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UNIVERSITY OF CALIFORNIA,
IRVINE

On the moduli space of locally conformally flat manifolds and collapsed constant scalar
manifolds

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Xiaokang Wang

Dissertation Committee:
Professor Jeff Viaclovsky, Chair
Professor Richard Schoen
Professor Zhiqin Lu
Professor Timothy Tait

DEDICATION

To my parents and my partner Wenlin, for their endless love, patience, and unwavering support.

TABLE OF CONTENTS

	Page
LIST OF FIGURES	vi
LIST OF TABLES	vii
ACKNOWLEDGMENTS	viii
VITA	ix
ABSTRACT OF THE DISSERTATION	xi
Notations	1
1 Introduction	3
1.1 Background in Riemannian Geometry	3
1.1.1 Group action and Quotients	4
1.1.2 ALE manifolds	9
1.1.3 Conformal change of scalar curvature	11
1.2 Background in PDE	12
1.3 Nilpotent group and actions	16
1.4 Gromov-Hausdorff Convergence	20
1.5 Cheeger-Gromov Convergence	23
1.6 Cheeger-Fukaya-Gromov Collapsing Theory	26
2 Special Metrics and Moduli Space	31
2.1 Locally Conformally Flat Metrics	31
2.1.1 Local Analytic Aspect for LCF manifolds	33
2.1.2 Global Geometric Aspect for LCF manifolds	40
2.2 Yamabe Metrics	45
2.3 Positive Mass Theorem	48
2.4 Examples and Classification Results for LCF Manifolds	51
2.4.1 Standard examples of LCF manifolds and their Yamabe class	52
2.4.2 Classification of LCF manifolds in the positive Yamabe class	53
2.4.3 Classification in dimension 3	54
2.4.4 Classification in dimension 4	55

2.4.5	Complete noncompact LCF manifolds with non-negative Ricci curvature	56
2.5	Moduli Space Analogy: Teichmüller Space	56
2.5.1	Teichmüller space of a surface	57
2.5.2	The LCF moduli space as a higher-dimensional analog	58
3	Locally Conformally Flat Moduli Space	60
3.1	The Moduli Space and Tian-Viaclovsky Compactification	60
3.2	Locally Conformally Flat ALE Manifolds	66
3.2.1	Totally umbilic submanifolds	67
3.2.2	Harmonic functions on ALE manifolds	69
3.2.3	LCF ALE manifolds are AE	70
3.2.4	Universal cover as a domain in $\mathbb{S}^n$	73
3.3	Structure Theory of LCF ALE Manifolds and Orbifolds	74
3.3.1	Injectivity theorem	74
3.3.2	Local isotropy groups	75
3.3.3	Finite manifold cover	76
3.4	The Good Orbifold Theorem	77
3.5	Classification Results	79
3.5.1	Conformal compactification and Yamabe invariant	79
3.5.2	Local isotropy and paired orbifold points	80
3.5.3	Dimension-specific classification	82
3.5.4	The Schwarzschild example and the pairs conjecture	85
3.6	Application to the Moduli Space: Refined Description	86
3.6.1	The oriented moduli space	87
3.6.2	Non-orientable limiting configurations	90
3.7	The positive mass theorem for LCF, ALE orbifolds	92
4	Collapsing Constant Scalar Curvature Metrics	97
4.1	The Question of Cheeger–Fukaya–Gromov	97
4.2	Cheeger–Fukaya–Gromov Theory with Finite Regularity	101
4.2.1	Finite-regularity fibration and connections	101
4.2.2	The $\mathcal{N}$ -invariant nearby metric	106
4.3	Convergence of Eigenvalues of the Laplacian	109
4.4	Proof Strategy: Invariant Metric and Scalar Curvature Estimates	111
4.4.1	The $\mathcal{N}$ -invariant nearby metric	112
4.4.2	Scalar curvature in the distributional norm	112
4.4.3	Laplacian on $\mathcal{N}$ -invariant functions	113
4.4.4	Weighted Sobolev spaces on the base	115
4.5	Linearized Equation Estimates	115
4.5.1	The operator P and its linearization	116
4.5.2	Calderón–Zygmund estimate	118
4.5.3	Uniform injectivity	119
4.6	Nonlinear Equation Solution	120
5	Further Discussions and Open Problems	124

5.1	A Gluing Construction of Constant Scalar Curvature Metrics	124
5.1.1	Joyce's gluing theorem	124
5.1.2	Collapsing factors and a new type of degeneration	125
5.2	An ε -Regularity Theorem in the Collapsing Setting	127
5.2.1	Failure in the collapsing setting	127
5.2.2	The conjecture	128
5.3	The Full Picture of the LCF Moduli Space	129
5.3.1	Collapsing LCF sequences and their limits	129
5.3.2	Scalar curvature is not preserved under collapsing	130
5.3.3	Open problems	131
Bibliography		133
Appendix A The proof of Cheeger-Fukaya-Gromov Theorem		138
A.1	Proof of Theorem 1.11	138
Appendix B Orbifold Points in Pairs		144

LIST OF FIGURES

	Page
2.1 Compact Riemann surfaces of genus $g = 0, 1$, and 2 . The Teichmüller space $\mathcal{T}_g$ parametrizes complex (equivalently, constant-curvature) structures on genus- g surfaces.	57
3.1 A sequence $(M_i, g_i) \in \mathcal{M}$ converging in the Gromov-Hausdorff topology to an LCF orbifold (M_∞, g_∞) . At each concentration point a rescaled bubble $(\hat{M}_\infty, \hat{g}_\infty)$, a non-compact LCF ALE manifold, detaches.	62
3.2 An ALE manifold (M, g) and its end E , diffeomorphic to $(\mathbb{R}^n \setminus B)/\Gamma$. The two colored curves are large coordinate spheres that become totally umbilic under the conformal map $\bar{\phi}: M \rightarrow \mathbb{R}^n$	72
3.3 The Schwarzschild ALE manifold (M, g) (left), its quotient $(M/\sigma, g)$ by the antipodal map (center), and the blow-up at the resulting single orbifold point, which produces the football metric $\mathbb{S}^n/\mathbb{Z}_2$ (bottom).	86
3.4 Two non-orientable GH limiting configurations. <i>Top row</i> (Example 3.3): the limit is a non-orientable cap, and one non-orientable Schwarzschild ALE bubble detaches. <i>Bottom row</i> (Example 3.4): the limit is the football metric $\mathbb{S}^4/\Gamma$, and two non-orientable Schwarzschild ALE pieces bubble off.	96

LIST OF TABLES

	Page
2.1 Comparison of the two proofs of the classical PMT.	50
2.2 Comparison of the classical and LCF Positive Mass Theorems.	51
2.3 Standard LCF manifolds by Yamabe class.	52
2.4 Dictionary between Teichmüller theory and LCF moduli theory.	59

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ABSTRACT OF THE DISSERTATION

On the moduli space of locally conformally flat manifolds and collapsed constant scalar manifolds

By

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Doctor of Philosophy in Mathematics

University of California, Irvine, 2026

Professor Jeff Viaclovsky, Chair

We prove several structure theorems for locally conformally flat, positive Yamabe orbifolds and nonnegative scalar curvature, ALE manifolds. These two kinds of spaces can be related by conformal blow-up and conformal compactification. For the orbifolds, we prove that such orbifolds admit a manifold cover. For the ALE manifolds, the homomorphism of the fundamental group for the ALE space induced by the embedding of the ALE end is always injective. Using these properties, several classifications of such ALE manifolds and orbifolds are given in low dimensions. As an application to the moduli space, we prove that the football orbifold $\mathbb{S}^4/\Gamma$ cannot be realized as the Gromov-Hausdorff limit. In addition, we prove the positive mass theorem of these ALE ends and give a simple proof for the optimal decay rate. Using the positive mass theorem, we can solve the orbifold Yamabe problem in the locally conformally flat case. Also, we prove that a sequence of constant scalar curvature (CSC) metrics which is collapsing with bounded curvature to a manifold (X, g) can be perturbed to a sequence of $\mathcal{N}$ -invariant collapsing CSC metrics, under a natural assumption involving the eigenvalues of the drift Laplacian on (X, g) . This answers a special case of a question of Cheeger-Fukaya-Gromov. We also give some natural conditions on the limiting metric-measure space under which the eigenvalue assumption is automatically satisfied.

Notations

This section collects the main notation used throughout the thesis.

(M, g) A smooth Riemannian manifold with metric g .

n The dimension of M .

dV_g The Riemannian volume element induced by g .

∇ The Levi-Civita connection of g .

Δ_g The Laplace-Beltrami operator with respect to g .

Ric_g The Ricci curvature tensor of g .

R_g The scalar curvature of g .

Rm_g The full Riemann curvature tensor of g .

$|\text{Rm}_g|$ The pointwise norm of Rm_g .

$\text{dist}_g(x, y)$ The geodesic distance between $x, y \in M$.

$B_r^g(x)$ The geodesic ball centered at x with radius r .

$\text{inj}_g(x)$ The injectivity radius at x .

$[g]$ The conformal class of the metric g .

$Y(M, [g])$ The Yamabe constant of the conformal class $[g]$.

$\mathcal{M}$ A moduli space of metrics (context-dependent).

LCF Locally conformally flat.

CSC Constant scalar curvature.

ALE Asymptotically locally Euclidean.

$C^{k,\alpha}$ Holder spaces with k derivatives and exponent $\alpha \in (0, 1)$.

$W^{k,p}$ Sobolev spaces with k weak derivatives in L^p .

ε A small positive constant used in perturbative arguments.

ω_n The volume of the unit ball in $\mathbf{R}^n$.

Chapter 1

Introduction

This chapter collects the background material required for the rest of the thesis. We do not prove any new results here. §1.1 reviews the basic language of Riemannian geometry, orbifolds, and ALE manifolds. §1.2 collects the elliptic estimates (Schauder, Calderón-Zygmund, Sobolev embedding) and the asymptotic expansion of harmonic functions that are used in later chapters. §1.3 review the fundamental theory of nilpotent group and the actions. §1.4–§1.6 develop the three convergence frameworks central to the thesis: Gromov-Hausdorff convergence, Cheeger-Gromov convergence for non-collapsing sequences, and the Cheeger-Fukaya-Gromov collapsing theory.

1.1 Background in Riemannian Geometry

This thesis studies Riemannian metrics with special curvature properties on compact manifolds, asymptotically locally Euclidean (ALE) manifolds, and orbifolds. We collect in this section the foundational definitions and fix notation.

Definition 1.1 (Riemannian manifold). *A **Riemannian manifold** (M^n, g) is a smooth n -dimensional manifold M equipped with a smooth, symmetric, positive-definite $(0, 2)$ -tensor*

field g called the **Riemannian metric**. In local coordinates $(x^1, \dots, x^n)$, we write

$$g = g_{ij} dx^i \otimes dx^j.$$

The metric g determines the following canonical structures:

- The **Levi-Civita connection** ∇ : the unique torsion-free connection satisfying $\nabla g = 0$.
- The **Riemann curvature tensor** Rm , with components R_{ijkl} , defined by $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$.
- The **Ricci tensor** $\text{Ric}_{ij} = R_{ikj}{}^k$ (trace of Rm).
- The **scalar curvature** $R_g = g^{ij} \text{Ric}_{ij}$ (trace of Ric).
- The **volume form** $dV_g = \sqrt{\det(g_{ij})} dx^1 \wedge \dots \wedge dx^n$.

A Riemannian manifold is **complete** if every geodesic extends to all of $\mathbb{R}$, equivalently, if (M, d_g) is complete as a metric space where d_g is the Riemannian distance.

Standard references for Riemannian geometry are do Carmo [23] and Petersen [50].

1.1.1 Group action and Quotients

In order to introduce symmetry and quotient of a manifold, we need to introduce group actions.

Definition 1.2 (Group action). *Let (M, g) be a Riemannian manifold and Γ a group. A **smooth action** $\Gamma \curvearrowright M$ is a homomorphism $\rho : \Gamma \rightarrow \text{Diff}(M)$. A **smooth isometric action** $\Gamma \curvearrowright M$ is a homomorphism $\rho : \Gamma \rightarrow \text{Isom}(M, g)$. The action is:*

- **properly discontinuous:** for every compact $K \subset M$, the set $\{\gamma \in \Gamma : \gamma(K) \cap K \neq \emptyset\}$ is finite;
- **free:** no $\gamma \neq \text{id}$ has a fixed point.

A free, properly discontinuous, isometric action yields a smooth Riemannian manifold M/Γ .

Now, we talk about several examples of constructing quotient manifolds.

Example 1.1 (Finite Group). Let $\Gamma \leq O(n+1)$ be a finite group acting freely and isometrically on the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Then the quotient

$$\mathbb{S}^n/\Gamma$$

is a compact Riemannian orbifold, and it is in fact a smooth manifold precisely when the action is free. In dimension three, a standard example is a lens space $L(p, q) = \mathbb{S}^3/\mathbb{Z}_p$, where $\mathbb{Z}_p$ acts by a finite subgroup of unitary rotations. Such quotients are called **spherical space forms**.

Example 1.2 (Discrete Group). Let $\Gamma = \mathbb{Z}^n$ act on $\mathbb{R}^n$ by translations:

$$k \cdot x = x + k, \quad k \in \mathbb{Z}^n, \quad x \in \mathbb{R}^n.$$

This action is free, properly discontinuous, and cocompact, and the quotient is the flat torus

$$\mathbb{R}^n/\mathbb{Z}^n \cong \mathbb{T}^n.$$

More generally, any lattice $\Gamma \leq \mathbb{R}^n$ gives a compact flat manifold $\mathbb{R}^n/\Gamma$. This is the basic model of a quotient by a discrete cocompact subgroup.

Example 1.3 (Continuous Group). Let S^1 act on the flat two-torus $\mathbb{T}^2 = S^1 \times S^1$ by rotation

in the first factor:

$$e^{i\theta} \cdot (z_1, z_2) = (e^{i\theta} z_1, z_2),$$

for $e^{i\theta} \in S^1$ and $(z_1, z_2) \in S^1 \times S^1$. This action is smooth, continuous, and free, so the orbit space is again a circle,

$$\mathbb{T}^2/S^1 \cong S^1.$$

This is the simplest example of a continuous group action on a compact torus. A similar phenomenon occurs for the standard S^1 -action on $S^3 \subset \mathbb{C}^2$, whose quotient is $\mathbb{S}^2$ via the Hopf fibration. This example shows that allowing a continuous group changes the geometry of the quotient fundamentally.

Example 1.4 (Infranilmanifold). Let H_3 denote the three-dimensional Heisenberg group, viewed as the simply-connected nilpotent Lie group of upper triangular matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R}.$$

The subgroup $H_3(\mathbb{Z})$ consisting of matrices with $x, y, z \in \mathbb{Z}$ is a cocompact lattice, and the quotient

$$H_3/H_3(\mathbb{Z})$$

is a compact nilmanifold. More generally, if $\mathcal{N}$ is a simply-connected nilpotent Lie group and $\Gamma \leq \mathcal{N}_L \rtimes \text{Aut}(\mathcal{N})$ is a discrete cocompact subgroup, then $\Gamma \backslash \mathcal{N}$ is called an **infranilmanifold**. In the collapsing theory of Chapter 4, these spaces appear as the typical fibers of

Cheeger–Fukaya–Gromov fibrations (see §1.6).

A properly discontinuous (but not necessarily free) isometric action yields a Riemannian orbifold M/Γ . Now, we can introduce this slightly generalization of manifolds.

Definition 1.3 (Riemannian orbifold). *A topological space M^n is an **orbifold** if every point $x \in M$ has a neighborhood U , an open set $\tilde{U} \subseteq \mathbb{R}^n$, a finite group G_x acting smoothly on $\tilde{U}$, and a G_x -equivariant homeomorphism $\tilde{U} \rightarrow U$ inducing a homeomorphism $U \cong \tilde{U}/G_x$.*

*A point x is **singular** (an **orbifold point**) if $G_x \neq \{1\}$; otherwise it is **regular**. A **Riemannian orbifold** (M, g) is an orbifold with a G_x -invariant Riemannian metric on each uniformizing chart, compatible on overlaps.*

In this thesis we consider orbifolds with **isolated singularities**: at each singular point x_i , the local group $G_i \leq O(n)$ acts on $\mathbb{R}^n$ with only the origin fixed, so a punctured neighborhood of x_i is diffeomorphic to $(\mathbb{R}^n \setminus \{0\})/G_i$. The link of the singularity is $\mathbb{S}^{n-1}/G_i$.

A prototypical example is $\mathbb{R}^n/\Gamma$ for a finite group $\Gamma \leq O(n)$: the origin is the unique singular point with isotropy group Γ , and the link $\mathbb{S}^{n-1}/\Gamma$ is a spherical space form. The football orbifold $\mathbb{S}^2/\mathbb{Z}_k$ (a sphere with two cone points of angle $2\pi/k$) is the simplest two-dimensional example (see §2.1).

Definition 1.4 (Orbifold fundamental group). *Let $\mathcal{O}$ be a connected good orbifold, written as a quotient*

$$\mathcal{O} = \tilde{M}/\Gamma,$$

where $\tilde{M}$ is simply connected and Γ acts properly discontinuously by orbifold isometries. The

orbifold fundamental group of $\mathcal{O}$ is the deck transformation group

$$\pi_1^{\text{orb}}(\mathcal{O}) := \Gamma.$$

Equivalently, it is the group of homotopy classes of orbifold loops, where loops are allowed to carry local isotropy data at singular points.

Example 1.5 (Football orbifold). Let $\Gamma \leq O(n) \subset O(n+1)$ be finite, and let it act on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ through the first n coordinates, fixing the north and south poles. Assume these are the only fixed points. Then

$$\mathcal{F} := \mathbb{S}^n / \Gamma$$

is a football orbifold with exactly two orbifold points.

Since $\mathbb{S}^n$ is the orbifold universal cover (for $n \geq 2$), the deck group is Γ , hence

$$\pi_1^{\text{orb}}(\mathcal{F}) \cong \Gamma.$$

On the other hand, the underlying topological quotient $|\mathcal{F}|$ is homeomorphic to $\mathbb{S}^n$ (the action only changes the two cone neighborhoods), so

$$\pi_1(|\mathcal{F}|) = 0 \quad (n \geq 2).$$

Thus the orbifold fundamental group detects isotropy information that the ordinary topological fundamental group does not see.

There is a very special way of constructing the orbifolds, where the orbifolds can be raised as a quotient of manifolds.

Definition 1.5 (Good orbifold). *An orbifold M is a **good orbifold** (or **developable orbifold**) if $M \cong \widetilde{M}/\Gamma$ for some smooth Riemannian manifold $\widetilde{M}$ and some group Γ acting properly discontinuously and isometrically on $\widetilde{M}$. An orbifold that does not arise this way is called **bad**.*

Remark 1.1. *A concrete model of the **teardrop orbifold** is the weighted projective line $\mathbb{CP}_{(1,k)}^1$, equivalently the 2-sphere with one cone point of order k . Topologically, its underlying space is S^2 ; a neighborhood of the singular point is modeled on $D^2/\mathbb{Z}_k$, where $\mathbb{Z}_k$ acts by rotations $z \mapsto e^{2\pi i/k}z$, and the rest of the orbifold is smooth. This orbifold is not developable, so it cannot be covered by a manifold. The teardrop is 2-dimensional, locally conformally flat, and admits a metric of positive curvature.*

Remark 1.2. *In this thesis, we have proved the following theorem: all compact locally conformally flat (LCF) orbifolds with isolated singularities and positive scalar curvature with dimension $n \geq 3$ are good orbifolds; see Theorem 3.13 in §3.3.*

1.1.2 ALE manifolds

Definition 1.6 (ALE manifold and ALE orbifold). *Let $\Gamma \leq O(n)$ be a finite subgroup. A complete Riemannian manifold (M^n, g) is **asymptotically locally Euclidean (ALE)** of **order** $\tau > 0$ if there exists a compact set $K \subset M$ and a diffeomorphism*

$$\Phi: M \setminus K \longrightarrow (\mathbb{R}^n \setminus B(0, R))/\Gamma$$

for some $R > 0$, such that in coordinates induced by Φ :

$$\begin{aligned} (\Phi_*g)_{ij}(x) &= \delta_{ij} + O(|x|^{-\tau}), \\ \partial^k(\Phi_*g)_{ij}(x) &= O(|x|^{-\tau-k}) \end{aligned}$$

for all $k \geq 1$ as $|x| \rightarrow \infty$. The group Γ is the **group at infinity** of the end. When $\Gamma = \{1\}$, we call (M, g) **asymptotically flat (AF)**.

A Riemannian orbifold (M, g) is an **ALE orbifold** if the same condition holds and M may have isolated orbifold singularities in its interior.

The model ALE end is $\mathbb{R}^n \setminus B(0, R)$ quotiented by Γ ; the trivial ALE manifold (with boundary) is $\mathbb{R}^n / \Gamma$ (minus a small ball). This definition, due to Bartnik [11], is the setting for the Positive Mass Theorem (§2.3) and the ALE orbifold moduli theory of Chapter 3.

Example 1.6. *The Eguchi–Hanson metric is the standard Ricci-flat ALE metric on the total space of the complex line bundle $\mathcal{O}_{\mathbb{CP}^1}(-2) \cong T^*\mathbb{S}^2$, which resolves the quotient singularity $\mathbb{C}^2 / \{\pm 1\}$. In coordinates (r, θ, ϕ, ψ) with $r \geq a > 0$, it can be written as*

$$g_{EH} = \left(1 - \frac{a^4}{r^4}\right)^{-1} dr^2 + \frac{r^2}{4} \left(1 - \frac{a^4}{r^4}\right) (d\psi + \cos \theta d\phi)^2 + \frac{r^2}{4} (d\theta^2 + \sin^2 \theta d\phi^2).$$

As $r \rightarrow \infty$, this metric approaches the flat metric on $\mathbb{R}^4 / \mathbb{Z}_2$, so its group at infinity is $\mathbb{Z}_2$.

Example 1.7. *The Schwarzschild metric is the standard scalar-flat AF metric on $\mathbb{R}^n \setminus \{0\}$. In polar coordinates (r, ω) on $\mathbb{R}^n \setminus \{0\}$, it is given by*

$$g_{Sch} = \left(1 + \frac{m}{2r^{n-2}}\right)^{\frac{4}{n-2}} (dr^2 + r^2 g_{\mathbb{S}^{n-1}}),$$

where $m > 0$ is the mass parameter and $g_{\mathbb{S}^{n-1}}$ is the round metric on the unit sphere. This metric is complete, has vanishing scalar curvature away from the origin, and is asymptotic to the Euclidean metric, so its group at infinity is trivial. It is the basic AF model in the study of the Positive Mass Theorem and appears as the simplest nontrivial scalar-flat exterior solution.

1.1.3 Conformal change of scalar curvature

In this thesis, we will talk about a special way of changing the metric while preserving certain geometric properties. We are particularly interested in the behavior of scalar curvature under conformal changes of metric, which is governed by the Yamabe equation. This is a nonlinear elliptic PDE that plays a central role in our analysis.

Definition 1.7 (Conformal change of scalar curvature). *Let (M^n, g) be a Riemannian manifold with $n \geq 3$. Under a conformal change $\tilde{g} = u^{4/(n-2)}g$ with $u > 0$ smooth, the scalar curvatures satisfy:*

$$R_{\tilde{g}} = u^{-\frac{n+2}{n-2}} \left(-\frac{4(n-1)}{n-2} \Delta_g u + R_g u \right). \quad (1.1)$$

The operator

$$L_g := -\frac{4(n-1)}{n-2} \Delta_g + R_g$$

*is the **conformal Laplacian** (or **Yamabe operator**). The equation $R_{\tilde{g}} = \lambda$ (constant scalar curvature) is equivalent to*

$$L_g u = \lambda u^{\frac{n+2}{n-2}}, \quad u > 0, \quad (1.2)$$

*known as the **Yamabe equation**. The conformal Laplacian satisfies the covariance identity $L_{\tilde{g}} v = \varphi^{-(n+2)/(n-2)} L_g(\varphi v)$ for $\tilde{g} = \varphi^{4/(n-2)}g$.*

Equation (1.1) is a standard computation via Christoffel symbols; see Lee-Parker [45] for details. This formula is the starting point for the Yamabe problem (§2.2) and for the Green's function method in the Positive Mass Theorem (§2.3).

The geometric objects defined in this section — curvature tensors, orbifold charts, ALE

asymptotics, and the conformal Laplacian — are the stage on which all subsequent analysis takes place. To extract information from them, we need the language of function spaces and elliptic estimates, which we collect in the next section.

1.2 Background in PDE

The geometric equations of this thesis — the Yamabe equation, the scalar curvature equation under conformal deformation, and the estimates for collapsing fibrations — are all second-order elliptic PDE. To state and prove these results with appropriate precision, we need a common framework of function spaces and elliptic estimates. We collect the relevant tools in this section; all manifolds are smooth and n -dimensional.

Definition 1.8 ($C^{k,\alpha}$ and $W^{k,p}$ spaces). *Fix a compact Riemannian manifold (M, g) with a finite atlas $\{(U_\ell, \phi_\ell)\}$ and a subordinate partition of unity $\{\rho_\ell\}$. For a function $u: M \rightarrow \mathbb{R}$, set $u_\ell = (\rho_\ell u) \circ \phi_\ell^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}$.*

1. **Hölder spaces:** for $k \geq 0$ and $\alpha \in (0, 1)$, the norm

$$\|u\|_{C^{k,\alpha}(M)} = \sum_{\ell} \|u_\ell\|_{C^{k,\alpha}(\mathbb{R}^n)}$$

defines the Banach space $C^{k,\alpha}(M)$, where $\|v\|_{C^{k,\alpha}(\mathbb{R}^n)}$ is the standard Hölder norm (supremum of v and its derivatives up to order k , plus the α -Hölder seminorm of the k -th order derivatives).

2. **Sobolev spaces:** for $k \in \mathbb{N}$ and $p \geq 1$, the norm

$$\|u\|_{W^{k,p}(M)} = \sum_{\ell} \|u_\ell\|_{W^{k,p}(\mathbb{R}^n)}$$

defines the Banach space $W^{k,p}(M)$, where $W^{k,p}(\mathbb{R}^n)$ is the standard Sobolev norm (sum

of L^p norms of weak derivatives up to order k).

These definitions are independent of the choice of atlas and partition of unity up to equivalent norms.

Theorem 1.1 (Sobolev embedding). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $k \geq 1$, $p \geq 1$.*

1. **Subcritical** ($kp < n$): $W^{k,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ continuously, where $p^* = np/(n - kp)$.
2. **Supercritical** ($kp > n$): $W^{k,p}(\Omega) \hookrightarrow C^{j,\beta}(\bar{\Omega})$ continuously for all $j + \beta < k - n/p$ with $j \in \mathbb{N}_0$, $\beta \in (0, 1)$.
3. **Critical** ($kp = n$): $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leq q < \infty$.

The embeddings in (1) and (2) are **compact** (Rellich-Kondrachov) when the target exponent is strictly below the critical threshold. The same embeddings hold for $W^{k,p}$ sections of vector bundles over compact Riemannian manifolds.

See Gilbarg-Trudinger [30] §7 or Evans [24] §5.6 for proofs.

Theorem 1.2 (Schauder estimates). *Let $L = a^{ij}(x)\partial_i\partial_j + b^i(x)\partial_i + c(x)$ be a second-order uniformly elliptic operator on $\Omega \subset \mathbb{R}^n$ with coefficients in $C^{k,\alpha}(\Omega)$, $k \geq 0$, $\alpha \in (0, 1)$. If $Lu = f$ with $f \in C^{k,\alpha}(\Omega)$, then for any $\Omega' \Subset \Omega$:*

$$\|u\|_{C^{k+2,\alpha}(\Omega')} \leq C(\|f\|_{C^{k,\alpha}(\Omega)} + \|u\|_{L^\infty(\Omega)}), \quad (1.3)$$

where C depends on n , k , α , the ellipticity constants, the $C^{k,\alpha}$ norms of the coefficients, and $\text{dist}(\Omega', \partial\Omega)$.

Under $C^{k+2,\alpha}$ boundary conditions on a $C^{k+2,\alpha}$ domain, the global estimate $\|u\|_{C^{k+2,\alpha}(\bar{\Omega})} \leq C(\|f\|_{C^{k,\alpha}} + \|u\|_{L^\infty})$ holds. The same estimate holds on compact Riemannian manifolds with

$C^{k,\alpha}$ metrics.

See Gilbarg-Trudinger [30] §6.1–6.2.

Theorem 1.3 (Calderón-Zygmund $W^{2,p}$ estimate). *Let $\Delta u = f \in L^p(\Omega)$ for $1 < p < \infty$ and $\Omega' \Subset \Omega \subset \mathbb{R}^n$. Then $u \in W_{\text{loc}}^{2,p}(\Omega)$ and*

$$\|u\|_{W^{2,p}(\Omega')} \leq C(n,p) (\|f\|_{L^p(\Omega)} + r^{-2} \|u\|_{L^p(\Omega)}), \quad (1.4)$$

where $r = \text{dist}(\Omega', \partial\Omega)$. More generally, for a uniformly elliptic operator L with VMO leading coefficients, if $Lu = f \in W^{k,p}$ then $u \in W^{k+2,p}$ with the analogous estimate.

See Gilbarg-Trudinger [30] §9.

As an example of geometric structure effecting the analysis result, we put the following harmonic function expansion result on $\mathbb{R}^n$ here. Latter, we will apply this result to geneal ALE manifolds.

Lemma 1.4 (Asymptotic expansion of harmonic functions on $\mathbb{R}^n$). *Let $n \geq 3$ and let u be harmonic on $\mathbb{R}^n \setminus B_1(0)$ with $u(x) = O(|x|^{2-n+\varepsilon})$ as $|x| \rightarrow \infty$ for some $\varepsilon > 0$. Then:*

$$u(x) = a|x|^{2-n} + b + O(|x|^{-(n-1)}) \quad (1.5)$$

as $|x| \rightarrow \infty$, for constants $a, b \in \mathbb{R}$.

Proof. Write u in polar coordinates $(r, \omega) \in (1, \infty) \times \mathbb{S}^{n-1}$ and expand $u(r, \cdot)$ in $L^2(\mathbb{S}^{n-1})$:

$$u(r, \omega) = \sum_{\ell=0}^{\infty} f_{\ell}(r) Y_{\ell}(\omega),$$

where Y_{ℓ} is a spherical harmonic of degree ℓ , satisfying $\Delta_{\mathbb{S}^{n-1}} Y_{\ell} = -\ell(\ell + n - 2)Y_{\ell}$.

Since $\Delta u = 0$, each radial coefficient f_ℓ satisfies the Euler ODE

$$f_\ell'' + \frac{n-1}{r} f_\ell' - \frac{\ell(\ell+n-2)}{r^2} f_\ell = 0.$$

The general solution is

$$f_\ell(r) = A_\ell r^{\alpha_\ell^+} + B_\ell r^{\alpha_\ell^-}$$

with exponents $\alpha_\ell^\pm = [-(n-2) \pm \sqrt{(n-2)^2 + 4\ell(\ell+n-2)}]/2$. For $\ell = 0$: $\alpha_0^+ = 0$, $\alpha_0^- = 2-n$, giving $f_0(r) = b + ar^{2-n}$. For $\ell \geq 1$: $\alpha_\ell^+ = \ell > 0$ and $\alpha_\ell^- = -(n-2+\ell) < 0$.

The hypothesis $u = O(|x|^{2-n+\varepsilon})$ forces $A_\ell = 0$ for all $\ell \geq 1$, since r^ℓ grows. Thus $f_\ell(r) = B_\ell r^{-(n-2+\ell)}$ for $\ell \geq 1$, giving

$$u(r, \omega) = b + ar^{2-n} + \sum_{\ell=1}^{\infty} B_\ell r^{-(n-2+\ell)} Y_\ell(\omega).$$

The first correction decays like $r^{-(n-1)}$, establishing (1.5). □

Remark 1.3. *The constant a is proportional to the capacity (or ADM mass) of the end; b is the asymptotic value. On an ALE manifold (M, g) with $g = g_{\mathbb{R}^n} + O(r^{-\tau})$, the same expansion holds for harmonic functions on the ALE end, and is used in §3.2 and §3.3.*

See Axler-Bourdon-Ramey [8] Ch. 5 for a more general treatment.

With these PDE tools in hand, we can now discuss how sequences of Riemannian manifolds converge to limit spaces. The interplay between curvature bounds and compactness — the subject of the next two sections — is what makes the moduli-space analysis in this thesis possible.

1.3 Nilpotent group and actions

We now make some notes on the nilpotent Lie group and their actions.

Definition 1.9 (Nilpotent group). A **nilpotent Lie group** is a Lie group, $\mathcal{N}$, which is simply connected and whose Lie algebra is $\mathfrak{g}$ is a nilpotent Lie algebra, that is

$$\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}_1], \dots \quad (1.6)$$

eventually vanishes, i.e., $\mathfrak{g}_k = 0$, for some k , finite.

Remark 1.4. A connected simply connected nilpotent Lie group $\mathcal{N}$ is diffeomorphic to $\mathbb{R}^n$, given by the exponential map. This can be seen from the Baker-Campbell-Hausdorff formula since it terminates in finitely many steps.

Definition 1.10 (Automorphism and Left action). The **automorphism group** of a nilpotent Lie group, $\text{Aut}(\mathcal{N})$ is the group of Lie group automorphisms of $\mathcal{N}$:

$$\text{Aut}(\mathcal{N}) = \{\phi : \mathcal{N} \rightarrow \mathcal{N} : \phi \text{ is a smooth group isomorphism}\}. \quad (1.7)$$

Let $h \in \mathcal{N}$, then we define the left action:

$$L_h : \mathcal{N} \rightarrow \mathcal{N}, \quad x \mapsto hx. \quad (1.8)$$

Then, we denote the group of **left translation**:

$$\mathcal{N}_L := \{L_h : h \in \mathcal{N}\}. \quad (1.9)$$

It can be easily seen that there is a natural isomorphism $\mathcal{N}_L \cong \mathcal{N}$. Similarly, we can define

the **right translation** group $\mathcal{N}_R \cong \mathcal{N}$.

Definition 1.11 (Left and right invariant vector fields). *For $V \in \mathfrak{n}$, the corresponding **left-invariant vector field** is*

$$V_g^L = (L_g)_* V, \quad (1.10)$$

*and the corresponding **right-invariant vector field** is*

$$V_g^R = (R_g)_* V, \quad (1.11)$$

where

$$L_g(h) = gh, \quad R_g(h) = hg. \quad (1.12)$$

It can be shown that left action is generated by a right invariant vector field

$$\tilde{V}_g = \left. \frac{d}{dt} \right|_{t=0} \exp(tV)g = (R_g)_* V = V_g^R. \quad (1.13)$$

There is a natural flat affine connection on $\mathcal{N}$. We now define it.

Definition 1.12 (Affine connection and affine structure). *The **canonical affine connection**, ∇^{can} , is the connection for which the left-invariant vector fields are parallel, i.e.,*

$$\nabla^{\text{can}} V^L = 0 \quad (1.14)$$

for every left-invariant vector field V^L .

Note that, ∇^{can} is flat, and, in general, not torsion free, in fact:

$$T^{\nabla^{can}}(X, Y) = -[X, Y]. \quad (1.15)$$

Since the left-invariant vector fields are parallel, so is $T^{\nabla^{can}}$.

Note also that, when ∇^{can} acts on right invariant vector field, we have:

$$\nabla_{W^R}^{can} V^R = [V, W]^R, \quad (1.16)$$

which is a constant right invariant vector field.

Definition 1.13. An **affine structure** on a smooth manifold is given by a choice of a flat, affine connection ∇ on TM with parallel torsion.

Two affine manifolds, (M, ∇^M) , (N, ∇^N) are **affine equivalent**, if there is a diffeomorphism $\phi : M \rightarrow N$, so that $\phi^* \nabla^N = \nabla^M$.

The **affine group**, $\text{Aff}(M, \nabla^M)$ (or just $\text{Aff}(M)$), of (M, ∇^M) , is defined to be the affine isomorphism group of (M, ∇^M) .

Note that $\text{Aff}(\mathcal{N})$ of a nilpotent Lie group $\mathcal{N}$ is equivalent to the group that preserves the left invariant vector fields. Thus,

$$\text{Aff}(\mathcal{N}) = \mathcal{N}_L \rtimes \text{Aut}(\mathcal{N}). \quad (1.17)$$

Now, we define the nilmanifolds and infranilmanifolds.

Definition 1.14 (nilmanifold and infranilmanifold). A compact **nilmanifold** is a quotient

$$M = \Lambda \backslash \mathcal{N}, \quad (1.18)$$

where $\Lambda \subset \mathcal{N}_L$ is a discrete cocompact subgroup.

A compact **infrnilmanifold** is a quotient

$$Z = \Gamma \backslash \mathcal{N}, \quad (1.19)$$

where Γ is a torsion-free discrete cocompact subgroup of $\mathcal{N}_L \rtimes \text{Aut}(\mathcal{N})$.

