PURSUING THE RELEVANCE OF WIGNER'S SEMICIRCLE LAW

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ABSTRACT

Around 1920, Wishart began researching how different sets of data could be connected. He focused on the eigenvalues of matrices, which are special numbers that can inform us about the structure of the data. This work laid the foundation for Random Matrix Theory, a field that combines ideas from linear algebra and probability.

In the 1950s, Wigner, an American physicist, used Random Matrix Theory to study the behavior of eigenvalues in large matrices. He found that as the size of the matrices became very large, the distribution of eigenvalues tended to form a semicircle shape. This unexpected result, known as Wigner's Semicircle Law, showed that the distribution of eigenvalues was not normal, but instead resembled a semicircle.

This project aims to prove Wigner's Semicircle Law using concepts from probability, linear algebra, and rigorous mathematical proofs. The goal is to provide a clear explanation of the proof and to understand the reasoning in why this happens. The aim of demonstrating the proof of the semicircle law is to present the work in a manner that is understandable to a wide variety of people, even those without a background in mathematics. By doing this, we hope to understand why the semicircle law holds and whether it remains true under different assumptions.

ACKNOWLEDGMENTS

There have been many people whom I have to thank for their help throughout this project. I am especially grateful to my mentor, Dr. Kevin Costello, for guiding me through his research interests and for his patience in answering my questions. I would not have been able to finish the project and acquire new mathematics skills from pure mathematics without his support. I truly thank him for that. Additionally, I would like to mention a graduate student, James Alcala, who was my TA in my first year, and another graduate student, Edwin Lin, who gave me some advice regarding designing a poster.

1. Introduction

In the realm of random matrix theory, Wigner's semicircle law stands as a cornerstone result, illuminating profound connections between probability theory, linear algebra, and combinatorics. At its core, the semicircle law describes the limiting behavior of the eigenvalue distribution of large symmetric (or Hermitian) random matrices. To appreciate this law's significance, we delve into the definitions of a semicircle, the eigenvalues of random matrices, and the Catalan numbers.

2. What is an Eigenvalue?

In the context of matrices, eigenvalues represent scalars that describe how a matrix stretches or compresses vectors in certain directions. When discussing the moments of the Wigner's semicircle distribution, these moments are connected to the moments of the eigenvalues. Specifically, the kth moment of the semicircle distribution relates to the kth moment of the eigenvalues.

3. Figure 1 description

Number of matrices and size: `num_matrices` specifies the number of random symmetric matrices to generate, and `matrix_size` specifies the size of each matrix (2000x2000 in this case).

Generating random symmetric matrices: For each matrix, it generates a random matrix A with elements between -1 and 1. It is the average over multiple random matrices.

Making A symmetric: It then creates a symmetric matrix A by setting the lower triangle of A equal to its transpose, ensuring symmetry.

Normalizing A: It normalizes A such that the maximum absolute value of its eigenvalues is 1.

