particular psychological interest because there are two main directions one can go in developing models which are more adequate than the one-element ones. The possibility most discussed by us is the development of multi-stimulus-element models which are first order Markov chains in the unobservable states of conditioning but not in the subject's responses. Another possibility is to develop higher order Markov chains which still have observable states.

The first order vs. second order hypothesis can be tested by computing the sum

$$\chi^2 = \sum_{i,j,k} n_{ij} \left(\frac{n_{ijk}}{n_{ij}} - \frac{n_{jk}}{n_j} \right)^2 \Big/ \left(\frac{n_{jk}}{n_j} \right) \tag{2.4.4}$$

where n_{ijk} is defined similarly to n_{ij}. If the null hypothesis is true, χ^2 has a limiting distribution with $m(m-1)^2$ degrees of freedom. It is straightforward to generalize (2.4.4) to a test of $r-1$ order vs. r order (see Anderson and Goodman [1957]). It may be noted that the various ratios which occur in these tests are actually the maximum likelihood estimates of the transition probabilities. Thus $\hat{p}_{ij} = n_{ij}/n_i$, $\hat{p}_{ijk} = n_{ijk}/n_{ij}$, and so on. Also, these order tests are predicated on the assumption of stationarity, although this is not necessary.

Goodness of fit. For all of the Markov chains considered in this book the transition matrix depends on at least one conditioning parameter. However, after the parameters have been estimated by the methods discussed in preceding sections, it is then possible to make a χ^2-test of the goodness of fit of the predicted transition matrix to the observed transition matrix. This test would seem to be restricted to one-element models, which are the only ones having observable states. However, in the next section it is shown that this is not the case, and that the transition numbers for multi-element models may be inferred from observed data even though they are not directly observable.

For simplicity of exposition, let $\hat{\Theta}$ be the estimated parameter, let $p_{ij}(\hat{\Theta})$ be the theoretical transition probabilities based on $\hat{\Theta}$. As before, let $\hat{p}_{ij}$ represent the observed transition probabilities as estimated from the transition numbers n_{ij}. Then under the null hypothesis

$$\chi^2 = \sum_{i,j} n_i(\hat{p}_{ij} - p_{ij}(\hat{\Theta}))^2 / p_{ij}(\hat{\Theta}) \tag{2.4.5}$$

has the usual limiting distribution with $m(m-1)-1$ degrees of freedom. (If q parameters are estimated, then there are $m(m-1)-q$ degrees of freedom.) When it is convenient, $p_{ij}(\hat{\Theta})$ in the denominator may be replaced by $\hat{p}_{ij}$ without seriously affecting the test. For example, if $\hat{p}_{ij} \neq 0$ and $p_{ij}(\hat{\Theta}) = 0$, such a substitution is convenient. This situation occurs in the transition matrix (1.4.5) for zero-sum two-person games.

Although this goodness of fit test has the virtue of providing an overall measure of the adequacy of a particular Markov chain with respect to a given experiment, it would be a mistake to construe it as providing a test of the goodness of fit of stimulus sampling theory to the data of the experiment. Consider, for example, the sequence of response random variables in the one-element model with noncontingent reinforcement. As we shall see in Chapter 10, the fit of this Markov chain to data may be exceptionally good, and yet the additional prediction from the theory that $P(A_{1,n+1} | E_{1,n} A_{1,n}) = 1$ will be clearly contradicted. The implication of this last prediction is that if we take the Markov chain whose states are the possible pairs of responses and reinforcements, the fit to data will not be as good as that of the Markov chain consisting only of the sequence of response random variables. On the other hand, poor fit of the sequence of response random variables in the one-element model does not entail rejection of the theory, for a multi-element model may fit considerably better. The point of these remarks is to caution against making too simple an interpretation of the relation between the χ^2 goodness of fit test and the fundamental theory of stimulus sampling.

Identical Processes. Often, a question of major interest in analyzing learning studies is whether or not the subjects in a particular experimental group can be considered equivalent. More specifically, whether the response protocols for different subjects can be viewed simply as a collection of samples from the same r^{th} order Markov chain. We will consider the test only for first order Markov chains but the generalization to r^{th} order chains is obvious.

Let $\hat{p}_{ij}^{(h)} = n_{ij}^{(h)}/n_i^{(h)}$ denote the maximum likelihood estimate of the first order transition probability $p_{ij}^{(h)}$ for the process from which sample h $(h = 1,2,\ldots,s)$ was obtained. We wish to test the null hypothesis that $p_{ij}^{(h)} = p_{ij}$ for $h = 1,2,\ldots,s$; that is, that the s processes are identical.

Again, the χ^2 test of homogeneity is appropriate; that is, to test this hypothesis, we calculate

$$\chi_i^2 = \sum_{h,j} n_i^{(h)} [\hat{p}_{ij}^{(h)} - \hat{p}_{ij}^{(\cdot)}]^2 / \hat{p}_{ij}^{(\cdot)} \tag{2.4.6}$$

where $n_{ij}^{(\cdot)} = \sum_h n_{ij}^{(h)}$ and $\hat{p}_{ij}^{(\cdot)} = n_{ij}^{(\cdot)} / \sum_j n_{ij}^{(\cdot)}$. χ_i^2 has the usual limiting distribution with $(s-1)(m-1)$ degrees of freedom. Finally,

$$\chi^2 = \sum_i \chi_i^2 \tag{2.4.7}$$

has a limiting χ^2-distribution with $m(m-1)(s-1)$ degrees of freedom.

*§2.5. Estimation of Transition Numbers in Multi-Element Models. It has already been remarked several times that the states of conditioning of stimuli are not observable when more than one stimulus is available for sampling. The literature of stimulus sampling theory would incline one to think that the transition numbers associated with these unobservable states of conditioning also are not identifiable. That is, that they are not uniquely determined by the observed response data. Fortunately this is not so for the models we consider in this book, and in fact the transition numbers for the conditioning states can be estimated independent of Θ .

*/ Starred sections may be omitted without loss of continuity by readers interested only in the main lines of development.

This is a particularly desirable state of affairs, for the estimated numbers may then themselves be used to estimate θ by the maximum likelihood methods of §2.2. (It is not difficult to show (see Anderson and Goodman [1957]) that these transition numbers form a set of sufficient statistics.)

We now carry through this anaylsis in detail for the two-element models in the noncontingent situation and also in the zero-sum, two-person situation. Naturally, the simpler of the two will be considered first, that is, the noncontingent case. We shall continually refer to the transition matrix (2.3.2) for the two-element model. For uniformity of notation, let n_{ij} be the transition numbers for the observed responses A_1 and A_2, and let m_{ij} be the transition numbers for the unobserved states 0, 1 and 2 of the Markov chain. Our first problem is to write an equation for each n_{ij} in terms of the m_{ij}. In order to do this it is necessary to compute the conditional probabilities $P(A_{i,n+1}A_{j,n}|C_{k,n+1}C_{\ell,n})$, where $i,j = 1,2$ and $k,\ell = 0,1,2$. For simplicity, we replace $C_{1,n}$ by 1_n, to designate the state with exactly one element conditioned to A_1, and similarly, for $C_{0,n}$ and $C_{2,n}$. Beginning with the transition number n_{11} for $A_{1,n+1}A_{1,n}$, we note that the state of conditioning on neither trial can be 0, whence

$$n_{11} = \sum_{i,j=1}^{2} m_{ij}P(A_{1,n+1}A_{1,n}|j_{n+1}i_n) \quad . \tag{2.5.1}$$

Clearly

$$P(A_{1,n+1}A_{1,n}|2_{n+1}2_n) = 1 \tag{2.5.2}$$

and by elementary probability theory

$$P(A_{1,n+1}A_{1,n}|1_{n+1}2_n) = \tag{2.5.3}$$

$$P(A_{1,n+1}|1_{n+1})P(1_{n+1}|A_{1,n}2_n)P(A_{1,n}|2_n) \big/ P(1_{n+1}|2_n) \quad .$$

Now $P(1_{n+1}|2_n)$ is given by the matrix (2.3.2), and by (2.3.8), $P(1_{n+1}|A_{1,n}2_n) = \theta(1-\pi)$, whence

$$P(A_{1,n+1}A_{1,n}|1_{n+1}2_n) = \frac{1}{2}\cdot\theta(1-\pi)\cdot 1 \big/ \theta(1-\pi) = \frac{1}{2} \quad . \tag{2.5.4}$$

On the other hand,

$$P(A_{1,n+1}A_{1,n}|2_{n+1}1_n) = 0 \ , \tag{2.5.5}$$

for in order to make an A_1 response on trial n the one element conditioned to this response must be sampled and thus the other element (which is not sampled and not conditioned to A_1) cannot change its conditioning. Therefore, $P(2_{n+1}|A_{1,n}1_n) = 0$, and (2.5.5) follows at once from this.

