Title
The representation theory of the exceptional Lie superalgebras F(4) and G(3)

Author
Martirosyan, Lilit

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The representation theory of the exceptional Lie superalgebras $F(4)$ and $G(3)$

by

Lilit Martirosyan

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Committee in charge:

Professor Vera Serganova, Chair
Professor Joseph Wolf, Co-chair
Professor Ori Ganor

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Abstract

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$F(4)$ and $G(3)$

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Professor Vera Serganova, Chair

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This thesis is a resolution of three related problems proposed by Yu. I. Manin and V. Kac for the so-called *exceptional* Lie superalgebras $F(4)$ and $G(3)$. The first problem posed by Kac (1978) is the problem of finding character and superdimension formulae for the simple modules. The second problem posed by Kac (1978) is the problem of classifying all indecomposable representations. The third problem posed by Manin (1981) is the problem of constructing the *superanalogue* of Borel-Weil-Bott theorem.

The representation theory of the exceptional Lie superalgebras $F(4)$ and $G(3)$

To my grandparents

Contents

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Chapter 1

Introduction

This thesis is within the area of Lie theory, algebraic geometry, and representation theory. Lie superalgebras and their representation theory are important in theoretical physics. They are used to describe the mathematics of *supersymmetry*, which is a theory originated in quantum physics that relates *bosons* and *fermions*. The study of representations of Lie superalgebras also has important applications in other branches of Lie theory and representation theory.

After classifying all finite-dimensional simple Lie superalgebras over $\mathbb{C}$ in 1977, V. Kac proposed the problem of finding character and superdimension formulae for the simple modules (see [10]).

Main result 1: The first main result in this thesis is solving this problem in full for the so-called *exceptional* Lie superalgebras $F(4)$ and $G(3)$.

The next problem, also posed by V. Kac in 1977, is the problem of classifying all indecomposable representations of classical Lie superalgebras (see [10]). Here we settle it as follows.

Main result 2: For the exceptional Lie superalgebras $F(4)$ and $G(3)$, we describe the *blocks* up to equivalence and find the corresponding *quivers*, which gives a full solution of this problem. We show that the blocks of atypicality 1 are *tame*, which together with Serganova's results for other Lie superalgebras proves a conjecture by J. Germoni.

In the geometric representation theory of Lie algebras, the Borel-Weil-Bott (BWB) theorem (see Theorem 2.3.1) plays a crucial role. This theorem describes how to construct families of representations from sheaf cohomology groups associated to certain vector bundles. It was shown by I. Penkov, that this theorem is not true for Lie superalgebras. In 1981, Yu. I. Manin proposed the problem of constructing a *superanalogue* of BWB theorem. The first steps towards the development of this theory were carried out by I. Penkov in [16].

Main result 3: One of my results (see Theorem 2.3.2 and Theorem 2.3.3) is an analogue of BWB theorem for the exceptional Lie superalgbras $F(4)$ and $G(3)$ for dominant weights.

Background: The *basic classical* Lie superalgebras that are not Lie algebras are:

(i) the series $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(m|n)$;
(ii) the exceptional Lie superalgbras $F(4)$ and $G(3)$; and
(iii) the family of exceptional Lie superalgebras $D(2, 1; \alpha)$.

In [10], Kac introduced the notions of *typical* and *atypical* irreducible representations. He classified the finite-dimensional irreducible representations for basic classical Lie superalgebras using highest weights and induced module constructions similar to Verma module constructions for simple Lie algebras. In [11], he found character formulae similar to the Weyl character formula for typical irreducible representations.

The study of atypical representations has been difficult and has been studied intensively over the past 40 years. Unlike the typical modules, atypical modules are not uniquely described by their central character. All simple modules with given central character form a *block* in the category of finite-dimensional representations.

Brief history: The problem of finding characters for simple finite-dimensional $\mathfrak{gl}(m|n)$-modules has been solved using a geometric approach by V. Serganova in [21] and [22], and later J. Brundan in [1] found characters using algebraic methods and computed extensions between simple modules.

More recently, this problem has been solved for all infinite series of basic classical Lie superalgebras in [9] by C. Gruson and V. Serganova. They compute the characters of simple modules using Borel-Weil-Bott theory and generalizing a combinatorial method of weight and cap diagrams developed first by Brundan and Stroppel for $\mathfrak{gl}(m|n)$ case.

In [5], J. Germoni solves Kac's problems for $D(2,1;\alpha)$. He also studies the blocks for $G(3)$ using different methods. I solve the above problems for $G(3)$ and $F(4)$, generalizing the methods of [9].

Strategies used for the exceptional Lie superalgebras $F(4)$ and $G(3)$: To solve the aformentioned problems of V. Kac and Yu. I. Manin for the superalgebras $F(4)$ and $G(3)$, we have used machinery from algebraic geometry, representation theory, and category theory.

The first step is to study the blocks and prove that up to equivalence there are two atypical blocks for $F(4)$, called *symmetric* and *non-symmetric*, and one block for $G(3)$. We find Ext^1 between atypical simple modules in a block. After finding the Ext^1 for simple modules in a block, we prove that the *quivers* corresponding to the atypical blocks are of type A_∞ and D_∞ for $F(4)$ and of type D_∞ for $G(3)$ (see Theorem 2.1.1 and Theorem 2.1.2). Using quiver theorem (Theorem 11.1.4), this leads to the classification of all indecomposable modules. We proved the formula for the *superdimension* for the atypical irreducible representations (see Theorem 10.1.1 and Theorem 10.1.2). We combined this with results in [23] to prove the Kac-Wakimoto conjecture (see Theorem 2.2.3) for $F(4)$ and $G(3)$.

Next, we studied the cohomologies of line bundles over flag supervarieties establishing a theorem that is a "superanalogue" of the Borel-Weil-Bott theorem for the classical Lie algebras for dominant weights (see Theorem 2.3.2 and Theorem 2.3.3). Unlike the Lie algebra case, the cohomology groups for atypical modules are not always simple modules and they may appear in several degrees. There are three special atypical simple modules L_{λ_0}, L_{λ_1}, L_{λ_2} in the symmetric block of $F(4)$ and the block of $G(3)$, and one special simple module L_{μ_0} for the non-symmetric block of $F(4)$. For other cases the cohomology groups vanish in positive degree and in zero degree have two simple quotients that are adjacent vertices of the quiver. The first cohomology appears only for sheaves $\mathcal{O}_{\lambda_1}$, $\mathcal{O}_{\lambda_2}$, and $\mathcal{O}_{\mu_0}$, and is equal to L_{λ_2}, L_{λ_1}, and L_{μ_0} correspondingly. Cohomology group in zero degree for $\mathcal{O}_\sigma$ is L_σ for $\sigma = \lambda_1, \lambda_2, \mu_0$. The cohomology of $\mathcal{O}_{\lambda_0}$ vanishes in positive degree and in zero degree it has three simple quotients L_{λ_i}, with $i = 0, 1, 2$.

Most complications in the proof were arising for the weights close to the walls of the Weyl chamber. The main difference from other classical cases was that there was no analogue of the standard module. Therefore, for the exceptional Lie superalgebras, one cannot move from any equivalent atypical block to another by the use of translation functor as in the infinite series of Lie superalgebras in [9]. Instead, the translation functor applies only in some cases. I use this in combination with other techniques including associated variety and fiber functor [3], geometric induction [19] and [21], V. Serganova's odd reflections method [20], and formulae for generic modules by Penkov and Serganova [18].

As for $\mathfrak{osp}(m|n)$ in [9], I use geometric induction for the study of highest weight modules, instead of the usual induction that are used in Bernstein-Gelfand-Gelfand category $\mathcal{O}$, or the Kac modules that were used for the case $\mathfrak{sl}(m|n)$. The cohomology groups $H^i(G/B, \mathcal{V}^*)^*$ of the induced vector bundle $\mathcal{V} = G \times_B V$ on the flag supervariety G/B are viewed as $\mathfrak{g}$-modules for an algebraic supergroup G and a Borel subgroup B. Penkov's method [16] is used to construct filtrations of $\mathfrak{g}$-modules by line bundles $\mathcal{O}_\lambda$ on flag supervariety G/B, which gives an upper bound on the multiplicities of simple modules L_μ in the cohomology groups $H^i(G/B, \mathcal{O}_\lambda^*)^*$.

Chapter 2

Main results

2.1 Classification of blocks

Let $\mathcal{C}$ denote the category of finite-dimensional $\mathfrak{g}$-modules. And let $\mathcal{F}$ be the full subcategory of $\mathcal{C}$ consisting of modules such that the parity of any weight space coincides with the parity of the corresponding weight.

The category $\mathcal{F}$ decomposes into direct sum of full subcategories called *blocks* $\mathcal{F}^{\chi}$, where $\mathcal{F}^{\chi}$ consists of all finite dimensional modules with (generalized) central character χ. A block having more than one element is called an *atypical* block. By F^{χ}, we denote the set of weights corresponding to central character χ.

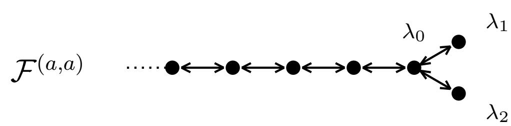
A *quiver diagram* is a directed graph that has vertices the finite-dimensional irreducible representations of $\mathfrak{g}$, and the number of arrows from vertex λ to the vertex μ is $dimExt^1_{\mathcal{A}}(L_\lambda, L_\mu)$.

Theorem 2.1.1 *The following holds for* $\mathfrak{g} = F(4)$*:*

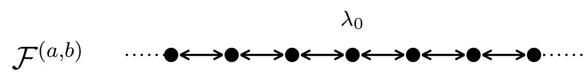
(1) The atypical blocks are parametrized by dominant weights μ *of* $\mathfrak{sl}(3)$*, such that* $\mu + \rho_l = a\omega_1 + b\omega_2$ *with* $a = 3n + b$*. Here,* $b \in \mathbb{Z}_{>0}$ *and* $n \in \mathbb{Z}_{\geq 0}$*;* ω_1 *and* ω_2 *are the fundamental weights of* $\mathfrak{sl}(3)$*;* ρ_l *is the Weyl vector for* $\mathfrak{sl}(3)$*.*

(2) There are two, up to equivalence, atypical blocks, corresponding to dominant weights μ *of* $\mathfrak{sl}(3)$*, such that* $\mu + \rho_l = a\omega_1 + b\omega_2$ *with* $a = b$ *or* $a \neq b$*. We call these blocks symmetric or non-symmetric and denote by* $\mathcal{F}^{(a,a)}$ *or* $\mathcal{F}^{(a,b)}$ *respectively.*

(3) For the symmetric block $\mathcal{F}^{(a,a)}$, we have the following quiver diagram, which is of type D_∞:



(4) For the non-symmetric block $\mathcal{F}^{(a,b)}$, we have the following quiver diagram, which is of type A_∞:

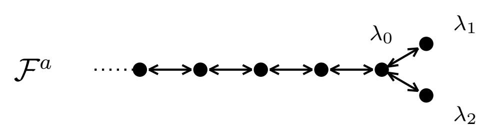


Theorem 2.1.2 *The following holds for $\mathfrak{g} = G(3)$:*

(1) The atypical blocks are parametrized by dominant weight μ of $\mathfrak{sl}(2)$, such that $\mu_l + \rho = a\omega_1$ with $a = 2n + 1$. Here, $n \in \mathbb{Z}_{\geq 0}$; ω_1 is the fundamental weight of $\mathfrak{sl}(2)$; ρ_l is the Weyl vector for $\mathfrak{sl}(2)$.

(2) There is one, up to equivalence, atypical block, corresponding to dominant weight μ of $\mathfrak{sl}(2)$, such that $\mu_l + \rho = a\omega_1$. Denote it by $\mathcal{F}^a$.

(3) For the block $\mathcal{F}^a$, we have the following quiver diagram, which is of type D_∞:



2.2 Character and superdimension formulae

The *superdimension* of a representation V is the number $sdimV = dimV_0 - dimV_1$ (see [14]).

Let $X = \{x \in \mathfrak{g}_{\bar{1}} | [x, x] = 0\}$ be the *self-commuting cone* in $\mathfrak{g}_{\bar{1}}$ studied in [3]. For $x \in X$, we denote by $\mathfrak{g}_x$ the quotient $C_{\mathfrak{g}}(x)/[x, \mathfrak{g}]$ as in [3], where $C_{\mathfrak{g}}(x) = \{a \in \mathfrak{g} | [a, x] = 0\}$ is the centralizer of x in $\mathfrak{g}$.

We proved the following superdimension formula for the exceptional Lie superalgebra $\mathfrak{g} = F(4)$.

Theorem 2.2.1 *Let $\mathfrak{g} = F(4)$. Let $\lambda \in F^{(a,b)}$ and $\mu + \rho_l = a\omega_1 + b\omega_2$. If $\lambda \neq \lambda_1$, λ_2, the following superdimension formula holds:*

$$sdim\, L_\lambda = \pm 2 dim\, L_\mu(\mathfrak{g}_x).$$

For the special weights, we have: $sdim\, L_{\lambda_1} = sdim\, L_{\lambda_2} = dim\, L_\mu(\mathfrak{g}_x)$. Here, $\mathfrak{g}_x \cong \mathfrak{sl}(3)$.

Similarly, we proved the following superdimension formula for the exceptional Lie superalgebra $\mathfrak{g} = G(3)$.

Theorem 2.2.2 *Let $\mathfrak{g} = G(3)$. Let $\lambda \in F^a$ and $\mu + \rho_l = a\omega_1$. If $\lambda \neq \lambda_1$, λ_2, the following superdimension formula holds:*

$$sdim\, L_\lambda = \pm 2 dim\, L_\mu(\mathfrak{g}_x).$$

For the special weights, we have: $sdim\, L_{\lambda_1} = sdim\, L_{\lambda_2} = dim\, L_\mu(\mathfrak{g}_x)$. Here, $\mathfrak{g}_x \cong \mathfrak{sl}(2)$.

A root α is called *isotropic* if $(\alpha, \alpha) = 0$. The *degree of atypicality* of the weight λ the maximal number of mutually orthogonal linearly independent isotropic roots α such that $(\lambda + \rho, \alpha) = 0$. The *defect* of $\mathfrak{g}$ is the maximal number of linearly independent mutually orthogonal isotropic roots. We use the above superdimension formulas and results in [23] to prove the following theorem, which is Kac-Wakimoto conjecture in [14] for $\mathfrak{g} = F(4)$ and $G(3)$.

Theorem 2.2.3 *Let $\mathfrak{g} = F(4)$ or $G(3)$. The superdimension of a simple module of highest weight λ is nonzero if and only if the degree of atypicality of the weight is equal to the defect of the Lie superalgebra.*

The following theorem gives a Weyl character type formula for the dominant weights. It was conjectured by Bernstein and Leites that formula 2.1 works for all dominant weights. However, we obtain a different character formula 2.2 for the special weights λ_1, λ_2 for $F(4)$ and $G(3)$.

Theorem 2.2.4 *Let $\mathfrak{g} = F(4)$ or $G(3)$. For a dominant weight $\lambda \neq \lambda_1, \lambda_2$, let $\alpha \in \Delta_{\bar{1}}$ be such that $(\lambda + \rho, \alpha) = 0$. Then*

$$chL_\lambda = \frac{D_1 \cdot e^\rho}{D_0} \cdot \sum_{w \in W} sign(w) \cdot w(\frac{e^{\lambda+\rho}}{(1+e^{-\alpha})}). \qquad (2.1)$$

For the special weights $\lambda = \lambda_i$ with $i = 1, 2$, we have the following formula:

$$ch(L_\lambda) = \frac{D_1 \cdot e^\rho}{2D_0} \cdot \sum_{w \in W} sign(w) \cdot w(\frac{e^{\lambda+\rho}(2+e^{-\alpha})}{(1+e^{-\alpha})}). \qquad (2.2)$$

2.3 Analogue of Borel-Weil-Bott theorem for Lie superalgebras $F(4)$ and $G(3)$.

Let $\mathfrak{g}$ be a Lie (super)algebra with corresponding (super)group G. Let $\mathfrak{b}$ be the distinguished Borel subalgebra of $\mathfrak{g}$ with corresponding (super)group B. Let V be a $\mathfrak{b}$-module.

Denote by $\mathcal{V}$ the induced vector bundle $G \times_B V$ on the flag (super)variety G/B. The space of sections of $\mathcal{V}$ has a natural structure of a $\mathfrak{g}$-module. The cohomology groups $H^i(G/B, \mathcal{V}^*)^*$ are $\mathfrak{g}$-modules.

Let C_λ denote the one dimensional representation of B with weight λ. Denote by $\mathcal{O}_\lambda$ the line bundle $G \times_B C_\lambda$ on the flag (super)variety G/B. Let L_λ denote the simple module with highest weight λ. See [19].

The classical result in geometric representation theory for finite-dimensional semisimple Lie algebra $\mathfrak{g}$ states:

Theorem 2.3.1 *(Borel-Weil-Bott) If $\lambda + \rho$ is singular, where λ is integral weight and ρ is the half trace of $\mathfrak{b}$ on its nilradical, then all cohomology groups vanish. If $\lambda + \rho$ is regular, then there is a unique Weyl group element w such that the weight $w(\lambda+\rho) - \rho$ is dominant and the cohomology groups $H^i(G/B, \mathcal{O}_\lambda^*)^*$ are non-zero in only degree $l = length(w)$ and in that degree they are equal to the simple module L_μ with highest weight $\mu = w(\lambda + \rho) - \rho$.*

We proved the following superanalogue for the exceptional Lie superalgebra $\mathfrak{g} = F(4)$ for the dominant weights and for specific choice of B.

Theorem 2.3.2 *Let $\mathfrak{g} = F(4)$.*

(1) For $\mu \in \mathcal{F}^{(a,a)}$ *with* $\mu \neq \lambda_1$, λ_2 *or* λ_0, *the group* $H^0(G/B, \mathcal{O}^*_\mu)^*$ *has two simple subquotients* L_μ *and* $L_{\mu'}$, *where* μ' *is the adjacent vertex to* μ *in the quiver* D_∞ *in the direction towards* λ_0.

At the branching point λ_0 *of the quiver, the group* $H^0(G/B, \mathcal{O}^*_{\lambda_0})^*$ *has three simple subquotients* L_{λ_0}, L_{λ_1}, *and* L_{λ_2}. *For* $i = 1, 2$, *we have* $H^0(G/B, \mathcal{O}^*_{\lambda_i})^* = L_{\lambda_i}$.

The first cohomology is not zero only at the endpoints λ_1 *and* λ_2 *of the quiver and* $H^1(G/B, \mathcal{O}^*_{\lambda_1})^* = L_{\lambda_2}$, $H^1(G/B, \mathcal{O}^*_{\lambda_2})^* = L_{\lambda_1}$. *All other cohomologies vanish.*

(2) For $\mu \in \mathcal{F}^{(a,b)}$, *the group* $H^0(G/B, \mathcal{O}^*_\mu)^*$ *has two simple subquotients* L_μ *and* $L_{\mu'}$, *where* μ' *is the adjacent vertex to* μ *in the quiver* A_∞ *in the direction towards* λ_0.

The first cohomology is not zero only in one particular point λ_0 *of the quiver and* $H^1(G/B, \mathcal{O}^*_{\lambda_0})^* = L_{\lambda_0}$. *Also,* $H^0(G/B, \mathcal{O}^*_{\lambda_0})^* = L_{\lambda_0}$. *All other cohomologies vanish.*

Similarly, we proved the following superanalogue of BWB theorem for the exceptional Lie superalgebra $\mathfrak{g} = G(3)$ for the dominant weights.

Theorem 2.3.3 *Let* $\mathfrak{g} = G(3)$.

For $\mu \in \mathcal{F}^a$ *with* $\mu \neq \lambda_1$, λ_2 *or* λ_0, *the group* $H^0(G/B, \mathcal{O}^*_\mu)^*$ *has two simple subquotients* L_μ *and* $L_{\mu'}$, *where* μ' *is the adjacent vertex to* μ *in the quiver* D_∞ *in the direction towards* λ_0.

At the brunching point, the group $H^0(G/B, \mathcal{O}^*_{\lambda_0})^*$ *has three simple subquotients* L_{λ_0}, L_{λ_1}, *and* L_{λ_2}. *For* $i = 1, 2$, *we have* $H^0(G/B, \mathcal{O}^*_{\lambda_i})^* = L_{\lambda_i}$.

The first cohomology is not zero only at the endpoints of the quiver and $H^1(G/B, \mathcal{O}^*_{\lambda_1})^* = L_{\lambda_2}$, $H^1(G/B, \mathcal{O}^*_{\lambda_2})^* = L_{\lambda_1}$. *All other cohomologies vanish.*

2.4 Germoni's conjecture and the indecomposable modules

The following theorem together with results in [9] for other Lie superalgebras proves a conjecture by J. Germoni (Theorem 2.4.2).

Theorem 2.4.1 *Let $\mathfrak{g} = F(4)$ or $G(3)$. The blocks of atypicality 1 are tame.*

Theorem 2.4.2 *Let $\mathfrak{g}$ be a basic classical Lie superalgebra. Then all tame blocks are of atypicality less or equal 1.*

The following theorem together with Theorem 11.1.4 gives a description of the indecomposable modules.

Theorem 2.4.3 *The quivers A_∞ and D_∞ are the ext-quiver for atypical blocks $\mathcal{F}^{(a,b)}$ and $\mathcal{F}^{(a,a)}$ of $F(4)$ and the quiver D_∞ is the ext-quiver for atypical block $\mathcal{F}^a$ of $G(3)$ with the following relations:*

For $\mathcal{F}^{(a,b)}$, we have:

$$d^+d^- + d^-d^+ = (d^+)^2 = (d^-)^2 = 0 \text{ ,where } d^\pm = \sum_{l\in\mathbb{Z}} d_l^\pm$$

For $\mathcal{F}^{(a,a)}$ or $\mathcal{F}^a$ we have the following relations:

$$d_l^- d_{l+1}^- = d_{l+1}^+ d_l^+ = 0, \text{ for } l \geq 3$$

$$d_1^- d_2^+ = d_2^- d_1^+ = d_0^+ d_2^+ = d_2^- d_0^- = d_0^- d_3^- = d_3^+ d_0^+ = d_1^- d_0^- = d_0^+ d_1^+ = 0$$

$$d_l^- d_l^+ = d_{l+1}^+ d_{l+1}^- \text{ for } l \geq 3$$

$$d_1^+ d_1^- = d_2^+ d_2^- = d_0^- d_0^+.$$

Chapter 3

Preliminaries

3.1 Lie superalgebras

The following preliminaries are taken from [13].

All vector spaces are over an algebraically closed field k of characteristic 0. By a *superspace* over k, we mean a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. By $p(a)$ we denote the degree of a homogeneous element a and we called a *even* or *odd* if $p(a)$ is 0 or 1 respectively.

A *Lie superalgebra* is a superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, with a bilinear map $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, satisfying the following axioms for all homogenous $a, b, c \in \mathfrak{g}$:

(a) $[a, b] = -(-1)^{p(a)p(b)}[b, a]$ (*anticommutatvity*);
(b) $[a, [b, c]] = [[a, b], c] + (-1)^{p(a)p(b)}[b, [a, c]]$ (*Jacobi identity*).

It follows from definition, that $\mathfrak{g}_{\bar{0}}$ is a Lie algebra and the multiplication on the left by elements of $\mathfrak{g}_{\bar{0}}$ determines a structure of $\mathfrak{g}_{\bar{0}}$-module on $\mathfrak{g}_{\bar{1}}$.

A bilinear form f on a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is called *invariant* if it satisfies the following conditions:

(a) $f(a, b) = (-1)^{p(a)p(b)} f(b, a)$ for all homogenous $a, b \in \mathfrak{g}$ (*supersymmetry*);
(b) $f(a, b) = 0$ if $p(a) \neq p(b)$ for all homogenous $a, b \in \mathfrak{g}$ (*consistency*);
(c) $f([a, b], c) = f(a, [b, c])$ for all $a, b, c \in \mathfrak{g}$ (*invariance*).

A bilinear form f on a Lie superalgebra $\mathfrak{g}$ is called *non-degenerate* if $f(a,b)=0$ for all $b \in \mathfrak{g}$ implies $a=0$. It is clear that on a simple Lie superalgebra, the invariant forms are either zero or non-degenerate, and any two invariant forms are proportional.

A simple Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is *basic classical* if $\mathfrak{g}_{\bar{0}}$ is a reductive subalgebra and if there is a non-degenerate invariant bilinear form for $\mathfrak{g}$. All simple finite-dimensional Lie superalgebras have been classified by Kac in [12]. The basic classical ones are all the simple Lie algebras, $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, $D(2,1;\alpha)$, $F(4)$, and $G(3)$.