It can be shown that $\Lambda = \Gamma \cap \mathcal{N}_L$, the pure translation subgroup, is a normal subgroup of Γ with finite index. Thus, an infrnilmanifold is finitely covered by a nilmanifold. Furthermore, the canonical affine connection descends to an affine connection on the quotient, so that the group $\text{Aff}(\Gamma \backslash \mathcal{N})$ is defined.

Finally, we will talk about the structure of affine group of an infrnilmanifold, $\text{Aff}(\Gamma \backslash \mathcal{N})$, and its identity component $\text{Aff}^0(\Gamma \backslash \mathcal{N})$.

Proposition 1.5. *Let $\Gamma \subseteq \mathcal{N}_L \rtimes \text{Aut}(\mathcal{N})$ be a cocompact lattice group, then we have*

$$\text{Aff}(\Gamma \backslash \mathcal{N}) \cong N_{\text{Aff}(\mathcal{N})}(\Gamma)/\Gamma \quad (1.20)$$

and

$$\text{Aff}^0(\Gamma \backslash \mathcal{N}) \cong C_{\text{Aff}(\mathcal{N})}(\Gamma)/\Gamma, \quad (1.21)$$

where $N_{\text{Aff}(\mathcal{N})}(\Gamma)$ and $C_{\text{Aff}(\mathcal{N})}(\Gamma)$ are normalizer and centralizer of Γ in $\text{Aff}(\mathcal{N})$, respectively.

Proof. For every element in $\text{Aff}(\Gamma \backslash \mathcal{N})$ can be lifted to $\text{Aff}(\mathcal{N})$ that makes every Γ -orbit to a Γ -orbit. Thus, the lift of $\text{Aff}(\Gamma \backslash \mathcal{N})$ can be identified with $N_{\text{Aff}(\mathcal{N})}(\Gamma)$. Thus, we have $\text{Aff}(\Gamma \backslash \mathcal{N}) \cong N_{\text{Aff}(\mathcal{N})}(\Gamma)/\Gamma$ after the quotient.

Note that since Γ is a discrete subgroup, and the identity map is in the centralizer. Thus,

we have $\text{Aff}^0(\Gamma \backslash \mathcal{N}) \cong C_{\text{Aff}(\mathcal{N})}(\Gamma)/\Gamma$. □

Remark 1.5. *It is natural to identify the centralizer of the left action group $\mathcal{N}_L$ as the right action subgroup $\mathcal{N}_R$. Indeed, for any $g \in \mathcal{N}_L$ and $\phi \in C_{\text{Aff}(\mathcal{N})}(\mathcal{N}_L)$, if $\phi(g(x)) = g\phi(x)$, for any $x \in \mathcal{N}$, then if we choose $x = e$, let $\phi(e) = h$, then $\phi(g) = gh$. Thus, we may view $C_{\text{Aff}(\mathcal{N})}(\Gamma)$ as a subgroup of $\mathcal{N}_R$, and acts trivially on the right-invariant vector fields.*

1.4 Gromov-Hausdorff Convergence

A fundamental tool in the study of moduli spaces of Riemannian metrics is convergence in the Gromov-Hausdorff sense. This notion measures how close two metric spaces are, allowing sequences of Riemannian manifolds to converge to a limit space that may not itself be a manifold.

Definition 1.15 (Gromov-Hausdorff distance and convergence). *The **Hausdorff distance** between two subsets A, B of a metric space (Z, d) is*

$$d_H(A, B) = \inf\{r > 0 : A \subseteq B_r \text{ and } B \subseteq A_r\},$$

where $A_r = \{z \in Z : d(z, A) < r\}$.

The **Gromov-Hausdorff (GH) distance** between compact metric spaces (X, d_X) and (Y, d_Y) is

$$d_{GH}(X, Y) = \inf\{d_H^Z(f(X), g(Y))\},$$

where the infimum is over all metric spaces Z and isometric embeddings $f: X \hookrightarrow Z$, $g: Y \hookrightarrow Z$. A sequence (X_i, d_i) **converges in the GH sense** to (X_∞, d_∞) if $d_{GH}(X_i, X_\infty) \rightarrow 0$.

An ε -**GH approximation** is a map $f: X \rightarrow Y$ satisfying

(i) $|d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)| < \varepsilon$ for all $x_1, x_2 \in X$, and

(ii) $f(X)$ is ε -dense in Y .

Example 1.8. *A guiding example: the flat tori $(\mathbb{T}^2, \varepsilon^2 d\theta_1^2 \oplus d\theta_2^2)$, as $\varepsilon \rightarrow 0$, GH-converge to a circle $\mathbb{S}^1$: the torus collapses by shrinking one $\mathbb{S}^1$ factor to a point. More precisely, write $\mathbb{T}^2 = \mathbb{S}_\varepsilon^1 \times \mathbb{S}_1^1$ where the first factor has length ε ; a ε -GH approximation is the projection $(\theta_1, \theta_2) \mapsto \theta_2$, since any two points over the same θ_2 are within distance ε of each other. As $\varepsilon \rightarrow 0$ this becomes an isometry in the limit. The limit $\mathbb{S}^1$ carries no information about the collapsed $\mathbb{S}^1$ factor. This illustrates a key phenomenon of collapsing geometry studied in §1.6.*

Example 1.9. *An anisotropic collapsing model is*

$$(\mathbb{T}^2, g_\varepsilon), \quad g_\varepsilon = \varepsilon^2 d\theta_1^2 + \varepsilon^4 d\theta_2^2,$$

with $\theta_1, \theta_2 \in \mathbb{R}/2\pi\mathbb{Z}$. Since both circle factors shrink to length $O(\varepsilon)$ and $O(\varepsilon^2)$, respectively, we have

$$\text{diam}(\mathbb{T}^2, g_\varepsilon) \rightarrow 0,$$

so $(\mathbb{T}^2, g_\varepsilon)$ GH-converges to a single point.

The same phenomenon occurs for nilmanifolds (Example 1.4): if $\Gamma \backslash \mathcal{N}$ is a compact nilmanifold with a fixed left-invariant metric g , then

$$(\Gamma \backslash \mathcal{N}, \varepsilon^2 g) \xrightarrow[\varepsilon \rightarrow 0]{GH} \{\text{pt}\}.$$

This is the basic “collapse to a point” model used repeatedly in collapsing theory.

Example 1.10. *A standard smooth-to-singular example is convergence to a metric cone.*

Fix $\alpha \in (0, 2\pi)$, and let

$$C_\alpha = ([0, \infty) \times S^1) / \sim,$$

where all points $(0, \theta)$ are identified, with cone metric

$$g_{\text{cone}} = dr^2 + \left(\frac{\alpha}{2\pi}\right)^2 r^2 d\theta^2.$$

The cone tip is singular (not a smooth manifold point). Now choose a sequence of smooth surfaces of revolution (M_i, g_i) that agree with g_{cone} outside $r \leq 1/i$ and smooth out the tip inside $r \leq 1/i$. Then

$$(M_i, g_i) \xrightarrow[i \rightarrow \infty]{GH} (C_\alpha, g_{\text{cone}}).$$

Thus a sequence of smooth manifolds can GH-converge to a space with a genuine singular point.

Although, there is not a priori regularity guaranteed by Gromov-Hausdorff convergence, there is a compactness theorem, which is closely related to the Riemannian structure, thanks to Gromov.

Theorem 1.6 (Gromov pre-compactness, Ricci bounded below). *Fix $n \geq 2$, $\kappa \in \mathbb{R}$, $D > 0$.*

The class of compact n -dimensional Riemannian manifolds satisfying

$$\text{Ric}(g) \geq -(n-1)\kappa \quad \text{and} \quad \text{diam}(M, g) \leq D$$

is pre-compact in the Gromov-Hausdorff topology: every sequence has a GH-convergent subsequence, and the limit is a compact metric space (X, d) .

Proof sketch. The key tool is the **Bishop-Gromov volume comparison**: for $\text{Ric} \geq -(n-$

$1)\kappa$, the ratio $\text{Vol}(B_r(x))/V_\kappa(r)$ is monotone non-increasing in r , where $V_\kappa(r)$ is the volume of an r -ball in the simply-connected space of constant curvature κ . This gives a uniform upper bound $N \leq N(n, \kappa, D, \varepsilon)$ on the cardinality of any ε -separated subset, i.e. a uniform covering number estimate. Pre-compactness in d_{GH} then follows from the completeness of the space of compact metric spaces under d_{GH} and the criterion that uniform total boundedness implies sequential compactness. $\square$

Remark 1.6. *No regularity of the limit is assumed: (X, d) need not be a manifold. In particular, collapsing sequences (injectivity radius $\rightarrow 0$) are permitted. The stronger assumption $|\text{Rm}| \leq C$ (used in the CFG theory of §1.6) additionally guarantees that X has integer Hausdorff dimension and admits a fibration structure.*

The moduli space manifolds with $|\text{Rm}| \leq C$ and diameter $\leq D$ satisfies the Ricci bound (since $|\text{Rm}| \leq C$ implies $|\text{Ric}| \leq (n-1)C$), so its GH closure is compact by Theorem 1.6. For LCF moduli space, an even weaker Riemannian curvature $L^{\frac{n}{2}}$ can guarantee the existence of GH limit. The Tian-Viaclovsky theorem (Chapter 3) further identifies the limit as an LCF orbifold.

Gromov-Hausdorff convergence is the weakest notion of convergence we use: it detects the correct limit metric space but provides no smooth structure. When injectivity radii are bounded below, the Cheeger-Gromov theory (§1.5) recovers a smooth limit manifold. When they collapse to zero, the finer CFG structure (§1.6) is needed.

1.5 Cheeger-Gromov Convergence

Gromov-Hausdorff convergence gives a limit metric space, but provides no smooth structure on the limit. When the sequence is *non-collapsing* (injectivity radius bounded below), one obtains a smooth limit manifold via Cheeger-Gromov convergence.

Definition 1.16 (Cheeger-Gromov convergence). *A sequence of pointed Riemannian manifolds $\{(M_i, g_i, p_i)\}$ converges in the pointed $C^{k,\alpha}$ Cheeger-Gromov sense to $(M_\infty, g_\infty, p_\infty)$ if there exist:*

- *an exhaustion $K_1 \subseteq K_2 \subseteq \dots$ of M_∞ by compact domains, and*
- *smooth embeddings $\Phi_i: K_j \hookrightarrow M_i$ (for $i \gg j$) with $\Phi_i(p_\infty) = p_i$,*

such that $\Phi_i^* g_i \rightarrow g_\infty$ in $C_{\text{loc}}^{k,\alpha}$ on M_∞ .

For a sequence of compact manifolds with global diffeomorphisms $\Phi_i: M_\infty \rightarrow M_i$, the convergence is $\Phi_i^* g_i \rightarrow g_\infty$ in $C^{k,\alpha}(M_\infty)$.

Theorem 1.7 (Cheeger's compactness theorem, $C^{k+1,\alpha}$ version). *Fix $n \geq 2$, $k \geq 0$, $\alpha \in (0, 1)$, constants $C_j > 0$ for $j = 0, \dots, k$, and $\iota_0 > 0$. Let $\{(M_i, g_i, p_i)\}$ be a sequence of pointed complete Riemannian n -manifolds satisfying*

$$|\nabla^j \text{Rm}(g_i)| \leq C_j \quad (j = 0, \dots, k), \quad \text{inj}(M_i, p_i) \geq \iota_0.$$

Then a subsequence converges in the pointed $C^{k+1,\alpha}$ Cheeger-Gromov sense to a complete pointed Riemannian manifold $(M_\infty, g_\infty, p_\infty)$.

Proof outline. Step 1 (Harmonic coordinates). By the injectivity radius bound $\iota_0 > 0$ and the curvature bound $|\text{Rm}| \leq C_0$, every $x \in M_i$ has a harmonic coordinate chart on $B_{r_0}(x)$ (for $r_0 = r_0(n, C_0, \iota_0) > 0$ uniform in i) in which the metric satisfies $\frac{1}{2}\delta_{ij} \leq g_{ij} \leq 2\delta_{ij}$ and $\|g_{ij}\|_{C^{k+1,\alpha}} \leq C$. The key point is that in harmonic coordinates, $\Delta_g g_{ij} = Q(g, \partial g)$ where the right side involves curvature; hence Schauder estimates (Theorem 1.2) convert bounds on $|\nabla^j \text{Rm}|$ into $C^{k+1,\alpha}$ bounds on g_{ij} .

Step 2 (Arzelà-Ascoli). In each harmonic chart, the metric components $g_{ij}^{(i)}$ form a sequence in $C^{k+1,\alpha}$ with uniform bounds. By the compact Rellich embedding $C^{k+1,\alpha} \hookrightarrow C^{k,\alpha'}$

(Theorem 1.1), a subsequence converges in $C^{k+1,\alpha'}$.

Step 3 (Diagonal subsequence). Applying this chart-by-chart and diagonalizing over an exhaustion of the limit manifold gives the global $C^{k+1,\alpha}$ convergence. $\square$

See Cheeger [16], Peters [49], and Greene-Wu for the foundational results.

Theorem 1.8 (Cheeger's compactness theorem, $W^{k+2,p}$ version). *Under the weaker hypothesis*

$$\|\nabla^j \text{Rm}(g_i)\|_{L^p} \leq C_j \quad (j = 0, \dots, k-1), \quad p > n/2, \quad \text{inj}(M_i, p_i) \geq \iota_0,$$

the same conclusion holds with $\Phi_i^* g_i \rightarrow g_\infty$ in $W_{\text{loc}}^{k+2,p}$.

The proof replaces Schauder estimates with Calderón-Zygmund estimates (Theorem 1.3) and Arzelà-Ascoli with Rellich-Kondrachov compactness (Theorem 1.1). See Anderson-Cheeger [5].

Remark 1.7. *The $W^{k+2,p}$ version is used in Chapter 4 (finite-regularity CFG theory), where the PDE estimates naturally produce L^p curvature bounds rather than pointwise C^k bounds. The $C^{k+1,\alpha}$ version is used throughout the non-collapsing compactification of Chapter 3, where pointwise curvature bounds are available.*

Definition 1.17 (Equivariant Cheeger-Gromov convergence). *Let $\{(M_i, g_i)\}$ be compact n -manifolds converging in GH to a compact d -manifold (X, g_∞) with $d \leq n$ (possibly collapsing). The convergence is **equivariant local** $C^{k,\alpha}$ if there is a uniform $r > 0$ such that for every sequence of points $x_i \in M_i \rightarrow x_\infty \in X$, the universal covers lift to equivariant convergence:*

$$(\tilde{B}_r(\tilde{x}_i), \tilde{g}_i, G_i) \xrightarrow{C^{k,\alpha}} (\tilde{B}_r(\tilde{x}_\infty), \tilde{g}_\infty, G_\infty),$$

where G_i, G_∞ are the deck transformation groups and the convergence is $C^{k,\alpha}$ Cheeger-

Gromov. In this setting, when $d < n$, then G_∞ is a nilpotent Lie group $\mathcal{N}$ acting by isometries, and the projection $\pi_\infty: \tilde{B}_r(\tilde{x}_\infty) \rightarrow B_r(x_\infty)$ is a Riemannian submersion.

This definition, due to Fukaya [26] and developed in CFG [19], makes precise the local structure of a collapsing sequence: upstairs on universal covers there is genuine non-collapsing $C^{k,\alpha}$ convergence, even though the original sequence collapses. The nilpotent group $\mathcal{N}$ that appears is the structure group of the CFG fibration constructed in §1.6.

The key limitation of Cheeger-Gromov theory is the *non-collapsing* hypothesis $\text{inj}(M_i, p_i) \geq \iota_0 > 0$: when injectivity radii collapse to zero, the harmonic coordinate construction breaks down entirely. The Cheeger-Fukaya-Gromov theory of §1.6 replaces the injectivity radius assumption with the fibration structure of the collapsing, enabling a parallel analysis in the collapsing regime.

1.6 Cheeger-Fukaya-Gromov Collapsing Theory

Let $\{(M_i^n, g_i)\}$ be a sequence of closed Riemannian manifolds with $|\text{Rm}(g_i)| \leq C_1$ and $\text{diam}(M_i, g_i) \leq C_2$, and assume $(M_i, g_i) \rightarrow (X^d, d_X)$ in the Gromov-Hausdorff sense with $d < n$. This is the collapsing regime where classical non-collapsed Cheeger-Gromov compactness is not directly applicable.

At an introductory level, the CFG mechanism used later in this thesis has three steps: (1) local non-collapsing after passing to local universal covers, (2) global fibration over the limit with almost-flat fibers, and (3) nilpotent algebraic structure and a canonical $\mathcal{N}$ -action.

We first start with a characterization of the limit when curvature is bounded.

Theorem 1.9 (GH compactness with bounded curvature). *Let $\{(M_i^n, g_i)\}$ be a sequence of closed n -manifolds with $|\text{Rm}(g_i)| \leq C_1$ and $\text{diam}(M_i, g_i) \leq C_2$. Then, after passing to a subsequence, (M_i, g_i) converges in GH to a compact metric space (X, d_X) . If the sequence*

collapses, Fukaya showed that the Hausdorff dimension of X is still an integer $d \leq n$ [29].

To fully understand the collapsing sequence and get regularity, we need the following steps.

Step 1: local universal covers are non-collapsed. The key point is that collapsing in the base does not force collapsing on small local universal covers. One uses a uniform lower bound for the conjugate radius, together with the lower bound for homotopically trivial closed geodesics.

Theorem 1.10 (Klingenberg’s lemma for contractible closed geodesics). *Let (M, g) be complete with sectional curvature $K_g \leq \Lambda^2$ for some $\Lambda > 0$. If γ is a homotopically trivial closed geodesic, then*

$$L(\gamma) \geq \frac{2\pi}{\Lambda}.$$

A standard reference for this estimate is Klingenberg’s injectivity-radius argument in comparison geometry; see, for example, [18, Ch. 6] and [23, Ch. 13]. In the collapsing context, this is one of the inputs in Fukaya’s local analysis of collapsing and equivariant limits [26, 28, 19]. Combined with a uniform lower conjugate-radius bound, Theorem 1.10 gives a uniform lower local injectivity-radius bound on the lifted balls, hence local non-collapsing there. Therefore one can pass to equivariant Cheeger-Gromov limits on local universal covers.

Definition 1.18 (Local convergence in the collapsing setting). *Suppose $(M_i^n, g_i) \rightarrow (X^d, g_\infty)$ in GH with $d < n$. We say the convergence is **local** $C^{k,\alpha}$ if there exists $r > 0$ (independent of i) such that for every $x_i \rightarrow x_\infty$ one has equivariant convergence*

$$\begin{array}{ccc} (\tilde{B}_{\tilde{x}_i}(r), \tilde{g}_i, G_i) & \xrightarrow{C^{k,\alpha}} & (\tilde{B}_{\tilde{x}_\infty}(r), \tilde{g}_\infty, G_\infty) \\ \downarrow \pi_i & & \downarrow \pi_\infty \\ (B_{x_i}(r), g_i) & \xrightarrow{GH} & (B_{x_\infty}(r), g_\infty). \end{array}$$

Step 2: global fibration and almost-flat fibers. After the local model is understood, one globalizes to a fibration over the limit space. In the bounded-curvature setting used here, this is the classical CFG fibration picture. The regularity is latter improved to $C^{2,\alpha}$ by Naber-Zhang [46] and related refinements such as [61]. We adapt their result so that the fiber structure can be easily derived since to make second fundamental form well defined, we need the fibration map to be at least C^2

$$A_{F_i^{-1}(x)}(V, W) = \frac{\text{Hess } F_i(V, W)}{|\nabla F_i|}.$$

Theorem 1.11 (Cheeger-Fukaya-Gromov fibration theorem [19]). *Let $\{(M_i^n, g_i)\}$ be a sequence of closed Riemannian n -manifolds with $|\text{Rm}(g_i)| \leq C_1$ and $\text{diam}(M_i, g_i) \leq C_2$, and suppose $(M_i, g_i) \rightarrow (X^d, g_\infty)$ in GH, where (X^d, g_∞) is a compact Riemannian manifold with $d < n$.*

Then for all sufficiently large i , and each fixed $\alpha \in (0, 1)$, there is a $C^{2,\alpha}$ fiber-bundle map

$$F_i: (M_i, g_i) \rightarrow (X, g_\infty)$$

such that:

1. F_i is a τ_i -GH approximation, with $\tau_i \rightarrow 0$.
2. $\|F_i\|_{C^{2,\alpha}} \leq C$ uniformly in i .
3. For each fiber $F_i^{-1}(x)$, the second fundamental form is uniformly bounded (equivalently, $\|A_{F_i^{-1}(x)}\|_{C^{0,\alpha}} \leq C_0$).
4. F_i is an almost-Riemannian submersion: for every horizontal vector v ,

$$(1 - \tau_i)|v|_{g_i} \leq |dF_i(v)|_{g_\infty} \leq (1 + \tau_i)|v|_{g_i}.$$

5. Each fiber is an infranilmanifold, namely $F_i^{-1}(x) \cong \Gamma \backslash \mathcal{N}$, where $\mathcal{N}$ is simply connected nilpotent and $\Gamma \leq \mathcal{N} \rtimes \text{Aut}(\mathcal{N})$ is cocompact.

6. With $\Lambda := \mathcal{N} \cap \Gamma$, one has $\Lambda \triangleleft \Gamma$ and $\#(\Lambda \backslash \Gamma) \leq w_0(n)$.

The proof of the fibration theorem can be found in Appendix A.

The fiber geometry is controlled quantitatively by combining bounded ambient curvature and bounded second fundamental form with the Gauss equation:

$$|\text{Rm}_{\text{fiber}}| \leq C(|\text{Rm}_{M_i}| + |A_{\text{fiber}}|^2).$$

Since fiber diameters go to zero in the collapsing regime, one obtains almost-flat fibers in the sense of Gromov-Ruh [32, 52].

Theorem 1.12 (Ruh's theorem on almost-flat manifolds [52]). *There exists $\varepsilon(n) > 0$ such that if (Z^n, h) is closed and satisfies*

$$\text{diam}(Z, h)^2 \sup_Z |\text{Rm}(h)| \leq \varepsilon(n),$$

Then we have $Z \simeq \Gamma \backslash \mathcal{N}$, where $\mathcal{N}$ is a simply-connected nilpotent Lie group, and Γ is a cocompact subgroup of $\mathcal{N}_L \rtimes \text{Aut}(\mathcal{N})$, where $\mathcal{N}_L \cong \mathcal{N}$ is the left translation subgroup, i.e., Z is an infranilmanifold. Also, $\Lambda = \mathcal{N} \cap \Gamma$ is normal in Γ with $\#(\Lambda \backslash \Gamma) \leq w_0$ for some constant $w_0 = w_0(n)$. Equivalently, Z admits a finite covering by a compact nilmanifold (Example 1.4).

The nilpotent manifold collapsing to a point in Example 1.9 can be viewed as a special case of Theorem 1.11 when $X = \{pt\}$.

Step 3: nilpotent algebraic structure and $\mathcal{N}$ -action. The almost-flat conclusion is promoted to a canonical nilpotent structure in CFG: locally over simply connected base sets,

the lifted fibration carries a nilpotent Lie group action that is compatible with changing trivializations.

Definition 1.19 (Affine structure, affine equivalence, and $\mathcal{N}$ -invariance). *An **affine structure** on a manifold Z is a flat torsion-free connection ∇ on TZ . Two affinely structured manifolds (Z, ∇^Z) and (W, ∇^W) are **affinely equivalent** if a diffeomorphism $\Phi: Z \rightarrow W$ satisfies $\Phi^* \nabla^W = \nabla^Z$.*

*Given a local CFG trivialization over simply connected $U \subseteq X$, $\widetilde{F_i^{-1}(U)} \cong U \times \mathcal{N}$, the $\mathcal{N}$ -**action** is the right action on the nilpotent factor. A tensor on $\widetilde{F_i^{-1}(U)}$ is **$\mathcal{N}$ -invariant** if it is invariant under this action.*

Theorem 1.13 (Existence of nilpotent action). *For each simply connected open set $U \subseteq X$, and all large i , the structure group of the fibration reduced to $\text{Aff}^0(\Gamma \backslash \mathcal{N}) \cong C_{\text{Aff}(\mathcal{N})}(\Lambda)/\Lambda \subset \text{Aff}(\Gamma \backslash \mathcal{N})$. Thus there exists a global $\mathcal{N}$ -action on M_i for i sufficiently large.*

Equivalently, each fiber has the form $\Gamma \backslash \mathcal{N}$ (up to finite extension), where $\mathcal{N}$ is simply connected nilpotent and $\Gamma \leq \mathcal{N}_L \rtimes \text{Aut}(\mathcal{N})$ is cocompact. This is the algebraic structure that later supports the invariant-geometry and regularity arguments in Chapter 4.

Chapter 2

Special Metrics and Moduli Space

This chapter introduces the two types of special metrics studied in this thesis — locally conformally flat (LCF) metrics, Yamabe metrics. §2.1 defines LCF manifolds, develops the Kleinian group framework via the developing map (Schoen-Yau [59]), and collects the key analytic estimates (Kato inequalities, Tian-Viaclovsky volume comparison) needed for the moduli theory. §2.2 reviews the resolution of the Yamabe problem [72, 67, 7, 53] and its orbifold analog. §2.3 presents the Positive Mass Theorem and its LCF variant, which is the main technical tool in Schoen’s solution of the Yamabe problem for LCF manifolds. §2.4 gives the standard examples and the known classification results for compact LCF manifolds.

2.1 Locally Conformally Flat Metrics

Locally conformally flat (LCF) metrics generalize constant-curvature surfaces to higher dimensions. In dimension $n = 2$, every Riemannian metric is locally conformally flat. For $n \geq 3$, conformal flatness is characterized by the Weyl and Cotton tensors.

Definition 2.1 (Locally conformally flat manifold). *A Riemannian manifold (M^n, g) is **locally conformally flat (LCF)** if for every $p \in M$, there exists a neighborhood U and a*

chart $\phi: B \subset \mathbb{R}^n \rightarrow U$ such that $\phi^*g = \lambda g_{\mathbb{R}^n}$ for some smooth positive function λ on B .

Definition 2.2 (Weyl and Cotton tensors). *Let (M^n, g) be a Riemannian manifold with Riemann tensor R_{ijkl} , Ricci tensor R_{ij} , and scalar curvature S . The **Weyl tensor** is defined by*

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}) \\ + \frac{S}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}).$$

Let

$$A_{ij} = \frac{1}{n-2} \left(R_{ij} - \frac{S}{2(n-1)}g_{ij} \right)$$

be the Schouten tensor. The **Cotton tensor** is

$$C_{ijk} = \nabla_k A_{ij} - \nabla_j A_{ik} = \nabla_k R_{ij} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{ij}\nabla_k S - g_{ik}\nabla_j S).$$

The next theorem gives the equivalence between these tensor identities and local conformal flatness.

Theorem 2.1 (Weyl-Schouten theorem [12, Ch. 1]). *Let (M^n, g) be a Riemannian manifold.*

1. *If $n \geq 4$, then (M, g) is locally conformally flat if and only if $W \equiv 0$.*
2. *If $n = 3$, then (M, g) is locally conformally flat if and only if $C \equiv 0$.*

A direct computation yields

$$\nabla^l W_{lijk} = \frac{n-3}{n-2} C_{ijk}. \tag{2.1}$$

Hence, if $W \equiv 0$ and the scalar curvature is constant (for example, for a Yamabe metric), then

$$\nabla_k R_{ij} - \nabla_j R_{ik} = 0, \quad (2.2)$$

which is the harmonic curvature condition.

This chain of implications is the analytic starting point for the Ricci estimates below.

2.1.1 Local Analytic Aspect for LCF manifolds

First, we will discuss the local analytic aspect of LCF manifolds and their consequence. In particular, we study the local curvature properties of LCF manifolds.

Because of curvature property of LCF, there is a very restrictive classification of the product structure that has LCF property.

Theorem 2.2 (LCF product classification [42]). *Let $(M^n, g) = (M_1^{n_1}, g_1) \times (M_2^{n_2}, g_2)$ be an LCF Riemannian product with $n = n_1 + n_2 \geq 3$. Then one of the following holds:*

1. **Cylinder:** (M_1, g_1) is a space form, and (M_2, g_2) is one dimensional, so (M, g) is a cylinder $M_1^{n_1} \times \mathbb{R}$ (or its quotient).
2. **Bi-space-form:** (M_1, g_1) and (M_2, g_2) are space forms of opposite curvatures: $\kappa_1 = -\kappa_2 \neq 0$, so the scalar curvature is zero.
3. **Flat:** (M, g) is flat.

Proof. For $n \geq 4$, (M, g) is LCF iff $W \equiv 0$; for $n = 3$, LCF is equivalent to $C \equiv 0$. We treat $n \geq 4$ in detail. The case $n = 3$ forces $(n_1, n_2) \in \{(1, 2), (2, 1)\}$ and falls into case (1); the analogous Cotton-tensor computation gives the same conclusion.

Step 1: Both factors are Einstein. Let a, b (resp. α, β) index an orthonormal frame for TM_1 (resp. TM_2). On a Riemannian product, cross-curvature and cross-Ricci tensors vanish, so the mixed Weyl component is

$$W_{a\alpha b\beta} = -\frac{1}{n-2}((Ric_1)_{ab}(g_2)_{\alpha\beta} + (Ric_2)_{\alpha\beta}(g_1)_{ab}) + \frac{S_1 + S_2}{(n-1)(n-2)}(g_1)_{ab}(g_2)_{\alpha\beta}.$$

Setting $W_{a\alpha b\beta} = 0$ and contracting with $(g_2)^{\alpha\beta}$ gives

$$(Ric_1)_{ab} = \left(\frac{S_1 + S_2}{n-1} - \frac{S_2}{n_2} \right) (g_1)_{ab},$$

so M_1 is Einstein. For $n_1 \geq 2$, Schur's lemma then forces $S_1 + S_2$, and hence S_2 , to be constant; a symmetric contraction with $(g_1)^{ab}$ shows M_2 is Einstein with constant S_1 .

Step 2: Both factors are space forms. With $Ric_i = \lambda_i g_i$ constant, the pure M_1 components of $W \equiv 0$ reduce to

$$(R_1)_{abcd} = \kappa_1((g_1)_{ac}(g_1)_{bd} - (g_1)_{ad}(g_1)_{bc}), \quad \kappa_1 = \frac{2(n-1)\lambda_1 - n_1\lambda_1 - n_2\lambda_2}{(n-1)(n-2)},$$

so M_1 is a space form of sectional curvature κ_1 . Analogously M_2 is a space form of curvature κ_2 .

Step 3: Classification. Substituting $Ric_i = (n_i - 1)\kappa_i g_i$ into the mixed Weyl condition yields

$$(n_1 - 1)(n_2 - 1)(\kappa_1 + \kappa_2) = 0.$$

- $n_1 = 1$ or $n_2 = 1$: the one-dimensional factor is flat; the other is any space form. This gives case (1).
- $n_1, n_2 \geq 2$ and $\kappa_1 + \kappa_2 = 0$, $\kappa_i \neq 0$: bi-space-form, case (2).

- $n_1, n_2 \geq 2$ and $\kappa_1 = \kappa_2 = 0$: (M, g) is flat, case (3). □

This theorem shows that LCF product metrics are essentially cylinders $M \times \mathbb{R}$ or bi-spherical spaces $\mathbb{S}^k(\kappa) \times \mathbb{H}^{n-k}(\kappa)$ (see §2.4).

When $W \equiv 0$ and R_g is constant (for example, for a Yamabe metric), (2.2) implies

$$\Delta \text{Ric} = \text{Ric} * \text{Ric} + \text{Ric} * \text{Rm}.$$

The following estimates for the Ricci curvature of LCF manifolds are used in the compactification theory of Chapter 3.

Lemma 2.3 (Kato's inequality for Ricci curvature). *For any tensor T : $|\nabla|T||^2 \leq |\nabla T|^2$ (**classical Kato inequality**). For a complete LCF manifold with constant scalar curvature, the harmonic curvature condition and $W \equiv 0$ imply:*

$$\Delta|\text{Ric}| + C|\text{Ric}|^2 \geq 0. \tag{2.3}$$

Proof. Assume (M, g) is LCF with constant scalar curvature. Then $\nabla S \equiv 0$, so from (2.2) we get the harmonic curvature identity

$$\nabla_j R_{ik} - \nabla_i R_{jk} = 0.$$

Using this identity and commuting covariant derivatives, one obtains the Ricci equation

$$\Delta \text{Ric} = \nabla^i (\nabla_i R_{jk}) = g^{il} \nabla_l \nabla_j R_{ik} = \text{Ric} * \text{Ric} + \text{Ric} * \text{Rm}.$$

Since $W \equiv 0$, the full curvature tensor is algebraically controlled by Ricci and scalar curva-

ture; because S is constant, this gives

$$|\text{Rm}| \leq C |\text{Ric}|.$$

Hence

$$|\Delta \text{Ric}| \leq C |\text{Ric}|^2.$$

Now apply the Bochner identity to Ric :

$$\begin{aligned} \frac{1}{2} \Delta |\text{Ric}|^2 &= \langle \Delta \text{Ric}, \text{Ric} \rangle + |\nabla \text{Ric}|^2 \\ &= |\text{Ric}| \Delta |\text{Ric}| + |\nabla |\text{Ric}||^2. \end{aligned}$$

Therefore

$$|\text{Ric}| \Delta |\text{Ric}| = \langle \Delta \text{Ric}, \text{Ric} \rangle + |\nabla \text{Ric}|^2 - |\nabla |\text{Ric}||^2.$$

By classical Kato inequality, $|\nabla |\text{Ric}||^2 \leq |\nabla \text{Ric}|^2$, so

$$|\text{Ric}| \Delta |\text{Ric}| \geq \langle \Delta \text{Ric}, \text{Ric} \rangle \geq -C |\text{Ric}|^3.$$

At points where $|\text{Ric}| > 0$, divide by $|\text{Ric}|$ to get

$$\Delta |\text{Ric}| + C |\text{Ric}|^2 \geq 0.$$

By standard barrier/approximation at the zero set, the same inequality holds weakly on all of M . □

A direct consequence of (2.3) is the following ϵ -regularity.

Lemma 2.4 (ε -regularity for Ricci curvature [64]). *Let (M^n, g) be a complete LCF, scalar-flat manifold with $Y(M, [g]) \geq \delta > 0$. There exist C and ε_0 (depending on n, δ) such that if $\int_{B_r(p)} |\text{Ric}|^{n/2} dV_g < \varepsilon_0$, then:*

$$\sup_{B_{r/2}(p)} |\text{Ric}| \leq \frac{C}{r^2} \left(\int_{B_r(p)} |\text{Ric}|^{n/2} dV_g \right)^{2/n}. \quad (2.4)$$

The proof applies Moser iteration to (2.3), using the Sobolev inequality provided by $Y(M, [g]) \geq \delta$. Since the proof is standard, we will not write the detailed in here and sketch the proof later in Lemma 3.2.

On the other hand, since the symmetry of the Ricci curvature is identical with the second fundamental form in [54], we have the following Refined Kato inequality.

Lemma 2.5 (Refined Kato inequality under LCF). *Let (M^n, g) be a complete LCF, scalar-flat manifold ($n \geq 3$). Under the harmonic curvature condition:*

$$\left(1 + \frac{2}{n}\right) |\nabla |\text{Ric}||^2 \leq |\nabla \text{Ric}|^2, \quad (2.5)$$

and consequently $\Delta |\text{Ric}|^{1-2/n} \geq -C |\text{Ric}|^{2-2/n}$.

Proof. Following the refined Kato strategy in the literature (cf. [10]), we work at a fixed point $p \in M$ and choose an orthonormal frame that diagonalizes Ric at p . Write $R_{ij} = \lambda_i \delta_{ij}$ at p . Then

$$\begin{aligned} 4 |\text{Ric}|^2 |\nabla |\text{Ric}||^2 &= |\nabla |\text{Ric}|^2|^2 = \sum_{k=1}^n \left(\sum_{i=1}^n 2R_{ii} \nabla_k R_{ii} \right)^2 \\ &\leq 4 |\text{Ric}|^2 \sum_{k=1}^n \sum_{i=1}^n (\nabla_k R_{ii})^2, \end{aligned}$$

so

$$|\nabla |\operatorname{Ric}||^2 \leq \sum_{k=1}^n \sum_{i=1}^n (\nabla_k R_{ii})^2.$$

Because $R_g \equiv 0$, we have $\sum_{j=1}^n R_{jj} = 0$ and hence $\nabla_i \sum_{j=1}^n R_{jj} = 0$. Therefore

$$\begin{aligned} \sum_{k,i} (\nabla_k R_{ii})^2 &= \sum_{i \neq k} (\nabla_k R_{ii})^2 + \sum_i (\nabla_i R_{ii})^2 \\ &= \sum_{i \neq k} (\nabla_k R_{ii})^2 + \sum_i \left(\sum_{j \neq i} \nabla_i R_{jj} \right)^2 \\ &\leq \sum_{i \neq k} (\nabla_k R_{ii})^2 + (n-1) \sum_i \sum_{j \neq i} (\nabla_i R_{jj})^2 \\ &= n \sum_{i \neq k} (\nabla_k R_{ii})^2. \end{aligned}$$

Under harmonic curvature (Codazzi),

$$\nabla_k R_{ii} = \nabla_i R_{ki} \quad (i \neq k),$$

so

$$\sum_{k,i} (\nabla_k R_{ii})^2 \leq n \sum_{i \neq k} (\nabla_i R_{ki})^2.$$

Combining this with the standard decomposition of $|\nabla \operatorname{Ric}|^2$ into diagonal and off-diagonal parts gives the refined estimate

$$\left(1 + \frac{2}{n}\right) |\nabla |\operatorname{Ric}||^2 \leq |\nabla \operatorname{Ric}|^2,$$

which is (2.5).