For the fourth term, we have

$$P(A_{1,n+1}A_{1,n}|1_{n+1}1_n) = P(A_{1,n+1}|1_{n+1})P(1_{n+1}|A_{1,n}1_n)P(A_{1,n}|1_n) \big/ P(1_{n+1}|1_n) \quad .$$

Now by virtue of (2.3.7)

$$P(1_{n+1}|A_{1,n}1_n) = \pi + (1-\theta)(1-\pi) ,$$

and we conclude:

$$(2.5.6) \qquad P(A_{1,n+1}A_{1,n}|1_{n+1}1_n) = \frac{1}{2}\cdot[\pi+(1-\theta)(1-\pi)]\cdot\frac{1}{2} \,/\, (1-\frac{1}{2}\theta) .$$

Combining (2.5.1) - (2.5.6), we obtain the equation for n_{11} and by similar arguments the equations for n_{12} , n_{21} and n_{22} . They are as follows:

$$(2.5.7) \quad \begin{cases} n_{11} = m_{22} + \frac{1}{2}m_{21} + \frac{1}{4}\left[\dfrac{\pi+(1-\theta)(1-\pi)}{1-\frac{1}{2}\theta}\right]m_{11} \\ n_{12} = \frac{1}{2}m_{21} + m_{10} + \frac{1}{4}\left[\dfrac{\pi+(1-\theta)(1-\pi)}{1-\frac{1}{2}\theta}\right]m_{11} \\ n_{21} = m_{12} + \frac{1}{2}m_{01} + \frac{1}{4}\left[\dfrac{1-\theta\pi}{1-\frac{1}{2}\theta}\right]m_{11} \\ n_{22} = m_{00} + \frac{1}{2}m_{01} + \frac{1}{4}\left[\dfrac{1-\theta\pi}{1-\frac{1}{2}\theta}\right]m_{11} \end{cases}$$

It is necessary to combine the four equations of (2.5.7) in order to eliminate θ . We obtain the four equations:

$$
(2.5.8) \quad \begin{cases} n_{11} + n_{21} = m_{22} + \frac{1}{2}m_{21} + m_{12} + \frac{1}{2}m_{01} + \frac{1}{2}m_{11} \\ n_{11} + n_{22} = m_{22} + \frac{1}{2}m_{21} + m_{00} + \frac{1}{2}m_{01} + \frac{1}{2}m_{11} \\ n_{12} + n_{21} = \frac{1}{2}m_{21} + m_{10} + m_{12} + \frac{1}{2}m_{01} + \frac{1}{2}m_{11} \\ n_{12} + n_{22} = \frac{1}{2}m_{21} + m_{10} + m_{00} + \frac{1}{2}m_{01} + \frac{1}{2}m_{11} \end{cases} .
$$

Secondly, note that for a large number of observations the number of transitions into a given row of (2.3.2) must approximate very closely the number of transitions out of the row, independent of the value of θ. Whence we have the two equations:

$$
(2.5.9) \quad \begin{cases} m_{01} = m_{10} \\ m_{12} = m_{21} \end{cases} .
$$

Thirdly, p_{10} and p_{12} stand in a ratio which is independent of θ. Thus

$$
(2.5.10) \quad m_{10} = \frac{(1-\pi)}{\pi} m_{12} .
$$

Finally, two additional linear relations can be obtained by comparing transition numbers of different rows. From (2.3.2) we see that

$$
\frac{p_{01}}{p_{12}} = 2 ,
$$

whence

$$(1) \qquad \frac{m_{01}}{m_{00}+m_{01}} = \frac{2m_{12}}{m_{10}+m_{11}+m_{12}} ,$$

and

$$\frac{p_{21}}{p_{10}} = 2 ,$$

whence

$$(2) \qquad \frac{m_{21}}{m_{21}+m_{22}} = \frac{2m_{10}}{m_{10}+m_{11}+m_{12}} .$$

From (1), (2), (2.5.9) and (2.5.10), we infer:

$$(1-\pi)(m_{10}+m_{11}+m_{12}) = 2\pi(m_{00}+m_{01})$$

$$(1-\pi)(m_{10}+m_{11}+m_{12}) = 2\pi(m_{21}+m_{22}) ,$$

which pair of equations is equivalent to:

$$(2.5.11) \qquad \begin{cases} m_{00}+m_{01} = m_{21}+m_{22} \\ (1-\pi)(m_{10}+m_{11}+m_{12} = 2\pi(m_{00}+m_{01}) \end{cases} .$$

The nine equations (2.5.8), (2.5.9), (2.5.10) and (2.5.11) in seven unknowns may be expressed by the matrix equation

(2.5.12)

$$\begin{pmatrix} & \frac{1}{2} & & \frac{1}{2} & 1 & \frac{1}{2} & 1 \\ 1 & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & 1 \\ & \frac{1}{2} & 1 & \frac{1}{2} & 1 & \frac{1}{2} & \\ 1 & \frac{1}{2} & 1 & \frac{1}{2} & & \frac{1}{2} & \\ & 1 & -1 & & & & \\ & & & & 1 & -1 & \\ & & \pi & & \pi-1 & & \\ 1 & 1 & & & & -1 & -1 \\ 2\pi & 2\pi & \pi-1 & \pi-1 & \pi-1 & & \end{pmatrix} \begin{pmatrix} m_{00} \\ m_{01} \\ m_{10} \\ m_{11} \\ m_{12} \\ m_{21} \\ m_{22} \end{pmatrix} = \begin{pmatrix} n_{11}+n_{21} \\ n_{11}+n_{22} \\ n_{12}+n_{21} \\ n_{12}+n_{22} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

We want to use the method of least squares to solve this system of equations for the overdetermined variables m_{ij} . However, the total number $N = \Sigma n_{ij} = \Sigma m_{ij}$ of transitions is not a number which is to be estimated in the least squares procedure for this number is known with probability one. In order to avoid having $\Sigma m_{ij} \neq N$, we replace the equation

$$n_{12}+n_{22} = \frac{1}{2}m_{21}+m_{10}+m_{00}+\frac{1}{2}m_{01}+\frac{1}{2}m_{11}$$

by

(2.5.13) $$N = \sum_{i,j} m_{ij} ,$$

either of which is a consequence of the other together with the remaining three equations of (2.5.8). We then use (2.5.13) to eliminate one of the seven variables, say m_{00} , and thereby guarantee exact satisfaction of (2.5.13) by the least squares estimate of the m_{ij} . The result is the following matrix equation

(2.5.14)

$$\begin{pmatrix} \frac{1}{2} & & \frac{1}{2} & 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & 1 & \frac{1}{2} & 1 & \frac{1}{2} & \\ \frac{1}{2} & 1 & \frac{1}{2} & 1 & \frac{1}{2} & \\ 1 & -1 & & & & \\ & & & 1 & -1 & \\ & \pi & & \pi-1 & & \\ & 1 & 1 & 1 & 2 & 2 \\ & \frac{1+\pi}{2\pi} & \frac{1+\pi}{2\pi} & \frac{1+\pi}{2\pi} & 1 & 1 \end{pmatrix} \begin{pmatrix} m_{01} \\ m_{10} \\ m_{11} \\ m_{12} \\ m_{21} \\ m_{22} \end{pmatrix} = \begin{pmatrix} n_{11}+n_{21} \\ N-n_{11}-n_{22} \\ n_{12}+n_{21} \\ 0 \\ 0 \\ 0 \\ N \\ N \end{pmatrix} ,$$

which can be written as

(2.5.15) $$QM = K ,$$

where Q is the 8×6 matrix, M the column vector of m_{ij} and K the column vector on the right of the equality sign of (2.5.14). For the moment it will be convenient to renumber the variables m_{ij} as $m_1, \ldots m_6$. Applying now the method of least squares, we seek to minimize

$$S(m_1,\ldots,m_6) = \sum_{i=1}^{8} [k_i - \sum_{j'=1}^{6} q_{ij'}m_{j'}]^2 \quad .$$

Now for $j = 1,\ldots,6$

$$\frac{\partial S}{\partial m_j} = -2 \sum_i (k_i - \sum_{j'} q_{ij'}m_{j'})q_{ij} \quad .$$

Settling these partial derivatives equal to zero, in order to minimize $S(m_1,\ldots,m_6)$ we solve the system of six equations which may be written

(2.5.16) $$BM = A$$

where $B = Q'Q$, $A = Q'K$, and Q' is the transpose of the matrix Q .