If the representation of $\mathfrak{g}_{\bar{0}}$ in $\mathfrak{g}_{\bar{1}}$ is irreducible and $\mathfrak{g}_{\bar{1}} \neq 0$, $\mathfrak{g}$ is said to be of *type II*, and if it is a direct sum of two irreducible representations, then it is of *type I*. The Lie superalgebras $F(4)$, $G(3)$ are called *exceptional*, because like the five exceptional Lie algebras they are unique and don't belong to the series. The exceptional algebras $F(4)$ and $G(3)$ are of type I. The Killing form $(a,b) = tr(\text{ad } a)(\text{ad } b)|_{\mathfrak{g}_{\bar{0}}} - tr(\text{ad } a)(\text{ad } b)|_{\mathfrak{g}_{\bar{1}}}$ on $\mathfrak{g}_{\bar{0}}$ is non-degenrate for $F(4)$, $G(3)$.

For the exceptional Lie superalgebras, there exist a distinguished $\mathbb{Z}$-gradation $\mathfrak{g} = \oplus \mathfrak{g}_i$ such that $\mathfrak{g}_i = 0$ for $|i| > 2$. This gradation is defined by Kac in [12].

Let $\mathfrak{h}$ be Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ Lie algebra, then $\mathfrak{g}$ had a weight decomposition $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$, with $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$. The set $\Delta = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\}$ is called the *set of roots* of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is the *root space corresponding* to root $\alpha \in \Delta$. For a regular $h \in \mathfrak{h}$, i.e. $Re\,\alpha(h) \neq 0 \ \forall \alpha \in \Delta$, we have a decomposition $\Delta = \Delta^+ \cup \Delta^-$. Here, $\Delta^+ = \{\alpha \in \Delta | Re\,\alpha(h) > 0\}$ is called the *set of positive roots* and $\Delta^- = \{\alpha \in \Delta | Re\,\alpha(h) < 0\}$ is called the *set of negative roots*.

The Lie superalgebra $\mathfrak{g}$ admits a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ with $\mathfrak{n}^\pm_\alpha = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha$, with $\mathfrak{n}^\pm$ nilpotent subalgebras of $\mathfrak{g}$. Then $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is a solvable Lie subsuperalgebra of $\mathfrak{g}$, which is called the *Borel subsuperalgebra* of $\mathfrak{g}$ with respect to the given triangular decomposition. Here, $\mathfrak{n}^+$ is an ideal of $\mathfrak{b}$.

We set $\Delta_0^\pm = \{\alpha \in \Delta^\pm | \mathfrak{g}_\alpha \subset \mathfrak{g}_{\bar{0}}\}$ and $\Delta_1^\pm = \{\alpha \in \Delta^\pm | \mathfrak{g}_\alpha \subset \mathfrak{g}_{\bar{1}}\}$. Then the set $\Delta_0^+ \cup \Delta_0^-$ called the set of *even roots* and the set $\Delta_1^+ \cup \Delta_1^-$ is called the set of *odd roots*.

The universal enveloping algebra of $\mathfrak{g}$ is defined to be the quotient $U(\mathfrak{g}) = T(\mathfrak{g})/R$, where $T(\mathfrak{g})$ is the tensor superalgebra over space $\mathfrak{g}$ with induced $\mathbb{Z}_2$-gradation and R is the ideal of $T(\mathfrak{g})$ generated by the elements of the form $[a,b] - ab + (-1)^{p(a)p(b)}ba$.

The following is the Lie superalgebra analogue of Poincar-Birkhoff-Witt (PBW) theorem: Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra, $a_1, \ldots, a_m$ be a basis of $\mathfrak{g}_{\bar{0}}$ and $b_1, \ldots, b_n$ be a basis of $\mathfrak{g}_{\bar{1}}$, then the elements of the form $a_1^{k_1} \cdots a_m^{k_m} b_{i_1} \cdots b_{i_s}$ with $k_i \geq 0$ and $1 \leq i_1 < \cdots < i_s \leq n$ form a basis of $U(\mathfrak{g})$.

By PBW theorem, $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+)$. Let $\theta : U(\mathfrak{g}) \to U(\mathfrak{h})$ be the projection with kernel $\mathfrak{n}^- U(\mathfrak{g}) \otimes U(\mathfrak{g})\mathfrak{n}^+$.

Let $Z(\mathfrak{g})$ to be the center of $U(\mathfrak{g})$. Then the restriction $\theta_{|Z(\mathfrak{g})} : Z(\mathfrak{g}) \to U(\mathfrak{h}) \cong S(\mathfrak{h})$ is a homomorphism of rings called *Harish-Chandra* map. Since $\mathfrak{h}$ is abelian, $S(\mathfrak{h})$ can be considered the algebra of polynomial functions of $\mathfrak{h}^*$.

The (generalized) *central character* is a map $\chi_\lambda : Z(\mathfrak{g}) \to k$ such that $\chi_\lambda(z) = \theta(z)(\lambda)$.

3.2 Weyl group and odd reflections

The *Weyl group* W of Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is the Weyl group of the Lie algebra $\mathfrak{g}_{\bar{0}}$. Weyl group is generated by *even reflections*, which are reflections with respect to even roots of $\mathfrak{g}$. Define parity ω on W, such that $\forall r \in W$, $\omega(r) = 1$ if ω can be written as a product of even number of reflections and $\omega(r) = -1$ otherwise.

A linearly independent set of roots Σ of a Lie superalgebra is called a *base* if for each $\beta \in \Sigma$ there are $X_\beta \in \mathfrak{g}_\beta$ and $Y_\beta \in \mathfrak{g}_{-\beta}$ such that X_β, Y_β, $\beta \in \Sigma$ and $\mathfrak{h}$ generate $\mathfrak{g}$ and for any distinct $\beta, \gamma \in \Sigma$ we have $[X_\beta, Y_\gamma] = 0$.

Let $\{X_{\beta_i}\}$ be a base and let $h_{\beta_i} = [X_{\beta_i}, Y_{\beta_i}]$. We have the following relations $[h, X_{\beta_i}] = \beta_i(h) X_{\beta_i}$, $[h, Y_{\beta_i}] = -\beta_i(h) Y_{\beta_i}$, and $[X_{\beta_i}, Y_{\beta_j}] = \delta_{ij} h_{\beta_i}$. We define the *Cartan matrix* of a base Σ to be matrix $A_\Sigma = (\beta_i(h_{\beta_j})) = (a_{\beta_i \beta_j})$.

A base where the number of odd roots is minimal is called a *distinguished* root base. In that case, the Cartan matrix is also called *distinguished* Cartan matrix.

In the given base Σ, let $\alpha \in \Sigma$ is such that $a_{\alpha\alpha} = 0$ and $p(\alpha) = 1$. An *odd reflection* r_α is defined in [20] by:

$r_\alpha(\alpha) = -\alpha$, $r_\alpha(\beta) = \beta$ if $\alpha \neq \beta$ and $a_{\alpha\beta} = a_{\beta\alpha} = 0$, and

$r_\alpha(\beta) = \beta + \alpha$ if $a_{\alpha\beta} \neq 0$ and $a_{\beta\alpha} \neq 0$, for all $\beta \in \Sigma$.

We call a root $\alpha \in \Sigma$ *isotropic*, if $a_{\alpha\alpha} = 0$; otherwise, it is called *non-isotropic*.

Lemma 3.2.1 *(Serganova, [20]) Let $\mathfrak{g}$ be any basic classical Lie superalgebra. For an isotropic $\alpha \in \Delta_1$, the set $r_\alpha(\Sigma) = \{r_\alpha(\beta) | \beta \in \Sigma\}$ is a base and every base of $\mathfrak{g}$ can be obtained from a given one by a sequence of even and odd reflections.*

From this lemma, we obtain different Cartan matrices for the same Lie superalgebra.

We also need the following lemma:

Lemma 3.2.2 *(Serganova, [20]) Let Π and Π' be two bases, and $\Delta^+(\Pi)$, $\Delta^+(\Pi')$ be the corresponding sets of positive roots. If $\Pi' = r_\alpha(\Pi)$, for some root $\alpha \in \Pi$. Then*

$$\Delta^+(\Pi') = \Delta^+(\Pi) \cap \{-\alpha\} \setminus \{\alpha\},$$

or

$$\Delta^+(\Pi') = \Delta^+(\Pi) \cap \{-\alpha, -2\alpha\} \setminus \{\alpha, 2\alpha\},$$

depending on whether 2α is a root.

3.3 Representations of Lie superalgebras

The following definitions and results can be found in [15].

A *linear representation* ρ of $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a superspace $V = V_0 \oplus V_1$, such that the graded action of $\mathfrak{g}$ on V preserves parity, i.e. $\mathfrak{g}_{\bar{i}}(V_j) \subset V_{i+j}$ for $i, j \in \mathbb{Z}_2$ and $[g_1, g_2]v = g_1(g_2(v)) - (-1)^{p(g_1)p(g_2)} g_2(g_1(v))$, where $g(v) := \rho(g)(v)$. Then, we call V a $\mathfrak{g}$-*module*.

A $\mathfrak{g}$-module is a *weight module*, if $\mathfrak{h}$ acts semisimply on V. Then we can write $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$, where $V_\mu = \{m \in V \mid hm = \mu(h)m, \forall h \in \mathfrak{h}\}$. The elements of $P(V) = \{\mu \in \mathfrak{h}^* \mid V_\mu \neq 0\}$ are called *weights* of V.

For a fixed Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we fix $\mathfrak{b}$ to be a Borel containing $\mathfrak{h}$. We have $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$.

Let $\lambda \in \mathfrak{h}^*$, we define one dimensional even $\mathfrak{b}$-module $C_\lambda = \langle v_\lambda \rangle$ by letting $h(v_\lambda) = \lambda(h)v_\lambda$, $\forall h \in \mathfrak{h}$ and $\mathfrak{n}^+(v_\lambda) = 0$ with $deg(v_\lambda) = \bar{0}$.

We define the *Verma module* with highest weight λ as the induced module

$$M_\lambda = Ind_{\mathfrak{b}}^{\mathfrak{g}} C_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C_\lambda.$$

The $\mathfrak{g}$-module M_λ has a unique maximal submodule I_λ. The module $L_\lambda = M_\lambda / I_\lambda$ is called an *irreducible representation with highest weight* λ. It is proven in [10] that L_{λ_1} and L_{λ_2} are isomorphic iff $\lambda_1 = \lambda_2$ and that any finite dimensional irreducible representation of $\mathfrak{g}$ is one of L_λ.

A weight $\lambda \in \mathfrak{h}^*$ is called *integral* if $a_i \in \mathbb{Z}$ for all $i \neq s$, where s corresponds to an odd isotropic root in a distinguished base.

We denote by Λ the lattice of integral weights. It is the same as the weight lattice of the $\mathfrak{g}_{\bar{0}}$. The root lattice will be a sublattice of Λ and is denoted by Q. We know that any simple finite-dimensional $\mathfrak{g}$-module that is simisimple over $\mathfrak{h}$ and has weights in Λ, is a quotient of the Verma module with highest weight $\lambda \in \Lambda$ by a maximal submodule. λ is called *dominant* if this quotient is finite-dimensional.

Thus, for every dominant weight, there are two simple modules, that can be obtained from each other by change of parity. In order to avoid "parity chasing", the *parity function* is defined $p : \Lambda \to \mathbb{Z}_2$, such that $p(\lambda + \alpha) = p(\lambda) + p(\alpha)$ for all $\alpha \in \Delta$ and extend it linearly to all weights.

For a $\mathfrak{g}$-module V, there is a functor π such that $\pi(V)$ is the module with shifted parity, i.e. $\pi(V)_0 = V_1$ and $\pi(V)_1 = V_0$. We have $\mathcal{C} = \mathcal{F} \oplus \pi(\mathcal{F})$, where $\mathcal{C}$ is the category of finite-dimensional representations of $\mathfrak{g}$ and $\mathcal{F}$ is the full subcategory of $\mathcal{C}$ consisting of modules such that the parity of any weight space coincides with the parity of the corresponding weight.

The *Dynkin labels* of a linear function $\lambda \in \mathfrak{h}^*$ are defined by $a_s = (\lambda, \alpha_s)$, if α_s is an odd isotropic root in a distinguished base and $a_i = \frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}$ for other roots in the distinguished base.

The following result from [12] is analogous to the theorem on the highest weights for finite-dimensional irreducible representations of Lie algebras. We state it only in the case $\mathfrak{g} = F(4)$ or $G(3)$:

Lemma 3.3.1 *(Kac,[12]) For a distinguished Borel subalgebra of $\mathfrak{g} = F(4)$ or $G(3)$, let e_i, f_i, h_i be standard generators of $\mathfrak{g}$. Let $\lambda \in \mathfrak{g}^*$ and $a_i = \lambda(h_i)$. Then the representation L_λ is finite dimensional if and only if the following conditions are satisfied:*

For $\mathfrak{g} = F(4)$,
1) $a_i \in \mathbb{Z}_+$;
2) $k = \frac{1}{3}(2a_1 - 3a_2 - 4a_3 - 2a_4) \in \mathbb{Z}_+$;
3) $k < 4$: $a_i = 0$ for all i if $k = 0$; $k \neq 0$; $a_2 = a_4 = 0$ for $k = 2$; $a_2 = 2a_4 + 1$ for $k = 3$.

For $\mathfrak{g} = G(3)$,
1) $a_i \in \mathbb{Z}_+$;
2) $k = \frac{1}{2}(a_1 - 2a_2 - 3a_3) \in \mathbb{Z}_+$ for $\mathfrak{g} = G(3)$;
3) $k < 3$: $a_i = 0$ for all i if $k = 0$; $k \neq 0$; $a_2 = 0$ for $k = 2$.

For a base Σ, we denote L^Σ_λ the simple $\mathfrak{g}$-module with highest weight λ corresponding to the triangular decomposition obtained from Σ.

Lemma 3.3.2 *(Serganova, [20]) Let $\alpha \in \mathfrak{h}^*$. Let $\Sigma = \rho_\alpha(\Pi)$ for some odd reflection, then $L^\Sigma_{\lambda'} \cong L^\Pi_\lambda$, where $\lambda' = \lambda - \alpha$ if $\lambda(h_\alpha) \neq 0$ and $\lambda' = \lambda$ if $\lambda(h_\alpha) = 0$.*

Lemma 3.3.3 *(Serganova, [20]) A weight λ is dominant integral if and only if for any base Σ obtained from Π by a sequence of odd reflections, and for any $\beta \in \Sigma$ such that $\beta(h_\beta) = 2$, we have $\lambda'(h_\beta) \in \mathbb{Z}_{\geq 0}$ if β is even and $\lambda'(h_\beta) \in 2\mathbb{Z}_{\geq 0}$ if β is odd. Here, $L^\Sigma_{\lambda'} \cong L^\Pi_\lambda$.*

An irreducible finite-dimensional representation of $\mathfrak{g}$ is called *typical* if it splits as a direct summand in any finite dimensional representation of $\mathfrak{g}$. Equivalently, a finite-dimensional irreducible representation is typical if the central character uniquely determines it. Also it is known that λ is a highest weight of a typical representation if $(\lambda + \rho, \alpha) \neq 0$ for any isotropic $\alpha \in \Delta^+_1$.

Chapter 4

Structure and blocks for the exceptional Lie superalgebras $F(4)$ and $G(3)$

4.1 Description of $F(4)$

Let $\mathfrak{g}$ be the exceptional Lie superalgebra $F(4)$. The structure, the roots, simple root systems with corresponding Cartan matrices and Dynkin Diagrams, the Weyl group, and the integral dominant weights have been studied by V. Kac in [12] and we describe them in this section. Generators and relations for $\mathfrak{g} = F(4)$ are taken from [2].

The Lie superalgebra $\mathfrak{g} = F(4)$ has dimension 40 and rank 4. The even part $\mathfrak{g}_{\bar{0}}$ is $B_3 \oplus A_1 = \mathfrak{o}(7) \oplus \mathfrak{sl}(2)$ and the odd part $\mathfrak{g}_{\bar{1}}$ is isomorphic to $\mathfrak{spin}_7 \otimes \mathfrak{sl}_2$ as a $\mathfrak{g}_{\bar{0}}$-module. Here $\mathfrak{spin}_7$ is the eight dimensional spinor representation of $\mathfrak{so}(7)$ and $\mathfrak{sl}_2$ is the two dimensional representation of $\mathfrak{sl}(2)$. The even part $\mathfrak{g}_{\bar{0}}$ has dimension 24. The odd part $\mathfrak{g}_{\bar{1}}$ has dimension 16.

Its root system can be written in the space $\mathfrak{h}^* = \mathbb{C}^4$ in terms of the basis vectors $\{\epsilon_1, \epsilon_2, \epsilon_3, \delta\}$ that satisfy the relations:

$$(\epsilon_i, \epsilon_j) = \delta_{ij} \ , \ (\delta, \delta) = -3 \ , \ (\epsilon_i, \delta) = 0 \text{ for all } i, j.$$

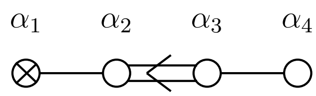
With respect to this basis, the root system $\Delta = \Delta_{\bar{0}} \oplus \Delta_{\bar{1}}$ is given by

$$\Delta_{\bar{0}} = \{\pm\epsilon_i \pm \epsilon_j; \pm\epsilon_i; \pm\delta\}_{i \neq j} \text{ and } \Delta_{\bar{1}} = \{\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \delta)\}.$$

For $F(4)$, we see that the isotropic roots are all odd roots.

We choose the simple roots to be $\Pi = \{\alpha_1 = \frac{1}{2}(-\epsilon_1 - \epsilon_2 - \epsilon_3 + \delta); \alpha_2 = \epsilon_3; \alpha_3 = \epsilon_2 - \epsilon_3; \alpha_4 = \epsilon_1 - \epsilon_2\}$. This will correspond to the following Dynkin diagram and Cartan matrix:

$F(4)$



$$\text{Cartan matrix} = A_\Pi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

We recall that the nodes $\circ$, $\otimes$, $\bullet$ are call respectively *white*, *gray*, and *black*, and they correspond respectively to even, odd isotropic, odd non-isotropic roots. The i-th and j-th nodes are not joined if $a_{ij} = a_{ji} = 0$ in the Cartan matrix and they are joined $max(|a_{ij}|, |a_{ji}|)$ times otherwise with arrows towards the i if $|a_{ij}| > |a_{ji}|$ and no arrows otherwise.

Up to W-equivalence, we have the following six simple root systems for $F(4)$ with Σ being the standard basis.

$\Sigma = \Pi = \{\alpha_1 = \frac{1}{2}(-\epsilon_1 - \epsilon_2 - \epsilon_3 + \delta); \alpha_2 = \epsilon_3; \alpha_3 = \epsilon_2 - \epsilon_3; \alpha_4 = \epsilon_1 - \epsilon_2\}$;
$\Sigma' = \{\alpha_1 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \delta); \alpha_2 = \frac{1}{2}(-\epsilon_1 - \epsilon_2 + \epsilon_3 + \delta); \alpha_3 = \epsilon_2 - \epsilon_3; \alpha_4 = \epsilon_1 - \epsilon_2\}$;
$\Sigma'' = \{\alpha_1 = \epsilon_3; \alpha_2 = \frac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 - \delta); \alpha_3 = \frac{1}{2}(-\epsilon_1 + \epsilon_2 - \epsilon_3 + \delta); \alpha_4 = \epsilon_1 - \epsilon_2\}$;
$\Sigma''' = \{\alpha_1 = \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta); \alpha_2 = \epsilon_2 - \epsilon_3; \alpha_3 = \frac{1}{2}(\epsilon_1 - \epsilon_2 + \epsilon_3 - \delta); \alpha_4 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 + \delta)\}$;
$\Sigma^{(4)} = \{\alpha_1 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \delta); \alpha_2 = \epsilon_2 - \epsilon_3; \alpha_3 = \epsilon_3; \alpha_4 = \delta\}$;
$\Sigma^{(5)} = \{\alpha_1 = \delta; \alpha_2 = \epsilon_2 - \epsilon_3; \alpha_3 = \epsilon_1 - \epsilon_2; \alpha_4 = \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3 - \delta)\}$.

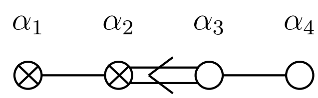
The following odd roots will be used later:
$\beta = \frac{1}{2}(-\epsilon_1 - \epsilon_2 - \epsilon_3 + \delta$;
$\beta' = \frac{1}{2}(-\epsilon_1 - \epsilon_2 + \epsilon_3 + \delta)$;
$\beta'' = \frac{1}{2}(-\epsilon_1 + \epsilon_2 - \epsilon_3 + \delta)$;
$\beta''' = \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta)$;

$\beta^{(4)} = \delta.$

The following are the Dynkin diagrams and Cartan matrices corresponding to above root systems:

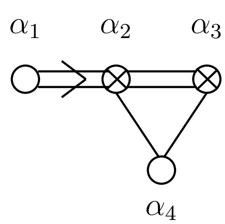
$$\text{Cartan matrix} = A_{\Sigma'} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$F(4)$



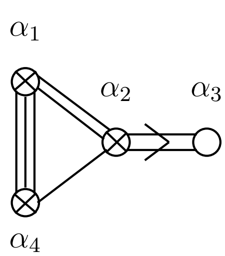
$$\text{Cartan matrix} = A_{\Sigma''} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 0 & 2 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

$F(4)$



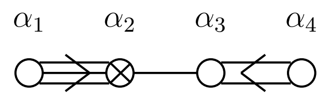
$$\text{Cartan matrix} = A_{\Sigma'''} = \begin{pmatrix} 0 & 3 & 2 & 0 \\ -3 & 0 & 1 & 0 \\ -2 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \end{pmatrix}$$

$F(4)$



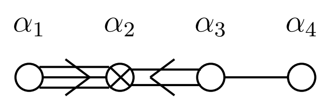
$$\text{Cartan matrix} = A_{\Sigma(iv)} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$F(4)$



$$\text{Cartan matrix} = A_{\Sigma(v)} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$F(4)$



With respect to the root system Σ, the positive roots are $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$, where $\Delta_0^+ = \{\delta, \epsilon_i, \epsilon_i \pm \epsilon_j \ |i < j\}$ and $\Delta_1^+ = \{\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 + \delta)\}$.

The Weyl vector is $\rho = \rho_0 - \rho_1 = \frac{1}{2}(5\epsilon_1 + 3\epsilon_2 + \epsilon_3 - 3\delta)$, where $\rho_0 = \frac{1}{2}(5\epsilon_1 + 3\epsilon_2 + \epsilon_3 + \delta)$ and $\rho_1 = 2\delta$.

The integral weight lattice, which is spanned by fundamental weights $\lambda_1 = \epsilon_1$, $\lambda_2 = \epsilon_1 + \epsilon_2$, $\lambda_3 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3)$, and $\lambda_4 = \frac{1}{2}\delta$ of $\mathfrak{g}_{\bar{0}}$, is $\Lambda = \frac{1}{2}\mathbb{Z}(\epsilon_1 + \epsilon_2 + \epsilon_3) \oplus \mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2 \oplus \frac{1}{2}\mathbb{Z}\delta$. Also, $\Lambda/Q \cong \mathbb{Z}_2$, where Q is the root lattice. We can define parity function on Λ, by setting $p(\frac{\epsilon_i}{2}) = 0$ and $p(\frac{\delta}{2}) = 1$.

Let T_i with $i = 1, 2, 3$, denote the generators of $\mathfrak{sl}(2)$. Let $M_{pq} = -M_{qp}$ with $1 \le p \ne q \le 7$ be generators of $\mathfrak{so}(7)$. Let $F_{\alpha\mu}$ with $\alpha = \pm 1$ and $1 \le \mu \le 8$ be the generators of $\mathfrak{g}_{\bar{1}}$. The bracket relations on $F(4)$ are given by:

$[T_l, T_m] = \mathbf{i}\epsilon_{lmk}T_k$; $[T_l, M_{pq}] = 0$;

$[M_{pq}, M_{rs}] = \delta_{qr}M_{ps} + \delta_{ps}M_{qr} - \delta_{pr}M_{qs} - \delta_{qs}M_{pr}$;

$[T_i, F_{\alpha\mu}] = \frac{1}{2}\sigma^i_{\beta\alpha}F_{\beta\mu}$; $[M_{pq}, F_{\alpha\mu}] = \frac{1}{2}(\gamma_p\gamma_q)_{\nu\mu}F_{\alpha\nu}$;

$[F_{\alpha\mu}, F_{\beta\nu}] = 2C^{(8)}_{\mu\nu}(C^{(2)}\sigma^i)_{\alpha\beta}T_i + \frac{1}{3}C^{(2)}_{\alpha\beta}(C^{(8)}\gamma_p\gamma_q)_{\mu\nu}M_{pq}$,

where σ^j with $j = 1, 2, 3$ are the Pauli matrices, $C^{(2)} = \mathbf{i}\sigma^2$ is the 2×2 charge conjugation matrix. The 8-dimensional matrices γ_p form a Clifford algebra $[\gamma_p, \gamma_p] = 2\delta_{pq}$ and $C^{(8)}$ is the 8×8 charge conjugation matrix.