For the differential inequality, apply the Kato-type Bochner computation from Lemma 2.3

and combine it with (2.5):

$$\Delta |\text{Ric}| \geq -C |\text{Ric}|^2 \implies \Delta |\text{Ric}|^{1-\frac{2}{n}} \geq -C |\text{Ric}|^{2-\frac{2}{n}}.$$

□

Follow the same type of estimate as in [10], we can prove the following decay estimate for LCF ALE manifolds.

Theorem 2.6 (ALE order from refined Kato's inequality [10]). *Let (M^n, g) be a complete LCF, scalar-flat ALE manifold satisfying the hypotheses of Lemma 2.5. Then for any $\varepsilon > 0$, [10] yields the pointwise decay estimates*

$$|\text{Ric}|^{1-\frac{2}{n}} = O(r^{-n+2+\varepsilon}),$$

$$|\text{Ric}| = O(r^{-n+\varepsilon}).$$

Consequently, (M, g) is an ALE space of order $n - 2 - \varepsilon$ (in the sense of Definition 1.6 in Chapter 1) for any $\varepsilon > 0$, i.e. the metric satisfies

$$g_{ij} = \delta_{ij} + O(r^{-n+2+\varepsilon}).$$

If we only assume $\int_M |\text{Ric}|^{n/2} dV_g \leq \Lambda < \infty$ for a complete non-compact, locally non-collapsing, LCF manifold, due to the lack of Bishop-Gromov volume comparison, we cannot directly get an ALE coordinate at infinity, like the Ricci flat case in [10]. We need the following theorem to determine the structure.

Theorem 2.7 (Tian-Viaclovsky volume comparison [64, 65]). *Let (M^n, g) be a complete LCF, scalar-flat manifold with Euclidean volume growth $\text{Vol}(B_t) \geq Vt^n$, $\int_M |\text{Ric}|^{n/2} dV_g \leq \Lambda < \infty$, and $Y(M, [g]) \geq \delta > 0$. Then $\text{Vol}(B_r(p)) \leq Cr^n$ for all p, r , and (M, g) has finitely*

many ends.

Thus, we have the desired volume upper bound to construct the ALE coordinate at infinity using [10]. Consequently, we have

Corollary 2.8. *Let (M^n, g) be a complete LCF, scalar-flat manifold with Euclidean volume growth $\text{Vol}(B_t) \geq Vt^n$, $\int_M |\text{Ric}|^{n/2} dV_g \leq \Lambda < \infty$, and $Y(M, [g]) \geq \delta > 0$. Then, (M, g) is an ALE space of order $n - 2 - \varepsilon$ (in the sense of Definition 1.6 in Chapter 1) for any $\varepsilon > 0$.*

Lemma 2.4 and Theorem 2.7 together are the analytic engine of the Tian-Viaclovsky compactification: the ε -regularity controls curvature away from finitely many concentration points, and the volume comparison prevents non-ALE end behavior. Thus, we have. These tools are applied in Chapter 3.

2.1.2 Global Geometric Aspect for LCF manifolds

The defining feature of a locally conformally flat manifold is the existence of local conformal charts into $\mathbb{R}^n$. A key insight is that these local charts can be analytically continued over the universal cover to produce a globally defined conformal map into $\mathbb{S}^n$. We now describe this construction and the additional structure it carries.

Definition 2.3 (Developing map). *Let (M^n, g) be a locally conformally flat manifold with $n \geq 3$. Since each point $p \in M$ admits a neighborhood U_p with a conformal diffeomorphism $\phi_p: (U_p, g) \rightarrow (V_p, g_{\mathbb{R}^n})$, via stereographic projection $\sigma: \mathbb{R}^n \hookrightarrow \mathbb{S}^n$ one regards each chart as a conformal map into $\mathbb{S}^n$. Performing analytic continuation along paths in the universal cover $\widetilde{M}$ — starting from an initial chart at a basepoint and propagating consistently along any path by composing overlapping charts — yields, since $\widetilde{M}$ is simply connected, a globally defined conformal local diffeomorphism*

$$\phi: \widetilde{M} \longrightarrow \mathbb{S}^n,$$

called the **developing map**. In general, ϕ is a conformal immersion but need not be injective or surjective.

Theorem 2.9 (Liouville's theorem). *Let $U, V \subset \mathbb{S}^n$ be open connected subsets with $n \geq 3$, and $f: U \rightarrow V$ a conformal diffeomorphism. Then f extends uniquely to a conformal automorphism of $\mathbb{S}^n$.*

Definition 2.4 (Holonomy representation). *Let (M^n, g) be a locally conformally flat manifold with developing map $\phi: \widetilde{M} \rightarrow \mathbb{S}^n$. Each deck transformation $\gamma \in \pi_1(M)$ satisfies an equivariance relation*

$$\phi \circ \gamma = \rho(\gamma) \circ \phi$$

for a unique element $\rho(\gamma) \in \text{Möb}(\mathbb{S}^n) \cong O^+(n+1, 1)$, given by Liouville's theorem (Theorem 2.9). The resulting group homomorphism

$$\rho: \pi_1(M) \longrightarrow \text{Möb}(\mathbb{S}^n)$$

is the **holonomy representation**. Its image $\Gamma = \rho(\pi_1(M))$ is the **Kleinian holonomy group**.

Under the additional assumption $R_g \geq 0$, Schoen and Yau showed that the developing map is in fact an embedding.

Theorem 2.10 (Schoen-Yau conformal embedding [59]). *Let (M^n, g) be a complete LCF manifold with $n \geq 3$ and $R_g \geq 0$. Then the developing map $\phi: \widetilde{M} \rightarrow \mathbb{S}^n$ is a conformal embedding.*

Remark 2.1. *When $Y(M, [g]) \geq 0$ (which implies $R_g \geq 0$ in the Yamabe class), Theorem 2.10 upgrades the developing map from a mere immersion to an embedding: $\phi(\widetilde{M})$*

is a proper open subdomain of $\mathbb{S}^n$. By Liouville's theorem, each deck transformation then acts on $\phi(\widetilde{M})$ by the restriction of a global Möbius transformation, so the holonomy ρ of Definition 2.4 is injective and $\Gamma = \rho(\pi_1(M))$ embeds as a discrete subgroup of $\text{Möb}(\mathbb{S}^n) \cong O^+(n+1, 1)$. The pair (ϕ, ρ) thus endows (M, g) with a **Kleinian structure**: the domain of discontinuity $\Omega \subset \mathbb{S}^n$ is the maximal open set on which Γ acts properly discontinuously, and the limit set $\Lambda = \mathbb{S}^n \setminus \Omega$ encodes the asymptotic geometry of M .

The pair (ϕ, ρ) endows (M, g) with the following structure.

Definition 2.5 (Kleinian structure [59]). *With $\Gamma = \rho(\pi_1(M)) \leq \text{Möb}(\mathbb{S}^n)$:*

- *The **domain of discontinuity** $\Omega \subseteq \mathbb{S}^n$ is the maximal open set on which Γ acts properly discontinuously.*
- *The **limit set** $\Lambda = \mathbb{S}^n \setminus \Omega$ is the minimal Γ -invariant closed subset of $\mathbb{S}^n$.*

(M, g) is conformally equivalent to $\phi(\widetilde{M})/\Gamma \subseteq \Omega/\Gamma$.

Definition 2.6 (Elliptic, parabolic, hyperbolic elements). *An element $\gamma \in \text{Möb}(\mathbb{S}^n) \cong \text{Isom}(\mathbb{H}^{n+1})$ is:*

- ***elliptic** if it has a fixed point in $\mathbb{H}^{n+1}$;*
- ***parabolic** if it has exactly one fixed point on $\mathbb{S}^n$;*
- ***hyperbolic** if it has two distinct fixed points on $\mathbb{S}^n$.*

Elements of finite order are elliptic; infinite-order elements acting freely on Ω are parabolic or hyperbolic.

The limit set Λ and the domain of discontinuity Ω encode the coarse structure of the LCF manifold: the smaller Λ is (in Hausdorff dimension), the closer (M, g) is to a spherical space form, and Schottky-type structure (with Λ a Cantor set) corresponds to multi-ended

connected sums. The classification results in §2.4 are organized by the geometry of Λ .

Definition 2.7 (Conformal connected sum [36]). *Let (M_1, g_1) and (M_2, g_2) be LCF orbifolds. Fix an orientation-preserving conformal embedding of an annulus $A \cong [0, 1] \times \mathbb{S}^{n-1} \hookrightarrow \mathbb{S}^n$, with complementary components $D_1, D_2 \cong B^n$. The **conformal connected sum** $(M_1, g_1) \#_C (M_2, g_2)$ is the LCF orbifold obtained by gluing $(M_j \setminus \psi_j(E_j), g_j)$ along $\psi_j(A)$ via orientation-preserving conformal maps $\psi_j: E_j \rightarrow M_j$ ($E_j = A \cup D_j$).*

Example 2.1 ($\mathbb{S}^3 \times \mathbb{S}^1 \# \mathbb{S}^3 \times \mathbb{S}^1$). *Start from two loxodromic Möbius transformations $\gamma_1, \gamma_2 \in \text{Möb}(\mathbb{S}^4)$ with disjoint isometric spheres, so that*

$$\Gamma = \langle \gamma_1, \gamma_2 \rangle \cong F_2$$

is a classical Schottky group. Its domain of discontinuity $\Omega \subset \mathbb{S}^4$ is connected, and the quotient

$$M = \Omega / \Gamma$$

is a compact LCF 4-manifold.

Topologically, this quotient is

$$M \cong (\mathbb{S}^3 \times \mathbb{S}^1) \# (\mathbb{S}^3 \times \mathbb{S}^1),$$

since each generator contributes one $\mathbb{S}^3 \times \mathbb{S}^1$ handle in the conformal connected-sum picture.

Its limit set

$$\Lambda = \mathbb{S}^4 \setminus \Omega$$

is a Cantor set: equivalently,

$$\Lambda = \bigcap_{m \geq 1} \bigcup_{|w|=m} B_w,$$

where w runs over reduced words in $\gamma_1^{\pm 1}, \gamma_2^{\pm 1}$ and B_w are the nested images of the fundamental round balls. This is a compact, perfect, totally disconnected set.

Regarding the conformal group structure. Note that for a manifold equipped with a Kleinian structure. The fundamental group can be viewed as a subgroup of $\text{Möb}(\mathbb{S}^n) \simeq O^+(n+1, 1)$. Thus, we can use the following Selberg's lemma to characterize these groups.

Theorem 2.11 (Selberg's lemma [60]). *A finitely generated subgroup of $GL(N, \mathbb{C})$ has a finite-index torsion-free subgroup, and in fact a normal finite-index torsion-free subgroup.*

In particular, any Kleinian LCF manifold (M, g) with $\pi_1(M)$ finitely generated has a finite orbifold cover that is a smooth manifold. This is used in §3.3.

Lastly, we will give the definition of a special type of orbifold that will appear a lot of places in this thesis.

Definition 2.8 (Football orbifold). *A **football orbifold** is $\mathbb{S}^n/\Gamma$ where $\Gamma \leq O(n) \subset O(n+1)$ acts with exactly two antipodal fixed points. The quotient metric is spherical, and $\pi_1^{\text{orb}}(\mathbb{S}^n/\Gamma) \cong \Gamma$. (cf. Example 1.5).*

In particular, football orbifolds are LCF. When n is odds, the only football orbifolds are $\mathbb{S}^n/\mathbb{Z}_2$, by a elementary eigenvalue argument.

The geometric tools above — developing maps, Kleinian groups, conformal connected sums — characterize the global topology of LCF manifolds. To analyze their metric properties and control their degeneration in moduli space, we need to combine analytic estimates and global structure of LCF manifolds.

The next canonical object in the LCF world is the Yamabe metric — the constant scalar curvature representative in a conformal class — whose existence is guaranteed by the Yamabe theorem.

2.2 Yamabe Metrics

The Yamabe problem asks whether every compact Riemannian manifold (M^n, g) admits a metric of constant scalar curvature in its conformal class. This was posed by Yamabe [72] and resolved by Trudinger, Aubin, and Schoen over the period 1968–1984. The resolution in the locally conformally flat case uses the Positive Mass Theorem (§2.3) and is central to this thesis.

Definition 2.9 (Yamabe constant and Yamabe invariant). *Let (M^n, g) be a compact Riemannian manifold with $n \geq 3$. Recall the Yamabe functional from (1.1):*

$$\mathcal{Y}_g(u) = \frac{\int_M u L_g u dV_g}{\|u\|_{L^{2n/(n-2)}(M)}^2}, \quad u \in W^{1,2}(M), \quad u \neq 0.$$

The **Yamabe constant** of the conformal class $[g]$ is

$$Y(M, [g]) = \inf_{0 \neq u \in W^{1,2}(M)} \mathcal{Y}_g(u) = \inf_{\tilde{g} \in [g]} \text{Vol}(\tilde{g})^{-(n-2)/n} \int_M R_{\tilde{g}} dV_{\tilde{g}}.$$

The functional is scaling-invariant, hence $Y(M, [g])$ depends only on the conformal class. A **Yamabe metric** is a metric $\tilde{g} \in [g]$ realizing the infimum. The **Yamabe invariant** of M is $Y(M) = \sup_{[g]} Y(M, [g])$.

For a Riemannian orbifold (M, g) , the **orbifold Yamabe constant** $Y_{\text{orb}}(M, [g])$ is defined by the same infimum over the orbifold conformal class.

Theorem 2.12 (Yamabe problem [72, 67, 7, 53]). *Let (M^n, g) be a compact Riemannian*

manifold without boundary, $n \geq 3$. Then every conformal class $[g]$ contains a smooth metric of constant scalar curvature (a Yamabe metric). Equivalently, the Yamabe equation

$$L_g u = \lambda u^{(n+2)/(n-2)}, \quad u > 0 \text{ smooth}, \quad \lambda = Y(M, [g]), \quad (2.6)$$

has a smooth positive solution.

Proof sketch. The proof was completed in four stages.

Step 1 (Yamabe 1960). Writing $\tilde{g} = u^{4/(n-2)}g$, the Yamabe equation (2.6) is the Euler-Lagrange equation for $\mathcal{Y}_g$. A minimizing sequence $\{u_k\}$ for $\mathcal{Y}_g$ is bounded in $W^{1,2}$, but the Sobolev embedding is only continuous (not compact) at the critical exponent $2^* = 2n/(n-2)$, so concentration phenomena may occur.

Step 2 (Trudinger 1968). Corrected Yamabe's argument; proved existence when $Y(M, [g]) \leq 0$ by a subcritical approximation: replace the exponent 2^* by $2^* - \varepsilon$, obtain subcritical minimizers u_ε by compact Sobolev embedding, and pass to the limit.

Step 3 (Aubin 1976). For $Y > 0$, bubbling may occur. The key criterion is $Y(M, [g]) < Y(\mathbb{S}^n)$: if this holds, a minimizing sequence cannot concentrate, so a minimizer exists. Aubin constructed test functions u_ε (cutoffs of the Euclidean bubble $(1 + |x|^2)^{-(n-2)/2}$ near a point p) and showed $\mathcal{Y}_g(u_\varepsilon) < Y(\mathbb{S}^n)$ when $n \geq 6$ and (M, g) is not LCF, using the Weyl tensor at p to gain a negative correction. In the LCF case $W \equiv 0$, this correction vanishes to all orders, so Aubin's argument does not apply and a fundamentally different tool is needed.

Step 4 (Schoen 1984). For the remaining cases ($n = 3, 4, 5$ and the LCF case, where Aubin's Weyl correction vanishes), Schoen used the **Positive Mass Theorem**: for a conformal factor G (the conformal Green's function at a point p), the metric $\hat{g} = G^{4/(n-2)}g$ is asymptotically flat, and $Y(M, [g]) < Y(\mathbb{S}^n)$ holds if and only if the ADM mass $m_{\text{ADM}}(\hat{g}) > 0$. By the PMT (§2.3), $m_{\text{ADM}} \geq 0$ with equality only on $(\mathbb{R}^n, g_{\text{Eucl}})$. This completes the proof. $\square$

Remark 2.2 (Sign of the Yamabe constant as conformal invariant). *The **sign** of $Y(M, [g])$ is a conformal invariant:*

- **Positive Yamabe class** ($Y > 0$): *the conformal Laplacian L_g is a positive operator, and $[g]$ contains a metric of positive scalar curvature. In this thesis, “PSC” always means $Y > 0$.*
- **Zero Yamabe class** ($Y = 0$): *$[g]$ contains a scalar-flat metric. Examples: flat tori $\mathbb{T}^n$ and balanced products $\mathbb{S}^k \times \mathbb{H}^{n-k}$.*
- **Negative Yamabe class** ($Y < 0$): *every metric in $[g]$ has total scalar curvature negative. Example: hyperbolic manifolds $\mathbb{H}^n/\Gamma$.*

Theorem 2.13 (Orbifold Yamabe existence [2, 1]). *Let (M, g) be a compact Riemannian orbifold with isolated singularities, local groups $G_i \leq O(n)$. Then:*

$$Y_{\text{orb}}(M, [g]) \leq Y(\mathbb{S}^n) \min_i |G_i|^{-2/n}. \quad (2.7)$$

If the inequality is strict, there exists a smooth conformal metric with constant scalar curvature (a Yamabe minimizer).

The factor $|G_i|^{-2/n}$ reflects that a test function concentrated near the singular point p_i sees the orbifold chart $\mathbb{R}^n/G_i$ and gains a volume factor $|G_i|^{-1}$. This theorem is used in §?? to solve the Yamabe problem on LCF orbifolds.

The Yamabe theorem guarantees a canonical representative — a CSC metric — in every conformal class, a fact exploited throughout Chapter 3 where we study moduli spaces of such representatives. The *sign* of the Yamabe constant (positive, zero, or negative) is a coarser but fundamental conformal invariant that determines which of the three broad classes a manifold belongs to. The key non-negativity ingredient in Schoen’s completion of Step 4 is

the Positive Mass Theorem, which we review next.

2.3 Positive Mass Theorem

The Positive Mass Theorem (PMT) is used in this thesis in two ways: (i) as the key ingredient in Schoen's completion of the Yamabe problem (§2.2, Step 4), and (ii) as the model for the LCF ALE orbifold PMT proved in Chapter 3 (§3.27).

Definition 2.10 (ADM mass [11]). *Let (M^n, g) be an ALE manifold with a single end, and let $\{x^i\}$ be the asymptotic coordinates on the end. The **ADM mass** of the end is*

$$m(g) = \lim_{r \rightarrow \infty} \frac{1}{2(n-1)\omega_n} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii}) \nu^j d\sigma, \quad (2.8)$$

where $S_r = \{|x| = r\}$, ν is the outward unit normal, and ω_n is the volume of the unit ball in $\mathbb{R}^n$. The integral converges when the scalar curvature R_g is integrable. For AF manifolds ($\Gamma = \{1\}$), this coincides with the classical general-relativistic definition.

Remark 2.3. *The ADM mass is well-defined for ALE orbifolds as well, and each end carries its own mass. For ALE manifolds with non-trivial group Γ at infinity, the PMT does not hold in general: LeBrun constructed anti-self-dual ALE metrics in dimension 4 with $R_g = 0$ and $m(g) < 0$. The PMT holds in the LCF ALE setting (Chapter 3, Theorem 3.27).*

Theorem 2.14 (Positive Mass Theorem [55]). *Let (M^n, g) be a complete asymptotically flat Riemannian manifold with $n \leq 7$, non-negative scalar curvature $R_g \geq 0$, and integrable R_g . Then $m(g) \geq 0$, with equality if and only if (M, g) is isometric to $(\mathbb{R}^n, g_{\text{Eucl}})$.*

Proof sketch (Schoen-Yau minimal surface method). The proof proceeds by contradiction. Assuming $m(g) < 0$, one perturbs g near infinity to produce an AF metric with $R_g \geq 0$ and $m(g) < 0$ that admits a **stable minimal hypersurface** Σ^{n-1} (existence via the Plateau

problem: the negative mass creates a barrier forcing the area functional to have a minimum on large spheres). The **second variation formula** for area gives

$$0 \leq \delta^2 \text{Area}(\Sigma) = - \int_{\Sigma} (|\text{II}|^2 + \text{Ric}(\nu, \nu)) f^2 d\sigma$$

for any variation function f . Combined with the Gauss equation and $R_g \geq 0$, this leads to a contradiction (the stability inequality forces the induced scalar curvature R_{Σ} to be non-negative, but a topological argument on Σ shows it must be zero, and then Σ is flat and the original manifold is flat). The dimension restriction $n \leq 7$ ensures Σ is smooth (no interior singularities). $\square$

The Schoen-Yau argument relies on the existence and regularity of stable minimal hypersurfaces, which fails in high dimensions ($n \geq 8$) due to possible interior singularities. Witten's spinorial proof, which we state next, bypasses this difficulty entirely.

Theorem 2.15 (Witten's spinorial proof [70]). *Let (M^n, g) be a complete spin asymptotically flat manifold with $R_g \geq 0$. Then $m(g) \geq 0$, with equality iff $(M, g) \cong (\mathbb{R}^n, g_{\text{Eucl}})$.*

Proof sketch. Since M is spin, the Dirac operator $\not{D}$ acts on spinors. Solve $\not{D}\psi = 0$ with $\psi \rightarrow \psi_0$ (constant spinor) at infinity. The **Weitzenböck formula** $\not{D}^2 = \nabla^* \nabla + R_g/4$ gives, on integration by parts:

$$m(g) \cdot |\psi_0|^2 = \int_M \left(|\nabla \psi|^2 + \frac{R_g}{4} |\psi|^2 \right) dV_g \geq 0.$$

Equality forces $\nabla \psi = 0$ and $R_g = 0$, implying (M, g) is flat. $\square$

The two proofs above assume the manifold is asymptotically flat (trivial group at infinity). In the LCF setting, the developing map provides a conformally natural way to handle the ALE case (non-trivial group at infinity), leading to a version of the PMT that applies directly

	Schoen-Yau (1979)	Witten (1981)
Method	Stable minimal hypersurfaces	Dirac operator / spinors
Spin structure	Not required	Required
Dimension	$n \leq 7$	All n
Rigidity	Isometric to $\mathbb{R}^n$	Parallel spinor $\Rightarrow$ flat

Table 2.1: Comparison of the two proofs of the classical PMT.

to the blow-up limits arising in the LCF moduli theory.

Theorem 2.16 (Schoen-Yau PMT for LCF manifolds [59]). *Let (M^n, g) be a complete LCF manifold with $n \geq 3$, $R_g \geq 0$, and an asymptotically Euclidean (AE) end. Suppose the developing map $\phi: \widetilde{M} \rightarrow \mathbb{S}^n$ is injective. Then $m(g) \geq 0$, with equality if and only if (M, g) is conformally equivalent to $(\mathbb{R}^n, g_{\text{Eucl}})$.*

The same conclusion holds for ALE orbifolds with group Γ at infinity, with equality iff (M, g) is the flat cone $\mathbb{R}^n/\Gamma$.

Proof sketch. **Step 1 (Conformal compactification).** Using the developing map, conformally compactify (M, g) to a compact LCF orbifold $(\hat{M}, \hat{g})$ by adding a point p_∞ at infinity.

Step 2 (Green's function). The conformal Laplacian $L_{\hat{g}}$ is non-negative (since $R_g \geq 0$ and $Y(\hat{M}, [\hat{g}]) > 0$). Let G_{p_∞} be the positive Green's function of $L_{\hat{g}}$ at p_∞ : $L_{\hat{g}}G_{p_\infty} = \delta_{p_\infty}$, $G_{p_\infty} > 0$ (maximum principle).

Step 3 (Mass identification). Near p_∞ (corresponding to $|x| \rightarrow \infty$ on M), by the harmonic expansion (Lemma 1.4):

$$G_{p_\infty}(x) = c(n)|x|^{2-n} + m + O(|x|^{-1}),$$

where m is proportional to $m_{\text{ADM}}(g)$.

Step 4 (Non-negativity). If $m < 0$, then G_{p_∞} would become negative for large $|x|$ (the

constant term m dominates $c(n)|x|^{2-n} \rightarrow 0$), contradicting $G_{p_\infty} > 0$. Hence $m \geq 0$.

Step 5 (Rigidity). If $m = 0$, the metric $G_{p_\infty}^{4/(n-2)} \hat{g}$ is scalar-flat and complete; by the classification of scalar-flat LCF AE manifolds, $(M, g) \cong (\mathbb{R}^n, g_{\text{Eucl}})$. $\square$

Remark 2.4. *This theorem is the direct predecessor of the LCF ALE orbifold PMT proved in §3.27, which extends it to the setting of ALE orbifolds with non-trivial group Γ at infinity. It is also the key step in Schoen’s resolution of the Yamabe problem for LCF manifolds: the strict positivity of the ADM mass (whenever $(M, g) \not\cong \mathbb{R}^n$) precisely gives $Y(M, [g]) < Y(\mathbb{S}^n)$, completing Step 4 of the proof of Theorem 2.12.*

	Classical PMT	LCF PMT (SY88)
Setting	AF spin manifolds	LCF manifolds/ALE orbifolds
Key tool	Minimal surfaces / Dirac	Developing map + Green’s function
Dimension	$n \leq 7$ (SY) or all n (Witten)	All $n \geq 3$
Rigidity	Isometric to $\mathbb{R}^n$	Conformally equivalent to $\mathbb{R}^n$

Table 2.2: Comparison of the classical and LCF Positive Mass Theorems.

With the Yamabe theorem and the Positive Mass Theorem in place, we have the two fundamental existence and non-negativity results for CSC metrics on LCF manifolds. The next section surveys the standard examples and known classification results, organized by Yamabe class, which motivate the moduli-space questions studied in Chapter 3.

2.4 Examples and Classification Results for LCF Manifolds

We survey the standard examples of LCF manifolds and the known classification results, organized by Yamabe class.

2.4.1 Standard examples of LCF manifolds and their Yamabe class

The following table summarizes the canonical LCF manifolds.

Manifold	Yamabe class	Developing map image	Limit set Λ
$\mathbb{R}^n$	$Y = 0$	$\mathbb{R}^n \subset \mathbb{S}^n$	$\{pt\}$
$(\mathbb{S}^n, g_{\text{round}})$	$Y > 0$ (max)	$\mathbb{S}^n$	$\emptyset$
$\mathbb{S}^n/\Gamma$, $\Gamma \leq O(n+1)$ finite	$Y > 0$	$\mathbb{S}^n$	$\emptyset$
$\mathbb{S}^{n-1} \times \mathbb{R}$	$Y > 0$	$\mathbb{S}^n \setminus \{N, S\}$	$\{N, S\}$
$\mathbb{S}^{n-1} \times \mathbb{S}^1$	$Y > 0$	$\mathbb{S}^n \setminus \{N, S\}$	$\{N, S\}$
$\mathbb{T}^n$ (flat)	$Y = 0$	$\mathbb{R}^n \subset \mathbb{S}^n$	$\{pt\}$
$\mathbb{R}^n/\Gamma$, $\Gamma \leq O(n)$ finite	$Y = 0$	$\mathbb{R}^n$	$\{pt\}$
$\mathbb{H}^n/\Gamma$, Γ cocompact	$Y < 0$	$\mathbb{B}^n \subset \mathbb{S}^n$	$\mathbb{S}^{n-1}$
$\mathbb{S}^k(\kappa) \times \mathbb{H}^{n-k}(\kappa)$ (balanced)	$Y = 0$	$\mathbb{S}^n \setminus \mathbb{S}^{n-k-1}$	$\mathbb{S}^{n-k-1}$
$\mathbb{S}^{n-1} \times \mathbb{S}^1 \# \mathbb{S}^{n-1} \times \mathbb{S}^1$	$Y > 0$	$\mathbb{S}^n \setminus \Lambda$	Cantor Set

Table 2.3: Standard LCF manifolds by Yamabe class.

- **Flat torus** $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$: Flat metric, $W = 0$ trivially; $Y(\mathbb{T}^n) = 0$. The developing map is the universal cover $\mathbb{R}^n \rightarrow \mathbb{R}^n$, with the limit set a single point.
- **Round sphere** $(\mathbb{S}^n, g_{\text{round}})$: Constant curvature $+1$, $W = 0$; the Yamabe constant $Y(\mathbb{S}^n)$ is the maximum over all compact manifolds.
- **Spherical space forms** $\mathbb{S}^n/\Gamma$: Constant curvature $+1$, $Y > 0$. The developing map is the covering $\mathbb{S}^n \rightarrow \mathbb{S}^n/\Gamma$. These are the building blocks of LCF manifolds with PSC (§2.4.2).
- **Round cylinder** $\mathbb{S}^{n-1} \times \mathbb{R}$: Product with curvatures $+1$ and 0 , LCF by Theorem 2.2 (Case 1). The scalar curvature is $R_g = (n-1)(n-2) > 0$ (positive Yamabe class). The developing map sends the cylinder to $\mathbb{S}^n \setminus \{N, S\}$ via stereographic projection from both poles.
- **Hyperbolic space forms** $\mathbb{H}^n/\Gamma$: Constant curvature -1 , $W = 0$; $Y < 0$. The developing map is the Poincaré ball inclusion $\mathbb{H}^n \hookrightarrow \mathbb{B}^n \subset \mathbb{S}^n$, with $\Lambda = \mathbb{S}^{n-1}$ (full boundary sphere).

- **Balanced products** $\mathbb{S}^{n_1}(\kappa) \times \mathbb{H}^{n_2}(\kappa)$ with $(n_1 - 1)\kappa = (n_2 - 1)\kappa$: LCF by Theorem 2.2 (Case 2), $R_g = 0$ (zero Yamabe class).
- **Flat cone** $\mathbb{R}^n/\Gamma$, $\Gamma \leq O(n)$ finite: The model ALE end; $Y = 0$. See Definition 1.6.

The Yamabe class of each example above is not a coincidence: it reflects the global curvature sign and the geometric complexity of the developing map. Among these, the positive Yamabe class is the most rigid and admits the most complete classification.

2.4.2 Classification of LCF manifolds in the positive Yamabe class

Now, we are at the point to state some existing results of classification of LCF manifolds in the positive Yamabe class. First, it is the Izeki's theorem of conformal connected sum.

Theorem 2.17 (Izeki [36]). *Let (M^n, g) be a compact LCF manifold with $\dim_{\mathcal{H}}(\Lambda) < 1$. The M is conformally equivalent to the finite quotient of*

$$\#_{j=1}^m (\mathbb{S}^{n-1} \times \mathbb{S}^1),$$

for some $m \geq 0$. In particular, when $m = 0$, M is conformally equivalent to a spherical space form.

The key analytic input is the size of the limit set Λ of Γ . For compact LCF manifolds with $Y > 0$, Schoen–Yau gives $\dim_{\mathcal{H}}(\Lambda) \leq (n - 2)/2$. In dimensions $n = 3, 4$, to obtain the strict estimate $\dim_{\mathcal{H}}(\Lambda) < 1$, one uses Brooks' non-amenability criterion for coverings (hence positive bottom spectrum) together with the Izeki/Sullivan dimension–spectrum relation [14, 36]; this is exactly the strict inequality needed in the low-dimensional classification step. The finite quotient is coming from the Selberg's Lemma (cf. Theorem 2.11).

Note that this classification also holds for good LCF orbifolds. All the LCF orbifolds with positive Yamabe class are good will be shown in Theorem 3.13. Hence, Theorem 2.17 holds

for orbifolds as well.

To actually classify M in dimension $n = 3$ or 4 , we need the following classification theorems.

2.4.3 Classification in dimension 3

In dimension 3, the topology of compact positive-scalar-curvature manifolds (orbifolds) is classified by Schoen–Yau and Geometrization (Perelman/Kleiner–Lott):

$$M^3 \cong \# (\mathbb{S}^3/\Gamma_i) \# \# (\mathbb{S}^2 \times \mathbb{S}^1).$$

This classification is not LCF-specific, but for compact LCF manifolds (orbifolds) with $Y > 0$, it is fully compatible with the Kleinian/Izeki description and confirms the same topological possibilities in dimension 3.

Theorem 2.18 (3D classification [40, 57]). *Let M^3 be a compact 3-manifold that admits an LCF metric with positive scalar curvature. Then M is conformally equivalent to a connected sum of spherical space forms and copies of $\mathbb{S}^2 \times \mathbb{S}^1$ with compact quotient.*

Proof sketch. The topology comes from the classification of PSC 3-manifolds:

1. By Schoen–Yau, PSC in dimension 3 excludes incompressible tori.
2. By Perelman’s geometrization (in the Kleiner–Lott formulation), compact 3-manifolds with PSC decompose as connected sums of spherical space forms and $\mathbb{S}^2 \times \mathbb{S}^1$ factors.
3. Therefore any compact 3-manifold carrying an LCF PSC metric has exactly this topology.
4. Use the fact that all the manifolds (orbifolds) in the classification admits a LCF metric with PSC.

5. The conformal connected sum is given by [37]. □

The compact three-dimensional topology in the PSC regime is well understood, while the noncompact case provides important models for ALE end geometry used later in Chapter 3.

2.4.4 Classification in dimension 4

In dimension 4, a compact LCF manifold with $Y > 0$ has $W = 0$ and $R > 0$, hence positive isotropic curvature (PIC). The Ricci-flow/PIC classification by Hamilton and Chen–Tang–Zhu, yields the following complete topological classification.

Theorem 2.19 (4D LCF classification [36, 33, 20]). *Let M^4 be a compact 4-manifold (orbifold) that admits an LCF metric with positive scalar curvature. Then M is conformally equivalent to a connected sum of quotients*

$$M \cong \mathbb{S}^4/\Gamma_1 \# \cdots \# \mathbb{S}^4/\Gamma_k \# (\mathbb{S}^3 \times \mathbb{S}^1)/\Gamma'_1 \# \cdots \# (\mathbb{S}^3 \times \mathbb{S}^1)/\Gamma'_m,$$

where Γ_i, Γ'_j are finite groups acting freely on $\mathbb{S}^4$ and $\mathbb{S}^3 \times \mathbb{S}^1$ respectively.

Proof sketch. The argument parallels the 3D case, combining PSC/PIC topology with the Kleinian framework.

1. An LCF metric with $Y > 0$ satisfies $W = 0$ and $R > 0$. In dimension 4, $W = 0$ and $R > 0$ together imply positive isotropic curvature (PIC).
2. By Hamilton’s theorem [33], a compact 4-manifold with PIC is diffeomorphic to a connected sum of $\mathbb{S}^4$ and $\mathbb{S}^3 \times \mathbb{S}^1$ factors. Chen–Tang–Zhu [20] completed the classification in the orbifold setting, using Ricci flow with surgery and the fact the PIC condition is preserved by Ricci flow.
3. [37] then identifies the conformal type with the connected sum of quotients of $\mathbb{S}^4$ and

$$\mathbb{S}^3 \times \mathbb{S}^1.$$

□

The four-dimensional case thus mirrors the three-dimensional picture: the PIC condition plays the role that geometrization plays in dimension 3, and the limit-set estimate $\dim_{\mathcal{H}}(\Lambda) < 1$ is the common bridge between the Kleinian description and the topological classification in both dimensions.

2.4.5 Complete noncompact LCF manifolds with non-negative Ricci curvature

Lastly, we conclude this section by the classification result with LCF and Ricci curvature nonnegative condition. This gives us a very limited case of LCF manifolds.

Theorem 2.20 (Carron-Herzlich[15]). *Let (M^n, g) be a complete non-compact LCF manifold with $\text{Ric}(g) \geq 0$. Then (M, g) is isometric (up to a finite quotient) to $\mathbb{R}^n$, $\mathbb{S}^{n-1} \times \mathbb{R}$, $\mathbb{S}^n$, or a quotient thereof, or globally conformally equivalent to $\mathbb{R}^n$ with a conformal non-flat metric with nonnegative Ricci curvature.*

This is the noncompact analog of the cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$. Together, the compact and non-compact examples in this section provide the palette of LCF manifolds that appear as limits and building blocks in the moduli space theory of Chapter 3.