Given these estimated transition numbers m_{ij} we may now make a maximum likelihood estimate of Θ . From the transition matrix (2.3.2), we see at once that

$$\frac{dL(\Theta)}{d\Theta} = -\frac{m_{00}\pi}{1-\Theta\pi} + \frac{(m_{01}+m_{10}+m_{12}+m_{21})}{\Theta} - \frac{m_{11}}{2-\Theta} - \frac{m_{22}(1-\pi)}{1-\Theta(1-\pi)} \quad .$$

Setting this derivative equal to zero, we obtain $\hat{\Theta}$ as a root of the following cubic equation:

$$(2.5.17) \quad (-m_{00} - m_{11} - m_{22} - m_{01} - m_{10} - m_{12} - m_{21})\pi(1-\pi)\theta^3 + [(3m_{00} + m_{22})\pi$$

$$+ (m_{01} + m_{10} + m_{12} + m_{21})(1 + 2\pi - 2\pi^2) + m_{11} + m_{22} - 2(m_{00} + m_{22})\pi^2]\theta^2$$

$$+ [2(m_{22} - m_{00})\pi - 3(m_{01} + m_{10} + m_{12} + m_{21}) - m_{11} - 2m_{22}]\theta$$

$$+ 2(m_{01} + m_{10} + m_{12} + m_{21}) = 0 \quad .$$

This estimate $\hat{\theta}$ may be compared with the first order pseudo-maximum likelihood estimate θ^* of §2.3. Moreover, the χ^2 goodness of fit test (2.4.5) may be applied to the 2×2 table of transition probabilities $P(A_{i,n+1} | A_{j,n})$ for $i,j = 1,2$ at asymptote. The theoretical values $p_{ij}(\hat{\theta})$ are given by (2.3.12) and (2.3.13). More interesting still is the application of the χ^2 test to the fit at asymptote of the theoretical probabilities $P(A_{i,n+1} | E_{k,n} A_{j,n})$ for which the one-element model is particularly bad because the re-occurrence of a reinforced response is predicted with probability one. Given $\hat{\theta}$, we may apply the χ^2 test to the following 4×2 table, whose entries are computed at asymptote by the methods of §2.3:

(2.5.18)

	A_1	A_2
A_1E_1	$\frac{1}{2} + \frac{\pi}{2}$	$\frac{1}{2} - \frac{\pi}{2}$
A_1E_2	$\frac{\pi}{2} + \frac{1-\theta}{2}$	$\frac{1}{2} + \frac{\theta}{2} - \frac{\pi}{2}$
A_2E_1	$\frac{1}{2}\pi + \frac{\theta}{2}$	$1 - \frac{1}{2}\pi - \frac{\theta}{2}$
A_2E_2	$\frac{\pi}{2}$	$1 - \frac{\pi}{2}$

The test is stringent, for neither the first nor fourth row have any probabilities which depend on θ . The results of this test are compared in Chapter 10 with a similar test for the generalized conditioning model discussed in the next section. In the two-element model the sequence of random variables $< \underline{A}_1\underline{E}_1, \underline{A}_2\underline{E}_2, \ldots, \underline{A}_n\underline{E}_n, \ldots >$ is not a Markov chain, but a chain of infinite order. Consequently the χ^2 goodness of fit test (2.4.5), when applied to the table (2.5.18) does not test the goodness of fit of the chain of infinite order. However, it does test the particular predictions given by (2.5.18).

The method which has just been presented for estimating the transition numbers m_{ij} in the noncontingent case may also be applied to the zero-sum, two-person situation. Here the method has more practical value because of the difficulty of obtaining expressions for the asymptotic probabilities of the conditioning states in the two-element model.

Corresponding to the transition matrix (1.4.5) for the one-element model, the transition matrix for the two-element model of the zero-sum, two-person case is as follows, where the state ij means player A has i stimuli conditioned to response A_1 and player B has j stimuli conditioned to response B_1 :

(2.5.19)

	22	21	20	12	11	10	02	01	00
22	$(1-\theta)$	θa_1		$\theta(1-a_1)$					
21	$\theta a_2/2$	$(1-\theta)$	$\theta a_1/2$		$\theta(2-a_1-a_2)/2$				
20		θa_2	$(1-\theta)$			$\theta(1-a_2)$			
12	$\theta(1-a_3)/2$			$(1-\theta)$	$\theta(a_1+a_3)/2$		$\theta(1-a_1)/2$		
11		$\theta(2-a_3-a_4)/4$		$\theta(a_2+a_4)/4$	$(1-\theta)$	$\theta(a_1+a_3)/4$		$\theta(2-a_1-a_2)/4$	
10			$\theta(1-a_4)/2$		$\theta(a_2+a_4)/2$	$(1-\theta)$			$\theta(1-a_2)/2$
02				$\theta(1-a_3)$			$(1-\theta)$	θa_3	
01					$\theta(2-a_3-a_4)/2$		$\theta a_4/2$	$(1-\theta)$	$\theta a_3/2$
00						$\theta(1-a_4)$		θa_4	$(1-\theta)$

On the basis of this matrix we proceed exactly as for the noncontingent case. Following (1.4.21) - (1.4.23), we introduce the events X_n, Y_n, Z_n, W_n:

$$(2.5.20)\qquad \begin{cases} X_n = A_{1,n}B_{1,n} \\ Y_n = A_{1,n}B_{2,n} \\ Z_n = A_{2,n}B_{1,n} \\ W_n = A_{2,n}B_{2,n} \end{cases} .$$

We then need to compute the conditional probabilities $P(X_{n+1}X_n|22_{n+1}22_n)$, etc., of which there are $4 \times 4 \times 9 \times 9 = 1296$, most of which are zero. To indicate the method we consider in detail the probabilities for $X_{n+1}X_n$ to obtain an expression for the observed transition number n_{XX} in terms of the unobserved $m_{ij,i'j'}$. Following (2.5.1), we have

$$(2.5.21)\qquad n_{XX} = \sum_{i,j=1}^{2} \sum_{i',j'=1}^{2} m_{ij,i'j'} P(X_{n+1}X_n|i'j'_{n+1}ij_n) ,$$

where summation over $i,j,i',j' = 0$ is omitted because these values prohibit an X response by the pair of players. Of the 16 conditional probabilities occurring in the summation on the right, 4 are zero because of zeros in the transition matrix (2.5.19), namely, the transitions 22_n11_{n+1}, 21_n12_{n+1}, 12_n21_{n+1}, and 11_n22_{n+1}. Moreover, the same argument which established (2.5.5) also may be used to show that $P(X_{n+1}X_n|22_{n+1}21_n) = P(X_{n+1}X_n|22_{n+1}12_n) = P(X_{n+1}X_n|21_{n+1}11_n) = P(X_{n+1}X_n|12_{n+1}11_n) = 0$. The remaining 8 transitions all contribute to n_{XX}, and we now proceed to compute them. The important and interesting observation is that all 8 conditional probabilities are independent of θ. First, it is immediately clear that

$$P(X_{n+1}X_n|22_{n+1}22_n) = 1$$

and

$$P(X_{n+1}X_n|12_{n+1}22_n) = P(X_{n+1}X_n|21_{n+1}22_n) = \frac{1}{2} .$$

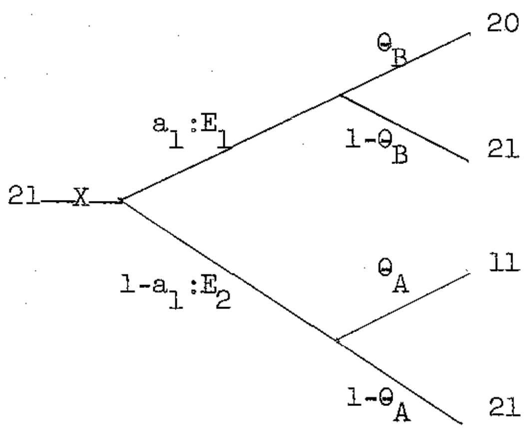
Now by elementary probability theory

$$P(X_{n+1}X_n|21_{n+1}21_n) = \frac{P(X_{n+1}|21_{n+1})P(21_{n+1}|X_n21_n)P(X_n|21_n)}{P(21_{n+1}|21_n)}$$

$$= \frac{\frac{1}{2}\cdot(1-\theta)\cdot\frac{1}{2}}{(1-\theta)}$$

$$= \frac{1}{4} ,$$

where the only computation needing remark is that $P(21_{n+1}|X_n21_n) = 1-\theta$, which may be explained by the following tree.