Let I be the 2×2 identity matrix, then γ_p can be chosen as follows:

$\gamma_1 = \sigma^1 \otimes \sigma^3 \otimes I$;
$\gamma_2 = \sigma^1 \otimes \sigma^1 \otimes \sigma^3$;
$\gamma_3 = \sigma^1 \otimes \sigma^1 \otimes \sigma^1$;
$\gamma_4 = \sigma^2 \otimes I \otimes I$;
$\gamma_5 = \sigma^1 \otimes \sigma^2 \otimes I$;
$\gamma_6 = \sigma^1 \otimes \sigma^1 \otimes \sigma^2$;
$\gamma_7 = \sigma^3 \otimes I \otimes I$.

The Weyl groups W is generated by six reflections that can be defined on basis vectors as follows: for an arbitrary permutation $(ijk) \in S_3$, we get three possible permutations $\sigma_i(e_i) = e_i$, $\sigma_i(e_j) = e_k$, $\sigma_i(e_k) = e_j$, and other three defined $\tau_i(e_i) = -e_i$, $\tau_i(e_j) = e_j$, $\tau_i(e_k) = e_k$ all six fixing δ, also one permutation $\sigma(e_i) = e_i$ for all i and $s(\delta) = -\delta$. The Weyl group in this case is $W = ((\mathbb{Z}/2\mathbb{Z})^3 \rtimes S_3) \oplus \mathbb{Z}/2\mathbb{Z}$.

Lemma 4.1.1 *A weight $\lambda = a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3 + a_4\delta \in X^+$ is dominant integral weight of $\mathfrak{g}$ if and only if $\lambda + \rho \in \{(b_1, b_2, b_3|b_4) \in \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \mid b_1 > b_2 > b_3 > 0;\ b_4 \geq -\frac{1}{2};\ b_1 - b_2 \in \mathbb{Z}_{>0}; b_2 - b_3 \in \mathbb{Z}_{>0};\ b_4 = -\frac{1}{2} \implies b_1 = b_2 + 1\ \&\ b_3 = \frac{1}{2};\ b_4 = 0 \implies b_1 - b_2 - b_3 = 0\}$.*

Proof. Let $\lambda = a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3 + a_4\delta \in X^+$.

Since the even roots in Π are $\beta = \epsilon_3$, $\epsilon_2 - \epsilon_3$, $\epsilon_1 - \epsilon_2$. The relations $\lambda(\beta) \in \mathbb{Z}_{\geq 0}$ imply $a_1 \geq a_2 \geq a_3 \geq 0$ or equivalently $b_1 > b_2 > b_3 > 0$ and $b_1 - b_2 \in \mathbb{Z}_{>0}$, $b_2 - b_3 \in \mathbb{Z}_{>0}$.

Using Lemma 3.3.3, we apply odd reflections with respect to odd roots $\beta, \beta', \beta'', \beta'''$ to λ we obtain conditions on a_4 or equivalently on b_4.

The following are the only possibilities:

(1) If $\lambda(\beta) \neq 0$, $\lambda'(\beta') \neq 0$, $\lambda''(\beta'') \neq 0$, $\lambda'''(\beta''') \neq 0$, $\lambda^{(4)}(\delta) = 2a_4 - 4 \in \mathbb{Z}_{\geq 0}$, then $a_4 \geq 2$ or $a_4 \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Or, $b_4 \geq \frac{1}{2}$.

(2) If $\lambda(\beta) \neq 0$, $\lambda'(\beta') \neq 0$, $\lambda''(\beta'') \neq 0$, $\lambda'''(\beta''') = 0$, $\lambda^4 = \lambda'''$ and $\lambda^{(4)}(\delta) = 2a_4 - 3 \in \mathbb{Z}_{\geq 0}$, implying $a_4 \geq \frac{3}{2}$ and $a_4 \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Only $a_4 = \frac{3}{2}$ is possible and we have $a_1 - a_2 - a_3 = -\frac{1}{2}$. Or, $b_4 = 0$ and $b_1 - b_2 - b_3 = 0$.

(2) $\lambda(\beta) \neq 0$, $\lambda'(\beta') \neq 0$, $\lambda''(\beta'') = 0$ and $\lambda''' = \lambda''$, $\lambda'''(\beta''') = 0$, $\lambda^4 = \lambda'''$ and $\lambda^{(4)}(\delta) = 2a_4 - 3 \in \mathbb{Z}_{\geq 0}$, implying $a_4 \geq 1$ and $a_4 \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Only $a_4 = 1$ is possible and we have $a_1 = a_2$ and $a_3 = 0$. Or, $b_4 = -\frac{1}{2}$ and $b_1 = b_2 + 1$, $b_3 = \frac{1}{2}$.

□

4.2 Description of $G(3)$

Let $\mathfrak{g}$ be the exceptional Lie superalgebra $G(3)$. The structure, the roots, simple root systems with corresponding Cartan matrices and Dynkin Diagrams, the Weyl group and integral dominant weights have been studied in [12] and we describe them in this section. Generators and relations for $\mathfrak{g} = G(3)$ are taken from [2].

The Lie superalgebra $\mathfrak{g} = G(3)$ is a 31-dimensional exceptional Lie superalgebra of defect 1. We have $\mathfrak{g}_{\bar{0}} = G_2 \oplus A_1$, where G_2 is the exceptional Lie algebra, and an irreducible $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ that is isomorphic to $\mathfrak{g}_2 \otimes \mathfrak{sl}_2$, where $\mathfrak{g}_2$ is the seven dimensional representation of G_2 and $\mathfrak{sl}_2$ is the two dimensional representation of $\mathfrak{sl}_(2)$. The $\mathfrak{g}_{\bar{0}}$ has dimension 17 and rank 3. And $\mathfrak{g}_{\bar{1}}$ has dimension 14.

We can realize its root system in the space $\mathfrak{h}^* = \mathbb{C}^3$ endowed with basis $\{\epsilon_1, \epsilon_2, \epsilon_3, \delta\}$ with $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ and with the bilinear form defined by:

$$(\epsilon_1, \epsilon_1) = (\epsilon_2, \epsilon_2) = -2(\epsilon_1, \epsilon_2) = -(\delta, \delta) = 2.$$

With respect to the above basis, the root system $\Delta = \Delta_{\bar{0}} \oplus \Delta_{\bar{1}}$ is given by

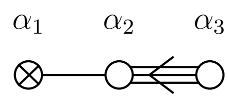
$$\Delta_{\bar{0}} = \{\pm\epsilon_i; \pm 2\delta; \epsilon_i - \epsilon_j\}_{i \neq j} \text{ and } \Delta_{\bar{1}} = \{\pm\delta; \pm\epsilon_i \pm \delta\}.$$

Up to W equivalence, there is are five systems of simple roots for $G(3)$ given by:

$\Pi = \{\alpha_1 = \epsilon_3 + \delta; \alpha_2 = \epsilon_1; \alpha_3 = \epsilon_2 - \epsilon_1\}$,
$\Pi' = \{-\epsilon_3 - \delta; -\epsilon_2 + \delta; \epsilon_2 - \epsilon_1\}$,
$\Pi'' = \{\epsilon_1; \epsilon_2 - \delta; -\epsilon_1 + \delta\}$,

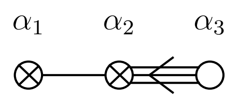
$\Pi''' = \{\delta; \epsilon_1 - \delta; \epsilon_2 - \epsilon_1\}$.

$G(3)$



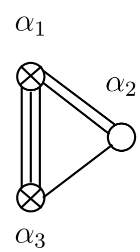
$$\text{Cartan matrix} = A_{\Pi} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$$

$G(3)$



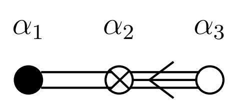
$$\text{Cartan matrix} = A_{\Sigma'} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 3 \\ 0 & -1 & 2 \end{pmatrix}$$

$G(3)$



$$\text{Cartan matrix} = A_{\Sigma''} = \begin{pmatrix} 0 & -3 & 2 \\ -3 & 0 & 1 \\ -2 & -1 & 2 \end{pmatrix}$$

$G(3)$



$$\text{Cartan matrix} = A_{\Sigma'''} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 0 & 3 \\ 0 & -1 & 2 \end{pmatrix}$$

The positive roots with respect to Σ are $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$, where

$$\Delta_0^+ = \{\epsilon_1;\ \epsilon_2;\ -\epsilon_3;\ 2\delta;\ \epsilon_2 - \epsilon_1;\ \epsilon_1 - \epsilon_3;\ \epsilon_2 - \epsilon_3\} \text{ and } \Delta_1^+ = \{\delta; \pm\epsilon_i + \delta\}.$$

The Weyl vector is $\rho = \rho_0 - \rho_1 = 2\epsilon_1 + 3\epsilon_2 - \frac{5}{2}\delta$, where $\rho_0 = 2\epsilon_1 + 3\epsilon_2 + \delta$ and $\rho_1 = \frac{7}{2}\delta$.

Let T_i with $i = 1, 2, 3$, denote the generators of $\mathfrak{sl}(2)$. Let $M_{pq} = -M_{qp}$ with $1 \leq p \neq q \leq 7$ be generators of $\mathfrak{so}(7)$. Let $F_{\alpha\mu}$ with $\alpha = \pm 1$ and $1 \leq \mu \leq 8$ be the generators of $\mathfrak{g}_{\bar{1}}$. The bracket relations on $F(4)$ are given by:

$[T_l, T_m] = \mathbf{i}\epsilon_{lmk}T_k$; $[T_l, M_{pq}] = 0$;

$[M_{pq}, M_{rs}] = \delta_{qr}M_{ps} + \delta_{ps}M_{qr} - \delta_{pr}M_{qs} - \delta_{qs}M_{pr} + \frac{1}{3}\xi_{pqu}\xi_{rsv}M_{uv}$;

$[T_i, F_{\alpha\mu}] = \frac{1}{2}\sigma^i_{\beta\alpha}F_{\beta\mu}$; $[M_{pq}, F_{\alpha r}] = \frac{2}{3}\delta_{qr}F_{\alpha p} - \frac{2}{3}\delta_{pr}F_{\alpha q} + \frac{1}{3}\xi_{pqrs}F_{\alpha s}$;

$[F_{\alpha p}, F_{\beta q}] = 2\delta_{pq}(C\sigma^i)_{\alpha\beta}T_i + \frac{2}{3}C_{\alpha\beta}M_{pq}$.

Here, σ^j with $j = 1, 2, 3$ are the Pauli matrices, $C = \mathbf{i}\sigma^2$ is the 2×2 charge conjugation matrix.

The embedding $G_2 \subset \mathfrak{so}(7)$ is obtained by imposing constrains on the generators M_{pq} given by $\xi_{ijk}M_{ij} = 0$, where ξ_{ijk} are completely antisymmetric and whose non-vanishing components are

$$\xi_{123} = \xi_{145} = \xi_{176} = \xi_{246} = \xi_{257} = \xi_{347} = \xi_{365} = 1.$$

The tensors ζ_{pars} are completely antisymmetric and whose non-vanishing components are given by

$$\zeta_{1247} = \zeta_{1265} = \zeta_{1364} = \zeta_{1375} = \zeta_{2345} = \zeta_{2376} = \zeta_{4576} = 1.$$

The Weyl groups W is the group $W = D_6 \oplus \mathbb{Z}/2\mathbb{Z}$, where D_6 is the dihedral group of order 12. It is generated by four reflections: for an arbitrary permutation $(ijk) \in S_3$, we get three $\sigma_i(e_i) = e_i$, $\sigma_i(e_j) = e_k$, $\sigma_i(e_k) = e_j$, one reflection defined by $\tau(e_i) = -e_i$ for all i and $\tau(\delta) = \delta$, also one reflection $\sigma(e_i) = e_i$ for all i and $\sigma(\delta) = -\delta$.

The integral weight lattice for $\mathfrak{g}_{\bar{0}}$ is $\Lambda = \mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2 \oplus \mathbb{Z}\delta$, which is the lattice spanned by the fundamental weights $\omega_1 = \delta$, $\omega_2 = \epsilon_1 + \epsilon_2$, $\omega_3 = \epsilon_1 + 2\epsilon_2$ of $\mathfrak{g}_{\bar{0}}$. Also,

$\Lambda/Q \cong \{1\}$, where Q is the root lattice.

We can define the parity function on Λ, by setting $p(\epsilon_i) = 0$ and $p(\delta) = 1$.

Using Lemma 3.3.3 and Lemma 3.3.1, it is convenient to write down dominant weights in terms of basis $\{\epsilon_1, \epsilon_2, \epsilon_3, \delta | \epsilon_1 + \epsilon_2 + \epsilon_3 = 0\}$:

Lemma 4.2.1 *A weight $\lambda = a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3 + a_4\delta \in X^+$ is a dominant integral weight of $\mathfrak{g} = G(3)$ if and only if $\lambda + \rho \in \{(b_1, b_2, b_3, b_4) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times (\frac{1}{2} + \mathbb{Z}) | b_2 > b_1 > b_3, 2b_1 - b_2 - b_3 > 0;$ either $b_4 > 0;$ or if $b_4 = -\frac{1}{2},$ then $2b_1 - b_2 - b_3 = 1;$ $b_4 \neq -\frac{3}{2};$ if $b_4 = -\frac{5}{2},$ then $b_1 - b_3 = 2$ and $b_2 - b_3 = 3\}$.*

Equivalently in terms of basis $\{\epsilon_1, \epsilon_2, \delta\}$, we can describe the dominant weights as $X^+ = \{\lambda = (a_1, a_2 | a_3) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \mid 2a_1 \geq a_2 \geq a_1 \geq 0;\ a_3 \geq 3;$ or if $a_3 = 2 \implies a_2 = 2a_1;\ a_3 \neq 1;\ a_3 = 0 \implies a_1 = a_2 = 0\}$.

Or equivalently, for $\lambda \in X^+$ if $\lambda + \rho \in \{(b_1, b_2, b_3) \in \mathbb{Z} \times \mathbb{Z} \times (\frac{1}{2} + \mathbb{Z}) | 2b_1 > b_2 > b_1 > 0;$ either $b_3 > 0;$ or if $b_3 = -\frac{1}{2},$ then $b_2 = 2b_1 - 1;$ $b_3 \neq -\frac{3}{2};$ if $b_3 = -\frac{5}{2},$ then $b_1 = 2$ and $b_2 = 3\}$.

Proof. Using Lemma 3.3.3 as in the case of $F(4)$. □

4.3 Associated variety and the fiber functor

Let G_0 be simply-connected connected Lie group with Lie algebra $\mathfrak{g}_{\bar{0}}$, for a Lie superalgebra $\mathfrak{g}$. Let $X = \{x \in \mathfrak{g}_{\bar{1}} | [x, x] = 0\}$. Then X is a G_0-invariant Zariski closed set in $\mathfrak{g}_{\bar{1}}$, called the *self-commuting cone* in $\mathfrak{g}_{\bar{1}}$, see [3].

Let S to be the set of subsets of mutually orthogonal linearly independent isotropic roots of Δ_1. So the elements of S are $A = \{\alpha_1, ..., \alpha_k | (\alpha_i, \alpha_j) = 0\}$. Let $S_k = \{A \in S|\ |A| = k\}$ and $S_0 = \emptyset$.

Lemma 4.3.1 *([3]) Every G_0-orbit on X contains an element $x = X_{\alpha_1} + \cdots + X_{\alpha_k}$ with $X_i \in \mathfrak{g}_{\alpha_i}$ for some set $\{\alpha_1, ..., \alpha_k\} \in S$.*

The number k in this lemma is called *rank* of x.

The following theorem is true for all contragradient Lie superalgebras.

Theorem 4.3.2 *([3]) There are finitely many G_0-orbits on X. These orbits are in one-to-one correspondence with W-orbits in S.*

The correspondence in the above theorem is given by taking an element $A = \{\alpha_1, ..., \alpha_k | (\alpha_i, \alpha_j) = 0\}$ of S into $G_0 x$, where $x = x_1 + \cdots + x_k \in X$ is such that $x_i \in \mathfrak{g}_{\alpha_i}$. This correspondence doesn't depend on the choice of x. For $\mathfrak{g} = F(4)$ or $G(3)$, k is equal to 1. Therefore, we have the following corollary.

Corollary 4.3.3 *For an exceptional Lie superalgebra $\mathfrak{g} = F(4)$ or $G(3)$, the rank of $x \in X \setminus \{0\}$ is 1. And every x is G_0 conjugate to some $X_\alpha \in \mathfrak{g}_\alpha$ for some isotropic root α with $[h, X_\alpha] = \alpha(h) X_\alpha$ for all $h \in \mathfrak{h}$.*

Proof. It follows from the proof of this theorem, that for exceptional Lie superalgebras, X has two $G_{\bar{0}}$-orbits: $\{0\}$ and the orbit of a highest vector in $\mathfrak{g}_{\bar{1}}$. The set S also consists of two W-orbits: $\emptyset$ and the set of all isotropic roots in Δ. For F_4, the set of all isotropic roots is Δ_1. For $G(3)$, this set is $\Delta_1 \setminus \{\delta\}$. □

Let $X_k = \{x \in X, rank x = k\}$. Then $X = \cup_{k \leq def \mathfrak{g}} X_k$ and $\bar{X} = \cup_{j \leq k} X_j$.

For a $\mathfrak{g}$-module M and for $x \in X$, define the fiber $M_x = Ker\, x / Im\, x$ as the cohomology of x in M as in [23]. The *associated variety* X_M of M is defined in [23] by setting $X_M = \{x \in X | M_x \neq 0\}$.

Lemma 4.3.4 *([3])X_M is a G_0-invariant Zariski closed subset of X, if M is finite dimensional.*

Lemma 4.3.5 *([3]) If M is finite dimensional $\mathfrak{g}$-module, then for all $x \in X$, $sdim M = sdim M_x$.*

We can assume that $x = \sum X_{\alpha_i}$ with $X_{\alpha_i} \in \mathfrak{g}_\alpha$ for $i = 1, \ldots, n$. Then, there is a base containing the roots α_i for $i = 1, \ldots, n$. We define quotient as in [3] by $\mathfrak{g}_x = C_{\mathfrak{g}}(x)/[x, \mathfrak{g}]$, where $C_{\mathfrak{g}}(x) = \{a \in \mathfrak{g} | [a, x] = 0\}$ is the centralizer of x in $\mathfrak{g}$, since $[x, \mathfrak{g}]$ is an ideal in $C_{\mathfrak{g}}(x)$. The superalgebra $\mathfrak{g}_x$ has a Cartan subalgra $\mathfrak{h}_x = (Ker\, \alpha_1 \cap \cdots \cap Ker\, \alpha_k)/(kh_{\alpha_1} \oplus \cdots \oplus kh_{\alpha_k})$ and a root system is equal to $\Delta_x = \{\alpha \in \Delta | (\alpha, \alpha_i) = 0 \text{ for } \alpha \neq \pm\alpha_i \text{ and } i = 1, ..., k\}$.

Since $Ker\, x$ is $C_{\mathfrak{g}}(x)$-invariant and $[x, \mathfrak{g}] Ker\, x \subset Im\, x$, M_x has a structure of a $\mathfrak{g}_x$-module. We can define $U(\mathfrak{g})^x$ to be subalgebra of ad_x-invariants. Then we have an isomorphism $U(\mathfrak{g}_x) \cong U(\mathfrak{g})^x / [x, U(\mathfrak{g})]$, which is given by $U(\mathfrak{g}_x) \to U(\mathfrak{g})^x \to U(\mathfrak{g}_x)/[x, U(\mathfrak{g})]$. The corresponding projection $\phi : U(\mathfrak{g})^x \to U(\mathfrak{g}_x)$ is such that $\phi(Z(\mathfrak{g})) \subset Z(\mathfrak{g}_x)$ and thus it can be restricted to a homomorphism of rings $\phi : Z(\mathfrak{g}) \to Z(\mathfrak{g}_x)$. The dual of this map is denoted by $\check{\phi} : Hom(Z(\mathfrak{g}_x), \mathbb{C}) \to Hom(Z(\mathfrak{g}), \mathbb{C})$.

Thus, $M \to M_x$ defines a functor from the category of $\mathfrak{g}$-modules to the category of $\mathfrak{g}_x$-modules, which is called the *fiber functor*. By construction, if central character of M is equal to χ, then the central character of M_x is in $\check{\phi}^{-1}\{\chi\}$.

Theorem 4.3.6 *([3]) For a finite dimensional $\mathfrak{g}$-module with central character χ and $at(\chi) = k$. $X_M \subset \bar{X}_k$*

For $x = \sum X_{\alpha_i}$ with $X_{\alpha_i} \in \mathfrak{g}_\alpha$ for $i = 1, \dots, n$, we can chose a base containing the roots α_i for $i = 1, \dots, n$. This gives $\mathfrak{h}_x^* = (\mathbb{C}\alpha_1 \oplus \dots \oplus \mathbb{C}\alpha_k)^\perp / (\mathbb{C}\alpha_1 \oplus \dots \oplus \mathbb{C}\alpha_k)$ and a natural projection $p : (\mathbb{C}\alpha_1 \oplus \dots \oplus \mathbb{C}\alpha_k)^\perp \to \mathfrak{h}_x^*$. Then ν, $\nu' \in p^{-1}(\mu)$ imply $\chi_\nu = \chi_{\nu'}$ and $\check{\phi}^{-1}(\chi_\mu) = \chi_\nu$, see [22].

4.4 Blocks

Let $\mathfrak{g} = F(4)$ or $G(3)$.

Consider a graph with vertices the elements of X^+ and arrows between each two vertices if they have a non-split extension. The connected components of this graph are called *blocks*. All the simple components of an indecomposable module belong to the same block, then we say that the indecomposable module itself belongs to this block.

For Lie superalgebras, the generalized central character may correspond to more than one simple $\mathfrak{g}$-module. The category $\mathcal{F}$ decomposes into direct sum of full subcategories called $\mathcal{F}^\chi$, where $\mathcal{F}^\chi$ consists of all finite dimensional modules with (generalized) central character χ. Let F^χ be the set of all weights λ such that $L_\lambda \in \mathcal{F}^\chi$. We will call the subcategories $\mathcal{F}^\chi$ blocks, since we will prove they are blocks in the above sense.

In this section, we describe all integral dominant weights in the *atypical* blocks, which are blocks containing more than one simple $\mathfrak{g}$-module.

Denote $\lambda^w := w(\lambda + \rho) - \rho$.

Lemma 4.4.1 *(Serganova, [22]) There is a set of odd roots $\alpha_1, \dots, \alpha_k \in \Delta_1$ and $\mu \in \mathfrak{h}^*$ a weight, such that $(\alpha_i, \alpha_j) = 0$ and $(\mu + \rho, \alpha_i) = 0$. Then $m_\chi = \{\mu \in \mathfrak{h}^* | \chi = \chi_\mu\} = \cup_{w \in W} (\mu + \mathbb{C}\alpha_1 + \dots + \mathbb{C}\alpha_k)^w$.*

If $k = 0$ in the above lemma, then χ is called *typical*, and if $k > 0$, then it is *atypical*. The number k is called *the degree of atypicality* of μ.

Lemma 4.4.2 *The degree of atypicality of any weight for* $\mathfrak{g}$ *is* ≤ 1.

Recall that X is the cone of self-commuting elements defined above. For a $\mathfrak{g}$-module M and for $x \in X$, recall that $M_x = Ker\, x/Im\, x$ is the fiber as the cohomology of x in M and $\mathfrak{g}_x = C_{\mathfrak{g}}(x)/[x, \mathfrak{g}]$.

Denote by ρ_l the Weyl vector and by $\omega_1 = \frac{1}{3}(2\beta_1 + \beta_2)$ and $\omega_2 = \frac{1}{3}(\beta_1 + 2\beta_2)$ the fundamental weights for $\mathfrak{sl}(3)$, where $\beta_1 = \epsilon_1 - \epsilon_2$ and $\beta_2 = \epsilon_2 - \epsilon_3$ are the simple roots of $\mathfrak{sl}(3)$.

The following lemma allows us to parametrize the atypical blocks by of $\mathfrak{g} = F(4)$ and label them $\mathcal{F}^{(a,b)}$.

Lemma 4.4.3 *If* $\mathfrak{g} = F(4)$ *and* $x \in X$, *then* $\mathfrak{g}_x \cong \mathfrak{sl}(3)$. *For any simple module* $M \in \mathcal{F}^{\chi}$ *for atypical* χ, *we have*

$$M_x \cong L_{a,b}^{\oplus m_1} \oplus L_{b,a}^{\oplus m_2} \oplus \Pi(L_{a,b})^{\oplus m_3} \oplus \Pi(L_{b,a})^{\oplus m_4},$$

where $L_{a,b}$ *is a simple* $\mathfrak{sl}(3)$*-module with highest weight* μ *of* $\mathfrak{sl}(3)$ *such that* $\mu + \rho_l = a\omega_1 + b\omega_2$. *Here,* $a = 3n + b$ *with* $(a, b) \in \mathbb{N} \times \mathbb{N}$ *and* $n \in \mathbb{Z}_{\geq 0}$ *such that* $a = b$ *or* $a > b$.