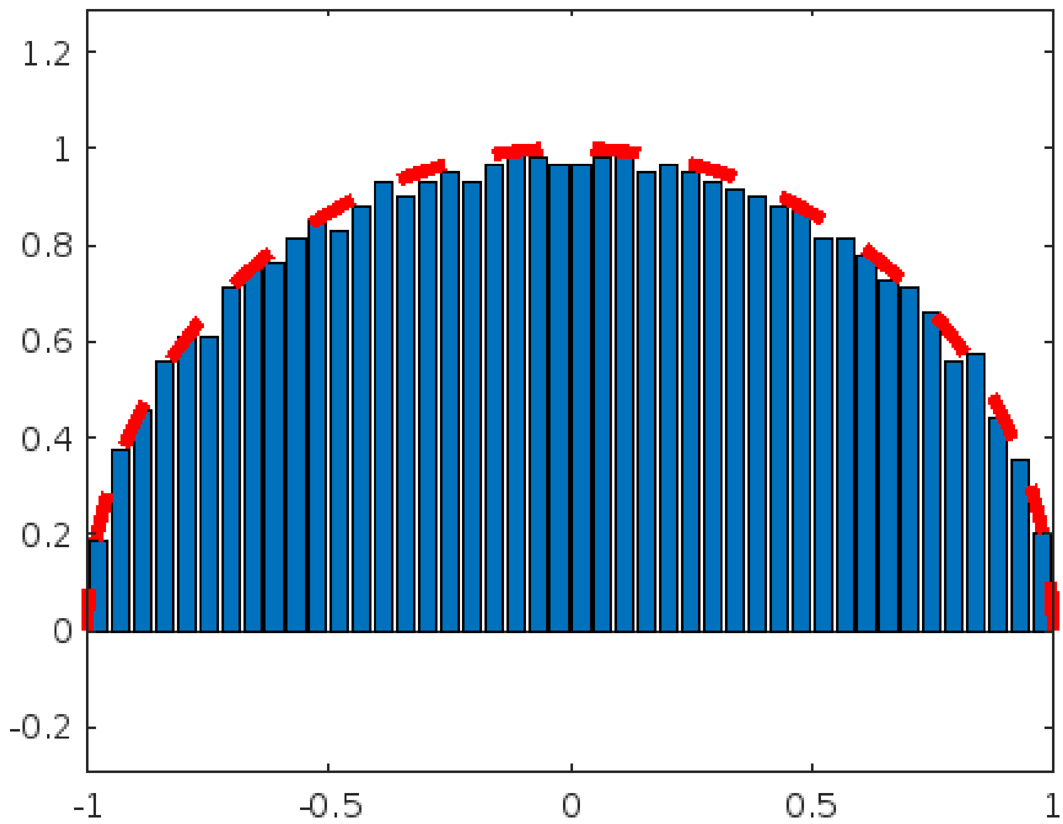


FIGURE 1. Eigenvalue distribution of a random matrix.

Computing the histogram: It computes the histogram of the eigenvalues of A, using `hist(eig(A), floor(sqrt(matrix_size)))`, where `floor(sqrt(matrix_size))` determines the number of bins for the histogram.

Plotting the histogram and semicircle: It plots the histogram of the eigenvalues (`num_eigenvalues`) normalized by the maximum count, along with the semicircle distribution (`plot(cos(t), sin(t), 'r--', 'LineWidth', 4)`), which is the expected distribution of eigenvalues for large random matrices.

Saving the figure: It saves the generated figure as a PNG file (`'my_figure.png'`).

Overall, this code provides a visualization of the distribution of eigenvalues of random symmetric matrices and compares it to the semicircle distribution, which is a theoretical distribution for such matrices.

A semicircle, in a mathematical context, refers to the shape of a half-circle arc. In probability theory, the semicircle distribution is characterized by its probability density function, resembling the upper half of a circle. The Semicircle distribution, also known as the Wigner semicircle distribution, is defined as probability distribution that describes the spectral density of certain random matrices. Its name comes from its shape, resembling a semicircle when plotted. This distribution is centered at the origin and extends symmetrically in both positive and negative directions, with a radius that can be adjusted.

Imagine drawing a semicircle on a graph, centered at (0, 0) and extending from -R to R on the x-axis. The probability density function (PDF) of the semicircle distribution is essentially the height of this semicircle, adjusted to fit the desired radius and center. The distribution's parameters, a (center) and r (radius), determine the specific shape and size of the semicircle.This distribution arises naturally in the study of random matrices, particularly in the context of symmetric matrices with independent and identically distributed entries. As the size of these matrices grows, their eigenvalues, representing key characteristics of the matrices, tend to follow the semicircle distribution. This convergence to the semicircle law is a remarkable phenomenon, illustrating the deep interplay between probability and linear algebra.

The theorem and its assumptions: Wigner's semicircle theorem is a result in random matrix theory that describes the distribution of eigenvalues in large symmetric random matrices. Specifically, the theorem states that as the size of the matrix becomes large, the density of eigenvalues near the origin converges to a semicircle shape in the complex plane.

Formally, let A be an $N \times N$ symmetric matrix with entries a_{ij} that are independent random variables. The entries are typically assumed to have a common distribution with zero mean and variance $1/N$, although other distributions are possible. The semicircle law states that the density of eigenvalues of A near the origin converges to a semicircle distribution as N approaches infinity. This semicircle distribution is given by the probability density function:

The main result of Wigner's semicircle theorem is the description of the limiting distribution of the eigenvalues of large symmetric random matrices. Specifically, as the size of the matrix approaches infinity, the density of eigenvalues near the origin converges to a semicircle shape in the complex plane. This semicircle distribution has a specific mathematical form, which can be derived from the assumptions mentioned above.

Wigner's semicircle Theorem and Assumptions

According to the introduction of the book "Introduction to Random Matrices" by Anderson G., Guionnet A., and Zeitouni O., it states that the theorem is

Theorem 1 (Wigner's Theorem)**.** *"For a Wigner matrix, the empirical measure L_N converges weakly, in probability, to the standard semicircle distribution. In greater detail, Theorem 2.1.1 asserts that for any $f \in C_b(\mathbb{R})$, and any $\varepsilon > 0$,*

$$\lim_{N\to\infty} P\left(|h(L_N, f_i) - h(\sigma, f_i)| > \varepsilon\right) = 0.$$

Start with two independent families of i.i.d., zero mean, real-valued random variables $\{X_{i,j}\}_{1\leq i<j}$ and $\{X_i\}_{1\leq i}$, such that $E[X_{1,2}^2] = 1$ and, for all integers $k \geq 1$, $r_k := \max\left(E|X_{1,2}|^k\right) < \infty$.

Consider the (symmetric) $N \times N$ matrix X_N with entries

$$X_N(j,i) = X_N(i,j) = \begin{cases} \frac{X_{i,j}}{\sqrt{N}}, & \text{if } i < j," (Anderson, 6-7) \end{cases}$$

Furthermore, the Catalan numbers, ubiquitous in combinatorics, unexpectedly find a role in the context of the semicircle law. These numbers count various structures, such as the number of ways to parenthesize an expression or the number of paths in a grid, and remarkably, they appear in the coefficients of the moments of the semicircle distribution. This connection underscores the profound unity in mathematics, where seemingly disparate concepts harmonize to reveal hidden symmetries and structures.

