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Current Opinion in
Colloid & Interface Science

Recent modeling advances in dense suspension rheology

Orhun Ayar[1], Bhargav Sriram Siddani[2], Ishan Srivastava[2] and Abhinendra Singh[3]

Dense suspensions of particles dispersed in liquids are central to industrial and geophysical processes and serve as model systems for out-of-equilibrium soft matter. At high particle concentrations, they exhibit stress-dependent rheology, including discontinuous shear thickening and shear jamming, arising from frictional contacts. Nonlinear physics arises from the interplay among direct contacts, interfacial chemistry, and fluid-mediated hydrodynamics. The relative importance of these mechanisms depends on the particle properties and flow conditions, making predictive modeling inherently multi-scale and, therefore, computationally challenging. Recent advances in computational methods have transformed our ability to simulate the physics of dense suspensions across different scales. In this perspective, we primarily focus on state-of-the-art simulation frameworks that integrate the mechanics of dry granular materials, mediated by contact friction, with suspension hydrodynamics to provide predictive models of dense suspension rheology. We highlight recent computational developments for simulating dense, predominantly non-Brownian suspensions in the viscous limit at varying levels of fidelity, ranging from particle-resolved to continuum models, as well as models that investigate their mesoscale organization during flow. Together, these approaches reveal a hierarchical structure in which microscale constraints give rise to mesoscale frictional networks that ultimately govern the macroscopic flow. While our emphasis is on dense suspensions governed by near-field hydrodynamics and frictional interactions, we also briefly discuss complementary grid-based methods that can account for complex geometries and particle properties, and also enable continuum-scale descriptions.

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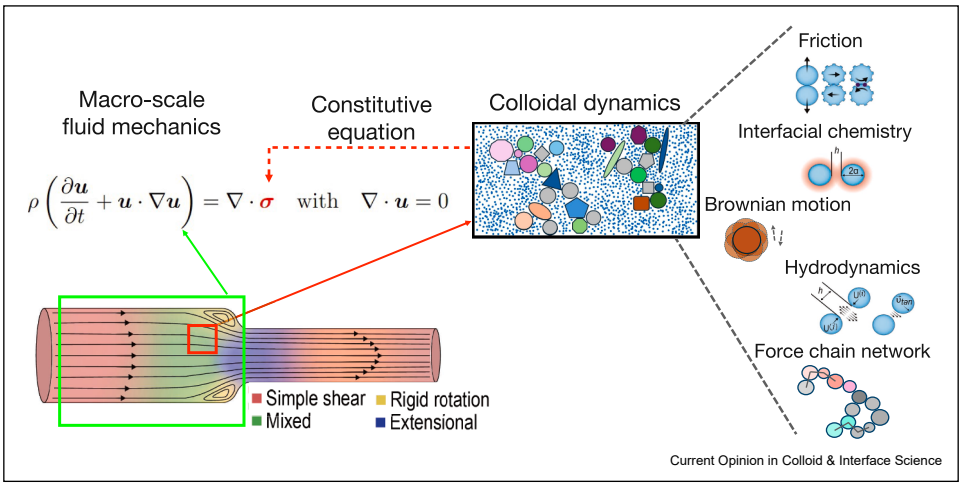
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Introduction

Dense suspensions are prototypical complex fluids whose flow behavior is challenging to predict yet is central to industrial, geotechnical, and biological processes such as cement transport, mudflow, and blood flow [1–3]. In the quiescent state, for dense and confined suspensions, particle motion exhibits pronounced hindrance and anomalous diffusion. Under shear, they display non-linear responses including large normal stresses, shear thickening, shear jamming (SJ), and yielding [4,5]. Because stress is shared between the fluid and particulate phases and is highly sensitive to particle concentration and evolving microstructure, predictive modeling remains fundamentally challenging. Although these behaviors have long been observed experimentally, their quantitative description requires the consistent integration of hydrodynamic interactions, contact mechanics, and multi-scale modeling strategies (Figure 1).

For classification purposes, we distinguish the suspension classes based on particle volume fraction ϕ and characteristic particle size a (Figure 2). In the dilute regime, long-range hydrodynamic interactions dominate. In the semi-dilute regime, near-field lubrication interactions become important, although persistent contacts remain infrequent. In the dense regime, which is the focus of this perspective, short-range interactions and near-contact physics govern stress transmission, as the mean interparticle separations become much smaller than the particle size [4,5]. Suspensions can also be distinguished by particle size a, where dimensionless numbers are helpful: the Reynolds number, $\mathrm{Re} = \rho_f a^2 \dot{\gamma}/\eta_0$, and the Stokes number, $St = \rho_p a^2 \dot{\gamma}/\eta_0$, which compare fluid and particle inertia, respectively, to viscous stresses. We focus on the viscous limit with $\mathrm{Re} \ll 1$ and $St \ll 1$. The Péclet number, $Pe =$

Figure 1



Multi-scale nature of suspension flow: The channel flow through a contraction offers a paradigmatic example Role of the constitutive equation in macro-scale continuum description of dense suspensions. The constitutive equation is governed by colloidal dynamics, which are in turn governed by the following forces and constraints at play: attraction–repulsion interactions, near-field hydrodynamics, Brownian motion, rolling and sliding friction constraints, and force chain networks.

$6\pi\eta_0 a^3\dot{\gamma}/k_BT$, quantifies the relative importance of shear to Brownian motion: $Pe \lesssim 1$ corresponds to colloidal suspensions, while $Pe \gg 1$ characterizes athermal non-colloidal systems. Granular suspensions may also involve gravitational and inertial effects.

The experimental characterization of suspensions in the dense limit is intrinsically challenging. Wall slip, shear-induced migration, free-surface instabilities, and sample ejection complicate measurements and obscure the internal stress state [1]. Simultaneous access to shear stress, particle pressure, and normal stress differences remains limited under strongly non-Newtonian conditions, and resolving particle-scale dynamics and contact organization in situ is challenging.

Therefore, computational simulations provide an essential complementary approach to experimental studies. By resolving particle motion under controlled conditions, simulations connect bulk rheology to particle-scale dynamics and evolving microstructure while enabling systematic variations in the interaction laws and loading conditions. In the dense regime, where lubrication, contact, and frictional interactions compete, particle-resolved computational frameworks have become indispensable for linking microscopic physics to emergent rheology.

In this perspective, we synthesize recent advances in modeling approaches for suspension rheology, with a primary focus on dense, predominantly non-Brownian suspensions in the viscous (low-Reynolds-number) limit. In this regime, nearly similar-sized particles dispersed in a Newtonian solvent interact through hydrodynamic, conservative, and contact forces, which together govern the rheological response. As a perspective, our goal is not to provide an exhaustive review of all suspension classes or modeling approaches. Instead, we emphasize a physically grounded framework in which microscale interactions give rise to constraints, constraints organize mesoscale frictional-contact networks, and these networks govern macroscopic flow behavior. Where appropriate, we briefly discuss extensions beyond this regime, including effects of particle inertia, non-spherical particles, confinement, and solvent complexity, as well as continuum–particle coupling approaches, to provide broader context and highlight emerging directions.

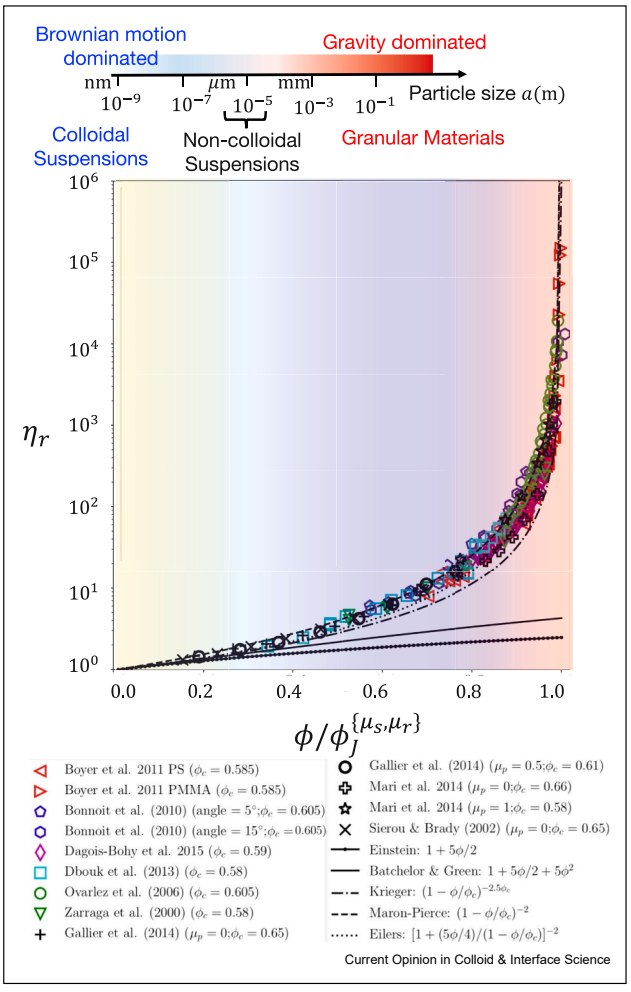
A rapid overview of methods

Modeling the hydrodynamic interactions between suspended particles and coupling them consistently with other relevant interactions, such as colloidal, contact, and conservative forces, is often the most challenging aspect of suspension simulations. This difficulty is further exemplified by the long-range nature of solvent-mediated interactions. In general, simulation approaches can be classified into implicit- and explicit-solvent methods, depending on the representation of the fluid phase.

Implicit solvent models

In implicit-solvent models, the effect of the surrounding fluid is incorporated through effective hydrodynamic forces acting between particles. In mesoscale systems, additional stochastic forces are included to represent the thermal fluctuations. These approaches are typically derived from a continuum description of the solvent. In

Figure 2



Definitions of flow regimes *Top: Regimes defined based on particle size a*: At the smallest length scale, Brownian motion and hydrodynamics dominate, while gravity and friction govern the physics in the granular regime; both friction and hydrodynamics are crucial for the non-colloidal suspensions, and particle inertia can be ignored. *Bottom: Flow regimes based on scaled particle concentration in rheology of colloidal and non-colloidal suspensions*: From the dilute (yellow shading) to semi-dilute (blue) and dense (red) limits based on the relative importance of interactions; full long-ranged hydrodynamic interactions are crucial in dilute limit; near-field lubrication interactions are important in the semi-dilute limit; frictional contact interactions dominate the physics in the dense limit. The bottom panel is adapted from Ref. [6].

the low-Re regime, the linearity of the Stokes equations allows hydrodynamic forces and torques to be linearly related to the particle velocities through a resistance or mobility tensor. Therefore, much of the development of implicit solvent methods has focused on constructing accurate and computationally tractable approximations of this tensor. Analytic representations are typically limited to simple solvent properties and particles with high shape symmetry, such as spheres with moderate size dispersity [7,8]. Consequently, the applicability of implicit methods is restricted when modeling more complex suspensions encountered in practice. Additional challenges arise from the numerical treatment of the mobility tensor, particularly in schemes that require evaluation of its divergence to maintain consistency with the fluctuation-dissipation theorem. These challenges are further exacerbated in dense suspensions, where particle contacts dominate.

Continuum–particle coupling approaches

A related class of methods couples particle dynamics to a continuum fluid solver, using the finite-volume or lattice-Boltzmann method, in which the solvent equations are solved on a grid and momentum is exchanged between the Eulerian fluid and Lagrangian particles. Immersed boundary (IB) [9–11], lattice-Boltzmann (LB) [12–14], and fictitious domain (FD) [15,16] methods represent approaches in which a particle's surface, and therefore volume, is adequately resolved to impose boundary conditions. The distinction among these approaches lies in the imposition of the fluid-particle boundary conditions. Unresolved approaches belonging to this class of computational methods include the Computational Fluid Dynamics–Discrete Element Method (CFD–DEM) methodology, which relies on closure relations for fluid-particle coupling. All these approaches offer flexibility in handling complex particle shapes, size distributions, and boundary conditions. However, incorporating dense suspension rheology and accurately resolving near-contact interactions remains an active area of research [17,18].