Proof. By Lemma 4.3.1 and Lemma 4.4.2, we can take $x = X_{\alpha_1}$ with $X_{\alpha_1} \in \mathfrak{g}_{\alpha_1}$ and $\alpha_1 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \delta) \in \Delta_1$. Then the root system for $\mathfrak{g}_x$ is $\Delta_x = \{\epsilon_i - \epsilon_j\}_{i \neq j}$, $i, j = 1, 2, 3$ and it correspond to the root system of $\mathfrak{sl}(3)$ proving the first part.

Let $M \in \mathcal{F}^{\chi}$ be the simple $\mathfrak{g}$-module with highest weight λ, then $(\lambda + \rho, \beta) = 0$ for some $\beta \in \Delta$. We choose $w \in W$, with $w(\beta) = \alpha_1$, then $(w(\lambda + \rho), \alpha_1) = 0$ and $w(\lambda + \rho) - \rho$ is dominant with respect to $\mathfrak{sl}(3)$.

Let $w(\lambda+\rho) = \lambda' + \rho = a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3 + a_4\delta$. Also, let $a + \frac{1}{2} = (w(\lambda+\rho), \beta_1) = a_1 - a_2$ and $b + \frac{1}{2} = (w(\lambda + \rho), \beta_2) = a_2 - a_3$.

Now we have $w(\lambda+\rho) = \lambda' + \rho = a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3 + a_4\delta = (a_1 + a_4, a_2 + a_4, a_3 + a_4|0) - 2a_4\alpha_1 = (\frac{2a+b}{3} + \frac{1}{2}, \frac{-a+b}{3}, -\frac{a+2b}{3} - \frac{1}{2}|0) - 2a_4\alpha_1 = a\omega_1 + b\omega_2 + \rho_l - 2a_4\alpha_1$.

Similarly, there is $\sigma \in W$ such that $\sigma(w(\lambda+\rho)) = \sigma(\lambda'+\rho) = \lambda''+\rho = \sigma(a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3 + a_4\delta) = (-a_3, -a_2, -a_1|a_4) = (-a_4-a_3, -a_4-a_2, -a_4-a_1|0) + 2a_4\alpha_1 = (\frac{2b+a}{3}+\frac{1}{2}, \frac{-b+a}{3}, -\frac{b+2a}{3}-\frac{1}{2}|0) + 2a_4\alpha_1 = b\omega_1 + a\omega_2 + \rho_l + 2a_4\alpha_1$.

This implies, $\lambda' \in p^{-1}(a\omega_1 + b\omega_2)$ or $\lambda'' \in p^{-1}(b\omega_1 + a\omega_2)$, which correspond to the dominant integral weights of $\mathfrak{g}_x = \mathfrak{sl}(3)$ since a and b are positive integers. Also, $a - b = -3(a_2 + a_4)$, implying that $a = 3n + b$.

From above, we have that $\lambda', \lambda'' \in p^{-1}(\mu)$, where $\mu = a\omega_1 + b\omega_2$ or $b\omega_1 + a\omega_2$ is a dominant integral weight of $\mathfrak{g}_x = \mathfrak{sl}(3)$ such that $a = 3n + b$. From Lemma 4.4.1, we have $\chi_\lambda = \chi_{\lambda'} = \chi_{\lambda''}$. By construction of $\check{\phi}$ above, the central character of M_x is in the set $\check{\phi}^{-1}\{\chi_\lambda\}$. Also, if $\lambda \in p^{-1}(\mu)$, then $\check{\phi}(\chi_\mu) = \chi_\lambda$. Therefore, M_x contains Verma modules over $\mathfrak{g}_x$ with highest weights in $p(\lambda')$ for any λ' such that $\chi_\lambda = \chi_{\lambda'}$, proving the lemma.

Conversely, for $(a, b) \in \mathbb{N} \times \mathbb{N}$ with $a - b = 3n$, there is a dominant weight $\lambda \in F^\chi$ with $\lambda + \rho = (a + b + 1)\epsilon_1 + (b + 1)\epsilon_2 + \epsilon_3 + (\frac{a+2b}{3} + 1)\delta$, such that $p(\lambda) = a\omega_1 + b\omega_2$. □

Similarly, denote by ρ_l is the Weyl vector and by $\omega_1 = \frac{1}{2}\beta_1$ be the fundamental weight of $\mathfrak{sl}(2)$, where $\beta_1 = \epsilon_1 - \epsilon_2$ is the simple root of $\mathfrak{sl}(2)$.

The following lemma allows parametrize the atypical blocks by of $\mathfrak{g} = G(3)$ by $a = 2n + 1$, with $n \in \mathbb{Z}_{\geq 0}$ and label them $\mathcal{F}^a$.

Lemma 4.4.4 *If $\mathfrak{g} = G(3)$ and $x \in X$, then $\mathfrak{g}_x \cong \mathfrak{sl}(2)$. For any simple $M \in \mathcal{F}^\chi$ for atypical χ, we have*

$$M_x \cong L_a^{\oplus m_1} \oplus \Pi(L_{a)})^{\oplus m_2},$$

where L_a is a simple $\mathfrak{sl}(2)$-module with dominant weight μ with $\mu + \rho_l = a\omega_1$. Here, $a = 2n + 1$ with $n \in \mathbb{Z}_{\geq 0}$.

Proof. Similarly, as for $F(4)$, by Lemma 4.3.1 and Lemma 4.4.2, we can choose $x = X_{\alpha_1}$ with $X_{\alpha_1} \in \mathfrak{g}_{\alpha_1}$ and $\alpha_1 = -\epsilon_3 + \delta \in \Delta_1^+$. Then the root system for $\mathfrak{g}_x$ is $\Delta_x = \{\epsilon_i - \epsilon_j\}_{i \neq j}$, $i, j = 1, 2$ and it correspond to the root system of $\mathfrak{sl}(2)$ proving the first part.

Let $M \in \mathcal{F}^\chi$ be the simple $\mathfrak{g}$-module with highest weight λ, then $(\lambda + \rho, \beta) = 0$ for some $\beta \in \Delta$. We choose $w \in W$, with $w(\beta) = \alpha_1$, then $(w(\lambda + \rho), \alpha_1) = 0$ and $w(\lambda + \rho) - \rho$ is dominant with respect to $\mathfrak{sl}(2)$.

Let $w(\lambda+\rho) = \lambda' + \rho = a_1\epsilon_1 + a_2\epsilon_2 + a_3\delta$. Also, let $a = (w(\lambda+\rho), \beta_1) = a_1 - a_2$.

Now we have $w(\lambda+\rho) = \lambda' + \rho = a_1\epsilon_1 + a_2\epsilon_2 + a_3\delta = (a_1 - a_3, a_2 - a_3|0) + 2a_3\alpha_1 = (\frac{a}{2} + \frac{1}{2}, -\frac{a}{2} - \frac{1}{2}|0) + a_3\alpha_1 = a\omega_1 + \rho_l + a_3\alpha_1$.

This implies, $\lambda' \in p^{-1}(a\omega_1)$, where $a\omega_1$ correspond to the dominant integral weights of $\mathfrak{g}_x = \mathfrak{sl}(2)$ since a is a positive integer. Also, $a = a_1 - a_2 = 2a_3 - 2a_2$,where $a_3 \in \frac{1}{2} + \mathbb{Z}$, implying that $a = 2n+1$ with $n \in \mathbb{Z}_{\geq 0}$.

From above, we have that $\lambda' \in p^{-1}(\mu)$, where $\mu = a\omega_1$ is a dominant integral weight of $\mathfrak{g}_x = \mathfrak{sl}(2)$ such that $a = 2n+1$. From Lemma 4.4.1, we have $\chi_\lambda = \chi_{\lambda'}$. By construction of $\check{\phi}$ above, the central character of M_x is in the set $\check{\phi}^{-1}\{\chi_\lambda\}$. Also, if $\lambda \in p^{-1}(\mu)$, then $\check{\phi}(\chi_\mu) = \chi_\lambda$. Therefore, M_x contains Verma modules over $\mathfrak{g}_x$ with highest weights in $p(\lambda')$ for any λ' such that $\chi_\lambda = \chi_{\lambda'}$, proving the lemma.

Conversely, for $a \in \mathbb{N}$ with $a = 2n+1$ and $a \geq 0$, there is a dominant weight $\lambda \in F^\chi$ with $\lambda + \rho = (a+1)\epsilon_1 + (2a+1)\epsilon_2 + \epsilon_3 + (\frac{3a}{2}+1)\delta$, such that $p(\lambda) = a\omega_1$. □

Now we can describe the dominant integral weights in the atypical blocks.

In the following two theorems, for every c, we describe a unique dominant weight λ_c in $\mathcal{F}^{(a,a)}$ (or $\mathcal{F}^a$), such that c is equal to the last coordinate of $\lambda_c + \rho$. For λ_c in $\mathcal{F}^{(a,b)}$ with $a \neq b$, c is equal to the last coordinate of $\lambda_c + \rho$ if c is positive and to the last coordinate of $\lambda_c + \rho$ with negative sign if c is negative.

For $F(4)$, we denote: $t_1 = \frac{2a+b}{3}, t_2 = \frac{a+2b}{3}, t_3 = \frac{a-b}{3}$. Note that if $a = b$, $t_1 = t_2 = a$ and $t_3 = 0$.

Theorem 4.4.5 *Let $\mathfrak{g} = F(4)$.*

(1) It is possible to parametrize the dominant weights λ with $L_\lambda \in \mathcal{F}^{(a,a)}$ by $c \in \frac{1}{2}\mathbb{Z}_{\geq -1} \setminus \{a, \frac{a}{2}, 0\}$ for $a > 1$ and by $c \in \frac{1}{2}\mathbb{Z}_{\geq 3} \cup \{\frac{-3}{2}\}$ for $a = 1$, such that $(\lambda + \rho, \delta) = 3c$.

(2) Similarly, it is possible to parametrize the dominant weights λ with $L_\lambda \in \mathcal{F}^{(a,b)}$ by $c \in \frac{1}{2}\mathbb{Z} \setminus \{t_2, \frac{t_1}{2}, t_3, -\frac{t_3}{2}, -\frac{t_2}{2}, -t_1, \}$, such that $(\lambda + \rho, \delta) = 3sign(c)\, c$.

Proof. To prove (1), take $\lambda + \rho = (2a+1)\epsilon_1 + (a+1)\epsilon_2 + \epsilon_3 + (a+1)\delta$, then $\lambda \in F^{(a,a)}$ by Lemma 4.4.3, and $(\lambda + \rho, \alpha) = 0$ for $\alpha = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta)$.

By Lemma 4.4.1 and Lemma 4.4.2, all dominant integral weights in $F^{(a,a)}$ are in $A = \{w(\lambda+\rho+k\alpha)-\rho | w \in W, k \in \mathbb{Z}\}$. The last coordinate of dominant integral weights in A is $c := a+1+\frac{k}{2} \in \frac{1}{2}\mathbb{Z}$.

Since for $c \in \{\pm a, \pm\frac{a}{2}, 0\}$, the element of A with $k = 2(c-a-1)$ is not dominant for any $w \in W$, we consider the following eight intervals for c: (1) $a < c$; (2)$\frac{a}{2} < c < a$; (3) $0 < c < \frac{a}{2}$; (4) $c = -\frac{1}{2}$; (5) $c < -a$; (6) $-a < c < -\frac{a}{2}$; (7) $-\frac{a}{2} < c < 0$; (8) $c = \frac{1}{2}$.

For every c, in the above intervals, we define corresponding Weyl group element as follows: (1) $w_c = id$, (2) $w_c = \tau_3$, (3) $w_c = \sigma_1\tau_3$, (4) $w_c = \sigma_1\sigma_2\tau_2\tau_3$; (5) $w_c = \sigma$, (6) $w_c = \tau_3\sigma$, (7) $w_c = \sigma_1\tau_3\sigma$, (8) $w_c = \sigma_1\sigma_2\tau_2\tau_3\sigma$. The last four cases give us same dominant weights as in the first four cases.

Since $\lambda+\rho = a(e_1-e_3)+(a+1)\alpha$, the dominant integral weight λ_c corresponding to this c can be written as follows:

For $c \in J_1 = (a, \infty)$, $\lambda_c+\rho = a(e_1-e_3)+2c\beta_1$, where $\beta_1 = \frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\delta) = w_c(\alpha)$;

For $c \in J_2 = (\frac{a}{2}, a)$, $\lambda_c+\rho = a(e_1+e_3)+2c\beta_2$, where $\beta_2 = \frac{1}{2}(\epsilon_1+\epsilon_2-\epsilon_3+\delta) = w_c(\alpha)$;

For $c \in J_3 = (0, \frac{a}{2})$, $\lambda_c+\rho = a(e_1+e_2)+2c\beta_3$, where $\beta_3 = \frac{1}{2}(\epsilon_1-\epsilon_2+\epsilon_3+\delta) = w_c(\alpha)$;

We also have the following cases:

Let $a = 1$. For $c = -\frac{3}{2}$, $\lambda_c+\rho = e_1-e_3-2c\beta_0$, where $\beta_0 = \frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3-\delta)$.

Let $a > 1$. For $c = -\frac{1}{2}$, $\lambda_c+\rho = a(e_1+e_2)-2c\beta_0$, where $\beta_0 = \frac{1}{2}(\epsilon_1-\epsilon_2+\epsilon_3-\delta)$.

For (2), we take $\lambda \in F^{(a,b)}$, such that $\lambda+\rho = t_1\epsilon_1+t_2\epsilon_2+t_3\epsilon_3$. By Lemma 4.4.3, $\lambda \in X^+$ and $(\lambda+\rho,\alpha) = 0$ for $\alpha = \frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3+\delta)$.

By Lemma 4.4.1 and Lemma 4.4.2, all dominant integral weights in $F^{(a,a)}$ are in $A = \{w(\lambda+\rho+\frac{k}{2}\alpha)-\rho | w \in W, k \in \mathbb{Z}\}$. Let $c := \frac{k}{2} \in \frac{1}{2}\mathbb{Z}$.

Since for $c \in \{t_2, \frac{t_1}{2}, t_3, -\frac{t_3}{2}, -\frac{t_2}{2}, -t_1, \}$, the element of A with $k = 2c$ is not dominant for any $w \in W$, we consider the following eight intervals for c:

(1) $t_2 < c$; (2)$\frac{t_1}{2} < c < t_2$; (3) $t_3 < c < \frac{t_1}{2}$; (4) $0 \leq c < t_3$; (5) $-\frac{t_3}{2} < c < 0$; (6) $-\frac{t_2}{2} < c < -\frac{t_3}{2}$; (7) $-t_1 < c < -\frac{t_2}{2}$; (8) $c < -t_1$.

For every c, in the above intervals, we define corresponding Weyl group element as follows: (1) $w_c = \tau_3\sigma_1\tau_3$, (2) $w_c = \sigma_1\tau_3$, (3) $w_c = \tau_3$, (4) $w_c = id$; (5) $w_c = \sigma$, (6) $w_c = \sigma_3\sigma$, (7) $w_c = \sigma_1\sigma_3\sigma$, (8) $w_c = \tau_3\sigma_1\sigma_3\sigma$.

Then, it is easy to check using Lemma 4.4.3 that $w_c \in W$ is the unique element such that $\lambda_c + \rho = w_c(\lambda + \rho + c\delta) \in X^+$.

In each case, we list the dominant integral weights in $F^{(a,b)}$, parametrized by c:

For $c \in I_1 = (t_2, \infty)$, $\lambda_c+\rho = t_1\epsilon_1 - t_3\epsilon_2 - t_2\epsilon_3 + 2c\beta_1$, where $\beta_1 = \frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\delta) = w_c(\alpha)$;

For $c \in I_2 = (\frac{t_1}{2}, t_2)$, $\lambda_c+\rho = t_1\epsilon_1 - t_3\epsilon_2 + t_2\epsilon_3 + 2c\beta_2$, where $\beta_2 = \frac{1}{2}(\epsilon_1+\epsilon_2-\epsilon_3+\delta) = w_c(\alpha)$;

For $c \in I_3 = (t_3, \frac{t_1}{2})$, $\lambda_c+\rho = t_1\epsilon_1 + t_2\epsilon_2 - t_3\epsilon_3 + 2c\beta_3$, where $\beta_3 = \frac{1}{2}(\epsilon_1-\epsilon_2+\epsilon_3+\delta) = w_c(\alpha)$;

For $c \in I_4 = [0, t_3)$, $\lambda_c+\rho = t_1\epsilon_1 + t_2\epsilon_2 + t_3\epsilon_3 + 2c\beta_4$, where $\beta_4 = \frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3+\delta) = w_c(\alpha)$;

For $c \in I_5 = (-\frac{t_3}{2}, 0)$, $\lambda_c + \rho = t_1\epsilon_1 + t_2\epsilon_2 + t_3\epsilon_3 - 2c\beta_5$, where $\beta_5 = \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta) = -w_c(\alpha)$;

For $c \in I_6 = (-\frac{t_2}{2}, -\frac{t_3}{2})$, $\lambda_c+\rho = t_2\epsilon_1 + t_1\epsilon_2 + t_3\epsilon_3 - 2c\beta_6$, where $\beta_6 = \beta_3 = -w_c(\alpha)$;

For $c \in I_7 = (-t_1, -\frac{t_2}{2})$, $\lambda_c+\rho = t_2\epsilon_1 + t_3\epsilon_2 + t_1\epsilon_3 - 2c\beta_7$, where $\beta_7 = \beta_2 = -w_c(\alpha)$;

For $c \in I_8 = (-\infty, -t_1)$, $\lambda_c+\rho = t_2\epsilon_1 + t_3\epsilon_2 - t_1\epsilon_3 - 2c\beta_8$, where $\beta_8 = \beta_1 = -w_c(\alpha)$.

The uniqueness of λ_c in both cases follows from Lemma 4.4.3.

If $\lambda \in F^{(a,b)}$ is a dominant integral weight, then, we can also write:

$\lambda + \rho \in \{(\frac{2a+b}{3} + c, -\frac{a-b}{3} + c, -\frac{a+2b}{3} + c|c)$ for $c \in I_1 = (\frac{a+2b}{3}, \infty)$;

$(\frac{2a+b}{3}+c, -\frac{a-b}{3}+c, \frac{a+2b}{3}-c|c)$ for $c \in I_2 = (\frac{2a+b}{6}, \frac{a+2b}{3})$;

$(\frac{2a+b}{3}+c, \frac{a+2b}{3}-c, -\frac{a-b}{3}+c|c)$ for $c \in I_3 = (\frac{a-b}{3}, \frac{2a+b}{6})$;

$(\frac{2a+b}{3}+c, \frac{a+2b}{3}-c, \frac{a-b}{3}-c|c)$ for $c \in I_4 = [0, \frac{a-b}{3})$;

$(\frac{2a+b}{3}+c, \frac{a+2b}{3}-c, \frac{a-b}{3}-c|-c)$ for $c \in I_5 = (\frac{b-a}{6}, 0)$;

$(\frac{a+2b}{3}-c, \frac{2a+b}{3}+c, \frac{a-b}{3}-c|-c)$ for $c \in I_6 = (-\frac{a+2b}{6}, \frac{b-a}{6})$;

$(\frac{a+2b}{3}-c, \frac{a-b}{3}-c, \frac{2a+b}{3}+c|-c)$ for $c \in I_7 = (\frac{2a+b}{3}, -\frac{a+2b}{6})$;

$(\frac{a+2b}{3}-c, \frac{a-b}{3}-c, -\frac{2a+b}{3}+c|-c)$ for $c \in I_8 = (-\infty, \frac{2a+b}{3})\}$.

□

Remark 4.4.6 *For every $\lambda_c \in F^{(a,a)}$ or $F^{(a,b)}$, such that $(\lambda+\rho,\delta) = 3c$ or $(\lambda+\rho,\delta) = -3c$ with $c \in J_i$ or I_i, we have corresponding $\beta_i \in \Delta^+$ is such that $(\lambda+\rho,\beta_i) = 0$. This $\beta_i = w_c(\alpha)$, where w_c's are defined in the above proof.*

Theorem 4.4.7 *Let $\mathfrak{g} = G(3)$.*

(1) It is possible to parametrize the dominant weights λ with $L_\lambda \in \mathcal{F}^1$ by $c \in (\frac{1}{2}+\mathbb{Z}_{\geq 2}) \cup \{-\frac{5}{2}\}$, such that $(\lambda+\rho,\delta) = 3c$.

(2) Similarly, for $a > 0$, it is possible to parametrize the dominant weights λ with $L_\lambda \in \mathcal{F}^a$ by $c \in (-\frac{1}{2}+\mathbb{Z}_{\geq 0}) \setminus \{0, \frac{a}{2}, \frac{3a}{2}\}$, such that $(\lambda+\rho,\delta) = 3c$.

Proof. (1) Let $a = 0$. In this case, take $\lambda+\rho = 2\epsilon_1 + 3\epsilon_2 + \frac{5}{2}\delta$, then $\lambda \in F^1$ by Lemma 4.4.4, and $(\lambda+\rho,\alpha) = 0$ for $\alpha = \epsilon_1+\epsilon_2+\delta$. By Lemma 4.4.1 and Lemma 4.4.2, all dominant integral weights in F^1 are in $A = \{w(\lambda+\rho+k\alpha)-\rho | w \in W, k \in \mathbb{Z}\}$. The last coordinate of dominant integral weights in A is $c := \frac{5}{2}+k \in \frac{1}{2}+\mathbb{Z}$, so $k \in \mathbb{Z}$.

Since for $c = \pm\frac{3}{2}$, the element of A with $k = c-\frac{5}{2}$ is not dominant for any $w \in W$, we consider the following intervals for c: (1) $\frac{3}{2} < c$; (2) $c = -\frac{5}{2}$; (3) $c < -\frac{3}{2}$; (4) $c = \frac{5}{2}$.

For every c, in the above intervals, we define corresponding Weyl group element as follows: (1) $w_c = id$, (2) $w_c = \sigma_3\tau$, (3) $w_c = \sigma_3\tau\sigma$, (4) $w_c = \sigma$. The last two cases correspond to the same dominant weights as in the first two cases.

Since $\lambda+\rho=\frac{1}{2}(e_2-e_1)+\frac{5}{2}\alpha$, the dominant integral weight λ_c with last coordinate c can be written as follows:

$c \in J_1=(\frac{3}{2},\infty)$, then $\lambda_c+\rho=\frac{1}{2}(e_2-e_1)+c\alpha$, $\beta=\epsilon_1+\epsilon_2+\delta=w_c(\alpha)$;

$c=-\frac{5}{2}$, $\lambda_c+\rho=(2,3,-\frac{5}{2})$, $\beta=-\epsilon_1-\epsilon_2+\delta=w_c(\alpha)$.

(2) Let $a>0$. In this case, take $\lambda=\epsilon_1+(a+1)\epsilon_2+(1+\frac{a}{2})\delta-\rho$, then $\lambda\in F^a$ by Lemma 4.4.4, and $(\lambda+\rho,\alpha)=0$ for $\alpha=\epsilon_1+\epsilon_2+\delta$. By Lemma 4.4.1 and Lemma 4.4.2, all dominant integral weights in F^a are in $A=\{w(\lambda+\rho+k\alpha)-\rho|w\in W,k\in\mathbb{Z}\}$. The last coordinate of dominant integral weights in A is $c:=\frac{a}{2}+1+k\in\frac{1}{2}+\mathbb{Z}$, so $k\in\mathbb{Z}$.

Since for $c\in\{\pm\frac{a}{2},\pm\frac{3a}{2}\}$, the element of A with $k=c-1-\frac{a}{2}$ is not dominant for any $w\in W$, we consider the following intervals for c: (1) $\frac{3a}{2}<c$; (2) $\frac{a}{2}<c<\frac{3a}{2}$; (3) $0<c<\frac{a}{2}$; (4) $c=-\frac{1}{2}$; (5) $c<-\frac{3a}{2}$; (6) $-\frac{3a}{2}<c<-\frac{a}{2}$; (7) $-\frac{a}{2}<c<0$; (8) $c=\frac{1}{2}$.

For every c, in the above intervals, we define corresponding Weyl group element as follows: (1) $w_c=id$, (2) $w_c=\sigma_1\tau$, (3) $w_c=\sigma_1\sigma_2\tau$, (4) $w_c=\sigma_2$; (5) $w_c=\tau\sigma_3\sigma$, (6) $w_c=\sigma_3\sigma_1\sigma$, (7) $w_c=\sigma_2\sigma$, (8) $w_c=\sigma_3\sigma_2\tau\sigma$. The last four cases correspond to the same dominant weights as in the first four cases.

