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# On the Effectiveness of Returns Policies in the Price-Dependent Newsvendor Model

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## Abstract

We study in this paper the price-dependent (PD) newsvendor model in which a manufacturer sells a product to an independent retailer facing uncertain demand and the retail price is endogenously determined by the retailer. We prove that for a zero salvage value and some expected demand functions, in equilibrium, the manufacturer may elect not to introduce buybacks. On the other hand, if buybacks are introduced in equilibrium, their introduction has an insignificant effect on channel efficiency improvement, but, by contrast, may significantly shift profits from the retailer to the manufacturer. We further demonstrate that the introduction of buybacks increases the wholesale price, retail price, and inventory level, as compared to the wholesale price-only contract, and that the corresponding vertically integrated firm offers the lowest retail price and highest inventory level.

## 1 Introduction

Manufacturers, whose products are subject to random demand, often accept returns of unsold goods for full or partial credit. For example, books, newspapers, recordings, CDs, dairy products, costume jewelry, fashion wear, computer products and peripherals, and perishable services, such as airline tickets and hotel rooms, are usually allowed to return to their source in North America for full or partial credit. In general, a supply chain composed of independent agents trying to maximize their own profits does not achieve channel coordination, see, e.g., Spengler (1950), and Pasternack (1985) was the first to show that buybacks can coordinate the basic price-independent newsvendor model, wherein a manufacturer ( $M$ ) offers a good to a retailer ( $R$ ) for a constant wholesale price and a constant buyback rate (linear pricing), and  $R$ , who faces a fixed retail price

and stochastic demand, needs to decide upon the optimal order quantity. Subsequently, other contracts, such as, e.g., quantity-flexibility (Tsay (1999)), sales-rebate (Taylor (2002)), and revenue-sharing (Pasternack (1999), Cachon and Lariviere (2005)) were also shown to be able to coordinate the basic newsvendor model. See also Lariviere (1999), Tsay et al. (1999) and Cachon (2004) for some excellent reviews of coordination mechanisms for the basic newsvendor model and related models.

As noted by Kandel (1996), the price-dependent (PD) newsvendor model, wherein the retail price is determined endogenously by  $R$ , is considerably more complicated. However, Emmons and Gilbert (E&G) (1998) have shown that if the wholesale price is large enough, both  $M$  and  $R$  would benefit from the introduction of buybacks when the expected demand function is linear. It was conjectured by Lariviere (1999), and it was proved by Bernstein and Federgruen (2005), that constant wholesale and buyback prices (i.e., independent of other decision variables) cannot, in general, lead to coordination in the PD-newsvendor model. By contrast, contracts which do not employ constant wholesale and buyback prices can induce coordination. Indeed, e.g., revenue-sharing contracts and the so-called “linear price discount sharing” scheme, were shown by Cachon and Lariviere (C&L) (2005) and by Bernstein and Federgruen (2005), respectively, that they could induce coordination in the PD-newsvendor model. We note, however, that as discussed by C&L, revenue-sharing contracts require the ability for  $M$  to verify *ex post*  $R$ 's revenue, which may be costly, and as noted by Bernstein and Federgruen (2005), the “linear price discount sharing” scheme bears close resemblance to the traditional “bill back” or “count-recount” schemes, which, unfortunately, are reported to be disliked by retailers (see, e.g., Blattberg and Neslin (1990), Chapter 11).

Marvel and Peck's (M&P) (1995) model, which assumes constant wholesale and buyback prices and is somewhat different than the traditional supply chain model in the Operations Management (OM) literature, incorporates two types of uncertainty: One with respect to product valuation and the other concerning the number of customers arriving to the retail store. They show that uncertainty *only* about product valuation leads to manufacturers' preference for a wholesale price-only contract, whereas uncertainty *only* about the number of arrivals induces manufacturers to offer buybacks in their contracts. Thus, valuation uncertainty leads to theoretically opposite results than those derived for arrival uncertainty, which, as M&P suggest, explains why return good systems are not more wide spread than observed. Note that if there is only arrival uncertainty, then M&P's model essentially reduces to the basic price-independent newsvendor model wherein the selling price coincides with a representative customer's product valuation. If there is only prod-

uct valuation uncertainty, then the equilibrium order quantity is either zero or equal to the known and fixed number of arriving customers.

In this paper we study the PD-newsvendor model with constant wholesale and buyback prices which, as stated by M&P, typifies manufacturer-distributor relations in many markets. Our objective is *not* to investigate channel coordination. Rather, our aim is to investigate possible factors that affect the introduction of returns. Thus, we address queries, such as that by Lariviere (1999, Section 8.6, second paragraph), as to why constant wholesale price and buyback rate contracts are not more prevalent: “Given the apparent power of returns policies, it is not surprising that they are common in industries such as publishing. Indeed, one may wonder why they are not even more common. Relatively little work has examined this issue...”.

We investigate the effect of buybacks for three different expected demand functions: linear, negative power and exponential. For a linear expected demand function and a uniformly distributed random component of the demand model, our PD-newsvendor model coincides with E&G’s model. Our main results, for a zero salvage value, are:

- (1) The manufacturer may elect not to offer buybacks. Indeed, buybacks are not introduced in equilibrium when the expected demand function is a negative power function of the retail price.
- (2) If buybacks are introduced in equilibrium, they have a relatively insignificant effect on channel efficiency improvement.
- (3) By contrast, if buybacks are introduced in equilibrium, they could have a rather dramatic effect on profit distribution. They could significantly increase  $M$ ’s expected profit and significantly decrease  $R$ ’s expected profit. For example, for a linear expected demand function and a uniformly distributed random component of demand, the introduction of buybacks is shown to increase  $M$ ’s expected profit by 12.5% to 23.94% and to decrease  $R$ ’s expected profit by 15.62% to 20.63%.
- (4) Our analysis demonstrates that the introduction of buybacks in equilibrium induces higher wholesale price, retail price and retail inventories than those obtained under wholesale price-only contracts.
- (5) In the PD-newsvendor model with buyback options, for a uniformly distributed random component of demand, the wholesale price, channel efficiency and profit distribution between  $M$  and  $R$  coincide with those in the corresponding model with deterministic demand.

It can also be shown that the introduction of a positive salvage value in the PD-newsvendor model may have a significant effect on the possible implementation of a returns policy. For example, for a positive and equal salvage value at  $M$ 's and  $R$ 's locations, buybacks are introduced for all three expected demand functions.

Our findings provide several answers to Lariviere's query as to why return good systems are not more common. Indeed, as it is the case for a negative power expected demand function and a zero salvage value, a manufacturer may prefer not to offer buybacks in equilibrium. Further, if buybacks are introduced, their insignificant effect on channel efficiency would be further diminished by the additional costs that would arise in a return system, which are not accounted for by the model and which will not be incurred by a wholesale price-only contract (see related discussions in, e.g., Lariviere (1999) and Lariviere and Porteus (2001)). Thus, the introduction of buybacks by manufacturers in the PD-newsvendor model could be viewed, perhaps correctly, by retailers as an attempt to grab additional channel profit at their expense. To the extent possible, therefore, retailers would object to the introduction of return good systems.

The remainder of this paper is organized as follows: §2 formally introduces the price-dependent (PD) newsvendor model. In §3 we analyze the PD-newsvendor model, as studied by E&G, wherein the expected demand function is linear in the retail price and the random component of demand is uniformly distributed. §4 extends the analysis to negative power and exponential expected demand functions. In §5 we discuss an extension to more general demand distributions and the effect of a positive salvage value on the implementation of the returns policies, and we reveal a surprising relationship between the PD-newsvendor model with buybacks and the corresponding deterministic model. Conclusions and future research are provided in §6.

## 2 Model Formulation

Consider the single-period PD-newsvendor model with buyback policies, wherein a manufacturer sells a single product to an independent retailer facing stochastic demand from the end-customer market. The decision sequence is as follows.  $M$ , who has unlimited production capacity and can produce the items at a fixed marginal cost  $c$ , is a Stackelberg leader.  $M$  initiates the process by offering a per unit (or linear) wholesale price  $w$ , at which items will be sold to  $R$  prior to the selling season, and a per unit (or linear) buyback rate  $b$ , at which she will buy back the unsold items at the end of the selling season. In response to the proposed  $w$  and  $b$ ,  $R$  commits to an order quantity  $Q$  prior to the selling season, and a per unit selling price  $p$ , at which to sell the items during the

season. Thereafter, demand is realized. At the end of the season,  $R$  returns all unsold inventory to  $M$ , receiving a refund of  $b$  for each unit returned.

It is assumed in this paper that unsatisfied demand is lost, there is no penalty cost for lost sales<sup>1</sup>, and that the salvage value of unsold inventory is zero for<sup>2</sup> both  $M$  and  $R$ . For feasibility, we assume: (i)  $c \leq w \leq p$  and (ii)  $0 \leq b \leq w$ .

The stochastic demand,  $X$ , that  $R$  faces is assumed to be of a multiplicative form  $X = D(p)\xi$ , which is a commonly used model in the Economics and OM literature.  $D(p)$  is the expected value of  $X$ , which decreases in the retail price  $p$ , and  $\xi$  ( $\xi \geq 0$ ) is the random part of  $X$ . The density function of  $X$  satisfies  $f(x|p) = \frac{1}{D(p)}\hat{f}(\epsilon|p)$ , where  $\epsilon = \frac{x}{D(p)}$ , and  $\hat{f}(\epsilon|p)(= \hat{f}(\epsilon))$  is the probability density function of  $\xi$  with mean normalized to one. The multiplicative demand model was initially proposed by Karlin and Carr (1962).

It would be interesting and challenging to extend our analysis to the additive demand model wherein  $X = D(p) + \xi$ . The additive model, which is also commonly used in the literature, would be an appropriate model wherein the variance of demand is unaffected by the expected demand level. By contrast, the multiplicative model is appropriate where the variance of demand increases with expected demand in a manner which leaves the coefficient of variation unaffected.

We note, however, that the additive model may lead to qualitatively different results than the multiplicative model (see, e.g., Mills (1959), E&G, Granot and Yin (2004a), Song et al. (2004), and, in particular, the excellent survey by Petruzzi and Dada (1999)). Moreover, it appears that it is less tractable than the multiplicative model (see, e.g., Padmanabhan and Png (1997), and Wang et al. (2004)). Indeed, even when  $\xi$  has a binary distribution and  $D(p)$  is linear in  $p$ , it is difficult to derive a closed-form expression for, e.g., the equilibrium value of  $w$ , in the PD-newsvendor problem with an additive demand model.

Finally, let us note the main differences between the multiplicative and additive demand models and M&P's demand model. In the multiplicative and additive models, the stochastic demand is

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<sup>1</sup>The zero penalty cost assumption is made mainly for tractability reasons. A positive penalty cost (or goodwill cost) of unmet demand (or lost sales) accounts for consumers' dissatisfaction and for potential business losses, especially in a multi-period setting. Generally speaking, incorporating a penalty cost for unsold inventory complicates the analysis significantly. Indeed, even in the price-independent newsvendor model, a goodwill cost will complicate the channel coordinating buyback contract (Pasternack (1985)) in the sense that it becomes demand distribution dependent and may not have a closed-form solution. When a goodwill cost is present in the PD-newsvendor model, closed-form expressions for equilibrium decisions and profits are not available for any of the expected demand functions considered in this paper. Nevertheless, we have conducted a numerical investigation of the PD-newsvendor model for linear and negative power expected demand functions. According to our findings, for a low goodwill cost, the results hold. That is, buybacks are introduced for the linear case but not for the negative power expected demand case. However, for a high enough goodwill cost, buybacks are introduced in both cases.