2.5 Moduli Space Analogy: Teichmüller Space

The study of LCF metrics on a fixed manifold M fits naturally into a broader tradition of parametrizing geometric structures up to equivalence. Before developing the LCF moduli space in Chapter 3, we recall the classical two-dimensional prototype — Teichmüller space — and draw the analogy that motivates the higher-dimensional theory.

2.5.1 Teichmüller space of a surface

Let Σ_g denote a closed orientable surface of genus $g \geq 2$. A *marked complex structure* on Σ_g is an equivalence class of pairs (Σ_g, J) , where J is a complex structure, under orientation-preserving diffeomorphisms isotopic to the identity.

Definition 2.11 (Teichmüller space [34, 25]). *The Teichmüller space $\mathcal{T}_g$ of genus $g \geq 2$ is the space of marked complex structures on Σ_g :*

$$\mathcal{T}_g = \{(\Sigma_g, J)\} / \text{Diff}_0(\Sigma_g),$$

where $\text{Diff}_0(\Sigma_g)$ denotes diffeomorphisms isotopic to the identity. The moduli space of Riemann surfaces is the further quotient

$$\mathcal{M}_g = \mathcal{T}_g / \text{MCG}(\Sigma_g),$$

where $\text{MCG}(\Sigma_g)$ is the mapping class group.

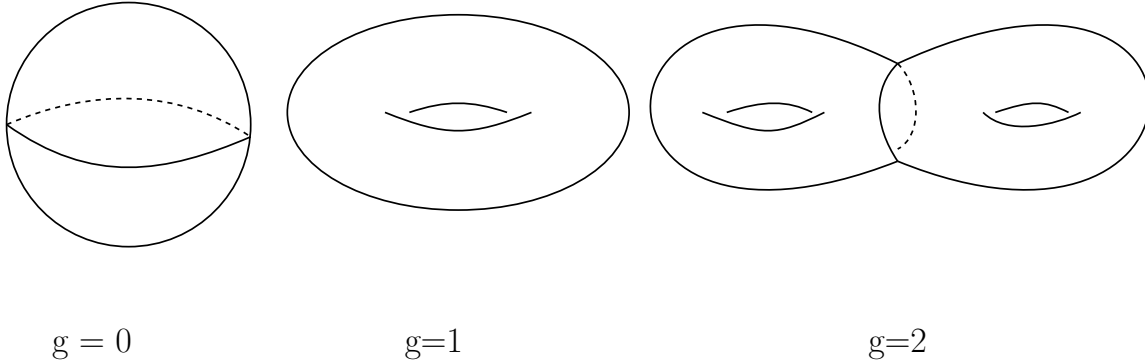


Figure 2.1: Compact Riemann surfaces of genus $g = 0, 1$, and 2 . The Teichmüller space $\mathcal{T}_g$ parametrizes complex (equivalently, constant-curvature) structures on genus- g surfaces.

The key metric-geometric content of $\mathcal{T}_g$ comes from the Uniformization theorem, which identifies complex structures with constant-curvature metrics.

Theorem 2.21 (Uniformization [41, 51]). *Every compact Riemann surface (Σ_g, J) admits a unique (up to scaling) Riemannian metric g_0 of constant curvature that is compatible with J :*

$$K(g_0) = \begin{cases} +1 & g = 0 \quad (\mathbb{S}^2), \\ 0 & g = 1 \quad (\mathbb{T}^2), \\ -1 & g \geq 2 \quad (\text{hyperbolic surface}). \end{cases}$$

In particular, within each conformal class $[g_0]$ on Σ_g there exists a unique unit-area constant-curvature representative.

Thus $\mathcal{T}_g$ can equivalently be viewed as parametrizing *unit-area hyperbolic metrics* on Σ_g (for $g \geq 2$), or *flat metrics* for $g = 1$. One of the central results of two-dimensional geometry is that $\mathcal{T}_g$ is homeomorphic to $\mathbb{R}^{6g-6}$ (for $g \geq 2$) and carries a natural complex structure [34].

The Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ is obtained by allowing *noded* (nodal) surfaces, i.e., surfaces in which simple closed geodesics degenerate to cusps of length zero. The boundary $\partial\overline{\mathcal{M}}_g$ thus consists of lower-complexity Riemann surfaces with orbifold singularities.

2.5.2 The LCF moduli space as a higher-dimensional analog

Remark 2.5 (Analogy to higher dimensions). *The LCF moduli space studied in Chapter 3 is the natural higher-dimensional analog of $\mathcal{T}_g$. The dictionary is as follows.*

The Uniformization theorem (Theorem 2.21) guarantees that within each conformal class on a Riemann surface there is a canonical constant-curvature metric. The Yamabe theorem (Theorem 2.12) plays exactly the same role in higher dimensions: within each conformal class $[g]$ on a compact Riemannian manifold (M^n, g) , there exists a metric of constant scalar curvature. This is why CSC metrics — rather than LCF metrics per se — are the preferred

2D (Riemann surfaces)		nD (LCF manifolds, $n \geq 4$)
Compact surface Σ_g	$\longrightarrow$	Compact LCF manifold $(M^n, [g])$
Complex structure J	$\longrightarrow$	LCF conformal structure $[g]$, $W \equiv 0$
Conformal class $[g_0]$	$\longrightarrow$	Conformal class $[g]$ with $Y(M, [g]) > 0$
Uniformization theorem	$\longrightarrow$	Yamabe theorem (Theorem 2.12)
Constant-curvature metric	$\longrightarrow$	CSC (Yamabe) metric in $[g]$
Teichmüller space $\mathcal{T}_g$	$\longrightarrow$	LCF moduli space $\mathcal{M}$
Deligne-Mumford compactification	$\longrightarrow$	TV orbifold compactification (Ch. 3)
Noded surface (boundary)	$\longrightarrow$	LCF orbifold with ALE bubbles

Table 2.4: Dictionary between Teichmüller theory and LCF moduli theory.

canonical representatives in the moduli space.

The Tian-Viaclovsky compactification theorem (Theorem 3.1, stated and proved in §3.1) is the analog of the Deligne-Mumford compactification: it asserts that the moduli space $\mathcal{M}$ of unit-volume positive-Yamabe LCF metrics with $L^{n/2}$ bounded curvature has compact closure in the Gromov-Hausdorff topology, with boundary consisting of compact LCF orbifolds. The bubble limits — the ALE manifolds that appear at concentration points — play the role of the nodal components in $\partial\overline{\mathcal{M}}_g$.

The analogy breaks down in one important respect: whereas $\mathcal{T}_g \cong \mathbb{R}^{6g-6}$ is a finite-dimensional smooth manifold, the LCF moduli space is infinite-dimensional as a naïve space of metrics. One imposes the curvature bound $\int_M |\text{Rm}|^{n/2} dV_g \leq C$ precisely to recover pre-compactness (via the ε -regularity Lemma 2.4 and the volume comparison Theorem 2.7), bringing the analogy back into focus.

Chapter 3

Locally Conformally Flat Moduli Space

This chapter presents the original results of the thesis, based on the paper *On the locally conformally flat ALE manifolds* [69]. The narrative arc is: set up the moduli space compactification problem (§3.1), observe that blow-up limits are ALE manifolds, study LCF ALE manifolds in depth (§3.2–§3.3), prove the main good-orbifold and classification theorems (§3.4–§3.5), apply those theorems back to describe the moduli space boundary (§3.6), and close with the Positive Mass Theorem and Yamabe problem for LCF orbifolds (§3.7).

3.1 The Moduli Space and Tian-Viaclovsky Compactification

Now, we begin to study the moduli space. Let a compact manifold M^n ($n \geq 3$) carrying a locally conformally flat (LCF) metric of positive scalar curvature. We study the following moduli space.

Definition 3.1 (LCF moduli space). *Let $[g]$ be an LCF conformal class on M with $Y(M, [g_0]) > 0$. For constants $C > 0$, set*

$$\mathcal{M}(n, \mu, C) = \{(M^n, g) \mid R(g) \equiv 1, Y(M, [g]) \geq \mu > 0, \|\text{Rm}_g\|_{L^{n/2}(M, g)} \leq C\}.$$

Elements of $\mathcal{M}$ are regarded up to pull-back by diffeomorphisms isotopic to the identity.

The $L^{n/2}$ curvature bound is the conformally invariant analogue of the L^2 energy in dimension 4; it controls both regularity and topology. The main compactification result is due to Tian-Viaclovsky [64, 65]. An improvement is done in [69].

Theorem 3.1 (Tian-Viaclovsky compactification [64, 65], Wang [69]). *The moduli space $\mathcal{M}(n, \mu_0, C)$, for some $\mu_0 > 0$, has compact closure in the Gromov-Hausdorff topology. Any GH limit is a compact LCF orbifold (M_∞, g_∞) with at most finitely many singular points and $Y(M_\infty, [g_\infty]) > 0$.*

Moreover, each singular point $p \in M_\infty$ has a neighborhood conformally diffeomorphic to a cone $\mathbb{R}^n/\Gamma_p$ for a finite group $\Gamma_p \leq O(n)$; in particular M_∞ is a good orbifold (Theorem 3.13).

Remark 3.1. *The good-orbifold assertion uses Theorem 3.13, proved in §3.4 below. The GH limit exists by the Cheeger-Gromov compactness theorem on the regular region provided by Lemma 3.2; its good-orbifold structure is the new contribution of this thesis.*

The proof of compactness rests on two analytic tools: the ε -regularity lemma and the gap lemma, which together control where curvature can concentrate.

Lemma 3.2 (ε -regularity for LCF metrics (cf. Lemma 2.4)). *There exist $\varepsilon_0, C_0 > 0$ depending*

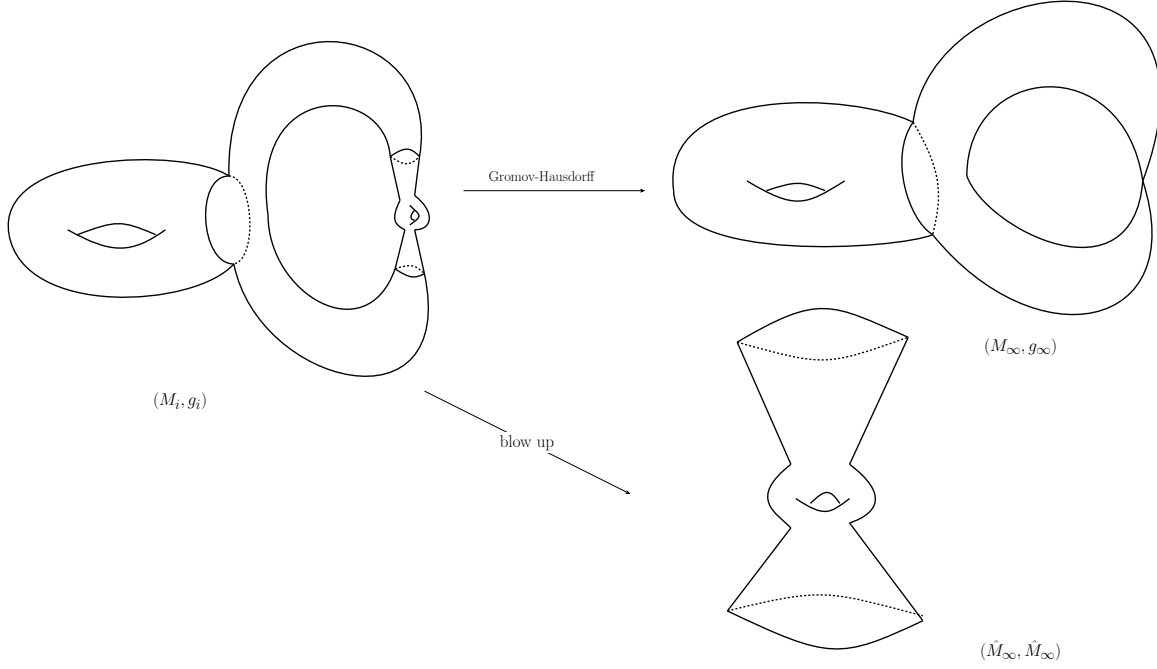


Figure 3.1: A sequence $(M_i, g_i) \in \mathcal{M}$ converging in the Gromov-Hausdorff topology to an LCF orbifold (M_∞, g_∞) . At each concentration point a rescaled bubble $(\hat{M}_\infty, \hat{g}_\infty)$, a non-compact LCF ALE manifold, detaches.

only on n such that: if (M^n, g) is a LCF manifold with $Y(M, [g]) > 0$ and

$$\int_{B_r(x)} |\text{Rm}_g|^{n/2} dV_g < \varepsilon_0,$$

then $\sup_{B_{r/2}(x)} |\text{Rm}_g| \leq C_0 r^{-2}$.

Proof sketch. The Kato inequality (Lemma 2.5) gives $\Delta |\text{Ric}| \geq -C |\text{Rm}| |\text{Ric}|$. Combined with a Moser iteration using the Calderón-Zygmund estimate (Theorem 1.3) on $L^{n/2}$ balls, this yields the pointwise bound. The LCF condition $W \equiv 0$ ensures $|\text{Rm}| \leq C |\text{Ric}|$, closing the iteration. See [64] for the complete argument. $\square$

Using Lemma 3.2, we can prove the following energy gap lemma.

Lemma 3.3 (Gap lemma). *There exists $\varepsilon_1 > 0$ such that: if (M^n, g) is a complete LCF*

manifold with $\int_M |\text{Rm}_g|^{n/2} dV_g < \varepsilon_1$, then g is flat.

Proof. Apply Lemma 3.2 with $r \rightarrow \infty$: the pointwise curvature bound $\sup_{B_r} |\text{Rm}| \leq C_0 r^{-2} \rightarrow 0$ forces $\text{Rm} \equiv 0$, hence g is flat. $\square$

By these two lemmas, curvature concentration in any sequence $g_i \in \mathcal{M}$ can occur at only finitely many points (where $\int_{B_r} |\text{Rm}|^{n/2} \geq \varepsilon_0$ for every $r > 0$). At each such concentration point the following blow-up analysis applies.

Lemma 3.4 (ALE blow-up lemma). *Let $(M_i, g_i) \in \mathcal{M}$ be a sequence with a concentration point $x_i \rightarrow x_\infty$. Set $r_i = |\text{Rm}_{g_i}(x_i)|^{-1/2} \rightarrow 0$. Then the rescaled sequence $(M_i, r_i^{-2} g_i, x_i)$ has a subsequence converging in C_{loc}^∞ to a complete non-compact LCF ALE orbifold $(\Sigma, g_\Sigma, x_\infty)$ with*

$$\int_\Sigma |\text{Rm}_{g_\Sigma}|^{n/2} dV_{g_\Sigma} < \infty, \quad |\text{Rm}_{g_\Sigma}(x_\infty)| = 1.$$

Proof. Rescaling normalizes curvature at x_i to 1. The $L^{n/2}$ bound and the TV volume comparison (Theorem 2.7) give volume non-collapse and ALE asymptotics at infinity. Cheeger-Gromov compactness (Theorem 1.7) then yields a C_{loc}^∞ limit, which inherits $W \equiv 0$. $\square$

At a different scaling scale, we can use these two lemmas to conclude the curvature concentration region will look like the ALE model.

Theorem 3.5 (Finite bubble tree structure). *Let $(M_i, g_i) \in \mathcal{M}$ be a sequence converging in the Gromov-Hausdorff sense to (M_∞, g_∞) , and let $S \subset M_\infty$ be the finite set of concentration points. There exists a finite rooted tree $\mathcal{T}$ — the bubble tree — with the following properties:*

- (i) *The root node corresponds to (M_∞, g_∞) .*
- (ii) *Each non-root node corresponds to a complete non-compact LCF ALE orbifold $(\Sigma_\alpha, g_\alpha)$*

with $\int_{\Sigma_\alpha} |\text{Rm}_{g_\alpha}|^{n/2} dV < \infty$.

(iii) Each leaf node corresponds to a complete non-compact smooth LCF ALE manifold.

(iv) Each edge of $\mathcal{T}$ connects a parent node to a child node via a neck region diffeomorphic to $(S^{n-1}/\Gamma) \times (0, 1)$ for a finite group $\Gamma \leq O(n)$.

(v) The total number of non-root nodes is at most C/ε_1 , where C is the $L^{n/2}$ curvature bound in $\mathcal{M}$ and ε_1 is the gap constant of Lemma 3.3.

(vi) In particular, a neighborhood of singular point is diffeomorphic to an ALE manifold.

Proof. We use the bubble tree scaling argument of Bando [9]; see also [66].

Step 1: First-level bubbles (children of the root). Fix $x \in S$ and let $x_i \rightarrow x$ with $B(x_i, r) \rightarrow B(x, r)$ in the Gromov-Hausdorff sense. Choose $r_\infty > 0$ small enough that

$$\sup_{B(x_i, r_\infty)} |\text{Rm}(g_i)| = |\text{Rm}(g_i)|(x_i) \rightarrow \infty, \quad \int_{B(x, r_\infty)} |\text{Rm}(g_\infty)|^{\frac{n}{2}} dv \leq \frac{\varepsilon_0}{2},$$

where ε_0 is the constant from Lemma 3.2.

Denote $A(r_1, r_2) = B(x, r_2) \setminus \overline{B(x, r_1)}$. Choose $r(i) < r_\infty$ such that

$$\int_{A_i(r(i), r_\infty)} |\text{Rm}(g_i)|^{\frac{n}{2}} dv = \varepsilon_0.$$

Since curvature concentrates at x , we have $r(i) \rightarrow 0$, so for i large and any fixed $0 < r_0 < r_\infty$ we have $r(i) < r_0$.

Neck region. For any fixed $r_0 \in (0, r_\infty)$ the annulus $A_i(r_0, r_\infty/2)$ lies entirely above scale $r(i)$, so $\int_{A_i(r_0, r_\infty/2)} |\text{Rm}|^{n/2} < \varepsilon_0$. Lemma 3.2 then gives a pointwise curvature bound on $A_i(r_0, r_\infty/2)$; the LCF condition $W \equiv 0$ promotes this via the elliptic system for Ricci

curvature to C^∞ convergence $(A_i(r_0, r_\infty/2), g_i) \xrightarrow{C^\infty} (A_\infty(r_0, r_\infty/2), g_\infty)$, and the neck is diffeomorphic to $(S^{n-1}/\Gamma_x) \times (0, 1)$.

First bubble. Consider the rescaled sequence $(M_i, r(i)^{-2} g_i, x_i)$. In the rescaled metric the annulus $A_i(r(i), r_\infty)$ becomes $A_i(1, r_\infty/r(i))$, which exhausts $A(1, \infty)$ as $r(i) \rightarrow 0$. By the choice of $r(i)$, the $L^{n/2}$ curvature energy in this region is exactly ε_0 ; applying Lemma 3.2 ball by ball outside B_1 gives a uniform curvature bound there, while the total $L^{n/2}$ energy bound on all of $(M_i, r(i)^{-2} g_i)$ and the TV volume comparison (Theorem 2.7) ensure volume non-collapse and ALE asymptotics at infinity. Thus $(M_i, r(i)^{-2} g_i, x_i)$ subconverges in C_{loc}^∞ to a complete LCF ALE orbifold $(\Sigma^{(1)}, g^{(1)})$ with $\int_{\Sigma^{(1)}} |\text{Rm}_{g^{(1)}}|^{n/2} dV < \infty$.

Step 2: Iterated blow-up. The first-level bubble $\Sigma^{(1)}$ may itself have curvature concentration points, corresponding to additional scales of blow-up inside $B(x_i, r(i))$ that are not resolved at scale $r(i)$. At any such concentration point $y \in \Sigma^{(1)}$, let $y_i \in M_i$ be a sequence of points with $y_i \rightarrow y$ in the rescaled metric $r(i)^{-2} g_i$. Define $\rho(i) < R_\infty$ (for a suitably chosen $R_\infty > 0$ in the rescaled coordinates) by

$$\int_{A_i^{(1)}(\rho(i), R_\infty)} |\text{Rm}(r(i)^{-2} g_i)|^{\frac{n}{2}} dv_{r(i)^{-2} g_i} = \varepsilon_0,$$

where $A_i^{(1)}$ denotes the annulus in the rescaled metric centered at y_i . Since $r(i)^{-2} g_i$ is itself a rescaling of g_i , the further rescaling $\rho(i)^{-2}(r(i)^{-2} g_i)$ is equivalent to working at the combined scale $r(i)\rho(i)$ in the original metric g_i . The same argument as Step 1 shows that $(M_i, (r(i)\rho(i))^{-2} g_i, y_i)$ subconverges to a second-level LCF ALE orbifold $(\Sigma^{(2)}, g^{(2)})$, connected to $\Sigma^{(1)}$ by a neck $(S^{n-1}/\Gamma_y) \times (0, 1)$. We continue inductively: at each level k , any concentration point of $\Sigma^{(k)}$ yields a scale $r^{(k)}(i) \rightarrow 0$ in the level- k rescaled metric, and the combined scale produces the next bubble $\Sigma^{(k+1)}$.

Step 3: Finiteness and smooth leaf nodes. By Lemma 3.3, every non-flat complete LCF manifold satisfies $\int |\text{Rm}|^{n/2} dV \geq \varepsilon_1$. Each bubble $\Sigma^{(k)}$ at any level is a distinct non-

flat limit contributing at least ε_1 to the total $L^{n/2}$ energy of the original sequence, and these contributions are disjoint (they arise from disjoint scales and regions). Since the total energy is bounded by C , the tree $\mathcal{T}$ has at most C/ε_1 non-root nodes, and the induction terminates in finitely many steps.

At a leaf node $\Sigma^{(k)}$, no further concentration points exist, so the rescaled sequence at that level has uniformly bounded curvature everywhere. In particular, there are no interior orbifold points (which would correspond to further blow-ups), and $\Sigma^{(k)}$ is a smooth complete LCF ALE manifold. $\square$

Lemma 3.4 shows that LCF ALE manifolds and orbifolds are the fundamental building blocks of the moduli space boundary. We now study their structure in depth. The key tools are the totally umbilic geometry of ALE ends, the harmonic expansion of the conformal factor, and the injectivity of the end group — all developed in §3.2–§3.3.

3.2 Locally Conformally Flat ALE Manifolds

The blow-up limits produced by Lemma 3.4 are non-compact LCF ALE manifolds (or orbifolds). In this section we analyze their end structure using the developing map as the central tool. Recall Definition 1.6: an ALE manifold of order $\tau > 0$ has an end diffeomorphic to $(\mathbb{R}^n \setminus B)/\Gamma$ with $\Gamma \leq O(n)$ finite and $g_{ij} = \delta_{ij} + O(r^{-\tau})$, $\partial^m g_{ij} = O(r^{-\tau-|m|})$.

Definition 3.2 (LCF ALE manifold). *A LCF ALE manifold is a complete Riemannian manifold (M^n, g) that is both locally conformally flat (Definition 2.1) and asymptotically locally Euclidean (Definition 1.6).*

3.2.1 Totally umbilic submanifolds

To study the conformal embedding, we need to understand the conformal invariant geometric objects. Motivated by this, we study the conformal invariant submanifolds.

Definition 3.3 (Totally umbilic submanifold). *A hypersurface $N^{n-1} \subset (M^n, g)$ is totally umbilic if $\Pi_x = \lambda(x) g|_N$ for some $\lambda \in C^\infty(N)$.*

Totally umbilic submanifolds are conformally invariant: under a conformal change $\hat{g} = e^{2\psi}g$, the second fundamental form transforms as

$$\hat{\Pi}(X, Y) = \Pi(X, Y) - \hat{g}(X, Y) \nabla \psi|_{\text{normal}},$$

so umbilic points are preserved.

It can be easily seen that the totally umbilic submanifolds are invariant under conformal change. Note that this invariance is pointwise, i.e., the umbilic points are invariant under conformal change.

For (M, g) , if there is a conformal embedding $\phi : M \rightarrow \mathbb{S}^n$, and (M, g) is not conformally equivalent to $(\mathbb{S}^n, g_{\mathbb{S}^n})$, then by the stereographic projection, we have a conformal embedding $\bar{\phi} : M \rightarrow \mathbb{R}^n$. Thus, all the totally umbilic submanifolds in (M, g) are totally umbilic in $\mathbb{R}^n$, which gives a very strong constraint.

Recall the classical theorem of totally umbilic submanifolds in $\mathbb{R}^n$. This is an old theorem that dates back to Cartan. Since it is difficult to find the precise reference, we include the proof as well.

Theorem 3.6 (Totally umbilic hypersurfaces in $\mathbb{R}^n$). *Let $n \geq 3$. The only closed totally umbilic hypersurfaces in $\mathbb{R}^n$ are round spheres $\mathbb{S}^{n-1}(r)$.*

Proof. Let $i : N \hookrightarrow \mathbb{R}^n$ be a closed totally umbilic hypersurface with $\text{II} = \lambda g|_N$.

Step 1: λ is constant. By the Codazzi equation, for tangential vector fields X, Y, Z on N :

$$(\nabla_X \text{II})(Y, Z) = (\nabla_Y \text{II})(X, Z).$$

Since $\text{II} = \lambda g|_N$, the left side equals $(X\lambda)g(Y, Z)$ and the right side equals $(Y\lambda)g(X, Z)$. Setting $Y = Z$ to a unit vector perpendicular to X gives $(X\lambda) = 0$ for all X tangent to N , so λ is constant on N .

Step 2: Identifying the center. For $\lambda > 0$, define

$$c := i(x) + \frac{1}{\lambda} \vec{\nu}(x),$$

where $\vec{\nu}$ is the unit inward normal. Then for any tangential X :

$$\nabla_X c = \nabla_X i + \frac{1}{\lambda} \nabla_X \vec{\nu} = X - \frac{1}{\lambda} A(X) = X - \frac{1}{\lambda} (\lambda X) = 0,$$

using $A(X) = \lambda X$ (since $\text{II}(X, \cdot) = \lambda g(X, \cdot)$). Hence c is a constant vector (the center), and $|i(x) - c| = 1/\lambda$ for all $x \in N$. Thus $i(N) \subset \mathbb{S}^{n-1}(1/\lambda)$. Since N is a closed connected hypersurface inside this sphere and has the same dimension, $i(N) = \mathbb{S}^{n-1}(1/\lambda)$. $\square$

Combined with Theorem 2.10, we have, if (M, g) is an LCF manifold with non-negative scalar curvature, then the only closed totally umbilic submanifolds in the universal covers are the conformal round spheres.

Remark 3.2. *In the ALE setting, the large coordinate spheres $\partial B_r/\Gamma \cong \mathbb{S}^{n-1}/\Gamma$ become asymptotically totally umbilic as $r \rightarrow \infty$. By the conformal invariance in Definition 3.3, their images under the developing map $\phi : \tilde{M} \rightarrow \mathbb{S}^n$ are asymptotically round spheres. This is the key geometric observation exploited throughout this section.*

The totally umbilic structure of large coordinate spheres is a *qualitative* constraint: it tells us the shape of the ALE end in the developing map image. To get *quantitative* control on the conformal factor and ultimately to read off the ADM mass, we need the asymptotic expansion of harmonic functions on ALE manifolds.

3.2.2 Harmonic functions on ALE manifolds

Now, we recall the theory of harmonic functions in ALE manifolds. This is very similar to the case of Lemma 1.4.

Theorem 3.7 (Harmonic expansion on ALE manifolds [22]). *Let (M^n, g) be an ALE manifold of order $\tau > 0$, and let u be a bounded function that satisfies $\Delta_g u = 0$ outside a compact set on one ALE end. Then u has an asymptotic expansion on that end:*

$$u(x) = c + A r^{2-n} + O(r^{2-n-\varepsilon}),$$

as $r = d(x, o) \rightarrow \infty$, for constants $c, A \in \mathbb{R}$ and some $\varepsilon > 0$. The same statement holds for the equation $\Delta u + \kappa S_g u = 0$ (with $S_g \in O(r^{-2-\tau})$).

Proof. We decompose $u = u_0 + u_1$ on the ALE end outside a compact set, where u_0 satisfies $\Delta_{\mathbb{R}^n/\Gamma} u_0 = 0$ and $u_1 = O(r^{-\varepsilon})$. This decomposition is achieved by solving the Poisson equation for the metric perturbation term in a weighted Sobolev space: since $g_{ij} - \delta_{ij} \in O(r^{-\tau})$, the operator $\Delta_g - \Delta_{\mathbb{R}^n}$ is a lower-order perturbation with decay, and the L^2 -theory on ALE spaces (using the weighted Sobolev embedding) gives $u_1 \in O(r^{-\varepsilon})$ for some $\varepsilon > 0$ depending on τ .

For the Euclidean part u_0 : bounded harmonic functions on $\mathbb{R}^n/\Gamma$ lift to bounded harmonic functions on $\mathbb{R}^n \setminus B$. By Lemma 1.4 (spherical harmonic expansion on $\mathbb{R}^n$), every bounded harmonic function outside a ball expands as $u_0 = c + \sum_{\ell \geq 0} h_\ell$, where h_ℓ is a spherical

harmonic of degree ℓ times $r^{2-n-\ell}$ (for $\ell \geq 1$) or a constant (for $\ell = 0$). The only term that can fail to decay as $r \rightarrow \infty$ with rate better than r^{2-n} is the $\ell = 0$ mode. Hence $u_0 = c + Ar^{2-n} + O(r^{2-n-1})$. Iterating the estimate on u_1 with the Calderón-Zygmund estimate (Theorem 1.3) on successive annuli gives the stated $\varepsilon > 0$. $\square$

The harmonic expansion of Theorem 3.7 provides the precise asymptotics of the conformal factor at each ALE end. Combining this with the totally umbilic sphere argument, we can now show that every LCF ALE manifold with non-negative scalar curvature is actually AE: the group Γ at each end must be trivial.

3.2.3 LCF ALE manifolds are AE

Now, we can prove the main theorem for LCF ALE manifolds structure.

Theorem 3.8 (LCF ALE manifolds are AE). *Let (M^n, g) be a LCF ALE manifold with $R_g \geq 0$. Suppose the universal cover $\tilde{M}$ admits a conformal embedding $\phi : \tilde{M} \hookrightarrow \mathbb{S}^n$. Then (M, g) is asymptotically Euclidean (AE): every ALE end has trivial group $\Gamma = \{e\}$.*

Proof. Without loss of generality assume (M, g) has one ALE end with group $\Gamma \leq O(n)$. We produce a totally umbilic $\mathbb{S}^{n-1}/\Gamma$ inside (M, g) for large scales, then apply Theorem 3.6 to force $\Gamma = \{e\}$.

Step 1: Conformal factor. Since $\phi(M) \subsetneq \mathbb{S}^n$, pick $p \in \mathbb{S}^n \setminus \phi(M)$ and apply stereographic projection $\pi_p : \mathbb{S}^n \setminus \{p\} \rightarrow \mathbb{R}^n$. The composition $\bar{\phi} = \pi_p \circ \phi : M \rightarrow \mathbb{R}^n$ is a conformal embedding with $\bar{\phi}^*(g_{\mathbb{R}^n}) = u^{4/(n-2)}g$ for some $u \in C^\infty(M)$, $u > 0$. Since $R_g \geq 0$, the function u satisfies

$$-\Delta_g u + a_n R_g u = 0, \quad a_n = \frac{n-2}{4(n-1)},$$

with u bounded away from a fixed base point. By Theorem 3.7 (applied to the equation

$\Delta u + a_n S u = 0$ with $S = O(r^{-2-\tau})$, on the ALE end:

$$u = r^{2-n} + O(r^{2-n-\varepsilon}), \quad \varepsilon > 0,$$

after normalizing so that the leading coefficient $A = 1$.

Step 2: Conformal comparison on an annulus. Fix $R \gg 1$ and consider the annular region in the ALE chart:

$$A_g(R, 4R) = \{x \in M : R \leq \text{dist}_g(x, z) \leq 4R\}.$$

In the inverted coordinate $\rho = 1/r$ and conformal metric $\hat{g} = r^{-4}g$, we have $A_g(R, 4R) = A_{\hat{g}}(1/(4R), 1/R)$. Adding the compactification point x_0 (corresponding to the end), the metric $g_0 = u^{4/(n-2)}g$ satisfies

$$g_0 = (1 + O(\rho^\varepsilon))\hat{g} \quad \text{near } x_0,$$

so $(B(x_0, \delta), g_0) \subset (\mathbb{R}^n/\Gamma, g_{\text{flat}})$ is part of the flat cone with vertex at x_0 .

Claim: For R large enough, there is a conformal embedding of $A(2R, 3R) \subset (\mathbb{R}^n/\Gamma, g_{\text{flat}})$ into $(A_g(R, 4R), g)$.

Proof of Claim: The ratio of the two distance functions $q(x) = \text{dist}_{g_0}(x, x_0)/\text{dist}_{\hat{g}}(x, x_0)$ is continuous on $B(x_0, \delta) \setminus \{x_0\}$ with $\lim_{x \rightarrow x_0} q(x) = 1$. Hence there exists $\eta(\delta) \rightarrow 0$ such that $(1 - \eta(\delta))\rho \leq \text{dist}_{g_0}(x, x_0) \leq (1 + \eta(\delta))\rho$ for $x \in B(x_0, \delta)$. For $R = 1/\delta$ large enough, the claim follows.

Step 3: Totally umbilic sphere. On the flat cone $([2R, 3R] \times \mathbb{S}^{n-1}/\Gamma, dt^2 + t^2 g_{\mathbb{S}^{n-1}/\Gamma})$,

the hypersurface $\{5R/2\} \times \mathbb{S}^{n-1}/\Gamma$ has second fundamental form

$$\Pi = \frac{2}{t} g|_{\{5R/2\} \times \mathbb{S}^{n-1}/\Gamma} = \frac{4}{5R} g|_{\{5R/2\} \times \mathbb{S}^{n-1}/\Gamma},$$

so it is totally umbilic. Since totally umbilic submanifolds are conformally invariant (Definition 3.3), the conformal embedding of Step 2 transfers this to a totally umbilic $N \cong \mathbb{S}^{n-1}/\Gamma$ inside $(A_g(R, 4R), g)$.

Step 4: Contradiction unless $\Gamma = \{e\}$. By the Schoen-Yau embedding (Theorem 2.10), $\phi(N)$ is a totally umbilic submanifold of $\mathbb{S}^n$, hence a round sphere $\mathbb{S}^{n-1}$ (the Möbius image of a totally umbilic sphere remains a sphere). Since $N \cong \mathbb{S}^{n-1}/\Gamma$ is the quotient of a sphere by Γ , and $\phi(N) \cong \mathbb{S}^{n-1}$, we need $\mathbb{S}^{n-1}/\Gamma \cong \mathbb{S}^{n-1}$, which forces $\Gamma = \{e\}$. Repeating for each ALE end completes the proof. $\square$

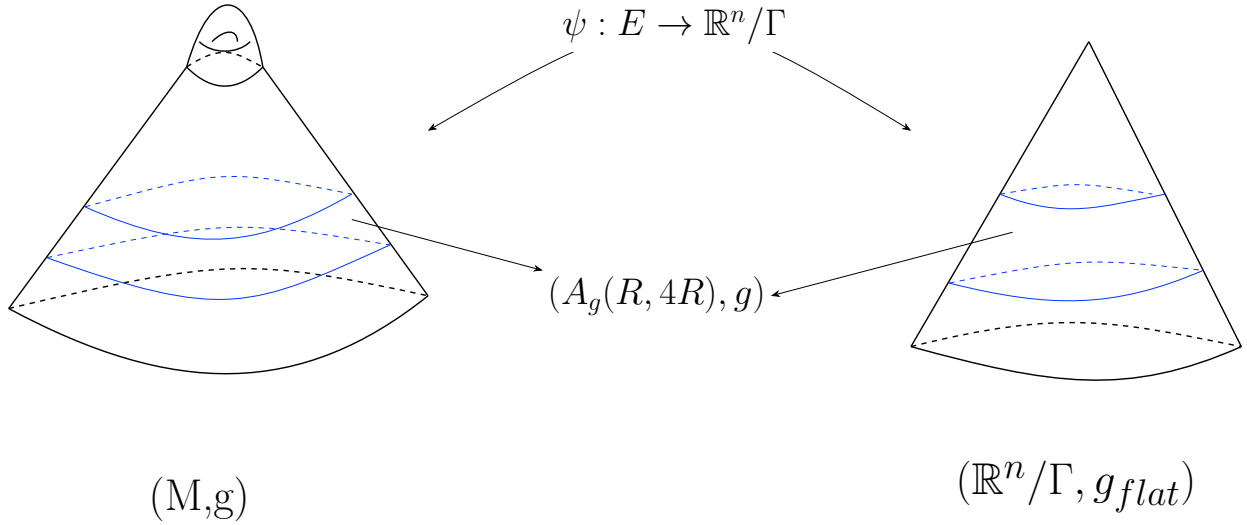


Figure 3.2: An ALE manifold (M, g) and its end E , diffeomorphic to $(\mathbb{R}^n \setminus B)/\Gamma$. The two colored curves are large coordinate spheres that become totally umbilic under the conformal map $\bar{\phi} : M \rightarrow \mathbb{R}^n$.