Note that by virtue of the tree $P(21_{n+1}|X_n 21_n) = a_1(1-\Theta_B) + (1-a_1)(1-\Theta_A)$, which reduces to $1-\Theta$ on the assumption that $\Theta_A = \Theta_B$. Also for reasons of symmetry

$$P(X_{n+1}X_n|12_{n+1}12_n) = \frac{1}{4} .$$

We now consider the sixth term.

$$P(X_{n+1}X_n|11_{n+1}21_n) = \frac{P(X_{n+1}|11_{n+1})P(11_{n+1}|X_n 21_n)P(X_n|21_n)}{P(11_{n+1}|21_n)}$$

$$= \frac{\frac{1}{4}\cdot\Theta(1-a_1)\cdot\frac{1}{2}}{\frac{1}{2}\Theta(2-a_1-a_2)}$$

$$= \frac{1}{4}(1-a_1)/(2-a_1-a_2) .$$

The fact that $P(11_{n+1}|X_n 21_n) = \Theta(1-a_1)$ may be seen from the tree just above and $P(11_{n+1}|21_n)$ may be read off directly from the transition matrix (2.5.19). Moreover, in exactly similar fashion it can be shown that

$$P(X_{n+1}X_n|11_{n+1}12_n) = \frac{1}{4}\left(\frac{a_1}{a_1 + a_3}\right) .$$

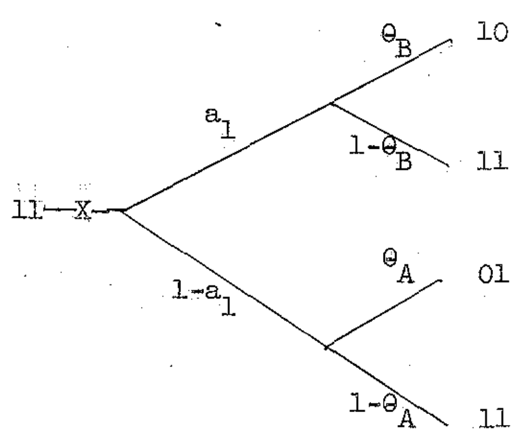
Finally for the last and eighth term we have:

$$P(X_{n+1}X_n|11_{n+1}11_n) = \frac{P(X_{n+1}|11_{n+1})P(11_{n+1}|X_n 11_n)}{P(11_{n+1}|11_n)} P(X_n|11_n)$$

$$= \frac{\frac{1}{4}\cdot(1-\Theta)\cdot\frac{1}{4}}{1-\Theta}$$

$$= \frac{1}{16} ,$$

where the tree for $P(11_{n+1}|X_n 11_n)$ is the following:



(The subscripts A and B have been placed on the Θ's to indicate which player is affected on a given branch by not being reinforced in his response.) Combining these eight results we then have for n_{XX} the following linear equation, which is independent of Θ and holds not only at asymptote but for all trials:

$$(2.5.22)\quad n_{XX} = m_{22,22} + \frac{1}{2}(m_{22,21} + m_{22,12}) + \frac{1}{4}m_{21,21} + \frac{(1-a_1)}{4(2-a_1-a_2)}m_{21,11}$$

$$+ \frac{1}{4}m_{12,12} + \frac{a_1}{4(a_1+a_3)}m_{12,11} + \frac{1}{16}m_{11,11} \quad .$$

By similar methods we obtain for the other 15 observed transitions in X, Y, Z and W the following linear equations:

$$(2.5.23)\quad n_{XY} = \frac{1}{2}m_{22,21} + \frac{1}{4}m_{21,21} + m_{21,20} + \frac{(1-a_1)}{4(2-a_1-a_2)}m_{21,11}$$

$$+ \frac{a_1}{4(a_1+a_3)}m_{12,11} + \frac{1}{16}m_{11,11} + \frac{a_1}{2(a_1+a_3)}m_{11,10}$$

$$n_{XZ} = \frac{1}{2}m_{22,12} + \frac{(1-a_1)}{4(2-a_1-a_2)}m_{21,11} + \frac{1}{4}m_{12,12} + \frac{a_1}{4(a_1+a_3)}m_{12,11}$$

$$+ m_{12,02} + \frac{1}{16}m_{11,11} + \frac{(1-a_1)}{2(2-a_1-a_2)}m_{11,01}$$

$$n_{XW} = \frac{(1-a_1)}{4(2-a_1-a_2)}m_{21,11} + \frac{a_1}{4(a_1+a_3)}m_{12,11} + \frac{1}{16}m_{11,11}$$

$$+ \frac{a_1}{2(a_1+a_3)}m_{11,10} + \frac{(1-a_1)}{2(2-a_1-a_2)}m_{11,01}$$

$$n_{YX} = m_{21,22} + \frac{1}{4}m_{21,21} + \frac{(1-a_2)}{4(2-a_1-a_2)}m_{21,11} + \frac{1}{2}m_{20,21}$$

$$+ \frac{a_2}{2(a_2+a_4)}m_{11,12} + \frac{1}{16}m_{11,11} + \frac{a_2}{4(a_2+a_4)}m_{10,11}$$

$$n_{YY} = \frac{1}{4}m_{21,21} + \frac{(1-a_2)}{4(2-a_1-a_2)}m_{21,11} + \frac{1}{2}m_{20,21} + m_{20,20}$$

$$+ \frac{1}{2}m_{20,10} + \frac{1}{16}m_{11,11} + \frac{a_2}{4(a_2+a_4)}m_{10,11} + \frac{1}{4}m_{10,10}$$

$$n_{YZ} = \frac{(1-a_2)}{4(2-a_1-a_2)}m_{21,11} + \frac{a_2}{2(a_2+a_4)}m_{11,12} + \frac{1}{16}m_{11,11}$$

$$+ \frac{(1-a_2)}{2(2-a_1-a_2)}m_{11,01} + \frac{a_2}{4(a_2+a_4)}m_{10,11}$$

$$n_{YW} = \frac{(1-a_2)}{4(2-a_1-a_2)}m_{21,11} + \frac{1}{2}m_{20,10} + \frac{1}{16}m_{11,11} + \frac{(1-a_2)}{2(2-a_1-a_2)}m_{11,01}$$

$$+ \frac{a_2}{4(a_2+a_4)}m_{10,11} + \frac{1}{4}m_{10,10} + m_{10,00}$$

$$n_{ZW} = m_{12,22} + \frac{1}{4}m_{12,12} + \frac{a_3}{4(a_1 + a_3)}m_{12,11} + \frac{(1-a_3)}{2(2-a_3-a_4)}m_{11,21}$$

$$+ \frac{1}{16}m_{11,11} + \frac{1}{2}m_{02,12} + \frac{(1-a_3)}{4(2-a_3-a_4)}m_{01,11}$$

$$n_{ZY} = \frac{a_3}{4(a_1 + a_3)}m_{12,11} + \frac{(1-a_3)}{2(2-a_3-a_4)}m_{11,21} + \frac{1}{16}m_{11,11}$$