<sup>2</sup>The implications of relaxing this assumption are considered in §5.2.

precisely the number of arrivals, which is a function of the retail price  $p$ . It can be assumed that *all* arriving customers in these models are familiar with the product, well informed about the retail price, and are interesting in buying it. If  $n$  customers were to arrive, then it would be optimal to order  $n$  items. However, in M&P's model, the number of arrivals is independent of the retail price. That is, the number of arriving customers is stochastically the same regardless whether the retail price is very high or very low. This could be interpreted as if the arriving customers are uninformed about the retail price. Once in the store, either all or none will buy the product, depending on whether a representative customer's product valuation exceeds the retail price. Thus, in M&P's model, if  $n$  customers were to arrive, by contrast with the multiplicative and additive demand models, it may be optimal to stock nothing.

For the multiplicative demand model, one can express  $M$ 's and  $R$ 's expected profit functions as follows:

$$E\Pi^M(w, b) = (w - c)Q - bE[Q - X]^+ \quad \text{and} \quad E\Pi^R(p, Q) = (p - w)Q - (p - b)E[Q - X]^+, \quad (1)$$

where  $E[Q - X]^+ = QF(\frac{Q}{D(p)}) - \int_0^{\frac{Q}{D(p)}} D(p)\epsilon \hat{f}(\epsilon|p)d\epsilon$  is the expected unsold inventory. In this paper, we adopt E&G's assumption that the random part of demand,  $\xi$ , follows a uniform distribution<sup>3</sup> on the interval  $[0, 2]$ , i.e.,  $\hat{f}(\epsilon|p) = 0.5$  on  $[0, 2]$ . Thus, we can simplify  $M$ 's and  $R$ 's expected profit functions, given by (1), to:

$$E\Pi^M(w, b) = (w - c)Q - b\frac{Q^2}{4D(p)} \quad \text{and} \quad E\Pi^R(p, Q) = (p - w)Q - (p - b)\frac{Q^2}{4D(p)}. \quad (2)$$

We analyze in the next section the effect of buybacks in the PD-newsvendor model with a linear expected demand function. In §4, we extend our study to two other expected demand functions.

### 3 Effect of Buybacks with Linear Expected Demand

We analyze in this section the effect of buybacks in the PD-newsvendor model wherein the expected demand function,  $D(p)$ , is linear of form<sup>4</sup>  $D(p) = 1 - p$ . Note that when  $p = 1$ , market demand

<sup>3</sup>The analysis can be easily extended to a uniform distribution of  $\xi$  on  $[0, K]$  for any  $K > 0$ , and see §5 where we briefly report on computational results with power and triangle distributions of  $\xi$ .

<sup>4</sup>Note that the analysis can be easily extended to a general linear expected demand function  $D(p) = a(k - p)$ , where  $a(> 0)$  and  $k(> 0)$  are constant, as was assumed in E&G. Indeed, for  $D(p) = a(k - p)$ , let  $p = k \cdot p'$ ,  $w = k \cdot w'$ ,  $b = k \cdot b'$ ,  $Q = ak^2 \cdot Q'$  and  $c = k \cdot c'$ . Then, it is not difficult to verify that the expected profit functions of  $M$  and  $R$ , given by (2), can be transformed to:  $E\Pi_M(w, b, p, Q, c) = ak^2 \cdot E\Pi'_M(w', b', p', Q', c')$  and  $E\Pi_R(w, b, p, Q, c) = ak^2 \cdot E\Pi'_R(w', b', p', Q', c')$ , where  $E\Pi'_M$  and  $E\Pi'_R$  are the expected profit functions of  $M$  and  $R$ , respectively, with respect to the expected demand function  $D(p') = 1 - p'$  and the marginal manufacturing cost  $c'$ . Thus, the analysis in a model with decisions  $(w, b, p, Q)$ , cost  $c$  and  $D(p) = a(k - p)$  coincides with that in a model with decisions  $(w', b', p', Q')$ , cost  $c'$  and  $D(p') = 1 - p'$ . Note that due to this normalization, the performance of the models with and without buybacks and the integrated system is independent of individual values of  $c$  and  $k$ , but is dependent on  $\frac{c}{k}$ , which can be referred to as the normalized marginal manufacturing cost.

is zero and both  $M$  and  $R$  gain zero expected profits. Thus, we assume that  $p < 1$  in the sequel, except as otherwise noted, and  $c \leq w \leq p < 1$ . We further note that, for any retail price  $p$ , the highest demand from the end-customer market is  $2D(p)$  since  $\xi \leq 2$ .

From (2) and for  $D(p) = 1 - p$ ,  $M$ 's and  $R$ 's expected profit functions can be simplified to:

$$E\Pi_L^M = (w - c)Q - b\frac{Q^2}{4(1-p)} \quad \text{and} \quad E\Pi_L^R = (p - w)Q - (p - b)\frac{Q^2}{4(1-p)}, \quad (3)$$

where the subscript “ $L$ ” stands for “linear expected demand”. The total expected channel profit,  $E\Pi_L^{Total}$ , is the sum of the expected profits of  $M$  and  $R$ .

According to  $R$ 's expected profit function, given by (3), and for any given pair  $(w, b)$ , E&G have shown that  $R$ 's optimal retail price and order quantity are:

$$p_L^* = \frac{3b + 1 + \sqrt{(1 + 8w - 9b)(1 - b)}}{4} \quad \text{and} \quad Q_L^* = \frac{2(1 - p_L^*)(p_L^* - w)}{p_L^* - b}. \quad (4)$$

Taking  $R$ 's reaction functions into account,  $M$ 's expected profit function becomes:

$$E\Pi_L^M = (w - c)Q_L^* - b\frac{(Q_L^*)^2}{4(1 - p_L^*)}. \quad (5)$$

Substituting  $w = c$  and  $b = 0$  into (4), we obtain the unique equilibrium values of  $p$  and  $Q$  in the corresponding integrated system<sup>5</sup>:

$$p_L^I = \frac{1 + \sqrt{1 + 8c}}{4} \quad \text{and} \quad Q_L^I = \frac{(3 - \sqrt{1 + 8c})^2}{4}. \quad (6)$$

Substituting  $p_L^I$  and  $Q_L^I$  into the expected integrated channel profit function:  $E\Pi_L^I = (p - c)Q - p\frac{Q^2}{4(1-p)}$ , and simplifying results with:

$$E\Pi_L^I = \frac{(3 - \sqrt{1 + 8c})^3(1 + \sqrt{1 + 8c})}{64}. \quad (7)$$

E&G have shown that for all wholesale prices  $w \in (w_T, 1)$ , where  $w_T$  is a threshold value less than 1, both  $M$  and  $R$  are better off when  $M$  offers a positive buyback rate (i.e.,  $b > 0$ ). However, by contrast with E&G, we are able to find closed-form expressions for  $w_T$ , the equilibrium wholesale price,  $w_L^*$ , and equilibrium buyback rate,  $b_L^*$ . We further show that when the expected demand function is linear, as assumed by E&G, the efficiency<sup>6</sup> of the PD-newsvendor model with buybacks is precisely 75%, and that the increased efficiency due to the introduction of buybacks is insignificant, and bounded by 3.16%. By contrast, we demonstrate that the introduction of buybacks has a

<sup>5</sup>Note that  $3 - \sqrt{1 + 8c} > 0$  since  $c < 1$ . Similarly for other expressions containing  $3 - \sqrt{1 + 8c}$  in the sequel.

<sup>6</sup>The efficiency of a supply chain is defined as the ratio of the equilibrium channel profit to the corresponding integrated channel profit.



significant effect on the distribution of the channel profit between  $M$  and  $R$ . Explicitly, we prove that the introduction of buybacks increases  $M$ 's expected profit by 12.5% to 23.94%, whereas,  $R$ 's expected profit decreases by 15.62% to 20.63%.

We start by providing a closed-form expression for  $w_T$ . All proofs are presented in the appendix.

**Proposition 3.1** *For any wholesale price in the interval  $(w_T \equiv \frac{2+30c+3\sqrt{6(1+5c)}}{50}, 1)$ , there exists a buyback rate  $b > 0$ , at which both  $M$  and  $R$  earn higher expected profits than when  $b = 0$ .*

Observe that neither Proposition 3.1 nor Proposition 2 in E&G implies that buybacks are used in equilibrium. Rather, they merely assert that when the wholesale price is large enough, both  $M$  and  $R$  benefit from the introduction of buybacks. To prove that buybacks are used in equilibrium, we need Proposition 3.2 and Lemma 3.3 below. In Proposition 3.2, we derive an explicit expression for the equilibrium wholesale price,  $\hat{w}_L^*$ , in the PD-newsvendor model under a wholesale price-only contract, wherein  $M$  first commits to a wholesale price  $w$ , and then  $R$  commits to a retail price  $p$  and an order quantity  $Q$ .

Now, substituting  $b = 0$  into the expected profit functions of  $M$  and  $R$  under a contract with buybacks, given by (3), we obtain  $M$ 's and  $R$ 's expected profit functions,  $E\hat{\Pi}_L^M$  and  $E\hat{\Pi}_L^R$ , respectively, in a wholesale price-only contract:

$$E\hat{\Pi}_L^M = (w - c)Q \quad \text{and} \quad E\hat{\Pi}_L^R = (p - w)Q - p \frac{Q^2}{4(1 - p)}. \quad (8)$$

**Proposition 3.2** *In the PD-newsvendor model under a wholesale price-only contract,  $M$ 's equilibrium wholesale price is:  $\hat{w}_L^* = \frac{5+32c+3\sqrt{17+64c}}{64}$ .*

The following relationship holds between  $w_T$  and  $\hat{w}_L^*$ .

**Lemma 3.3**  $w_T < \hat{w}_L^*$ .

In view of Proposition 3.1 and Lemma 3.3, we have:

**Corollary 3.4** *Buybacks are introduced in equilibrium in the PD-newsvendor model with a linear expected demand function.*

For  $b = 0$  and knowing  $\hat{w}_L^*$ , we are able to calculate  $\hat{p}_L^*$  and  $\hat{Q}_L^*$  in a wholesale price-only contract by substituting  $b = 0$  and  $\hat{w}_L^*$  into  $p_L^*$  and  $Q_L^*$ , given in (4):

$$\hat{p}_L^* = \frac{7 + \sqrt{17 + 64c}}{16} \quad \text{and} \quad \hat{Q}_L^* = \frac{(9 - \sqrt{17 + 64c})^2}{64}. \quad (9)$$

Substituting the resulting  $\hat{p}_L^*$  and  $\hat{Q}_L^*$  further into  $E\hat{\Pi}_L^M$  and  $E\hat{\Pi}_L^R$ , given by (8), and simplifying provides us with  $M$ 's and  $R$ 's equilibrium expected profits in the PD-newsvendor model under a wholesale price-only contract:

$$E\hat{\Pi}_L^{M*} = t(3 + \sqrt{17 + 64c}) \quad \text{and} \quad E\hat{\Pi}_L^{R*} = t\left(\frac{7}{2} + \frac{1}{2}\sqrt{17 + 64c}\right), \quad (10)$$

and the equilibrium total expected channel profit under a wholesale price-only contract is:

$$E\hat{\Pi}_L^{Total*} = t\left(\frac{13}{2} + \frac{3}{2}\sqrt{17 + 64c}\right), \quad (11)$$

where<sup>7</sup>  $t = \frac{(9 - \sqrt{17 + 64c})^3}{8192}$ .