Theorem 3.8 shows that every universal cover of an ALE manifold has trivial group: each ALE end is actually AE. It follows that the AE ends of the universal cover, under the developing map, all converge to isolated points in $\mathbb{S}^n$. We now use this to give a global

description of the universal cover.

3.2.4 Universal cover as a domain in $\mathbb{S}^n$

In addition, we can prove the following structure theorem for the universal cover of the LCF, ALE manifolds.

Proposition 3.9 (Universal cover of a LCF ALE manifold). *Let (M^n, g) be a LCF ALE manifold with $R_g \geq 0$. The universal cover $\tilde{M}$ is diffeomorphic to $\mathbb{S}^n \setminus (\Lambda \cup I)$, where $\Lambda \subset \mathbb{S}^n$ is the limit set of $\Gamma = \rho(\pi_1(M))$ and I is a discrete set of points corresponding to the AE ends of $\tilde{M}$.*

Proof. By Theorem 2.10, the developing map gives a conformal embedding $\phi : \tilde{M} \hookrightarrow \mathbb{S}^n$. By Theorem 3.8, every ALE end of $\tilde{M}$ is actually AE (trivial group at infinity).

Step 1: One-point compactification of each end. Fix an AE end. For each $k \geq 1$, consider the annulus

$$A_k = \{x \in \tilde{M} : kr \leq \text{dist}(x, z) \leq (k+1)r\}.$$

By the AE condition and the ALE chart, the slice $\{(k + \frac{1}{2})r\} \times \mathbb{S}^{n-1}$ is $C^{1,\alpha}$ -close to a round sphere for large k . Applying the totally umbilic argument from the proof of Theorem 3.8, there exist totally umbilic spheres $N_k \cong \mathbb{S}^{n-1}$ in A_k . Let $C_k \subset \tilde{M}$ be the compact region bounded by N_k (containing the base point z); then $C_1 \subset C_2 \subset \dots$ exhausts the end.

Step 2: Convergence to a point. Under ϕ , the images $S_k = \phi(N_k) \cong \mathbb{S}^{n-1}$ are pairwise disjoint round spheres in $\mathbb{S}^n$ with $\phi(C_k) \subset \phi(C_{k+1})$. By Theorem 2.7 of [59] applied to the developing map, $\lim_{k \rightarrow \infty} \text{rad}(S_k) = 0$. Hence $\bigcap_k \overline{\phi(C_k^c)}$ is a single point $\{x_i\} \in \mathbb{S}^n$: the AE end converges to x_i .

Step 3: Compactified universal cover. Adding the compactification points, set

$$\overline{M} = \phi(\tilde{M}) \cup \{x_i : i \in I\},$$

with the spherical metric. By van Kampen's theorem and $n \geq 3$, adding finitely many isolated points does not change simple connectivity, so $\overline{M}$ is simply connected. The deck transformation group Γ extends to a properly discontinuous action on $\overline{M}$.

Since Γ acts properly discontinuously on $\mathbb{S}^n \setminus \Lambda$, we have $\phi(\tilde{M}) \subset \mathbb{S}^n \setminus \Lambda$. Each compactification point x_i satisfies $x_i \notin \Lambda$: if $x_i \in \Lambda$, it would be a limit of orbit points, but the AE end converges to x_i non-periodically, contradicting proper discontinuity. Hence $\tilde{M} \cong \mathbb{S}^n \setminus (\Lambda \cup I)$, where $I = \{x_i\}$ is discrete. $\square$

3.3 Structure Theory of LCF ALE Manifolds and Orbifolds

Having embedded the universal cover $\tilde{M}$ as a domain in $\mathbb{S}^n$ (Proposition 3.9), we now analyze the Kleinian group action on that domain. The main results are the injectivity theorem (Theorem 3.10) and the manifold cover (Proposition 3.12).

3.3.1 Injectivity theorem

Theorem 3.10 (Injectivity of the end group [69]). *Let (M^n, g) be a complete LCF ALE manifold with $R_g \geq 0$, and let E_k be an ALE end with group $\Gamma_k \leq O(n)$. Then the inclusion $i_k : E_k \hookrightarrow M$ induces an injective homomorphism*

$$(i_k)_* : \Gamma_k \hookrightarrow \pi_1(M).$$

Proof. Consider

$$S_r = \{x : x \text{ in the } k\text{-th end, } \text{dist}(x, z) = r\}$$

to be the distance sphere. For r large enough, $(S_r, j^*g/r^2)$ and $(\mathbb{S}^{n-1}/\Gamma_k, g_{\mathbb{S}^{n-1}/\Gamma_k})$ are $C^{1,\alpha}$ -close by the ALE condition, so $S_r \cong \mathbb{S}^{n-1}/\Gamma_k$ and $\pi_1(S_r) \cong \Gamma_k$. (Here j^*g denotes the pullback metric under $j : S_r \hookrightarrow M$ in the tangential directions.)

Let $(\tilde{M}, p^*g)$ be the Riemannian universal cover with covering map $p : \tilde{M} \rightarrow M$. Choose $z_0 \in p^{-1}(z)$; by lifting geodesics from z_0 , we obtain a lift $\tilde{S}_r \subset \tilde{M}$ of S_r , and $p|_{\tilde{S}_r} : \tilde{S}_r \rightarrow S_r$ is a covering map. By Proposition 3.9, $\tilde{M}$ has only AE ends (trivial group), so for r large, $\tilde{S}_r$ lies inside an AE end of $\tilde{M}$ and is $C^{1,\alpha}$ -close to the round sphere; hence $\tilde{S}_r \cong \mathbb{S}^{n-1}$. Since $\pi_1(\tilde{S}_r) = \{e\}$, $\tilde{S}_r$ is the universal cover of S_r .

Now let $\gamma \in \ker(i_k)_* \leq \Gamma_k \cong \pi_1(S_r)$. Since $(i_k)_*(\gamma)$ is trivial in $\pi_1(M)$, any loop $\ell \subset S_r$ representing γ is null-homotopic in M . Hence γ acts trivially as a deck transformation of $\tilde{M}$, and the lift $\tilde{\ell} \subset \tilde{S}_r$ of ℓ is a closed loop. Since $\pi_1(\tilde{S}_r) = \pi_1(\mathbb{S}^{n-1}) = \{e\}$ for $n \geq 3$, $\tilde{\ell}$ is null-homotopic in $\tilde{S}_r$, so ℓ is null-homotopic in S_r , giving $[\gamma] = e$ in $\pi_1(S_r) = \Gamma_k$. Therefore $(i_k)_*$ is injective. $\square$

3.3.2 Local isotropy groups

In particular, for each $\gamma \in \Gamma_k < \pi_1(M)$, acting as a deck transformation, it fixes the compactification point x_k . It therefore induces a conformal map on a small ball B_ϵ centered at x_k , and by the Liouville theorem (Theorem 2.9) this map extends uniquely to a Möbius transformation fixing x_k . Thus:

Corollary 3.11. *For each $\Gamma_k < \pi_1(M)$, the holonomy image satisfies $\rho(\Gamma_k) < C(n)$, and $\rho(\Gamma_k)$ fixes $\{x_k\}$, where $C(n)$ is the conformal group of $\mathbb{S}^n$.*

Note that the action $\rho(\pi_1(M))$ extends to the conformal compactification $\overline{M}$. Moreover, this action is properly discontinuous, so

$$M' := \overline{M}/\rho(\pi_1(M))$$

is a compact orbifold (cf. Definition 1.3). In particular,

$$M' = M \cup \{x_1, \dots, x_m\},$$

which is the one-point compactification of each end of M . Since M' is a quotient of the manifold $\overline{M}$, it is a good orbifold with each local group Γ_i injecting into $\pi_1(M)$.

Since $M = \Omega/\Gamma$ for some open set $\Omega \subseteq \mathbb{S}^n$ and a discrete group $\Gamma \leq C(n)$ (Definition 2.5), we say that M' is a **Kleinian orbifold** (see §2.1). Thus, M' is a Kleinian orbifold.

3.3.3 Finite manifold cover

Now, apply Selberg's lemma (Theorem 2.11), we have the following fact about Kleinian orbifold.

Proposition 3.12 (Existence of a manifold cover). *Let M' be a good Kleinian orbifold. Then there exists a compact Kleinian manifold N such that*

$$p : N \rightarrow M'$$

is a finite covering map.

Proof. Since $M = \Omega/\Gamma$ for some open set $\Omega \subseteq \mathbb{S}^n$ and $\Gamma \leq C(n)$, by Theorem 2.11 there exists a normal subgroup $H \trianglelefteq \Gamma$ of finite index that is torsion-free. Note that a torsion-free element of $C(n)$ acts without fixed points (Definition 2.6). Thus H acts freely on Ω , and the

quotient $N \cong \Omega/H$ is a compact smooth manifold whose fundamental group is isomorphic to H . The covering map $N \rightarrow M' \cong \Omega/\Gamma$ has finite degree $[\Gamma : H] < \infty$. $\square$

3.4 The Good Orbifold Theorem

Now, we link the geometry of LCF orbifolds with positive scalar curvature with ALE manifolds of nonnegative scalar curvature. In the following, we show that the conformal compactification of the nonnegative scalar curvature, LCF, ALE manifolds is a compact LCF orbifold with positive Yamabe invariant, assembling the results of §3.2–§3.3.

Theorem 3.13 (Good orbifold theorem). *Every compact LCF orbifold (M^n, g) with $R_g > 0$ and $n \geq 3$ is a good orbifold: there exists a smooth compact manifold $\tilde{M}$ and a finite group Γ acting smoothly and freely on $\tilde{M}$ with $M \cong \tilde{M}/\Gamma$.*

Proof. Let $\{x_i\}_{i=1}^m$ be the orbifold singular points of M .

Step 1: Conformal blow-up. Since $R_g > 0$, the conformal Laplacian $L_g = -a_n \Delta_g + R_g$ (with $a_n = 4(n-1)/(n-2)$) is positive. For each orbifold singular point x_i , there exists a positive Green's function G_i for L_g with pole at x_i . The superposition $G = \sum_i G_i > 0$ gives a conformal metric

$$\tilde{g} = G^{\frac{4}{n-2}} g$$

on $M \setminus \{x_i\}$. Near each x_i , the Green's function has a pole; pulling back to the local orbifold chart $\mathbb{R}^n/\Gamma_{x_i}$, the blown-up metric $(B(x_i, \varepsilon) \setminus \{x_i\}, \tilde{g})$ is a complete ALE orbifold end with group Γ_{x_i} and order $n-2$ (see Corollary 3.14).

Thus $(M \setminus \{x_i\}, \tilde{g})$ is a multi-ALE manifold of order $n-2$, and $\pi_1^{\text{orb}}(M) \cong \pi_1(M \setminus \{x_i\})$.

Step 2: Injectivity of local isotropy. By Proposition 3.9 applied to the multi-ALE man-

ifold $(M \setminus \{x_i\}, \tilde{g})$, the universal cover $\widetilde{M \setminus \{x_i\}}$ embeds as a domain in $\mathbb{S}^n$. By Theorem 3.10 and Corollary 3.11, each local isotropy group Γ_{x_i} injects into $\pi_1(M \setminus \{x_i\}) \cong \pi_1^{\text{orb}}(M)$.

Step 3: Kleinian good orbifold. By the orbifold goodness criterion [63], an orbifold is good if and only if each local isotropy group injects into the orbifold fundamental group. Step 2 provides exactly this injectivity, so M is a good Kleinian orbifold: $M = \Omega/\Gamma$ for a domain $\Omega \subset \mathbb{S}^n$ and a discrete group $\Gamma \leq C(n)$.

Step 4: Smooth manifold cover. By Proposition 3.12 (Selberg lemma), there exists a finite-index torsion-free normal subgroup $H \trianglelefteq \Gamma$, and the quotient $N = \Omega/H$ is a smooth manifold covering M with finite degree $[\Gamma : H] < \infty$. $\square$

Corollary 3.14 (Optimal ALE decay). *Let (M^n, g) be a LCF scalar-flat ALE manifold of order $\tau > 0$. Then there exist charts at infinity for every end in which*

$$g_{ij} = \delta_{ij} + O(r^{-(n-2)}), \quad \partial^m g_{ij} = O(r^{-(n-2)-|m|}),$$

i.e. (M, g) is ALE of order $n-2$. If (M, g) has $R_g \geq 0$, then a conformal metric $\tilde{g} = u^{4/(n-2)}g$ exists that is scalar-flat and ALE of order $n-2$.

Proof. Since $R_g = 0$ and (M, g) is LCF, the conformal Laplacian is $L_g = -a_n \Delta_g$. By Theorem 3.15 (proved below), (M, g) conformally compactifies to a compact LCF orbifold $(\hat{M}, \hat{g})$ with $Y(\hat{M}, [\hat{g}]) > 0$.

On $(\hat{M}, \hat{g})$, the conformal Green's function G_p at a singular point p admits the expansion (using the LCF condition to eliminate the logarithmic term; see Lemma 6.4 of [45]):

$$G_p = r^{2-n} + A + O(r), \quad r = \text{dist}_{\hat{g}}(\cdot, p),$$

without any $\log r$ correction. The conformal factor $u = G_p^{1/(n-2)}$ then satisfies $u = r^{(2-n)/(n-2)}(1 +$

$O(r^2)$), which translates back to an ALE decay rate of order $n - 2$ in the original chart at infinity. $\square$

3.5 Classification Results

We now classify LCF ALE manifolds and compact LCF orbifolds using the structure theory of §3.2–§3.4. The strategy in each case is the same: use the good-orbifold theorem and conformal compactification to reduce to a compact problem, then apply known results from topology and conformal geometry. The three main results are: conformal compactification (Theorem 3.15), the pairing of orbifold points (Theorem 3.17), and the dimension-specific diffeomorphism type.

3.5.1 Conformal compactification and Yamabe invariant

Theorem 3.15 (Conformal compactification). *Let (M^n, g) be a complete LCF ALE manifold with $R_g \geq 0$. Then (M, g) conformally compactifies to a compact LCF orbifold $(\hat{M}, \hat{g})$ with $Y_{\text{orb}}(\hat{M}, [\hat{g}]) > 0$.*

Proof. Without loss of generality assume (M, g) has one ALE end with group $\Gamma \leq O(n)$. We construct the compactification in two steps.

Step 1: Inverted metric. Fix $z \in M$ and let $r(x) = \text{dist}_g(x, z)$. Define the inverted metric $\hat{g} = r^{-4}g$; in the inverted coordinate $\rho = 1/r$, the ALE condition gives $\hat{g} = \delta + O(\rho^\tau)$ near the compactification point $x_0 = \{r = \infty\}$. Thus a neighborhood of x_0 in $(M \cup \{x_0\}, \hat{g})$ is isomorphic to a ball $B(0, \varepsilon)/\Gamma \subset \mathbb{R}^n/\Gamma$ with an isolated orbifold point at the origin.

Step 2: Positive Yamabe invariant. We use the PMT for LCF ALE orbifolds (Theorem 3.27, proved in §3.7): $m_{\text{ADM}}(M, g) \geq 0$.

The conformal compactification $\hat{M} = M \cup \{x_0\}$ carries the LCF conformal structure $[\hat{g}]$.

For the Yamabe invariant, choose the conformal Green's function G_{x_0} for $L_{\hat{g}}$ at x_0 . Its expansion near x_0 reads $G_{x_0} = c\rho^{2-n} + m_{\text{ADM}} + O(\rho)$ (after normalizing by a positive constant c depending on n and Γ). Since $m_{\text{ADM}} \geq 0$ and $G_{x_0} > 0$, the Schoen argument (as in [59]) shows:

$$Y_{\text{orb}}(\hat{M}, [\hat{g}]) \geq 0.$$

Strict positivity $Y_{\text{orb}} > 0$ follows from $m_{\text{ADM}} > 0$ (which holds unless (M, g) is flat, in which case $\hat{M}$ is a spherical space form with $Y > 0$ anyway) or by a direct check using the positivity of $R_{\hat{g}}$. $\square$

The conformal compactification theorem reduces all questions about ALE ends to questions about orbifold points of a compact LCF orbifold with positive Yamabe invariant. The next key fact about such orbifold points is their pairing structure, which follows from the orientability of the developing map image.

3.5.2 Local isotropy and paired orbifold points

To study the local isotropy subgroups, we first show that we can conjugate the local group Γ_i to be a subgroup of $O(n)$.

Lemma 3.16 (Local isotropy conjugates into $O(n)$). *Let (M^n, g) be a compact LCF orbifold with $Y > 0$ and let $p \in M$ be an orbifold point with local isotropy group G_p . Then $\rho(G_p) \leq C(n)$ is conjugate to a finite subgroup of $O(n)$.*

Proof. By Corollary 3.11, $\rho(G_p)$ fixes a single point $p' \in \mathbb{S}^n$. The stabilizer in $C(n)$ of a point $p' \in \mathbb{S}^n$ is, after conjugating so that $p' = \infty$ (the north pole in the stereographic model), the

group of similarities:

$$C(n)_{p'} = \{x \mapsto \alpha Ux + b : \alpha > 0, U \in O(n), b \in \mathbb{R}^n\} \cong O(n) \ltimes (\mathbb{R}^{>0} \times \mathbb{R}^n).$$

A finite subgroup of $C(n)_{p'}$ has no elements of infinite order, so its dilation part α must equal 1 (otherwise $\alpha^k \rightarrow 0$ or ∞ , giving an infinite-order element). Hence $\rho(G_p)$ maps to $O(n) \ltimes \mathbb{R}^n$ with $\alpha = 1$. But a finite subgroup of the Euclidean group $O(n) \ltimes \mathbb{R}^n$ with a fixed point (here p' , the pre-image of ∞) must be conjugate into $O(n)$ (since fixing $b = 0$ without loss of generality after translating the fixed point to the origin). $\square$

For a subgroup $\Gamma \leq O(n)$ acting on $\mathbb{S}^n$, if it has a fixed point, it fixes the antipodal point as well. It seems that the non-trivial orbifold points must appear in pairs. This will be the case if this subgroup does not contain parabolic elements.

Theorem 3.17 (Orbifold points come in pairs). *Let (M^n, g) be an oriented compact LCF orbifold with $Y > 0$. Assume there are no parabolic elements in Γ and $\dim_{\mathcal{H}} \Lambda < (n - 2)/2$. Then the orbifold singular points decompose into pairs of antipodal fixed-point sets; in particular the total number of orbifold points is even.*

Proof. Proved in Appendix B. $\square$

Corollary 3.18 (Orientable ALE ends come in pairs). *Let (M^n, g) be an orientable LCF ALE manifold. If $\pi_1(M)$ contains no parabolic element, then the ALE ends with non-trivial group $\Gamma \neq \{1\}$ appear in pairs.*

Proof. Let M' be the conformal compactification of M . By Theorem 3.15, $Y_{\text{orb}}(M') > 0$. By Proposition 3.12, there exists a compact manifold cover $N \rightarrow M'$, and the positive orbifold Yamabe constant passes to $Y(N) > 0$. By Proposition 4.7 of [59], $\dim_{\mathcal{H}}(\Lambda) \leq (n - 2)/2$, and since $Y(N) > 0$, Corollary 3.4 of [47] gives the strict inequality $\dim_{\mathcal{H}}(\Lambda) < (n - 2)/2$. Since

$\pi_1(M)$ has no parabolic element, Theorem 3.17 applies, and the orbifold singular points of M' —which correspond to the non-trivial ALE ends of M —appear in pairs. $\square$

With the pairing result in hand, we can now state the dimension-specific diffeomorphism classification. In dimensions 3 and 4, the LCF and PSC conditions together leave only a small list of possible topological types, determined by Perelman’s geometrization (dimension 3) and Hamilton’s PIC classification (dimension 4).

3.5.3 Dimension-specific classification

Now, we obtain some classification theorems of the orbifolds in low dimensions. Thus, we also classify the ALE spaces in these cases.

Theorem 3.19 (3D classification of compact LCF orbifolds with PSC). *Let (M^3, g) be a compact LCF orbifold with $Y > 0$. Then M is a good orbifold covered by a spherical space form $\mathbb{S}^3/\Gamma$ or a connected sum $\#_k(\mathbb{S}^2 \times \mathbb{S}^1)$. In particular, it is conformal equivalent to the connected sum of the quotients $\mathbb{S}^3/\Gamma$ and $\mathbb{S}^1 \times \mathbb{S}^2/\Gamma$, i.e.,*

$$M \cong \mathbb{S}^3/\Gamma_1 \# \mathbb{S}^3/\Gamma_2 \# \dots \mathbb{S}^3/\Gamma_k \# \mathbb{S}^2 \times \mathbb{S}^1/\Gamma'_1 \# \dots \mathbb{S}^2 \times \mathbb{S}^1/\Gamma'_m,$$

where all the orbifold points are $\mathbb{Z}_2$ -quotient singularities.

Proof. By Theorem 3.13, M is a good orbifold with a finite smooth manifold cover $\tilde{M}^3$. Since $\tilde{M}$ has positive scalar curvature (inherited via the finite cover), by Theorem 2.18 (the 3D LCF classification of §2.4.3), $\tilde{M}$ is diffeomorphic to a spherical space form or a connected sum $\#_k(\mathbb{S}^2 \times \mathbb{S}^1)$. Quotienting by the finite deck group gives the stated orbifold forms. For the conformal connected sum: note the above decomposition can be viewed as the connected sum of orbifolds near the manifold points. Thus, we can apply Theorem 2.7 from [35]. $\square$

Theorem 3.20 (4D classification of compact LCF orbifolds with PSC). *Let (M^4, g) be a compact LCF orbifold with $Y > 0$. Then a finite smooth manifold cover of M is diffeomorphic to $\mathbb{S}^4$ or a connected sum $\#_k(\mathbb{S}^1 \times \mathbb{S}^3)$ for some $k \geq 0$. In particular, it is conformal equivalent to the connected sum of the quotients $\mathbb{S}^4/\Gamma$ and $\mathbb{S}^1 \times \mathbb{S}^3/\Gamma$, i.e.,*

$$M \cong \mathbb{S}^4/\Gamma_1 \# \mathbb{S}^4/\Gamma_2 \# \dots \mathbb{S}^4/\Gamma_k \# \mathbb{S}^3 \times \mathbb{S}^1/\Gamma'_1 \# \dots \mathbb{S}^3 \times \mathbb{S}^1/\Gamma'_m.$$

Proof. Theorem 3.13 provides a finite smooth manifold cover $\tilde{M}^4$ with $R > 0$. In dimension 4, the LCF condition ($W = 0$) combined with $R > 0$ implies positive isotropic curvature (PIC); see [33]. Hamilton's classification of closed 4-manifolds with PIC [33] states that any such manifold is diffeomorphic to $\mathbb{S}^4$ or a connected sum $\#_k(\mathbb{S}^1 \times \mathbb{S}^3)$, $k \geq 1$ (cf. Theorem 2.19). In particular, $\mathbb{CP}^2$ does not arise, since it does not admit a metric of positive isotropic curvature. The rest of the argument just follows the 3D case. $\square$

As a consequence of our 4D result, we can refine the result of [20].

Corollary 3.21 (Positive isotropic curvature in 4D). *A 4D compact LCF orbifold with positive isotropic curvature and $Y > 0$ has a finite manifold cover diffeomorphic to a connected sum $\#_k(\mathbb{S}^1 \times \mathbb{S}^3)$, $k \geq 0$.*

Proof. By Theorem 3.13 and Proposition 3.12, there is a finite smooth manifold cover $\tilde{M}^4$. Positive isotropic curvature is preserved under finite covers. Hamilton's theorem [33] classifies closed 4-manifolds with PIC as connected sums $\#_k(\mathbb{S}^1 \times \mathbb{S}^3)$ (after passing to a finite cover if necessary; in our case the cover is already smooth). $\square$

Remark 3.3. *The corollary a priori seems to have nothing to do with LCF metrics. The connection is pointed out by [20], namely: a compact four-orbifold with at most isolated singularities admits positive isotropic curvature if and only if it admits an LCF metric with positive scalar curvature (Corollary 2 in [20]). Since the conclusion is topological, we work*

on its LCF metric.

In low dimension, we have the following corollary about the ALE ends in dimension 3 or 4.

Corollary 3.22 (ALE ends in dimensions 3 and 4). *For $n = 3$ or 4 , the ends of an orientable LCF ALE manifold (M^n, g) with $R_g \geq 0$ are either groups for the ALE ends are trivial (trivial group) or appear in antipodal pairs corresponding to orbifold points of the conformal compactification. In particular, when $n = 3$, orientable LCF ALE manifold with $R_g \geq 0$ is Asymptotically flat (trivial group).*

Proof. Apply Theorem 3.15 to compactify (M, g) to an oriented compact LCF orbifold $\hat{M}$ with $Y > 0$. The non-trivial ends are orbifold points of $\hat{M}$. By $n = 3$ or 4 , we have $\dim_{\mathcal{H}}(\Lambda) < 1$. We can apply Theorem 6.2 in [36] and Selberg's lemma (Theorem 2.11). There is a torsion-free normal subgroup of $\rho(\pi_1(M))$ containing no parabolic elements. By Corollary 3.18 (which follows from the Appendix theorem), these appear in antipodal pairs. Note, when n is odd, the local isotropy group is isomorphic to $\mathbb{Z}_2 \subseteq O(n)$, which is orientation-reversing. So by Theorem 3.10, when M is orientable, then it has no such orbifold points. $\square$

It is natural to make the following conjecture in higher dimension.

Conjecture 3.23 (Pairs conjecture in high dimensions). *For $n \geq 5$, the non-trivial ALE ends of any orientable LCF ALE manifold (M^n, g) with $R_g \geq 0$ appear in antipodal pairs.*

The pairing conjecture in dimensions $n \geq 5$ is the natural extension of the result proved in Appendix B under the limit set dimension hypothesis. The Schwarzschild example illustrates why orientability is essential for the pairing.

3.5.4 The Schwarzschild example and the pairs conjecture

Orientable ALE manifolds with more than one end can be easily constructed. An example can be constructed as follows: on $\mathbb{S}^n/\Gamma$, where $\Gamma \leq O(n)$ has two fixed points, s and $-s$, with the standard spherical metric. Consider the Green's function metric blown up at s and $-s$, the resulting manifold $(M, G^{\frac{4}{n-2}} g_{\mathbb{S}^n})$ is diffeomorphic to the standard Schwarzschild metric modulo Γ . We call such a manifold the **Schwarzschild ALE** manifold. In low dimension, the non-trivial ALE ends of an orientable ALE space appear in pairs.

Example 3.1 (Schwarzschild ALE metric). *The Schwarzschild ALE metric on $\mathbb{R}^n \setminus \{0\}/\Gamma$ is*

$$g_{\text{Schw}} = \left(1 + \frac{m}{r^{n-2}}\right)^{\frac{4}{n-2}} g_{\text{flat}}, \quad m > 0, \quad r = |x|.$$

This metric is LCF (conformal to flat), ALE of order $n - 2$, and satisfies $R_{g_{\text{Schw}}} = 0$.

The *non-orientable cousin* is the quotient of the same metric by the antipodal map $\sigma : x \mapsto -x$ on the $\mathbb{S}^{n-1}$ at infinity. This yields a non-orientable LCF ALE manifold $(M/\sigma, g_{\text{Schw}})$ with a single non-trivial end (group $\mathbb{Z}_2$), showing that the pairing result (Theorem 3.17) fails in the non-orientable case.

Example 3.2. *If the dimension n is even, then the antipodal map is an orientation-reversing map. Then $\mathbb{S}^n/\{-Id\}$ is non-orientable. Consider the universal cover $\tilde{M} \cong \mathbb{S}^n \setminus \{x, -x\}$, blown up at x and $-x$, then $\tilde{M}/\Gamma$ is a Schwarzschild ALE manifold. If Γ contains no $\mathbb{Z}_2$ -rotation, we can further do a quotient (otherwise, we fix the equator). Then $(M, g) \cong (\mathbb{R}P^n \setminus \{p\}, g_{\mathbb{R}P^n})/\Gamma$, which is an LCF, ALE manifold with one end. But it is not orientable. Such a one-end, non-orientable ALE is illustrated in Figure 3.3. We call such a metric a **non-orientable Schwarzschild ALE**.*

Note that the same construction does not hold when n is odd. In this case, the only non-

trivial Γ is the $\mathbb{Z}_2$ action, which leaves the resulting quotient contains edge singularities.

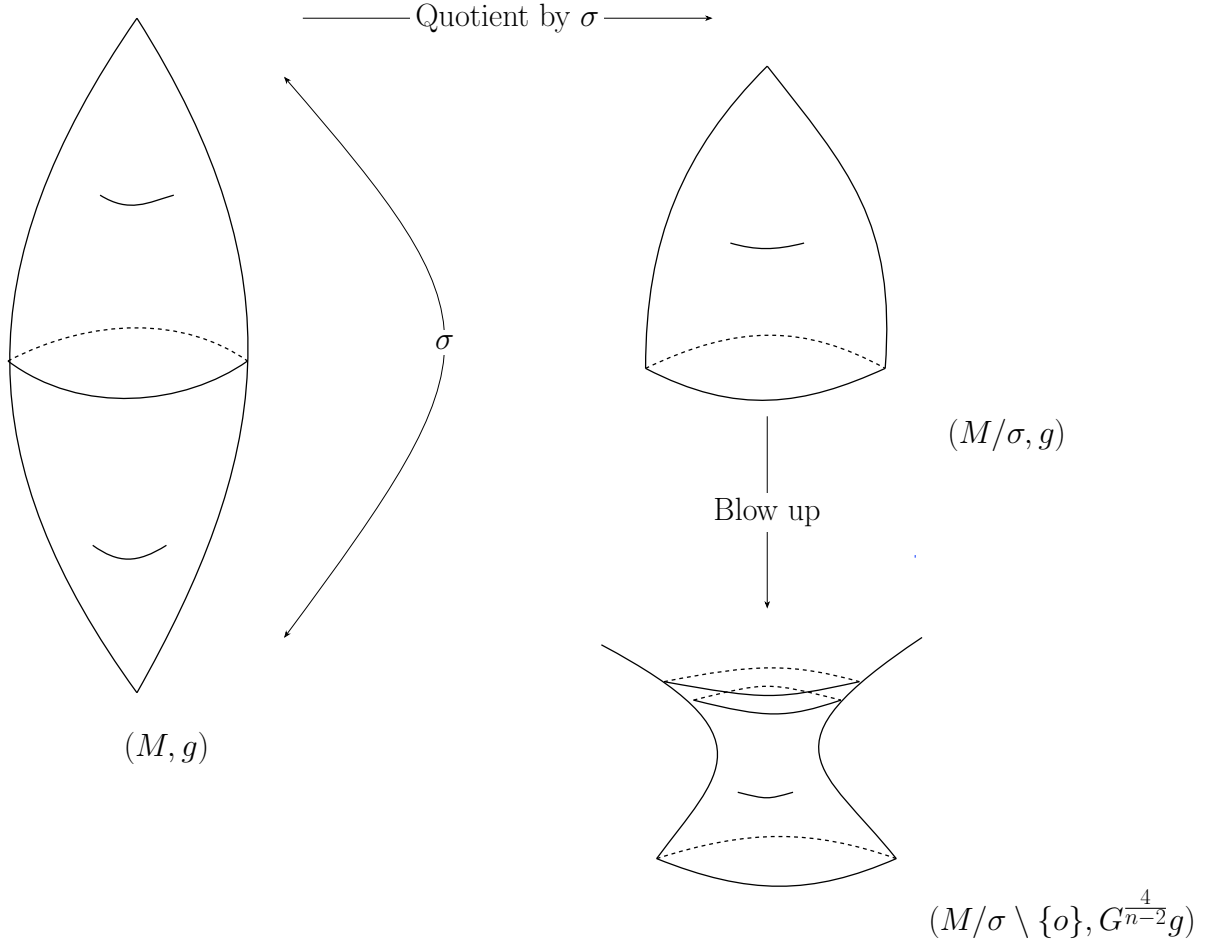


Figure 3.3: The Schwarzschild ALE manifold (M, g) (left), its quotient $(M/\sigma, g)$ by the antipodal map (center), and the blow-up at the resulting single orbifold point, which produces the football metric $\mathbb{S}^n/\mathbb{Z}_2$ (bottom).

3.6 Application to the Moduli Space: Refined Description

We now apply the structural results of §3.1–§3.5 to give a refined description of the Gromov-Hausdorff boundary of the LCF moduli space $\mathcal{M}(n, \mu_0, C)$ in the oriented setting, for some $\mu_0 > 0$. Throughout this section $n = 3$ or 4 . We write $\mathcal{M}^+ = \mathcal{M}^+(n, \mu_0, C)$ for the subspace

of $\mathcal{M}$ consisting of oriented metrics.

Recall from Theorem 3.1 that any Gromov-Hausdorff limit of a sequence in $\mathcal{M}$ is a compact LCF good orbifold with $Y_{\text{orb}} \geq 0$, and from Theorem 3.5 that the curvature concentration at the singular points is organized into a finite bubble tree of complete LCF ALE orbifolds.

3.6.1 The oriented moduli space

Lemma 3.24 (Conformal blow-up of ALE orbifold singularities). *Let (M, g) be a complete LCF ALE orbifold with $R_g \geq 0$ and finitely many isolated orbifold singularities $\{q_1, \dots, q_s\}$. Then there exists a positive function H on $M \setminus \{q_i\}$, harmonic with respect to $L_g = -a_n \Delta_g + R_g$, such that $(M \setminus \{q_i\}, H^{\frac{4}{n-2}} g)$ is a smooth LCF ALE manifold with $R \geq 0$.*

Proof. By the existence theory for the Green's function on ALE spaces [31], for each q_i there exists a positive Green's function G_i for L_g with pole at q_i . The superposition $H = \sum_i G_i > 0$ is a positive solution of $L_g H = 0$ on $M \setminus \{q_i\}$, with the leading singularity $G_i \sim c_i r^{2-n}$ near each q_i resolving the orbifold point into an ALE end. Since $R_g \geq 0$ and $L_g H = 0$ with $H > 0$, the conformal change formula gives $R_{H^{4/(n-2)}g} = H^{-(n+2)/(n-2)} L_g H = 0$, so the blown-up metric is scalar-flat (in particular $R \geq 0$). An alternative construction is given by Lemma 2.1 of [39]. $\square$

Theorem 3.25 (GH limits in the oriented moduli space). *Let $n = 3$ or 4 , and let (M_i^n, g_i) be a sequence of closed, oriented, LCF manifolds in $\mathcal{M}^+(n, \mu_0, C)$. Let (M_∞, g_∞) be any Gromov-Hausdorff limit, with finite singular set $S = \{x_1, \dots, x_k\} \subset M_\infty$ as provided by Theorem 3.1. Then:*

1. (M_∞, g_∞) is a compact oriented LCF good orbifold with $Y_{\text{orb}}(M_\infty, [g_\infty]) > 0$.

2. At each singular point $x \in S$, the tangent cone decomposes as

$$T_x M_\infty = \bigsqcup_{j=1}^m C(\mathbb{S}^{n-1}/\Gamma_j),$$

and the number of non-trivial factors (those with $\Gamma_j \neq \{1\}$) is even.

3. When $n = 3$, the singular set S is empty; that is, (M_∞, g_∞) is a smooth manifold.

Proof. **Part (1).** Theorem 3.1 provides (M_∞, g_∞) as a compact LCF orbifold with $Y_{\text{orb}} \geq 0$. Since each (M_i, g_i) is oriented, the limit is oriented away from S ; as S has codimension ≥ 2 the orientation extends globally. Theorem 3.13 shows M_∞ is a good orbifold.

Part (2). We argue by induction on the depth of the bubble tree (Theorem 3.5). The bubble tree has finite depth because, by Lemma 3.3, every non-flat LCF manifold carries at least $\varepsilon_1 > 0$ of $L^{n/2}$ -curvature energy, and the total energy is bounded by C .

Base case. The leaf nodes of the bubble tree are complete smooth LCF ALE manifolds (N_ℓ, g_ℓ) (Theorem 3.5(iii)). Each (N_ℓ, g_ℓ) inherits $R_{g_\ell} \geq 0$ from the sequence and is oriented. By Corollary 3.18, the non-trivial ends of (N_ℓ, g_ℓ) appear in pairs. Via the neck structure of the bubble tree (Theorem 3.5(iv)), each end of (N_ℓ, g_ℓ) corresponds to a cone factor in the tangent cone of the parent node at the blow-up point. Hence the non-trivial cone factors contributed by (N_ℓ, g_ℓ) to its parent are even in number.