$$+ \frac{a_3}{2(a_1 + a_3)}m_{11,10} + \frac{(1-a_3)}{4(2-a_3-a_4)}m_{01,11}$$

$$n_{ZZ} = \frac{1}{4}m_{12,12} + \frac{a_3}{4(a_1 + a_3)}m_{12,11} + \frac{1}{16}m_{11,11} + \frac{1}{2}m_{02,12} + m_{02,02}$$

$$+ \frac{1}{2}m_{02,01} + \frac{(1-a_3)}{4(2-a_3-a_4)}m_{01,11} + \frac{1}{4}m_{01,01}$$

$$n_{ZW} = \frac{a_3}{4(a_1 + a_3)}m_{12,11} + \frac{1}{16}m_{11,11} + \frac{a_3}{2(a_1 + a_3)}m_{11,10} + \frac{1}{2}m_{02,01}$$

$$+ \frac{(1-a_3)}{4(2-a_3-a_4)}m_{01,11} + \frac{1}{4}m_{01,01} + m_{01,00}$$

$$n_{WX} = \frac{(1-a_4)}{2(2-a_3-a_4)}m_{11,21} + \frac{a_4}{2(a_2 + a_4)}m_{11,12} + \frac{1}{16}m_{11,11}$$

$$+ \frac{a_4}{4(a_2 + a_4)}m_{10,11} + \frac{(1-a_4)}{4(2-a_3-a_4)}m_{01,11}$$

$$n_{WY} = \frac{(1-a_4)}{2(2-a_3-a_4)} m_{11,21} + \frac{1}{16} m_{11,11} + m_{10,20} + \frac{a_4}{4(a_2+a_4)} m_{10,11}$$

$$+ \frac{1}{4} m_{10,10} + \frac{(1-a_4)}{4(2-a_3-a_4)} m_{01,11} + \frac{1}{2} m_{00,10}$$

$$n_{WZ} = \frac{a_4}{2(a_2+a_4)} m_{11,12} + \frac{1}{16} m_{11,11} + \frac{a_4}{4(a_2+a_4)} m_{10,11}$$

$$+ \frac{(1-a_4)}{4(2-a_3-a_4)} m_{01,11} + m_{01,02} + \frac{1}{4} m_{01,01} + \frac{1}{2} m_{00,01}$$

$$n_{WW} = \frac{1}{16} m_{11,11} + \frac{a_4}{4(a_2+a_4)} m_{10,11} + \frac{1}{4} m_{10,10} + \frac{(1-a_4)}{4(2-a_3-a_4)} m_{01,11}$$

$$+ \frac{1}{4} m_{01,01} + \frac{1}{2} m_{00,10} + \frac{1}{2} m_{00,01} + m_{00,00} .$$

Next, from the necessary approximate equality between the number of transitions in and out of any row, we have immediately from the transition matrix (2.5.19), the following 9 linear equations:

(2.5.24) $m_{12,22} + m_{21,22} = m_{22,21} + m_{22,12}$

$$m_{11,21} + m_{20,21} + m_{22,21} = m_{21,22} + m_{21,20} + m_{21,11}$$

$$m_{10,20} + m_{21,20} = m_{20,21} + m_{20,10}$$

$$m_{02,12} + m_{11,12} + m_{22,12} = m_{12,22} + m_{12,11} + m_{12,02}$$

$$m_{01,11} + m_{10,11} + m_{12,11} + m_{21,11} = m_{11,21} + m_{11,12} + m_{11,10} + m_{11,01}$$

$$m_{00,10} + m_{11,10} + m_{20,10} = m_{10,20} + m_{10,11} + m_{10,00}$$

$$m_{01,02} + m_{12,02} = m_{02,12} + m_{02,01}$$

$$m_{00,01} + m_{02,01} + m_{11,01} = m_{01,11} + m_{01,02} + m_{01,00}$$

$$m_{01,00} + m_{10,00} = m_{00,10} + m_{00,01} \quad .$$

Finally, we have 15 linear equations like (2.5.10) which arise from linear relationships within a given row and are independent of Θ .

(2.5.25)

$$a_1 m_{22,12} = (1-a_1) m_{22,21}$$

$$a_2 m_{21,20} = a_1 m_{21,22}$$

$$a_2 m_{21,11} = (2-a_1-a_2) m_{21,22}$$

$$a_2 m_{20,10} = (1-a_2) m_{20,21}$$

$$(1-a_3) m_{12,11} = (a_1 + a_3) m_{12,22}$$

$$(1-a_3) m_{12,02} = (1-a_1) m_{12,22}$$

$$(2-a_3-a_4) m_{11,12} = (a_2 + a_4) m_{11,21}$$

$$(2-a_3-a_4) m_{11,10} = (a_1 + a_3) m_{11,21}$$

$$(2-a_3-a_4) m_{11,01} = (2-a_1-a_2) m_{11,21}$$

$$(1-a_4)m_{10,11} = (a_2 + a_4)m_{10,20}$$

$$(1-a_4)m_{10,00} = (1-a_2)m_{10,20}$$

$$(1-a_3)m_{02,01} = a_3 m_{02,12}$$

$$(2-a_3-a_4)m_{01,02} = a_4 m_{01,11}$$

$$(2-a_3-a_4)m_{01,00} = a_3 m_{01,11}$$

$$(1-a_4)m_{00,01} = a_4 m_{00,10} \quad .$$

In addition, to these 40 equations there are a large number of nonlinear ones which arise from ratio comparisons between rows (cf. 2.5.11). However, the 40 are adequate to give a reasonably good least squares fit, and further consideration is restricted to them. In making the least squares fit, one variable is eliminated, as in the noncontingent case, to guarantee that $\Sigma m_{ij} = \Sigma n_{ij}$.

Given the estimated m_{ij} , we may then make a maximum likelihood estimate of θ , which surprisingly is simpler than that for the noncontingent case. It is the following:

(2.5.26) $$\hat{\theta} = \frac{\delta}{\delta + \delta'}$$

where

$$\delta' = m_{22,22} + m_{21,21} + m_{20,20} + m_{12,12} + m_{11,11} + m_{10,10} + m_{02,02} + m_{01,01} + m_{00,00}$$

$$\delta = m_{22,21} + m_{22,12} + m_{21,22} + m_{21,20} + m_{21,11} + m_{20,21} + m_{20,10} + m_{12,22} + m_{12,11}$$
$$+ m_{12,02} + m_{11,21} + m_{11,12} + m_{11,10} + m_{11,01} + m_{10,20} + m_{10,11} + m_{10,00} + m_{02,12}$$
$$+ m_{02,01} + m_{01,11} + m_{01,02} + m_{01,00} + m_{00,10} + m_{00,01} \ .$$

Given $\hat{\theta}$, we test the goodness of fit of asymptotic transition probabilities in the two-element model to the observed values. The particular transition probabilities of interest are $P_\infty(A_{i,n+1}B_{j,n+1}|E_{k,n}A_{i',n}B_{j',n})$ where $E_{k,n}$ denotes the reinforcing event for Player A (in the zero-sum case the reinforcing event given one player uniquely determines the reinforcing event given the other player). As was already remarked in Chapter 1, in the zero-sum case the one-element models make far too many response predictions with probability one, and so this extension is of particular interest. These theoretical probabilities are functions of the asymptotic probabilities u_{ij} of the states ij in the Markov chain whose transition matrix is (2.5.19). Consequently, we solve (2.5.19) numerically for u_{ij}, compute the theoretical probabilities $P_\infty(A_{i,n+1}B_{j,n+1}|E_{k,n}A_{i',n}B_{j',n})$, and proceed to a χ^2 goodness of fit test. The remarks made after (2.5.18) about a similar test in the noncontingent case also apply here. These theoretical probabilities may be arranged in a 8×4 matrix, but due to the length of the expressions it is only practical to write the 32 equations separately.