Since $\lambda_c+\rho=\frac{a}{2}(e_2-e_1)+c\alpha$, the dominant integral weight λ_c corresponding to c can be written as follows:

For $c\in J_1=(\frac{3}{2}a,\infty)$, $\lambda_c+\rho=(c-\frac{a}{2},c+\frac{a}{2},c)$, $\beta=\epsilon_1+\epsilon_2+\delta=w_c(\alpha)$;

For $c\in J_2=(\frac{1}{2}a,\frac{3}{2}a)$, $\lambda_c+\rho=(a,c+\frac{a}{2},c)$, $\beta=\epsilon_2+\delta=w_c(\alpha)$;

For $c\in J_3=(0,\frac{1}{2}a)$, $\lambda_c+\rho=(c+\frac{a}{2},a,c)$, $\beta=\epsilon_1+\delta=w_c(\alpha)$;

For $c=-\frac{1}{2}$, $\lambda_c+\rho=(\frac{a}{2}+\frac{1}{2},a,-\frac{1}{2})$, $\beta=-\epsilon_1+\delta=w_c(\alpha)$.

The uniqueness of λ_c in both cases follows from Lemma 4.4.4.

□

Remark 4.4.8 *For every $\lambda_c \in F^1$ or F^a, such that $(\lambda_c + \rho, \delta) = 2c$, we have corresponding $\beta = w_c(\alpha) \in \Delta^+$ is such that $(\lambda_c + \rho, \beta) = 0$, where w_c's are defined in the above proof.*

We have the following theorem:

Theorem 4.4.9 *For $\mathfrak{g} = F(4)$, the atypical blocks are parametrized by dominant weights μ of $\mathfrak{sl}(3)$, such that $\mu + \rho_l = a\omega_1 + b\omega_2$ with $a = 3n + b$. Here, $b \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{\geq 0}$; ω_1 and ω_2 are the fundamental weights of $\mathfrak{sl}(3)$. We labeled blocks by $\mathcal{F}^{(a,b)}$.*

For $\mathfrak{g} = G(3)$, the atypical blocks are parametrized by dominant weights μ of $\mathfrak{sl}(2)$, such that $\mu + \rho_l = a\omega_1$ with $a = 2n+1$. Here, $n \in \mathbb{Z}_{\geq 0}$; ω_1 is the fundamental weight of $\mathfrak{sl}(2)$. We labeled blocks by $\mathcal{F}^a$.

Proof. Follows from Lemma 4.4.3 and Lemma 4.4.4. $\square$

Chapter 5

Geometric induction and translation functor

5.1 Geometric induction

We fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, and let V be a $\mathfrak{b}$-module. Denote by $\mathcal{V}$ the vector bundle $G \times_B V$ over the generalized grassmannian G/B. The space of sections of $\mathcal{V}$ has a natural structure of a $\mathfrak{g}$-module, in other words the sheaf of sections of $\mathcal{V}$ is a $\mathfrak{g}$-sheaf.

Let C_λ denote the one dimensional representation of B with weight λ. Denote by $\mathcal{O}_\lambda$ the line bundle $G \times_B C_\lambda$ on the flag (super)variety G/B. See [8].

The functor Γ_i from the category of $\mathfrak{b}$-modules, to the category of $\mathfrak{g}$-modules was defined by $\Gamma_i(G/B, V) = \Gamma_i(G/B, \mathcal{V}) := (H^i(G/B, \mathcal{V}^*))^*$ in [19].

Denote by $\varepsilon(\lambda)$ the *Euler characteristic* of the sheaf $\mathcal{O}_\lambda$ belonging to the category $\mathcal{F}$:

$$\varepsilon(\lambda) = \sum_{i=0}^{dim(G/B)_0} (-1)^i [\Gamma_i(G/B, \mathcal{O}_\lambda) : L_\mu][L_\mu].$$

The following properties of this functor will be useful and have been studied in [9]:

Lemma 5.1.1 *([9]) If*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

is a short exact sequence of B-modules, then one has the following long exact sequence

$$\ldots \longrightarrow \Gamma_1(G/B, W) \longrightarrow \Gamma_0(G/B, U) \longrightarrow \Gamma_0(G/B, V) \longrightarrow \Gamma_0(G/B, W) \longrightarrow 0$$

Lemma 5.1.2 *([9]) The module $\Gamma_0(G/B, \mathcal{V})$ is the maximal finite-dimensional quotient of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} V$.*

Lemma 5.1.3 *([17]) For λ typical weight, Theorem 2.3.1 holds.*

Corollary 5.1.4 *([9]) For every dominant weight λ, the module L_λ is a quotient of $\Gamma_0(G/B, \mathcal{O}_\lambda)$ with $[\Gamma_0(G/B, \mathcal{O}_\lambda) : L_\lambda] = 1$.*

Lemma 5.1.5 *([9]) If L_μ occurs in $\Gamma_i(G/B, \mathcal{O}_\lambda)$ with non-zero mulitiplicity, then $\mu + \rho = w(\lambda + \rho) - \sum_{\alpha \in I} \alpha$ for some $w \in W$ of length i and $I \subset \Delta_1^+$.*

Lemma 5.1.6 *([17]) Assume for an even root γ in the base of B, $\beta + \rho = r_\gamma(\alpha + \rho)$. Then $\Gamma^i(G/B, \mathcal{O}_\alpha) \cong \Gamma^{i+1}(G/B, \mathcal{O}_\beta)$.*

Lemma 5.1.7 *If $L_\lambda \in \mathcal{F}^\chi$, then*

$$\sum_i (-1)^i sdim \Gamma_i(G/B, \mathcal{O}_\lambda) = 0.$$

Proof. We follow similar argument as in lemma 5.2 in [23]. Let $\lambda \in F^\chi$, then for $t \in \mathbb{Z}$, the weight $\lambda + t\delta$ is integral. The weight $\lambda + t\delta$ is typical for almost all t. From Lemma 5.1.3, we have $\Gamma_i(G/B, L_\lambda) = 0$ for $i > 0$ and $\Gamma_0(G/B, L_\lambda) = L_\lambda$. Also, since $\lambda + t\delta$ is typical we have:

$$\sum_i (-1)^i sdim \Gamma_i(G/B, L_{\lambda+t\delta}) = sdim(\lambda + t\delta) = 0.$$

On the other hand, we have $chL_{\lambda+t\delta} = e^{t\delta} chL_\lambda$. Therefore, from Theorem 6.1.3 we have:

$$\sum_i (-1)^i sdim \Gamma_i(G/B, L_{\lambda+t\delta}) = p(t)$$

for some polynomial $p(t)$. We have $p(t) = 0$ for almost all $t \in \mathbb{Z}$. Thus, $p(0) = 0$. □

Lemma 5.1.8 *([9]) If M is a $\mathfrak{g}$-module and V is a B-module, the following holds:*

$$\Gamma_i(G/B, V \otimes M) = \Gamma_i(G/B, V) \otimes M.$$

Lemma 5.1.9 *([8]) For every dominant weight λ, let $p(w)$ be the parity of w, then*

$$\sum_i (-1)^i ch\Gamma_i(G/B, \mathcal{O}_\lambda) = (-1)^{p(w)} \sum_i (-1)^i ch\Gamma_i(G/B, \mathcal{O}_{w(\lambda+\rho)-\rho})$$

Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, and $\mathfrak{n}$ is the nilpotent part of $\mathfrak{b}$. Consider the projection

$$\phi : U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n}) \to U(\mathfrak{n}^-)U(\mathfrak{h})$$

with kernel $U(\mathfrak{g})\mathfrak{n}$. The restriction of ϕ to $Z(\mathfrak{g})$ induces the injective homomorphism of centers $Z(\mathfrak{g}) \to Z(\mathfrak{h})$. Denote the dual map by $\Phi : Hom(Z(\mathfrak{h}), \mathbb{C}) \to Hom(Z(\mathfrak{g}), \mathbb{C})$.

Lemma 5.1.10 *([9]) If V is an irreducible $\mathfrak{b}$-module admitting a central character χ, then the $\mathfrak{g}$-module $\Gamma_i(G/B, V)$ admits the central character $\Phi(\chi)$.*

Let $M^\chi = \{m \in M | (z - \chi(z))^N v = 0, z \in Z\}$.

Corollary 5.1.11 *([9]) For any finite-dimensional $\mathfrak{g}$-module M, let M^χ denote the component with generalized central character χ. Then*

$$\Gamma_i(G/B, (V \otimes M)^{\Phi^{-1}(\chi)}) = (\Gamma_i(G/B, V) \otimes M)^\chi.$$

5.2 Translation functor

A *translation functor* $T_{\chi,\tau} : \mathcal{F}^\chi \to \mathcal{F}^\tau$ is defined by

$$T_{\chi,\tau}(V) = (V \otimes \mathfrak{g})^\tau, \text{ for } V \in \mathcal{F}^\chi.$$

Here, $(M)^\tau$ denotes the projection of M to the block $\mathcal{F}^\tau$. Since $\mathfrak{g} \cong \mathfrak{g}^*$, the left adjoint functor of $T_{\chi,\tau}$ is defined by

$$T^*_{\chi,\tau}(V) = (V \otimes \mathfrak{g})^\chi, \text{ for } V \in \mathcal{F}^\tau.$$

For convenience, when its clear, we will denote $T := T_{\chi,\tau}$.

For $\lambda \in \mathcal{F}^\chi$, we also define

$$T_{\chi,\tau}(\mathcal{O}_\lambda) = (\mathcal{O}_\lambda \otimes \mathfrak{g})^{\Phi^{-1}(\chi)},$$

where $V^{\Phi^{-1}(\chi)}$ is the component with generalized character lying in $\Phi^{-1}(\chi)$.

Lemma 5.2.1 *([9]) We have* $\Gamma^i(G/B, T(\mathcal{O}_\lambda)) = T(\Gamma^i(G/B, \mathcal{O}_\lambda))$*, where* T *is a translation functor.*

Lemma 5.2.2 *Assume* $T(\mathcal{O}_\lambda)$ *has a filtration with quotients* $\mathcal{O}_{\sigma_i}$*,* $i = 1, 2$ *with* σ_1 *is dominant and* σ_2 *acyclic. Then for all* $i \geq 0$*, we have* $\Gamma_i(G/B, T(\mathcal{O}_\lambda)) = \Gamma_i(G/B, \mathcal{O}_{\sigma_1})$*.*

Proof. We have an exact sequence of vector bundles:

$$0 \to \mathcal{O}_{\sigma_2} \to T(\mathcal{O}_{\lambda_1}) \to \mathcal{O}_{\sigma_1} \to 0,$$

Since σ_2 is acyclic, $\Gamma_i(G/B, \mathcal{O}_{\sigma_2}) = 0$ for all $i \geq 0$. Thus $\Gamma_i(G/B, T(\mathcal{O}_\lambda)) = \Gamma_i(G/B, \mathcal{O}_{\sigma_1})$.

□

Lemma 5.2.3 *Let* X *is an indecomposable* $\mathfrak{g}$*-module with unique simple quotient* L_λ*, such that if* L_σ *is a subquotient of* X *implies* $\sigma < \lambda$*, then there is a surjection* $\Gamma_0(G/B, \mathcal{O}_\lambda) \to X$*.*

Proof. Follows from Lemma 5.1.2. □

Lemma 5.2.4 *For* $\lambda \in F^{(a,b)}$ *(or* F^a*) let* $T(L_\lambda) = L_{\lambda'}$ *and* T *an equivalence of categories* $\mathcal{F}^{(a,b)}$ *and* $\mathcal{F}^{(a+1,b+1)}$ *(or* F^a *and* F^{a+2}*) preserving the order on weights. We have* $T(\Gamma_0(G/B, \mathcal{O}_\lambda)) = \Gamma_0(G/B, \mathcal{O}_{\lambda'})$*.*

Proof. From Lemma 5.1.2, $\Gamma_0(G/B, \mathcal{O}_\lambda)$ is a maximal indecomposable module with quotient L_λ. Since T is an equivalence of categories, $T(\Gamma_0(G/B, \mathcal{O}_\lambda))$ is an indecomposable module with quotient L'_λ. All other simple subquotients of $T(\Gamma_0(G/B, \mathcal{O}_\lambda))$ are L_σ with $\sigma < \lambda'$.

By Lemma 5.2.3, we have a surjection $\Gamma_0(G/B, \mathcal{O}_{\lambda'}) \to T(\Gamma_0(G/B, \mathcal{O}_\lambda))$. In a similar way we have a surjection $\Gamma_0(G/B, \mathcal{O}_\lambda) \to T^*(\Gamma_0(G/B, \mathcal{O}_{\lambda'}))$. This proves the equality. □

Lemma 5.2.5 *([23]) For any* $\mathfrak{g}$*-modules* M *and* N*, we have* $(M \otimes N)_x = M_x \otimes N_x$*.*

Lemma 5.2.6 *Let* $T = T_{\chi,\tau}$*. For* $\mathfrak{g} = F(4)$*, let* $\chi = (a, b)$ *and* $\tau = (a+1, b+1)$ *and for* $\mathfrak{g} = G(3)$*, let* $\chi = a$ *and* $\tau = a + 2$*. Then* $T(L_\lambda) \neq 0$ *for any* $\lambda \in \mathcal{F}^\chi$*.*

Proof. From definition of translation functor, we have $T(L_\lambda) = (L_\lambda \otimes \mathfrak{g})^\tau$. From Lemma 5.2.5, we have $(M \otimes \mathfrak{g})_x = M_x \otimes \mathfrak{g}_x$ for any $\mathfrak{g}$-module M. Thus, $T(L_\lambda)_x = (L_\lambda \otimes \mathfrak{g})^\tau_x = ((L_\lambda)_x \otimes \mathfrak{g}_x)^{\Phi^{-1}(\tau)}$, where $(L_\lambda)_x$ is an $\mathfrak{g}_x$-module. And $\mathfrak{g}_x \cong sl(3)$ or $sl(2)$. This implies $T(L_\lambda) \neq 0$. □

Lemma 5.2.7 *Let $\lambda \in F^\chi$ be dominant. Assume there is exactly one dominant weight $\mu \in F^\tau$ of the form $\lambda + \gamma$ with $\gamma \in \Delta$. Then we have $T(L_\lambda) = L_\mu$.*

Proof. By definition, $T(L_\lambda) = (L_\lambda \otimes \mathfrak{g})^\tau$.

By assumption, μ is the only $\mathfrak{b}$-singular weight in $T(L_\lambda)$. Since $T(L_\lambda)$ is contregradient, $T(L_\lambda) = L_\lambda \oplus M$. If $M \neq 0$, it must have another $\mathfrak{b}$-singular vector. Hence, $M = 0$ and the statement follows.

□

Theorem 5.2.8 *Assume, for every $L_\lambda \in \mathcal{F}^\chi$, there is a unique $L_{\lambda'} = T(L_\lambda) \in \mathcal{F}^\tau$. Also assume for each $L_{\lambda'} \in \mathcal{F}^\tau$, there are at most two weights λ_1 and λ_2 in F^χ such that $\lambda' + \gamma = \lambda_i$, $i = 1, 2$ with $\lambda_1 = \lambda > \lambda_2$ and $\mathfrak{g} \in \Delta$. Then the categories $\mathcal{F}^\chi$ and $\mathcal{F}^\tau$ are equivalent.*

Proof. We show that translation functor T defined by $T(L_\lambda) = (L_\lambda \otimes \mathfrak{g})^\tau$ is an equivalence of categories $\mathcal{F}^\chi$ and $\mathcal{F}^\tau$.

It is sufficient to show that we have exact and mutually adjoint functors T and T^*, which induce bijection between simple modules. Since we already have that T maps simple modules in $\mathcal{F}^\chi$ to simple modules in $\mathcal{F}^\tau$, we just need to show that T^* also maps simple modules to simple modules such that $T \cdot T^* = id_{\mathcal{F}^\tau}$ and $T^* \cdot T = id_{\mathcal{F}^\chi}$.

Thus, we just show that $T^*(L_{\lambda'}) = L_\lambda$ for each $\lambda' \in F^\tau$.

We have $Hom_{\mathfrak{g}}(T^*(L_{\lambda'}), L_\mu) = Hom_{\mathfrak{g}}(L_{\lambda'}, T(L_\mu)) = Hom_{\mathfrak{g}}(L_{\lambda'}, L_{\mu'}) = \mathbb{C}$ for $\lambda = \mu$ and 0 otherwise.

Similarly, we have $Hom_{\mathfrak{g}}(L_\mu, T^*(L_{\lambda'})) = Hom_{\mathfrak{g}}(T(L_\mu), L_{\lambda'}) = Hom_{\mathfrak{g}}(L_{\mu'}, L_{\lambda'}) = \mathbb{C}$ for $\mu = \lambda$ and 0 otherwise.

The $\mathfrak{b}$-singular vectors in $T^*(L_{\lambda'})$ have weights of the form $\lambda = \lambda' + \gamma$ with $\gamma \in \Delta$.

By assumption of the theorem, all $\mathfrak{b}$-singular vectors in $T^*(L_{\lambda'})$ are less than or equal to λ in the standard order. Since $T^*(L_{\lambda'})$ is contragradient and the multiplicity of L_λ in $T^*(L_{\lambda'})$ is one, we must have $T^*(L_{\lambda'}) = L_\lambda \oplus M$ for some module M. Since $Hom(M, L_\xi) = 0$ for any $\xi \in F^\chi$, we have $M = 0$.

□

Lemma 5.2.9 *For the distinguished Borel B and dominant weight λ, we have*

$$\Gamma_i(G/B, \mathcal{O}_\lambda) = 0 \textit{ for } i > 1.$$

Proof. Consider the bundle $\pi : G/B \to G/P$, where P is the parabolic subgroup obtained from B by adding all negative even simple roots. The even dimension of G/P equals 1.

On the other hand, $\pi_*^0(\mathcal{O}_\lambda) = L_\lambda(\mathfrak{p})$ and $\pi_*^i(\mathcal{O}_\lambda) = 0$, since λ is dominant. Hence, by Leray spectral sequence (see [9]), we have

$$\Gamma_i(G/B, \mathcal{O}_\lambda) = \Gamma_i(G/P, L_\lambda(\mathfrak{p})) = 0,$$

where $\mathfrak{p}$ is the corresponding parabolic subalgebra. □

Chapter 6

Generic weights

6.1 Character and superdimension formulae for generic weights

For $\mathfrak{g}$-module M_λ and $V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu$ the weight decomposition of its quotient, we define the *character* of V by

$$chV = \sum_{\mu\in P(V)} (dimV_\mu)e^\mu.$$

If $\lambda \in \Lambda^+$ is a typical weight, then the following character formula is proven by Kac and it holds for the exceptional Lie superalgebras:

$$chL_\lambda = \frac{D_1 \cdot e^\rho}{D_0} \cdot \sum_{w\in W} signw \cdot e^{w(\lambda+\rho)},$$

where $D_0 = \prod_{\alpha\in\Delta_0^+}(1-e^{-\alpha})^{dim\mathfrak{g}_\alpha}$ and $D_1 = \prod_{\alpha\in\Delta_1^+}(1+e^{-\alpha})^{dim\mathfrak{g}_\alpha}$.

The *generic* weights are defined in [16] to be the weights far from the walls of the Weyl chamber. Here is a more precise definition:

Definition 6.1.1 *We define $\lambda_c \in \mathcal{F}^\chi$ with $\chi = (a,b)$ or $\chi = a$ to be a generic weight if $c > \frac{a+2b}{3} + \frac{3}{2}$ or $c < -\frac{3}{2} - \frac{2a+b}{3}$ for $F(4)$ and if $c > \frac{3a}{2} - 2$ for $G(3)$.*

The following theorems will be used later in the proofs:

Theorem 6.1.2 *(Penkov, [16]) For a generic weight λ, the following formula holds:*

$$chL_\lambda = S(\lambda) = \frac{D_1 \cdot e^\rho}{D_0} \cdot \sum_{w \in W} sign\, w \cdot w(\frac{e^{\lambda+\rho}}{\prod_{\alpha \in A(\lambda)}(1+e^{-\alpha})}),$$

where $A(\lambda)$ is the maximal set of mutually orthogonal linearly independent real isotropic roots α such that $(\lambda+\rho, \alpha) = 0$. The set $A(\lambda)$ is one-element set for $F(4)$ and $G(3)$.

Theorem 6.1.3 *(Penkov, [16]) For a finite-dimensional $\mathfrak{b}$-module V, the following formula holds:*

$$\sum_i (-1)^i ch(H^i(G/B, V^*)^*) = \frac{D_1 \cdot e^\rho}{D_0} \cdot \sum_{w \in W} sign\, w \cdot w(ch(V)e^\rho),$$

We first prove the following theorem for generic weights. In later section, we establish it for all weights.

Theorem 6.1.4 *Let $\mathfrak{g} = F(4)$ (or $G(3)$). Let $\lambda \in F^{(a,b)}$ (or F^a) be a generic dominant weight and and $\mu + \rho_l = a\mu_1 + b\mu_2$ (or and $\mu + \rho_l = a\mu_1$), then following superdimension formula holds:*

$$sdim\, L_\lambda = (-1)^{p(\mu)} 2 dim\, L_\mu(\mathfrak{g}_x).$$

Proof. From Theorem 6.1.2, we have:

$$chL_\lambda = S(\lambda) = \sum_{\mu \in S} (-1)^{p(\mu)} chL_\mu(\mathfrak{g}_{\bar{0}}),$$

where $S = \{\mu = \lambda - \sum \alpha | \alpha \in \Delta_{\bar{1}}^+ , \alpha \neq \beta = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta)\}$.

Computing using definition of $sdimV$, the above formula and the classical Weyl character formula we get

$$sdimL_\lambda = \sum_{\mu \in S} (-1)^l dimL_\mu(\mathfrak{g}_{\bar{0}}),$$

where $S = \{\mu = \lambda - \sum \alpha | \alpha \in \Delta_{\bar{1}}^+ , \alpha \neq \beta = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta)\}$ and l is the number of roots α in the expression of μ. This is true since for all generic λ, we have $(\lambda + \rho, \beta) = 0$.

Computing the formula above, using computer program (see Appendix), we have:

$$sdim\, L_\lambda = (-1)^{p(\mu)} 2 dim\, L_\mu(\mathfrak{g}_x).$$

□

6.2 Cohomology groups for generic weights for $F(4)$ and $G(3)$

Lemma 6.2.1 *For a generic weight $\lambda \in F^\chi$, there is a unique $\alpha \in \Delta^+$ such that $\lambda - \alpha \in F^\chi$ and $(\lambda + \rho, \alpha) = 0$.*

Proof. From Theorem 4.4.5 and Theorem 4.4.7, there is a unique c corresponding to λ. Since $\lambda - \alpha$ will correspond to $c - \frac{1}{2}$ for $F(4)$ and to $c - 1$ for $G(3)$, there is a unique such possible $\lambda - \alpha$. We take $\alpha = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}|\frac{1}{2})$ for $\mathfrak{g} = F(4)$ and $\alpha = (1, 1|1)$ for $\mathfrak{g} = G(3)$, then it follows from Theorem 4.4.5 and Theorem 4.4.7 that $\lambda - \alpha \in F^\chi$. □

Lemma 6.2.2 *Let $\lambda \in F^{(a,b)}$ (or F^a) be generic weight and $\alpha \in \Delta^+$ such that $\lambda - \alpha \in F^{(a,b)}$ (or F^a) and $(\lambda + \rho, \alpha) = 0$, then*

$$[\Gamma_0(G/B, \mathcal{O}_\lambda) : L_{\lambda-\alpha}] \leq 1 \text{ and}$$

$$[\Gamma_0(G/B, \mathcal{O}_\lambda) : L_\xi] = 0 \text{ if } \xi \neq \lambda - \alpha.$$

For $i > 0$, we have $\Gamma_i(G/B, \mathcal{O}_\lambda) = 0$.

Proof. If λ is a generic weight, than the only weights obtained in the form $\mu + \rho = w(\lambda + \rho) - \sum \alpha$ are λ and $\lambda - \alpha$. One can see this from Lemma 4.4.5 and Lemma 4.4.7.

Thus, the lemma follows from Lemma 5.1.5 and Lemma 6.2.1, since there is a unique root $\alpha = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}|\frac{1}{2})$ with $\lambda - \alpha \in F^{(a,b)}$ (or F^a). And $w = id$ is the only possibility. □

Lemma 6.2.3 *Let $\lambda \in F^{(a,b)}$ (or F^a) be generic weight, then we have the exact sequence*

$$0 \longrightarrow L_{\lambda-\alpha} \longrightarrow \Gamma_0(G/B, \mathcal{O}_\lambda) \longrightarrow L_\lambda \longrightarrow 0$$

for $\alpha \in \Delta_1^+$ such that $(\lambda + \rho, \alpha) = 0$.

