Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

Accelerator & Fusion Research Division

Presented at the Thirteenth International Free Electron Laser Conference, Santa Fe, NM, August 25–30, 1991, and to be published in the Proceedings

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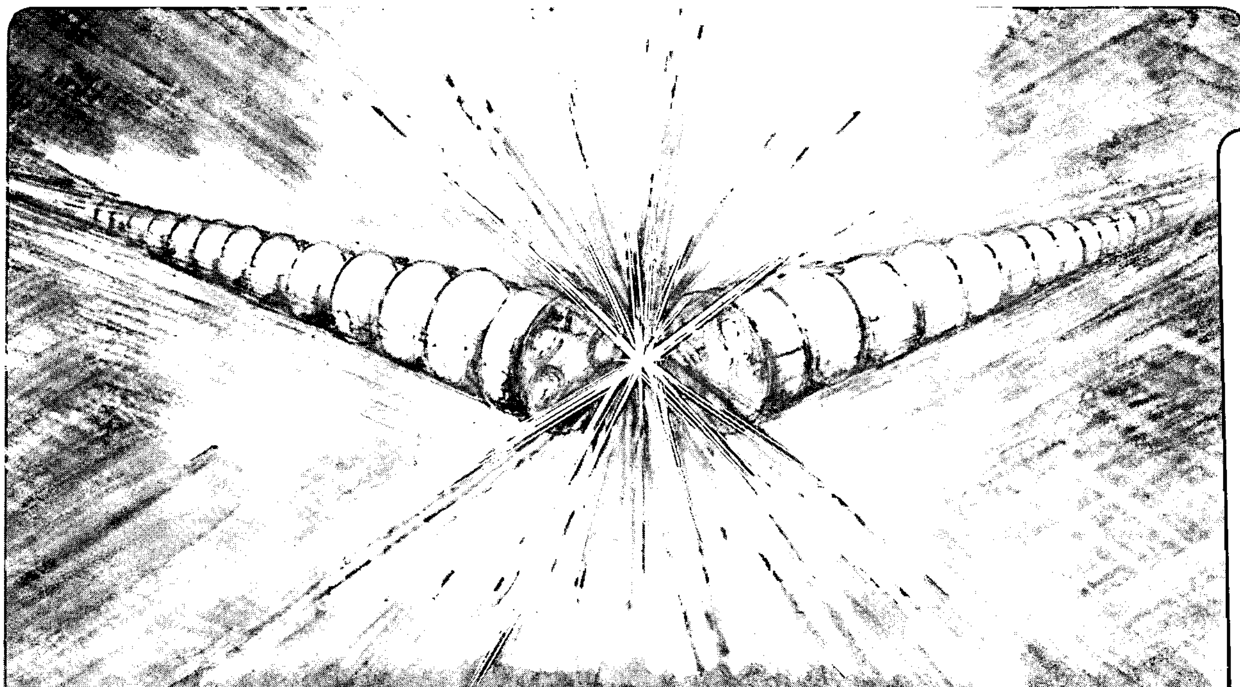
August 1991



Prepared for the U.S. Department of Energy under Contract Number DE-AC03-76SF00098

LBL-30628
ESG Note-128

FEL Gain Taking into Account Diffraction and Electron Beam Emittance; Generalized Madey's Theorem*

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Abstract

We derive a formula for the free electron laser gain in the small-signal, low-gain regime which resembles closely the 1-D formula but taking into account the effect of wave diffraction and electron beam divergence and betatron motion. The formula is cast in a form which exhibits clearly the role of the transverse phase space distribution of photons and electrons.

1. Introduction

We obtain a simple expression for the free electron laser (FEL) gain in the low gain regime that takes into account the three dimensional effects due to wave diffraction, and electron beam divergence and betatron motion. The analysis is based on a perturbation theory of the coupled Maxwell-Vlasov's equations derived earlier [1].