Explicit solvent models

In explicit solvent models, the fluid is represented as a collection of interacting particles whose collective dynamics gives rise to hydrodynamic behavior. Popular examples include dissipative particle dynamics [19,20] and multi-particle collision dynamics [21,22]. These methods are attractive for their relative ease of implementation and ability to handle complex geometries and particle shapes. Similarly, accurately reproducing incompressible-solvent behavior and achieving realistic transport coefficients, particularly at large Schmidt numbers, remains challenging.

The selection of a particular method depends on the application, as each method class involves a trade-off between physical fidelity, computational cost, and applicability to the dense suspensions. In the remainder of this perspective, we focus primarily on implicit-solvent, particle-based approaches that are most commonly used to probe the rheology and microstructure of dense suspensions.

Simulation frameworks for suspensions

The simulation of suspensions has traditionally been approached from a fluid-mechanics perspective, in

which the particle motion is governed by hydrodynamic interactions among all particles, mediated by the viscous fluid (solvent). Although this provides a natural starting point for describing suspension mechanics, recent discoveries have shown that it becomes insufficient in the dense limit, thereby motivating the development of new physics. In this section, we first discuss the fluid-mechanical framework for suspensions, followed by the latest developments in near-contact physics that are required for the accurate modeling of highly dense suspensions.

In any particle-based system, the particle motion is governed by the balance of hydrodynamic, contact, and other interparticle forces and torques, which can be written schematically as

$$\boldsymbol{M}\frac{d}{dt}\begin{pmatrix}\boldsymbol{U}\\ \boldsymbol{\Omega}\end{pmatrix} = \sum_{\alpha}\begin{pmatrix}\boldsymbol{F}_{\alpha}\\ \boldsymbol{T}_{\alpha}\end{pmatrix}, \tag{1}$$

where $\boldsymbol{U}$ and $\boldsymbol{\Omega}$ are the translational and rotational velocities, $\boldsymbol{F}_{\alpha}$ and $\boldsymbol{T}_{\alpha}$ denote the forces and torques, respectively, and $\boldsymbol{M}$ is the mass and moment-of-inertia matrix. In suspensions, the bulk stress (excluding the isotropic fluid pressure) consists of a solvent contribution associated with the imposed rate-of-strain tensor $\boldsymbol{E}^{\infty}$ and a particle contribution arising from interparticle interactions, which can be expressed as

$$\boldsymbol{\Sigma} = 2\eta_0\boldsymbol{E}^{\infty} + \boldsymbol{\Sigma}^{p}, \qquad \boldsymbol{\Sigma}^{p} = \boldsymbol{\Sigma}^{H} + \boldsymbol{\Sigma}^{C} + \boldsymbol{\Sigma}^{\mathrm{cons}}, \tag{2}$$

where the particle stress includes hydrodynamic, contact, and conservative contributions, each obtained from volume-averaged stresslets, $\boldsymbol{\Sigma}^{\alpha} = V^{-1}\sum \boldsymbol{S}^{\alpha}$, where $\alpha \in \{H, C, \mathrm{cons}\}$.

From $\boldsymbol{\Sigma}$, standard rheological observables can be defined: shear stress $\sigma_{xy} = \Sigma_{12}$, particle pressure $\Pi = -(\Sigma_{11} + \Sigma_{22} + \Sigma_{33})/3$, and normal stress differences $N_1 = \Sigma_{11} - \Sigma_{22}$ and $N_2 = \Sigma_{22} - \Sigma_{33}$. The relative viscosity is $\eta_r = \sigma_{xy}/(\eta_0\dot{\gamma})$, where indices 1, 2, and 3 denote the flow, gradient, and vorticity directions, respectively.

Stokesian dynamics

For typical particle sizes of colloidal and non-colloidal suspensions, both liquid and particle inertia are negligible, i.e., $St = 0$. In the overdamped limit, hydrodynamic interactions between particles arise from the disturbance flows generated by their motion in a viscous solvent. These interactions are long-ranged and many-body in nature, and they decay slowly with increasing interparticle separation. The Stokesian dynamics (SD) modeling framework was developed to systematically capture these effects by exploiting the linearity of the Stokes equation and relating particle velocities to hydrodynamic forces and torques via a grand resistance (or mobility) matrix [7,23]. Given $St = 0$, the equation of motion (Eq [1]) reduces to:

$$0 = \boldsymbol{R}\begin{pmatrix}\boldsymbol{U} - \boldsymbol{U}^{\infty}\\ \boldsymbol{\Omega} - \boldsymbol{\Omega}^{\infty}\end{pmatrix} + \begin{pmatrix}\boldsymbol{F}_{\mathrm{nH}}\\ \boldsymbol{T}_{\mathrm{nH}}\end{pmatrix}, \tag{3}$$

where $\boldsymbol{X}$ denotes the particle positions, $\boldsymbol{R}$ is the hydrodynamic resistance matrix, $\boldsymbol{U}^{\infty}$ and $\boldsymbol{\Omega}^{\infty}$ denote the imposed background flow, and $\boldsymbol{F}_{\mathrm{nH}}$ and $\boldsymbol{T}_{\mathrm{nH}}$ represent all non-hydrodynamic forces and torques acting on the particles, respectively. The resistance matrix incorporates both far-field hydrodynamic interactions, which are typically represented through multipole expansions, and near-field lubrication forces obtained from asymptotic two-body solutions [24,25]. SD has been remarkably successful in describing the microstructure and rheology of dilute and semi-dilute suspensions by accurately modeling key suspension physics such as shear thinning and weak shear thickening associated with the so-called 'hydroclusters', which are transient particle aggregates held together by dissipative lubrication forces [23].

Computational advances within hydrodynamic frameworks

The long-ranged nature of hydrodynamic interactions and the resistance matrices used in SD makes these interactions computationally expensive to compute. This involves the calculation of the $\mathcal{O}(N^2)$ far-field mobility matrix and the inversion of the matrix $\mathcal{O}(N^3)$. Accelerated Stokesian dynamics (ASD) and related Ewald-based approaches reduce the computational cost of far-field hydrodynamics to $\mathcal{O}(N\log N)$, allowing the simulation of significantly larger systems while retaining accurate near-field lubrication forces [26]. This is achieved by resolving long-range interactions by calculating the Ewald-summed wave-space contribution as a Fourier-transform sum and by inverting the now-sparse resistance matrix iteratively [26], which at the time enabled to simulate the short-time self-diffusion coefficient with $\mathcal{O}(10^3)$ particles. More recent formulations, such as Fast Stokesian dynamics (FSD), recast the hydrodynamic problem to avoid the explicit construction and inversion of resistance matrices, thus extending the accessible system sizes [27]. FSD avoids explicit inversion of the hydrodynamic operators, drastically reducing the computation to $\mathcal{O}(\mathcal{N})$, whereas a graphics-processing-unit enabled version of the FSD code simulated $\mathcal{O}(10^5)$ particle system. These computational advances, such as graphics processing units enabled Verlet list approaches [28], have reduced computational costs and enabled large-scale simulations of suspensions within hydrodynamic-only frameworks.

Lubrication breakdown and near-contact physics

For two smooth hard spheres, the normal lubrication resistance diverges as $F_{\mathrm{lub}} \sim \frac{1}{h}$, where h is the surface separation [24,25]. As such, in a purely continuum description of suspensions, such a divergence implies that perfectly smooth particles should never touch or make contact. In their seminal paper, Melrose and Ball

demonstrated [29] that this leads to unphysical behavior as particle surfaces reach separations far below molecular or asperity length scales. For example, these models predict unphysical pathological jamming at volume fractions well below the experimentally observed jamming conditions or even random close packing (RCP ≈ 0.64) [30]. This observation implies lubrication breakdown, which can allow the formation of frictional contacts, an effect that is especially relevant under dense conditions, as discussed next.

Near-hard-sphere viewpoint and missing physics

A consensus that has emerged in the last decade is that real particles are not perfectly smooth hard spheres; they exhibit surface roughness, finite compliance, and short-range surface forces that are dominant at separations much smaller than the particle radius $h \approx 10^{-3}a$ [4,31]. The near-hard-sphere (NHS) modeling framework incorporates this by allowing lubrication interactions to be regularized at small separations, leading to contact friction upon physical contact between particles [31]. Recent simulations have extensively demonstrated that by incorporating lubrication breakdown and contact friction, both discontinuous shear thickening (DST) and SJ can be successfully reproduced with near-quantitative accuracy [1,8,34–36].

Few studies have examined whether shear thickening in dense suspensions can be captured within modified hydrodynamic frameworks without explicitly introducing new contact physics. Jamali and Brady [37] modeled surface roughness by resolving discrete asperities using dissipative particle dynamics, demonstrating that lubrication interactions between asperities can dominate energy dissipation at very small separations and give rise to the DST. In a related study [38], it was shown that enhancing the tangential divergence of lubrication resistance, using parameters related to asperity geometry, can also reproduce DST within a hydrodynamic description. Although successful in reproducing DST, SJ remains a challenge for such hydrodynamic treatments, given that fluid-mediated forces vanish in the limit of zero shear rate.

Taken together, these observations motivate the NHS framework, in which particles interact through lubrication forces together with short-range conservative and contact interactions that become relevant at small surface separations, thus providing a minimal description of dense suspension rheology [5,31,34].

Integrating hydrodynamics with contact friction in dense suspensions

Following Melrose and Ball [29], hydrodynamic interactions are typically restricted to single-particle Stokes drag and pairwise lubrication forces, which dominate dissipation in the crowded environments. Once the particles approach a small cutoff separation $h \leq 10^{-3}a$, a distance that mimics the small asperities on the surface of the particle, the lubrication forces are regularized ($F_{\mathrm{lub}} \sim \frac{1}{h+\Delta}$), allowing frictional contacts between the particles. Thus, Stokes flow (Eq [3]) can be rewritten as

$$\vec{0} = \vec{F}_{\mathrm{H}}(\vec{X}, \vec{U}) + \vec{F}_{\mathrm{C}}(\vec{X}) + \vec{F}_{\mathrm{cons}}(\vec{X}), \qquad (4)$$

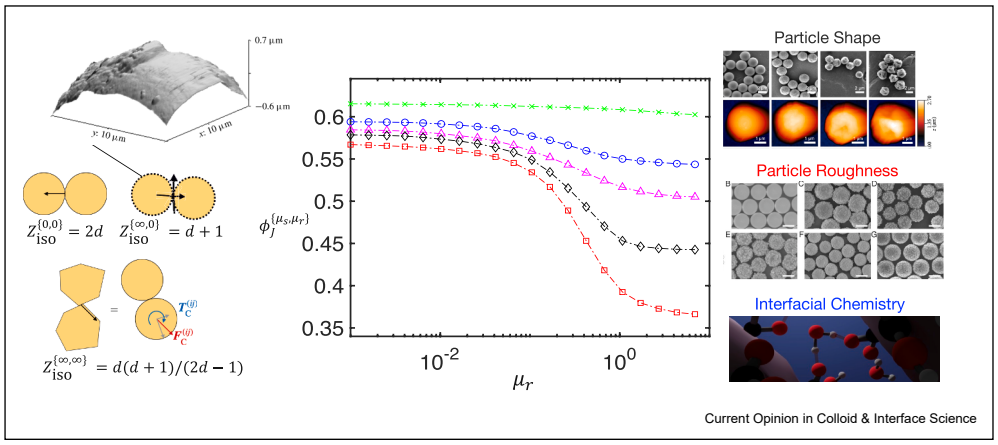
where $\vec{F}_{\mathrm{H}}$, $\vec{F}_{\mathrm{C}}$, and $\vec{F}_{\mathrm{cons}}$ denote the hydrodynamic (including lubrication), contact, and conservative forces, respectively.