**Lemma 3.5** *In the PD-newsvendor model under a wholesale price-only contract, in equilibrium, the ratio of  $M$ 's and  $R$ 's expected profits is bounded between 1.28 and 1.5.*

We are now able to calculate the channel efficiency with wholesale price-only contracts.

**Proposition 3.6** *The channel efficiency with a wholesale price-only contract is strictly increasing in  $c$  and is bounded between 71.84% and 74.07%.*

A possible explanation for the increased efficiency as a function of  $c$  is that as  $c$  increases, the range for  $w$  decreases since  $c \leq w \leq p < 1$ . Thus, an increase in  $c$  decreases the possibility for double marginalization. See also Granot and Yin (2004a) for a similar behavior of channel efficiency in decentralized systems under decision postponement.

In the PD-newsvendor model with buybacks, E&G had to resort to a numerical and graphical investigation to analyze the equilibrium expected profits of  $M$ ,  $R$  and the overall channel as a function of  $w$ , for parameter values  $(c, a, k) = (1, -3, 5)$ . Fortunately, we are able to derive closed-form expressions for the equilibrium values of  $w_L^*$  and  $b_L^*$ , and therefrom to derive explicit expressions for  $M$ 's and  $R$ 's equilibrium expected profits.

**Proposition 3.7** *In the PD-newsvendor model with buybacks, the equilibrium values of  $M$ 's decision variables are:  $(w_L^* = \frac{1+c}{2}, b_L^* = \frac{1}{2})$ , and in equilibrium,*

$$E\Pi_L^{M*} = 2E\Pi_L^{R*} = \frac{(3 - \sqrt{1 + 8c})^3(1 + \sqrt{1 + 8c})}{128}. \quad (12)$$

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<sup>7</sup>Note that  $9 - \sqrt{17 + 64c} > 0$  since  $c < 1$ . Similarly for other expressions which contain  $9 - \sqrt{17 + 64c}$  in the sequel.

Having the equilibrium wholesale and buyback prices, we are able to calculate  $R$ 's equilibrium retail price and order quantity:

$$p_L^* = \frac{5 + \sqrt{1 + 8c}}{8} \quad \text{and} \quad Q_L^* = \frac{(3 - \sqrt{1 + 8c})^2}{8}. \quad (13)$$

Further, having the equilibrium expected profits of  $M$  and  $R$  in contracts with buybacks, given by (12), and the integrated channel profit, given by (7), we derive the following conclusion.

**Proposition 3.8** *The channel efficiency of the PD-newsvendor model with buybacks is 75%.*

Propositions 3.6 and 3.8 imply<sup>8</sup> that as compared to the wholesale price-only contract, the improvement in channel efficiency due to the introduction of buybacks is decreasing in  $c$ , and it is quite insignificant, at most 3.16% for  $c = 0$ . This result should be contrasted with the significant effect of buybacks on channel efficiency improvement in the basic newsvendor model, wherein the retail price is exogenously determined. Indeed, Lariviere and Porteus (2001) have studied the basic newsvendor model under a wholesale price-only contract, and they have shown, e.g., that the channel efficiency under such a contract is only 75% when demand follows a uniform distribution. But, as shown by Pasternack (1985), the channel can be perfectly coordinated when buybacks are introduced in the basic newsvendor model, which implies that buybacks can increase channel efficiency by 25% for uniformly distributed demand.

From the above discussion we conclude that channel efficiency improvement is unlikely to be the motivation behind the introduction of buybacks to the PD-newsvendor model. The following two propositions suggest another motivation for their introduction in this model.

**Proposition 3.9** *In the PD-newsvendor model, the percent improvement in  $M$ 's equilibrium expected profit due to the introduction of buybacks is strictly decreasing in  $c$  and is bounded between 23.94%, for  $c = 0$ , and 12.5%, for  $c \rightarrow 1$ .*

**Proposition 3.10** *In the PD-newsvendor model, the percent deterioration of  $R$ 's equilibrium expected profit due to the introduction of buybacks is strictly decreasing in  $c$  and is bounded between 20.63%, for  $c = 0$ , and 15.62%, for  $c \rightarrow 1$ .*

A possible explanation for the decreased improvement in  $M$ 's equilibrium expected profit (Proposition 3.9) and the decreased deterioration in  $R$ 's equilibrium expected profit (Proposition 3.10) as a function of  $c$  is similar to that for Proposition 3.6. That is, as  $c$  increases, there is less room for  $M$  to manipulate  $w$  to improve her welfare.

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<sup>8</sup>Recall that the channel efficiency in the wholesale price-only contract is increasing in  $c$ .

In view of Propositions 3.9 and 3.10, we may conclude that a possible motivation for the introduction (by  $M$ ) of buyback policies to the PD-newsvendor model is the significant and favorable, for  $M$ , effect it has on the distribution of the channel profit.

Propositions 3.9 and 3.10 are consistent with E&G's findings for the specific instance of the PD-newsvendor model they have studied, wherein  $(c, a, k) = (1, -3, 5)$ . Indeed, in their specific example,  $c' = c/k = 1/5 = 0.2$ , and there is an 18.92% increase in  $M$ 's expected profit and a 19.26% decrease in  $R$ 's expected profit, due to the introduction of buybacks.

Proposition 3.11 below reveals the relationships among the equilibrium wholesale and retail prices and the order (or production) quantities in supply contracts with and without buybacks and in the vertically integrated channel.

**Proposition 3.11** *In the PD-newsvendor model:*

- (i)  $\hat{w}_L^* < w_L^*$ ,
- (ii)  $p_L^I < \hat{p}_L^* < p_L^*$  and
- (iii)  $\hat{Q}_L^* < Q_L^* < Q_L^I$ .

It follows from Proposition 3.11 that, as expected, the integrated channel would be preferred by the end customers to a decentralized supply channel with or without buybacks, in the sense that it offers a lower retail price and makes a larger amount of the product available to customers. But, while the retail price with buybacks is strictly higher than that without buybacks, the quantity available for the end customers in a supply chain with buybacks is strictly larger than that without buybacks. We note that the results derived in Proposition 3.11 are consistent with those derived by M&P for their demand model.

## 4 Effect of Buybacks with Other Expected Demand Functions

In this section, we maintain the assumption that  $\xi$  follows a uniform distribution, and we investigate the robustness of our results, presented in §3, for other expected demand functions. Specifically, in §4.1 the expected demand function is a negative power function of the retail price, and in §4.2, the expected demand function is exponential.

### 4.1 Negative power expected demand

In this subsection, we study the PD-newsvendor model with a negative power expected demand function of the retail price  $p$ ,  $D(p) = p^{-q}$ , where  $q > 1$  and  $w \leq p < \infty$ . The restriction  $q > 1$  is

used to ensure that  $R$ 's optimal retail price will be finite. The analysis can be easily extended to a general  $D(p) = ap^{-q}$ , where  $a > 0$ .

According to (2),  $M$ 's and  $R$ 's expected profit functions, denoted as  $E\Pi_N^M$  and  $E\Pi_N^R$ , in the PD-newsvendor model with  $D(p) = p^{-q}$  and buyback options are:

$$E\Pi_N^M = (w - c)Q - b\frac{Q^2}{4p^{-q}} \quad \text{and} \quad E\Pi_N^R = (p - w)Q - (p - b)\frac{Q^2}{4p^{-q}}, \quad (14)$$

where the subscript “ $N$ ” stands for “negative power demand”. Let

$$p_N^* = \frac{qw + qb + w - 2b + J}{2(q - 1)} \quad \text{and} \quad Q_N^* = \frac{2(p_N^*)^{-q}(p_N^* - w)}{p_N^* - b}, \quad (15)$$

where  $J \equiv \sqrt{(q + 1)^2w^2 - 2(q^2 - q + 2)wb + (q - 2)^2b^2}$ .

**Proposition 4.1** *In the PD-newsvendor model with buybacks and  $D(p) = p^{-q}$ , for any given  $(w, b)$ ,  $R$ 's optimal reaction functions are given by (15).*

Substituting  $w = c$  and  $b = 0$  into  $p_N^*$  and  $Q_N^*$ , we obtain the unique equilibrium  $p_N^I$  and  $Q_N^I$  in the corresponding integrated system:

$$p_N^I = \frac{q + 1}{q - 1}c \quad \text{and} \quad Q_N^I = \frac{4(q - 1)^q}{(q + 1)^{q+1}}c^{-q}.$$

Substituting  $p_N^I$  and  $Q_N^I$  into the integrated channel profit function:  $E\Pi_N^I = (p - c)Q - p\frac{Q^2}{4p^{-q}}$ , and simplifying results with:

$$E\Pi_N^I = \frac{4(q - 1)^{q-1}}{(q + 1)^{q+1}}c^{1-q}. \quad (16)$$

Having  $R$ 's reaction functions,  $p_N^*$  and  $Q_N^*$ , given by (15),  $M$ 's expected profit function becomes:

$$E\Pi_N^M = (w - c)Q_N^* - b\frac{(Q_N^*)^2}{4(p_N^*)^{-q}}. \quad (17)$$

Similar to the case when the expected demand function is linear (Proposition 3.1), we find a range of wholesale prices in which exercising buybacks can benefit both  $M$  and  $R$ .

**Proposition 4.2** *In the PD-newsvendor model with  $D(p) = p^{-q}$ , for any wholesale price  $w$ , where  $w > w_T^N \equiv \frac{qc}{q-1}$ , exercising a buyback option benefits both  $M$  and  $R$ .*

$M$ 's equilibrium decision variables,  $(w_N^*, b_N^*)$ , and the equilibrium expected profits of  $M$  and  $R$  are as follows:

**Proposition 4.3** *In the PD-newsvendor model with buybacks and  $D(p) = p^{-q}$ , the equilibrium values of  $M$ 's decision variables are:  $(w_N^* = \frac{qc}{q-1}, b_N^* = 0)$ , and, in equilibrium,*

$$E\Pi_N^{M*} = \frac{4(q-1)^{2q-1}}{c^{q-1}q^q(q+1)^{q+1}}, \quad E\Pi_N^{R*} = \frac{4(q-1)^{2q-2}}{c^{q-1}q^{q-1}(q+1)^{q+1}} \quad \text{and} \quad \frac{E\Pi_N^{M*}}{E\Pi_N^{R*}} = \frac{q-1}{q}. \quad (18)$$

Proposition 4.3 implies that when the expected demand function is a negative power of the retail price,  $M$  elects *not* to offer a buyback option in equilibrium. Thus, as suggested earlier, Proposition 4.2, which proves the existence of a range of wholesale prices at which both  $M$  and  $R$  would benefit from the implementation of buybacks, is not sufficient for the introduction of buybacks in equilibrium. Rather, a sufficient condition for the introduction of buybacks is that the equilibrium wholesale price, for a wholesale price-only contract, falls in the interval of wholesale prices at which both  $M$  and  $R$  would benefit from buybacks. Indeed, in the linear case, this wholesale price falls in that interval (Lemma 3.3), and thus, in equilibrium, buybacks are used. However, for the negative power demand case, this wholesale price is not in that interval (Propositions 4.2 and 4.3), and, indeed, in equilibrium, buybacks are not used.