Inductive step. Fix a level- ℓ non-leaf node (N_ℓ, g_ℓ) in the bubble tree; it is a complete LCF ALE orbifold with finitely many isolated orbifold singularities $S_\ell = \{q_1, \dots, q_s\}$. Assume the claim holds for all bubbles at levels below ℓ : for each $q_j \in S_\ell$, the bubble attached at q_j has an even number of non-trivial ends, so an even number of non-trivial cone factors appear in $T_{q_j} N_\ell$.

By Lemma 3.24, applied to the LCF ALE orbifold (N_ℓ, g_ℓ) with singularities S_ℓ , there exists

a positive conformal factor H on $N_\ell \setminus S_\ell$, producing a smooth LCF ALE manifold

$$(X_\ell, \tilde{g}_\ell) = (N_\ell \setminus S_\ell, H^{\frac{4}{n-2}} g_\ell)$$

with $R_{\tilde{g}_\ell} \geq 0$. The ends of $(X_\ell, \tilde{g}_\ell)$ fall into two groups:

- (a) the original non-trivial ends of (N_ℓ, g_ℓ) at spatial infinity, and
- (b) the new ends opened at each $q_j \in S_\ell$ by the conformal blow-up.

Since $(X_\ell, \tilde{g}_\ell)$ is an oriented LCF ALE manifold with $R \geq 0$, Corollary 3.18 implies the total number of non-trivial ends is even. By the inductive hypothesis, the ends in group (b) arising at each q_j already account for an even count in aggregate. Therefore the ends in group (a) are also even in number; translated back through the neck structure, the non-trivial cone factors in $T_x M_\infty$ contributed by (N_ℓ, g_ℓ) appear in pairs.

Applying the induction from the deepest level up to the root M_∞ completes the proof of Part (2).

Part (3). Every orbifold singular point $x \in S$ has local isotropy group $\Gamma_x \leq O(n)$ acting freely on $\mathbb{S}^{n-1}$. Since M_∞ is oriented, $\Gamma_x \leq SO(n)$. For $n = 3$: every non-identity element of $SO(3)$ is a rotation, which fixes exactly two antipodal points on $\mathbb{S}^2$, hence cannot act freely. Thus $\Gamma_x = \{1\}$ for all $x \in S$, so $S = \emptyset$ and M_∞ is a smooth manifold. $\square$

Corollary 3.26. *Let $n = 4$ and $\Gamma \leq SO(4)$ be a non-trivial finite group acting freely on $\mathbb{S}^3$. The football orbifold $\mathbb{S}^4/\Gamma$ (Definition 2.8) cannot be realized as a Gromov-Hausdorff limit of any sequence $(M_i^4, g_i) \in \mathcal{M}^+(4, \mu_0, C)$.*

Proof. The football orbifold $\mathbb{S}^4/\Gamma$ has exactly two orbifold singular points (the north and south poles), each with tangent cone $C(\mathbb{S}^3/\Gamma)$ — a single non-trivial cone factor. Theo-

rem 3.25(2) requires the number of non-trivial cone factors at each singular point to be even, but 1 is odd. Hence $\mathbb{S}^4/\Gamma$ cannot arise as a GH limit. $\square$

Remark 3.4. *Corollary 3.26 is the oriented LCF analogue of results in the Einstein setting. Anderson [4] raised the general question of which Einstein orbifolds can be desingularized by Einstein metrics. Biquard [13] identified analytic obstructions to such desingularizations, and Ozuch [48] proved that $\mathbb{S}^4/\Gamma$ cannot be approximated by a sequence of Einstein manifolds with Eguchi-Hanson bubbles. Our result is an analogue for oriented LCF manifolds: the single non-trivial cone at each pole of the football is an obstruction to its appearance as an oriented LCF GH limit.*

3.6.2 Non-orientable limiting configurations

Outside the oriented setting, the pairing constraint of Theorem 3.25(2) fails, and new limiting configurations appear. We describe two explicit constructions, both in dimension $n = 4$, showing that non-orientable irreducible multifolds can be realized as non-collapsing Gromov-Hausdorff limits of non-orientable LCF 4-manifolds. The gluing and perturbation techniques are standard and follow [38]; we omit the analytic details and focus on the geometric content. A slight difference from $\mathcal{M}^+(n, \mu_0, C)$ is that the metrics constructed are only constant scalar curvature (not necessarily Yamabe minimizers), but all moduli space results in §3.1 hold in the same form for the moduli space of constant scalar curvature metrics with bounded Sobolev constant and bounded diameter.

Definition 3.4 (Non-orientable cap). *Let n be even, $\Gamma \leq SO(n)$ a finite group acting freely on $\mathbb{S}^{n-1}$, and extend the Γ -action to $\mathbb{R}^{n+1}$ via the first n coordinates. Assume the antipodal map σ of $\mathbb{S}^{n-1}$ does not belong to Γ . Let $\tilde{\Gamma} = \langle \Gamma, \tilde{\sigma} \rangle$ where $\tilde{\sigma}$ denotes the antipodal map of $\mathbb{S}^n$. The non-orientable cap is the orbifold $\mathbb{S}^n/\tilde{\Gamma}$.*

Remark 3.5. *Taking the Green's function metric of the non-orientable cap at its unique*

orbifold point recovers the non-orientable Schwarzschild ALE manifold of Example 3.1: a complete LCF ALE manifold with a single non-trivial end with group Γ at infinity. Being non-orientable, it does not contradict Corollary 3.18.

Example 3.3 (Non-orientable sequence converging to non-orientable cap). *Let (N_1, h_1) be the non-orientable Schwarzschild ALE manifold with group Γ at infinity (Remark 3.5), and let (N_2, h_2) be the non-orientable cap $\mathbb{S}^4/\tilde{\Gamma}$, which has a single irreducible Γ -singularity. For each $t > 0$, glue the truncated ALE manifold (N_1, h_1) to (N_2, h_2) along the collar neighborhood of the singularity with gluing parameter t . Using the perturbation techniques of [38], for t sufficiently small one can conformally perturb the glued metric to a positive constant scalar curvature LCF metric g_t on the resulting non-orientable 4-manifold M_t .*

As $t \rightarrow 0$, the ALE piece (N_1, h_1) bubbles off and the sequence (M_t, g_t) converges in the Gromov-Hausdorff sense to (N_2, h_2) . This is illustrated in the top row of Figure 3.4.

In contrast to Corollary 3.26, the football metric does arise as a Gromov-Hausdorff limit of a non-orientable sequence:

Example 3.4 (Non-orientable sequence converging to football metric). *Let (N_1, h_1) be the non-orientable Schwarzschild ALE manifold of Example 3.1, with Γ containing no $\mathbb{Z}_2$ -rotation. Let (N_2, h_2) be the football orbifold $\mathbb{S}^4/\Gamma$ equipped with the spherical quotient metric; note (N_2, h_2) is orientable and has exactly two irreducible Γ -singularities (the north and south poles).*

For each $t > 0$, glue two copies of the truncated ALE manifold (N_1, h_1) to (N_2, h_2) , one at each of the two singularities, with the same gluing parameter t . For t sufficiently small, the techniques of [38] produce a positive constant scalar curvature LCF metric g_t on the resulting non-orientable 4-manifold M_t .

As $t \rightarrow 0$, the two ALE pieces bubble off and (M_t, g_t) converges in the Gromov-Hausdorff

sense to (N_2, h_2) . This is illustrated in the bottom row of Figure 3.4.

Remark 3.6. *The non-orientable one-ended ALE metric (N_1, h_1) plays the same role here as the caps C_Γ^σ in [20]: in both cases, a non-orientable ALE piece is glued onto an orbifold singularity to produce a smooth (or smoother) manifold carrying a canonical curvature condition. Chen-Zhu cap off singularities to run the Ricci flow; here we cap off singularities to produce a converging sequence in the moduli space.*

3.7 The positive mass theorem for LCF, ALE orbifolds

Finally, we discuss the positive mass theorem. First, we recall the ADM mass:

Definition 3.5. *Let (M, g) be an ALE manifold. Let the coordinate on one end be $\{x^i\}$ for r large. Then **the ADM mass** of this end is the following functional:*

$$m(g) := \lim_{r \rightarrow \infty} \frac{1}{2(n-1)\omega} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii}) \partial_j \lrcorner dv,$$

where ω is the volume of the unit ball in $\mathbb{R}^n$.

The positive mass theorem states that in asymptotically flat manifolds with non-negative scalar curvature, with the condition that the scalar curvature is integrable, the mass is non-negative, with zero mass if and only if the asymptotically flat manifold is isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$. This was proved by Schoen-Yau [56], [58] using minimal surfaces, and by Witten [71] using spinors. This was generalized to AE ends of orbifolds in [39], and to AE ends of spaces with isolated conical singularities in [21].

It is natural to ask the same question on ALE manifolds. However, counterexamples were found by LeBrun [43]. Thus, a naive extension of the positive mass theorem is not true. Note that LeBrun's counterexamples are anti-self-dual in real dimension 4.

Now, we prove the positive mass theorem in the LCF case:

Theorem 3.27 (PMT for LCF ALE orbifolds). *Let (M^n, g) be a complete LCF ALE orbifold with $n \geq 3$, group $\Gamma \leq O(n)$, and $R_g \geq 0$ with $R_g \in L^1$. Then $m(g) \geq 0$, with equality if and only if (M, g) is isometric to $(\mathbb{R}^n/\Gamma, g_{\text{flat}})$.*

Proof of Theorem 3.27. Our method is an indirect method from [59].

By our assumption, the scalar curvature is integrable hence the mass is well-defined. From Lemma 3.4, we can conformally blow-up the orbifold singularities and form an ALE manifold. From Theorem 3.15, we can choose the conformal Green's function so that the conformal metric is scalar-flat, and the coordinate such that the ADM mass is well-defined.

Let $(M, u^{\frac{4}{n-2}}g)$ to be the above conformal scalar-flat, ALE metric. We want to relate the mass between $(M, u^{\frac{4}{n-2}}g)$ and (M, g) . Note that we can adjust the scaling such that u solves:

$$\begin{cases} -\Delta u + c(n)Su = 0 \\ u \rightarrow 1 \text{ as } |x| \rightarrow \infty \end{cases}$$

Note that $S \geq 0$. The asymptotic of u is:

$$u = 1 + Ar^{2-n} + O(r^{2-n-\epsilon})$$

as $r = |x| \rightarrow \infty$. Using Green's formula, Theorem 3.7, and the maximal principle, we have $A = -C \int_M u S dv$, for some $C \geq 0$. In particular, $A \leq 0$. Then, the mass $m(u^{\frac{4}{n-2}}g) = m(g) + (n-1)A \leq m(g)$. Thus, it is enough to prove the mass is positive for scalar-flat ALE metrics.

By the previous discussion, consider the conformal compactification, (M', g') , of (M, g) . Consider the manifold cover, N , (Proposition 3.12), with the pull-back metric on N is

$$\bar{g} = P^*g.$$

Consider $x_i \in M'$, which is an orbifold point with the local group Γ_i . Let the minimal Green's function at x_i is G_i . Now, we pull back G_i to N . Denote $\bar{G}_i = P^*G_i$. As in [59], choose $s \in P^{-1}(x_i)$, there exists a minimal Green's function on N , G_s , with a singular point at s . Thus, $\bar{G}_i \geq G_s$. By the maximal principle of the conformal Laplacian with positive Yamabe constant, we have either $\bar{G}_i > G_s$ or $\bar{G}_i = G_s$.

If $\bar{G}_i > G_s$, then, in the inverted coordinate at s , near infinity,

$$\bar{G}_i(x) = 1 + E_i|x|^{2-n} + O(|x|^{1-n})$$

and

$$G_s(x) = 1 + E_s|x|^{2-n} + O(|x|^{1-n}),$$

then we know $E_i > E_s$. Since P is a local isometry, then $m(G_i^{\frac{4}{n-2}}g) > m(G_s^{\frac{4}{n-2}}\bar{g})$. By N being an LCF manifold and by Proposition 4.3 in [59], we have $m(G_s^{\frac{4}{n-2}}\bar{g}) \geq 0$, thus $m(G_i^{\frac{4}{n-2}}g) > 0$.

If $\bar{G}_i = G_s$, then $P^{-1}(x_i) = s$. Thus, all the deck transformations on N fix s . Thus, $\Gamma_i \cong \pi_1(M)/\pi_1(N)$. By the previous case, if $m(G_s^{\frac{4}{n-2}}\bar{g}) > 0$, then it is done. Otherwise, $m(G_s^{\frac{4}{n-2}}\bar{g}) = 0$, then by Proposition 4.3 in [59], $\pi_1(N) = \{e\}$. Thus, $\pi_1(M) \cong \Gamma_i$, which means $N \cong \mathbb{S}^n$ since there is no limiting set. Thus, by Lemma 3.16, (M', g') is conformal to $(\mathbb{S}^n, g_{\mathbb{S}^n})/\Gamma_i$, the football orbifold with the spherical quotient metric. If Γ_i is not trivial, (M, g) is either conformal to $(\mathbb{R}^n \setminus \{0\}, g_{\text{Sch}})/\Gamma_i$, the Schwarzschild ALE manifolds, in the manifold case; or conformal to $(\mathbb{R}^n, g_{\text{flat}})/\Gamma_i$, the flat cone metric, in the orbifold case. By direct computation: In the manifold case: $m(g) > 0$. Otherwise, $M' \cong \mathbb{S}^n$. If $m(g) = 0$, it is true if and only if $(M, g) \cong (\mathbb{R}^n, g_{\mathbb{R}^n})$. In the orbifold case: $m(g) = 0$ and $(M, g) \cong (\mathbb{R}^n, g_{\mathbb{R}^n})/\Gamma$. $\square$

A direct corollary is a special case of the orbifold Yamabe problem.

Corollary 3.28. *Let (M^n, g) be a compact LCF good orbifold with $Y_{\text{orb}}(M, [g]) > 0$. Then M admits a smooth constant scalar curvature metric in $[g]$.*

Proof of Corollary 3.28. If (M, g) is not conformally equivalent to the football orbifold with the spherical metric, then by Theorem 3.27, the mass for the conformal Green's function metric at each point $m(\hat{g}_x) > 0$.

Now, we choose a singularity $x \in M$. Since (M, g) is LCF, the local distortion constant is purely determined by the mass (See Lemma 9.7 in [45]), we can use Schoen's test function in [53] and conclude that $Y_{\text{orb}}(M, [g]) < Y(\mathbb{S}^n)|\Gamma_x|$, where Γ_x is the orbifold group. Since this strict inequality holds for any orbifold singularity, we can apply Theorem 2.13.

For the case where (M, g) is conformally equivalent to the football orbifold with the spherical metric, then it is obvious that it contains a metric which is of constant scalar curvature. $\square$

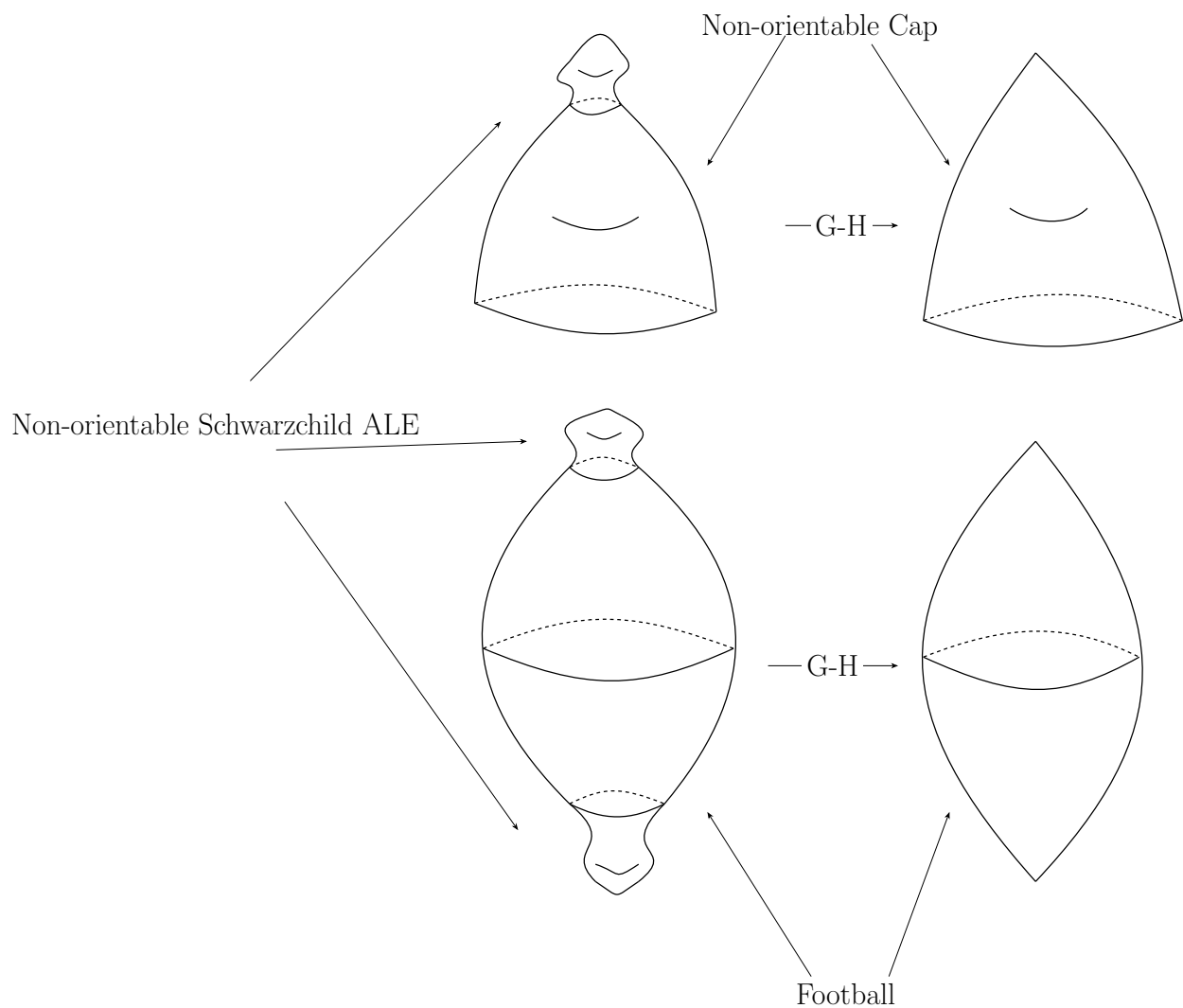


Figure 3.4: Two non-orientable GH limiting configurations. *Top row* (Example 3.3): the limit is a non-orientable cap, and one non-orientable Schwarzschild ALE bubble detaches. *Bottom row* (Example 3.4): the limit is the football metric $\mathbb{S}^4/\Gamma$, and two non-orientable Schwarzschild ALE pieces bubble off.

Chapter 4

Collapsing Constant Scalar Curvature Metrics

4.1 The Question of Cheeger–Fukaya–Gromov

The Cheeger–Fukaya–Gromov (CFG) collapsing theory [19, 26, 29, 28] describes the structure of sequences of Riemannian manifolds that collapse with bounded sectional curvature to a lower-dimensional limit. A central consequence is the existence of an almost-free nilpotent group action $\mathcal{N}$ on the collapsing manifolds whose orbits are precisely the collapsing fibers. A metric is $\mathcal{N}$ -invariant if it is invariant under this nilpotent group action.

Assumption 1. *Let $\{(M_i^n, g_i)\}$ be a sequence of closed n -dimensional Riemannian manifold such that:*

- *The Riemannian curvature is uniformly bounded:*

$$\| \text{Rm}(g_i) \|_{C^0} \leq C_0, \tag{4.1}$$

- The diameter is bounded from above: $d(M_i, g_i) \leq C$.
- The sequence collapses to a low dimensional manifold, i.e.:

$$(M_i^n, g_i) \rightarrow (X^d, g_\infty) \quad (4.2)$$

in the Gromov-Hausdorff sense to a manifold X , where $d < n$.

For the collapsing sequence satisfying Assumption 1, the limit space (X^d, g_∞) carries a natural *metric-measure structure* $(X^d, g_\infty, d\nu)$, where $d\nu = \chi dV_{g_\infty}$ and

$$\chi(x) := \lim_{i \rightarrow \infty} \frac{\text{vol}(F_i^{-1}(x))}{\text{vol}(M_i)} \quad (4.3)$$

is the *relative volume function*. The natural self-adjoint Laplacian on $L^2(X, d\nu)$ is the *drift Laplacian*

$$\Delta_\chi h := \Delta_X h + \langle \nabla \log \chi, \nabla h \rangle_X, \quad (4.4)$$

which is the Laplacian for $\mathcal{N}$ -invariant functions on M_i in the collapsing limit (see Lemma 4.10).

The main theorem of this chapter is the following.

Theorem 4.1 (Collapsing CSC metrics are $\mathcal{N}$ -invariant up to perturbation [68]). *Let $\{(M_i^n, g_i)\}$ be a sequence of closed n -dimensional Riemannian manifolds satisfying:*

1. **Bounded curvature:** $\|\text{Rm}(g_i)\|_{C^0} \leq C_0$;
2. **Bounded diameter:** $d(M_i, g_i) \leq C$;
3. **Constant scalar curvature:** $S(g_i) \equiv v$ for a fixed constant v ;
4. **Collapse:** $(M_i^n, g_i) \rightarrow (X^d, g_\infty)$ in the Gromov-Hausdorff sense, where $d < n$;

5. **Spectral condition** (only when $v > 0$): the drift Laplacian Δ_χ defined in (4.4) does not have $\frac{v}{n-1}$ as an eigenvalue on (X, g_∞) , i.e. there are no nonzero solutions $h \in W^{2,p}(X)$ of

$$-\Delta_\chi h = \frac{v}{n-1} h. \quad (4.5)$$

Then, for all sufficiently large i , there exist smooth metrics g'_i on M_i such that:

- (i) $\|g_i - g'_i\|_{W^{1,p}} \rightarrow 0$ for any $p > n$;
- (ii) g'_i is $\mathcal{N}$ -invariant;
- (iii) $S(g'_i) \equiv v$ if $v \neq 0$; or $S(g'_i) \equiv v_i$ with $v_i \rightarrow 0$ if $v = 0$;
- (iv) $(M_i, g'_i) \rightarrow (X, g_\infty)$ in the Gromov–Hausdorff sense.

Remark 4.1. For $v \leq 0$, no spectral condition is imposed. When $v = 0$, we cannot achieve $S(g'_i) = 0$ exactly since $0 \in \text{Spec}(-\Delta_\chi)$ always (constants are in the kernel); instead we obtain $S(g'_i) = v_i \rightarrow 0$.

Remark 4.2. The spectral condition (4.5) is used to ensure that the linearized conformal scalar curvature operator is uniformly invertible in the limit (Proposition 4.13).

Remark 4.3. The convergence in (i) is in the Sobolev norm $W^{1,p}$ rather than the pointwise $C^{k,\alpha}$ norm. This is because the CFG averaging procedure, starting from only bounded curvature ($k = 0$), produces an $\mathcal{N}$ -invariant metric with $C^{0,\alpha}$ and $W^{1,p}$ control, but not higher regularity (see Theorem 4.6).

Proof strategy

The proof of Theorem 4.1 has three steps.

Step 1: $\mathcal{N}$ -invariant metric with finite regularity. Using the CFG fibration theorem [19] and the finite-regularity estimates of §4.2, with only the bounded curvature hypothesis $\|\text{Rm}(g_i)\|_{C^0} \leq C_0$ (no Ricci flow smoothing), we obtain an $\mathcal{N}$ -invariant nearby metric $\hat{g}_i$ satisfying

$$\|\hat{g}_i - g_i\|_{C^{0,\alpha}} \leq C\tau_i, \quad \|\hat{g}_i - g_i\|_{W_{loc}^{1,p}} \leq C\tau_i, \quad (4.6)$$

where $\tau_i \rightarrow 0$ is the collapse scale. At this stage, the scalar curvature satisfies only a distributional estimate:

$$\|S(\hat{g}_i) - v\|_{W^{-1,p}(X, \nu_i)} \leq C\tau_i, \quad (4.7)$$

in the negative Sobolev space on the base (Proposition 4.11).

Step 2: Analysis on the base X in Sobolev spaces. Since $\hat{g}_i$ is $\mathcal{N}$ -invariant, conformal perturbations by $\mathcal{N}$ -invariant factors reduce to equations on the base X . We work in the weighted Sobolev spaces $W^{1,p}(X, \nu_i)$ and their duals $W^{-1,p}(X, \nu_i)$, where $d\nu_i = \chi_i \, \text{dvol}(g_{X,i})$. The conformal perturbation operator $P_{\hat{g}_i} : W^{1,p}(X, \nu_i) \rightarrow W^{-1,p}(X, \nu_i)$ is analyzed via its linearization $\hat{L}_{\hat{g}_i}$ (Proposition 4.11).

Step 3: Conformal perturbation via implicit function theorem. We conformally perturb $g'_i = (1 + h_i)^{4/(n-2)} \hat{g}_i$ to make the scalar curvature exactly constant, by solving $P_{\hat{g}_i}(h_i) = 0$ (or $P_{0,\hat{g}_i}(h_i) = 0$ when $v = 0$) via the implicit function theorem (Lemma 4.14). The uniform invertibility of the linearized operator on $\mathcal{N}$ -invariant functions (Proposition 4.13), which uses the eigenvalue convergence theorem (§4.3), provides the required estimates. This is carried out in §4.5–§4.6.

4.2 Cheeger–Fukaya–Gromov Theory with Finite Regularity

The original CFG theory [19] works with C^∞ regularity. For our purposes we need quantitative finite-regularity versions ($C^{k,\alpha}$ and $W^{k,p}$ estimates) of the fibration theorem, the flat-connection construction, and the invariant nearby metric.

4.2.1 Finite-regularity fibration and connections

In our application, we focus mainly on finite Hölder and Sobolev regularity. Thus, we will state the finite regularity version of the theorem with the additional Riemannian curvature C^k assumption.

Assumption 2. *Let $\{(M_i^n, g_i)\}$ be a sequence of closed n -dimensional Riemannian manifold such that:*

- *The Riemannian curvature is uniformly bounded:*

$$\|\mathrm{Rm}(g_i)\|_{C^k} \leq C_k, \tag{4.8}$$

for some $k \geq 0$,

- *The diameter is bounded from above: $d(M_i, g_i) \leq C$.*

Corollary 4.2 (CFG fibration with finite regularity). *Let $\{(M_i^n, g_i)\}$ and (X^d, g_∞) be as in the CFG Fibration Theorem [19], with fibration maps $F_i : M_i \rightarrow X$. If in addition the Assumption 2 is satisfied, then*

1. *for all large i :*

$$\|F_i\|_{C^{k+2,\alpha}} \leq C(n, C_k, X), \quad \|A_{F_i^{-1}(x)}\|_{C^{k,\alpha}} \leq C(n, C_k, X),$$

for any $0 < \alpha < 1$.

2. for all large i :

$$\|F_i\|_{W^{k+3,p}} \leq C(n, C_{k,p}, X), \quad \|A_{F_i^{-1}(x)}\|_{W^{k+1,p}} \leq C(n, C_{k,p}, X).$$

for any $p > 1$.

Proof. From the convergence theory, around each $x_i \in M_i$ the local universal cover converges in $C^{k+1,\alpha}$ (resp. $W^{k+1,p}$) in the local equivariant Gromov–Hausdorff sense.

Hölder case: the metric is locally $C^{k+1,\alpha}$, so the harmonic functions constructed in the CFG fibration proof admit uniform $C^{k+2,\alpha}$ Schauder estimates. The fibration regularity follows.

Sobolev case: using $\Delta(\nabla u) = \nabla \Delta u + \text{Ric}(\nabla u)$ and $W^{k+2,p}$ estimates together with the Banach algebra structure of $W^{k,p}$ for large p , one obtains $W^{k+3,p}$ control. $\square$

Theorem 4.3 (CFG connection difference regularity). *Under the Assumption 2, for any open $U \subseteq X$ and for all large i , there exists a family of flat connections $\{\nabla_i^x\}_{x \in U}$ on the fiber $Z_i^x = F_i^{-1}(x)$ such that:*

1. $\{\nabla_i^x\}$ is a uniform $C^{k+1,\alpha}$ (resp. $W^{k+2,p}$) family in $x \in U$.
2. The difference from the Levi-Civita connection satisfies

$$\|\nabla_i^x - \nabla_i^{LC}\|_{C^{k,\alpha}} \leq C(n, C_k, X) \cdot \tau_i \quad (\text{resp. } \|\cdot\|_{W^{k+1,p}} \leq C\tau_i),$$

where $\tau_i \rightarrow 0$ is the collapse scale.

3. For each fiber Z_i^x , there is an affine isomorphism $\psi_x : \Gamma \backslash \mathcal{N} \rightarrow Z_i^x$ with uniform $C^{k+2,\alpha}$ (resp. $W^{k+3,p}$) estimate, depending smoothly on x in $C^{k+1,\alpha}$ (resp. $W^{k+2,p}$).

Proof. Parts (1) and (2) follow from the Section 4 of Cheeger-Fukaya-Gromov [19] construction of local invariant pure nilpotent structures. The bounded curvature condition $\|\text{Rm}(g_i)\|_{C^k} \leq C_k$ ensures that, locally, the geometry is controlled in $C^{k+1,\alpha}$ or $W^{k+2,p}$. The affine connection ∇_i^x is constructed in [19] by an averaging process from the construction of Theorem 1.12 in the z -variable, which involves integration of the $g^{q,q'}\nabla^{q'}$, which have $C^{k,\alpha}$ or $W^{k+1,p}$ regularity (this takes one derivative of the $g^{q,q'}$), against the invariant measure, which has $C^{k+1,\alpha}$ or $W^{k+2,p}$ regularity, since it only depends on the underlying Riemannian structure. The difference tensor $S_i = \nabla_i^x - \nabla_i^{LC}$ satisfies the elliptic estimates bounded by the scale of the collapse τ_i , yielding the uniform bounds in $C^{k,\alpha}$ (or $W^{k+1,p}$) on each fiber, which gives Part (2). Since we are averaging with respect to the Riemannian volume form, the connection ∇_i^x itself has $C^{k+1,\alpha}$ (or $W^{k+2,p}$) regularity in the parameter x , which gives us Part (1).

For part (3), we must construct the affine isomorphism $\psi_x : \Gamma \backslash \mathcal{N} \rightarrow Z_i^x$ and establish its regularity. First, we need to apply Malcev's Theorem. For any simply connected open subset $U \subseteq X$, by choosing a local trivialization, consider $F_i^{-1}(U) \simeq U \times \Gamma \backslash \mathcal{N}$ and $F_i^{-\tilde{1}}(U) \simeq U \times \mathcal{N}$. This induces a family of homomorphism from $\pi_1(Z_i^x)$ to Γ . As in [19], By Malcev's Theorem, any finitely generated torsion-free nilpotent group Γ uniquely embeds as a uniform (cocompact) lattice in a unique simply connected nilpotent Lie group $\mathcal{N}$. The real form of $\mathcal{N}$ is uniquely determined by the rational completion of Γ . This gives us our standard model space $\Gamma \backslash \mathcal{N}$.

As in [19], The connection ∇_i^x on Z_i^x constructed in part (1) is flat and has parallel torsion. Therefore, Z_i^x is an affinely flat manifold. We consider the universal cover $\tilde{Z}_i^x$, equipped with the pullback connection $\tilde{\nabla}_i^x$. Because $\tilde{\nabla}_i^x$ is flat and has parallel torsion, we can construct a developing map $D_x : \tilde{Z}_i^x \rightarrow \mathcal{N}$. By the [19], the holonomy group of ∇_i^x is trivial, and the monodromy representation maps $\pi_1(Z_i^x) = \Gamma$ into the affine group $\mathcal{N}_L \rtimes \text{Aut}(\mathcal{N})$. The developing map is an equivariant affine diffeomorphism between $\tilde{Z}_i^x$ and $\mathcal{N}$ hence this map

descends to the quotient by the action of Γ

$$\psi_x : \Gamma \backslash \mathcal{N} \rightarrow Z_i^x. \quad (4.9)$$

The regularity of the map ψ_x is determined by the regularity of the affine connection ∇_i^x from which the developing map D_x is constructed. The developing map is essentially obtained by fixing a frame and integrating the parallel transport equation. In particular, it solves

$$\frac{\partial^2(\psi_x)^a}{\partial u^b \partial u^c} + (\Gamma_i^x)_{bc}^k \frac{\partial(\psi_x)^a}{\partial u^k} = 0 \quad (4.10)$$

where u are the local coordinates on $\Gamma \backslash \mathcal{N}$. It immediately follows that if the connection ∇_i^x is in $C^{k,\alpha}$, then ψ_x is uniformly bounded in $C^{k+2,\alpha}$ with respect to the fiber variables. Similarly, if $\nabla_i^x \in W^{k+1,p}$, the resulting map ψ_x is in $W^{k+3,p}$.

Finally, the dependence of ψ_x on the base point $x \in U$ is determined by the variation of the connections ∇_i^x with respect to x , which is of $C^{k+1,\alpha}$ (or $W^{k+2,p}$). $\square$

Lemma 4.4 (Local trivialization regularity). *Under the Assumption 2, for each small simply connected open $U \subseteq X$ there exists a local trivialization*

$$\Phi_{U,i} : F_i^{-1}(U) \xrightarrow{\sim} U \times \Gamma \backslash \mathcal{N}$$

with uniform $C^{k+2,\alpha}$ (resp. $W^{k+3,p}$) estimate.

Proof. With the assumption above, by Corollary 4.2, the fibration map $F_i : M_i \rightarrow X$, satisfies uniform $\|F_i\|_{C^{k+2,\alpha}}$ or $\|F_i\|_{W^{k+3,p}}$ estimate. Let $U \subseteq X$ be a simply connected open subset. Since F_i are submersions, by implicit function theorem, there exists a local section $\sigma_i : U \rightarrow F_i^{-1}(U) \subseteq M_i$, so that $F_i \circ \sigma_i = \text{Id}$ on U with uniform $\|\sigma_i\|_{C^{k+2,\alpha}}$ or $\|\sigma_i\|_{W^{k+3,p}}$ estimate respectively.

On the local universal cover, $F_i^{-\tilde{1}}(U)$, we have the fiber-wise affine isomorphism:

$$\tilde{\psi}_x : \mathcal{N} \rightarrow \tilde{Z}_i^x. \quad (4.11)$$

With a little abuse of notation, we denote $\tilde{\psi}_x : \mathcal{N}_L \rightarrow \text{Aut}(\tilde{Z}_i^x)$ as the isomorphism of group action. By Theorem 1.13, the fiber-wise group action generated by the right invariant vector field on $\mathcal{N}$ extend to a group action on $F_i^{-\tilde{1}}(U)$. For each fiber $\tilde{Z}_i^{x'}$, we have the map

$$\phi_{x'} : \text{Aut}(Z_i^x) \rightarrow \text{Aut}(Z_i^{x'}). \quad (4.12)$$

Thus, we define the fiber-wise action on x' by: $\psi_{x'} := \phi_{x'}^{-1} \circ \tilde{\psi}_x$. which gives us the action as

$$\tilde{\psi}_i : \mathcal{N}_L \rightarrow \text{Aut}(F_i^{-\tilde{1}}(U)) \quad (4.13)$$

with the property that $F_i(\tilde{\psi}_i(g)p) = F_i(p)$, for any $p \in M_i$, $g \in \mathcal{N}_L$. By (3) of Theorem 4.3 and Theorem 1.13, the mapping $\tilde{\psi}_i$ has a uniform $C^{k+2,\alpha}$ or $W_{loc}^{k+3,p}$ estimate, respectively.

Thus, we can define our local trivialization

$$\Phi_{U,i}^{-1} : U \times \mathcal{N} \rightarrow F_i^{-\tilde{1}}(U) \quad (4.14)$$

$$(x, g) \mapsto \tilde{\psi}_i(g) \cdot \tilde{\sigma}_i(x), \quad (4.15)$$

where $\tilde{\sigma}_i$ is a lift to $F_i^{-\tilde{1}}(U)$. Note that, by Theorem 1.13, $\tilde{\psi}_i$ is invariant under Γ . Thus, $\Phi_{U,i}^{-1}$ descends to $U \times \Gamma \backslash \mathcal{N}$, which is the desired local trivialization with regularity. $\square$

Remark 4.4. *The construction of this local trivialization is different from the one from [19]. In [19], they use the normal exponential map to construct the tubular neighborhood of the center fiber. In our case, we use the improved regularity given by the harmonic coordinate, which is optimal. Also, instead of using the family of affine isomorphism of the fiber using*

Riemannian structure, we directly use the fiber-wise isomorphism given by Malcev's Theorem, which does not lose regularity.

4.2.2 The $\mathcal{N}$ -invariant nearby metric

Since the definition of right invariant vector field is independent the choice of the local trivialization, we have the following regularity for the right invariant vector field.