Contacts are modeled using granular physics, following the discrete element method (DEM) originally introduced by Cundall and Strack [39]. When two particles come into contact, i.e., overlap $\delta^{(i,j)} \geq 0$, a normal contact force $\boldsymbol{F}_{C,\mathrm{n}}^{(i,j)} = k_n \delta^{(i,j)} \boldsymbol{n}_{ij}$ is introduced along their line of centers. Here, k_n is the normal spring stiffness and $\boldsymbol{n}_{ij}$ is the unit vector connecting the particle centers. Tangential contact forces are modeled through an elastic spring that accumulates tangential displacement as $\boldsymbol{F}_{C,\mathrm{t}}^{(i,j)} = k_t \boldsymbol{\xi}_{ij}$ with k_t and $\boldsymbol{\xi}_{ij}$ being tangential spring stiffness and the spring stretch, respectively. These forces are subjected to the Coulomb friction criterion $|\boldsymbol{F}_{C,\mathrm{t}}^{(i,j)}| \leq \mu |\boldsymbol{F}_{C,\mathrm{n}}^{(i,j)}|$. In real-world suspensions with adhesive surface chemistries, the particles are often rough, with asperities, and have faceted shapes, which lead to additional friction (constraints on relative motion), i.e., rolling and twisting modes [34,40]. In the dry granular literature, it has long been recognized that contacts between non-spherical shapes can be modeled using spherically symmetric shapes with additional constraints, and thus, torques [41].

This framework can be extended to regimes with small but finite particle inertia, allowing a unified treatment of viscous and inertial contributions within the same numerical approach [35,42]. In this case, inertia enters as an additional time scale without altering the underlying role of the near-contact interactions and frictional contacts. Although the simplest implementations assume a constant inter-particle friction, recent studies have shown that μ can depend on the history of deformation or contact load, reflecting the elastic and plastic deformation of the surface asperities [43–45]. Incorporating such deformation-dependent friction laws provides a more realistic description of particle contacts, particularly at high stresses, and remains an active research topic.

Together, these developments establish particle-based simulations that explicitly resolve near-contact interactions and frictional forces as the minimal framework for understanding the physics of dense suspension flow beyond purely hydrodynamic descriptions.

Figure 3



Connecting jamming and particle features. *Left: Different types of constraints on relative particle motion*: Hard sphere, frictionless $\{\mu_s, \mu_r\} = \{0, 0\}$ having isostatic condition $Z_{\rm iso}^{\{0,0\}} = 2d$ in d – dimension leading to $\phi_J^{\{0,0\}} \approx 0.65$ in 3-d; infinite sliding friction $\{\mu_s, \mu_r\} = \{\infty, 0\}$ having isostatic condition $Z_{\rm iso}^{\{\infty,0\}} = d+1$ in d – dimension leading to $\phi_J^{\{0,0\}} \approx 0.57$ in 3-d; infinite sliding and rolling frictions $\{\mu_s, \mu_r\} = \{\infty, \infty\}$ having isostatic condition $Z_{\rm iso}^{\{\infty,\infty\}} = d(d+1)/(2d-1)$ in d – dimension leading to $\phi_J^{\{0,0\}} \approx 0.365$ in 3-d. *Middle: Jamming phase diagram:* Jamming volume fraction $\phi_J^{\{\mu_s,\mu_r\}}$ extracted by fitting η_r in the lubricated and frictional states to the functional form $\eta_r = (1-\phi/\phi_J^{\{\mu_s,\mu_r\}})^{-2}$ plotted as a function of μ_r for several values of μ_s. *Right: Control knobs of jamming by tuning particle properties*: (top) particle shape, (middle) roughness, and (bottom) interfacial chemistry. The middle panel is adapted from Ref. [46].

Friction, constraints, and a minimal framework for the rheology of dense suspensions

Once frictional contacts form between particles in dense suspensions, the central question shifts from how these interactions are modeled to how they reorganize stress transmission and affect flow, which we discuss below.

Constitutive expectations for dense suspensions

Dense suspensions are commonly studied under simple steady shear, either at a fixed volume fraction ϕ or particle pressure Π [6,47]. In a volume-controlled geometry, steady-state stresses take the form

$$\sigma_{xy} = \eta_0 \dot{\gamma}\, \eta_s(\phi), \qquad \Pi = \eta_0 \dot{\gamma}\, \eta_n(\phi), \tag{5}$$

where η_0 is the suspending-fluid viscosity and η_s and η_n are the shear and normal viscosities that diverge as ϕ approaches the jamming volume fraction ϕ_J [47].

In a pressure-controlled geometry, the rheology is conveniently expressed in terms of the viscous number $J \equiv \eta_0 \dot{\gamma}/\Pi$, as follows:

$$\mu = \mu(J), \qquad \phi = \phi(J), \tag{6}$$

with $\phi \to \phi_J$ and $\mu \to \mu_c$ as $J \to 0$ [47,48]. Here, the macroscopic friction coefficient $\mu \equiv \sigma_{xy}/\Pi$ and volume fraction ϕ are the order parameters. These constitutive forms closely parallel dry granular rheology, highlighting the central role of particle contacts in dense suspensions [5,47]. However, they do not predict any rate dependence, which is striking: simulations with hydrodynamic-only and those that combine hydrodynamics with contact friction both produce steady states that are invariant under rescaling of the imposed shear rate [5]. This is *counterintuitive*, as particles of different shapes, sizes, and chemistries dispersed in different fluids exhibit DST (see the review [1] and the references therein).

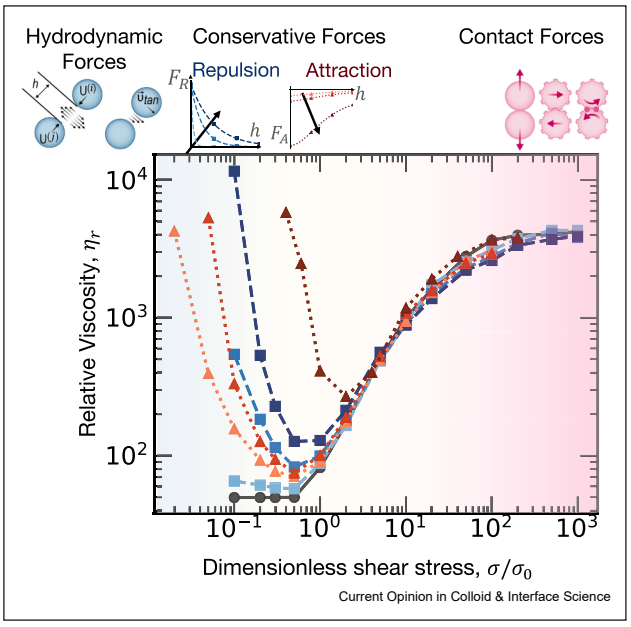
Introducing a force scale: Stress-activated frictional contacts

Hydrodynamics and friction alone are insufficient to produce a rate-dependent steady-state rheology. Thus, to obtain shear thickening, an additional physical ingredient, a force scale, must be introduced to establish the transition from lubricated (hydrodynamic) interactions to direct frictional contact.

In colloidal suspensions, Brownian motion provides this force scale. The colloidal force $F_B \sim k_B T/a$, competes with the imposed flow: at low Pe, the colloidal forces maintain particle separation, and the suspension remains in a lubricated state; at high Pe, the particle contacts become frictional [1,37,49].

However, for non-colloidal suspensions, no intrinsic force scale exists. This implies that shear thickening arises from forces that *protect* the particle surfaces at low stress. It can be numerically implemented in two ways:

Figure 4



Connecting interparticle forces with rheology. *Representative shear-thickening response of a dense suspension*, showing relative viscosity η_r as a function of dimensionless shear stress σ/σ_0. The background shading indicates the dominant force scale at a given stress: hydrodynamic and conservative interactions (low-to-intermediate stress) and frictional contact forces (high stress). Different symbols denote simulation models incorporating hydrodynamics and friction with varying conservative interactions. Circles (black solid line) correspond to the critical load model (CLM). Squares represent electrostatic repulsion, with darker shades indicating increasing Debye length κ^{-1}. Triangles denote simulations including both repulsive and attractive forces, with darker shades corresponding to a larger Hamaker constant A.

(i) the electrostatic repulsion model (ERM) and (ii) the critical load model (CLM).

Electrostatic repulsion model (ERM): The ERM originates from electrostatic double layers, steric stabilization, or other interfacial chemistry, and is often used for non-colloidal suspensions to prevent aggregation [50]. For two particles or radii a_i and a_j, it is modeled as

$$|\overrightarrow{F}_R^{i,j}| = \left\{ \begin{matrix} F_0 a_i a_j/(a_i+a_j)\exp(-\kappa h)\boldsymbol{n}_{ij} & \text{if } h^{(i,j)} \geq 0 \\ F_0 a_i a_j/(a_i+a_j)\boldsymbol{n}_{ij} & : \text{otherwise} \end{matrix} \right\}, \tag{7}$$

where κ^{-1} is the Debye length.

Critical load model (CLM): The so-called CLM is a minimal model that provides additional scaling for stress-dependence. In this model, a critical normal force F_0 is required to activate a frictional contact between two particles. When the normal force F_n is smaller than the F_0, the particle interactions are frictionless, beyond which finite friction μ becomes active.

$$|\overrightarrow{F}_{C,tan}^{i,j}| \leq \left\{ \begin{matrix} \mu(|\overrightarrow{F}_{C,norm}^{i,j}| - F_{CL}) & : |\overrightarrow{F}_{C,norm}^{i,j}| \geq F_{CL} \\ 0 & : \text{otherwise} \end{matrix} \right\} \tag{8}$$

Both models introduce a characteristic force scale F_0, which sets a corresponding stress scale $\sigma_0 \sim F_0/a^2$. The particles remain separated by lubrication films when $\sigma \ll \sigma_0$, i.e., the rheology is frictionless. For $\sigma/\sigma_0 \gg 1$, almost all contacts are frictional, thus, the rheology is frictional. The two limits $\sigma/\sigma_0 \ll 1$ and $\sigma/\sigma_0 \gg 1$ are still quasi-Newtonian: the flow is rate-independent with finite normal stress differences [5].

From friction to constraints–a minimal framework for dense suspension rheology

Once stress-activated frictional contacts are allowed, the central question becomes how these *nanoscopic* interactions or particle scale features, such as roughness, shape, and surface chemistry, give rise to *macroscopic* rheology. A useful and unifying perspective is to view friction as *constraints* on relative particle motion [34,40,46] that restrict specific degrees of freedom (DOFs), thus reducing the available DOFs. Mechanical stability follows from a balance between constraints and force and torque balance conditions. Maxwell−type counting arguments define the isostatic coordination number Z_{iso}, which determines the jamming volume fraction for each type of constraint [51,52].

In a $d-$ dimensional system, for frictionless particles, only normal forces are transmitted, and the torque balance is trivially satisfied, leading to $Z_{\text{iso}} = 2d$ and $\phi_J^{(0,0)} \approx 0.64$ in $3d$. Constraining the sliding motion reduces the isostatic condition to $Z_{\text{iso}} = d+1$, shifting to $\phi_J^{(\mu_s=\infty,\mu_r=0)} \approx 0.57$ in $3d$ [33−35,53]. However, an additional rolling constraint further reduces $Z_{\text{iso}} = d(d+1)/(2d-1)$ and $\phi_J^{(\mu_s=\infty,\mu_r=\infty)} \approx 0.365$ in $3d$ [34]. Dense suspensions [34] or, in general, jamming of particulate systems can be understood in terms of constraint-dependent $\phi_J^{\{\mu_s,\mu_r\}}$. the premise of the approach being that the loss in DOFs is more important than the exact physics/chemistry that leads to it (Ref. Figure 3).

The connection between jamming and particle properties is summarized in Figure 3. In this framework, the jamming point $\phi_J^{\{\mu_s,\mu_r\}}$ is extracted by fitting the steady-state viscosity in the lubricated and frictional states to the functional form $\eta_r = (1-\phi/\phi_J^{\{\mu_s,\mu_r\}})^{-\alpha}$, where most studies assume $\alpha \sim 2$ [33,54,55]; however, the exponents are also reported to depend on particle properties [53].