Proposition 4.3 should be compared with the result derived by M&P, according to which  $M$  would always prefer to offer buybacks in equilibrium in the presence of uncertainty only with respect to the number of arrivals. However, this specific result appears to be implied by their model. Indeed, when the uncertainty is only with respect to the number of arrivals,  $M$  and  $R$  know with certainty the customers' valuation of the product. By requesting a wholesale price equal to the customers' valuation,  $M$ , in M&P model, is able to secure the entire channel profit by implementing a complete consignment contract (full return for full credit). In fact, as noted in the introduction section, M&P's model with only arrival uncertainty essentially coincides with the price-independent newsvendor model wherein the retail price is exogenously fixed. Thus, M&P's result that full-credit buybacks are offered when there is only arrival uncertainty is consistent with the literature on channel coordination through buybacks in the price-independent newsvendor model (Pasternack (1985), Kandel (1996)), wherein  $M$  is able to secure the entire channel profit by setting  $w = b = p$ . Observe that in the presence of uncertainty both with respect to valuation and arrivals, but when there is not very much arrival uncertainty,  $M$  may not offer buybacks in M&P's model (M&P (1995)).

Having the equilibrium expected profits of  $M$  and  $R$ , given by (18), and the integrated channel profit, given by (16), we can derive the channel efficiency with (or without) buybacks.

**Proposition 4.4** *The channel efficiency of the PD-newsvendor model with (or without) buybacks and with  $D(p) = p^{-q}$ , is  $\frac{(q-1)^{q-1}(2q-1)}{q^q}$ , where  $q > 1$ , and it is strictly decreasing in  $q$ .*

Note that price elasticity of a negative power expected demand  $D(p) = p^{-q}$  is  $-\frac{dD(p)/dp}{D(p)/p} = q$  ( $> 1$ ). Thus, the larger  $q$  is, the more sensitive customers are to changes in the retail price, and the more severe are the effects of double marginalization. This may explain the decrease of channel efficiency as a function of  $q$ .

Similar to the linear expected demand case, one can show that the following relationships hold among the equilibrium decision variables for an integrated firm and a decentralized channel with a negative power expected demand function: (i)  $p_N^I < p_N^*$ , and (ii)  $Q_N^* < Q_N^I$ . That is, the integrated firm charges a lower retail price and produces a larger quantity than the decentralized system.

## 4.2 Exponential expected demand

We consider in this subsection the PD-newsvendor model with an exponential expected demand function<sup>9</sup>, i.e.,  $D(p) = e^{-p}$ , where  $w \leq p < \infty$ .

For  $D(p) = e^{-p}$ ,  $M$ 's and  $R$ 's expected profit functions with buybacks, given by (2), reduce to:

$$E\Pi_E^M = (w - c)Q - b\frac{bQ^2}{4e^{-p}} \quad \text{and} \quad E\Pi_E^R = (p - w)Q - (p - b)\frac{Q^2}{4e^{-p}}, \quad (19)$$

where the subscript “ $E$ ” stands for “exponential expected demand”. Similar to the negative power expected demand case (Proposition 4.1), according to  $R$ 's expected profit function, given in (19), it can be shown that  $R$ 's reaction functions for any given  $w$  and  $b$  are:

$$p_E^* = \frac{w + b + 1 + \sqrt{(w - b)^2 + 6(w - b) + 1}}{2} \quad \text{and} \quad Q_E^* = \frac{2e^{-p_E^*}(p_E^* - w)}{p_E^* - b}. \quad (20)$$

Substituting  $w = c$  and  $b = 0$  into  $p_E^*$  and  $Q_E^*$ , we obtain the unique equilibrium  $p_E^I$  and  $Q_E^I$  in the corresponding integrated system:  $p_E^I = \frac{c+1+H}{2}$  and  $Q_E^I = \frac{2e^{-p_E^I}(p_E^I - c)}{p_E^I}$ , where  $H \equiv \sqrt{c^2 + 6c + 1}$ . Substituting  $p_E^I$  and  $Q_E^I$  into the corresponding integrated channel profit function:  $E\Pi_E^I = (p - c)Q - p\frac{Q^2}{4e^{-p}}$ , and simplifying results with:

$$E\Pi_E^I = \frac{(c + 3 - H)(-c + 1 + H)}{4} e^{-\frac{c+1+H}{2}}. \quad (21)$$

Let us first consider the model under a wholesale price-only contract. Substituting  $b = 0$  into  $R$ 's reaction functions  $p_E^*$  and  $Q_E^*$  for contracts with buybacks, given by (20), and simplifying

<sup>9</sup>Similar to the linear expected demand case, the analysis can be easily extended to a general  $D(p) = ae^{-s \cdot p}$ , where  $a(> 0)$  and  $s(> 0)$  are constant. As in the linear expected demand case, due to this normalization, the performance of the models with and without buybacks and the integrated system is independent of individual values of  $c$  and  $s$ , but is dependent on  $s \cdot c$ .

provides us with  $R$ 's reaction functions in the wholesale price-only contract:

$$\hat{p}_E^* = \frac{w+1+Z}{2} \quad \text{and} \quad \hat{Q}_E^* = e^{-\frac{w+1+Z}{2}}(w+3-Z), \quad (22)$$

where  $Z = \sqrt{w^2 + 6w + 1}$ . By substituting the resulting  $\hat{Q}_E^*$  further into  $M$ 's expected profit function under a wholesale price-only contract,  $E\hat{\Pi}_E^M = (w-c)Q$ , we obtain:

$$E\hat{\Pi}_E^M = (w-c)(w+3-Z)e^{-\frac{w+1+Z}{2}}. \quad (23)$$

**Proposition 4.5** *In the PD-newsvendor model under a wholesale price-only contract with  $D(p) = e^{-p}$ , the equilibrium wholesale price,  $\hat{w}_E^*(\geq c)$ , is the unique solution to the nonlinear equation:*

$$(w+5)(w-c) + (w-c-2)\sqrt{w^2 + 6w + 1} = 0. \quad (24)$$

Using Proposition 4.5, we are able to show:

**Lemma 4.6** *In the PD-newsvendor model under a wholesale price-only contract with  $D(p) = e^{-p}$ , in equilibrium,  $M$ 's expected profit is strictly smaller than  $R$ 's expected profit.*

We are also able to obtain explicit expressions for the equilibrium wholesale price,  $w_E^*$ , and buyback rate,  $b_E^*$ , in the PD-newsvendor model with buybacks and exponential expected demand.

**Proposition 4.7** *In the PD-newsvendor model with buybacks and  $D(p) = e^{-p}$ ,  $M$ 's expected profit is globally maximized at  $(w_E^* = 1+c, b_E^* = 1)$ , and in equilibrium,*

$$E\Pi_E^{M*} = E\Pi_E^{R*} = \frac{(c+3-H)(-c+1+H)}{4} e^{-\frac{3+c+H}{2}}, \quad (25)$$

where  $H = \sqrt{c^2 + 6c + 1}$ .

Having derived the equilibrium expected profits of  $M$  and  $R$ , given by (25), and the integrated channel profit, given by (21), we are able to calculate the channel efficiency of the PD-newsvendor model with buybacks.

**Proposition 4.8** *The channel efficiency of the PD-newsvendor model with buybacks and  $D(p) = e^{-p}$  is  $\frac{2}{e} \approx 73.58\%$ .*

By Proposition 4.5, the equilibrium wholesale price under a price-only contract,  $\hat{w}_E^*$ , is implicitly given by (24), and it seems unlikely that a closed-form expression for  $\hat{w}_E^*$  as a function of  $c$  can be found. Fortunately though, Maple 6 is able to solve for  $\hat{w}_E^*$  for any given value of  $c$ . Indeed,



$c$	Percentage (%) improvement			Equilibrium decision variables								
	$M$ 's profit	$R$ 's profit	Channel efficiency	Integrated		Buybacks				No buybacks		
				$p_E^I$	$Q_E^I$	$w_E^*$	$b_E^*$	$p_E^*$	$Q_E^*$	$\hat{w}_E^*$	$\hat{p}_E^*$	$\hat{Q}_E^*$
0.0	11.14	-4.33	2.02	1.00	0.7358	1.00	1.00	2.00	0.2707	0.56	1.86	0.2175
0.1	9.22	-4.16	1.51	1.18	0.5602	1.10	1.00	2.18	0.2061	0.69	2.03	0.1741
0.5	5.54	-3.23	0.70	1.78	0.2424	1.50	1.00	2.78	0.0892	1.16	2.61	0.0816
1.0	3.65	-2.43	0.38	2.41	0.1048	2.00	1.00	3.41	0.0385	1.72	3.25	0.0365
2.0	2.06	-1.54	0.17	3.56	0.0249	3.00	1.00	4.56	0.0092	2.79	4.42	0.0089
5.0	0.74	-0.63	0.04	6.74	0.0006	6.00	1.00	7.74	0.0002	5.87	7.64	0.0002

Table 1: Supply chain performance due to buybacks with  $D(p) = e^{-p}$

in Table 1 we present the equilibrium values of the decision variables for the integrated channel, and of the channel with and without buybacks, as well as the effect of buybacks on the equilibrium expected profits of  $M$  and  $R$  and the channel efficiency. Recall that by Proposition 4.8, the channel efficiency under a buyback contract is  $\frac{2}{c} \approx 73.58\%$ .

Based on Table 1, we can make the following observations.

**Observation 4.9** *The percentage increase in channel efficiency due to buybacks is decreasing in  $c$ , and is maximized at  $c = 0$  for which it is 2.02%.*

**Observation 4.10** *In equilibrium, due to the introduction of buybacks,  $M$ 's expected profit increases at a decreasing rate in  $c$  and  $R$ 's expected profit decreases at a decreasing rate in  $c$ .*

Table 1 and Observations 4.9 and 4.10 imply that for an exponential expected demand function, the introduction of buybacks in equilibrium has an insignificant effect on channel efficiency. However, by contrast, it may have a relatively large and favorable, for  $M$ , effect on the distribution of the channel profit. For example, when  $c = 0$ ,  $M$ 's expected profit increases by 11.14%, while  $R$ 's expected profit decreases by 4.33%. These results are consistent with those obtained in the linear expected demand case.

Further based on Table 1 we can make the following observation.