(2.5.27)
$$P_\infty(X|E_1X) = [u_{22}x + \tfrac{1}{2}u_{21}z + \tfrac{1}{4}u_{12}x + \tfrac{1}{8}u_{11}z] / P_\infty(X)$$

$$P_\infty(Y|E_1X) = [u_{22}w + \tfrac{1}{2}u_{21}y + \tfrac{1}{4}u_{12}w + \tfrac{1}{8}u_{11}y] / P_\infty(X)$$

$$P_\infty(Z|E_1X) = [\tfrac{1}{4}u_{12}x + \tfrac{1}{8}u_{11}z] / P_\infty(X)$$

$$P_\infty(W|E_1X) = [\tfrac{1}{4}u_{12}w + \tfrac{1}{8}u_{11}y] / P_\infty(X)$$

$$P_\infty(X|E_2X) = [u_{22}x + \tfrac{1}{4}u_{21}x + \tfrac{1}{2}u_{12}z + \tfrac{1}{8}u_{11}z] / P_\infty(X)$$

$$P_\infty(Y|E_2X) = [\tfrac{1}{4}u_{21}x + \tfrac{1}{8}u_{11}z] / P_\infty(X)$$

$$P_\infty(Z|E_2X) = [u_{22}w + \tfrac{1}{4}u_{21}w + \tfrac{1}{2}u_{12}y + \tfrac{1}{8}u_{11}y] / P_\infty(X)$$

$$P_\infty(W|E_2X) = [\tfrac{1}{4}u_{21}w + \tfrac{1}{8}u_{11}y] / P_\infty(X)$$

$$P_\infty(X|E_1Y) = [\tfrac{1}{2}u_{21}y + u_{20}w + \tfrac{1}{8}u_{11}y + \tfrac{1}{4}u_{10}w] / P_\infty(Y)$$

$$P_\infty(Y|E_1Y) = [\tfrac{1}{2}u_{21}z + u_{20}x + \tfrac{1}{8}u_{11}z + \tfrac{1}{4}u_{10}x] / P_\infty(Y)$$

$$P_\infty(Z|E_1Y) = [\tfrac{1}{8}u_{11}y + \tfrac{1}{4}u_{10}w] / P_\infty(Y)$$

$$P_\infty(W|E_1Y) = [\tfrac{1}{8}u_{11}z + \tfrac{1}{4}u_{10}x] / P_\infty(Y)$$

$$P_\infty(X|E_2Y) = [\tfrac{1}{4}u_{21}x + \tfrac{1}{8}u_{11}z] / P_\infty(Y)$$

$$P_\infty(Y|E_2Y) = [\tfrac{1}{4}u_{21}x + u_{20}x + \tfrac{1}{8}u_{11}z + \tfrac{1}{2}u_{10}z] / P_\infty(Y)$$

$$P_\infty(Z|E_2Y) = [\tfrac{1}{4}u_{21}w + \tfrac{1}{8}u_{11}y]/P_\infty(Y)$$

$$P_\infty(W|E_2Y) = [\tfrac{1}{4}u_{21}w + u_{20}w + \tfrac{1}{8}u_{11}y + \tfrac{1}{2}u_{10}y]/P_\infty(Y)$$

$$P_\infty(X|E_2Z) = [\tfrac{1}{4}u_{12}x + \tfrac{1}{8}u_{11}z]/P_\infty(Z)$$

$$P_\infty(Y|E_2Z) = [\tfrac{1}{4}u_{12}w + \tfrac{1}{8}u_{11}y]/P_\infty(Z)$$

$$P_\infty(Z|E_2Z) = [\tfrac{1}{4}u_{12}x + \tfrac{1}{8}u_{11}z + u_{02}x + \tfrac{1}{2}u_{01}z]/P_\infty(Z)$$

$$P_\infty(W|E_2Z) = [\tfrac{1}{4}u_{12}w + \tfrac{1}{8}u_{11}y + u_{02}w + \tfrac{1}{2}u_{01}y]/P_\infty(Z)$$

$$P_\infty(X|E_1Z) = [\tfrac{1}{2}u_{12}y + \tfrac{1}{8}u_{11}y + u_{02}w + \tfrac{1}{4}u_{01}w]/P_\infty(Z)$$

$$P_\infty(Y|E_1Z) = [\tfrac{1}{8}u_{11}y + \tfrac{1}{4}u_{01}w]/P_\infty(Z)$$

$$P_\infty(Z|E_1Z) = [\tfrac{1}{2}u_{12}z + \tfrac{1}{8}u_{11}z + u_{02}x + \tfrac{1}{4}u_{01}x]/P_\infty(Z)$$

$$P_\infty(W|E_1Z) = [\tfrac{1}{8}u_{11}z + \tfrac{1}{4}u_{01}x]/P_\infty(Z)$$

$$P_\infty(X|E_2W) = [\tfrac{1}{8}u_{11}y + \tfrac{1}{4}u_{10}w]/P_\infty(W)$$

$$P_\infty(Y|E_2W) = [\tfrac{1}{8}u_{11}z + \tfrac{1}{4}u_{10}x]/P_\infty(W)$$

$$P_\infty(Z|E_2W) = [\tfrac{1}{8}u_{11}y + \tfrac{1}{4}u_{10}w + \tfrac{1}{2}u_{01}y + u_{00}w]/P_\infty(W)$$

$$P_\infty(W|E_2W) = [\tfrac{1}{8}u_{11}z + \tfrac{1}{4}u_{10}x + \tfrac{1}{2}u_{01}z + u_{00}x]/P_\infty(W)$$

$$P_{\infty}(X|E_1W) = [\tfrac{1}{8}u_{11}y + \tfrac{1}{4}u_{01}w] / P_{\infty}(W)$$

$$P_{\infty}(Y|E_1W) = [\tfrac{1}{8}u_{11}y + \tfrac{1}{2}u_{10}y + \tfrac{1}{4}u_{01}w + u_{00}w] / P_{\infty}(W)$$

$$P_{\infty}(Z|E_1W) = [\tfrac{1}{8}u_{11}z + \tfrac{1}{4}u_{01}x] / P_{\infty}(W)$$

$$P_{\infty}(W|E_1W) = [\tfrac{1}{8}u_{11}z + \tfrac{1}{2}u_{10}z + \tfrac{1}{4}u_{01}x + u_{00}x] / P_{\infty}(W)$$

where $x = (1-\theta) + \frac{1}{2}\theta$, $y = \theta + \frac{1}{2}(1-\theta)$, $w = \frac{1}{2}\theta$, $z = \frac{1}{2}(1-\theta)$ and

(2.5.28)
$$P_{\infty}(X) = u_{22} + \tfrac{1}{2}(u_{21} + u_{12}) + \tfrac{1}{4}u_{11}$$

$$P_{\infty}(Y) = u_{20} + \tfrac{1}{2}(u_{21} + u_{10}) + \tfrac{1}{4}u_{11}$$

$$P_{\infty}(Z) = u_{02} + \tfrac{1}{2}(u_{12} + u_{01}) + \tfrac{1}{4}u_{11}$$

$$P_{\infty}(W) = u_{00} + \tfrac{1}{2}(u_{10} + u_{01}) + \tfrac{1}{4}u_{11} \quad .$$

*§2.6. Generalized Conditioning Model. When we consider the goodness of fit test discussed in §2.4, the mathematical advantages of one-element models are partially offset by some of the unrealistic predictions they generate. An example is the set of four zeros in the transition matrix (1.4.5) for the zero-sum two-person game. It is virtually certain that transitions will be observed for these cells and this expectation is supported by data in Chapters 3 and 4. The source of the difficulty is

the one mentioned at the end of the preceding section. Namely, in the one-element model

$$(2.6.1) \qquad P(A_{i,n+1} | E_{i,n} A_{i,n}) = 1 .$$

That is, conditioning to the A_i response cannot change when that response is reinforced. In the two-person zero-sum situation only one subject of the pair has his actual response reinforced, and thus, by virtue of (2.6.1), both subjects cannot change conditioning states. Consequently, the anti-diagonal of the transition matrix must be uniformly zero. It has been remarked that this difficulty may be avoided by assuming a multi-element model, but it has also been noted that serious mathematical and statistical difficulties ensue from this shift.

There is, fortunately a second alternative which we pursue in this section. The essence of this alternative is to generalize the assumptions about conditioning embodied in Axiom C2. For simplicity, this more general axiom, designated C2' , will be stated for only the two response case.

C2'. <u>If a stimulus element is sampled on a trial, and if response</u> A_i <u>is made and then followed by reinforcement</u> E_j , <u>there is a probability</u> c_{ij} <u>that the stimulus is conditioned to</u> A_1 .