Stress-activated constraints and shear thickening

The stress-dependent activation of constraints naturally leads to shear thickening. As the shear stress increases beyond σ_0, the constraints are activated, reducing the effective jamming volume fraction $\phi_J^{\{\mu_s,\mu_r\}}$ at a fixed ϕ. The stress-activated constraint picture provides a minimal constitutive framework for shear thickening in dense suspensions. In the Wyart–Cates framework [56], the fraction of frictional contacts $f(\sigma)$ is a *stress-activated* order parameter. Based on this, the effective jamming volume fraction can be expressed as

$$\phi_m(\sigma) = \phi_J^{(\mu_s,\mu_r)} f(\sigma) + \phi_J^{(0,0)}[1 - f(\sigma)], \tag{9}$$

where $\phi_J^{(0,0)}$ and $\phi_J^{(\mu_s,\mu_r)}$ are the jamming volume fractions for the unconstrained and constrained states, respectively.

The common functional form for $f(\sigma)$ is:

$$f(\sigma) = \exp(-\sigma_0/\sigma). \tag{10}$$

Thus, the relative viscosity can be expressed as the distance from the stress-dependent jamming point as follows:

$$\eta_r \sim (1 - \phi/\phi_m(\sigma))^{-2}. \tag{11}$$

Similar expressions can also be written for the normal stresses and particle pressure [33,36]. This framework unifies the continuous shear thickening (CST), DST, and SJ in terms of stress-activated friction/constraints. CST is observed for $\phi \ll \phi_J^{(\mu_s,\mu_r)}$, which becomes stronger as ϕ increases. At a critical volume fraction $\phi_c \lesssim \phi_J^{(\mu_s,\mu_r)}$, DST is observed, whereas SJ occurs at high stresses for $\phi > \phi_J^{(\mu_s,\mu_r)}$.

Beyond stress-activated friction – realistic steady-state rheology

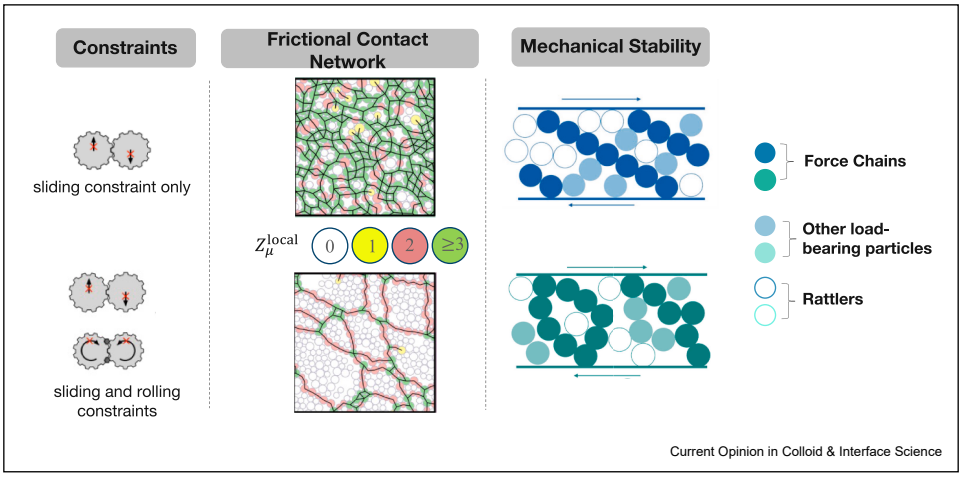
Experimentally observed flow curves often exhibit shear thinning and yield stresses preceding shear thickening, implying the need for additional interparticle interactions. In simulations, this is captured by the inclusion of short-range repulsive or attractive forces in the force balance, as follows:

$$\overrightarrow{0} = \boldsymbol{F}_{\mathrm{H}}(\boldsymbol{X},\boldsymbol{U}) + \boldsymbol{F}_{\mathrm{C}}(\boldsymbol{X}) + \boldsymbol{F}_{\mathrm{cons}}(\boldsymbol{X}), \tag{12}$$

where $\boldsymbol{F}_{\mathrm{cons}} = F_R + F_A$, where F_R and F_A are the repulsive and attractive forces, respectively. The attractive force is often modeled as a van der Waals form $F_A = A\bar{a}/12(h^2 + \Delta)$, where $\bar{a}$ is the harmonic mean radius, A is the Hamaker constant and Δ is used to avoid the singularity (so the contacts are allowed) [44,57–60]. Attraction arising from van der Waals forces or depletion effects can lead to strong shear thinning and even a finite yield stress [57–60]. In these cases, the suspension is an unyielded soft solid for $\sigma < \sigma_y$, and it shear-thins from infinite viscosity and shear-thickens at even higher stresses. At higher attraction strength, the attractive interactions can eventually obscure shear-thickening behavior [57–59,61] (see Figure 4)

Several mean-field models have been proposed to capture this physics: (i) the combination of Herschel–Bulkley with the WC model [57], and (ii) a more physics-based model of stress-released adhesive

Figure 5



From microscopic constraints to mesoscale frictional contact network *Top: Sliding constraints only*: Considering sliding constraints only leads to a highly branched frictional contact network (FCN), with particles having local frictional coordination number $Z_\mu^{\mathrm{local}} \geq 3$ (green colored) dominating, with a few particles with $Z_\mu^{\mathrm{local}}<2$ and a minimal (*yet finite*) numbers of rattlers ($Z_\mu^{\mathrm{local}} = 0$). Strong force chains in the primary (compressive) axis need orthogonal support to maintain mechanical stability. *Bottom: Sliding and rolling constraints*: A suspension in which particles interact with both sliding and rolling constraints can exhibit a similar rheology (viscosity η_r) but a very distinct FCN. The FCN is primarily composed of particles with ($Z_\mu^{\mathrm{local}} \leq 2$) and a very few particles with ($Z_\mu^{\mathrm{local}} \geq 3$), and is composed of many rattlers. The mechanical stability for particles interacting with both sliding and rolling constraints does not require orthogonal support. The figure is adapted from Ref. [74].

constraints [62,63]. The latter picture considers adhesive interactions that constrain relative rotational motion and are released as the stress increases. Thus, *stress-released* constraints lead to yielding and shear-thinning, whereas stress-activated constraints lead to shear thickening. Adding the two together leads to the picture described above.

Role of inertia

The discussion above assumes Stokes flow. When particle inertia becomes finite, the rheology depends on the Stokes number St, and the system crosses over to an inertial regime [35,64–66]. A minimal representation is as follows:

$$\Sigma \sim (\phi_J - \phi)^{\alpha}\dot{\gamma}^2 + (\phi_J - \phi)^{\beta}\dot{\gamma}. \tag{13}$$

This framework attempts to integrate both viscous and inertial suspension physics, though it is important to note that the majority of experimental literature on shear thickening is in the viscous regime with St, $Re_p \ll 1$ [67,68].

A note on model parameters

Even though these simulation frameworks have semi-quantitatively reproduced experimentally observed steady-state rheology in the dense limit, several implementation choices require careful consideration:

- **Near-hard-sphere limit:** As mentioned, a finite contact overlap (or compliance) is required to generate large-scale structures, i.e., perfectly rigid particles, or the strict rigid limit, cannot reproduce features such as DST, as shown by Ref. [69]. Finite stiffness is essentially a numerical choice, with the maximum overlap decreasing with k_n. Previous simulation studies have shown that a maximum dimensionless overlap of approximately 2–3% of the particle radius is generally sufficient to reproduce behavior consistent with experiments [32–34,53]. Together with the normal stiffness, the tangential and rolling stiffnesses, k_t and k_r, also require care to ensure that the spring stretches remain small compared to the particle size. Although no universal choice exists, commonly used ratios include $k_t/k_n = 2/7$ [64] or 1/2 [33,34,53].
- **Lubrication regularization and friction:** Following the NHS idea, lubrication interactions need to be regularized, i.e., ($F_{\text{lub}} \sim \frac{1}{h+\Delta}$). The choice of Δ requires care, as it effectively represents particle asperities or unresolved surface roughness, and typical values of $\Delta \sim 10^{-2} - 10^{-3}$ are commonly employed. The volume fractions for DST and SJ are directly related to the friction coefficients $\{\mu_s, \mu_r\}$. The families of friction coefficients $\{\mu_s, \mu_r\} = \{1, 0\}$, $\{0.5, 0.07\}$ have been shown to reproduce rheological trends that are *semi-quantitatively* consistent with experiments, although these friction values are somewhat larger than the experimentally measured ones.
- **Stokes number:** SD [7] or related simulations integrating lubrication with contact friction simulate overdamped dynamics. These methods involve inversion of the hydrodynamic resistance matrix, which is numerically expensive. Other related methods that simulate small (but finite) Stokes numbers avoid explicit matrix inversion and instead use analytical expressions for pairwise lubrication interactions (see Refs. [35,42,64,70,71]). Finally, these studies generally suggest that relatively small Stokes numbers, ($St \sim \mathcal{O}(10^{-2})$), are sufficient for inertial effects to remain negligible.

Recent developments: Beyond mean-field description

The mean-field models discussed so far, by construction, consider the globally averaged rheological description in terms of globally averaged order parameters, such as the mean frictional coordination number Z or the fraction of frictional contacts $f(\sigma)$ (see review articles [1,4,5,8]). However, by definition, they provide little insight into stress heterogeneity, contact and force networks, and fluctuations in the rheological responses, often associated with features of a critical point in theoretical physics.

Network science insights into dense suspension rheology

Insights from disordered particulate systems [52,72,73] have long suggested that mechanical stability and jamming are controlled by the emergence of system-spanning force chains and their organization under load. Here, we provide a brief overview of the recent progress along these lines (see Figure 5).

This shifts the focus from traditional structural descriptors in real (grain) space, such as $g(r)$ or $S(q)$, to frictional contact or force network space [5,53]. At low stress, the frictional contact network (FCN) is not present. As σ approaches σ_0, sparse frictional contacts are formed primarily along the compressive direction. At higher stress, contacts also develop in orthogonal directions, stabilizing force transmission and giving rise to looped and branched network structures that closely resemble the force-chain organization in dry granular systems [72,75].

Gameiro et al. [76] showed that the persistence of looped structures in the FCN is correlated with the viscosity in two- and three-dimensional systems, over a range of packing fractions and applied stresses. Much of this understanding, particularly of network topology, has been developed using two-dimensional simulations. Under simple shear, structural variations are primarily

confined to the flow and velocity-gradient directions, motivating the use of two-dimensional simulations. Consequently, much of the current understanding of FCN topology has been developed within this framework.

From a network- and graph-theoretic standpoint, particular attention has been given to loops as simple motifs in the FCN. Of particular importance are third-order loops, whose emergence has been linked to frustrated particle rotation and enhanced constraint [77]. Recent computational studies suggest that η_r can collapse onto a master curve when plotted against the number of third-order loops n_3 across a wide parameter space of (ϕ, σ, μ_s) [78]. These observations suggest that the FCN can serve as an important *mesoscale* descriptor of *macroscale* rheology, including DST and SJ. Using the pebble-game algorithm, Naald et al. [79] analyzed the locally 'minimally' rigid clusters that are formed as the suspension is sheared. *Nearly* system-spanning rigid clusters were found in the vicinity of jamming; however, the onset of rigidity ϕ_{rig} was found to be close to, and yet distant from ϕ_{DST} (packing fraction corresponding to DST).

Further studies suggest that the bulk rigidity stems from the local rigidity of FCNs [80]. CST and DST have also been shown to be distinguished based on the degree of constraint and rigidity at the cluster level [81]. Collectively, these results suggest that the emergence of rigidity may play an important role in the onset of SJ. However, the effect of many parameters, such as finite particle softness, system size, and protocol dependence, remains unexplored.

More recent studies have further generalized the network picture by explicitly accounting for different types of constraints, including sliding-only or sliding-and-rolling [74]. Sliding-only systems exhibit highly branched networks that require orthogonal support for stability, as proposed by Cates *et al.* [72]. However, with rolling constraints, stability can be achieved through long self-supporting chains, even with a large fraction of rattlers. These results suggest that the network organization and mechanical stability depend not only on the number of contacts but also on the *type* of constraints. These findings are consistent with experimental work by Hsiao and coworkers, who experimentally reported that the contact network is strongly affected by particle roughness [82].