**Observation 4.11** *From Table 1:*

- (i)  $\hat{w}_E^* < w_E^*$ ,
- (ii)  $p_E^I < \hat{p}_E^* < p_E^*$  and
- (iii)  $\hat{Q}_E^* < Q_E^* < Q_E^I$ .

Observation 4.11 is consistent with the corresponding results derived in §3 and §4.1 for the linear and negative power expected demand functions.

## 5 Discussion and Extensions

In this section we reveal a close relationship between the PD-newsvendor model with buybacks and the corresponding deterministic model, discuss extensions for other distributions of  $\xi$  and examine the effect of introducing a positive salvage value for unsold inventory.

### 5.1 The PD-newsvendor model and the corresponding deterministic model

In the deterministic model,  $M$ , in Stage 1, offers a wholesale price  $w$  to  $R$ , who then determines the selling price  $p$  in Stage 2, which induces demand  $D(p)$  that coincides with the expected demand function in the newsvendor model. Obviously in the deterministic case,  $R$  would order a quantity that is exactly equal to the deterministic demand. Thus, no buybacks are necessary in the deterministic model. Let us take the linear demand as an example and analyze the deterministic model with  $D(p) = a(k - p)$ , where  $a > 0$  and  $k > 0$ .

**Deterministic model with  $D(p) = a(k - p)$ .** In Stage 2, for any given  $w$ ,  $R$  determines  $p$  to maximize  $\Pi^R(D) = (p - w)D(p)$ , where  $D(p) = a(k - p)$ . Clearly,  $\Pi^R$  is concave in  $p$ , which results with  $p_L^*(D) = \frac{k+w}{2}$ . Taking  $R$ 's reaction function  $p_L^*(D)$  into consideration,  $M$ 's profit in Stage 1 becomes:  $\Pi^M = (w - c)D(p) = \frac{a}{2}(w - c)(k - w)$ , which is, again, concave in  $w$ . Thus,  $w_L^*(D) = \frac{k+c}{2}$ , and accordingly,  $p_L^*(D) = \frac{3k+c}{4}$  and  $\Pi_L^{M*}(D) = 2\Pi_L^{R*}(D) = \frac{a(k-c)^2}{8}$ . For the integrated channel in the deterministic model, we substitute  $w = c$  into  $\Pi^R = a(p - w)(k - p)$ . Then,  $p_L^I(D) = \frac{k+c}{2}$  and the corresponding integrated channel profit is  $\Pi_L^I(D) = \frac{a(k-c)^2}{4}$ .

A similar analysis can be carried out for the other two expected demand functions. The results are summarized in the following table.

	$\Pi^{M*}(D)$	$\Pi^{R*}(D)$	Profit distribution	$w^*(D)$	$\Pi^I(D)$	Channel efficiency
$D(p) = a(k - p)$	$\frac{a(k-c)^2}{8}$	$\frac{a(k-c)^2}{16}$	2 : 1	$\frac{k+c}{2}$	$\frac{a(k-c)^2}{4}$	75%
$D(p) = p^{-2}$	$\frac{1}{16c}$	$\frac{1}{8c}$	1 : 2	$2c$	$\frac{1}{4c}$	75%
$D(p) = ae^{-sp}$	$\frac{a}{s}e^{-sc-2}$	$\frac{a}{s}e^{-sc-2}$	1 : 1	$\frac{sc+1}{s}$	$\frac{a}{s}e^{-sc-1}$	$\frac{2}{e} \approx 73.58\%$

Table 2: Supply chain performance in the deterministic model

The results derived for the PD-newsvendor model with a uniform  $\xi$ , presented in §3 and §4, and those given in Table 2 reveal a remarkable connection between the multiplicative PD-newsvendor model with buybacks and the corresponding deterministic model, which are summarized below.

**Theorem 5.1** *In the PD-newsvendor model with a uniform  $\xi$  and buyback options, when the expected demand function is either linear, negative power or exponential, the wholesale price, the channel efficiency and the profit distribution between  $M$  and  $R$  coincide with those in the corresponding deterministic model.*

In fact, to ascertain the robustness of the results derived for a uniform  $\xi$ , we have carried out a numerical investigation for two families of distributions of  $\xi$ : power distributions with a non-negative exponent ( $\hat{f}(\epsilon) = \lambda(\epsilon)^i, i \in [0, \infty)$ ) and triangle distributions on the interval  $[r, 2 - r]$  for any  $r \in [0, 1)$ . The results are available in Granot and Yin (2004b), and for space consideration, we only very briefly summarize them below. Specifically, the numerical study reveals that the results derived analytically for a uniform  $\xi$  are quite robust. More explicitly, for the power and triangle families of distributions of  $\xi$ :

- (1) In equilibrium, buybacks are introduced for linear and exponential expected demand functions, while they are not used for a negative power expected demand function.
- (2) The increase in channel efficiency due to buybacks is relatively insignificant, if at all.
- (3) Buybacks essentially shift the channel profit from  $R$  to  $M$ .
- (4) Buybacks increase the equilibrium retail price and inventory level.

Thus, based on the results obtained for the PD-newsvendor model for power and triangle distributions of  $\xi$ , we can make the following observation:

**Observation 5.2** *In the PD-newsvendor model with buyback options, for power and triangle distributions of  $\xi$  and linear, negative power and exponential expected demand functions:*

- (i) *The equilibrium wholesale and buyback prices are independent of the distribution of  $\xi$ .*
- (ii) *The channel profit distribution between  $M$  and  $R$  and the channel efficiency are independent of the distribution of  $\xi$ . Further, for linear and exponential expected demand functions, the profit distribution and channel efficiency are independent of the model parameters ( i.e.,  $(c, a, k)$  for the model with  $D(p) = a(k - p)$  and  $(c, a, s)$  for  $D(p) = ae^{-sp}$ ).*

Observation 5.2 suggests that for an arbitrary distribution of  $\xi$  ( $> 0$ ) and  $D(p) = a(k - p)$ ,  $w_L^*$  and  $b_L^*$  can be derived from Proposition 3.7,  $E\Pi_L^{M*} = 2E\Pi_L^{R*}$  and the channel efficiency is 75% for any  $(c, a, k)$ ; for  $D(p) = p^{-q}$ ,  $w_N^*$  and  $b_N^*$  can be derived from Proposition 4.3,  $E\Pi_N^{M*} = \frac{q-1}{q}E\Pi_N^{R*}$  and the channel efficiency is  $\frac{(q-1)^{q-1}(2q-1)}{q^q}$ , and for  $D(p) = ae^{-s \cdot p}$ ,  $w_E^*$  and  $b_E^*$  can be derived from Proposition 4.7,  $E\Pi_E^{M*} = E\Pi_E^{R*}$  and the channel efficiency is  $\frac{2}{e} \approx 73.68\%$  for any  $(c, a, s)$ .

Thus, in view of the above results we make the following conjecture:

**Conjecture 5.3** *In the PD-newsvendor model with buyback options:*

- (i) *For a general distribution of  $\xi$  ( $\geq 0$ ), the wholesale price, the channel efficiency and the profit distribution between  $M$  and  $R$  coincide with those in the corresponding deterministic model.*
- (ii) *The buyback rate is independent of the distribution of  $\xi$ .*

Conjecture 5.3 implies<sup>10</sup> that the addition of buybacks to a wholesale price-only contract model increases the channel efficiency up to the efficiency of the corresponding deterministic model. This explains why buybacks are not implemented in the negative power expected demand case, wherein the channel efficiency under a wholesale price-only contract coincides with the efficiency of the corresponding deterministic model.

Naturally, Conjecture 5.3 implies a significant reduction in the computational burden associated with solving the PD-newsvendor model with a buyback option. Indeed, the equilibrium wholesale price, efficiency and profit allocation are derived from the corresponding deterministic model. The increase in efficiency due to buybacks is available once the efficiency of the wholesale price-only contract is found, and the equilibrium buyback rate for an arbitrary  $\xi$  can be found by solving the model for a simple form of  $\xi$ , such as, e.g., a uniform  $\xi$ .

## 5.2 Positive salvage value of unsold inventory

A zero salvage value for both  $M$  and  $R$  was assumed for unsold inventory. There are cases, however, where some salvage value can be generated from unsold inventory. In this subsection, we will briefly consider the effect of a positive salvage value on the possible implementation of buybacks in equilibrium for a uniformly distributed  $\xi$ .

We denote by  $S_M$  (respectively,  $S_R$ ) the salvage value at  $M$ 's (respectively,  $R$ 's) location, and we will consider the following cases: (i)  $S_M = S_R = S$ , (ii)  $S_M > S_R$  and (iii)  $S_M < S_R$ . It is

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<sup>10</sup>After the completion of essentially the current version of this paper, and motivated by essentially the current and previous versions of this paper, Song, Ray and Li (2004) have managed to verify Conjecture 5.3 for linear, negative power and exponential expected demand functions.

reasonable to assume that  $\max(S_M, S_R) < c$  to avoid a situation of producing for salvaging. We briefly discuss the three cases below.<sup>11</sup>

(i)  $S_M = S_R = S$ . In this case, without loss of generality, we can assume<sup>12</sup> that  $b \geq S$ . Then,  $M$ 's and  $R$ 's expected profit functions can be written as:

$$E\Pi_M = (w - c)Q - b\frac{Q^2}{4D(p)} + S\frac{Q^2}{4D(p)} \quad \text{and} \quad E\Pi_R = (p - w)Q - (p - b)\frac{Q^2}{4D(p)}, \quad (26)$$

where  $R$ 's expected profit function coincides with his expected profit function in the case of no salvage value.

Following the same steps as in the model with a zero salvage value, one can show that  $M$ 's equilibrium decision variables are:

- $(w_L^* = \frac{1+c}{2}, b_L^* = \frac{1+S}{2})$  for a linear expected demand function  $D(p) = 1 - p$ ;
- $(w_N^* = \frac{q}{q-1}c, b_N^* = \frac{q}{q-1}S)$  for a negative power expected demand function  $D(p) = p^{-q}$ , and
- $(w_E^* = 1 + c, b_E^* = 1 + S)$  for an exponential expected demand function  $D(p) = e^{-p}$ .

Thus, the introduction of a positive salvage value at  $M$ 's and  $R$ 's locations in the PD-newsvendor model with buybacks does not affect the equilibrium wholesale price, and it can be further shown that it has no impact on channel efficiency and the profit distribution between  $M$  and  $R$ . Thus, Theorem 5.1 holds for a positive salvage value, where  $S_M = S_R$ . However, by contrast with the case of a zero salvage value, buybacks are implemented for a negative power expected demand function when  $S_M = S_R > 0$ . Apparently, the introduction of a positive salvage value is enough to make a buyback option attractive for  $M$ .

(ii)  $S_M > S_R$ . As compared to Case (i), a higher salvage value for  $M$  would provide her with an additional incentive to buy back unsold inventory. Indeed, returns are introduced in all three expected demand cases.

(iii)  $S_M < S_R$ . If  $R$  has an advantage salvaging unsold inventory, no returns may occur for all three expected demand functions. Indeed, if  $S_R - S_M$  is large enough,  $M$  will prefer  $b = 0$  on  $b > S_R$ .