Corollary 4.5 (Right-invariant vector field regularity). *Let V be a right-invariant vector field on $\widetilde{F_i^{-1}(U)} \cap B_{\tilde{x}_i}(2\tau_i)$. Then*

$$\|V\|_{C^{k+1,\alpha}} \leq C(n, C_k, X) \cdot |V(\tilde{x}_i)| \quad (\text{resp. } \|V\|_{W^{k+2,p}} \leq C|V(\tilde{x}_i)|).$$

Proof. Note, the right-invariant vector field on $F_i^{-\tilde{1}}(U)$ is given by the pull-back of the right-invariant vector field using $\tilde{\Phi}_{U,i}$. Then, the C^0 (L^∞) norm of V in both cases is uniformly bounded by $|V(\tilde{x}_i)|$, depends on the C^0 norm of the Riemannian metric.

Let $S^x := \nabla_i^x - \nabla_i^{LC}$ on the fiber. Denote A^x as the second fundamental form of the fiber of $F_i^{-1}(x)$, which we regard as a tensor on the universal cover $F_i^{-\tilde{1}}(U)$. By Theorem 1.11, $\|A^x\|_{C^0}$ is uniformly bounded. Since V is a right invariant vector field, by (1.16), we have:

$$\|\nabla V\|_{C^0} \leq \|S^x V\|_{C^0} + \|A^x V\|_{C^0} + \|\nabla_i^x V\|_{C^0} \leq C\|V\|_{C^0}, \quad (4.16)$$

by ODE comparison, we have the desired C^1 estimate. For the higher order estimate, since $\nabla_i^x V$ produces a constant right invariant vector field (recall (1.16)), we have

$$\|\nabla^k V\|_{C^0} \leq \|(\nabla^{k-1} S^x) V\|_{C^0} + \|(\nabla^{k-1} A^x) V\|_{C^0}. \quad (4.17)$$

Thus, the estimate follows from Corollary 4.2 and Theorem 4.3. □

Now, we are ready to construct the $\mathcal{N}$ -invariant metric

Definition 4.1. *The tensor T_i on M_i is called $\mathcal{N}$ -**invariant** if for every $x \in X$, there is a simply connected neighborhood $U \subseteq X$ such that $F_i^{-1}(U) \simeq U \times \mathcal{N}/\Gamma$. Then on the universal cover $F_i^{-1}(\tilde{U}) \simeq U \times \mathcal{N}$, the pull-back tensor $\tilde{T}_i$ is invariant under $\mathcal{N}$ -action.*

Follow [19], we can construct the nearby $\mathcal{N}$ -invariant metric using average method. The upshot is that we have the finite regularity estimate.

Theorem 4.6 ($\mathcal{N}$ -invariant nearby metric estimate). *Under the finite regularity assumption, for all large i there exists an $\mathcal{N}$ -invariant metric $\hat{g}_i$ on M_i such that, for every $U \subseteq X$ and any compact $K \subseteq \widetilde{F_i^{-1}(U)}$:*

$$\|\hat{g}_i - g_i\|_{C^{k,\alpha}} \leq C(n, C_k, X) \cdot \tau_i \quad (\text{resp. } \|\hat{g}_i - g_i\|_{W_{\text{loc}}^{k+1,p}(K)} \leq C(n, C_{k,p}, X, K) \cdot \tau_i).$$

Proof. The construction follows the averaging method in the section 4 of [19]. By Theorem 1.11, $\Lambda = \mathcal{N}_L \cap \Gamma$ is normal in Γ with uniform finite index, by passing to the finite cover, it is enough to consider the case $\Gamma \subseteq \mathcal{N}_L$.

On $F_i^{-1}(\tilde{U}) \simeq U \times \mathcal{N}$. Let $h \in \mathcal{N}_L$, then we regard $h \in \text{Aut}(F_i^{-1}(\tilde{U}))$. Consider $h \mapsto h^*g_i$. Note that this map is constant on the Γ orbit. Since the group $\mathcal{N}_L$ is nilpotent, it is unimodular. Therefore, the space $\Gamma \backslash \mathcal{N}_L$ admits a canonical left-invariant measure $d\mu$ of total volume 1. Thus, we construct the metric:

$$\hat{g}_i = \int_{\Gamma \backslash \mathcal{N}} h^* g_i d\mu. \tag{4.18}$$

Thus, $\hat{g}_i$ is a left-invariant metric on $F_i^{-1}(\tilde{U})$.

For each $h \in \mathcal{N}_L$, there exists a right invariant vector field V_h so that $h = \exp(V_h)$. By the

fundamental theorem of calculus, we have:

$$h^*g_i - g_i = \int_0^1 \frac{d}{dt} (\exp^*(tV_h)g_i) dt = \int_0^1 (\exp^*(tV_h))(\mathcal{L}_{V_h}g_i) dt. \quad (4.19)$$

Thus, we have:

$$\|\hat{g}_i - g_i\| \leq C_n \max_{h \in \Gamma \setminus \mathcal{N}} (\|D \exp(V_h)\| \cdot \|\nabla V_h\|). \quad (4.20)$$

Now by the collapsing assumption, the diameter of each fiber is less than τ_i . For every $x \in U$, choose a lift $\tilde{x}_i$ on $\tilde{F}_i^{-1}(U)$. Then $|V_h(\tilde{x}_i)| \leq C \cdot \tau_i$. By the expansion of Jacobi field (see, for example, Section 5 of [45]), k -th nilpotency of group $\mathcal{N}$, and Theorem 4.3,

$$\|D \exp(V_h)\| = \left\| \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \text{ad}_{V_h}^n \right\| \leq \left\| \sum_{n=0}^k \frac{(-1)^n}{(n+1)!} \text{ad}_{V_h}^n \right\| \leq \|(1 + C\tau_i) \text{Id}\| \leq C. \quad (4.21)$$

By Corollary 4.5, $\|\nabla V_h\| \leq C \cdot \tau_i$, so we are done. $\square$

Corollary 4.7 (Curvature estimate for invariant nearby metric). *With the notation of Theorem 4.6, for $k \geq 2$:*

$$\|\text{Rm}(\hat{g}_i) - \text{Rm}(g_i)\|_{C^{k-2,\alpha}} \leq C(n, C_k, X) \cdot \tau_i,$$

and similarly in $W_{\text{loc}}^{k-1,p}$.

Proof. Immediate from Theorem 4.6 and the fact that Rm is a second-order differential operator in the metric. $\square$

4.3 Convergence of Eigenvalues of the Laplacian

The key spectral input for the proof of Theorem 4.1 is Fukaya's theorem [27] on eigenvalue convergence under measured Gromov–Hausdorff collapse. We state the relevant definitions and the theorem precisely.

Definition 4.2 (Measured Gromov–Hausdorff convergence). *Let μ be a Borel measure on X . We say*

$$(X_\alpha, d_\alpha, \mu_\alpha) \rightarrow (X, d_\infty, \mu)$$

in the measured Gromov–Hausdorff (mGH) sense if there exists a Borel measurable ε_α -approximation $\psi_\alpha : X_\alpha \rightarrow X$ with $\varepsilon_\alpha \rightarrow 0$ such that the pushed-forward measures converge weakly:

$$\int_X u d((\psi_\alpha)_* \mu_\alpha) \rightarrow \int_X u d\mu \quad \text{for all } u \in C^\infty(X).$$

In the collapsing setting, set $\mu_i = \text{dvol}_{g_i} / \text{vol}(M_i)$. The collapse $(M_i, g_i) \rightarrow (X, g_\infty)$ then gives the mGH convergence $(M_i, g_i, \mu_i) \rightarrow (X, g_\infty, \mu)$, where $\mu = \chi \text{dvol}_{g_\infty}$ and χ is the *relative volume function*

$$\chi(x) = \lim_{i \rightarrow \infty} \frac{\text{vol}(F_i^{-1}(x))}{\text{vol}(M_i)}.$$

By the improved regularity of the fibration F_i (Theorem ??), $\chi_i \rightarrow \chi$ in $C^{1,\alpha}$ and $W^{2,p}$ for any $0 < \alpha < 1$ and $p > 1$.

Theorem 4.8 (Eigenvalue convergence under collapsing [27]). *In the collapsing setting $(M_i, g_i) \rightarrow (X, g_\infty)$ with $d = \dim X \geq 1$, let $\lambda_k(M_i, g_i)$ denote the k -th eigenvalue of the Laplacian on (M_i, g_i) . Then:*

1. $\lim_{i \rightarrow \infty} \lambda_k(M_i, g_i) = \lambda_k$ exists for each $k \geq 0$.
2. λ_k is the k -th eigenvalue of the drift Laplacian Δ_χ on $L^2(X, \mu)$, defined weakly by the bilinear form

$$Q(f_1, f_2) = \int_X \langle \nabla f_1, \nabla f_2 \rangle_{g_\infty} d\mu,$$

or explicitly

$$\Delta_\chi = \Delta_{g_\infty} + \langle \nabla \log \chi, \nabla \cdot \rangle_{g_\infty}.$$

Proof. Take the mGH convergence $(M_i, g_i, \mu_i) \rightarrow (X, g_\infty, \mu)$. The relative volume functions χ_i are uniformly bounded and equicontinuous by the CFG fibration theorem, so by Arzelà–Ascoli, $\chi_i \rightarrow \chi$ in $C^{1,\alpha}$. Integration by parts on the eigenfunction equation gives the weak formula for Δ_χ :

$$\int_X \langle \nabla f_1, \nabla f_2 \rangle d\mu = \int_X \langle \nabla f_1, \nabla f_2 \rangle \chi \, \text{dvol}_{g_\infty} = - \int_X (\Delta_{g_\infty} f_1 + \langle \nabla \log \chi, \nabla f_1 \rangle) f_2 d\mu.$$

The eigenvalue convergence $\lambda_k(M_i, g_i) \rightarrow \lambda_k$ is the content of [27]. □

Remark 4.5. Applying this to our setting: under the bounded curvature assumption, $\chi_i \rightarrow \chi$ in $C^{1,\alpha}$ and $W^{2,p}$. By the averaging construction of $\hat{g}_i$ in Theorem 4.6, the fiber volume is unchanged, hence $\chi_i(g_i) = \chi_i(\hat{g}_i)$, and we also have $\chi(\hat{g}_i) \rightarrow \chi$ in $C^{1,\alpha}$ and $W^{2,p}$.

Remark 4.6. Theorem 4.8 does not assert pointwise convergence of Δ_{g_i} to Δ_χ . It does imply: if $v/(n-1) \notin \text{Spec}(-\Delta_\chi)$, then for all large i the value $v/(n-1)$ is also not an eigenvalue of the relevant operator on (M_i, g_i) . This is the spectral condition needed in Theorem 4.1 and applied in the uniform injectivity estimate (Proposition 4.13).

Note that the spectral condition is only required when $v > 0$. When $v \leq 0$, the linearized

operator is automatically invertible (resp. invertible on mean-value-zero functions when $v = 0$) without any additional assumption.

4.4 Proof Strategy: Invariant Metric and Scalar Curvature Estimates

In this section we carry out Steps 1 and 2 of the proof of Theorem 4.1: producing the $\mathcal{N}$ -invariant metric $\hat{g}_i$ close to g_i in $W^{1,p}$, and establishing the scalar curvature estimate in the distributional norm $W^{-1,p}$ on the base. We work throughout under Assumption 3 with only $\|\text{Rm}(g_i)\|_{C^0} \leq C_0$.

Assumption 3. *Let $\{(M_i^n, g_i)\}$ be a sequence of closed n -dimensional Riemannian manifold such that:*

- *The Riemannian curvature is uniformly bounded:*

$$\|\text{Rm}(g_i)\|_{C^0} \leq C_0, \tag{4.22}$$

- *The diameter is bounded from above: $d(M_i, g_i) \leq C$.*
- *The scalar curvature $S(g_i) \equiv v$, where v is a constant;*
- *The sequence collapses to a low dimensional manifold, i.e.:*

$$(M_i^n, g_i) \rightarrow (X^d, g_\infty) \tag{4.23}$$

in the Gromov-Hausdorff sense to a manifold X , where $d < n$.

- If $v > 0$, the operator

$$\Delta_\chi h \equiv \Delta_X h + \nabla \log \chi \cdot \nabla_X h \quad (4.24)$$

does not have $\frac{v}{n-1}$ as an eigenvalue, where χ is defined above, i.e, there are no non-zero solutions on X of

$$-\Delta_\chi h = \frac{v}{n-1} h. \quad (4.25)$$

4.4.1 The $\mathcal{N}$ -invariant nearby metric

By Theorem 4.6 applied with $k = 0$, for all sufficiently large i there exists an $\mathcal{N}$ -invariant metric $\hat{g}_i$ on M_i satisfying

$$\|\hat{g}_i - g_i\|_{C^{0,\alpha}} \leq C(n, C_0, X) \cdot \tau_i, \quad (4.26)$$

$$\|\hat{g}_i - g_i\|_{W_{loc}^{1,p}(K)} \leq C(n, C_0, p, X, K) \cdot \tau_i \quad (4.27)$$

for any $0 < \alpha < 1$, $p > n$, and compact $K \subseteq \widetilde{F_i^{-1}(U)}$.

Remark 4.7. *Unlike the higher-regularity case, when only $\|\text{Rm}(g_i)\|_{C^0}$ is bounded, the curvature tensor of $\hat{g}_i$ is not necessarily close to that of g_i . This is because the CFG averaging procedure controls the metric only in $C^{0,\alpha}$ and $W^{1,p}$.*

4.4.2 Scalar curvature in the distributional norm

Since $\hat{g}_i$ only has $W^{1,p}$ bounds, its scalar curvature $S(\hat{g}_i)$ is not pointwise bounded. However, the difference $S(\hat{g}_i) - S(g_i)$ is controlled in a negative Sobolev norm.

Recall the Lee–LeFloch formula [44]: for two metrics $g, h \in W_{loc}^{1,\infty}(M)$ with difference tensor

$$\Upsilon = \Gamma(g) - \Gamma(h),$$

$$S(g) - S(h) = \operatorname{div}_h T + Q, \quad (4.28)$$

where $T^k = g^{ij}\Upsilon_{ij}^k - g^{ik}\Upsilon_{ji}^j$ and Q is quadratic in Υ . Integrating by parts, $S(g) - S(h)$ defines a bounded functional on $W_0^{1,p}(M)$ via

$$\langle S(g) - S(h), \phi \rangle := \int_M \left(-T \cdot \nabla^h \left(\phi \frac{d\operatorname{vol}(g)}{d\operatorname{vol}(h)} \right) + Q \phi \frac{d\operatorname{vol}(g)}{d\operatorname{vol}(h)} \right) d\operatorname{vol}(h). \quad (4.29)$$

In particular,

$$\|S(g) - S(h)\|_{W^{-1,p}(K)} \leq C(n, p, K) \cdot \|g - h\|_{W^{1,p}(K)}. \quad (4.30)$$

Applying (4.30) to $g = \hat{g}_i$ and $h = g_i$ with $S(g_i) \equiv v$, and using (4.27):

Theorem 4.9. *Let $\{(M_i^n, g_i)\}$ satisfy Assumption 3. Then on $\widetilde{F_i^{-1}(U)}$, for every $p > n$ and compact K ,*

$$\|S(\hat{g}_i) - v\|_{W^{-1,p}(K)} \leq C(n, C_0, X, K) \cdot \tau_i. \quad (4.31)$$

4.4.3 Laplacian on $\mathcal{N}$ -invariant functions

Lemma 4.10 (Laplacian on $\mathcal{N}$ -invariant functions [62]). *Let $u \in C^\infty(M_i)$ be $\mathcal{N}$ -invariant and $\hat{g}_i$ $\mathcal{N}$ -invariant. Using the second fundamental form decomposition $\nabla_{\partial_a} \partial_b = \bar{\nabla}_{\partial_a} \partial_b + A(\partial_a, \partial_b)$,*

$$\Delta_{\hat{g}_i} u = \Delta_{g_{X,i}} u + \langle H_i, \nabla_X u \rangle, \quad (4.32)$$

where $g_{X,i} = \hat{g}_i|_{\ker dF_i^\perp}$ is the base metric and $H_i = \text{tr } A_i$ is the mean curvature vector of the fibers.

Moreover, $g_{X,i} \rightarrow g_\infty$ in the $W^{1,p}$ sense on X , and $H_i \rightarrow \nabla_{g_\infty} \log \chi$ in $W^{1,p}$.

Consequently, defining $d\nu_i = \chi_i d\text{vol}(g_{X,i})$ and the drift Laplacian $\Delta_{\nu_i} := \Delta_{g_{X,i}} + \langle \nabla \log \chi_i, \nabla \cdot \rangle$, for $\mathcal{N}$ -invariant functions,

$$\Delta_{\hat{g}_i} u = \Delta_{\nu_i} \Pi_i(u), \quad (4.33)$$

and in the limit,

$$\Delta_{\nu_i} \xrightarrow{i \rightarrow \infty} \Delta_\chi := \Delta_{g_\infty} + \langle \nabla \log \chi, \nabla \cdot \rangle_{g_\infty} \quad \text{in } W^{1,p}\text{-sense.} \quad (4.34)$$

Proof. The splitting (4.32) follows from the definition of the second fundamental form and $\mathcal{N}$ -invariance: since u is constant on fibers, ∇u is horizontal, and the fiber Laplacian contributes only $\langle H_i, \nabla u \rangle$.

For the convergences: by the equivariant Gromov-Hausdorff diagram

$$\begin{array}{ccc} (\widetilde{F_i^{-1}(U)}, \tilde{g}_i, \Gamma_i, \tilde{p}_i) & \xrightarrow{eq \ GH} & (\tilde{U}_\infty, \tilde{g}_\infty, \mathcal{N}, \tilde{p}_\infty) \\ \downarrow \pi_i & & \downarrow \pi_\infty \\ (F_i^{-1}(U), g_i) & \xrightarrow{F_i} & (U, g_\infty) \end{array}$$

with F_i of uniform $W^{3,p}$ bound (Corollary 4.2), the inner products of horizontal lifts converge in $W^{1,p}$, giving $g_{X,i} \rightarrow g_\infty$.

For the mean curvature: the $\mathcal{N}$ -invariant metric decomposes as $\hat{g}_i = F_i^* g_{X,i} + (c_i \chi_i)^{2/(n-d)} g_{\mathcal{N}}(\omega_i(\cdot), \omega_i(\cdot))$, so $H_i = \nabla_{g_i} \log \chi_i$. Since the fibration converges in $W^{3,p}$, $A_i \rightarrow A_\infty$ in $W^{1,p}$, hence $H_i \rightarrow \nabla_{g_\infty} \log \chi$ in $W^{1,p}$. $\square$

4.4.4 Weighted Sobolev spaces on the base

Since $\hat{g}_i$ is $\mathcal{N}$ -invariant, every $\mathcal{N}$ -invariant function u on M_i corresponds to a function $\Pi_i(u)$ on X . We transfer the analysis to X using the weighted measure $d\nu_i = \chi_i d\text{vol}(g_{X,i})$.

Definition 4.3. Define $L^p(X, \nu_i)$ with norm $\|f\|_{L^p(X, \nu_i)}^p = \int_X |f|^p d\nu_i$, and $W^{1,p}(X, \nu_i)$ with norm $\|f\|_{W^{1,p}(X, \nu_i)} = \|\nabla f\|_{L^p(X, \nu_i)} + \|f\|_{L^p(X, \nu_i)}$.

Define the dual space $W^{-1,p}(X, \nu_i) = (W^{1,q}(X, \nu_i))^*$ with $1/p + 1/q = 1$.

When $v = 0$, define the mean-value-zero subspaces:

$$W_0^{1,p}(X, \nu_i) := \left\{ \phi \in W^{1,p}(X, \nu_i) : \int_X \phi d\nu_i = 0 \right\}, \quad (4.35)$$

$$W_0^{-1,p}(X, \nu_i) := \{f \in W^{-1,p}(X, \nu_i) : f(1) = 0\}. \quad (4.36)$$

The identification Π_i intertwines the $W_{loc}^{1,p}$ -norm on $\widetilde{F_i^{-1}(U)}$ with the $W^{1,p}(X, \nu_i)$ -norm, and similarly for $W^{-1,p}$. In particular, Theorem 4.9 gives

$$\|\Pi_i(S(\hat{g}_i)) - v\|_{W^{-1,p}(X, \nu_i)} \leq C\tau_i. \quad (4.37)$$

4.5 Linearized Equation Estimates

In Step 3 of the proof of Theorem 4.1, we seek a conformal factor $\psi_i = 1 + h_i$ (an $\mathcal{N}$ -invariant function on X) such that $g'_i = \psi_i^{4/(n-2)} \hat{g}_i$ has exactly constant scalar curvature. This section establishes the analytic estimates on the base $(X, g_{X,i}, d\nu_i)$ needed to apply the implicit function theorem.

4.5.1 The operator P and its linearization

Proposition 4.11 (P -operator in the Sobolev setting). *Under Assumption 3, let $\hat{g}_i$ be the $\mathcal{N}$ -invariant metric from Theorem 4.6. Set $\Theta_i := v - S(\hat{g}_i)$.*

For $\mathcal{N}$ -invariant ϕ , via the projection $\Pi_i(\phi) \in C^\infty(X)$ and the drift Laplacian $\Delta_{\nu_i} = \Delta_{g_{X,i}} + \langle \nabla \log \chi_i, \nabla \cdot \rangle$, define the conformal perturbation operator on the base:

$$P_{\hat{g}_i}(\phi) := -4 \frac{n-1}{n-2} \Delta_{\nu_i} \phi + S(\hat{g}_i)(1 + \phi) - v(1 + \phi)^{(n+2)/(n-2)}, \quad (4.38)$$

viewed as a map $P_{\hat{g}_i} : W^{1,p}(X, \nu_i) \rightarrow W^{-1,p}(X, \nu_i)$ for $p > d$.

Write $P_{\hat{g}_i}(h) = P_{\hat{g}_i}(0) + \hat{L}_{\hat{g}_i}h + Q_{\hat{g}_i}(h)$ where

$$\hat{L}_{\hat{g}_i} := -4 \frac{n-1}{n-2} \Delta_{\nu_i} - \left(v \cdot \frac{4}{n-2} - \Theta_i \right). \quad (4.39)$$

Then:

(i) **Initial error:** *for $i \gg 0$,*

$$\|P_{\hat{g}_i}(0)\|_{W^{-1,p}(X, \nu_i)} = \|\Theta_i\|_{W^{-1,p}(X, \nu_i)} \leq C\tau_i. \quad (4.40)$$

(ii) **Linearization:** *the linearized operator at $h = 0$ is $\hat{L}_{\hat{g}_i}$.*

(iii) **Quadratic remainder:** *for $h_j \in W^{1,p}(X, \nu_i)$ with $\|h_j\|_{W^{1,p}} < s_0$ sufficiently small, there exists $C_2 = C_2(s_0)$ so that*

$$\|Q_{\hat{g}_i}(h_1) - Q_{\hat{g}_i}(h_2)\|_{W^{-1,p}(X, \nu_i)} \leq C_2(\|h_1\|_{W^{1,p}} + \|h_2\|_{W^{1,p}})\|h_1 - h_2\|_{W^{1,p}}. \quad (4.41)$$

Proof. Part (i): $P_{\hat{g}_i}(0) = \Theta_i = v - S(\hat{g}_i)$. By (4.37), $\|\Theta_i\|_{W^{-1,p}(X, \nu_i)} \leq C\tau_i$.

Part (ii): Direct computation; differentiating the Yamabe equation $-a_n \Delta_{\hat{g}_i}(1+\phi) + S(\hat{g}_i)(1+\phi) = S(g'_i)(1+\phi)^{(n+2)/(n-2)}$ (with $a_n = 4(n-1)/(n-2)$) at $\phi = 0$ gives $\hat{L}_{\hat{g}_i}$.

Part (iii): The nonlinear term $Q_{\hat{g}_i}(h) = -v[(1+h)^{(n+2)/(n-2)} - 1 - \frac{n+2}{n-2}h]$ is analytic in h . The Sobolev embedding $W^{1,p}(X, \nu_i) \hookrightarrow C^0(X)$ (for $p > d$) and Taylor's theorem give the Lipschitz estimate (4.41). $\square$

In the case $v = 0$, we need to consider the space $W_0^{1,p}(X, \nu_i) \subseteq W^{1,p}(X, \nu_i)$ defined as

$$\int_X \phi d\nu_i = 0 \quad (4.42)$$

for $\phi \in W_0^{1,p}(X, \nu_i)$. We define the nonlinear functional:

$$v_i = v_i(\phi) = \frac{\int_X S(\hat{g}_i)(1+\phi) d\nu_i}{\int_X (1+\phi)^{\frac{n+2}{n-2}} d\nu_i}. \quad (4.43)$$

Thus

$$v_i(0) = \int_X S(\hat{g}_i) d\nu_i \quad (4.44)$$

Then define

$$P_{0,\hat{g}_i} : W_0^{1,p}(X, \nu) \rightarrow W_0^{-1,p}(X, \nu) \quad (4.45)$$

by

$$\phi \mapsto -4 \frac{n-1}{n-2} \Delta_{\hat{g}_i} \phi + S(\hat{g}_i) \cdot (1+\phi) - v_i(\phi) \cdot (1+\phi)^{\frac{n+2}{n-2}}, \quad (4.46)$$

The linearized operator can be computed as

$$\hat{L}_{0,\hat{g}_i}h = -4\frac{n-1}{n-2}\Delta_{\nu_i}h + \left(\Theta_i - \frac{n+2}{n-2}v_i(0)\right)h - \int_X \left(\Theta_i - \frac{n+2}{n-2}v_i(0)\right)h d\nu'_i. \quad (4.47)$$

We also have

$$Q_{0,\hat{g}_i}(h) = P_{0,\hat{g}_i}(h) - P_{0,\hat{g}_i}(0) - \hat{L}_{0,\hat{g}_i}h, \quad (4.48)$$

and it can be computed as

$$Q_{0,\hat{g}_i}(h) = -v_i(h) \cdot (1+h)^{\frac{n+2}{n-2}} + \left(\frac{n+2}{n-2}\right)v_i(0) \cdot h + \int_X \left(\Theta_i - \left(\frac{n+2}{n-2}\right)v_i(0)\right)h d\nu_i + v_i(0). \quad (4.49)$$

The analog of Proposition 4.11 holds for $P_{0,\hat{g}_i}$, we do not restate it here.

4.5.2 Calderón-Zygmund estimate

Proposition 4.12 (Calderón-Zygmund estimate on the base). *For $p > d$ and $i \gg 1$, there exists a constant C , independent of i , such that for any $h \in W^{1,p}(X, \nu_i)$:*

$$\|h\|_{W^{1,p}(X, \nu_i)} \leq C \left(\|\hat{L}_{\hat{g}_i}h\|_{W^{-1,p}(X, \nu_i)} + \|h\|_{W^{-1,p}(X, \nu_i)} \right). \quad (4.50)$$

Proof. Since $g_{X,i} \rightarrow g_\infty$ in $C^{0,\alpha}$ (Lemma 4.10), the operators $\hat{L}_{\hat{g}_i}$ converge to $\hat{L}_\infty = -4\frac{n-1}{n-2}\Delta_X - v \cdot \frac{4}{n-2}$ in an appropriate sense. These are non-collapsing elliptic operators on X with uniformly bounded coefficients (in $W^{1,p}$), so standard Calderón-Zygmund theory applies. The constant C is independent of i for large i . $\square$

4.5.3 Uniform injectivity

Proposition 4.13 (Uniform injectivity of $\hat{L}_{\hat{g}_i}$). *Under Assumption 3 with limiting manifold (X, g_∞) .*

Case $v \neq 0$: *For i sufficiently large, the map $\hat{L}_{\hat{g}_i} : W^{1,p}(X, \nu_i) \rightarrow W^{-1,p}(X, \nu_i)$ is uniformly injective for $p > d$: there exists $\delta_0 > 0$ independent of i such that*

$$\|\hat{L}_{\hat{g}_i} h\|_{W^{-1,p}(X, \nu_i)} \geq \delta_0 \|h\|_{W^{1,p}(X, \nu_i)} \quad (4.51)$$

for all $h \in W^{1,p}(X, \nu_i)$.

When $v > 0$, this uses the **spectral condition** (4.5): $v/(n-1) \notin \text{Spec}(-\Delta_\chi)$. When $v < 0$, the estimate holds without any spectral condition.

Case $v = 0$: *The map $\hat{L}_{0, \hat{g}_i} : W_0^{1,p}(X, \nu_i) \rightarrow W_0^{-1,p}(X, \nu_i)$ satisfies the same uniform injectivity estimate (4.51) for $h \in W_0^{1,p}(X, \nu_i)$. Moreover, $\text{Ker}(\hat{L}_{0, \hat{g}_i}) \cong \text{Coker}(\hat{L}_{0, \hat{g}_i}) \cong \mathbb{R}$.*

Proof. **Case $v \neq 0$:** By contradiction. If (4.51) fails, there exist $h_i \in W^{1,p}(X, \nu_i)$ with $\|h_i\|_{W^{1,p}(X, \nu_i)} = 1$ and $\varepsilon_i := \hat{L}_{\hat{g}_i} h_i \rightarrow 0$ in $W^{-1,p}(X, \nu_i)$.

Since $p > d$, the Sobolev embedding gives $h_i \in C^{0,\alpha}(X)$ ($\alpha = 1 - d/p$) with uniform bound. By Arzelà-Ascoli, passing to a subsequence, $h_i \rightarrow h_\infty$ in $C^{0,\beta}$ for $\beta < \alpha$.

Since $\chi_i \rightarrow \chi_\infty$ in $C^{1,\alpha}$ and $d\nu_i \rightarrow d\nu_\infty$ weakly, for any test function $\psi \in W^{1,p^*}(X, g_\infty)$ (where $1/p + 1/p^* = 1$):

$$\langle \hat{L}_{\hat{g}_i} h_i, \psi \rangle = \int_X 4 \frac{n-1}{n-2} \langle \nabla h_i, \nabla \psi \rangle d\nu_i - \frac{4}{n-2} v h_i \psi d\nu_i + \Theta_i(h_i \psi) = \varepsilon_i(\psi) \rightarrow 0. \quad (4.52)$$

Taking $i \rightarrow \infty$, using $\|\Theta_i\|_{W^{-1,p}} \leq C\tau_i \rightarrow 0$:

$$\int_X 4 \frac{n-1}{n-2} \langle \nabla h_\infty, \nabla \psi \rangle d\nu_\infty - \frac{4}{n-2} v h_\infty \psi d\nu_\infty = 0, \quad (4.53)$$

i.e. h_∞ solves $-\Delta_\chi h_\infty = \frac{v}{n-1} h_\infty$ weakly on X .

When $v > 0$: by the spectral condition (4.5), $h_\infty \equiv 0$. When $v < 0$: the operator $-\Delta_\chi - \frac{v}{n-1}$ is coercive (since $-v/(n-1) > 0$), so again $h_\infty \equiv 0$.

By the Calderón-Zygmund estimate (Proposition 4.12), $\|h_i\|_{W^{1,p}} \rightarrow 0$, contradicting $\|h_i\|_{W^{1,p}} = 1$.

Case $v = 0$: The proof is identical. To confirm $h_\infty \in W_0^{1,p}$: since $\|h_i\|_{L^\infty} \leq C$ (Sobolev) and $h_i \in W_0^{1,p}$ (by assumption), the dominated convergence theorem gives $\int_X h_\infty d\nu_\infty = 0$. Then $-\Delta_\chi h_\infty = 0$ with mean value zero implies $h_\infty \equiv 0$. $\square$

4.6 Nonlinear Equation Solution

We now combine the estimates of the preceding sections to solve the nonlinear equation via the implicit function theorem, completing the proof of Theorem 4.1.

Lemma 4.14 (Implicit function theorem for Banach spaces). *Let $H : E \rightarrow F$ be a smooth map between Banach spaces. Write $Q = H - H(0) - H'(0)$. Assume:*

(1) *There exist $C_1 > 0$ and $s_0 > 0$ such that*

$$\|Q(x) - Q(y)\|_F \leq C_1(\|x\|_E + \|y\|_E)\|x - y\|_E \quad \text{for all } x, y \in B_E(0, s_0).$$

(2) *The linearized operator $H'(0) : E \rightarrow F$ is an isomorphism with $\|[H'(0)]^{-1}\|_{F \rightarrow E} \leq C_2$.*

If $s < \min(s_0, 1/(2C_1C_2))$ and $\|H(0)\|_F < s/(2C_2)$, then there is a unique $x \in B_E(0, s)$

solving $H(x) = 0$.

Proof. Rewrite $H(x) = 0$ as $x = -[H'(0)]^{-1}(H(0) + Q(x))$. Under the hypotheses, the right-hand side defines a contraction on $B_E(0, s)$ by the Banach fixed-point theorem. $\square$

Theorem 4.15 (Existence of $\mathcal{N}$ -invariant CSC perturbation). *Under Assumption 3 with $\|\text{Rm}(g_i)\|_{C^0} \leq C_0$, for all $i \gg 0$ there exists $h_i \in C^\infty(X)$ with*

$$\|h_i\|_{W^{1,p}(X, \nu_i)} \leq C\tau_i, \quad \tau_i \rightarrow 0, \quad (4.54)$$

solving:

- $P_{\hat{g}_i}(h_i) = 0$ if $v \neq 0$;
- $P_{0, \hat{g}_i}(h_i) = 0$ if $v = 0$, where

$$P_{0, \hat{g}_i}(\phi) := -4 \frac{n-1}{n-2} \Delta_{\nu_i} \phi + S(\hat{g}_i)(1+\phi) - v_i(\phi)(1+\phi)^{(n+2)/(n-2)}, \quad (4.55)$$

$$\text{and } v_i(\phi) := \frac{\int_X S(\hat{g}_i)(1+\phi) d\nu_i}{\int_X (1+\phi)^{(n+2)/(n-2)} d\nu_i}, \text{ maps } W_0^{1,p}(X, \nu_i) \rightarrow W_0^{-1,p}(X, \nu_i).$$

Consequently, the conformal metric $g'_i := (1 + h_i)^{4/(n-2)} \hat{g}_i$ satisfies:

- (i) g'_i is $\mathcal{N}$ -invariant;
- (ii) $\|g_i - g'_i\|_{W^{1,p}} \rightarrow 0$ for any $p > n$;
- (iii) $S(g'_i) \equiv v$ if $v \neq 0$; or $S(g'_i) \equiv v_i(h_i)$ with $v_i(h_i) \rightarrow 0$ if $v = 0$;
- (iv) $(M_i, g'_i) \rightarrow (X, g_\infty)$ in the Gromov-Hausdorff sense.

This completes the proof of Theorem 4.1.

Proof. **Case $v \neq 0$:** Apply Lemma 4.14 with $E = W^{1,p}(X, \nu_i)$, $F = W^{-1,p}(X, \nu_i)$, and

$$H = P_{\hat{g}_i}.$$

- *Quadratic estimate:* Proposition 4.11(iii) provides the constant C_1 for estimate on $\mathcal{Q}_{\hat{g}_i}$.
- *Invertibility:* By Proposition 4.13, $\hat{L}_{\hat{g}_i}$ is uniformly injective. Combined with Proposition 4.12 (Calderón-Zygmund estimate), $\hat{L}_{\hat{g}_i}$ is an isomorphism with uniform bound C_2 independent of i .
- *Smallness of $H(0)$:* By Proposition 4.11(i), $\|P_{\hat{g}_i}(0)\|_{W^{-1,p}} \leq C\tau_i \rightarrow 0$.

Since $\tau_i \rightarrow 0$, all conditions of Lemma 4.14 are satisfied for large i . The unique solution $h_i \in W^{1,p}(X, \nu_i)$ satisfies (4.54). Smoothness $h_i \in C^\infty(X)$ follows from standard elliptic regularity.

Case $v = 0$: Use $E = W_0^{1,p}(X, \nu_i)$, $F = W_0^{-1,p}(X, \nu_i)$, and $H = P_{0,\hat{g}_i}$. The analog of Proposition 4.11 holds for $P_{0,\hat{g}_i}$, and Proposition 4.13 (Case $v = 0$) gives the uniform injectivity of the modified linearized operator $\hat{L}_{0,\hat{g}_i} : W_0^{1,p} \rightarrow W_0^{-1,p}$. The argument is otherwise identical.

Verification of conclusions:

(i): g'_i is $\mathcal{N}$ -invariant because $\hat{g}_i$ is $\mathcal{N}$ -invariant and h_i is $\mathcal{N}$ -invariant (it descends from a function on X).

(ii): By (4.54), $\|h_i\|_{W^{1,p}(X, \nu_i)} \leq C\tau_i \rightarrow 0$. The conformal change $(1 + h_i)^{4/(n-2)} \rightarrow 1$ in $W^{1,p}$, giving $\|g'_i - \hat{g}_i\|_{W^{1,p}} \leq C\|h_i\|_{W^{1,p}} \rightarrow 0$. Combined with $\|\hat{g}_i - g_i\|_{W^{1,p}} \leq C\tau_i$, we get (ii).