* This work was supported by the Director, Office of Energy Research, Office of Basic Energy Sciences, Materials Sciences Division, of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

Section 2 establishes the notation and provides a brief summary of the steps leading to the coupled equations. Section 3 contains the main results of this paper. The formula for the gain is given by eq. (23), where $\overline{F}$ is the electron distribution function, U_ν is the amplitude of the undulator radiation, and A_ν is the amplitude of the incoming laser; The gain is an energy derivative of the undulator radiation intensity emitted into the incoming laser mode — averaged over the electron beam distribution. Thus it is in a form closely analogous to the 1-D Madey's theorem [2]. When the betatron focusing effect can be neglected, the gain formula can be cast into a form which brings out the effects of the electrons' phase space distribution and the wave diffraction very clearly. This is equation (28), which is a convolution of three phase space distribution functions; $\overline{F}$ for the electrons, B_A for the incoming radiation and B_U for the undulator radiation. Here B_A and B_U are an example of the generalized brightness function which plays the role of the phase space distribution function for radiation field [3].

2. <u>FEL Equations Taking into Account 3-D Effects and Electron Beam Betatron Motion</u>

We consider electrons traveling in the z-direction through an undulator, interacting with the radiation field. The notations are as follows: The average energy of electrons = $mc^2\gamma_o$ (m = electron mass, c = velocity of light), length of the undulator period = λ_u, peak undulator magnetic field = B_o, the fundamental resonant frequency $\omega_1 = ck_1 = 2\pi c/\lambda_1 = 2\,k_u c\gamma_0^2/(1 + K^2/2)$,

$K = eB_o/mc\,k_u$, e = electron charge, $k_u = 2\pi/\lambda_u$. (MKS units are used throughout this paper). We consider the case of a planar undulator with the magnetic field in the y-direction. Denoting the transverse coordinate by $\mathbf{x}$ = (x, y), the x-component electric field is represented by

$$E(\mathbf{x}, z, t) = \frac{1}{\sqrt{2\pi}} \int d\omega\, a_\nu(\mathbf{x},z)\, e^{-i\Delta\nu k_u z}\, e^{i\nu\omega_1(t-z/c)} \tag{1}$$

In FELs, the complex amplitude $a_\nu(\mathbf{x}, z)$ is a slowly varying function of z and peaked around narrow regions of $\nu = \omega/\omega_1 = k/k_1$ around odd integers n, i.e.,

$$\Delta\nu \equiv \nu - n << 1 \quad . \tag{2}$$

The distance z will be regarded as the independent variable. The dependent variables describing the motion of an electron in the longitudinal direction are

$$\theta = k_u z - \omega_1 (\bar{t}(z) - z/c), \quad \eta = (\gamma - \gamma_o)/\gamma_o \quad . \tag{3}$$

Here $\bar{t}(z)$ is the time at which the electron passes through z, averaged over one undulator period. The variables in the transverse direction are

$$\mathbf{x} = (x, y) \quad , \quad \dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dz} = \left(\frac{dx}{dz}, \frac{dy}{dz}\right) \ . \tag{4}$$

The pair $(\mathbf{x},\dot{\mathbf{x}})$ describes the betatron motion of the electron trajectory. The equations of motion which generalize the one-dimensional pendulum equation are

$$\frac{d\theta}{dz} = 2\eta k_u - \frac{k_1}{2}\left(\dot{\mathbf{x}}^2 + k_\beta^2\, \mathbf{x}^2\right) \ , \quad \frac{d\eta}{dz} = \frac{e\omega_1}{2\sqrt{2\pi}\, mc^2\, \gamma_0^2} \sum_n K_n \int_{\nu\sim n} d\nu\; a_\nu(\mathbf{x},z) e^{-i\nu\theta} \tag{5}$$

$$\frac{d\mathbf{x}}{dz} = \dot{\mathbf{x}} \quad , \quad \frac{d\dot{\mathbf{x}}}{dz} = - k_\beta^2 \mathbf{x} \quad . \tag{6}$$