The results above imply that the existence of a positive salvage value (and perhaps other costs associated with a returns policy) may have a significant effect on the possible implementation of a returns policy. We note, however, that our results are consistent with those presented in Kandel (1996) for the basic price-independent newsvendor model. Specifically, as noted in Kan-

<sup>11</sup>The detailed analysis is available upon request.

<sup>12</sup>For  $b < S$ , there are clearly no returns. For  $b = S$ , there is no difference between returns and no returns, and considering the extra costs that are possibly associated with returns, we can assume no returns. For  $b > S$ , actual returns may take place.

del (1996), milk and flowers seem to have different returns policies. Unsold milk is usually returned to the milk processing plants, while unsold flowers are often disposed at the retail store by price discounting. The allowance for returns of unsold milk is due to the fact that milk processing plants (i.e.,  $M$ ) can use it to produce other dairy items, while a grocery store (i.e.,  $R$ ) does not have such a capability. On the other hand, it is more economic for a flower retailer to sell unsold flowers at a discount price than to return them to the flower suppliers. See further Kandel (1996) for other industrial examples, e.g., apparels and produce, where different returns policies are implemented for unsold items due to differences in salvage values.

Finally, in addition to the form of the expected demand function and the salvage value, factors, such as transportation cost or new product introduction consideration could also affect returns policies. For example, in the textbook publishing industry, publishers are willing, sometime even trying hard, to buy back used or unsold textbooks in order to promote and increase revenues from a new edition of the textbook.

## 6 Conclusions and Further Research

We have studied in this paper the PD-newsvendor problem with a multiplicative probabilistic demand model. We have investigated the desirability of introducing buybacks and their effect on the equilibrium values of decision variables, channel efficiency and profit distribution for three commonly used expected demand functions: linear, negative power and exponential. Initially, we have assumed a zero salvage value. For this case, we have demonstrated that in equilibrium, buybacks will be introduced for linear and exponential expected demand functions, but they are not introduced for a negative power expected demand function. In those cases where buybacks are introduced, we have shown that their introduction has an insignificant effect on channel efficiency improvement. By contrast, their introduction in those cases may significantly increase  $M$ 's expected profit, and significantly decrease  $R$ 's expected profit. Thus, we suggest that in the absence of the salvage value, the introduction of buybacks to the PD-newsvendor model is probably not motivated by a desire to increase channel efficiency. Rather, it is more likely motivated by the significantly favorable, for  $M$ , effect it has on the distribution of the channel profit. These results partially explain why returns policies are not more common.

It is interesting to note that whenever buybacks are implemented in equilibrium in the PD-newsvendor model, the wholesale price, profit distribution between  $M$  and  $R$  and channel efficiency coincide with those values in the corresponding deterministic model. Since a return system involves

costs not incorporated in this model (see, e.g., Lariviere (1999) and Lariviere and Porteus (2001)), buybacks will not be introduced when a wholesale price-only contract is relatively efficient. Indeed, as we have shown, buybacks are not introduced in the negative power expected demand function case with a zero salvage value, wherein the channel efficiency under a wholesale price-only contract coincides with that in the deterministic model.

However, we have also shown that the existence of a positive salvage value may have a significant effect on the introduction of buybacks. For example, for a positive and equal salvage value at  $M$ 's and  $R$ 's locations, buybacks will be introduced for all three expected demand functions. Thus, if the salvage value at  $M$  is positive and larger than that at  $R$ ,  $M$  has an additional incentive to introduce buybacks. These results may explain why some industries implement a return system, and are consistent with the related discussion in Kandel (1996).

Several natural extensions of our results could be pursued. For example, it would be useful to study other expected demand functions, and it would be interesting to extend the analysis to the PD-newsvendor model with an additive demand model. As suggested earlier, however, (see also Emmons and Gilbert (1998), Mills (1959), and Petruzzi and Dada (1999)), the additive model could produce results which are different from those derived in the multiplicative demand model. Indeed, one can verify that Conjecture 5.3 is not valid for the additive model.

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## References

- [1] Bernstein, F. and A. Federgruen. Decentralized Supply Chains with Competing Retailers under Demand Uncertainty. *Management Science*. 51(1) (2005), 18-29.
- [2] Cachon, G. Supply Chain Coordination with Contracts. *Supply Chain Management: Design, Coordination and Operation* (Chapter 6). S. Graves and T. de Kok (Eds.). *Handbooks in Operations Research and Management Science*, Vol. 11. Elsevier. (2004).
- [3] Cachon, G. and M. Lariviere. Supply Chain Coordination with Revenue-Sharing Contracts: Strengths and Limitations. *Management Science*. 51(1) (2005), 30-44.
- [4] Emmons, H. and S. Gilbert. Note. The Role of Returns Policies in Pricing and Inventory Decisions for Catalogue Goods. *Management Science*. 44(2) (1998), 276-283.
- [5] Granot, D. and S. Yin. Price and Order Postponement in a Decentralized Newsvendor Model with Price-Dependent Demand. Working Paper. The University of British Columbia. (2004a).
- [6] Granot, D. and S. Yin. On the Effectiveness of Returns Policies in the Price-Dependent Newsvendor Model. Working Paper. The University of British Columbia. (September 2004b).
- [7] Kandel, E. The Right to Return. *Journal of Law and Economics*. 39 (1996), 329-356.
- [8] Karlin, S. and C. R. Carr. Prices and Optimal Inventory Policy. *Studies in Applied Probability and Management Science* (Chapter 11). K. J. Arrow, S. Karlin and H. Scarf (Eds.), Stanford Press. (1962), 159-172.
- [9] Lariviere, M. Supply Chain Contracting and Coordination with Stochastic Demand. *Quantitative Models for Supply Chain Management* (Chapter 8). S. Tayur, M. Magazine and R. Ganeshan (Eds.). Kluwer. (1999).
- [10] Lariviere, M. and E. Porteus. Selling to the Newsvendor: An Analysis of Price-Only Contracts. *Manufacturing and Service Operations Management*. 3(4) (2001), 293-305.
- [11] Lee, E. and R. Staelin. Vertical Strategic Interaction: Implications for Channel Pricing Strategy. *Marketing Science*. 16(3) (1997), 185-207.
- [12] Marvel, H. P. and J. Peck. Demand Uncertainty and Returns Policies. *International Economics Review*. 36(3) (1995), 691-714.
- [13] Mills, E. S. Uncertainty and Price Theory. *The Quarterly Journal of Economics*. 73 (1959), 116-130.
- [14] Pasternack, B. A. Optimal Pricing and Return Policies for Perishable Commodities. *Marketing Science*. 4(2) (1985), 166-176.



- [15] Pasternack, B. A. Using Revenue Sharing to Achieve Channel Coordination for A Newsboy Type Inventory Model. Supply Chain Management: Models, Applications, and Research Directions (Chapter 6). J. Geunes, P.M. Pardalos and H.E. Romeijn (Eds.). Applied Optimization, Vol. 62. Kluwer. (2002).
- [16] Petruzzi, N. and M. Dada. Pricing and the Newsvendor Problem: A Review with Extensions. Operations Research. 47(2) (1999), 183-194.
- [17] Song, Y., S. Ray and S. Li. Analysis of Buy-Back Contracts Under Price Sensitive Stochastic Demand for a Serial Two-Echelon Supply Chain. Faculty of Management. McGill University, Canada (November 12, 2004).
- [18] Spengler, J. Vertical Integration and Antitrust Policy. Journal of Political Economy. 58 (1950), 347-352.
- [19] Taylor, T. A. Supply Chain Coordination Under Channel Rebates with Sales Effort Effects. Management Science. 48(8) (2002), 992-1007.
- [20] Tsay, A. Quantity Flexibility Contract and Supplier-Customer Incentives. Management Science. 45(10) (1999), 1339-1358.
- [21] Tsay, A., S. Nahmias and N. Agrawal. Modeling Supply Chain Contracts: A Review. Quantitative Models for Supply Chain Management (Chapter 10). S. Tayur, M. Magazine and R. Ganeshan (Eds.). Kluwer. (1999).
- [22] Wang, Y., L. Jiang and Z. Shen. Channel Performance Under Consignment Contract with Revenue Sharing. Management Science. 50(1) (2004), 34-47.

## Appendix:

**Proof of Proposition 3.1.** As pointed out by E&G, for a given  $w$ ,  $E\Pi_L^R$  is non-decreasing in  $b$ . Thus, the proof will follow if we are able to show that  $\frac{\partial E\Pi_L^M}{\partial b}(b=0) > 0$  for any  $w \in (w_T, 1)$ . For convenience, we present derivatives of  $p_L^*$  and  $Q_L^*$  with respect to  $w$  and  $b$  in Table A-1. Recall that  $p_L^*$  and  $Q_L^*$  are given by (4). The derivation of the partial derivative expressions is straightforward.

Expressions corresponding to $p_L^*$	Expressions corresponding to $Q_L^*$
$p_L^* = \frac{1}{4}(1 + 3b + \sqrt{(1 + 8w - 9b)(1 - b)})$	$Q_L^* = 4 + 2w - 6p_L^*$
$\frac{\partial p_L^*}{\partial w} = \frac{1-b}{\sqrt{(1+8w-9b)(1-b)}}$	$\frac{\partial Q_L^*}{\partial w} = 2 - \frac{6(1-b)}{\sqrt{(1+8w-9b)(1-b)}}$
$\frac{\partial p_L^*}{\partial b} = -\frac{5+4w-9b-3\sqrt{(1+8w-9b)(1-b)}}{4\sqrt{(1+8w-9b)(1-b)}}$	$\frac{\partial Q_L^*}{\partial b} = -6\frac{\partial p_L^*}{\partial b} = \frac{3(5+4w-9b-3\sqrt{(1+8w-9b)(1-b)})}{2\sqrt{(1+8w-9b)(1-b)}}$

Table A-1: Summary of some partial derivatives

It follows from (5) that

$$\frac{\partial E\Pi_L^M}{\partial b} = (w - c) \frac{\partial Q_L^*}{\partial b} - \frac{1}{4} Q_L^* (1 - p_L^*)^{-2} [(1 - p_L^*)(Q_L^* + 2b \frac{\partial Q_L^*}{\partial b}) + b Q_L^* \frac{\partial p_L^*}{\partial b}]. \quad (\text{A-1})$$

By evaluating  $p_L^*$ ,  $Q_L^*$ ,  $\frac{\partial p_L^*}{\partial b}$  and  $\frac{\partial Q_L^*}{\partial b}$ , given in Table A-1, at  $b = 0$ , substituting them and  $b = 0$  into (A-1) and simplifying, we obtain that:  $\frac{\partial E\Pi_L^M}{\partial b}(b = 0) = \frac{(-3 + \sqrt{1+8w})^2}{16\sqrt{1+8w}}(20w + 1 - 12c - 3\sqrt{1+8w})$ .