In 3D, network-based measures of rigidity are less well established, and few structural descriptors have been investigated. Three-dimensional analysis of FCNs using k-core decompositions, in which k-core clusters are defined by particles with at least k frictional contacts, suggests that simple contact percolation occurs in both CST and DST [83]. Examining subnetworks of particles with fixed numbers of frictional contacts indicates that the percolation of particles with four frictional contacts coincides with the onset of the DST and is accompanied by large stress fluctuations that exhibit signatures of critical scaling [71].

Network-based analysis has emerged as a promising, still-evolving framework for understanding dense suspension rheology. These approaches suggest that rheological transitions, from DST to SJ, are closely linked to the reorganization of FCNs, providing a mesoscale perspective that complements traditional constitutive descriptions. It is now well established that the emergence of frictional contacts is closely related to continuous shear thickening (CST), while the formation of loop-like structures, corresponding to frictional contacts spanning both compressive and extensional directions, stabilizes the FCN and enables load-bearing stress transmission associated with DST and SJ. Identifying universal topological descriptors governing rigidity, nonlocality, and SJ—particularly in three-dimensional systems—remains an active area of research.

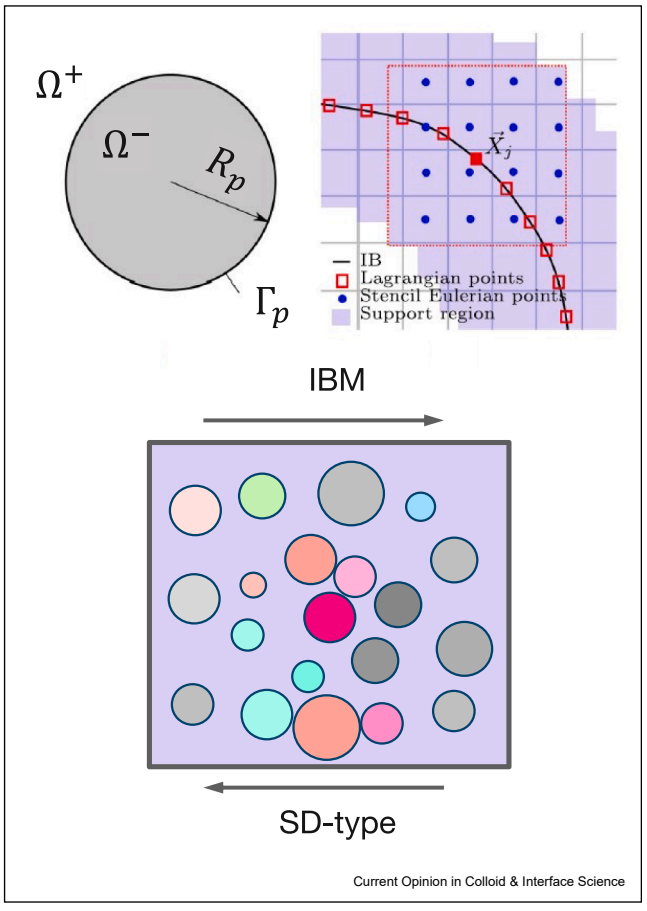
Emergence of fluctuations and viewing discontinuous shear thickening/shear jamming as phase-transition like behavior

DST has long been associated with the emergence of large fluctuations in both viscosity and normal stresses [84,85]. Experiments on highly concentrated, nearly monodisperse suspensions have shown that the viscosity can fluctuate by over an order of magnitude as the system approaches DST. Motivated by the formation of frictional contacts, recent efforts have focused on understanding the microscopic origins of these fluctuations [53,71,86,87]. A common observation is that viscosity fluctuations are relatively small in both the lubricated and frictional regimes, where well-defined mean values exist (though they may differ significantly), whereas large temporal fluctuations emerge near the onset of DST, where the system intermittently samples both states. This behavior can also be interpreted in terms of viscosity distributions: unimodal in the quasi-Newtonian regimes and becoming bimodal near the onset of DST. These observations have motivated connections to ideas from critical phenomena, where mean-field descriptions become insufficient and fluctuation-dominated behavior emerges. More recent studies have explored scaling descriptions inspired by critical phenomena, suggesting that shear thickening may be associated with distinct underlying regimes and crossover behavior that can be characterized using scaling frameworks [88–90].

Beyond Stokes flow, simple shear, and viscometric flows

Most simulation and theoretical descriptions of dense suspension rheology have focused on steady simple

Figure 6



Spatial discretization of a suspended body in surrounding fluid *Top: Particle with radius R_p and a solid boundary Γ_p 'immersed' in a fluid. The solid boundary of the particle is marked by Lagrangian points to define it. Bottom: Simple shear flow of polydisperse particles. Lagrangian points that define the particle boundaries interact with the fluid, external Lagrangian points that belong to other suspended bodies and the boundaries of the system.* The top panel is adapted from Ref. [11].

shear in periodic domains, where the roles of hydrodynamic interactions, conservative forces, and frictional contacts can be isolated. However, real flows involve non-viscometric deformations, transients, confinement, and pressure-controlled conditions. A central question is therefore which physical mechanisms identified in simple shear persist under more general loading conditions.

Extension of lubrication and friction approaches beyond Lees–Edwards imposed simple shear

Extensional flows: Particle-based simulations have extended dense-suspension modeling beyond simple shear to planar, uniaxial, and biaxial extension [91,92]. Frictionless suspensions exhibit nearly flow-type-independent jamming, consistent with approximately isotropic microstructure, whereas frictional systems show a pronounced dependence of the jamming volume fraction on the deformation geometry.

More recently, generalized periodic boundary conditions, such as the generalized Kraynik–Reinelt boundary conditions [93], have been implemented to facilitate non-shear simulations of particulate materials, such as extensional flows, up to large strains. A recent study implemented these boundary conditions to simulate dense granular flows with varying deformation geometries, ranging from triaxial extension to compression and pure shear, and found a significant dependence of the stress response on the deformation geometry [94]. The effects of the geometry of deformation are expected to be important in dense granular suspensions, suggesting the need for a complex tensorial rheological description, such as that formulated in a recent theoretical framework [95].

Pressure-imposed flows. Beyond volume-controlled shear flow simulations, pressure-controlled simulations allow the system to compact or dilate as necessary in response to externally imposed flow. Wall-bounded simulation frameworks with imposed particle pressure allow the solid volume fraction ϕ to adjust dynamically under shear [96,97]. These methods provide direct access to transient rheology and the coupled evolution of shear, particle pressure, and microstructure. They demonstrate that dense suspension flow is controlled by proximity to mechanical instability rather than by shear rate alone, reinforcing the connections with granular-flow physics.

In addition to the imposed pressure, simulations of dense suspensions can also be conducted under applied shear stress rather than the customary applied shear rate. Previously, such stress-controlled simulations have successfully identified the physics of flow–arrest transitions in colloidal suspensions [48] and dry granular materials [98,99]. In stress-controlled simulations, the system under applied stress can flow or not, allowing precise quantification of the yield stress, the divergence of properties such as viscosity near the yield stress, and the stochasticity associated with the flow–arrest transition.

Wall-bound flows: The presence of a wall modifies hydrodynamic interactions and stress transmission, since particle-wall lubrication forces and long-range wall-mediated flows alter microstructure and diffusion. Incorporating these effects into particle-resolved simulations presents significant computational challenges. Swan *et al.* extended SD to account for confinement by constructing wall-corrected mobility formulations, demonstrating how particle–wall interactions influence rheology and short-time diffusion [100,101]. Earlier pressure-driven simulations by Nott and Brady revealed shear-induced particle migration and accumulation in

the center of the channels [102]. More recent algorithmic developments have incorporated curved boundaries and wall-induced traction effects, enabling quantitative predictions of confinement-dependent transport and rheology [103]. These studies collectively show that boundaries fundamentally alter microstructure, diffusion, and stress heterogeneity, and must be explicitly resolved to connect dense-suspension mechanics with realistic flow geometries.

Grid-based frameworks

This perspective has so far focused primarily on SD-like approaches for dense suspensions, particularly in the dense limit where near-contact hydrodynamics and frictional interactions dominate the rheology. In this section, we briefly discuss simulation methods beyond the SD family, many of which have traditionally been applied to dilute and semi-dilute suspensions, though recent developments have extended some of these approaches toward denser regimes. The methods discussed below are grid-based (see Figure 6 for comparison of methods) and offer additional flexibility in treating particle characteristics, such as shape and size distribution, boundary conditions (including confinement), and solvent rheology, ranging from Newtonian to non-Newtonian.

Force coupling method (FCM) and fictitious domain method (FDM): FCM represent particles as distributed force multipoles embedded within a fixed fluid grid, avoiding explicit inversion of the resistance-matrix while retaining hydrodynamic interactions [104]. Sub-grid lubrication corrections enable accurate microstructural and rheological predictions with favorable $\mathcal{O}(N \log N)$ scaling. Force-coupling-based approaches have enabled simulations of non-inertial suspensions with $\mathcal{O}(10^3)$ particles while resolving many-body far-field hydrodynamic interactions [105], primarily in the dilute regime ($\phi \lesssim 0.2$). These studies also highlighted the importance of lubrication corrections for accurately capturing particle interactions at very small interparticle separations. Addressing this, Dance and Maxey [106] incorporated lubrication interactions into the force-coupling method to study non-Brownian suspensions of inertialess particles in Stokesian Poiseuille flow. Using analytical near-field solutions, lubrication corrections were introduced for both particle−particle and particle−wall interactions, enabling more accurate treatment of near-contact hydrodynamics.

FDMs enforce rigid-body motion through constraint-based formulations and fractional-step solvers without remeshing [43,107]. Combined with frictional contacts using DEM and implicit time integration, these approaches capture the transition from hydrodynamic-dominated behavior in semi-dilute systems to contact-dominated rheology in the dense regime. Recent implementations within open-source frameworks have extended these methods to pressure-driven and heterogeneous flows, revealing non-local stress effects in confined geometries [108]. Building on such particle-resolved approaches, pressure-driven flow of dense non-Brownian suspensions with volume fractions up to 50% was investigated in heterogeneous channel geometries [109]. These studies examined spatial variations in stress, volume fraction, and velocity profiles, revealing distinct wall-layering regions, a central plug-like region, and an intermediate bulk region. While the intermediate region showed good agreement with predictions from the local suspension balance model (SBM), significant deviations were observed near the walls and in the central region, suggesting the importance of non-local stress contributions and highlighting the challenges of applying purely local constitutive descriptions to heterogeneous dense suspension flows. Using both force-coupling and fictitious-domain methods, bidisperse non-Brownian suspensions under Couette flow were investigated to examine the role of particle-size variation and migration in heterogeneous flows [110]. Comparisons with a modified SBM for bidisperse systems showed that the continuum description was capable of reproducing migration trends of the smaller particles in reasonable agreement with particle-resolved simulations. While the overall rheology exhibited relatively weak dependence on size bidispersity for the conditions studied, the work highlighted the growing need for continuum frameworks that incorporate particle-scale heterogeneity and migration dynamics.

Immersed boundary method (IBM): The IBM is well-suited for complex particle shapes and confined geometries, as both particles and boundaries are represented using discrete Lagrangian nodes. Combining the IBM with SD-type approaches enables the study of flexible fiber suspensions in Newtonian fluids, modeling fibers as inextensible slender bodies dynamically coupled to the surrounding flow [111].

To address the challenges associated with property discontinuities in two-phase formulations, volume-filtered Navier−Stokes approaches have been developed [112], introducing analytical viscous closures to maintain Galilean invariance and allowing coarse direct numerical simulations (DNS) in the dense limit. Fully particle-resolved DNS frameworks have also coupled IBM with Newton−Euler dynamics and DEM-based contact models to simulate dense suspensions in confined shear flows [113]. These studies demonstrate that particle-resolved DNS can capture the coupled roles of hydrodynamics, lubrication, and contact interactions across various concentration regimes.