In this paper, we explore the intricacies of Wigner's semicircle law, examining its origins, implications, and connections to other areas of mathematics. Through this exploration, we aim to illuminate the beauty and depth of mathematics, where patterns emerge from randomness, and structures intertwine across seemingly distinct domains.

Wigner's semicircle theorem is a result in random matrix theory that describes the distribution of eigenvalues in large symmetric random matrices. Specifically, the theorem states that as the size of the matrix becomes large, all the eigenvalues converge to a semicircle.

- **Symmetric Matrices**: Wigner's semicircle theorem applies to large symmetric matrices. These matrices have real entries and satisfy the property that $A = A^T$, where A^T denotes the transpose of A.
- **Randomness/independence**: The above diagonal entries are typically assumed to be independent random variables, meaning that the value of one entry does not depend on the values of the other entries.
- **Large Matrix Size**: The theorem is concerned with the behavior of the eigenvalues as the size of the matrix becomes large. In the limit of large matrix size, certain statistical properties of the eigenvalues converge to a specific distribution.
- **Variance/Mean**: We assume that the variance Var(X_{ij}) = 1, and the Expectation E$(X_{ij}) = 0$ for each entry.
- **Moments**: For each $k > 0$, there is a constant C_k such that $E(X_{ij}^k) \leq C_k$.

2.1.1 Why are the moments of the semicircle distribution given by the Catalan numbers?

We will be showing why this is true.

Reasoning:

The kth moment of the semicircle distribution is defined by the integral

$$m_{2k} := \frac{1}{2\pi} x^k \sqrt{4 - x^2}\, dx.$$

When considering the moment m_{2k}, where k is a positive integer, it is important to note that this moment is related to the standard semicircle distribution, denoted by σ. The standard semicircle distribution is characterized by its density function, which is symmetric about the origin. When the moment m_{2k} is computed, it involves integrating a function that is symmetric about the origin over a symmetric interval.

When k is odd, the integral and $x^k \sqrt{4 - x^2}$ is an odd function, symmetric about the origin, leading to a net integral of zero over the symmetric interval. This symmetry causes equal and opposite contributions to the integral on each side of the origin, resulting in cancellation.

On the other hand, for even k, the integrand becomes an even function, which is not symmetric about the origin but symmetric about the y-axis. This symmetry ensures that the positive contributions on one side are exactly balanced by equal positive contributions on the other side, leading to a non-zero integral.

Therefore, for odd k, the integral m_{2k} evaluates to zero, while for even k, it evaluates to a non-zero value. This property is essential for understanding the moments of the semicircle distribution and its connections to other mathematical concepts, such as the Catalan numbers.

If k is even, our first step will be to show why

$m_{2k} = \frac{2^{2k}}{2k\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k}(\theta) \cos^2(\theta)\, d\theta$ and how this connects with the Catalan numbers.

2.1.1 The Semicircle distribution, Catalan numbers, and Dyck Paths

The Semicircle Distribution, Catalan Numbers, and Dyck Paths

Define the moments $m_k = \langle \sigma, x^k \rangle$. Recall the Catalan numbers:

$$C_k = \frac{\binom{2k}{k}}{k+1} = \frac{(2k)!}{(k+1)!k!}.$$

We now check that for all integers k,

$$m_{2k} = C_k, \quad m_{2k+1} = 0$$

Indeed, $m_{2k+1} = 0$ by symmetry, while

$$m_{2k} = \int_{-2}^{2} x^{2k} \sigma(x) dx = \frac{2 * 2^k}{2k\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k}(\theta) \cos^2(\theta) d\theta$$

We define the standard semicircle distribution as the probability distribution $\sigma(x)\, dx$ on $\mathbb{R}$ with density as:

$$\sigma(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

We will now being to prove how we got from left to right:

$$m_{2k} = \int_{-2}^{2} x^{2k} \sigma(x) dx = \frac{2 * 2^k}{2k\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k}(\theta) \cos^2(\theta) d\theta$$

We know that $\sigma(x)$ on $\mathbb{R}$ is given by:

$$\sigma(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

Now we plug it in

$$\begin{aligned} m_{2k} &= \int_{-2}^{2} x^{2k} \frac{1}{2\pi} \sqrt{4-x^2}\, dx \\ &= \frac{1}{2\pi} \int_{-2}^{2} x^{2k} \sqrt{4-x^2}\, dx \end{aligned}$$

Through trig identities, we now substitute

$$x = 2\sin(\theta),$$

$$dx = 2\cos(\theta)\, d\theta.$$

$$= \frac{1}{2\pi} \int_{-2}^{2} [2\sin(\theta)^{2k}] \sqrt{4(2\sin(\theta)^2)}\, 2\cos(\theta)\, d\theta$$