Let  $A_1(w) \equiv 20w + 1 - 12c - 3\sqrt{1+8w}$ .  $\frac{\partial E\Pi_L^M}{\partial b}(b = 0) > 0$  if and only if  $A_1(w) > 0$ . Note that  $A_1(w)$  is convex in  $w$  and it has two stationary points,  $w_1$  and  $w_2$ , where  $w_1 = \frac{2+30c-3\sqrt{6(1+5c)}}{50} < c < \frac{2+30c+3\sqrt{6(1+5c)}}{50} = w_2 < 1$ . The last two strict inequalities follow since  $c < 1$ . Let  $w_T = w_2$ . Since  $A_1(w) > 0$  when  $w > w_T$ , we have verified that for any  $w \in (w_T, 1)$ ,  $\frac{\partial E\Pi_L^M}{\partial b}(b = 0) > 0$ .  $\square$

**Proof of Proposition 3.2.** By substituting  $b = 0$  into (4), we derive  $R$ 's reaction functions in a wholesale price-only contract:  $\hat{p}_L^* = \frac{1+\sqrt{1+8w}}{4}$  and  $\hat{Q}_L^* = \frac{(3-\sqrt{1+8w})^2}{4}$ . Upon their substitution in  $M$ 's expected profit function, given by (8), and simplifying, we obtain:  $E\hat{\Pi}_L^M = \frac{(w-c)(3-\sqrt{1+8w})^2}{4}$ . The first derivative of  $E\hat{\Pi}_L^M$  with respect to  $w$  is:  $\frac{dE\hat{\Pi}_L^M}{dw} = \frac{(3-\sqrt{1+8w})}{4\sqrt{1+8w}}(-16w + 3\sqrt{1+8w} - 1 + 8c)$ . Let  $A_2(w) \equiv -16w + 3\sqrt{1+8w} - 1 + 8c$ . Then, since  $c \leq w < 1$ ,  $\frac{dA_2(w)}{dw} = -16 + \frac{12}{\sqrt{1+8w}} < 0$ . Thus,  $A_2(w)$  is strictly decreasing in  $w$ . Let  $\bar{w} = \frac{5+32c+3\sqrt{17+64c}}{64}$ . It follows from the definition of  $A_2(w)$  that  $A_2(w) > 0$  for  $c \leq w < \bar{w}$ ,  $A_2(w) < 0$  for  $\bar{w} < w < 1$  and  $A_2(w) = 0$  for  $w = \bar{w}$ . Since  $\frac{dE\hat{\Pi}_L^M}{dw} = \frac{(3-\sqrt{1+8w})}{4\sqrt{1+8w}}A_2(w)$  and  $\frac{(3-\sqrt{1+8w})}{4\sqrt{1+8w}} > 0$ , we conclude that  $\frac{dE\hat{\Pi}_L^M}{dw} > 0$  for  $c \leq w < \bar{w}$ ,  $\frac{dE\hat{\Pi}_L^M}{dw} < 0$  for  $\bar{w} < w < 1$  and  $\frac{dE\hat{\Pi}_L^M}{dw} = 0$  for  $w = \bar{w}$ . Therefore,  $E\hat{\Pi}_L^M$  is pseudo-concave in  $w \in [c, 1)$  and is uniquely maximized at  $\hat{w}_L^* = \bar{w} = \frac{5+32c+3\sqrt{17+64c}}{64}$ .  $\square$

**Proof of Lemma 3.3.** Simply compare  $w_T = \frac{2+30c+3\sqrt{6(1+5c)}}{50}$  and  $\hat{w}_L^* = \frac{5+32c+3\sqrt{17+64c}}{64}$ .  $\square$

**Proof of Lemma 3.5.** From (10) the ratio,  $F$ , of  $M$ 's and  $R$ 's equilibrium expected profits can be simplified to  $F \equiv \frac{E\hat{\Pi}_L^{M*}}{E\hat{\Pi}_L^{R*}} = \frac{6+2\sqrt{17+64c}}{7+\sqrt{17+64c}}$ , which is strictly increasing in  $c$ . Thus, the ratio of  $M$ 's and  $R$ 's equilibrium expected profits,  $F$ , is bounded between  $F(c = 0) \approx 1.28$  and  $F(c \rightarrow 1) = 1.5$ .  $\square$

**Proof of Proposition 3.6.** By (7) and (11), the efficiency of a wholesale price-only contract is:

$$\frac{E\hat{\Pi}_L^{Total*}}{E\Pi_L^I} * 100\% = \frac{(9 - \sqrt{17 + 64c})^3(13 + 3\sqrt{17 + 64c})}{256(3 - \sqrt{1 + 8c})^3(1 + \sqrt{1 + 8c})} = \frac{2(13 + 3\sqrt{17 + 64c})(3 + \sqrt{1 + 8c})^3}{(1 + \sqrt{1 + 8c})(9 + \sqrt{17 + 64c})^3} * 100\%,$$

which increases in  $c$ , and thus it is bounded between 71.84% for  $c = 0$  and 74.07% for  $c \rightarrow 1$ .  $\square$

**Proof of Proposition 3.7.** It follows from (5) that

$$\frac{\partial E\Pi_L^M}{\partial w} = Q_L^* + (w - c) \frac{\partial Q_L^*}{\partial w} - \frac{1}{4} b Q_L^* (1 - p_L^*)^{-2} [2(1 - p_L^*) \frac{\partial Q_L^*}{\partial w} + Q_L^* \frac{\partial p_L^*}{\partial w}]. \quad (\text{A-2})$$

Substituting  $p_L^*$ ,  $Q_L^*$ ,  $\frac{\partial p_L^*}{\partial w}$  and  $\frac{\partial Q_L^*}{\partial w}$ , given in Table A-1, into (A-2), and simplifying gives us:  $\frac{\partial E\Pi_L^M}{\partial w} = \frac{(U-3V)}{4UV} [(U-3V)(UV+3b) + 8(w-c)V]$ , where  $V \equiv \sqrt{1-b}$  and  $U \equiv \sqrt{1+8w-9b}$ . Since  $b \leq w < 1$ ,  $U - 3V < 0$ . Thus, the first-order condition of  $E\Pi_L^M$  with respect to  $w$  implies:

$$(U - 3V)(UV + 3b) + 8V(w - c) = 0. \quad (\text{A-3})$$

Similarly, by substituting  $p_L^*$ ,  $Q_L^*$ ,  $\frac{\partial p_L^*}{\partial b}$  and  $\frac{\partial Q_L^*}{\partial b}$ , given in Table A-1, into (A-1), and simplifying, the first-order condition of  $E\Pi_L^M$  with respect to  $b$  yields:

$$(U - 3V)[2V(UV + 6b) + b(U - 3V)] + 24V^2(w - c) = 0. \quad (\text{A-4})$$

Solving (A-3) and (A-4) reveals that  $(w_L^* = \frac{1+c}{2}, b_L^* = \frac{1}{2})$  is the unique stationary point of  $M$ 's expected profit function, and the Hessian matrix at this stationary point is:

$$\begin{vmatrix} \frac{\partial^2 E\Pi_M}{\partial w^2} & \frac{\partial^2 E\Pi_M}{\partial w \partial b} \\ \frac{\partial^2 E\Pi_M}{\partial b \partial w} & \frac{\partial^2 E\Pi_M}{\partial b^2} \end{vmatrix} = \frac{4}{z(-3+z)^3} \begin{vmatrix} Mww & Mwb \\ Mbw & Mbb \end{vmatrix},$$

where  $z \equiv \sqrt{1+8c}$ ,  $Mww \equiv 8c^3z - 72c^3 + 648c^2 - 24zc^2 - 462zc + 1350c + 261 - 251z$ ,  $Mbb \equiv 8(8c^2 + 56c - 12zc + 17 - 15z)$  and  $Mwb = Mbw \equiv 6(-60c^2 + 4zc^2 - 150c + 46zc - 33 + 31z)$ . Since  $c \in [0, 1)$ , it is not difficult to verify that  $z < 3$ ,  $Mww > 0$ ,  $Mbb > 0$  and  $Mwb = Mbw < 0$ . Furthermore, we have  $Mww \cdot Mbb - Mwb \cdot Mbw > 0$ , which implies that the Hessian matrix at  $(w^* = \frac{1+c}{2}, b^* = \frac{1}{2})$  is negative definite. Thus, this point is the global maximizer of  $M$ 's problem. Accordingly, we have:  $E\Pi_L^{M*} = 2E\Pi_L^{R*} = \frac{(3-\sqrt{1+8c})^3(1+\sqrt{1+8c})}{128}$ .  $\square$

**Proof of Proposition 3.9.** From (10) and (12), after some simplifications, the percent improvement of  $M$ 's equilibrium expected profit due to the introduction of buybacks reduces to:

$$\frac{E\Pi_L^{M*} - E\hat{\Pi}_L^{M*}}{E\hat{\Pi}_L^{M*}} * 100\% = \left( \frac{(1 + \sqrt{1+8c})(9 + \sqrt{17+64c})^3}{8(3 + \sqrt{17+64c})(3 + \sqrt{1+8c})^3} - 1 \right) * 100\%, \quad (\text{A-5})$$

which decreases in  $c$ , and thus it is bounded between 12.5% for  $c \rightarrow 1$  and 23.94% for  $c = 0$ .  $\square$

**Proof of Proposition 3.10.** From (10) and (12), after some simplifications, the percent deterioration of  $R$ 's equilibrium expected profit due to the introduction of buybacks reduces to:

$$\frac{E\Pi_L^{R*} - E\hat{\Pi}_L^{R*}}{E\hat{\Pi}_L^{R*}} * 100\% = \left( \frac{(1 + \sqrt{1+8c})(9 + \sqrt{17+64c})^3}{8(7 + \sqrt{17+64c})(3 + \sqrt{1+8c})^3} - 1 \right) * 100\%, \quad (\text{A-6})$$

which increases in  $c$ , and thus it is bounded between -20.63% for  $c = 0$  and -15.62% for  $c \rightarrow 1$ .  $\square$

**Proof of Proposition 3.11.** (i)  $\hat{w}_L^* < w_L^*$ . By Propositions 3.2 and 3.7, we know that  $\hat{w}_L^* = \frac{5+32c+3\sqrt{17+64c}}{64}$  and  $w_L^* = \frac{1+c}{2}$ . Thus, we have  $w_L^* - \hat{w}_L^* = \frac{3}{64}(9 - \sqrt{17+64c}) > 0$  since  $c < 1$ .