A limitation of classical SD-type methods is their restriction to Newtonian solvents. To overcome this, Shaqfeh *et al.* [114,115] developed particle-resolved DNS frameworks employing body-fitted and

immersed-boundary grids to study suspensions in polymeric solvents modeled using Oldroyd-B and Giesekus constitutive equations. The evolution equations for the polymer stress tensor were coupled to the momentum balance via a source term in a finite-volume solver, enabling fully resolved two-way coupling between the particle motion and viscoelastic stresses. In the dilute-to-semidilute regime, these simulations showed that while the per-particle contribution may exhibit shear thickening, the overall suspension rheology can transition to shear thinning as the particle proximity suppresses particle-induced fluid stresses. Using a Giesekus constitutive model, the effects of flow conditions, solvent rheology, and particle concentration on drag forces and normal stress differences can be systematically investigated for suspensions of fixed spherical particles [116].

Continuum modeling

In engineering practice and natural phenomena, dense suspensions typically undergo a transient evolution through complex geometries. Predicting the fluid mechanics of dense suspensions in realistic flow scenarios requires a general, predictive 3D continuum model that can be implemented and solved numerically using computational fluid dynamics (CFD). Most previous studies on dense suspensions primarily in the physics literature have focused on steady simple shearing flows [1,8,32–34]; however, these materials exhibit rich rheological features that need to be correctly accounted for in a constitutive model to accurately capture their dynamics in real applications.

Continuum modeling of dense suspensions can be broadly classified into two categories. In a first class of continuum models, the two phases of the suspension, the particle phase and the fluid phase, are modeled individually as two interacting continua where each phase has its own continuum fields of velocity and volume fraction. Here, the particles are homogenized into a single continuum 'granular' phase. In this formulation, a set of coupled partial differential equations (PDEs) can be written by considering basic conservation laws of mass, momentum, energy, and entropic inequality for each phase [117,118]. The resulting system of PDEs is generally not closed, and constitutive closures (models) that encode the physics of suspensions are required to model drag and buoyancy forces in the momentum exchange, fluid-phase pore pressure, and stresses in the fluid and granular phases. Such 'two-fluid' modeling is typically important in scenarios where the velocities of the two phases can be very different, such as in fluidized bed reactors [119]. While some of the constitutive closures are well calibrated from experiments and simulations, the models for granular stresses represent key challenges in continuum modeling of dense suspensions. As discussed in detail above, this results from our recent understanding that, in dense suspensions, frictional contacts strongly mediate suspension viscosity, particularly at very high solid volume fractions where CST, DST, and SJ are observed. In recent works [118,120], the "two-fluid" representation was extended to model dense suspensions by expressing local suspension viscosity and granular pressure as a function of the fraction of frictional granular contacts [33,56], which is included as an auxiliary field variable that locally modulates the volume fraction at which the suspension viscosity rapidly diverges, thus capturing the phenomena of CST, DST, and SJ.

In a second class of continuum models, the suspension is modeled as a mixture, instead of two individual phases, through conservation PDEs of the mass, momentum, and energy of the whole mixture. In these models, the migration or 'drift' of particles relative to the mixture requires additional closure modeling. For example, the diffusive flux of particles relative to the mixture is typically dependent on the spatial gradients of volume fraction and the strain-rate tensor [121], or can be expressed through a relationship with the divergence of the granular-phase pressure, such as in the well-known 'suspension balance models' [102,122]. These types of models are usually valid when the drift velocities are small, which is often the case in steady flows of dense suspensions. Furthermore, higher-order rheological effects, such as normal stress differences have also been incorporated into such models of dense suspension in a frame-indifferent manner [123,124]. These effects play a key role in secondary flow features such as curvature in free-surface flows, the rod-climbing effect, and particle migration due to normal-stress-difference gradients [6].

The above continuum models do not explicitly account for the granular microstructure, and, as such, are well-suited for steady flows where the microstructure evolves smoothly and steadily. However, in highly transient flows such as shear reversal and oscillatory shear, an explicit accounting of the microstructure in the continuum modeling is essential to capture the complex interactions between the suspension rheology and microstructure [125]. In such cases, the microstructure is typically represented as a tensor, such as the fabric tensor, and its dynamic evolution is represented by a PDE that is coupled with the conservation PDEs [126,127]. The key challenge in such continuum models is the accurate calibration of the model parameters representing the microstructural evolution, which are typically obtained from extensive DEM simulations [91,92,128].

Take home messages and outlook

Dense suspensions are at the intersection of colloidal science, contact mechanics, continuum theory, and

network physics, and remain central to industrial, technological, and geophysical processes. Over the past decade, major advances in particle-resolved simulations have transformed our understanding of the shear thickening, dilatancy, and SJ. Phenomena first reported nearly a century ago [129] can now be reproduced with *near-quantitative* accuracy, and their underlying mechanisms, hydrodynamic interactions, lubrication breakdown, frictional contacts, and mesoscale organization, can be systematically interrogated.

The computational advances discussed in this perspective have not only clarified the physics of stress-activated transitions but have also enabled the development of rational strategies to control and mitigate shear thickening, including orthogonal deformation protocols, particle deformation, and polymer-mediated interactions. Looking ahead, several challenges and opportunities remain.

1. *Going beyond nearly-similar-sized spherical particles*: Most existing simulation frameworks focus on nearly equal-sized spherical particles, with limited treatment of strong polydispersity and non-spherical geometries [89,130]. However, real materials exhibit broad size distributions and complex shapes. Extending computational methods to highly polydisperse and anisotropic particles is essential for industrial applications.
2. *Boundary conditions and realistic geometries*: Many state-of-the-art models assume idealized simple shear conditions without explicitly resolving wall slip, localization, free surfaces, or complex boundary conditions. Incorporating such boundary effects into particle-resolved and continuum descriptions is critical for theoretical rheological models and application-scale flows.
3. *Coupling different scales*: It is abundantly clear that the microscopic details of dense suspensions, such as particle characteristics and fluid rheology, strongly impact their macroscale behavior. Recent evidence also indicates that many intriguing physical phenomena in dense suspensions manifest at the mesoscale, such as the evolution of force chains and correlated particle motion. Therefore, multiscale modeling of these materials is required, enabling microscopic details to be reliably upscaled to macroscale continua while preserving the mesoscale characteristics of the system.
4. *Transients and dilatancy*: Although the physics of dense suspensions is well established in the steady-shear regime, practical applications of these flows routinely encounter more complex settings such as transient dilatancy, shear reversal, and non-shear deformation. Constitutive models capable of describing such complex flow histories with microstructural fidelity are still under active development.
5. *Beyond the mean-field description and statistical mechanics*: Increasing evidence indicates that dense suspensions cannot be fully characterized by scalar viscosity or mean-field closures alone. Incorporating network topology, force-chain statistics, and higher-order structural descriptors, particularly in three dimensions, offers a promising path toward predictive modeling near mechanical instability.
6. *Incorporating machine learning tools in modeling dense suspension mechanics:* Rapidly growing machine-learning technologies provide exciting avenues to accelerate the simulations of dense suspensions, and consequently the discovery of physical mechanisms that underlie their mechanics [131,132]. For example, machine-learning methods such as graph neural networks can be used to construct microstructural surrogates of the contact friction network, which has recently been shown to be promising in predicting the near-jamming physics of dense particulate systems [133–136]. Data-driven machine-learning methods can be used to train complex rheological closures [137–139] with appropriate physical constraints that upscale particle-level information into continuum descriptions capable of handling confinement, strong polydispersity, and spatio-temporal stress heterogeneity. Such training can also be done adaptively in-situ in a hybrid particle-continuum simulation with uncertainty quantification, as was successfully demonstrated recently for dense granular flows [140]. While the rapid increase in computational power and algorithmic advances will provide rich simulations datasets of dense suspensions, crucial for the training of machine learning surrogates, the increasing availability of these surrogates will, in turn, accelerate the current capabilities of numerical simulations to length and time scales that were previously unachievable.

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

References

Papers of particular interest, published within the period of review, have been highlighted as:

* of special interest
** of outstanding interest

1. Morris JF: **Shear thickening of concentrated suspensions: recent developments and relation to other phenomena**. *Annu Rev Fluid Mech* 2020, **52**:121–144.

2. Jerolmack DJ, Daniels KE: **Viewing earth's surface as a soft-matter landscape**. *Nat Rev Phys* 2019, **1**:716–730.

3. Hodgson DJ, Hermes M, Blanco E, Poon WC: **Granulation and suspension rheology: a unified treatment**. *J Rheol* 2022, **66**: 853–858.

4. Ness C, Seto R, Mari R: **The physics of dense suspensions**. *Annu Rev Condens Matter Phys* 2022, **13**:97–117.

5. Clavaud C, Singh A: **Quick starch guide: a perspective on shear thickening in dense non-brownian suspensions**. *Rheol Acta* 2025, **64**:563–582.

6. Guazzelli É, Pouliquen O: **Rheology of dense granular suspensions**. *J Fluid Mech* 2018, **852**.

7. ** Brady JF, Bossis G: **The rheology of concentrated suspensions of spheres in simple shear flow by numerical simulation**. *J Fluid Mech* 1985, **155**:105–129.
Seminal paper that introduced Stokesian dynamics as a computational framework that combines far-field hydrodynamics with near-field lubrication interactions for particulate suspensions. The methodology continues to underpin many modern approaches for simulating dense suspension rheology.

8. Morris JF: **Lubricated-to-frictional shear thickening scenario in dense suspensions**. *Phys Rev Fluids* 2018, **3**, 110508.

9. Uhlmann M: **An immersed boundary method with direct forcing for the simulation of particulate flows**. *J Comput Phys* 2005, **209**:448–476.

10. Breugem W-P: **A second-order accurate immersed boundary method for fully resolved simulations of particle-laden flows**. *J Comput Phys* 2012, **231**:4469–4498.

11. Abdol Azis MH, Evrard F, van Wachem B: **An immersed boundary method for flows with dense particle suspensions**. *Acta Mech* 2019, **230**:485–515.

12. Ladd AJ, Verberg R: **Lattice-boltzmann simulations of particle-fluid suspensions**. *J Stat Phys* 2001, **104**:1191–1251.

13. Maxey M: **Simulation methods for particulate flows and concentrated suspensions**. *Annu Rev Fluid Mech* 2017, **49**: 171–193.

14. Aidun CK, Clausen JR: **Lattice-boltzmann method for complex flows**. *Annu Rev Fluid Mech* 2010, **42**:439–472.

15. Patankar NA, Singh P, Joseph DD, Glowinski R, Pan T-W: **A new formulation of the distributed lagrange multiplier/fictitious domain method for particulate flows**. *Int J Multiphas Flow* 2000, **26**:1509–1524.

16. Glowinski R, Pan T-W, Hesla TI, Joseph DD, Periaux J: **A fictitious domain approach to the direct numerical simulation of incompressible viscous flow past moving rigid bodies: application to particulate flow**. *J Comput Phys* 2001, **169**:363–426.

17. Biegert E, Vowinckel B, Meiburg E: **A collision model for grain-resolving simulations of flows over dense, mobile, polydisperse granular sediment beds**. *J Comput Phys* 2017, **340**: 105–127.

18. Burgisser A, Bergantz GW, Breidenthal RE: **Addressing complexity in laboratory experiments: the scaling of dilute multiphase flows in magmatic systems**. *J Volcanol Geoth Res* 2005, **141**:245–265.

19. Martys NS: **Study of a dissipative particle dynamics based approach for modeling suspensions**. *J Rheol* 2005, **49**:401–424.

20. Boromand A, Jamali S, Maia JM: **Viscosity measurement techniques in dissipative particle dynamics**. *Comput Phys Commun* 2015, **196**:149–160.

21. Howard MP, Nikoubashman A, Palmer JC: **Modeling hydrodynamic interactions in soft materials with multiparticle collision dynamics**. *Curr Opin Chem Eng* 2019, **23**:34–43.