Upper bound:

$$2 = 2\sin(\theta) \quad \text{(divide both sides by 2)}$$

$$1 = \sin(\theta)$$

$$\theta = \frac{\pi}{2}$$

Lower bound:

$$-2 = 2\sin(\theta) \quad \text{(divide both sides by 2)}$$

$$-1 = \sin(\theta)$$

$$\theta = -\frac{\pi}{2}$$

Simplification:

$$\sqrt{4-4\sin^2(\theta)} = \sqrt{4(1-\sin^2(\theta))} \quad (\text{since } 1-\sin^2(\theta)=\cos^2(\theta))$$
$$= 2\cos(\theta)$$

Incorporate the new bounds and the simplification

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(2\sin(\theta))^{2k}2\cos(\theta)\,2\cos(\theta)\,d\theta$$
$$= \frac{1}{2\pi}\cdot 2\cdot 2\cdot(2^{2k})\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin(\theta))^{2k}\cos^2(\theta)\,d\theta \qquad = \frac{2(2^{2k})}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin^{2k}(\theta))\cos^2(\theta)\,d\theta$$

Therefore, we have just proven that:

$$m_{2k} = \int_{-2}^{2} x^{2k}\sigma(x)dx = \frac{2(2^{2k})}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^{2k}(\theta)\cos^2(\theta)d\theta$$

Now we want to prove:

$$\frac{2(2^{2k})}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin^{2k}(\theta))\cos^2(\theta)\,d\theta = \frac{2(2^{2k})}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^{2k}(\theta)\,d\theta - (2k+1)m_{2k}$$

Prove it from left to right:

$$\cos^2\theta = 1-\sin^2\theta$$
$$m_{2k} = \int \sin^{2k}(\theta)(1-\sin^2(\theta))\,d\theta$$
$$= \int \sin^{2k}(\theta)\,d\theta - \int \sin^{2k+2}(\theta)\,d\theta$$

Using Integration by parts

$$u = \sin^{2k+1}(\theta),\ v = -\sin(\theta) :$$

$$du = (2k+1)\sin^{2k}(\theta)\cos(\theta),\ dv = -\cos(\theta)\, d\theta,$$

$$= \int \sin^{2k}(\theta)\, d\theta + \left[-\cos(\theta)\sin^{2k+1}(\theta) \Big|_{-\pi/2}^{\pi/2} - \int -(2k+1)\sin^{2k}(\theta)\cos^2(\theta)\, d\theta \right]$$

When plugging in $\frac{\pi}{2}$ for θ, we know that $-\cos\left(\frac{\pi}{2}\right) = 0$, so it cancels out.

This leaves us with $\int \sin^{2k}(\theta)\, d\theta - \left(\int (2k+1)\sin^{2k}(\theta)\cos^2(\theta)\, d\theta\right)$

$$m_{2k} = \tfrac{2^{2k}}{\pi(2k+2)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k}(\theta)\, d\theta$$

Using u-substitution and integration by parts with

$$u = \sin^{2k-1}(\theta),\ v = -\cos(\theta),$$

$$du = (2k-1)\sin^{2k-2}(\theta)\cos(\theta)\, d\theta, dv = \sin(\theta),$$

uv- $\int v\, du = \left[\frac{2 \cdot 2^{2k}}{\pi} \sin^{2k-1}(\theta)(-\cos(\theta)) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-\cos(\theta)) \cdot (2k-1)\sin^{2k-2}(\theta)\cos(\theta)\, d\theta$

$$m_{2k} = \frac{2 \cdot 2^{2k}}{(2k+2)\pi} \cdot (2k-1) \cdot \frac{\pi m_{2k-2}}{2 \cdot 2^{2k-2}}$$

cancel out 2 and π and reduce

$$m_{2k-2} = \frac{2 \cdot 2^{2k-2}}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k-2}(\theta)\cos(\theta)\,d\theta$$

multiply by π and divide by $2 \cdot 2^{k-2}$

$$= \frac{\pi \cdot m_{2k-2}}{2 \cdot 2^{2k-2}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k-2}(\theta)\cos(\theta)\,d\theta$$

$$= \frac{2 \cdot 2^{2k} \cdot (2k-1)}{\pi \cdot (2k+2)} \cdot \frac{\pi}{2 \cdot 2^{2k-2}} \cdot m_{2k-2}$$

Cancel π and 2^k and 2^{k-2} can be factored out.

$$= \frac{4(2k-1)}{2k+2} \cdot m_{2k-2} \tag{1}$$

USING INDUCTION FROM THE FORMULA TO THE CATALAN NUMBERS

Base Case: For $k = 1$, we have $M_2 = \frac{4(2(1)-1)}{2(1)+2} \cdot M_0 = 2 \cdot M_0$. If we have $M_0 = \sigma(x) = \frac{1}{2\pi}\sqrt{4-x^2}$, we know that the area of $\sqrt{4-x^2}$ is 2π. Therefore, multiplying this with $\frac{1}{2\pi}$ gives us 1, which satisfies the base case, for k=1. Since $M_0 = 1$ (base case for Catalan numbers), we get $M_2 = 2$. Now, let's express M_2 in terms of Catalan numbers.