In the above, k_β is the strength of the betatron focussing due to, for example, a shaping of the undulator poles, and

$$K_n = (-1)^{\frac{n+1}{2}} K\left\{J_{\frac{n+1}{2}}\left(\frac{nK^2}{4 + 2K^2}\right) - J_{\frac{n+1}{2}}\left(\frac{nK^2}{4 + 2K^2}\right)\right\} \ . \tag{7}$$

The Klimontovitch distribution function describing the microscopic distribution of the electrons is

$$\tilde{F}(\theta,\eta,\mathbf{x},\dot{\mathbf{x}};z) = \left(k_1 / \frac{dN_e}{dz}\right) \sum_i \delta\left(\theta - \theta_i(z)\right) \delta(\eta - \eta_i(z))\ \delta^{(2)}(\mathbf{x} - \mathbf{x}_i(z))\, \delta^{(2)}(\dot{\mathbf{x}} - \mathbf{x}_i(z)) \ , \tag{8}$$

where dN_e/dz is the average number of electrons per unit longitudinal length, assumed in this paper to be constant. The distribution function is separated into the ensemble averaged smooth background $\overline{F}$ and the deviation from it δf:

$$\tilde{F}(\theta,\eta,x,\dot{x};z) = \overline{F}(\eta,x,\dot{x};z) + \delta f(\theta,\eta,x,\dot{x};z) \quad .$$

Note that $\overline{F}$ is independent of θ, since we are assuming that the beam density is uniform in the longitudinal direction. The normalization is such that

$$\int \overline{F}(\eta,x,\dot{x};z)\, d\eta\, dx\, d\dot{x} = 1 \quad .$$

The function δF contains the FEL dynamics, and will be regarded small compared to $\overline{F}$. It is convenient to introduce the Fourier transform:

$$\delta F_\nu(\eta,x,\dot{x};z) = \frac{1}{\sqrt{2\pi}} \int d\theta\, e^{i\nu\theta}\, \delta f(\theta,\eta,x,\dot{x};z) \quad .$$

With these definitions, the Maxwell equation becomes

$$\left(\frac{\partial}{\partial z} - i\Delta\nu k_u + \frac{i}{2k}\nabla_\perp^2\right) a_\nu(x;z) = - g_n \int d\eta\, d\dot{x}\, \delta F_\nu(\eta,x,\dot{x};z) \quad , \tag{9}$$

where

$$g_n = \frac{e\, K_n}{4\varepsilon_o \gamma_o c k_1} \frac{dN_e}{dz} \quad . \tag{10}$$

Here, ε_o is the vacuum dielectric constant. The continuity equation for the electron distribution function can be separated into two parts. The equation for δF_ν in the lowest order is

$$\left(\frac{\partial}{\partial z} - 2i\nu\eta k_u + i\frac{k}{2}w(x,\dot{x}) + \dot{x}\frac{\partial}{\partial \dot{x}} - k_\beta^2\, x\frac{\partial}{\partial \dot{x}}\right) \delta F_\nu(\eta,x,\dot{x};z)$$
$$= - h_n\, a_\nu(x;z) \frac{\partial}{\partial \eta} \overline{F}(\eta,x,\dot{x};z) \quad , \tag{11}$$

where

$$w(x,\dot{x}) = \dot{x}^2 + k_\beta^2 x^2 \ , \tag{12}$$

$$h_n = \frac{e\, k_1\, K_n}{2\, mc\, \gamma_0^2} \ . \tag{13}$$

The equation describing the evolution of $\overline{F}$, obtained by taking the ensemble- and the θ-averaging of the continuity equation is

$$\left(\frac{\partial}{\partial z} + \dot{x}\frac{\partial}{\partial x} - k_\beta^2 x \frac{\partial}{\partial \dot{x}}\right)\overline{F}(\eta,x,\dot{x};z) + \frac{e}{2\,\gamma_0^2\, mc\, \ell_z}\sum_n K_n \int_{\nu\sim n} d\nu\, a_\nu(x;z)\frac{\partial}{\partial\eta}\delta F_\nu(\eta,x,\dot{x};z) = 0 \ . \tag{14}$$

Here ℓ_z is the length of the electron bunch, assumed to be sufficiently long.