22. Wani YM, Kovakas PG, Nikoubashman A, Howard MP: **Diffusion and sedimentation in colloidal suspensions using multiparticle collision dynamics with a discrete particle model**. *J Chem Phys* 2022, **156**.

23. Bossis G, Brady JF: **The rheology of Brownian suspensions**. *J Chem Phys* 1989, **91**:1866–1874.

24. Jeffrey DJ, Onishi Y: **Calculation of the resistance and mobility functions for two unequal rigid spheres in low-reynolds-number flow**. *J Fluid Mech* 1984, **139**:261–290.

25. Kim S, Karrila SJ: *Microhydrodynamics: principles and selected applications*. Boston: Butterworth-Heinemann; 1991.

26. Sierou A, Brady JF: **Accelerated stokesian dynamics simulations**. *J Fluid Mech* 2001, **448**:115–146.

27. Fiore AM, Swan JW: **Fast stokesian dynamics**. *J Fluid Mech* 2019, **878**:544–597.

28. Howard MP, Anderson JA, Nikoubashman A, Glotzer SC, Panagiotopoulos AZ: **Efficient neighbor list calculation for molecular simulation of colloidal systems using graphics processing units**. *Comput Phys Commun* 2016, **203**:45–52.

29. Melrose JR, Ball RC: **The pathological behaviour of sheared hard spheres with hydrodynamic interactions**. *Europhys Lett* 1995, **32**:535–540.

30. O'Hern CS, Silbert LE, Liu AJ, Nagel SR: **Jamming at zero temperature and zero applied stress: the epitome of disorder**. *Phys Rev E* 2003, **68**, 011306.

31. Morris JF: **Progress and challenges in suspension rheology**. *Rheol Acta* 2023, **62**:617–629.

32. ** Seto R, Mari R, Morris JF, Denn MM: **Discontinuous shear thickening of frictional hard-sphere suspensions**. *Phys Rev Lett* 2013, **111**, 218301.
Introduced one of the first particle-resolved simulation frameworks that combines hydrodynamics and contact friction, demonstrating that stress-activated frictional contacts reproduce discontinuous shear thickening (DST) in dense suspensions.

33. ** Singh A, Mari R, Denn MM, Morris JF: **A constitutive model for simple shear of dense frictional suspensions**. *J Rheol* 2018, **62**:457–468.
Introduced a constitutive framework for dense frictional suspensions based on stress-activated frictional contacts, allowing prediction of the full stress tensor under shear flow. The model helped to distinguish discontinuous shear thickening from shear jamming and highlighted the central role of friction in controlling suspension rheology.

34. Singh A, Ness C, Seto R, de Pablo JJ, Jaeger HM: **Shear thickening and jamming of dense suspensions: the "roll" of friction**. *Phys Rev Lett* 2020, **124**, 248005.

35. Ness C, Sun J: **Shear thickening regimes of dense Non-Brownian suspensions**. *Soft Matter* 2016, **12**:914–924.

36. More R, Ardekani A: **A constitutive model for sheared dense suspensions of rough particles**. *J Rheol* 2020, **64**: 1107–1120.

37. * Jamali S, Brady JF: **Alternative frictional model for discontinuous shear thickening of dense suspensions: hydrodynamics**. *Phys Rev Lett* 2019, **123**, 138002.
Presented an alternative hydrodynamic perspective on discontinuous shear thickening in dense suspensions, highlighting the role of lubrication interactions and hydrodynamic clustering. The work contributed to broader discussions on the relative roles of hydrodynamic and frictional mechanisms in shear thickening.

38. Wang M, Jamali S, Brady JF: **A hydrodynamic model for discontinuous shear-thickening in dense suspensions**. *J Rheol* 2020, **64**:379–394.

39. Cundall PA, Strack ODL: **A discrete numerical model for granular assemblies**. *Geotechnique* 1979, **29**:47–65.

40. Santos AP, Bolintineanu DS, Grest GS, Lechman JB, Plimpton SJ, Srivastava I, Silbert LE: **Granular packings with sliding, rolling, and twisting friction**. *Phys Rev* 2020, **102**, 032903.

41. Estrada N, Azéma E, Radjai F, Taboada A: **Identification of rolling resistance as a shape parameter in sheared granular media**. *Phys Rev E* 2011, **84**, 011306.

42. Ness C: **Simulating dense, rate-independent suspension rheology using lammps**. *Computational Particle Mechanics* 2023, **10**(6):2031–2037.

43. ** Gallier S, Lemaire E, Peters F, Lobry L: **Rheology of sheared suspensions of rough frictional particles**. *J Fluid Mech* 2014, **757**:514–549.
Numerical study of the Couette flow of non-Brownian suspensions with the fictitious domain method, while modeling particle contacts with a discrete element method framework to include roughness effects. Showed that including particle surface asperities and thus the roughness associated, is crucial for reproduction of relative-viscosity behavior.

44. More RV, Ardekani AM: **Unifying disparate rate-dependent rheological regimes in non-brownian suspensions**. *Phys Rev E* 2021, **103**, 062610.

45. Ruiz-Lopez JA, Prasanna Kumar SS, Vazquez-Quesada A, de Vicente J, Ellero M: **Tribological variable-friction coefficient models for the simulation of dense suspensions of rough polydisperse particles**. *J Rheol* 2023, **67**:541–558.

46. Singh A, Jackson GL, van der Naald M, de Pablo JJ, Jaeger HM: **Stress-activated constraints in dense suspension rheology**. *Phys Rev Fluids* 2022, **7**, 054302.

47. ** Boyer F, Guazzelli É, Pouliquen O: **Unifying suspension and granular rheology**. *Phys Rev Lett* 2011, **107**, 188301.
A seminal experimental study that unifies dense suspension and dry granular rheology near jamming, proposing a constitutive framework that bridges constant-pressure (granular) and constant-volume (suspension).

48. * Wang M, Brady JF: **Constant stress and pressure rheology of colloidal suspensions**. *Phys Rev Lett* 2015, **115**, 158301.
Presented a simulation scheme for dense suspensions using Brownian dynamics with constant stress and pressure. Showed a unifying connection between dry granular media and dense colloidal suspensions through the ratio of shear to normal stress.

49. Mari R, Seto R, Morris JF, Denn MM: **Discontinuous shear thickening in brownian suspensions by dynamic simulation**. *Proc Natl Acad Sci* 2015, **112**:15326–15330.

50. Israelachvili JN: *Intermolecular and surface forces*. Academic Press; 2011.

51. van Hecke M: **Jamming of soft particles: geometry, mechanics, scaling and isostaticity**. *J Phys Condens Matter* 2009, **22**, 033101.

52. Behringer RP, Chakraborty B: **The physics of jamming for granular materials: a review**. *Rep Prog Phys* 2018, **82**, 012601.

53. Mari R, Seto R, Morris JF, Denn MM: **Shear thickening, frictionless and frictional rheologies in Non-Brownian suspensions**. *J Rheol* 2014, **58**:1693–1724.

54. Guy BM, Hermes M, Poon WCK: **Towards a unified description of the rheology of hard-particle suspensions**. *Phys Rev Lett* 2015, **115**, 088304.

55. Royer JR, Blair DL, Hudson SD: **Rheological signature of frictional interactions in shear thickening suspensions**. *Phys Rev Lett* 2016, **116**, 188301.

56. ** Wyart M, Cates ME: **Discontinuous shear thickening without inertia in dense non-Brownian suspensions**. *Phys Rev Lett* 2014, **112**, 098302.
Developed the influential mean-field model using the fraction of frictional contacts as an order parameter, unifying continuous shear thickening (CST), DST, and shear jamming within a single theoretical framework.

57. Singh A, Pednekar S, Chun J, Denn MM, Morris JF: **From yielding to shear jamming in a cohesive frictional suspension**. *Phys Rev Lett* 2019, **122**, 098004.

58. Pednekar S, Chun J, Morris JF: **Simulation of shear thickening in attractive colloidal suspensions**. *Soft Matter* 2017, **13**: 1773–1779.

59. Rathee V, Monti A, Rosti ME, Shen AQ: **Shear thickening behavior in dense repulsive and attractive suspensions of hard spheres**. *Soft Matter* 2021, **17**:8047–8058.

60. Wang J, Pan D: **Effect of dlvo interactions on the rheology and microstructure of non-brownian suspensions**. *Acta Mech Sin* 2023, **39**, 322469.

61. Gopalakrishnan V, Zukoski C: **Effect of attractions on shear thickening in dense suspensions**. *J Rheol* 2004, **48**: 1321–1344.

62. ** Guy BM, Richards J, Hodgson D, Blanco E, Poon WCK: **Constraint-based approach to granular dispersion rheology**. *Phys Rev Lett* 2018, **121**, 128001.
Proposed a constraint-based description of dense suspension rheology in which flow behavior is governed by stress-dependent activation and release of interparticle constraints. This framework helped to unify discontinuous shear thickening, yielding, and jamming within a common phenomenological picture.

63. Richards JA, Guy BM, Blanco E, Hermes M, Poy G, Poon WC: **The role of friction in the yielding of adhesive non-brownian suspensions**. *J Rheol* 2020, **64**:405–412.

64. Ge Z, Brandt L: *Implementation note on a minimal hybrid lubrication/granular dynamics model for dense suspensions*. 2020, 12755. arXiv preprint arXiv:2005.

65. Trulsson M, Andreotti B, Claudin P: **Transition from the viscous to inertial regime in dense suspensions**. *Phys Rev Lett* 2012, **109**, 118305.

66. Trulsson M, Bouzid M, Kurchan J, Clément E, Claudin P, Andreotti B: **Athermal analogue of sheared dense Brownian suspensions**. *Europhys Lett* 2015, **111**, 18001.

67. Denn MM, Morris JF: **Rheology of non-Brownian suspensions**. *Annu Rev Chem Biomol Eng* 2014, **5**.

68. Mewis J, Wagner NJ: *Colloidal suspension rheology*. Cambridge University Press; 2011.

69. Townsend AK, Wilson HJ: **Frictional shear thickening in suspensions: the effect of rigid asperities**. *Phys Fluids* 2017, **29**.

70. Monti A, Rathee V, Shen AQ, Rosti ME: **A fast and efficient tool to study the rheology of dense suspensions**. *Phys Fluids* 2021, **33**.

71. Goyal A, Martys NS, Del Gado E: **Flow induced rigidity percolation in shear thickening suspensions**. *J Rheol* 2024, **68**:219–228.

72. Cates ME, Wittmer JP, Bouchaud J-P, Claudin P: **Jamming, force chains, and fragile matter**. *Phys Rev Lett* 1998, **81**: 1841–1844.

73. Hsiao LC, Newman RS, Glotzer SC, Solomon MJ: **Role of isostaticity and load-bearing microstructure in the elasticity of yielded colloidal gels**. *Proc Natl Acad Sci* 2012, **109**: 16029–16034.

74. Sharma S, Sharma A, Singh A: **Frictional contact network in dense suspension flow**. *Newton* 2026, **2**.

75. Radjai F, Wolf DE, Jean M, Moreau J-J: **Bimodal character of stress transmission in granular packings**. *Phys Rev Lett* 1998, **80**:61.

76. Gameiro M, Singh A, Kondic L, Mischaikow K, Morris JF: **Interaction network analysis in shear thickening suspensions**. *Phys Rev Fluids* 2020, **5**, 034307.

77. Papadopoulos L, Porter MA, Daniels KE, Bassett DS: **Network analysis of particles and grains**. *J Complex Networks* 2018, **6**: 485–565.

78. D'Amico A, Tu S, Singh A: **Topological insights into dense frictional suspension rheology: Third-order loops drive discontinuous shear thickening**. *J Chem Phys* 2025, **162**, 214906.

79. ** van der Naald M, Singh A, Eid TT, Tang K, de Pablo JJ, Jaeger HM: **Minimally rigid clusters in dense suspension flow**. *Nat Phys* 2024:1–7.
Demonstrated the emergence of nearly system-spanning rigid clusters in two-dimensional frictional suspensions, identifying rigidity as a precursor to shear jamming (SJ) rather than DST.