(iii): $P_{\hat{g}_i}(h_i) = 0$ is exactly the CSC equation $S(g'_i) = v$ when $v \neq 0$. For $v = 0$: $P_{0,\hat{g}_i}(h_i) = 0$ gives $S(g'_i) = v_i(h_i)$; since $|\int_X S(\hat{g}_i) d\nu_i| \leq \|S(\hat{g}_i) - v\|_{W^{-1,p}} \leq C\tau_i$ and $\int_X (1 + h_i)^{(n+2)/(n-2)} d\nu_i \rightarrow 1$, we get $v_i(h_i) \rightarrow 0$.

(iv): The Gromov–Hausdorff convergence follows from (4.54) and Sobolev embedding: $\|h_i\|_{C^0} \leq C\|h_i\|_{W^{1,p}} \leq C\tau_i \rightarrow 0$, so g'_i is uniformly close to g_i in C^0 and converges to the same GH

limit. □

Proof of Theorem 4.1. By Assumption 3, there exists an $\mathcal{N}$ -invariant nearby metric $\hat{g}_i$ from Theorem 4.6. By Theorem 4.15, the metric $g'_i = (1 + h_i)^{4/(n-2)} \hat{g}_i$ satisfies all the required conclusions:

- (1) When $v < 0$: no eigenvalue assumption; $S(g'_i) \equiv v$;
 - (2) When $v = 0$: no eigenvalue assumption; $S(g'_i) \equiv v_i \rightarrow 0$;
 - (3) When $v > 0$: the eigenvalue assumption guarantees uniform injectivity via Proposition 4.13; $S(g'_i) \equiv v$.
-

Chapter 5

Further Discussions and Open Problems

5.1 A Gluing Construction of Constant Scalar Curvature Metrics

The main theorem of Chapter 4 produces, from a collapsing CSC sequence, a nearby $\mathcal{N}$ -invariant CSC sequence. A complementary direction is to *construct* collapsing CSC sequences from scratch by gluing. In this section we describe Joyce’s gluing theorem for connected sums [38] and a generalization due to a new paper by Viaclovsky–Wang [68] that allows one of the factors to be itself collapsing.

5.1.1 Joyce’s gluing theorem

Theorem 5.1 (Constant scalar curvature metrics on connected sums [38]). *Let (M_1^n, g_1) and (M_2^n, g_2) be compact Riemannian manifolds, each with constant scalar curvature $S(g_j) = v > 0$. Let $p_j \in M_j$ be base points. Then for all sufficiently small gluing parameter $t > 0$, the*

connected sum $M_1 \# M_2$ admits a constant scalar curvature metric g_t with $S(g_t) = v_t \rightarrow v$ as $t \rightarrow 0$.

In the LCF setting (both factors locally conformally flat), g_t can be chosen LCF as well.

Construction sketch. The gluing uses the *Green's function metric*: on (M_j, g_j) with $S(g_j) = v > 0$, the conformal Green's function G_{p_j} at p_j satisfies $(-a_n \Delta + S)G_{p_j} = \delta_{p_j}$. Setting $\hat{g}_j = G_{p_j}^{4/(n-2)} g_j$ on $M_j \setminus \{p_j\}$ gives an asymptotically flat (AF) manifold. The gluing identifies the annular region near p_1 with the annular region near infinity in the AF manifold from M_2 , via the identification ι scaled by t . An implicit function theorem argument (invertibility of the linearized scalar curvature operator on each factor) perturbs the approximate metric to exact constant scalar curvature; see [38] for the complete details. $\square$

5.1.2 Collapsing factors and a new type of degeneration

Joyce's construction requires both factors to be non-collapsing (positive injectivity radius at the gluing point). The Viaclovsky–Wang generalization allows one factor to be collapsing.

Definition 5.1 (Collapsing factor). *A family (Z_t^n, g_t) parametrized by $t \in (0, t_0)$ is a collapsing factor (at base point $z_t \in Z_t$) if:*

1. $|\text{Rm}(g_t)| \leq C_1$ uniformly;
2. $\text{inj}_{g_t}(z_t) \rightarrow 0$ as $t \rightarrow 0$;
3. $(Z_t, g_t) \rightarrow (X^d, g_\infty)$ in the Gromov–Hausdorff sense with $d < n$, with smooth equivariant convergence on local universal covers.

The quintessential example is a family of metrics on a fixed manifold M^n fibering over X^d with infranilmanifold fibers, where the fiber directions have diameter $O(t)$.

Theorem 5.2 (VW gluing theorem: collapsing factors [68]). *Let (Z_t^n, g_t) be a collapsing*

factor with $S(g_t) \equiv v$ and base point z_t . Let $(N^n, \hat{g})$ be an asymptotically flat manifold with $S(\hat{g}) = 0$. Suppose the spectral condition of Theorem 4.1 holds on the limit (X, g_∞) .

Then, for t sufficiently small and suitable gluing parameters $a = a(t)$, $b = b(t)$ with $b \ll \text{inj}_{g_t}(z_t)$, the connected sum $Z_t \# M$ (where $N = M \setminus \{p\}$, $p \in M$) admits a constant scalar curvature metric g'_t such that:

- (i) $S(g'_t) = a^2 b^2 v + O(a^{2+\delta})$ for some $\delta > 0$;
- (ii) g'_t is $\mathcal{N}$ -invariant;
- (iii) As $t \rightarrow 0$, the family $(Z_t \# M, g'_t)$ degenerates: the Z_t -factor collapses while the M -factor remains at fixed scale.

Construction sketch. The gluing identifies:

- the annular region $A_t(b_t, 2b_t) \subseteq Z_t$ around z_t (at scale $b_t \ll \text{inj}_{g_t}(z_t)$), and
- the annular region $A_N(a_t^{-1}, 2a_t^{-1}) \subseteq N$ near infinity,

via $\iota(x) = a_t b_t x$. This produces an approximate metric $g_{(a,b)}$ on $Z_t \# M$.

In the gluing (damage) zone, the curvature of the collapsing factor contributes $|\text{Rm}(\iota^*(a^{-2}b^{-2}g_Z))| = O(a^2b^2)$, and $|S(g_{(a,b)})| = O(a^2b^2)$ in the damage zone.

The perturbation $g'_t = (1 + \phi)^{4/(n-2)} g_{(a,b)}$ is found by solving the nonlinear scalar curvature equation on $Z_t \# M$. The linearized operator $L = -\frac{4}{n-2}((n-1)\Delta + S)$ is uniformly invertible on $\mathcal{N}$ -invariant functions by Proposition 4.13 (which applies because the spectral condition is inherited from the collapsing factor). The implicit function theorem (Lemma 4.14) gives the perturbation ϕ with $\|\phi\|_{C^{k,\alpha}} \leq C a^2 b^2$. $\square$

Remark 5.1. *The VW construction produces new types of degenerations not visible in*

Joyce's original framework: one factor collapses (fibers shrink to zero) while the other (the AF piece) remains non-degenerate. These are natural boundary components of the moduli space of collapsing CSC manifolds, parallel to the ALE bubbles that appear at the boundary of the non-collapsing moduli space (Chapter 3).

5.2 An ε -Regularity Theorem in the Collapsing Setting

The ε -regularity lemma (Lemma 3.2) is a cornerstone of the non-collapsing moduli space theory: a small $L^{n/2}$ -norm of curvature on a ball implies pointwise curvature control on the half-ball. In the collapsing setting this lemma fails in its naive form, but may hold in the special cases.

5.2.1 Failure in the collapsing setting

Remark 5.2 (Contrast between collapsing and non-collapsing). *In the non-collapsing case (Tian–Viaclovsky [64, 65], Anderson [3], Bando–Kasue–Nakajima [10]): if (M^n, g) has $Y(M, [g]) \geq \mu_0 > 0$ (giving a uniform Sobolev constant), then*

$$\int_{B_r(x)} |\text{Ric}|^{n/2} dV < \varepsilon_0 \implies \sup_{B_{r/2}(x)} |\text{Ric}| \leq \frac{C}{r^2} \left(\int_{B_r} |\text{Ric}|^{n/2} dV \right)^{2/n}.$$

The Yamabe constant lower bound provides a uniform Sobolev inequality, which is the key input to the Moser iteration.

In the collapsing setting, the Sobolev constant degenerates (volume ratios $\text{vol}(B_r)/r^n \rightarrow 0$), so the $L^{n/2}$ -norm can be small simply because the volume is small — not because the curvature is pointwise small. See Example 5.1 for an explicit counterexample.

Example 5.1 (Counterexample to naive ε -regularity with $\mu_0 = 0$). *Let (Σ, h) be a complete non-compact hyperbolic surface with finite total volume ($S(h) = -1$) and let $(\mathbb{S}^2, g)$ be the*

standard sphere ($S(g) = 2$). The product $(\mathbb{S}^2 \times \Sigma, g \times h)$ is LCF with scalar curvature $S = 2 - 2 = 0$.

Rescale by $a > 0$: under $a^2(g \times h)$, we have $|\text{Ric}| \equiv 2a^{-2}$. Since Σ is non-compact with finite volume, for any $\varepsilon > 0$ there exist points $x \in \mathbb{S}^2 \times \Sigma$ near a cusp such that $\text{vol}_{a^2(g \times h)}(B_1(x)) \leq \frac{\varepsilon}{4}a^4$. Therefore

$$\int_{B_1(x)} |\text{Ric}|^{n/2} dV \leq 2^{n/2} a^{-n} \cdot \frac{\varepsilon}{4} a^4 = \frac{2^{n/2} \varepsilon}{4} a^{4-n},$$

which is less than ε_0 for small a (when $n = 4$). Yet $|\text{Ric}| = 2a^{-2} \rightarrow \infty$, so the pointwise curvature is large. The naive ε -regularity fails.

5.2.2 The conjecture

We aim to compactify the moduli space $\mathcal{M}(n, 0^+, C_0)$ by assuming positive Yamabe and to fully describe the degeneration behavior as metrics approach the boundary of the moduli space, analogous to the K3 case treated in [61]. A central analytic input is an ε -regularity statement for LCF manifolds assuming only a positive Yamabe invariant.

Conjecture 5.3. (*3D ε -regularity for LCF, positive Yamabe*) *There exist universal constants C and $\varepsilon_0 > 0$ with the following property. If $(M, g) \in \mathcal{M}(3, 0^+, C_0)$ and for some geodesic ball B_r one has*

$$\int_{B_r} |\text{Ric}|^{\frac{3}{2}} dv < \varepsilon_0,$$

then the pointwise bound

$$\sup_{B_{r/2}} |\text{Ric}| \leq \frac{C}{r^2} \left(\int_{B_r} |\text{Ric}|^{\frac{3}{2}} dv \right)^{\frac{2}{3}}$$

holds.

The planned approach is a contradiction / blow-up argument following the strategy in [61]. The main difficulties arise from possible noncompact bubble limit: in dimension four certain bad limits (for example $\mathbb{S}^2 \times \mathbb{H}^2$) cannot be easily excluded and require additional input. This will be showed in the futuer work.

5.3 The Full Picture of the LCF Moduli Space

The Tian–Viaclovsky compactification (Chapter 3) describes the *non-collapsing* part of the boundary of the LCF moduli space $\mathcal{M}$: GH limits are compact LCF orbifolds with ALE bubble trees. The work of this thesis raises the natural question: what happens at the *collapsing* part of the boundary, where injectivity radii degenerate? In this section we present examples of new degeneration types and discuss the open problems they raise.

5.3.1 Collapsing LCF sequences and their limits

Example 5.2 (Collapsing to an orbifold limit). *Consider $\mathbb{S}^3 \times \mathbb{R}$ with the standard product metric $g = dr^2 + g_{\mathbb{S}^3}$. Via the conformal embedding $\phi : \mathbb{S}^3 \times \mathbb{R} \rightarrow \mathbb{R}^4$ with pull-back metric $\phi^*g_{\mathbb{R}^4} = dt^2 + t^2g_{\mathbb{S}^3}$ (where $t = e^r$), define the conformal maps $\gamma_k : x \mapsto e^{1/k}Ax$ on $\mathbb{R}^4$ with*

$$A = \text{diag}(1, 1, -1, -1) \in O(4).$$

The lift $\phi^*\gamma_k : (x, t) \mapsto (Ax, t + \frac{1}{k})$ is a free isometry on $\mathbb{S}^3 \times \mathbb{R}$. The quotient

$$(M_k, g_k) = (\mathbb{S}^3 \times \mathbb{R} / \langle \phi^*\gamma_k \rangle, \text{ quotient metric}) \cong \mathbb{S}^3 \times_A \mathbb{S}^1$$

is a mapping torus. Since each (M_k, g_k) is LCF (the conformal structure descends from $\mathbb{R}^4$) with $|\text{Rm}(g_k)| \leq C$, and as $k \rightarrow \infty$:

$$(M_k, g_k) \xrightarrow{\text{GH}} (\mathbb{S}^3/A, g_{\mathbb{S}^3/A}),$$

where the limit is an orbifold (the fixed-point set of A on $\mathbb{S}^3$ is an $\mathbb{S}^1$, giving a codimension-2 singular set).

This shows that LCF manifolds can collapse with bounded curvature to an LCF orbifold — a different mechanism from the ALE bubbling in the Tian–Viadlovsky theory.

Example 5.3 (Collapsing to a manifold with boundary). Define $\gamma_k : x \mapsto e^{1/k} A_k x$ on $\mathbb{R}^4$ with

$$A_k = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos \frac{2\pi}{k} & -\sin \frac{2\pi}{k} \\ & & \sin \frac{2\pi}{k} & \cos \frac{2\pi}{k} \end{pmatrix}.$$

In spherical coordinates $\{r, \phi, \theta, \gamma\}$ on $\mathbb{R}^4$, A_k rotates $\gamma \mapsto \gamma + 2\pi/k$. The quotient $(M_k, g_k) = (\mathbb{S}^3 \times \mathbb{R} / \langle \phi^* \gamma_k \rangle, \text{quotient})$ is again LCF with bounded curvature.

As $k \rightarrow \infty$, $A_k \rightarrow \text{Id}$, so the γ -direction is collapsed. The limiting metric on the remaining coordinates (ϕ, θ) is $d\phi^2 + \sin^2 \phi d\theta^2$, isometric to a hemisphere. Thus

$$(M_k, g_k) \xrightarrow{\text{GH}} (D^2, g_{\mathbb{S}^2}),$$

where $D^2 \cong \mathbb{S}^2_+$ is the hemisphere — a manifold with boundary. This type of limit is entirely absent from the Tian–Viadlovsky compactification (which gives compact orbifolds without boundary).

5.3.2 Scalar curvature is not preserved under collapsing

One severe problem is that the lower dimensional limit may have different scalar curvature. Here is one example in the zero Yamabe case.

Example 5.4 (Scalar curvature can jump). *Consider $(\mathbb{S}^2 \times \mathbb{H}^2, g \times h)$ where g is the standard spherical metric (curvature $+1$) and h is the hyperbolic metric (curvature -1) in the upper half-plane model. Write $h = e^{-2t}dx^2 + dt^2$ ($y = e^t$), so that $g \times h = g + e^{-2t}dx^2 + dt^2$ and $S(g \times h) = 2 - 2 = 0$.*

Form the quotient (M_k, g_k) by the isometry $\gamma_k : (\theta_1, \theta_2, x, t) \mapsto (\theta_1, \theta_2, x + 1/k, t)$. As $k \rightarrow \infty$, the x -direction collapses and

$$(M_k, g_k) \xrightarrow{\text{GH}} (\mathbb{S}^2 \times \mathbb{R}, g + dt^2).$$

But $S(g_k) \equiv 0$ while $S(g + dt^2) = S(g) = 2 \neq 0$. Scalar curvature is not preserved under collapsing.

This phenomenon is explained by the drift Laplacian: the limit scalar curvature involves the relative volume density χ (via the drift Laplacian eigenvalue condition in Theorem 4.1), and the relationship between the scalar curvatures of the collapsing metrics and the base is governed by this non-trivial coupling.

5.3.3 Open problems

The examples above reveal three new boundary components of the LCF moduli space that lie *beyond* the Tian–Viaclovsky compactification:

1. **Collapsing limits as orbifolds** (Example 5.2): the limit is an LCF orbifold, but reached by collapsing rather than by ALE bubbling.
2. **Manifolds with boundary as limits** (Example 5.3): the limit space has boundary, a qualitatively new phenomenon.
3. **Scalar curvature jumps** (Example 5.4): CSC sequences can collapse to non-CSC limits.

These lead naturally to the following open problems:

1. **Compactification of the collapsing moduli space:** describe a compact closure of the moduli space of collapsing LCF CSC metrics with bounded curvature, analogous to the Tian–Viaclovsky compactification for the non-collapsing case.
2. **ε -regularity in the collapsing setting:** prove Conjecture 5.3, which would be the key analytic tool for any such compactification.
3. **Full boundary description:** classify all Gromov–Hausdorff limits of sequences $(M_i^n, g_i) \in \mathcal{M}$ (allowing both collapsing and non-collapsing degenerations). The non-collapsing boundary is described by Chapters 3 of this thesis; the collapsing boundary remains an open problem. If Conjecture 5.3 is established, curvature concentration can occur only at finitely many points, and one expects that the collapsing and gluing phenomena constructed in Section §5.1 capture the worst-case degenerations in the boundary of $\mathcal{M}(n, 0^+, C_0)$.

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Appendix A

The proof of Cheeger-Fukaya-Gromov Theorem

A.1 Proof of Theorem 1.11

We will first construct such $C^{2,\alpha}$ fibration map under the assumption, follows the idea of the proof of Proposition 6.6 of [46].

Step 1: the existence of local fibration map.

We first fix a scale $\epsilon > 0$ and consider the rescaled metric $g'_i := \epsilon^{-2}g_i$ converge to $g'_\infty = \epsilon^{-2}g_\infty$. By the assumption, for any sequence $\{x_i\}$ with $x_i \in M_i^n$ and $x_i \rightarrow x \in X^d$ and the ball $(B_4(x_i), g'_i) \rightarrow (B_4(x), g'_\infty)$ in the Gromov-Hausdorff sense. We can choose the scaling parameter ϵ so small such that

- (1) The Gromov-Hausdorff distance $(B_4(x), g'_\infty)$ with the Euclidean ball $B_4(0^d)$ is less than ϵ ;
- (2) The Ricci curvature $\text{Ric}(g_i) \geq -(n-1)\epsilon$.

(3) The harmonic radius of g'_∞ is greater than 3.

Then, we can use the construction of harmonic function from Section 6 of [17]. For every i large, there exists a harmonic map

$$\Phi_{x_i}^i : B_2(x_i) \rightarrow \mathbb{R}^d \quad (\text{A.1})$$

such that $\Phi_{x_i}^i : B_2(x_i) \rightarrow B_2(0^d)$ is a $\tau(\epsilon)$ -Gromov-Hausdorff approximation with the following gradient estimate

$$|\nabla \Phi_{x_i}^{i,\alpha}|(p) \leq 1 + \tau(\epsilon), \quad (\text{A.2})$$

and the integral estimate

$$\frac{1}{\text{vol}(B_2(x_i))} \left(\sum_{\alpha,\beta=1}^d \int_{B_2(x_i)} |\langle \nabla \Phi_{x_i}^{i,\alpha}, \nabla \Phi_{x_i}^{i,\beta} \rangle - \delta_{\alpha,\beta}| d\text{vol}_{g_i} + \sum_{\alpha=1}^d \int_{B_2(x_i)} |\text{Hess } \Phi_{x_i}^{i,\alpha}|^2 d\text{vol}_{g_i} \right) < \tau(\epsilon). \quad (\text{A.3})$$

To get pointwise $C^{2,\alpha}$ estimate for $\Phi_{x_i}^i$, consider the pull-back metric $(B_2(0^n), \hat{g}'_i)$, where we define $\hat{g}'_i := \exp_{x_i}^* g'_i$ as the pull-back metric on the local universal cover $B_2(0^n) \subseteq \mathbb{R}^n$. Then, the pull-back function $\hat{\Phi}_{x_i}^i := \Phi_{x_i}^i \circ \exp_{x_i}$ is harmonic. By our curvature assumption, the conjugate radius is bounded from below and thus the injective radius of $(B_2(0^n), \hat{g}'_i)$ is bounded from below. Thus, we have the metric, under a suitable coordinate, is uniformly $C^{1,\alpha}$ bounded. Thus, the coefficient of the Laplacian $\Delta_{\hat{g}'_i}$ is uniform $C^{0,\alpha}$ bounded. Thus, we can apply the standard Schauder estimate and conclude pointwise $C^{2,\alpha}$ estimate

$$\|\Phi_{x_i}^{i,\alpha}\|_{C^{2,\alpha}(B_1(x_i))} < C(n). \quad (\text{A.4})$$

To check the nondegeneracy, we need to show that there is no critical point in $B_1(x_i)$. In

fact, we want to show the following pointwise estimate:

$$\sum_{\alpha, \beta=1}^d |\langle \nabla \Phi_{x_i}^{i, \alpha}, \nabla \Phi_{x_i}^{i, \beta} \rangle - \delta_{\alpha, \beta}| < \tau(\epsilon). \quad (\text{A.5})$$

This can be proved by combining uniform relative volume comparison and a compactness argument.

On the limit X . By our assumption, X is a compact Riemannian manifold. Let $B_3(x_i)$ converges to $B_3(x_\infty) \subseteq X$. Then, there exists harmonic coordinate system

$$H_{x_\infty} = (u_1, \dots, u_d) : B_3(x_\infty) \rightarrow \mathbb{R}^d \quad (\text{A.6})$$

with the point-wise estimate $\sum_{\alpha, \beta=1}^d |\langle \nabla u_\alpha, \nabla u_\beta \rangle - \delta_{\alpha, \beta}| < \tau(\epsilon)$, and uniform $C^{2, \alpha}$ estimate. Thus, we can define a local map $F_{x_i} = H_{x_\infty}^{-1} \circ \Phi_{x_i}^i : B_1(x_i) \rightarrow B_1(x_\infty)$, with point-wise $C^{2, \alpha}$ estimate. By choosing ϵ small enough, there is no critical point and thus defines a local fiber bundle with fiber of dimension $n - d$.

Step 2: get a global fibration map via patching.

Again, we fix the scaling ϵ as well. Now, we want to define a partition of unit for each M_i when i is large with uniform $C^{2, \alpha}$ estimate. First, we need to construct a set of cut off function centered at any $p_i \in M_i$ with the cutting radius 1. In another, we want to find ψ_{p_i} so that:

$$\psi_{p_i}(x) = \begin{cases} 1, & x \in B_{\frac{1}{2}}(p_i) \\ 0, & x \in M_i \setminus B_1(p_i) \end{cases} \quad (\text{A.7})$$

for every p_i with $\|\psi_{p_i}\|_{C^{2, \alpha}} \leq C$, which is independent of i . This cut-off function is constructed

in Theorem 6.33 of [17]. First, we construct the following functions:

$$\begin{cases} \Delta k_{p_i} = \kappa, & x \in B_1(p_i) \setminus B_{\frac{1}{2}}(p_i) \\ k_{p_i} = 1, & x \in \partial B_{\frac{1}{2}}(p_i) \\ k_{p_i} = 0, & x \in \partial B_1(p_i). \end{cases} \quad (\text{A.8})$$

For some $\kappa > 0$, which only depends on the dimension. Take $\Psi : \mathbb{R} \rightarrow [0, 1]$ as a smooth cut-off function. Then, we construct $\psi_{p_i} = \Psi \circ k_{p_i}$. It is the desired cut-off function by [17]. We care about its regularity. By our curvature boundedness assumption, the coefficient of the Laplacian is uniform $C^{0,\alpha}$ bounded, so by Schauder estimate, we can improve the regularity of such cut-off function to $C^{2,\alpha}$, follow the argument of [17].

We can construct the partition of unity with uniform $C^{2,\alpha}$ estimate now. We choose a cover $\{B_1(q^k)\}_{k=1}^N$ so that $M_i \subseteq \cup_{k=1}^N B_1(q^k)$ and we define the partition of unity subordinate to this cover by:

$$\phi_k = \frac{\psi_{q^k}}{\sum_{j=1}^N \psi_{q^j}}. \quad (\text{A.9})$$

Note that N is uniformly bounded by bounded curvature of the sequence and non-collapsing assumption of X .

To patch local fibration maps together, we need to introduce the following function on X to determine the image of the fibration. Define the distance-like function:

$$\eta_{q_\infty} : B_1(q_\infty) \times B_1(q_\infty) \rightarrow \mathbb{R}^+, \quad (x, y) \mapsto \sum_{\alpha=1}^d |u_\alpha(x) - u_\alpha(y)|^2 \quad (\text{A.10})$$

and the global distance-like function $\eta : M_i \times X \rightarrow \mathbb{R}^+$:

$$\eta(p_i, p_\infty) = \sum_{k=1}^N \phi_k \eta_{q_\infty^k}(F_{q_k}(p_i), p_\infty). \quad (\text{A.11})$$

On each local chart $H_{p_\infty^k}$, by the harmonic coordinate estimate, $|\eta_{q_\infty} - d_{g'_\infty}^2| \leq \tau(\epsilon)$. Thus, it can be easily seen that $\eta(p_i, \cdot)$ is a convex function on X , hence has a unique minimum, $\rho(p_i)$. Thus, we can define the global map:

$$F_i : M_i \rightarrow X, \quad p_i \mapsto \rho(p_i). \quad (\text{A.12})$$

Now, we examine the regularity. Note η_{q_∞} is smooth, then $\eta(p_i, \cdot)$ is smooth function, $C^{2,\alpha}$ in p_i variable. Consider $G(p_i, y) = \nabla_y \eta(p_i, y)$. By convexity, $\nabla_y G(p_i, y) > 0$. Thus, by the implicit function theorem, the map $p_i \mapsto \rho(p_i)$ is $C^{2,\alpha}$. It is a well-defined map, with $d_{g'_\infty}(F_i(p_i), F_{x_i}(p_i)) \leq \tau(\epsilon)$ and

$$|\langle \nabla \Phi_{x_i}^{i,\alpha}, \nabla \Phi_{x_i}^{i,\beta} \rangle - \langle \nabla(F_i \circ H_{x_\infty})^\alpha, \nabla(F_i \circ H_{x_\infty})^\beta \rangle| \leq \tau(\epsilon). \quad (\text{A.13})$$

Similarly, F_i is a $C^{2,\alpha}$ fibration map with fiber of dimension $n - d$.

Now, we shall prove the rest of the theorem. To show the $C^{0,\alpha}$ bound for the fiber second fundamental form. Note that we have:

$$A_{F_i^{-1}(x)}(V, W) = \frac{\text{Hess } F_i(V, W)}{|\nabla F_i|} \quad (\text{A.14})$$

for $V, W \in \ker(dF_i)$. Thus, we have:

$$\|A_{F_i^{-1}(x)}\|_{C^{0,\alpha}} \leq C(n) \|F_i\|_{C^{2,\alpha}}. \quad (\text{A.15})$$

To show that F_i is an almost-Riemannian submersion, we can use the contradiction argument given in [61]. Suppose not, there exists a sequence of vector $v_i \in T_{x_i} M_i$, perpendicular to the fiber direction, and a constant $\tau_0 \geq 0$ so that $|\frac{|dF_i(v_i)|_{g_\infty}}{|v_i|_{g_i}} - 1| \geq \tau_0$. We may assume $|v_i|_{g_i} = 1$. Consider the local equivariant Gromov-Hausdorff convergence of local universal cover

$$\begin{array}{ccc}
(\tilde{B}_{\tilde{x}_i}(r), \tilde{g}_i, G_i) & \xrightarrow{C^{1,\alpha}} & (\tilde{B}_{\tilde{x}_\infty}(r), \tilde{g}_\infty, G_\infty) \\
\downarrow \pi_i & & \downarrow \pi_\infty \\
(B_{x_i}(r), g_i) & \xrightarrow{\text{GH}} & (B_{x_\infty}(r), g_\infty)
\end{array}$$

where π_i is the local universal cover map, π_∞ is a Riemannian submersion. Thus, $\tilde{F}_i = F_i \circ \pi_i$ converges to π_∞ in $C^{2,\beta}$. Let $\tilde{v}_i$ be a lift of v_i . Thus, $\tilde{v}_i$ converges to $\tilde{v}_\infty$ with $|d\pi_\infty \tilde{v}_\infty| = 1$. This violates the assumption $|\frac{|dF_i(v_i)|_{g_\infty}}{|v_i|_{g_i}} - 1| \geq \tau_0$.

Appendix B

Orbifold Points in Pairs

This appendix proves Theorem 3.17: in an oriented compact LCF orbifold with positive Yamabe invariant, the orbifold singular points appear in antipodal pairs. The proof uses the following lemma from the theory of discrete groups.

B.1 A Lemma on Hyperbolic Elements

Lemma B.1 ([6], p. 52). *Let $g_1, g_2 \in G \leq C(n)$, where g_1 is hyperbolic (has exactly two fixed points on $\mathbb{S}^n$ and expansion factor $\alpha \neq 1$). If g_1 and g_2 have precisely one common fixed point, then G is not discrete.*

Proof. Without loss of generality let the common fixed point be ∞ , and write $\mathbb{S}^n = \mathbb{R}^n \cup \{\infty\}$. Since g_1 is hyperbolic with $g_1(\infty) = \infty$ and $g_1(0) = 0$ (the second fixed point of g_1 , which is different from ∞ by assumption), and $g_2(\infty) = \infty$ but $g_2(0) \neq 0$, we may write

$$g_1(x) = \alpha U_1 x, \quad g_2(x) = \beta U_2 x + b,$$

where $\alpha \in (0, 1)$ (replacing g_1 by g_1^{-1} if necessary), $\beta > 0$, $U_1, U_2 \in O(n)$, and $b \in \mathbb{R}^n \setminus \{0\}$.

Consider the sequence of Möbius transformations

$$h_k = g_2 g_1^k g_2^{-1} g_1^{-k}.$$

A direct computation gives

$$h_k(x) = U_2 U_1^k U_2^{-1} U_1^{-k}(x) - \alpha^k U_2 U_1^k U_2^{-1}(b) + b.$$

Case 1: $U_1^m = \text{Id}$ for some $m \geq 1$. Setting $k = jm$:

$$h_{jm}(x) = x - \alpha^{jm}b + b \xrightarrow{j \rightarrow \infty} x + b.$$

The limit $x \mapsto x + b$ is a parabolic (non-identity) Möbius transformation. Hence G contains elements h_{jm} accumulating at $x \mapsto x + b \neq \text{Id}$, so G is not discrete.

Case 2: U_1 has infinite order. For any $\varepsilon > 0$, there exists $k_\varepsilon \geq 1$ with $|U_1^{k_\varepsilon}(y) - y| < \varepsilon$ for all $y \in B(0, r)$ (by the pigeonhole principle for the compact group $\overline{\langle U_1 \rangle} \leq O(n)$). Then $|h_{k_\varepsilon}(x) - (x + b)| \leq |\alpha^{k_\varepsilon}| |b| + C\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ (taking $k_\varepsilon \rightarrow \infty$ simultaneously). Again $h_{k_\varepsilon} \rightarrow x + b$, so G is not discrete. $\square$

B.2 Proof of Theorem 3.17

Theorem B.2 (Fixed points appear in pairs; Theorem 3.17 restated). *Let $G \leq C(n)$ be a discrete group acting on $\mathbb{S}^n$ properly discontinuously, with limit set Λ and no parabolic elements. Assume $\dim_{\mathcal{H}} \Lambda < (n - 2)/2$. Let $\Gamma \leq G$ be a local isotropy subgroup with isolated fixed points. Then the fixed points of Γ appear in pairs unless the quotient $\mathbb{S}^n/G$ is non-orientable.*

Proof. Let $s \in \mathbb{S}^n$ be a fixed point of some non-trivial $\gamma \in \Gamma$. By Lemma 3.16, Γ is (conjugate

to) a finite subgroup of $O(n)$. For any $A \in O(n)$ acting on $\mathbb{S}^n$, if it has an isolated fixed point s , then $-s$ is also a fixed point (since $A \in O(n)$ and the eigenvectors come in $\pm$ pairs over $\mathbb{R}$ for the action on $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ with $\mathbb{S}^{n-1}$ the unit sphere). Explicitly: if $As = s$, $s \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$, then $A(-s) = -As = -s$. So Γ also fixes $-s \in \mathbb{S}^n$.

It remains to show that s and $-s$ lie in *different* G -orbits (i.e., they correspond to two distinct orbifold points of $M = \Omega/G$) unless the quotient is non-orientable. Suppose $-s$ is in the same G -orbit as s . We treat two cases.

Case 1: $-s \notin \Lambda$ (so $-s \in \Omega$, the domain of discontinuity).

Then there exists $\gamma_0 \in G$ with $\gamma_0(s) = -s$. Set $s = 0$ and $-s = \infty$ in the stereographic chart. By Corollary 1.8 of [6], since γ_0 maps $0 \mapsto \infty$ and preserves coordinate spheres centered at s (because the local group Γ fixes s and γ_0 maps $s \mapsto -s$), we can write $\gamma_0(v) = r^2 Uv/|v|^2$ for some $U \in O(n)$ (the term $q' = 0$ since γ_0 maps round spheres S_t to round spheres S_R centered at ∞).

Then $\gamma_0^2(v) = U^2v$, so γ_0^2 fixes $s = 0$. Since γ_0 has no fixed points in Ω (it maps $0 \mapsto \infty$ with $0, \infty \notin \Omega$), $\gamma_0^2|_\Omega$ is a deck transformation fixing s . By assumption $\gamma_0^2 \in \Gamma$ (a finite group), so $\gamma_0^{2m} = \text{Id}$ for some m . Since $\dim_{\mathcal{H}}(\mathbb{S}^n \setminus \Omega) = \dim_{\mathcal{H}} \Lambda < (n-2)/2$, by the Liouville theorem (Theorem 2.9) a conformal map agreeing with the identity on a set of positive Hausdorff dimension $(n-2)/2$ must be the identity globally; hence $\gamma_0^{2m} = \text{Id}$ as a Möbius transformation. We may assume $\gamma_0^2 = \text{Id}$, so $U^2 = \text{Id}$ and $U = A^{-1}D_k A$ for some $A \in O(n)$ and diagonal D_k with k entries $+1$ and $n-k$ entries -1 .

The element γ_0 fixes a $\mathbb{S}^k \subset \mathbb{S}^n$; the condition $\dim_{\mathcal{H}} \Lambda < (n-2)/2$ requires $k < (n-2)/2$.

- If $D_k = -\text{Id}$ ($k = -1$, i.e. $U = -\text{Id}$): For odd n , the composition $\gamma_0 \circ \alpha$ (where α is a generator of Γ) fixes a $\mathbb{S}^{n-1}$, creating a non-isolated fixed point — contradicting that Γ has only isolated fixed points. For even n , γ_0 is orientation-reversing, so the quotient

$\mathbb{S}^n/G$ is non-orientable.

- If $k \geq 0$: The fixed $\mathbb{S}^k$ lies in Λ , violating $\dim_{\mathcal{H}} \Lambda < (n-2)/2$ for $k \geq (n-2)/2$.

For smaller k , a similar parity argument produces either a non-isolated fixed point (contradiction) or an orientation-reversing element (non-orientable quotient).

Case 2: $-s \in \Lambda$.

We claim $-s$ is not a hyperbolic fixed point of G . Suppose otherwise: let $h \in G$ be hyperbolic with $\lim_{k \rightarrow \infty} h^k(x) = -s$ for all $x \in \Omega$. Since $s \notin \Lambda$ by assumption, the other fixed point of h is some $s_0 \neq s$ with $\lim_{k \rightarrow \infty} h^{-k}(x) = s_0$.

For any $\delta \in \Gamma$, the conjugate $\delta^{-1}h\delta \in G$ is also hyperbolic with fixed points $-s$ (preserved since Γ fixes s and $-s$) and $\delta^{-1}(s_0)$. Since $\delta(s) = s$, we have $\delta^{-1}(s_0) \neq s$ in general. Hence h and $\delta^{-1}h\delta$ are both hyperbolic with the single common fixed point $-s$. By Lemma B.1, G is not discrete — contradiction.

Therefore $-s$ is not a hyperbolic fixed point of G , and by the classification of elements of $C(n)$ (elliptic, parabolic, hyperbolic), $-s$ cannot be in Λ as an attracting fixed point of any hyperbolic element. Since G has no parabolic elements by assumption and the only remaining option (elliptic fixed points with $-s \in \Lambda$) leads to $\dim_{\mathcal{H}} \Lambda \geq (n-2)/2$ (a contradiction with the hypothesis), Case 2 cannot occur.

In both cases, the assumption that s and $-s$ are in the same G -orbit leads to a contradiction (or a non-orientable quotient). Hence in the orientable case, s and $-s$ lie in different G -orbits, giving a pair of distinct antipodal orbifold points. Since the antipodal involution is a fixed-point-free pairing on the set of orbifold singular points, that set has even cardinality. $\square$