Proof. We know $\Gamma_0(G/B, \mathcal{O}_\lambda)$ is the maximal finite dimensional quotient of the Verma module M_λ with highest weight λ. Therefore, $[\Gamma_0(G/B, \mathcal{O}_\lambda) : L_\lambda] = 1$. By Lemma 6.2.2, we have $[\Gamma_0(G/B, \mathcal{O}_\lambda) : L_{\lambda-\alpha}] \leq 1$. To prove the exact sequence, it is enough to show $[\Gamma_0(G/B, \mathcal{O}_\lambda) : L_{\lambda-\alpha}] \neq 0$.

From Lemma 6.2.2, we have $0 = sdim \Gamma_0(G/B, \mathcal{O}_\lambda) = sdim L_\lambda + [\Gamma_0(G/B, \mathcal{O}_\lambda) : L_{\lambda-\alpha}] sdim L_{\lambda-\alpha}$. From Lemma 6.1.4, since λ is generic we have that $sdim L_\lambda \neq 0$. Thus, $[\Gamma_0(G/B, \mathcal{O}_\lambda) : L_{\lambda-\alpha}] \neq 0$.

□

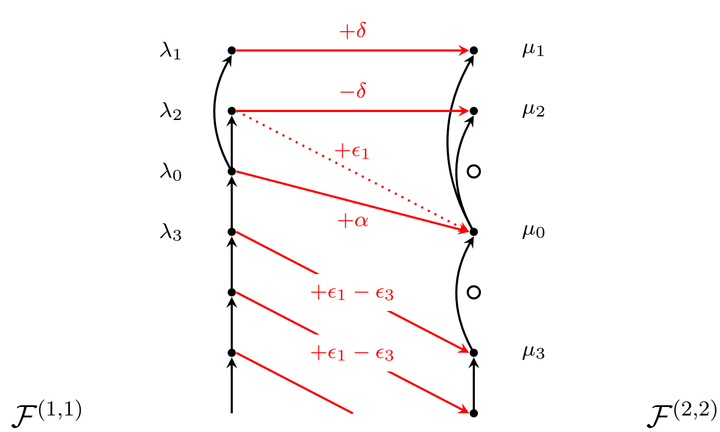
Chapter 7

Equivalence of symmetric blocks in $F(4)$

7.1 Equivalence of blocks $\mathcal{F}^{(1,1)}$ and $\mathcal{F}^{(2,2)}$

Let $\mathfrak{g} = F(4)$. We prove the equivalence of the symmetric blocks $\mathcal{F}^{(1,1)}$ and $\mathcal{F}^{(2,2)}$ as the first step of mathematical induction in a of proving the equivalence of the symmetric blocks $\mathcal{F}^{(a,a)}$ and $\mathcal{F}^{(a+1,a+1)}$.

The following is the picture of translator functor from block $\mathcal{F}^{(1,1)}$ to $\mathcal{F}^{(2,2)}$. It is defined by $T(L_\lambda) = (L_\lambda \otimes \mathfrak{g})^{(2,2)}$. The non-filled circles represent the non-dominant weights in the block occurring on the walls of the Weyl chamber. The filled circles represent dominant weights in the block. The horizontal arrows are maps $\lambda \mapsto \lambda + \gamma$, with $\gamma \in \Delta$ is the root above the arrow.



In this section, we will show that the solid arrows represent the maps $L_\lambda \mapsto T(L_\lambda)$ and use this to prove the equivalence of symmetric blocks.

In the above picture $\alpha = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}| - \frac{1}{2})$ and $\lambda_1 + \rho = (\frac{5}{2}, \frac{3}{2}, \frac{1}{2}| - \frac{3}{2})$; $\lambda_2 + \rho = (\frac{5}{2}, \frac{3}{2}, \frac{1}{2}|\frac{3}{2})$; $\lambda_0 + \rho = (3, 2, 1|2)$; $\lambda_3 + \rho = (\frac{7}{2}, \frac{5}{2}, \frac{3}{2}|\frac{5}{2})$; $\mu_1 + \rho = (\frac{5}{2}, \frac{3}{2}, \frac{1}{2}| - \frac{1}{2})$; $\mu_2 + \rho = (\frac{5}{2}, \frac{3}{2}, \frac{1}{2}|\frac{1}{2})$; $\mu_0 + \rho = (\frac{7}{2}, \frac{3}{2}, \frac{1}{2}|\frac{3}{2})$; $\mu_3 + \rho = (\frac{9}{2}, \frac{5}{2}, \frac{1}{2}|\frac{5}{2})$.

(Note that the indices here are different from the index c, which corresponds to the last coordinate of $\lambda + \rho$.)

Lemma 7.1.1 *Any dominant weight $\lambda \in F^{(1,1)}$ with $\lambda \neq \lambda_1$ and λ_2 can be obtained from λ_0 by adding root $\beta = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}|\frac{1}{2})$ finitely many times.*

Proof. From Theorem 4.4.5, if $a = 1$, then $J_2, J_3 = \emptyset$. Since $c \neq \pm\frac{3}{2}$, we have $\lambda = \lambda_0 + c\beta$, where $\beta = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta)$. □

Lemma 7.1.2 *For a dominant weight $\lambda \in F^{(1,1)}$ with $\lambda \neq \lambda_i$ for $i = 1, 2$, we have $\Gamma_i(G/B, \mathcal{O}_\lambda) = 0$ for $i > 0$.*

Proof. Assume $\lambda \neq \lambda_s$ for $s = 1, 2$ and $\Gamma_i(G/B, \mathcal{O}_\lambda) \neq 0$ for $i > 0$. There is $\mu \in F^{(1,1)}$ dominant weight such that L_μ occurs in $\Gamma_i(G/B, \mathcal{O}_\lambda)$ with non-zero multiplicity.

For $\lambda \neq \lambda_s$ for $s = 1, 2$, we have by Lemma 7.1.1, $\lambda + \rho = \lambda_0 + \rho + n\beta = (3 + \frac{n}{2}, 2 + \frac{n}{2}, 1 + \frac{n}{2}|2 + \frac{n}{2})$. By Lemma 5.1.5, we have $\mu + \rho = w(\lambda + \rho) - \sum_{\alpha \in I} \alpha$ for $w \in W$ of length i. The last coordinate of $\mu + \rho$ is in

$$[\frac{n}{2} - 2, \frac{n}{2} + 2] \cap \frac{1}{2}\mathbb{Z}_{\geq 4} \text{ or } \pm\frac{3}{2}.$$

Assume $n = 0$. The last coordinate of $\mu + \rho$ is 2 or $\pm\frac{3}{2}$. By Theorem 4.4.5 and computation there are only three possibilities $\mu = \lambda_i$ with $i = 0, 1, 2$ and in each case $w = id$. This implies $\Gamma_i(G/B, \mathcal{O}_{\lambda_0}) = 0$ for $i > 0$.

Assume $n = 1$. The last coordinate of $\mu + \rho$ is

$$2, \frac{5}{2}, \pm\frac{3}{2}.$$

By computation there are only four possibilities $\mu = \lambda_i$ with $i = 0, 1, 2, 3$ and in each case either $w = id$ or doesn't exist. This implies $\Gamma_i(G/B, \mathcal{O}_{\lambda_3}) = 0$ for $i > 0$.

Assume $n \geq 1$. The last coordinate of $\mu + \rho$ is in

$$[\frac{n}{2} - 2, \frac{n}{2} + 2] \cap \frac{1}{2}\mathbb{Z}_{\geq 4}.$$

By computation, only $w = id$ is possible when $\mu + \rho$ has last coordinate equal the last coordinate of $\lambda + \rho$ minus $\frac{1}{2}$. Thus, $\Gamma_i(G/B, \mathcal{O}_\lambda) = 0$ for $i > 0$. ☐

Lemma 7.1.3 *For a dominant weight $\lambda \in F^{(1,1)}$ with $\lambda \neq \lambda_i$ for $i = 0, 1, 2$, we have $[\Gamma_0(G/B, \mathcal{O}_\lambda) : L_{\lambda-\alpha}] = 1$ for a unique $\alpha \in \Delta$ such that $\lambda - \alpha \in F^{(1,1)}$.*

Also, we have $[\Gamma_0(G/B, \mathcal{O}_\lambda) : L_\mu] = 0$ for $\mu \neq \lambda$ and $\mu \neq \lambda - \alpha$.

Proof. As in the previous lemma, by Lemma 7.1.1, $\lambda + \rho = \lambda_0 + \rho + n\beta = (3 + \frac{n}{2}, 2 + \frac{n}{2}, 1 + \frac{n}{2} | 2 + \frac{n}{2})$.

The first part of the lemma follows from Lemma 6.2.2.

Assume $n = 0$. The last coordinate of $\mu + \rho$ is 2 or $\pm\frac{3}{2}$. By computation, there are only three possibilities $\mu = \lambda_i$ with $i = 0, 1, 2$ and in each case $w = id$. This implies $[\Gamma_0(G/B, \mathcal{O}_{\lambda_0}), L_\mu] = 0$ for $\mu \neq \lambda_0 - \alpha$.

Assume $n = 1$. The last coordinate of $\mu + \rho$ is $2, \frac{5}{2}, \pm\frac{3}{2}$. By computation there are only four possibilities $\mu = \lambda_i$ with $i = 0, 1, 2, 3$. For $i = 0, 2$, there is unique possible $w = id$ and set I. This implies $[\Gamma_0(G/B, \mathcal{O}_{\lambda_3}), L_\mu] = 0$ for $\mu \neq \lambda_3 - \alpha$.

Assume $n > 1$. The last coordinate of $\mu + \rho$ is in $[\frac{n}{2} - 2, \frac{n}{2} + 2] \cap \frac{1}{2}\mathbb{Z}_{\geq 4}$. By computation and Lemma 4.4.5, only $w = id$ is possible when $\mu + \rho$ has last coordinate equal the last coordinate of $\lambda + \rho$ minus $\frac{1}{2}$ or $\mu = \lambda$, in each case there is a unique set I. Thus, $[\Gamma_0(G/B, \mathcal{O}_\lambda) : L_\mu] = 0$ for $\mu \neq \lambda$ and $\mu \neq \lambda - \alpha$ for any $\alpha \in \Delta_{\bar{1}}$. ☐

Lemma 7.1.4 *For a dominant weight $\lambda \in F^{(1,1)}$, we have $sdim L_\lambda = \pm 2$ if $\lambda \neq \lambda_i$ for $i = 1, 2$.*

Proof. We prove this by induction starting with a generic weight $\lambda \in F^{(1,1)}$. From generic formula for superdimension, we have $sdim L_\lambda = a$ with $a = \pm 2$. The weights in $F^{(1,1)}$ can be obtained successively from λ by subtracting odd root β from Lemma 7.1.1.

By Lemma 5.1.7 and Lemma 7.1.2, we have

$$0 = sdim\Gamma_0(G/B, \mathcal{O}_\lambda) = sdimL_\lambda + [\Gamma_0(G/B, \mathcal{O}_\lambda) : L_{\lambda-\alpha}]sdimL_{\lambda-\alpha}.$$

Since $sdimL_\lambda = \pm 2$ and $[\Gamma_0(G/B, \mathcal{O}_\lambda) : L_{\lambda-\alpha}] \leq 1$ from proof of previous lemma, we must have $[\Gamma_0(G/B, \mathcal{O}_\lambda) : L_{\lambda-\alpha}] = 1$ and $sdimL_{\lambda-\alpha} = \mp 2$. By induction, this way from generic weight we obtain L_{λ_0}. Thus, $sdimL_{\lambda_0} = \pm 2$. □

Lemma 7.1.5 *We have* $\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) = L_{\lambda_1}$.

Proof. From Lemma 5.1.5 and Theorem 4.4.5, if L_σ occurs in $\Gamma_0(G/B, \mathcal{O}_{\lambda_1})$, then $\mu \leq \lambda$. Thus, $[\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) : L_\sigma] = 0$ for $\sigma \neq \lambda_1$.

We know $[\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) : L_{\lambda_1}] = 1$ from Lemma 5.1.4. □

Lemma 7.1.6 *We have* $\Gamma_1(G/B, \mathcal{O}_{\lambda_1}) = L_{\lambda_2}$.

Proof. We have

$$0 = sdim\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) - sdim\Gamma_1(G/B, \mathcal{O}_{\lambda_1})$$

and

$$sdim\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) = sdimL_{\lambda_1} = 1.$$

This implies that $sdim\Gamma_1(G/B, \mathcal{O}_{\lambda_1}) = 1$. Thus, we either have $\Gamma_1(G/B, \mathcal{O}_{\lambda_1}) = L_{\lambda_1}$ or $\Gamma_1(G/B, \mathcal{O}_{\lambda_1}) = L_{\lambda_2}$. This is true since $\Gamma_1(G/B, \mathcal{O}_{\lambda_1}) : L_\sigma] = 0$ for all $\sigma \neq \lambda_1, \lambda_2$.

We have

$$ch\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) - ch\Gamma_1(G/B, \mathcal{O}_{\lambda_1}) = \frac{D_1 e^\rho}{D_0} \sum_{w \in W} sgn(w) e^{w(\lambda_1+\rho)}.$$

The expression on the right is not zero, since one can compute that the lowest degree term in the numerator is not zero. This implies $\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) \neq \Gamma_1(G/B, \mathcal{O}_{\lambda_1})$. Thus, $\Gamma_1(G/B, \mathcal{O}_{\lambda_1}) = L_{\lambda_2}$.

□

Lemma 7.1.7 *We have* $sdimL_{\lambda_1} = sdimL_{\lambda_2} = 1$.

Proof. This follows from previous two lemmas and since

$$sdim\Gamma_0(G/B,\mathcal{O}_{\lambda_1}) = sdim\Gamma_1(G/B,\mathcal{O}_{\lambda_1}).$$

□

Lemma 7.1.8 *The cohomology group $\Gamma_0(G/B,\mathcal{O}_{\lambda_0})$ has a filtration with quotients L_{λ_0}, L_{λ_1}, and L_{λ_2}.We know that L_{λ_0} is a quotient of $\Gamma_0(G/B,\mathcal{O}_{\lambda_0})$. The kernel of that quotient has a filtration with subquotients L_{λ_1}, L_{λ_2}. Also, $sdimL_{\lambda_0} = -2$.*

Proof. From previous lemmas, we have $sdimL_{\lambda_0} = \pm 2$, $sdimL_{\lambda_1} = sdimL_{\lambda_2} = 1$. We also know from Lemma 7.1.3, $[\Gamma_0(G/B,\mathcal{O}_{\lambda_0}) : L_\sigma] = 0$, unless $\sigma = \lambda_i$ with $i = 0, 1, 2$. From Lemma 5.1.5, we have $[\Gamma_0(G/B,\mathcal{O}_{\lambda_0}) : L_{\lambda_0}] = 1$, $[\Gamma_0(G/B,\mathcal{O}_{\lambda_0}) : L_{\lambda_1}] \leq 1$, $[\Gamma_0(G/B,\mathcal{O}_{\lambda_0}) : L_{\lambda_2}] \leq 1$.

We have

$$0 = sdim\Gamma_0(G/B,\mathcal{O}_{\lambda_0}) = sdimL_{\lambda_0} + [\Gamma_0(G/B,\mathcal{O}_{\lambda_0}) : L_{\lambda_1}]sdimL_{\lambda_1} +$$

$$+[\Gamma_0(G/B,\mathcal{O}_{\lambda_0}) : L_{\lambda_2}]sdimL_{\lambda_2}.$$

This implies that $[\Gamma_0(G/B,\mathcal{O}_{\lambda_0}) : L_{\lambda_1}] = [\Gamma_0(G/B,\mathcal{O}_{\lambda_0}) : L_{\lambda_2}] = 1$, and $sdimL_{\lambda_0} = -2$. □

Lemma 7.1.9 *We have $\Gamma_0(G/B,\mathcal{O}_{\lambda_2}) = L_{\lambda_2}$ and $\Gamma_1(G/B,\mathcal{O}_{\lambda_2}) = L_{\lambda_1}$.*

Proof. From Lemma 5.1.5, we have $[\Gamma_0(G/B,\mathcal{O}_{\lambda_2}) : L_\sigma] = 0$ for $\sigma \neq \lambda_i$ with $i = 1, 2$. We know $[\Gamma_0(G/B,\mathcal{O}_{\lambda_2}) : L_{\lambda_2}] = 1$ from Lemma 5.1.4. We need to show $[\Gamma_0(G/B,\mathcal{O}_{\lambda_2}) : L_{\lambda_1}] = 0$.

From Lemma 5.1.9, since $\lambda_2 = w(\lambda_1 + \rho) - \rho$, with w reflection with respect to root δ, we have

$$ch\Gamma_0(G/B,\mathcal{O}_{\lambda_1}) - ch\Gamma_1(G/B,\mathcal{O}_{\lambda_1}) = -ch\Gamma_0(G/B,\mathcal{O}_{\lambda_2}) + ch\Gamma_1(G/B,\mathcal{O}_{\lambda_2}).$$

From Lemma 9.1.5, we have $\Gamma_0(G/B,\mathcal{O}_{\lambda_1}) = L_{\lambda_1}$. From Lemma 9.1.6, we have $\Gamma_1(G/B,\mathcal{O}_{\lambda_1}) = L_{\lambda_2}$. From Lemma 5.1.5, we know that $[\Gamma_1(G/B,\mathcal{O}_{\lambda_2}) : L_{\lambda_2}] = 0$. We also know that $[\Gamma_0(G/B,\mathcal{O}_{\lambda_2}) : L_{\lambda_2}] = 1$. The above equation gives

$$[\Gamma_1(G/B,\mathcal{O}_{\lambda_2}) : L_{\lambda_1}] - [\Gamma_0(G/B,\mathcal{O}_{\lambda_2}) : L_{\lambda_1}] = 1.$$

We show that $\Gamma_1(G/B, \mathcal{O}_{\lambda_2}) = L_{\lambda_1}$, which together with previous equality implies $[\Gamma_0(G/B, \mathcal{O}_{\lambda_2}) : L_{\lambda_1}] = 0$ and proves the lemma.

Consider the typical weight μ, with $\mu + \rho = (3, 2, 1|1)$. The module $(L_\mu \otimes \mathfrak{g})^{(1,1)}$ has a filtration with quotients $\mathcal{O}_\lambda$ with $\lambda = \lambda_i$ with $i = 0, 2$. As $\lambda_2 < \lambda_0$, we have an exact sequence:

$$0 \to \mathcal{O}_{\lambda_0} \to (\mathcal{O}_\mu \otimes \mathfrak{g})^{\Phi^{-1}(\chi)} \to \mathcal{O}_{\lambda_2} \to 0.$$

Applying Lemma 5.1.1, gives the following long exact sequence:

$$0 \to \Gamma_1(G/B, \mathcal{O}_{\lambda_2}) \to \Gamma_0(G/B, \mathcal{O}_{\lambda_0}) \to (L_\mu \otimes \mathfrak{g})^\chi \to \Gamma_0(G/B, \mathcal{O}_{\lambda_2}) \to 0.$$

From previous lemma, we have $[\Gamma_0(G/B, \mathcal{O}_{\lambda_0}) : L_{\lambda_1}] = 1$. From the long exact sequence we have $[\Gamma_1(G/B, \mathcal{O}_{\lambda_2}) : L_{\lambda_1}] \leq [\Gamma_0(G/B, \mathcal{O}_{\lambda_0}) : L_{\lambda_1}] = 1$. Since $sdim\Gamma_1(G/B, \mathcal{O}_{\lambda_2}) = sdim\Gamma_0(G/B, \mathcal{O}_{\lambda_2}) \neq 0$, we have $[\Gamma_1(G/B, \mathcal{O}_{\lambda_2}) : L_{\lambda_1}] \neq 0$. This proves the lemma. □

Lemma 7.1.10 *We have $T(L_{\lambda_i}) = L_{\mu_i}$, for all $i \neq 2$.*

Proof. By definition, $T(L_{\lambda_i}) = (L_{\lambda_i} \otimes \mathfrak{g})^{(2,2)}$. For each $i \neq 2$, there is a unique dominant weight μ_i in the block $\mathcal{F}^{(2,2)}$ of the form $\lambda_i + \gamma$ with $\gamma \in \Delta$ as its shown in the picture. Thus, the lemma follows from Lemma 5.2.7.

□

Lemma 7.1.11 *We have $T(L_{\lambda_2}) = L_{\mu_2}$.*

Proof. By definition, $T(L_{\lambda_2}) = (L_{\lambda_2} \otimes \mathfrak{g})^{(2,2)}$. The only dominant weights in $F^{(2,2)}$ of the form $\lambda_2 + \gamma$ with $\gamma \in \Delta$ are μ_2 and μ_0.

It suffices to prove that $T(L_{\lambda_2})$ does not have a subquotient L_{μ_0}.

We know that L_{λ_0} is a quotient of $\Gamma_0(G/B, \mathcal{O}_{\lambda_0})$ from Lemma 5.1.4. The kernel of that quotient has a filtration with subquotients L_{λ_1}, L_{λ_2} (see Lemma 7.1.8). We have the following exact sequence:

$$0 \to S \to \Gamma_0(G/B, \mathcal{O}_{\lambda_0}) \to L_{\lambda_0} \to 0.$$

Since T is an exact functor, we get the following exact sequence:

$$0 \to T(S) \to T(\Gamma_0(G/B, \mathcal{O}_{\lambda_0})) \to T(L_{\lambda_0}) \to 0.$$

From Lemma 7.1.10, we have $T(L_{\lambda_0}) = L_{\mu_0}$. The kernel $T(S)$ of that quotient has a filtration with subquotients $T(L_{\lambda_1})$, $T(L_{\lambda_2})$. By Lemma 5.2.1 and Lemma 5.2.2, we have $T(\Gamma_0(G/B, \mathcal{O}_{\lambda_0})) = \Gamma_0(G/B, T(\mathcal{O}_{\lambda_0})) = \Gamma_0(G/B, \mathcal{O}_{\mu_0})$. The later module has a unique quotient L_{μ_0}. Therefore, $T(S)$ has no simple subquotient L_{μ_0}. Hence, $T(L_{\lambda_2})$ also does not have a subquotient L_{μ_0}.

□

Corollary 7.1.12 *For any $\lambda \in F^{(1,1)}$, the module $T(L_\lambda) \in F^{(2,2)}$ is irreducible of highest weight $\lambda + \alpha$ for some $\alpha \in \Delta$. Conversely, any irreducible module in $F^{(2,2)}$ is obtained this way.*

Proof. For any dominant weight $\lambda \in F^{(1,1)}$, with $\lambda \neq \lambda_2$, there is a unique $\alpha \in \Delta$ with dominant weight $\lambda + \alpha \in F^{(2,2)}$. Thus, $T(L_\lambda)$ is an irreducible with highest weight $\lambda + \alpha$. From previous lemma, the corollary follows.

□

Theorem 7.1.13 *The blocks $\mathcal{F}^{(1,1)}$ and $\mathcal{F}^{(2,2)}$ are equivalent as categories.*

Proof. From above corollary, for each $\lambda_i \in F^{(1,1)}$, let $L_{\mu_i} = T(L_{\lambda_i})$ be the simple module with highest weight $\mu_i \in F^{(2,2)}$. We show that $T^*(L_{\mu_i}) = L_{\lambda_i}$ for each $\mu_i \in F^{(2,2)}$.

For all $\mu \neq \mu_0$, we have a unique $\gamma \in \Delta$, such that $\mu + \gamma \in F^{(1,1)}$. For $\mu = \mu_0$, there are two possible $\gamma \in \Delta$ such that $\mu_0 + \gamma \in F^{(1,1)}$. From the picture above, we have $\gamma = -(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}| - \frac{1}{2})$ or $\gamma = -\epsilon_1$, such that $\mu_0 + \gamma = \lambda_0$ or λ_1.

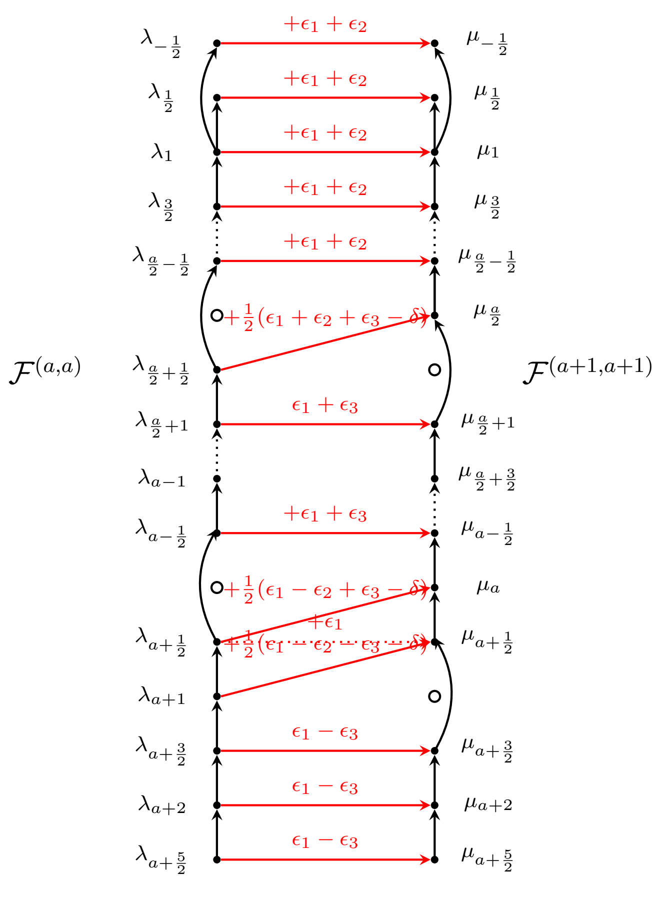
The theorem follows from Theorem 5.2.8.