For the equation $\sigma(x)\,dx$ on $\mathbb{R}$ with density as:

$$\sigma(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

Inductive Hypothesis: Assume that the formula holds for $k-1$, i.e., $M_{2(k-1)} = C_{k-1}$.

Inductive Step: Now, let's prove the formula for k:

$$M_{2k} = \frac{4(2k-1)}{2k+2} \cdot M_{2k-2}$$

$$C_{k-1} = \frac{(2(k-1))!}{k!(k-1)!}$$

With our inductive hypothesis, we assumed:

$$C_{k-1} = M_{2k-2}.$$

$$M_{2k} = \frac{4(2k-1)}{2k+2} \cdot \frac{(2(k-1))!}{k!(k-1)!} = \frac{4(2k-1)}{2k+2} \cdot \frac{(2k-2)!}{k!(k-1)!}$$

Simplifying further:

$$M_{2k} = \frac{4((2k-1)!)}{2(k+2)k!(k-1)!} \cdot \frac{2k}{2k}$$

Simplify: (2k-1)! · (2k) = 2k! and (k-1)! · k = k! and (2k+2) can be rewritten as 2(k+1)

$$M_{2k} = \frac{4(2k)!}{2(k+1)(k)!(k)! \cdot 2}$$

Simplify: cancel 2 =

$$M_{2k} = \frac{2(2k)!}{2(k+1)(k)!(k)!}$$

Simplify: cancel 2 again =

$$M_{2k} = \frac{(2k)!}{(k+1)(k)!(k)!}$$

Simplify (k+1) · (k)! = (k+1)! Finally the solution is

$$M_{2k} = \frac{(2k)!}{(k+1)!(k)!}$$

4. Special Vs. Unspecial Walks.

The expected kth power of eigenvalues can be related to a sum over closed walks of length k.

$$E(\lambda_i^k) = \frac{1}{n}E(\text{Tr}(A^k)) = \frac{1}{n}\sum_{\text{Walks}}\left(\prod_{\text{edges }(i,j)} X_{(i,j)}\right)$$

1. Sum over unspecial group $\rightarrow 0$
2. Sum over special group $\rightarrow$ Catalan

Consider a graph whose vertices are given by the rows of the matrix, with each pair (i, j) connected by an edge labelled by the entry $X_{i,j}$.

(1) First Equation:
- $E(\lambda_i^k)$: This represents the expected value of the kth power of the ith eigenvalue (λ_i) of a random matrix.
- $E(\text{Tr}(A^k))$: This represents the expected value of the trace of the kth power of a random matrix A. The trace of a matrix is the sum of its diagonal elements.
- $\sum_{\text{Closed walks of length } k}$ edges (i, j): This part relates to the concept of closed walks in a graph associated with the matrix A. The sum is taken over all closed walks of length k, and for each walk, you take the product of the corresponding matrix elements.

In the context of Wigner's semicircle theorem, this equation might be used to relate the statistical properties of the eigenvalues of certain random matrices to the properties of the matrices themselves. The sum over closed walks captures the contributions of different paths in the matrix, which can be related to the behavior of eigenvalues.

The sum of the walks in this equation includes many walks, with three examples being three different types of walks

We classify the walks into three groups

- Group 1: Walks which have some edge used exactly once
- Group 2: Walks which use every edge exactly twice and involve exactly $\frac{k}{2}+1$ vertices
- Group 3: Walks which involve fewer than $\frac{k}{2}+1$ vertices

”

(1) **Unspecial group 1:** Some edge is used only once. The sum representing the k^{th} moment involves a considerable number of terms. We will provide separate limits on the number of terms and the magnitude of each term.

Number of terms: To describe our walk, we must specify (1) which vertices are visited (at any point) in the walk and (2) which of those vertices is visited at each specific step in the walk. (3) The denominator/scaling = $\mathbf{n}^{k+1}$:

$k =$ The number of vertices visited.The denominator is denoted as n^{k+1}

Expectation will be 0

(2) **Special:** $k/2+1$ vertices, where each edge is used exactly twice. For each edge, each such walk contributes to 1 because from our assumptions the expectation squared = 1. In addition, it remains to compute the number of terms. Each walk corresponds to picking $\frac{k}{2}+1$ vertices (This is where $P(n, \frac{k}{2}+1)$ comes from) (in order) and choosing which steps are forward and backward (where $C_{\frac{k}{2}}$ comes from). Furthermore, We divide by $n^{\frac{k}{2}+1}$ to

account for, where we are scaling down the entries. Also, we are dividing it by n because we are averaging over the eigenvalues.

List of what's explained:

$P(n, \frac{k}{2}+1)$ Choose $k/2+1$ vertices.

$C_{k/2} =$ the number of walks.

$1 =$ Expectation of the walk.

$n^{k/2+1} = \frac{1}{\sqrt{n}}$ scaling down entries.

$\frac{1}{n} =$ averaging over the eigenvalues.

In conclusion: $\frac{P(n,\frac{k}{2}+1)}{n^{k/2+1}} \cdot C_{k/2} \cdot 1 \longrightarrow \frac{n^{k/2+1}}{n^{k/2+1} \cdot C_{k/2}} = C_{k/2}$.