Equations (9), (11) and (14) are the basis of a self-consistant, quasi-linear theory of FEL including the three dimensional effect. In this paper, we neglect the evolution of $\overline{F}$, taking it to be the unperturbed distribution. The coupled set, Eqs. (9) and (11), then describes the FEL amplification in the small signal regime.

We solve Eq. (11) by the technique of the integration over the unperturbed trajectory [4] One obtains

$$\delta F_\nu(\eta,x,\dot{x};z) = \delta F_\nu^0(\eta,x,\dot{x};z) - h_n \int_{-L/2}^{z} ds\, a_\nu(x_0;s)\frac{\partial}{\partial\eta}\overline{F}(\eta,x_0,\dot{x}_0;s)\times \exp - i\nu\left\{2\,k_u\eta(s-z) - \frac{k_1}{2}\int_z^s ds'\, w(x_0', \dot{x}_0')\right\} \ , \tag{15}$$

where δF^0_ν is δF_ν in the absence of the FEL interaction, L is the length of the undulator, and

$$x_0 \equiv x_0[x,\dot{x},z;s] \quad , \dot{x}_0 \equiv \dot{x}_0[x,\dot{x},z;s] \quad ,$$

are the solutions of the trajectory equation, Eq. (6), with the initial condition $x_0 = x$, $\dot{x}_0 = \dot{x}$ at s=z. Also x_0' and $\dot{x}_0'$ represent x_0 and $\dot{x}_0$ in the above with the independent variable s replaced by s'. We note that

$$\overline{F}(\eta,x_0,\dot{x}_0;s) = \overline{F}(\eta,x,\dot{x};z) \quad . \tag{16}$$

Inserting Eq. (15) into Eq. (9), we obtain

$$\left(\frac{\partial}{\partial z} - i\Delta\nu k_u + \frac{i}{2k}\nabla^2\right) a_\nu(x;z) = - g_n \int d\eta\, d\dot{x}\; \delta F^0_\nu(\eta,x,\dot{x},z) \; - g_n h_n \int d\eta\, d\dot{x}\; \overline{F}(\eta,x,\dot{x};z)$$

$$\frac{\partial}{\partial \eta} \int_{-L/2}^{z} ds\; a_\nu(x_0,s)\; \exp\text{-}i\nu\left\{2k_u\eta(s-z) - \frac{k_1}{2}\int_z^s ds'\, w(x_0',\dot{x}_0')\right\} \tag{17}$$

The effect of the first term in the R.H.S. is to produce the spontaneous radiation. The second term contains the FEL gain, and reduces the well known formula for the case electrons are parallel [5]. The Laplace-transformed version of the equation was first reported in Ref [1]. The equation was solved by a variational approximation to obtain the exponential growth rate in the high-gain regime [6]. Recently, a more general approach to study the exponential gain regime is being developed based on orthogonal function expansion of δF_ν [7].

3. Analysis in the Low Gain Regime

To solve Eq (17) in perturbation series, it is convenient to introduce the angular representation:

$$A_\nu(\phi;z) = \frac{1}{\lambda^2}\int a_\nu(x;z)e^{ikx\cdot\phi}\, dx \quad . \tag{18}$$