□

7.2 Equivalence of blocks $\mathcal{F}^{(a,a)}$ and $\mathcal{F}^{(a+1,a+1)}$

In this section, we prove the inductive step of the equivalence of all the symmetric blocks. Let V be a finite-dimensional $\mathfrak{g}$-module. We define translator functor $T(V)_{\chi,\tau} : F_\chi \to F_\tau$ by $T(V)_{\chi,\tau}(M) = (M \otimes V)^\tau$ as before.

The following is the picture of translator functor from block $\mathcal{F}^{(a,a)}$ to $\mathcal{F}^{(a+1,a+1)}$. It is defined by $T(L_\lambda) = (L_\lambda \otimes \mathfrak{g})^{(a+1,a+1)}$. The non-filled circles represent the non-dominant weights in the block occurring on the walls of the Weyl chamber. The filled circles represent dominant weights in the block. The horizontal arrows are maps $\lambda \mapsto \lambda + \gamma$, with $\gamma \in \Delta$ is the root above the arrow. In this section, we will show that the solid arrows represent the maps $L_\lambda \mapsto T(L_\lambda)$.



Lemma 7.2.1 *For $\lambda \in F^{(a,a)}$, let T be an equivalence of categories $\mathcal{F}^{(a,a)}$ and $\mathcal{F}^{(a+1,a+1)}$ and $T(L_\lambda) = L_{\lambda'}$, then $\Gamma_i(G/B, \mathcal{O}_{\lambda'})$ has a subquotients $L_{\lambda'_s}$ with*

$$[\Gamma_i(G/B, \mathcal{O}_{\lambda'}) : L_{\lambda'_s}] = [\Gamma_i(G/B, \mathcal{O}_\lambda) : L_{\lambda_s}].$$

Proof. Assume $i = 0$. Then $\Gamma_0(G/B, \mathcal{O}_{\lambda'}) = T(\Gamma_0(G/B, \mathcal{O}_\lambda)$ from Lemma 5.2.4.

Assume $i > 0$. For $\lambda \neq \lambda_t$ with $t = 1, 2$, we have $\Gamma_i(G/B, \mathcal{O}_\lambda) = 0$ for $i > 0$ from computation using Lemma 5.1.5.

For $t = 1, 2$, we know from Lemma 5.2.4, $\Gamma_0(G/B, \mathcal{O}_{\lambda_t}) = L_{\lambda_t}$ since all other submodules in $\Gamma_0(G/B, \mathcal{O}_{\lambda_t})$ have highest weight $< \lambda_t$ and this is impossible.

Thus, we have $sdim\Gamma_1(G/B, \mathcal{O}_{\lambda_t}) = sdim\Gamma_0(G/B, \mathcal{O}_{\lambda_t}) = sdimL_{\lambda_t}$.

For $s \neq 1, 2$, $sdimL_{\lambda_s} > sdimL_{\lambda_1}$, which implies $\Gamma_1(G/B, \mathcal{O}_{\lambda_t}) = L_{\lambda_k}$ for $t, k = 1, 2$.

We have

$$ch\Gamma_0(G/B, \mathcal{O}_{\lambda_i}) - ch\Gamma_1(G/B, \mathcal{O}_{\lambda_i}) = \frac{D_1 e^\rho}{D_0} \sum_{w \in W} sgn(w) e^{w(\lambda_i + \rho)}.$$

The expression on the right is not zero, since one can compute that the lowest degree term in the numerator is not zero.

Thus, $ch\Gamma_1(G/B, \mathcal{O}_{\lambda_i}) \neq ch\Gamma_0(G/B, \mathcal{O}_{\lambda_i})$ and we must have $\Gamma_1(G/B, \mathcal{O}_{\lambda_i}) = L_{\lambda_s}$ with $s \neq i$. This proves the lemma.

□

Lemma 7.2.2 *Let $\lambda \in F^{(a,a)}$ be dominant, then there is unique $\gamma \in \Delta$ such that $\lambda + \gamma \in \mathcal{F}^{(a+1,a+1)}$ is dominant, unless $\lambda + \rho = (2a + \frac{1}{2}, a + \frac{1}{2}, \frac{1}{2}|a + \frac{1}{2})$.*

Proof. From Lemma 4.4.5, for given $c \geq -\frac{1}{2}$, there is at most one dominant $\lambda \in F^{(a,a)}$ with $\lambda + \rho = (b_1, b_2, b_3|c)$. Assume $\gamma \in \Delta$ is such that $\lambda + \gamma \in F^{(a+1,a+1)}$, then $\lambda + \rho + \gamma$ must have last coordinate $c \pm 1$, $c \pm \frac{1}{2}$, or c.

Thus in generic cases, the last coordinate of $\lambda + \gamma + \rho$ and $\lambda + \rho$ are in the same interval J_i. The few exceptional cases, when the last coordinates are in the distinct intervals, occur around walls of the Weyl chamber, when $c = a + \frac{1}{2}$, $a + 1$, $\frac{a}{2} + \frac{1}{2}$, $\frac{a}{2} + 1$. And only for $c = a + \frac{1}{2}$, there are two possible γ.

We show that the last coordinates of $\lambda + \gamma + \rho$ and $\lambda + \rho$ are the same in generic cases, and thus, there is at most one such γ, proving the uniqueness.

Note that for generic λ, $(\lambda+\rho,\alpha)=0$ and $(\lambda+\gamma+\rho,\alpha)=0$ are true for the same $\alpha\in\Delta_{\bar{1}}^+$ (see Remark 4.4.6 above). That implies $(\gamma,\alpha)=0$. This is impossible for $\gamma=\delta$. If γ is odd then $(\gamma,\alpha)=0$ implies $\gamma=\pm\alpha$, which is impossible for λ and $\lambda+\gamma$ would be in the same block. For even root $\gamma\neq\delta$ the statement is clear.

For the existence, for each λ_c the root γ described in the picture above above each arrow. □

Lemma 7.2.3 *We have $T(L_{\lambda_i})=L_{\lambda_i+\gamma}$, for all $i\neq a+\frac{1}{2}$ and for the unique $\gamma\in\Delta$ in the previous lemma.*

Proof. By definition, $T(L_{\lambda_i})=(L_{\lambda_i}\otimes\mathfrak{g})^{(a+1,a+1)}$. For each λ_i, there is a unique dominant weight μ_i in the block $\mathcal{F}^{(a+1,a+1)}$ of the form $\lambda_i+\gamma$ with $\gamma\in\Delta$. Thus, the lemma follows from Lemma 5.2.7.

□

Lemma 7.2.4 *Assume for each $\lambda\in F^{(a,a)}$, $T(L_\lambda)$ is a simple module in $\mathcal{F}^{(a+1,a+1)}$. Then categories $\mathcal{F}^{(a,a)}$ and $\mathcal{F}^{(a+1,a+1)}$ are equivalent.*

Proof. By hypothesis, for each $\lambda_i\in F^{(a,a)}$, $T(L_{\lambda_i})$ is a simple module in $\mathcal{F}^{(a+1,a+1)}$, we denote $L_{\mu_i}=T(L_{\lambda_i})$ the simple module with highest weight $\mu_i\in F^{(a+1,a+1)}$. We show that $T^*(L_{\mu_i})=L_{\lambda_i}$ for each $\mu_i\in F^{(a+1,a+1)}$.

For all $\mu\neq\mu_{a+\frac{1}{2}}$, we have a unique $\gamma\in\Delta$, such that $\mu+\gamma\in F^{(a,a)}$. For $\mu=\mu_{a+\frac{1}{2}}$, there are two possible $\gamma\in\Delta$ such that $\mu+\gamma\in F^{(a,a)}$. From the picture picture above, we have $\gamma=-(\frac{1}{2},-\frac{1}{2},-\frac{1}{2}|-\frac{1}{2})$ or $\gamma=-\epsilon_1$, such that $\mu_{a+\frac{1}{2}}+\gamma=\lambda_{a+\frac{1}{2}}$ or λ_{a+1}.

The statement follows from Theorem 5.2.8

□

Lemma 7.2.5 *Let $\mathfrak{g}=F(4)$ and $\lambda\in F^{(a,a)}$ such that $\lambda=(2a+\frac{1}{2},a+\frac{1}{2},\frac{1}{2}|a+\frac{1}{2})-\rho$. If $a=1$, let $\alpha=\delta$, and if $a>1$, let $\alpha=(-\frac{1}{2},\frac{1}{2},-\frac{1}{2}|\frac{1}{2})$. Then $T(L_\lambda)=L_{\lambda-\alpha}$.*

Proof. We will assume that blocks $\mathcal{F}^{(c,c)}$ for $c\leq a$ are all equivalent. Then using this assumption we will prove the lemma. This lemma implies the equivalence of $\mathcal{F}^{(a,a)}$ and $\mathcal{F}^{(a+1,a+1)}$. Thus, we use a complicated induction in a.

For $a=1$, we have the statement from Lemma 7.1.11. Let $a>1$. From our assumption and Lemma 7.2.1, we obtain all cohomology groups for $\mathcal{F}^{(a,a)}$, since we know them for $\mathcal{F}^{(1,1)}$ from previous section.

From definition, we have $\lambda = \lambda_{a+\frac{1}{2}}$ and $T(L_{\lambda_{a+\frac{1}{2}}}) = (L_{\lambda_{a+\frac{1}{2}}} \otimes \mathfrak{g})^{(a+1,a+1)}$. Thus, the only dominant weights in $\mathcal{F}^{(a+1,a+1)}$ of the form $\lambda_{a+\frac{1}{2}} + \gamma$ with $\gamma \in \Delta$ are $\mu_{a+\frac{1}{2}}$ and μ_a as its shown in the picture.

It will suffice to prove that $T(L_{\lambda_{a+\frac{1}{2}}})$ does not have a subquotient $L_{\mu_{a+\frac{1}{2}}}$. Thus, $T(L_{\lambda_{a+\frac{1}{2}}}) = L_{\mu_a}$ as required.

We know that $L_{\lambda_{a+1}}$ is a quotient of $\Gamma_0(G/B, \mathcal{O}_{\lambda_{a+1}})$. From inductive assumption, Lemma 7.1.3, and Lemma 7.2.1, we have the following exact sequence:

$$0 \to L_{\lambda_{a+\frac{1}{2}}} \to \Gamma_0(G/B, \mathcal{O}_{\lambda_{a+1}}) \to L_{\lambda_{a+1}} \to 0.$$

Since T is an exact functor, we obtain the following exact sequence:

$$0 \to T(L_{\lambda_{a+\frac{1}{2}}}) \to T(\Gamma_0(G/B, \mathcal{O}_{\lambda_{a+1}})) \to T(L_{\lambda_{a+1}}) \to 0.$$

From Lemma 7.2.3, we have $T(L_{\lambda_{a+1}}) = L_{\mu_{a+\frac{1}{2}}}$. By Lemma 5.2.1 and Lemma 5.2.2, we have

$$T(\Gamma_0(G/B, \mathcal{O}_{\lambda_{a+1}})) = \Gamma_0(G/B, T(\mathcal{O}_{\lambda_{a+1}})) = \Gamma_0(G/B, \mathcal{O}_{\mu_{a+\frac{1}{2}}}).$$

The module $\Gamma_0(G/B, \mathcal{O}_{\mu_{a+\frac{1}{2}}})$ has a unique quotient $L_{\mu_{a+\frac{1}{2}}}$. Hence, $T(L_{\lambda_{a+\frac{1}{2}}})$ does not have a subquotient $L_{\mu_{a+\frac{1}{2}}}$. □

Theorem 7.2.6 *The categories $\mathcal{F}^{(a,a)}$ and $\mathcal{F}^{(a+1,a+1)}$ are equivalent for all $a \geq 1$.*

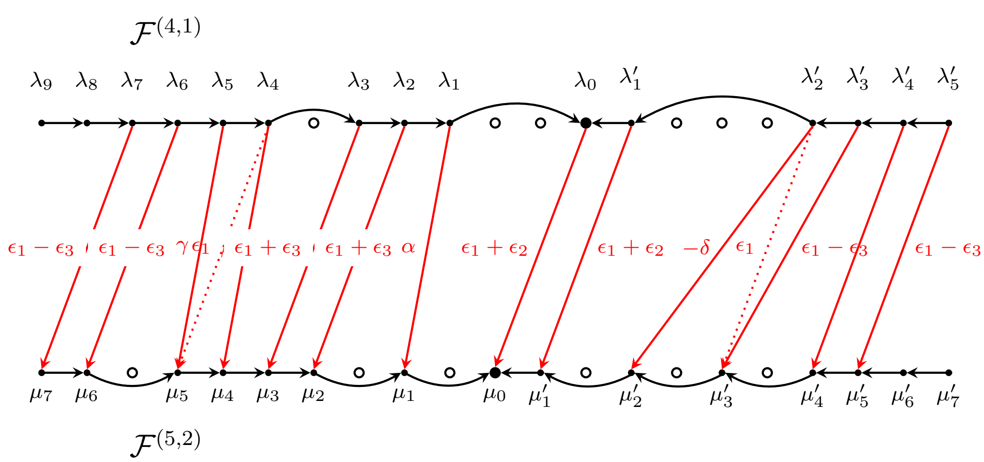
Proof. This follows from Theorem 5.2.8 together with Lemma 7.2.3 and Lemma 7.2.5. □

Chapter 8

Equivalence of non-symmetric blocks in $F(4)$

8.1 Equivalence of blocks $\mathcal{F}^{(4,1)}$ and $\mathcal{F}^{(5,2)}$

Let $\mathfrak{g} = F(4)$. The following is the picture of translator functor from block $\mathcal{F}^{(4,1)}$ to $\mathcal{F}^{(5,2)}$. It is defined by $T(L_\lambda) = (L_\lambda \otimes \mathfrak{g})^{(2,2)}$. The non-filled circles represent the non-dominant weights in the block occurring on the walls of the Weyl chamber. The filled circles represent dominant weights in the block. The vertical arrows are maps $\lambda \mapsto \lambda + \gamma$, with $\gamma \in \Delta$ is the root above the arrow. In this section, we will show that the solid arrows represent the maps $L_\lambda \mapsto T(L_\lambda)$.



In the above picture $\gamma = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}| - \frac{1}{2})$ and $\lambda'_4 + \rho = (\frac{13}{2}, \frac{5}{2}, \frac{3}{2}|\frac{7}{2})$; $\lambda'_3 + \rho = (6, 2, 1|3)$; $\lambda'_2 + \rho = (\frac{11}{2}, \frac{3}{2}, \frac{1}{2}|\frac{5}{2})$; $\lambda'_1 + \rho = (\frac{7}{2}, \frac{3}{2}, \frac{1}{2}|\frac{1}{2})$; $\lambda_0 + \rho = (3, 2, 1|0)$;

$\lambda_1+\rho=(\frac{7}{2},\frac{5}{2},\frac{3}{2}|\frac{3}{2})$; $\lambda_2+\rho=(4,3,1|2)$; $\lambda_3+\rho=(\frac{9}{2},\frac{7}{2},\frac{1}{2}|\frac{5}{2})$; $\lambda_4+\rho=(\frac{11}{2},\frac{9}{2},\frac{1}{2}|\frac{7}{2})$; $\lambda_5+\rho=(6,5,1|4)$; $\lambda_6+\rho=(\frac{13}{2},\frac{11}{2},\frac{3}{2}|\frac{9}{2})$; $\mu'_4+\rho=(\frac{15}{2},\frac{5}{2},\frac{1}{2}|\frac{7}{2})$; $\mu'_3+\rho=(\frac{13}{2},\frac{3}{2},\frac{1}{2}|\frac{5}{2})$; $\mu'_2+\rho=(\frac{11}{2},\frac{3}{2},\frac{1}{2}|\frac{3}{2})$; $\mu'_1+\rho=(\frac{9}{2},\frac{5}{2},\frac{1}{2}|\frac{1}{2})$; $\mu_0+\rho=(4,3,1|0)$; $\mu_1+\rho=(4,3,2|1)$; $\mu_2+\rho=(5,3,2|2)$; $\mu_3+\rho=(\frac{11}{2},\frac{7}{2},\frac{3}{2}|\frac{5}{2})$; $\mu_4+\rho=(6,4,1|3)$; $\mu_5+\rho=(\frac{13}{2},\frac{9}{2},\frac{1}{2}|\frac{7}{2})$; $\mu_6+\rho=(\frac{15}{2},\frac{11}{2},\frac{1}{2}|\frac{9}{2})$.

Note that the indices for λ above are different from the index c which represents the last coordinate of $\lambda+\rho$.

Lemma 8.1.1 *For a dominant weight $\lambda \in F^{(4,1)}$ with $\lambda=\lambda_c$ such that $c>\frac{5}{2}$ or $c<-\frac{7}{2}$, we have $[\Gamma_0(G/B,\mathcal{O}_\lambda):L_{\lambda-\alpha}]=1$ for a unique $\alpha\in\Delta$ such that $\lambda-\alpha\in F^{(4,1)}$.*

Also, we have $[\Gamma_0(G/B,\mathcal{O}_\lambda):L_\mu]=0$ for $\mu\neq\lambda$ and $\mu\neq\lambda-\alpha$.

Proof. Follows from Lemma 5.1.5 and Lemma 6.2.2. □

Lemma 8.1.2 *We have $T(L_{\lambda_i})=L_{\mu_i}$, for all $i\neq 4$ and $T(L_{\lambda'_i})=L_{\mu'_i}$, for all $i\neq 2$.*

Proof. By definition, $T(L_{\lambda_i})=(L_{\lambda_i}\otimes\mathfrak{g})^{(5,2)}$. For each λ_i, we have a unique dominant weight μ_i in the block $\mathcal{F}^{(5,2)}$ of the form $\lambda_i+\gamma$ with $\gamma\in\Delta$. Thus, the lemma follows from Lemma 5.2.7. □

Lemma 8.1.3 *We have $T(L_{\lambda_4})=L_{\mu_4}$ and $T(L_{\lambda'_2})=L_{\mu'_2}$.*

Proof. By definition, $T(L_{\lambda_4})=(L_{\lambda_4}\otimes\mathfrak{g})^{(5,2)}$. The only dominant weights in the block $\mathcal{F}^{(5,2)}$ of the form $\lambda_4+\gamma$ with $\gamma\in\Delta$ are μ_4 and μ_5, as its shown in the picture above.

From Lemma 5.1.4, L_{λ_5} is a quotient of $\Gamma_0(G/B,\mathcal{O}_{\lambda_5})$. From Lemma 8.1.1, we have the following exact sequence:

$$0\to L_{\lambda_4}\to\Gamma_0(G/B,\mathcal{O}_{\lambda_5})\to L_{\lambda_5}\to 0$$

Since T is an exact functor, we obtain the following exact sequence:

$$0\to T(L_{\lambda_4})\to T(\Gamma_0(G/B,\mathcal{O}_{\lambda_5}))\to T(L_{\lambda_5})\to 0.$$

From Lemma 8.1.2, we have $T(L_{\lambda_5})=L_{\mu_5}$. By lemma Lemma 5.2.1 and Lemma 5.2.2, we have:

$$T(\Gamma_0(G/B,\mathcal{O}_{\lambda_5})) = \Gamma_0(G/B, T(\mathcal{O}_{\lambda_5})) = \Gamma_0(G/B,\mathcal{O}_{\mu_5}).$$

The module $\Gamma_0(G/B,\mathcal{O}_{\mu_5})$ has a unique quotient L_{μ_5}. Therefore, $T(L_{\lambda_4})$ has no simple subquotient L_{μ_5}. This is sufficient to prove the lemma.

The proof of the second part of the lemma is exactly the same. □

Theorem 8.1.4 *We have an equivalence of categories $\mathcal{F}^{(4,1)}$ and $\mathcal{F}^{(5,2)}$.*

Proof. From previous lemma, for each $\lambda_i \in F^{(4,1)}$, $T(L_{\lambda_i})$ is a simple module in $\mathcal{F}^{(5,2)}$, we denote $L_{\mu_i} = T(L_{\lambda_i})$ the simple module with highest weight $\mu_i \in F^{(5,2)}$.

For all $\mu \neq \mu_5, \mu_3'$, we have a unique $\gamma \in \Delta$, such that $\mu + \gamma \in F^{(4,1)}$. For $\mu = \mu_5, \mu_3'$, there are two possible $\gamma \in \Delta$ such that $\mu+\gamma \in F^{(4,1)}$. From the picture above, we obtain two roots $\gamma = -\epsilon_1$ and $\gamma = -(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}| -\frac{1}{2})$ such that $\mu_5 + \gamma = \lambda_5$ or λ_4 and two roots $\gamma = -\epsilon_1$ and $\gamma = -(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}| -\frac{1}{2})$ for $\mu_3' + \gamma = \lambda_3'$ or λ_2'.

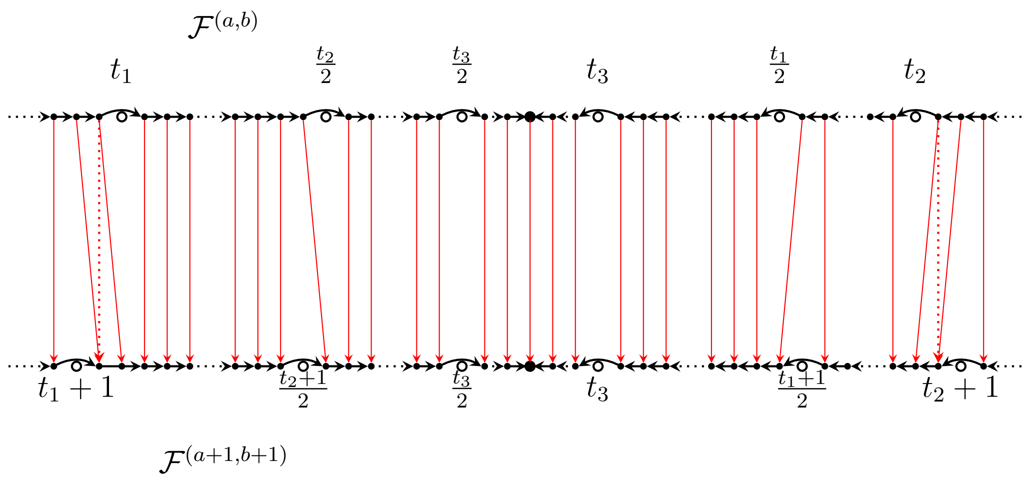
The theorem now follows from Theorem 5.2.8.

□

8.2 Equivalence of blocks $\mathcal{F}^{(a,b)}$ and $\mathcal{F}^{(a+1,b+1)}$

Let $\mathfrak{g} = F(4)$. The following is the picture of translator functor from block $\mathcal{F}^{(a,b)}$ to $\mathcal{F}^{(a+1,b+1)}$. It is defined by $T(L_\lambda) = (L_\lambda \otimes \mathfrak{g})^{(a+1,b+1)}$. The non-filled circles represent the non-dominant weights in the block occurring on the walls of the Weyl chamber. The filled circles represent dominant weights in the block. The vertical arrows are maps $\lambda \mapsto \lambda + \gamma$, with $\gamma \in \Delta$ is the root above the arrow.

In the picture below, t_i are as defined before and represent the indices c corresponding to acyclic weights. Let λ_i denote the starting vertices of the vertical arrows and μ_i be the corresponding end vertices. In this section, we will show that the solid arrows represent the maps $L_\lambda \mapsto T(L_\lambda)$.



Lemma 8.2.1 *For a weight $\lambda \in F^{(a,b)}$, if $\lambda+\gamma \in F^{(a+1,b+1)}$, then the correspondingc$_\lambda$ and $c_{\lambda+\gamma}$ are either in the same interval J_j or adjacent ones.*

Proof. Given $c \in \frac{1}{2}\mathbb{Z}$, by theorem 6.5, there is at most one dominant $\lambda \in F^{(a,b)}$, with $\lambda_c = \lambda$, for $c \in I_i$. We want to show that both c and $c+\gamma_4$ in the same or adjacent intervals I_j.