With this Final equation we see that with $\frac{P(n,\frac{k}{2}+1)}{n^{k/2+1}}$ as it goes to infinity it will equal 1. Therefore, with 1 $\cdot C_{k/2} \cdot 1 = C_{k/2}$

(3) **Unspecial group 2:**

Everything in unspecial group 2 is very similar to the special group case we have just explained. However, there are some differences.

Question: There are 4 terms, but for each of those 4 terms is it the same or different??

The expectation of a single walk wouldn't be 1 anymore compared to the special walks, rather it would be a constant. When being scaled, the expectation goes to 0 as n goes to infinity. Now, since the the number of vertices is smaller, (specifically exponent of n smaller). The denominator stays the same as the special walk. However, the numerator is the small power of n goes to 0.

Notes:

Fewer than $k/2+1$ vertices are used.

The vertices $< k/2+1$, i.e., $\leq \frac{k+1}{2}$.

Numerator: $\leq n^{\left(\frac{k+1}{2}\right)}$.

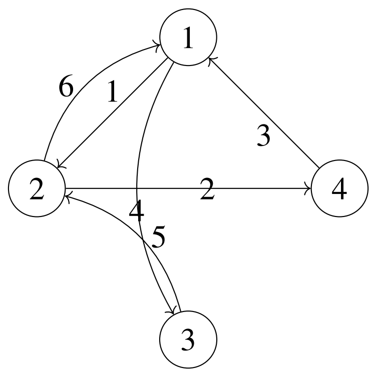
Denominator: $n^{\left(\frac{k}{2}+1\right)}$.

$C_{k/2}$: $\leq \left(\frac{k+1}{2}\right)^k$, k steps = this represents the number of walks $\leq k/2$ choices per step.

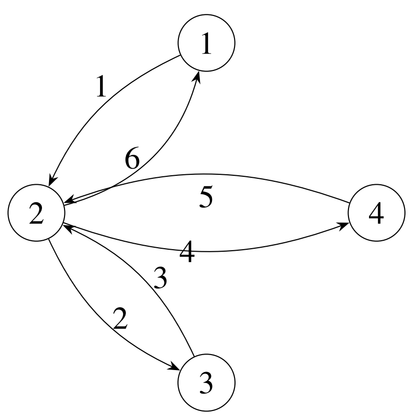
$E(\text{walk}) \leq C$.

In the end, we get $\frac{n^{k+\frac{1}{2}}}{n^{\frac{k}{2}+1}} \cdot C_{k/2} \cdot C$ Therefore, in $\frac{n^{k+\frac{1}{2}}}{n^{\frac{k}{2}+1}}$ the numerator is smaller then the denominator, making this go to zero. So we are left with 0 $\cdot C_{k/2} \cdot C$.

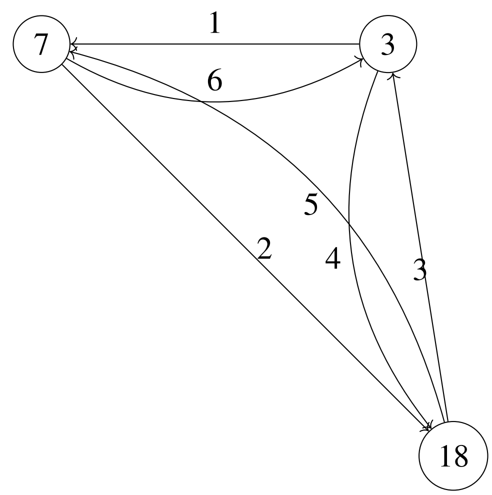
Example 1:



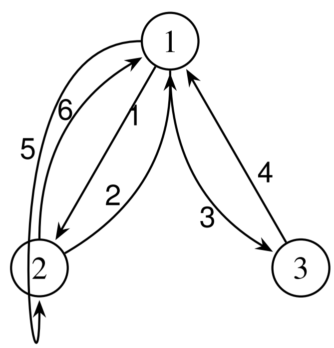
Example 2:



Example 3:



Example 4:



The sum representing the k^{th} moment involves a considerable number of terms. We will provide separate limits on the number of terms and the magnitude of each term.

Number of terms: To describe our walk, we must specify (1) which vertices are visited (at any point) in the walk and (2) which of those vertices is visited at each specific step in the walk. (3) The denominator/scaling = $\mathbf{n}^{k+1}$:

$k =$ The number of vertices visited.The denominator is denoted as n^{k+1}

Looking at Example 1: Concerning (1): We assume our walk visits at most 4 vertices. Since there are a total of n vertices, the number of choices for those vertices is at most $n(n-1)(n-2)(n-3)$.

Regarding (2): The walk consists of 5 steps. In each step, we have at most 4 choices for which vertex to visit (it must be a vertex from (1)).