Equation (17) can be converted to an integral equation for $A_\nu(\phi;z)$, which can be iterated to obtain a perturbation series. The lowest order terms are

$$A_\nu(\phi;L/2) = e^{i(\Delta\nu k_u + k\phi^2/2)L/2} \Big\{ A_\nu^0(\phi) - \sum_j e^{i\nu\theta_j}\, U_\nu(\phi,\eta,x_j,\dot{x}_j)$$

$$+ \frac{h_n g_n}{\lambda^2} A_\nu^1(\phi) + \ldots \Big\} \tag{19}$$

In the above, the first, the second and the third terms in the bracket represent respectively the incident radiation, the spontaneous radiation usually referred to as the undulator radiation, and the first order amplification. The undulator radiation is given by

$$U_\nu(\phi,\eta,x,\dot{x}) = u_n\, e^{ik\phi\cdot x} \int_{-L/2}^{L/2} ds\, e^{-i\int_0^s ds'\, \xi_\nu(\phi,\eta,x,\dot{x};s')} \quad ,$$

$$\xi_\nu(\phi,\eta,x,\dot{x};s) = (\Delta\nu - 2\eta\nu)\, k_u + \frac{k}{2}\left[(\phi - \dot{x}(s))^2 + k_\beta^2 x^2(s)\right] \quad , \tag{20}$$

where $u_n = e\, K_n/4\sqrt{2\pi}\, \varepsilon_0\, \gamma_0\, c\, \lambda^2$, and $x(s)$ and $\dot{x}(s)$ are the solution of the trajectory equation with the initial condition $x(0) = x$ and $\dot{x}(0) = \dot{x}$. The first order amplification term is given by

$$A_\nu^1(\phi) = -\int d\eta\, dx\, d\dot{x}\, d\phi'\ \overline{F}(\eta,x,\dot{x};0)\frac{\partial}{\partial\eta}\int_{-L/2}^{L/2} ds\ e^{-ik\phi\cdot x}\ e^{-i\int_0^s ds_1\, \xi_\nu(\phi,\eta,x,\dot{x};s_1)} \ \times$$

$$\int_{-L/2}^{s} ds'\ e^{-ik\phi'\cdot x}\ e^{i\int_0^{s'} ds_1\, \xi_\nu(\phi',\eta,x,\dot{x};s_1)}\ A_\nu^0(\phi') \quad . \tag{21}$$

The gain is defined as

$$G = \int d\phi \left(|A^1_\nu(\phi)|^2 - |A^0_\nu(\phi)|^2 \right) / \int d\phi \, |A^0_\nu(\phi)|^2 \ . \tag{22}$$

To the lowest order, it then follows from Eq. (21) that

$$G = -c_n \int d\eta \, dx \, d\dot{x} \ \overline{F}(\eta,x,\dot{x};0) \frac{\partial}{\partial\eta}$$

$$\times \left| \int d\phi \ U^*_\nu(\phi,\eta,x,\dot{x}) \, A^0_\nu(\phi) \right|^2 / \int d\phi \, |A^0_\nu(\phi)|^2 \ , \tag{23}$$

where $c_n = h_n \, g_n/u_n^2 \, \lambda^2$, and $U_\nu{}^*$ is the complex conjugate of U_ν. Equation (23) is one of the main results of the present paper. Recall that $U_\nu(\phi,\eta,x,\dot{x})$ is the amplitude of the (spontaneous) undulator radiation, at $\omega = \nu\omega_1$, by an electron travelling the entire length of the undulator with energy $\gamma = \gamma_0(1 + \eta)$ and with the transverse position x and angle $\dot{x}$ at z=0. The integral $\int d\phi \ U^*_\nu(\phi,..) \ A^0_\nu(\phi)$ can be interpreted as the projection of the undulator radiation amplitude into the mode of the incoming radiation. Equation (23) closely resembles the 1-D Madey's theorem [2]. The mode with the maximal gain can be found from the eigenvalue equation

$$- c_n \int d\eta \, dx \, d\dot{x} \, d\phi' \ \overline{F}(\eta,x,\dot{x};0) \frac{\partial}{\partial\eta} \left\{ U_\nu(\phi,\eta,x,\dot{x}) \, U^*_\nu(\phi',\eta,x,\dot{x}) \right\} A^0_\nu(\phi')$$

$$= G \, A^0_\nu(\phi) \ . \tag{24}$$

Equations (23) and (24) are a generalization of those derived by Moore [5] for the case the electrons move parallel to each other without the betatron motion (no focusing).