(ii)  $p_L^I < \hat{p}_L^* < p_L^*$ . By (6) and (9), we have  $\hat{p}_L^* - p_L^I = \frac{7+\sqrt{17+64c}}{16} - \frac{1+\sqrt{1+8c}}{4}$ . We simplify it to  $\frac{1}{16}(3 + \sqrt{17+64c} - 4\sqrt{1+8c})$ , which can be shown to be positive for any  $c < 1$ . Thus,  $p_L^I < \hat{p}_L^*$ . Similarly, by (9) and (13), we have  $p_L^* - \hat{p}_L^* = \frac{5+\sqrt{1+8c}}{8} - \frac{7+\sqrt{17+64c}}{16} = \frac{1}{16}(3 + 2\sqrt{1+8c} - \sqrt{17+64c})$ . It can be shown that  $\hat{p}_L^* < p_L^*$  for any  $c < 1$ . Thus,  $p_L^I < \hat{p}_L^* < p_L^*$ . Similarly for the proof of (iii).  $\square$

**Proof of Proposition 4.1.** For any given  $w$ ,  $b$  and  $p$ ,  $E\Pi_N^R$ , given in (14), is concave in  $Q$ . Thus, at optimality,  $Q_N^* = \frac{2p^{-q}(p-w)}{p-b}$ . By substituting  $Q_N^*$  into  $E\Pi_N^R$ , we obtain  $E\Pi_N^R = \frac{p^{-q}(p-w)^2}{p-b}$ . By

employing the same proof method as that used in Proposition 3.2 to prove that  $E\hat{\Pi}_L^M$  is pseudo-concave in  $w$ , we can prove that  $E\Pi_N^R$  is pseudo-concave in  $p$ . Thus,  $E\Pi_N^R$  is uniquely maximized at  $p_N^* = \frac{qw+qb+w-2b+J}{2(q-1)}$ , where  $J \equiv \sqrt{(q+1)^2w^2 - 2(q^2 - q + 2)wb + (q-2)^2b^2}$ .  $\square$

**Proof of Proposition 4.2.** As in the proof of Proposition 3.1, for any  $w$ ,  $R$ 's expected profit is non-decreasing in  $b$ . Thus, the proof will follow if we can show that  $WT \equiv \frac{\partial E\Pi_N^M}{\partial b}(b=0) > 0$  for any  $w > w_T^N$ , where  $E\Pi_N^M$  is given by (17). By using Maple 6, we get that  $WT = -\frac{4(q-1)^q}{w^{q+1}(q+1)^{q+2}}(-w(q-1) + qc)$ , where  $-\frac{4(q-1)^q}{w^{q+1}(q+1)^{q+2}} < 0$  since  $q > 1$ . Thus,  $WT > 0$  for any  $w > \frac{qc}{q-1}$ .  $\square$

**Proof of Proposition 4.3.** Using Maple 6 to solve, simultaneously, the first-order conditions of  $E\Pi_N^M$  with respect to  $w$  and  $b$ , i.e.,  $\frac{\partial E\Pi_N^M}{\partial w} = 0$  and  $\frac{\partial E\Pi_N^M}{\partial b} = 0$ , results with a unique solution ( $w_N^* = \frac{qc}{q-1}, b_N^* = 0$ ). (For brevity, we don't present the intermediate results here.) The Hessian matrix at this stationary point is:

$$\begin{vmatrix} \frac{\partial^2 E\Pi_M}{\partial w^2} & \frac{\partial^2 E\Pi_M}{\partial w \partial b} \\ \frac{\partial^2 E\Pi_M}{\partial b \partial w} & \frac{\partial^2 E\Pi_M}{\partial b^2} \end{vmatrix} = \begin{vmatrix} Mww = -4\left[\frac{(q-1)^2}{cq(q+1)}\right]^{q+1} & Mwb \\ Mbw & Mbb = -\frac{8(q-1)^{2q+2}}{c^{q+1}q^q(q+1)^{q+4}} \end{vmatrix},$$

where  $Mwb = Mbw \equiv \frac{4(q-1)^{2q+2}}{(cq)^{q+1}(q+1)^{q+2}}$ . Since  $q > 1$ , we have  $Mww \cdot Mbb - Mwb \cdot Mbw > 0$ , which implies that the Hessian matrix at the unique stationary point is negative definite. Thus, the unique stationary point ( $w^* = \frac{qc}{q-1}, b^* = 0$ ) is the global maximizer of  $M$ 's problem in Stage 1. Upon substitution of this point, and the corresponding  $p_N^*$  and  $Q_N^*$ , given by (15), into  $M$ 's and  $R$ 's expected profit functions, given by (14), we obtain that in the PD-newsvendor model with a negative power expected demand function,  $E\Pi_N^{M*} = \frac{4(q-1)^{2q-1}}{c^{q-1}q^q(q+1)^{q+1}}$ ,  $E\Pi_N^{R*} = \frac{4(q-1)^{2q-2}}{c^{q-1}q^{q-1}(q+1)^{q+1}}$ , and  $\frac{E\Pi_N^{M*}}{E\Pi_N^{R*}} = \frac{q-1}{q}$ .  $\square$

**Proof of Proposition 4.5.** By (23), we have  $\frac{\partial E\hat{\Pi}_E^M}{\partial w} = \frac{e^{-\frac{w+1+Z}{2}}(w+3-Z)}{2Z}B(w)$ , where  $B(w) \equiv -wZ + cZ + 2Z - w^2 + wc - 5w + 5c$  and  $Z = \sqrt{w^2 + 6w + 1}$ . Note that  $B(w)$  can be transformed to:  $B(w) = (-w + c + 2)\sqrt{w^2 + 6w + 1} - (w + 5)(w - c)$ , which is concave in  $w$  since  $\frac{\partial^2 B(w)}{\partial w^2} < 0$ . Since  $B(w=c) > 0$  and  $B(w) < 0$  for  $w$  large enough, there exists a unique  $\bar{w}$  such that  $B(w) > 0$  for  $c \leq w < \bar{w}$ ,  $B(w) < 0$  for  $\bar{w} < w$  and  $B(w) = 0$  for  $w = \bar{w}$ . Since  $\frac{\partial E\hat{\Pi}_E^M}{\partial w} = \frac{e^{-\frac{w+1+Z}{2}}(w+3-Z)}{2Z}B(w)$  and  $\frac{e^{-\frac{w+1+Z}{2}}(w+3-Z)}{2Z} > 0$ , we conclude that  $\frac{\partial E\hat{\Pi}_E^M}{\partial w} > 0$  for  $c \leq w < \bar{w}$ ,  $\frac{\partial E\hat{\Pi}_E^M}{\partial w} < 0$  for  $\bar{w} < w$  and  $\frac{\partial E\hat{\Pi}_E^M}{\partial w} = 0$  for  $w = \bar{w}$ . Therefore,  $E\hat{\Pi}_E^M$  is pseudo-concave in  $w \in [c, \infty)$  and uniquely maximized at  $\hat{w}_E^*$ , where  $\hat{w}_E^*$  is the unique solution to  $B(w) = 0$  such that  $\hat{w}_E^* \geq c$ .  $\square$

**Proof of Lemma 4.6.** Substituting  $b = 0$  into  $M$ 's and  $R$ 's expected profit functions under a wholesale price-only contract, given by (19), results with  $E\hat{\Pi}_E^M = (w-c)Q$  and  $E\hat{\Pi}_E^R = (p-w)Q - \frac{pQ^2}{4e^{-p}}$ . To prove  $E\hat{\Pi}_E^{M*} < E\hat{\Pi}_E^{R*}$ , we need to show that in equilibrium,  $(w-c)Q < (p-w)Q - \frac{pQ^2}{4e^{-p}}$ , i.e.,

$0 < Q < \frac{4e^{-p(p+c-2w)}}{p}$ . By substituting  $\hat{p}_E^*$ , given by (22), into  $\bar{Q} \equiv \frac{4e^{-p(p+c-2w)}}{p}$ , and simplifying, we obtain:  $\bar{Q} = \frac{4e^{-\frac{w+1+Z}{2}}(-3w+1+Z+2c)}{w+1+Z}$ , where  $Z = \sqrt{w^2 + 6w + 1}$ . Thus, we need to compare, at  $w = \hat{w}_E^*$ , the equilibrium order quantity in the wholesale price-only contract,  $\hat{Q}_E^*$ , given by (22), and  $\bar{Q}$ , where  $\hat{w}_E^*$  is the equilibrium wholesale price, which satisfies (24) in Proposition 4.5. Now, at  $w = \hat{w}_E^*$ ,  $\hat{Q}_E^* < \bar{Q}$  if  $(w + 3 - Z)(w + 1 + Z) < 4(-3w + 1 + Z + 2c)$ , which holds if and only if  $5w - 4c - 1 < Z$ . By Proposition 4.5, in equilibrium,  $c = \frac{1+3w^2+18w-(w+5)Z}{2(6+w)}$ , and upon substituting  $c$  into  $5w - 4c - 1 < Z$  and simplifying, we obtain that  $\hat{Q}_E^* < \bar{Q}$  if  $(4+w)\sqrt{w^2 + 6w + 1} < w^2 + 7w + 8$ , which holds for any  $w(\geq c)$ . Thus, we conclude that in equilibrium,  $\hat{Q}_E^* < \bar{Q}$ , which completes the proof of Lemma 4.6.  $\square$

**Proof of Proposition 4.7.** Having derived  $R$ 's reaction functions,  $p_E^*$  and  $Q_E^*$ , given by (20),  $M$ 's expected profit function becomes:  $E\Pi_E^M = (w - c)Q_E^* - \frac{b(Q_E^*)^2}{4e^{-p_E^*}}$ .

Using Maple 6 to solve, simultaneously, the first-order conditions of  $E\Pi_E^M$  with respect to  $w$  and  $b$ , i.e.,  $\frac{\partial E\Pi_E^M}{\partial w} = 0$  and  $\frac{\partial E\Pi_E^M}{\partial b} = 0$ , results with a unique stationary point ( $w_E^* = 1 + c, b_E^* = 1$ ). The Hessian matrix (for brevity, again, we don't present the intermediate results) at this point is:

$$\begin{vmatrix} \frac{\partial^2 E\Pi_M}{\partial w^2} & \frac{\partial^2 E\Pi_M}{\partial w \partial b} \\ \frac{\partial^2 E\Pi_M}{\partial b \partial w} & \frac{\partial^2 E\Pi_M}{\partial b^2} \end{vmatrix} = \begin{vmatrix} Mww & Mwb \\ Mbw & Mbb \end{vmatrix},$$

where

$$\begin{aligned} Mww &\equiv -\frac{16\beta(2c^4 + 18c^3 + 2c^3H + 12c^2H + 42c^2 + 14cH + 23c + 3 + 3H)}{H(1 + c + H)^4}, \\ Mbb &\equiv -\frac{8c^3 + 11c^2 - c^2H + 8cH + 17c + 3 + 3H)\beta}{H(1 + c + H)^4}, \\ Mwb = Mbw &= \frac{16(2c^3 + 2c^2H + 15c^2 + 7cH + 13c + 2 + 2H)\beta}{H(1 + c + H)^4}, \end{aligned}$$

where  $\beta = e^{-\frac{3+c+H}{2}}$ , and, as we recall,  $H = \sqrt{c^2 + 6c + 1}$ . One can verify that since  $\beta > 0$  and  $Mww < 0, Wbb < 0$  and  $Mww \cdot Mbb - Mwb \cdot Mbw > 0$ , which implies that the Hessian matrix at the unique stationary point is negative definite, and  $M$ 's expected function is globally maximized at ( $w^* = 1 + c, b^* = 1$ ). Thus, the equilibrium expected profits of  $M$  and  $R$  are:  $E\Pi_E^{M*} = E\Pi_E^{R*} = \frac{(c+3-H)(-c+1+H)}{4}e^{-\frac{3+c+H}{2}}$ , where  $H = \sqrt{c^2 + 6c + 1}$ .  $\square$