Regarding (3): The denominator would be n^5 since there are 4 vertices.
Combining (1) and (2), we find that the number of walks is at most $4^5 \cdot n \cdot (n-1) \cdot (n-2) \cdot (n-3)$.
Additionally, combining the denominator we get = $\frac{n(n-1)(n-2)(n-3)}{n^5} \times 4^5$

The expression $\frac{(n^4)}{(n^5)} \times 4^5$ simplifies to $\frac{1}{n} \times 4^5$, which indeed approaches zero as n approaches infinity.

$$1 \to 2 \to 4 \to 1 \to 3 \to 2 \to 1$$

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$$

$$E(X_{12}) = 0, \quad E(X_{24}) = 0, \quad E(X_{41}) = 0, \quad E(X_{23}^2) = 1$$

$$E(X_{12}X_{24}X_{41}X_{23}^2) = E(X_{12})E(X_{24})E(X_{41})E(X_{23}^2) = (0 \cdot 0 \cdot 0 \cdot 1) = 0$$

Therefore, the expectation is zero.

Just like Example 1, any walk 1 has 0 expectation in general.
In conclusion, from Example 1, we see that some edges are used exactly once, which gives us that the expectation is 0. This is known as an **Unspecial group 1**.

Looking at example 2 (Generality):

Special:

$$\textbf{Vertices} \approx n^{k/2}+1$$

$$\textbf{Scaling/denominator:} \quad \frac{1}{n^{\frac{k}{2}+1}}$$

$$\textbf{choices F vs. B:} \quad C_{\frac{k}{2}}$$

Group 2 (Example 2) (k=6)

$$1 \to 2 \to 3 \to 2 \to 4 \to 2 \to 1$$

F F B F B B

$$E(X_{12}^2)=1, \quad E(X_{23}^2)=1, \quad E(X_{24}^2)=1$$

$$E(X_{12}X_{23}X_{23}X_{24}X_{24}X_{12}) = E(X_{12}^2)E(X_{23}^2)E(X_{24}^2) = 1$$

$$(1\cdot 1\cdot 1)=1$$

$$E(X_{12}^2)=1, \quad E(X_{23}^2)=1, \quad E(X_{24}^2)=1$$

$$E(X_{12}X_{23}X_{23}X_{24}X_{24}X_{12}) = E(X_{12}^2)E(X_{23}^2)E(X_{24}^2) = 1$$

$$(1\cdot 1\cdot 1)=1$$

Therefore, our expectation is 1.

Choose F vs. B $= C_3$

Finding the Sum: $= \frac{n(n-1)(n-2)(n-3)\cdot C_3}{n^4} \approx \frac{n^4}{n^4}\cdot (C_3) \longrightarrow C_3$ (Can relate to Catalan walks)

Since in this case we use each edge exactly twice this. Then we chose $k/2+1$ vertices and it gives us the Catalan numbers. Therefore, is an example of a special walk.

Example 3:

We see in Example 3 there are only 3 verticies. The vertices are 3, 7 ,18. Since there are only 3 vertices, the numerator will be smaller then the denominator

$$= \frac{n(n-1)(n-2)(3^7)\cdot \text{Constant}}{n^4}$$

Note: the power of the n's:

$$= \frac{n^3}{n^4}$$

Therefore, since the bottom power is bigger the expectation heads to zero.

In conclusion, if we use fewer than 4 vertices it heads to zero. This is known as **Unspecial group 2**.

Example 4:

$$1 \to 2 \to 1 \to 3 \to 1 \to 2 \to 1$$

1 2 3 4 5 6

$$E(X_{12}^2 X_{23}^4 X_{24}^2) = E(X_{12}^2)E(X_{23}^4)E(X_{24}^2)$$

Given the assumptions $E(X_{12}^2) = 1$ and $E(X_{24}^2) = 1$,

$$= E(X_{23}^4) \leq D$$

Since we don't have an assumption that gives us the expectation beyond the second power, we let it equal a constant, denoted by D in this case.

Vertices: n(n-1)(n-2)(n-3); n^4

Scaling/Denominator $= \frac{1}{n^5}$

Choose F vs. B $= C_4$

Expectation Equation:

= $\frac{n(n-1)(n-2)(n-3) \cdot C_4}{n^4}$

= $\frac{n^4}{n^4} \cdot (C_4) \longrightarrow C_4$ (Catalan numbers) Since in this case we use each edge exactly twice this and chose $k/2 + 1$ vertices this means that it is also a special walk.

Why is Wigner's semicircle Law relevant??

Wigner's semicircle law is a pivotal result in random matrix theory, a branch of mathematics focused on matrices whose entries are random variables. This law provides a profound insight into the distribution of eigenvalues in large random symmetric matrices, offering a simplified model for understanding complex systems. Originating from the work of Eugene Wigner and Freeman Dyson in the mid-20th century, Wigner's semicircle law has become a fundamental concept with wide-ranging applications in mathematics and physics.