In the one-element model the state of conditioning may be identified with the response to which the single stimulus is conditioned. Consequently, for every trial n :

(2.6.2)
$$\begin{cases} c_{ij} = P(A_{1,n+1} \mid E_{j,n} A_{i,n}) \\ 1 - c_{ij} = P(A_{2,n+1} \mid E_{j,n} A_{i,n}) \end{cases} .$$

The θ formulation of Axiom C2 may be expressed as the following special case of C2':

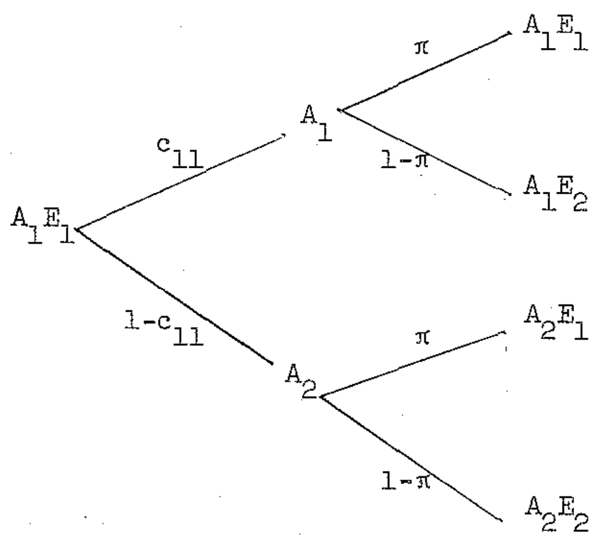
(2.6.3)
$$\begin{cases} c_{11} = 1 \\ c_{12} = (1-\theta) \\ c_{21} = \theta \\ c_{22} = 0 \end{cases} .$$

Before we examine the formal consequences of this generalized axiom, it is pertinent to consider what psychological arguments can be proposed to support its introduction. In the first place, the experimenter-defined events E_k are not necessarily events which reinforce the possible responses for the subject in the manner intended by the experimenter. The notation introduced in Estes and Suppes [1959a], [1959b] makes this point explicit. A distinction is drawn between the observable experimenter-defined outcomes O_j and the unobservable subjective reinforcing events E_k. (A similar differentiation was made earlier by Bush and Mosteller [1955] but not actually much used in their formal developments.) If it be granted that the detailed nature of conditioning is not yet well understood, then there are advantages to a model which permits direct estimation of the coefficients c_{ij} without the major constraints imposed by (2.6.3). There is, for instance, the possibility that conditioning of the stimulus is more affected by the occurrence on the preceding trial of the response or reinforcement which has on the average occurred more often. In succeeding chapters we scrutinize the data for precisely this effect. The ability of the generalized conditioning model to analyze such an effect has the virtue of incorporating into a one-stage Markov chain what is very possibly an important time-dependent "historical" phenomenon which accumulates over trials.

We turn now to formal development of the generalized conditioning model. In order to set forth the central ideas without encumbering details, we begin, as was done in Chapter 1, with the noncontingent case. It is to be emphasized that in the discussion of this case (and all others for the generalized conditioning model) we always assume for simplicity that each subject has only one stimulus element available for sampling.

It is obvious that the 2×2 transition matrix (1.3.7) for the noncontingent case may be rewritten in terms of the coefficients c_{ij}. However, it is more to the point to use the generalized conditioning axiom to analyze the modifications in (2.6.1), that is, the probability that an A_1 response will occur on trial $n+1$ given an E_1 reinforcement and an A_1 response on trial n. It has already been remarked that the prediction of this probability does not follow from the Markov chain which consists of the sequence of response random variables, and this fact suggests that the reinforcement random variables be included in the chain. With this inclusion the states of the chain are the ordered pairs A_iE_j representing possible response and reinforcement combinations on a given trial. For all the one-element models considered in this book, it may be shown (see Estes and Suppes [1959b]) that the inclusion of the reinforcing events in the states of the chain does not disturb its Markovian character. A typical tree for the noncontingent case is the following:

(2.6.4)



The three other trees are similar in form, together they yield the following transition matrix:

(2.6.5)

	A_1E_1	A_1E_2	A_2E_1	A_2E_2
A_1E_1	$c_{11}\pi$	$c_{11}(1-\pi)$	$(1-c_{11})\pi$	$(1-c_{11})(1-\pi)$
A_1E_2	$c_{12}\pi$	$c_{12}(1-\pi)$	$(1-c_{12})\pi$	$(1-c_{12})(1-\pi)$
A_2E_1	$c_{21}\pi$	$c_{21}(1-\pi)$	$(1-c_{21})\pi$	$(1-c_{21})(1-\pi)$
A_2E_2	$c_{22}\pi$	$c_{22}(1-\pi)$	$(1-c_{22})\pi$	$(1-c_{22})(1-\pi)$

The rows indicate the response and reinforcing event on trial n, and the columns the response and reinforcing event on trial $n+1$.

The maximum likelihood estimates of the coefficients of conditioning c_{ij} assume a particularly simple form, for the partial derivative with respect to any c_{ij} of the likelihood function $L(c_{11}, c_{12}, c_{21}, c_{22})$, corresponding to $L(\theta)$ of (2.2.3), is a function only of c_{ij}. For example,

$$\frac{\partial L}{\partial c_{11}} = \frac{n_{11} + n_{12}}{c_{11}} - \frac{n_{13} + n_{14}}{1 - c_{11}} \quad . \tag{2.6.6}$$

From the three other equations like (2.6.6) we conclude that the maximum likelihood estimates of the c_{ij}'s are as follows:

$$
(2.6.7)\quad \begin{cases} \hat{c}_{11} = (n_{11} + n_{12}) / \sum_j n_{1j} \\ \hat{c}_{12} = (n_{21} + n_{22}) / \sum_j n_{2j} \\ \hat{c}_{21} = (n_{31} + n_{32}) / \sum_j n_{3j} \\ \hat{c}_{22} = (n_{41} + n_{42}) / \sum_j n_{4j} , \end{cases}
$$

where $j = 1,\ldots,4$. Moreover, an important observation about these estimates is that in tabulating the transition numbers n_{ij} , it is actually unnecessary to record the reinforcing event which is part of the state j . Thus, it is sufficient to tabulate the data in a 4×2 matrix which has the simple theoretical form:

(2.6.8)

	A_1	A_2
A_1E_1	c_{11}	$1-c_{11}$
A_1E_2	c_{12}	$1-c_{12}$
A_2E_1	c_{21}	$1-c_{21}$
A_2E_2	c_{22}	$1-c_{22}$

The maximum likelihood estimates $\hat{c}_{ij}$ are then just the estimated conditional probabilities $P(A_1 | E_j A_i)$ which are constant over trials.

Moreover, the tabulations indicated for (2.6.8) are sufficient for the χ^2 goodness of fit test, provided we assume that

$$(2.6.9) \qquad \begin{cases} \pi(n_{i1} + n_{i2}) = n_{i1} \\ \pi(n_{i3} + n_{i4}) = n_{i3} \end{cases},$$

for $i = 1,\ldots,4$. The assumption of these equalities implies that the maximum likelihood estimate of the probability of reinforcement is π. Obviously, for properly selected reinforcement sequences this assumption can be violated only by a small experimental error. To clarify this point concerning the goodness of fit test we need consider only the computation for the first two cells of the 4×4 matrix, i.e., for the n_{11} and n_{12} transition numbers. Following (2.4.5), the contribution η of these two cells to the value of χ^2 is:

$$(2.6.10) \qquad \eta = \frac{n_1\left[\frac{n_{11}}{n_1} - c_{11}\pi\right]^2}{c_{11}\pi} - \frac{n_1\left[\frac{n_{12}}{n_1} - c_{11}(1-\pi)\right]^2}{c_{11}(1-\pi)},$$

and using (2.6.9), we have:

$$(2.6.11) \qquad \eta = \frac{n_1\left[\frac{\pi(n_{11}+n_{12})}{n_1} - c_{11}\pi\right]^2}{c_{11}\pi} + \frac{n_1\left[\frac{(1-\pi)(n_{11}+n_{12})}{n_1} - c_{11}(1-\pi)\right]^2}{c_{11}(1-\pi)}$$