When the focusing can be neglected, we have [3]

$$U_\nu(\phi,\eta,x,\dot{x}) = e^{ik\phi\cdot x} \ U^0_\nu(\phi-\dot{x},\eta) \ , \tag{25}$$

where $U_v^o(\phi,\eta) \equiv U_v(\phi,\eta,0,0)$ is the radiation amplitude of an ideal trajectory with $x = \dot{x} = 0$. From Eq. (25), we have

$$U_v^o(\phi,\eta) = u_n L \sin \Phi/\Phi \quad , \tag{26}$$

where $\Phi = \left(\left(\Delta\nu - 2\eta\nu\right) k_u + k\phi^2/2\right) L/2$ is sometimes known as the detuning parameter. Inserting Eq. (25) into Eq. (23), we obtain the following gain formula when the focussing can be neglected:

$$G = -\left(e^2 K_n^2/8mc^3 \varepsilon_o \gamma_o^3\right) dN_e/dz \int d\eta \, dx \, d\dot{x} \, \overline{F}(\eta,x,\dot{x};0) \frac{\partial}{\partial\eta}$$
$$\times \left| \frac{1}{u_n\lambda} \int d\phi \, e^{-ik\phi\cdot x} \, U_v^{o*}(\phi-\dot{x},\eta) \, A_v^o(\phi) \right|^2 \quad . \tag{27}$$

In the 1-D limit without energy spread, we set $\overline{F} = \delta(\eta)\, \delta^{(2)}(\dot{x})/\Sigma$, where Σ is the transverse area, and $\left|A^o(\phi)\right|^2 = \delta^{(2)}(\phi)$. Equation (27) then becomes the well-known 1-D formula.

Given an electric field amplitude $E(\phi)$ in angular representation, we can introduce the corresponding brightness distribution function

$$B(x,\phi) = 2\, \varepsilon_0\, c \int d\xi < E^*(\phi + \xi/2)\, E(\phi - \xi/2) > e^{-ikx\cdot\xi} \quad .$$

The real function B can be interpreted as the phase space density of the spectral flux of photons, much like $\overline{F}(\eta,x,\dot{x};0)$ is the phase space density of electrons [3]. Denoting B_A and B_U be the brightness functions corresponding to the input radiation $A_v^o(\phi)$ and the undulator radiation $U_v^o(\phi;\eta)$, we can show from (Eq. (27) that the gain in the case of no transverse focussing can be written as

$$G = -\frac{\lambda^2}{mc\,\gamma_o} \frac{dN_e}{dz} \frac{\int d\eta \, dx \, d\dot{x} \, dy \, d\phi \;\; \overline{F}(x,\dot{x},\eta;0)\, B_A(y,\phi) \frac{\partial}{\partial\eta} B_U(y-x,\, \phi-\dot{x})}{\int B_A(y,\phi)\, dy d\phi} \quad . \tag{28}$$

Thus the gain is essentially a convolution of the three phase-space distribution functions, $\overline{F}$, B_U and B_A. Equation (28) exhibits very clearly the role of the phase space distribution of electrons and photons in the FEL gain.

The three distribution functions in Eq. (28) are those evaluated at z=0. The distribution functions, for electrons as well as photons, at $z \neq 0$ are related to those at z=0 by the coordinate transformation of the form $\mathbf{x} \rightarrow \mathbf{x} - z\phi$, $\phi \rightarrow \phi$ [3]. Equation (28) is invariant under this transformation.

Numerical study of the formula derived in this paper wil be presented elsewhere. A previous work on the subject of this paper can be found in Ref. [8].

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