At the core of Wigner's semicircle law is the behavior of eigenvalues in large symmetric matrices with independent and identically distributed entries. As the size of the matrix grows infinitely large, the density of eigenvalues follows a semicircular shape when properly normalized. This distribution, known as the semicircle distribution, indicates that eigenvalues are more likely to cluster around the center of the semicircle and less likely to be found at the edges. One of the most remarkable aspects of Wigner's semicircle law is its universality. This means that the semicircle distribution emerges in various random matrix ensembles, regardless of the specific distribution of matrix entries. This universality property allows researchers to apply the insights from random matrix theory to analyze complex systems in physics, statistics, and beyond.

In quantum mechanics, Wigner's semicircle law plays a crucial role in understanding the distribution of energy levels in certain quantum systems. For example, in nuclear physics, the energy levels of atomic nuclei can be approximated using random matrix ensembles, providing insights into nuclear structure and behavior. The semicircle law also appears in statistical physics, where it describes the density of states of disordered systems such as amorphous solids or glasses. Moreover, Wigner's semicircle law has intriguing connections to number theory. Specifically, the distribution of zeros of the Riemann zeta function, a central object in number theory, is conjectured to be related to the eigenvalue distribution of certain random matrix ensembles. This connection highlights the deep interplay between seemingly unrelated areas of mathematics and physics, showcasing the broad impact of Wigner's semicircle law. In conclusion, Wigner's semicircle law stands as a cornerstone of random matrix theory with profound implications across various disciplines.

Its universality, connection to other areas of mathematics and physics, and ability to simplify the study of complex systems make it a powerful and versatile tool for researchers. Whether analyzing the energy levels of atomic nuclei or investigating the distribution of zeros in number theory, Wigner's semicircle law continues to inspire new discoveries and insights in mathematics and physics. I have tried to proof this law and show it's relevancy in the real world.

Motivation

Wigner's development of the semicircle law in the 1950s was motivated by the need for a mathematical framework to describe the energy levels of heavy nuclei, which exhibited statistical regularities despite the complexity of nuclear interactions. His key insight was to model the interactions between nucleons in a nucleus using a random symmetric matrix, where each matrix element represented the interaction strength between two nucleons. By analyzing the eigenvalues of these random matrices, Wigner aimed to uncover the underlying distribution that governed the behavior of nuclear energy levels. Through meticulous calculations and pioneering mathematical techniques, Wigner showed that as the size of the random matrices grew to infinity, the distribution of eigenvalues converged to a semicircular shape. This semicircle law was a remarkable discovery, as it provided a universal description of the spectral properties of large random matrices, independent of the specific details of the matrix entries. The significance of Wigner's semicircle law extends far beyond its original application in nuclear physics. It has become a fundamental result in random matrix theory, with implications in diverse fields such as quantum chaos, statistical physics, and number theory. The universality of the semicircle law has been confirmed in various random matrix ensembles, demonstrating its broad applicability and deep connections to other areas of mathematics and physics. In quantum chaos, the semicircle law provides insights into the quantum behavior of classically chaotic systems, where the energy levels exhibit statistical patterns analogous to those of random matrices. In statistical physics, the semicircle law describes the density of states in disordered systems, shedding light on the macroscopic properties of complex materials. Furthermore, the semicircle law has sparked new developments in number theory, particularly in the study of the Riemann zeta function. The connection between the distribution of zeros of the Riemann zeta function and the eigenvalue distribution of random matrices has led to conjectures about the universality of certain number-theoretic phenomena. Overall, Wigner's creation of the semicircle law was a transformative moment in mathematics and physics, providing a powerful tool for understanding the behavior of complex systems with random interactions. Its universality and deep connections to other areas of science underscore its significance and ensure its enduring legacy in the scientific community.

Future Research topics from this

Expanding on the research into Wigner's semicircle law offers a wealth of intriguing avenues for further exploration. One compelling direction involves investigating generalizations of the semicircle law to different random matrix ensembles, such as the Gaussian Unitary Ensemble (GUE) or the Gaussian Orthogonal Ensemble (GOE). This exploration would delve into how various matrix ensembles impact the distribution of eigenvalues and whether the concept of universality extends to these ensembles. Another fascinating path could involve studying non-symmetric random matrices, such as those in the GUE, to analyze how eigenvalue distributions and spacing statistics differ from those of symmetric matrices.

Moreover, delving into the applications of random matrix theory in quantum information theory could yield valuable insights into entangled states, quantum channels, and quantum algorithms. This could lead to advancements in quantum computing and communication. Furthermore, exploring the use of random matrix theory in financial mathematics could offer new perspectives on modeling stock price movements, portfolio optimization, and risk management strategies. Additionally, investigating the connections between random matrix theory and quantum chaos could provide deeper insights into the spectral statistics of chaotic quantum systems.

Another avenue worth exploring is the application of random matrix theory in machine learning, particularly in analyzing the performance of deep learning models and neural networks. Additionally, studying the connections between random matrix theory and random graph theory could lead to a better understanding of complex networks and large-scale systems. Furthermore, exploring the connections between random matrix theory and number theory, particularly in relation to the Riemann zeta function and the distribution of prime numbers, could uncover new insights at the intersection of these fields. Finally, developing computational methods for efficiently simulating and analyzing large random matrices could open up new possibilities for exploring the properties of these matrices in high-dimensional systems.

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