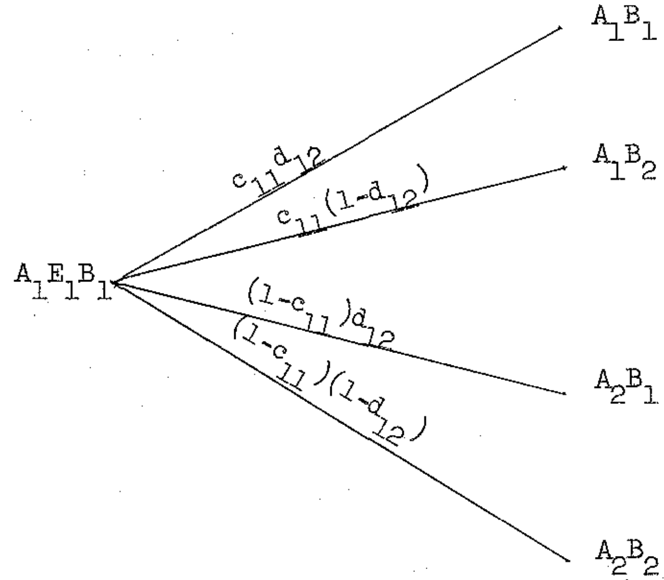
$$= \frac{n_1}{c_{11}}\left[\pi\left(\frac{n_{11}+n_{12}}{n_1} - c_{11}\right)^2 + (1-\pi)\left(\frac{n_{11}+n_{12}}{n_1} - c_{11}\right)^2\right]$$

$$= \frac{n_1\left[\frac{(n_{11}+n_{12})}{n_1} - c_{11}\right]^2}{c_{11}} ,$$

which is just the expression for computing the contribution to a χ^2 test of the first cell of (2.6.8). This line of argument directly establishes that the χ^2 goodness of fit (2.4.5) for the 4×4 transition matrix (2.6.5) may be replaced by a similar test for the 4×2 matrix (2.6.8). However by now the reader may have realized that this particular goodness of fit test is vacuous, for the four estimated parameters c_{ij} guarantee that the fit is exact, each row being exactly fitted by one estimated c_{ij}. The upshot of this is that without the imposition of constraints which specify relations between the c_{ij}, a test of the goodness of fit of the generalized conditioning model for the noncontingent case is not provided by the χ^2 test (2.4.5).

This same situation does not obtain in the two-person situations, to which we now turn. As in the case of Chapter 1 we shall restrict ourselves here to the zero-sum case. In analogy with the transition matrix (2.6.5) for the noncontingent case, the chain for the two-person situation which includes the reinforcing events in its states has 16 states. However, in the zero-sum case this number may be reduced to 8 , for on a given trial the responses of both subjects and the reinforcing event given one subject uniquely determine the reinforcing event given the other subject; that is, as remarked earlier, exactly one of the two subjects has his actual response reinforced. Also, the arguments just given to justify a rectangular 4 x 2 matrix for the noncontingent case apply here <u>mutatis mutandis</u>. So we need consider only an 8 x 4 matrix. The tree for the first row is given below; E_1 designates the reinforcement given subject A , c_{ij} and d_{ij} are the conditioning parameters for subjects A and B , respectively.

(2.6.12)



The 8 x 4 matrix has the following form:

(2.6.13)

	A_1B_1	A_1B_2	A_2B_1	A_2B_2
$A_1E_1B_1$	$c_{11}d_{12}$	$c_{11}(1-d_{12})$	$(1-c_{11})d_{12}$	$(1-c_{11})(1-d_{12})$
$A_1E_1B_2$	$c_{11}d_{21}$	$c_{11}(1-d_{21})$	$(1-c_{11})d_{21}$	$(1-c_{11})(1-d_{21})$
$A_1E_2B_1$	$c_{12}d_{11}$	$c_{12}(1-d_{11})$	$(1-c_{12})d_{11}$	$(1-c_{12})(1-d_{11})$
$A_1E_2B_2$	$c_{12}d_{22}$	$c_{12}(1-d_{22})$	$(1-c_{12})d_{22}$	$(1-c_{12})(1-d_{22})$
$A_2E_1B_1$	$c_{21}d_{11}$	$c_{21}(1-d_{11})$	$(1-c_{21})d_{11}$	$(1-c_{21})(1-d_{11})$
$A_2E_1B_2$	$c_{21}d_{22}$	$c_{21}(1-d_{22})$	$(1-c_{21})d_{22}$	$(1-c_{21})(1-d_{22})$
$A_2E_2B_1$	$c_{22}d_{12}$	$c_{22}(1-d_{12})$	$(1-c_{22})d_{12}$	$(1-c_{22})(1-d_{12})$
$A_2E_2B_2$	$c_{22}d_{21}$	$c_{22}(1-d_{21})$	$(1-c_{22})d_{21}$	$(1-c_{22})(1-d_{21})$

Let the transition numbers n_{ij} be for this 8 x 4 matrix, and let $L(c_{11},\ldots,d_{22})$ be the likelihood function corresponding to (2.2.5). Then

$$\frac{\partial L}{\partial c_{11}} = \frac{n_{11}+n_{12}+n_{21}+n_{22}}{c_{11}} - \frac{n_{13}+n_{14}+n_{23}+n_{24}}{1-c_{11}} \quad . \tag{2.6.14}$$

Similar expressions obtain for the other conditioning parameters. Thus, we have for the maximum likelihood estimates the following:

$$(2.6.15)\quad \left\{\begin{aligned}
\hat{c}_{11} &= \frac{n_{11} + n_{12} + n_{21} + n_{22}}{n_1 + n_2} \\
\hat{c}_{12} &= \frac{n_{31} + n_{32} + n_{41} + n_{42}}{n_3 + n_4} \\
\hat{c}_{21} &= \frac{n_{51} + n_{52} + n_{61} + n_{62}}{n_5 + n_6} \\
\hat{c}_{22} &= \frac{n_{71} + n_{72} + n_{81} + n_{82}}{n_7 + n_8} \\
\hat{d}_{11} &= \frac{n_{31} + n_{33} + n_{51} + n_{53}}{n_3 + n_5} \\
\hat{d}_{12} &= \frac{n_{11} + n_{13} + n_{71} + n_{73}}{n_1 + n_7} \\
\hat{d}_{21} &= \frac{n_{21} + n_{23} + n_{81} + n_{83}}{n_2 + n_8} \\
\hat{d}_{22} &= \frac{n_{41} + n_{43} + n_{61} + n_{63}}{n_4 + n_6} ,
\end{aligned}\right.$$

where as before $n_i = \sum_j n_{ij}$.

Given these eight estimated parameters, the χ^2 goodness of fit test (2.4.5) applied to the matrix (2.6.13) has $8 \cdot 3 - 8 = 16$ degrees of freedom and obviously the test for this two-person situation provides a real check on the empirical adequacy of the model. In subsequent chapters we shall compare the fit of the model with these eight conditioning parameters to the fit when it is assumed that $c_{ij} = d_{ij}$. On this latter assumption, the maximum likelihood estimates of the c_{ij}'s are, of course, different from those given above; these new estimates are presented below:

$$
(2.6.16) \quad \begin{cases}
\hat{c}_{11} = \dfrac{n_{11} + n_{12} + n_{21} + n_{22} + n_{31} + n_{33} + n_{51} + n_{53}}{n_1 + n_2 + n_3 + n_5} \\[2ex]
\hat{c}_{12} = \dfrac{n_{11} + n_{13} + n_{31} + n_{32} + n_{41} + n_{42} + n_{71} + n_{73}}{n_1 + n_3 + n_4 + n_7} \\[2ex]
\hat{c}_{21} = \dfrac{n_{21} + n_{23} + n_{51} + n_{52} + n_{61} + n_{62} + n_{81} + n_{83}}{n_2 + n_5 + n_6 + n_8} \\[2ex]
\hat{c}_{22} = \dfrac{n_{41} + n_{43} + n_{61} + n_{63} + n_{71} + n_{72} + n_{81} + n_{82}}{n_4 + n_6 + n_7 + n_8}
\end{cases}
$$

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