80. Santra A, Orsi M, Chakraborty B, Morris JF: **Rigid clusters in shear-thickening suspensions: a nonequilibrium critical transition**. *Phys Rev Res* 2025, **7**, 013275.

81. Nabizadeh M, Singh A, Jamali S: **Structure and dynamics of force clusters and networks in shear thickening suspensions**. *Phys Rev Lett* 2022, **129**, 068001.

82. * Pradeep S, Nabizadeh M, Jacob AR, Jamali S, Hsiao LC: **Jamming distance dictates colloidal shear thickening**. *Phys Rev Lett* 2021, **127**, 158002.
Experimental study that provides direct evidence for the formation of frictional contact networks in shear-thickening suspensions. The results helped to connect shear thickening and proximity to jamming across different particle systems through the evolution of stress-bearing contact networks.

83. Sedes O, Makse HA, Chakraborty B, Morris JF: **K-core analysis of shear-thickening suspensions**. *Phys Rev Fluids* 2022, **7**, 024304.

84. Lootens D, Van Damme H, Hébraud P: **Giant stress fluctuations at the jamming transition**. *Phys Rev Lett* 2003, **90**, 178301.

85. Boersma WH, Baets PJM, Laven J, Stein HN: **Time-dependent behavior and wall slip in concentrated shear thickening dispersions**. *J Rheol* 1991, **35**:1093–1120.

86. Heussinger C: **Shear thickening in granular suspensions: inter-particle friction and dynamically correlated clusters**. *Phys Rev E* 2013, **88**, 050201.

87. Boromand A, Jamali S, Grove B, Maia JM: **A generalized frictional and hydrodynamic model of the dynamics and structure of dense colloidal suspensions**. *J Rheol* 2018, **62**: 905–918.

88. Ramaswamy M, Griniasty I, Liarte DB, Shetty A, Katifori E, Del Gado E, Sethna JP, Chakraborty B, Cohen I: **Universal scaling of shear thickening transitions**. *J Rheol* 2023, **67**:1189–1197.

89. Malbranche N, Chakraborty B, Morris JF: **Shear thickening in dense bidisperse suspensions**. *J Rheol* 2023, **67**:91–104.

90. Ramaswamy M, Griniasty I, Sethna JP, Chakraborty B, Cohen I: **Universal scaling framework for controlling phase behavior in thickening and jamming suspensions**. *Phys Rev Lett* 2025, **134**, 058203.

91. Cheal O, Ness C: **Rheology of dense granular suspensions under extensional flow**. *J Rheol* 2018, **62**:501–512.

92. Seto R, Giusteri GG, Martiniello A: **Microstructure and thickening of dense suspensions under extensional and shear flows**. *J Fluid Mech* 2017, **825**:R3.

93. Hunt TA: **Periodic boundary conditions for the simulation of uniaxial extensional flow of arbitrary duration**. *Mol Simul* 2016, **42**:347–352.

94. Clemmer JT, Srivastava I, Grest GS, Lechman JB: **Shear is not always simple: Rate-dependent effects of flow type on granular rheology**. *Phys Rev Lett* 2021, **127**, 268003.

95. Giusteri GG, Seto R: **A theoretical framework for steady-state rheometry in generic flow conditions**. *J Rheol* 2018, **62**:713–723.

96. Athani S, Metzger B, Forterre Y, Mari R: **Transients in shear thickening suspensions: when hydrodynamics matters**. *Phys Rev Fluids* 2025, **10**, 043301.

97. Dong J, Trulsson M: **Analog of discontinuous shear thickening flows under confining pressure**. *Phys Rev Fluids* 2017, **2**, 081301.

98. Srivastava I, Silbert LE, Grest GS, Lechman JB: **Flow-arrest transitions in frictional granular matter**. *Phys Rev Lett* 2019, **122**, 048003.

99. Srivastava I, Silbert LE, Lechman JB, Grest GS: **Flow and arrest in stressed granular materials**. *Soft Matter* 2022, **18**:735–743.

100. Swan JW: *Colloids in confined geometries: hydrodynamics, simulation and rheology*. California Institute of Technology; 2010. Ph.D. thesis.

101. Swan JW, Brady JF: **The hydrodynamics of confined dispersions**. *J Fluid Mech* 2011, **687**:254–299.

102. Nott PR, Brady JF: **Pressure-driven flow of suspensions: simulation and theory**. *J Fluid Mech* 1994, **275**:157–199.

103. Aponte-Rivera C, Su Y, Zia RN: **Equilibrium structure and diffusion in concentrated hydrodynamically interacting suspensions confined by a spherical cavity**. *J Fluid Mech* 2018, **836**:413–450.

104. Yeo K, Maxey MR: **Simulation of concentrated suspensions using the force-coupling method**. *J Comput Phys* 2010, **229**: 2401–2421.

105. Abbas M, Climent E, Simonin O: **Fully coupled simulations of non-colloidal monodisperse sheared suspensions**. *Chem Eng Res Des* 2007, **85**:778–791.

106. * Dance S, Maxey M: **Incorporation of lubrication effects into the force-coupling method for particulate two-phase flow**. *J Comput Phys* 2003, **189**:212–238.
Presented a sub-grid lubrication correction framework for a force coupling method simulation, enabling an accurate account of near-field hydrodynamics.

107. Gallier S, Lemaire E, Lobry L, Peters F: **A fictitious domain approach for the simulation of dense suspensions**. *J Comput Phys* 2014, **256**:367–387.

108. Orsi M, Lobry L, Peters F: **Frame-invariant sub-grid corrections to the fictitious domain method for the simulation of particulate suspensions in nonlinear flows using openfoam**. *J Comput Phys* 2023, **474**, 111823.

109. Orsi M, Lobry L, Lemaire E, Peters F: **Mass and momentum balance during particle migration in the pressure-driven flow of frictional non-brownian suspensions**. *J Fluid Mech* 2024, **998**:A16.

110. Howard AA, Maxey MR, Gallier S: **Bidisperse suspension balance model**. *Phys Rev Fluids* 2022, **7**, 124301.

111. * Khan M, More RV, Banaei AA, Brandt L, Ardekani AM: **Rheology of concentrated fiber suspensions with a load-dependent friction coefficient**. *Phys Rev Fluids* 2023, **8**, 044301.
The first numerical study to extend the stress-activated friction framework beyond spherical particles.

112. Hausmann M, Elmestikawy H, van Wachem B: **Physically consistent immersed boundary method: a framework for predicting hydrodynamic forces on particles with coarse meshes**. *J Comput Phys* 2024, **519**, 113448.

113. Peerbooms W, Nadorp T, van der Heijden A, Breugem W-P: **Interparticle friction in sheared dense suspensions: comparison of the viscous and frictional rheology descriptions**. *J Rheol* 2024, **68**:263–283.

114. Zhang A, Shaqfeh ESG: **Rheology of non-brownian particle suspensions in viscoelastic solutions. part 1: effect of the polymer concentration**. *J Rheol* 2023, **67**:499–516.

115. Zhang A, Shaqfeh ESG: **Rheology of non-brownian particle suspensions in viscoelastic solutions. part ii: effect of a shear thinning suspending fluid**. *J Rheol* 2023, **67**:517–540.

116. Ayar O, Fernandes C, Ferrás LL, Alves MA: **Numerical simulations of suspensions of rigid spheres in shear-thinning viscoelastic fluids**. *Phys Fluids* 2023, **35**, 113327.

117. Jackson R: *The dynamics of fluidized particles*. Cambridge University Press; 2000.

118. * Baumgarten AS, Kamrin K: **A general fluid–sediment mixture model and constitutive theory validated in many flow regimes**. *J Fluid Mech* 2019, **861**:721–764.
Introduced a predictive three-dimensional continuum model for shear-thickening suspensions based on mixture theory, coupling fluid, and granular phases to reproduce steady, transient, and impact-activated jamming responses.

119. Agrawal K, Loezos PN, Syamlal M, Sundaresan S: **The role of meso-scale structures in rapid gas–solid flows**. *J Fluid Mech* 2001, **445**:151–185.

120. Baumgarten AS, Kamrin K: **A general constitutive model for dense, fine-particle suspensions validated in many geometries**. *Proc Natl Acad Sci* 2019, **116**:20828–20836.

121. Mills P, Snabre P: **Rheology and structure of concentrated suspensions of hard spheres. shear induced particle migration**. *J Phys II* 1995, **5**:1597–1608.

122. * Nott PR, Guazzelli E, Pouliquen O: **The suspension balance model revisited**. *Phys Fluids* 2011, **23**.
Important revisitation of the suspension balance model framework, providing physical information on stress-driven particle migration and heterogeneous flow behavior in concentrated suspensions.

123. Morris JF, Boulay F: **Curvilinear flows of noncolloidal suspensions: the role of normal stresses**. *J Rheol* 1999, **43**: 1213–1237.

124. Badia A, D'angelo Y, Peters F, Lobry L: **Frame-invariant modeling for non-brownian suspension flows**. *J Non-Newtonian Fluid Mech* 2022, **309**, 104904.

125. Morris JF: **A review of microstructure in concentrated suspensions and its implications for rheology and bulk flow**. *Rheol Acta* 2009, **48**:909–923.

126. Gillissen JJJ, Ness C, Peterson JD, Wilson HJ, Cates ME: **Constitutive model for time-dependent flows of shear-thickening suspensions**. *Phys Rev Lett* 2019, **123**, 214504.

127. Chacko RN, Mari R, Fielding SM, Cates ME: **Shear reversal in dense suspensions: the challenge to fabric evolution models from simulation data**. *J Fluid Mech* 2018, **847**:700–734.

128. Srivastava I, Silbert LE, Grest GS, Lechman JB: **Viscometric flow of dense granular materials under controlled pressure and shear stress**. *J Fluid Mech* 2021, **907**, A18.

129. Freundlich H, Roder HL: **Dilatancy and its relation to thixotropy**. *Trans Faraday Soc* 1938, **34**:308–316.

130. Singh A, Ness C, Sharma AK, de Pablo JJ, Jaeger HM: **Rheology of bidisperse non-brownian suspensions**. *Phys Rev* 2024, **110**, 034901.

131. Jackson NE, Webb MA, de Pablo JJ: **Recent advances in machine learning towards multiscale soft materials design**. *Curr Opin Chem Eng* 2019, **23**:106–114.

132. Barrat J-L, Del Gado E, Egelhaaf SU, Mao X, Dijkstra M, Pine DJ, Kumar SK, Bishop K, Gang O, Obermeyer A, *et al.*: **Soft matter roadmap**. *J Phys: Materials* 2023, **7**, 012501.

133. Aminimajd A, Maia J, Singh A: **Robust prediction of frictional contact network in near-jamming suspensions employing deep graph neural networks**. *Phys Fluids* 2025, **37**.

134. Aminimajd A, Maia J, Singh A: **Scalability of graph neural network in accurate prediction of frictional contact network in suspensions**. *Soft Matter* 2025, **21**:2826–2835.

135. Aminimajd A, Maia J, Singh A: *Graph neural network for multi-task prediction of rheological and microstructural behavior in suspensions*. 2026. arXiv preprint arXiv:2602.07296.

136. Mandal R, Casert C, Sollich P: **Robust prediction of force chains in jammed solids using graph neural networks**. *Nat Commun* 2022, **13**:1–7.

137. Mangal D, Jha A, Dabiri D, Jamali S: **Data-driven techniques in rheology: developments, challenges and perspective**. *Curr Opin Colloid Interface Sci* 2025, **75**, 101873.

138. Brunton SL, Noack BR, Koumoutsakos P: **Machine learning for fluid mechanics**. *Annu Rev Fluid Mech* 2020, **52**:477–508.

139. Lennon KR, McKinley GH, Swan JW: **Scientific machine learning for modeling and simulating complex fluids**. *Proc Natl Acad Sci* 2023, **120**, e2304669120.

140. Siddani B, Zhang W, Nonaka A, Bell J, Srivastava I: **An adaptive, data-driven multiscale approach for dense granular flows**. *Comput Methods Appl Mech Eng* 2025, **446**, 118294.