Say $\lambda + \rho = (b_1, b_2, b_3|b_4)$, then $b_4 = c$ if $i = 1, 2, 3, 4$ and $b_4 = -c$ if $i = 5, 6, 7, 8$.

Assume $b_4 = c \in I_i$ with $i = 1, 2, 3, 4$, we claim that there is no $\gamma \in \Delta$ such that $\lambda + \gamma \in F^{(a+1,b+1)}$ and $-(b_4 + \gamma_4) \in I_i$ with $i = 6, 7, 8$. If $b_4 \in I_i$ $i = 1, 2, 3, 4$, then $b_1 - b_4 = \frac{2a+b}{3}$, while if $b_4 \in I_i$ $i = 6, 7, 8$, then $b_1 - b_4 = \frac{a+2b}{3}$. Now, if such γ exists, we will have $b_1 + \gamma_1 - b_4 - \gamma_4 = \frac{2a+b}{3} + (\gamma_1 - \gamma_4) = \frac{a+2b}{3} + 1$, which implies $\gamma_1 - \gamma_4 = \frac{-a+b}{3} + 1 = -n + 1$. The last number must be in the interval $[-1, 1]$, since $\gamma \in \Delta$. But this is only possible if $n = 1$.

Similarly, if $-b_4 = c \in I_i$ with $i = 6, 7, 8$ and γ is such that $b_4 + \gamma_4 \in I_i$ with $i = 1, 2, 3, 4$ we have $b_1 - b_4 = \frac{a+2b}{3}$ and $b_1 + \gamma_1 - b_4 - \gamma_4 = \frac{2a+b}{3} + 1$, and we get $\gamma_1 - \gamma_4 = \frac{2a+b}{3} + 1 - \frac{a+2b}{3} = \frac{a-b}{3} + 1 = n + 1$. This is a contradiction since $\gamma_1 - \gamma_4 \in [-1, 1]$, for $\gamma \in \Delta$. It is also not possible to have $\lambda \in F^{(a,b)}$ with $\lambda + \rho = (b_1, b_2, b_3|b_4)$ and $-b_4 \in I_5$ and $\gamma \in \Delta$ with $\lambda + \gamma \in F^{(a+1,b+1)}$ and $b_4 + \gamma_4 \in I_i$ with $i = 1, 2, 3$, since if $-b_4 \in I_5$ implies $0 < b_4 < \frac{a-b}{6} < \frac{a-b}{3}$ implying $b_4 + \gamma_4 \in I_4$ or I_5.

The case $n = 1$ can be checked separately. □

The following lemma justifies the above picture.

Lemma 8.2.2 *For $\lambda \in F^{(a,b)}$, there is a unique $\gamma \in \Delta$ such that $\lambda + \gamma \in \mathcal{F}^{(a+1,b+1)}$ is dominant, unless $\lambda + \rho = (a + b + \frac{1}{2}, b + \frac{1}{2}, \frac{1}{2}|\frac{a+2b}{3} + \frac{1}{2})$ or $\lambda + \rho = (a + b + \frac{1}{2}, a + \frac{1}{2}, \frac{1}{2}|\frac{2a+b}{3} + \frac{1}{2})$.*

Proof. Assume $\gamma \in \Delta$ is such that $\lambda + \gamma \in F^{(a+1,b+1)}$. We first show that the c corresponding to $\lambda + \gamma + \rho$ and $\lambda + \rho$ is the same in generic cases. By Remark 4.4.6, this will imply that there is at most one such γ, proving the uniqueness.

Assume that the last coordinate of $\lambda + \rho$ is c. Then $\lambda + \rho + \gamma$ must have last coordinate $c \pm 1$, $c \pm \frac{1}{2}$, or c.

Thus for generic λ, $(\lambda + \rho, \alpha) = 0$ and $(\lambda + \gamma + \rho, \alpha) = 0$ are true for the same $\alpha \in \Delta_{\bar{1}}^+$ (see Remark 4.4.6 above). That implies $(\gamma, \alpha) = 0$. Thus, $\gamma \neq \delta$. If γ is odd then $(\gamma, \alpha) = 0$ implies $\gamma = \pm\alpha$, which is impossible, since then λ and $\lambda + \gamma$ correspond to the same central character from Lemma 4.4.1. If $\gamma \neq \delta$ is even the statement is clear.

The few exceptional cases occur around walls of the Weyl chamber, when $c = \frac{a+2b}{3} + 1$, $\frac{a+2b}{3} + \frac{1}{2}$, $\frac{2a+b}{6} + \frac{1}{2}$, $\frac{a+2b}{6} + \frac{1}{2}$, $\frac{2a+b}{3} + \frac{1}{2}$, and $\frac{2a+b}{3} + 1$. We can see that only in the second and fifth places there are two such γ. □

Lemma 8.2.3 *We have $T(L_{\lambda_i}) = L_{\mu_i}$, for all $\lambda_i \neq \lambda_c$ with $c = \frac{a+2b}{3} + \frac{1}{2}$ or $\frac{2a+b}{3} + \frac{1}{2}$.*

Proof. By definition, $T(L_{\lambda_i}) = (L_{\lambda_i} \otimes \mathfrak{g})^{(a+1,b+1)}$. As one can see from the picture above, for each i, there is a unique dominant weight μ_i in $\mathcal{F}^{(a+1,b+1)}$ of the form $\lambda_i + \gamma$ with $\gamma \in \Delta$. Thus, the lemma follows from Lemma 5.2.7. □

Lemma 8.2.4 *For $c = \frac{a+2b}{3} + \frac{1}{2} = t_1 + \frac{1}{2}$, we have $T(L_{\lambda_c}) = L_{\mu_{c'}}$ with $c' = \frac{a+2b}{3} = t_1$. Similarly, for $c = \frac{2a+b}{3} + \frac{1}{2} = t_2 + \frac{1}{2}$, we have $T(L_{\lambda_c}) = L_{\mu_{c'}}$ with $c' = \frac{2a+b}{3} = t_2$.*

Proof. By definition, $T(L_{\lambda_c}) = (L_{\lambda_c} \otimes \mathfrak{g})^{(a+1,b+1)}$. The only dominant weights with central character corresponding to block $\mathcal{F}^{(a+1,b+1)}$ of the form $\lambda_c + \gamma$ with $\gamma \in \Delta$ are μ_c and $\mu_{c'}$.

Let $c'' = c + \frac{1}{2}$. We know that $L_{\lambda_{c''}}$ is a quotient of $\Gamma_0(G/B, \mathcal{O}_{\lambda_{c''}})$ from Lemma 5.1.4. From Lemma 6.2.2 and Lemma 5.1.5, we obtain the following exact sequence:

$$0 \to L_{\lambda_c} \to \Gamma_0(G/B, \mathcal{O}_{\lambda_{c''}}) \to L_{\lambda_{c''}} \to 0$$

Since T is an exact functor, we have the following exact sequence:

$$0 \to T(L_{\lambda_c}) \to T(\Gamma_0(G/B, \mathcal{O}_{\lambda_{c''}})) \to T(L_{\lambda_{c''}}) \to 0.$$

From Lemma 8.2.3, we have $T(L_{\lambda_{c''}}) = L_{\mu_c}$. By lemma Lemma 5.2.1 and Lemma 5.2.2, we have

$$T(\Gamma_0(G/B, \mathcal{O}_{\lambda_{c''}})) = \Gamma_0(G/B, T(\mathcal{O}_{\lambda_{c''}})) = \Gamma_0(G/B, \mathcal{O}_{\mu_c}).$$

The module $\Gamma_0(G/B, \mathcal{O}_{\mu_c})$ has a unique quotient L_{μ_c}. Therefore, $T(L_{\lambda_c})$ has no simple subquotient L_{μ_c}, which is sufficient to prove the lemma.

□

Theorem 8.2.5 *We have an equivalence between categories $\mathcal{F}^{(a,b)}$ and $\mathcal{F}^{(a+1,b+1)}$.*

Proof. From previous lemma, for each $\lambda_i \in F^{(a+1,b+1)}$, $T(L_{\lambda_i})$ is a simple module in $\mathcal{F}^{(a+1,b+1)}$, we denote $L_{\mu_i} = T(L_{\lambda_i})$ the simple module with highest weight $\mu_i \in F^{(a+1,b+1)}$. We show that the the conditions of Theorem 5.2.8 are satisfied.

For all $\mu \neq \lambda_c \in F^{(a+1,b+1)}$ with $c = t_2 + \frac{1}{2}$ or $t_1 + \frac{1}{2}$, we have a unique $\gamma \in \Delta$, such that $\mu + \gamma \in F^{(a,b)}$.

From the picture above, for $\mu = \lambda_c \in F^{(a+1,b+1)}$ with $c = t_2 + \frac{1}{2}$ or $t_1 + \frac{1}{2}$, there are two possible $\gamma \in \Delta$ such that $\mu + \gamma \in F^{(a,b)}$.

Here, $\mu + \gamma = \lambda_{t_2+\frac{1}{2}}$ and λ_{t_2+1} or $\mu + \gamma = \lambda_{t_1+\frac{1}{2}}$ and λ_{t_1+1} correspondingly such that $\lambda_{t_2+\frac{1}{2}} < \lambda_{t_2+1}$ and $\lambda_{t_1+\frac{1}{2}} < \lambda_{t_1+1}$.

The theorem follows from Theorem 5.2.8.

□

8.3 Cohomology groups in the block $F^{(a,b)}$ with $a = b + 3$.

We let $b = 1$. In the block $F^{(4,1)}$, the dominant weights close to the walls of the Weyl chamber are denoted:

$\lambda_7 + \rho = (\frac{11}{2}, \frac{3}{2}, \frac{1}{2}|\frac{5}{2})$;
$\lambda_6 + \rho = (\frac{7}{2}, \frac{3}{2}, \frac{1}{2}|\frac{1}{2})$;
$\lambda_0 + \rho = (3, 2, 1|0)$;

$\lambda_1+\rho=(\frac{7}{2},\frac{5}{2},\frac{3}{2}|\frac{3}{2})$;
$\lambda_2+\rho=(4,3,1|2)$;
$\lambda_3+\rho=(\frac{9}{2},\frac{7}{2},\frac{1}{2}|\frac{5}{2})$;
$\lambda_4+\rho=(\frac{11}{2},\frac{9}{2},\frac{1}{2}|\frac{7}{2})$.

(Note that indices for λ above are different from the index c that corresponds to the last coordinate of $\lambda+\rho$.)

Lemma 8.3.1 *For all $\lambda\in F^{(4,1)}$ such that $\lambda\neq\lambda_0$, we have $\Gamma_1(G/B,\mathcal{O}_\lambda)=0$.*

Proof. For generic weights, this follows from Lemma 6.2.2. For weights close to the walls of the Weyl chamber, we compute from Lemma 5.1.5 in a similar way as for $\mathcal{F}^{(1,1)}$ in Lemma 7.1.2 or for generic weights. □

Lemma 8.3.2 *For non-generic weight $\lambda=\lambda_4\in F^{(4,1)}$, we have an exact sequence:*

$$0\longrightarrow L_{\lambda_4}\longrightarrow\Gamma_0(G/B,\mathcal{O}_{\lambda_4})\longrightarrow L_{\lambda_3}\longrightarrow 0$$

Proof. From Lemma 5.1.5, we have $[\Gamma_0(G/B,\mathcal{O}_{\lambda_4}):L_{\lambda_3}]\leq 1$ and $[\Gamma_0(G/B,\mathcal{O}_{\lambda_4}):L_{\lambda_\sigma}]=0$ for $\sigma\neq\lambda_3,\lambda_4$.

Also, we have:

$$0=sdim\Gamma_0(G/B,\mathcal{O}_{\lambda_4})=sdimL_{\lambda_4}+[\Gamma_0(G/B,\mathcal{O}_{\lambda_4}):L_{\lambda_3}]sdimL_{\lambda_3}. \qquad (8.1)$$

Since, starting with generic weight, we have $sdimL_{\lambda_4}\neq 0$, this imp less $[\Gamma_0(G/B,\mathcal{O}_{\lambda_4}):L_{\lambda_3}]\neq 0$, proving the lemma. □

Lemma 8.3.3 *For non-generic weight $\lambda=\lambda_7\in F^{(4,1)}$, we have an exact sequence:*

$$0\longrightarrow L_{\lambda_7}\longrightarrow\Gamma_0(G/B,\mathcal{O}_{\lambda_7})\longrightarrow L_{\lambda_6}\longrightarrow 0$$

Proof. From Lemma 5.1.5, we have $[\Gamma_0(G/B,\mathcal{O}_{\lambda_7}):L_{\lambda_6}]\leq 1$ and $[\Gamma_0(G/B,\mathcal{O}_{\lambda_7}):L_{\lambda_\sigma}]=0$ for $\sigma\neq\lambda_6,\lambda_7$.

Also, we have:

$$0=sdim\Gamma_0(G/B,\mathcal{O}_{\lambda_7})=sdimL_{\lambda_7}+[\Gamma_0(G/B,\mathcal{O}_{\lambda_7}):L_{\lambda_6}]sdimL_{\lambda_6}. \qquad (8.2)$$

Since, starting with generic weight, we have $sdimL_{\lambda_7} \neq 0$, this implies $[\Gamma_0(G/B, \mathcal{O}_{\lambda_7}) : L_{\lambda_6}] \neq 0$, proving the lemma.

□

Lemma 8.3.4 *For non-generic weights $\lambda = \lambda_3, \lambda_6 \in F^{(4,1)}$, we have the following exact sequences:*

$$0 \longrightarrow L_{\lambda_3} \longrightarrow \Gamma_0(G/B, \mathcal{O}_{\lambda_3}) \longrightarrow L_{\lambda_2} \longrightarrow 0$$

$$0 \longrightarrow L_{\lambda_6} \longrightarrow \Gamma_0(G/B, \mathcal{O}_{\lambda_6}) \longrightarrow L_{\lambda_0} \longrightarrow 0$$

Proof. From Lemma 5.1.5, we have $[\Gamma_0(G/B, \mathcal{O}_{\lambda_3}) : L_{\lambda_2}] \leq 1$ and $[\Gamma_0(G/B, \mathcal{O}_{\lambda_3}) : L_{\lambda_\sigma}] = 0$ for $\sigma \neq \lambda_3, \lambda_2$. Similarly, we have $[\Gamma_0(G/B, \mathcal{O}_{\lambda_6}) : L_{\lambda_0}] \leq 1$ and $[\Gamma_0(G/B, \mathcal{O}_{\lambda_6}) : L_{\lambda_\sigma}] = 0$ for $\sigma \neq \lambda_6, \lambda_0$.

Also, we have

$$0 = sdim\Gamma_0(G/B, \mathcal{O}_{\lambda_3}) = sdimL_{\lambda_3} + [\Gamma_0(G/B, \mathcal{O}_{\lambda_3}) : L_{\lambda_2}]sdimL_{\lambda_2}.$$

From equation 8.1, it follows that $sdimL_{\lambda_3} \neq 0$, since we have $sdimL_{\lambda_4} \neq 0$. Thus, we have

$$[\Gamma_0(G/B, \mathcal{O}_{\lambda_3}) : L_{\lambda_2}] \neq 0,$$

proving the first exact sequence. Similarly, we have the second exact sequence using equation 8.2. □

Lemma 8.3.5 *For non-generic weight $\lambda = \lambda_0 \in F^{(4,1)}$, we have $\Gamma_0(G/B, \mathcal{O}_{\lambda_0}) = L_{\lambda_0}$ and $\Gamma_1(G/B, \mathcal{O}_{\lambda_0}) = L_{\lambda_0}$.*

Proof. From Lemma 5.1.5 and Lemma 5.1.2, we have $[\Gamma_0(G/B, \mathcal{O}_{\lambda_0}) : L_{\lambda_0}] = 1$ and $[\Gamma_0(G/B, \mathcal{O}_{\lambda_0}) : L_{\lambda_\sigma}] = 0$ for $\sigma \neq \lambda_0$. Also, that $\Gamma_i(G/B, \mathcal{O}_{\lambda_0}) = 0$ for $i > 1$.

Also, we have $0 = sdim\Gamma_0(G/B, \mathcal{O}_{\lambda_0}) - sdim\Gamma_1(G/B, \mathcal{O}_{\lambda_0})$. This implies $\Gamma_1(G/B, \mathcal{O}_{\lambda_0}) \neq 0$. Since Lemma 5.1.5 implies that any simple subquotient of $\Gamma_1(G/B, \mathcal{O}_{\lambda_0})$ has highest weight less than λ_0, we must have $\Gamma_1(G/B, \mathcal{O}_{\lambda_0}) = L_{\lambda_0}$.

□

It remains to understand the cohomology groups for the dominant non-generic weights λ_1, λ_2. These cases are more complicated and we first prove the following lemma:

Lemma 8.3.6 *For all $\lambda \in F^{(4,1)}$, we have $sdim L_\lambda = \pm d$, where $d = dim\, L_\mu(\mathfrak{g}_x)$, where μ is from theorem Theorem 4.4.9.*

Proof. Starting with generic weights λ and using the Theorem 6.1.4 for generic weight, we have $sdim L_\lambda = \pm d$ for generic weight. For the weights close to the walls of the Weyl chamber, we use the above lemmas and exact sequences to show this.

From exact sequences in Lemma 8.3.2, Lemma 8.3.3, Lemma 8.3.4, we know that $sdim L_i = \pm d$ for $i = 6, 0, 2$. Since, in each case we know that $\Gamma_0(G/B, \mathcal{O}_{\lambda_i}) = 0$ and $sdim L_j = \pm d$ for the other L_j in the exact sequence.

To prove that $sdim L_1 = \pm d$ is more challenging. We first apply translation functor T to the dominant weights λ_0, λ_1, λ_2, λ_6 twice to get dominant weights λ'_0, λ'_1, λ'_2, λ'_6 in the equivalent block $\mathcal{F}^{(6,3)}$.

The categories $\mathcal{F}^{(4,1)}$ and $\mathcal{F}^{(6,3)}$ are equivalent from Theorem 8.2.5. Thus, by Lemma 5.2.4, we have $[\Gamma_0(G/B, \mathcal{O}_{\lambda'}) : L_{\lambda'}] = [\Gamma_0(G/B, \mathcal{O}_\lambda) : L_\lambda]$.

We apply odd reflections with respect to odd roots $\beta, \beta', \beta'', \beta'''$ to obtain dominant weights λ''_0, λ''_1, λ''_2, λ''_6 with respect to another Borel subalgebra B''.

We get the following:

$\lambda'_6 + \rho = (\frac{11}{2}, \frac{7}{2}, \frac{1}{2}|\frac{1}{2})$;
$\lambda'_0 + \rho = (5, 4, 1|0)$;
$\lambda'_1 + \rho = (5, 4, 2|1)$;
$\lambda'_2 + \rho = (\frac{11}{2}, \frac{7}{2}, \frac{5}{2}|\frac{3}{2})$;
$\lambda'_3 + \rho = (\frac{13}{2}, \frac{7}{2}, \frac{5}{2}|\frac{5}{2})$;
$\lambda'_4 + \rho = (7, 4, 2|3)$.

After applying the odd reflections we get the following dominant weights with respect to the new Borel B'':

$\lambda''_6 + \rho'' = (\frac{11}{2}, \frac{7}{2}, \frac{1}{2}|\frac{1}{2})$;
$\lambda''_0 + \rho'' = (\frac{9}{2}, \frac{9}{2}, \frac{3}{2}|\frac{1}{2})$;
$\lambda''_1 + \rho'' = (5, 4, 2|1)$;
$\lambda''_2 + \rho'' = (\frac{11}{2}, \frac{7}{2}, \frac{5}{2}|\frac{3}{2})$;
$\lambda''_3 + \rho'' = (\frac{13}{2}, \frac{7}{2}, \frac{5}{2}|\frac{5}{2})$;

$\lambda''_4 + \rho'' = (7, 4, 2|3)$.

From Lemma 3.2.2, the positive odd roots with respect to the new Borel B'' are all the odd roots with first coordinate $\frac{1}{2}$.

From Lemma 5.1.5 with respect to B'', we have $[\Gamma_0(G/B'', \mathcal{O}_{\lambda''_2}) : L_{\lambda''_1}] \leq 1$ and $[\Gamma_0(G/B'', \mathcal{O}_{\lambda''_2}) : L_{\lambda''_\sigma}] = 0$ for all $\sigma \neq \lambda''_1, \lambda''_2$.

We also have

$$0 = sdim\Gamma_0(G/B'', \mathcal{O}_{\lambda''_2}) =$$
$$= sdimL_{\lambda''_2} + [\Gamma_0(G/B'', \mathcal{O}_{\lambda''_2}) : L_{\lambda''_1}]sdimL_{\lambda''_1}$$

implying that $[\Gamma_0(G/B'', \mathcal{O}_{\lambda''_2}) : L_{\lambda''_1}] = 1$ and $sdimL_{\lambda''_1} = \pm d$. Now we have $sdimL_{\lambda'_1} = sdimL_{\lambda''_1} = \pm d$. □

Lemma 8.3.7 *For non-generic weight $\lambda = \lambda_1 \in F^{(4,1)}$, we have an exact sequence:*

$$0 \longrightarrow L_{\lambda_1} \longrightarrow \Gamma_0(G/B, \mathcal{O}_{\lambda_1}) \longrightarrow L_{\lambda_0} \longrightarrow 0$$

Proof. From computation using Lemma 5.1.5, it follows that $[\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) : L_{\lambda_0}] = [\Gamma_0(G/B, T(\mathcal{O}_{\lambda_1})) : T(L_{\lambda_0})] \leq 2$ and $[\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) : L_{\lambda_\sigma}] = 0$ for $\sigma \neq \lambda_0, \lambda_1$.

We also have

$$0 = sdim\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) = sdimL_{\lambda_1} + [\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) : L_{\lambda_0}]sdimL_{\lambda_0}.$$

From Lemma 8.3.6, we know that $sdimL_{\lambda_1} = -sdimL_{\lambda_0} = \pm d$. We must have $[\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) : L_{\lambda_0}] = 1$, proving the lemma. □

We will call an odd reflection r *typical* with respect to the weight λ if $r(\lambda) = \lambda$.

Lemma 8.3.8 *([17]) If an odd reflection r is typical with respect to the weight λ, then $\Gamma_0(G/r(B), \mathcal{O}_{r(\lambda)}) = \Gamma_0(G/B, \mathcal{O}_\lambda)$.*

Lemma 8.3.9 *For non-generic weight $\lambda = \lambda_2 \in F^{(4,1)}$, we have an exact sequence:*

$$0 \longrightarrow L_{\lambda_2} \longrightarrow \Gamma_0(G/B, \mathcal{O}_{\lambda_2}) \longrightarrow L_{\lambda_1} \longrightarrow 0$$

Proof. Follows from computation using Lemma 5.1.5, that $[\Gamma_0(G/B, \mathcal{O}_{\lambda_2}) : L_{\lambda_1}] \leq 1$, $[\Gamma_0(G/B, \mathcal{O}_{\lambda_2}) : L_{\lambda_6}] \leq 1$, and $[\Gamma_0(G/B, \mathcal{O}_{\lambda_2}) : L_{\lambda_\sigma}] = 0$ for $\sigma \neq \lambda_1, \lambda_2, \lambda_6$.

We also know that

$$0 = sdim\Gamma_0(G/B, \mathcal{O}_{\lambda_2}) = sdimL_{\lambda_2}+$$

$$+[\Gamma_0(G/B, \mathcal{O}_{\lambda_2}) : L_{\lambda_1}]sdimL_{\lambda_1} + [\Gamma_0(G/B, \mathcal{O}_{\lambda_2}) : L_{\lambda_6}]sdimL_{\lambda_6}.$$

From Lemma 8.3.6, we know that $sdimL_{\lambda_2} = \pm sdimL_{\lambda_1} = \pm sdimL_{\lambda_6} = \pm d \neq 0$. This implies that one of the numbers $[\Gamma_0(G/B, \mathcal{O}_{\lambda_2}) : L_{\lambda_1}]$ or $[\Gamma_0(G/B, \mathcal{O}_{\lambda_2}) : L_{\lambda_6}]$ is one and another is zero.

We prove $[\Gamma_0(G/B, \mathcal{O}_{\lambda_2}) : L_{\lambda_6}] = 0$.

The odd reflections with respect to the weight λ_2' are typical, which means that the weight doesn't change. From Lemma 8.3.8, this implies that $\Gamma_0(G/B, \mathcal{O}_{\lambda_2'}) = \Gamma_0(G/B'', \mathcal{O}_{\lambda_2''})$. The later module has subquotients $L_{\lambda_1''} = L_{\lambda_1'}$ and $L_{\lambda_0''} = L_{\lambda_0'}$. Thus, $[\Gamma_0(G/B, \mathcal{O}_{\lambda_2'}) : L_{\lambda_6'}] = 0$.

Since T is an equivalence of categories from Theorem 8.1.4, from Lemma 5.2.4 we have $[\Gamma_0(G/B, \mathcal{O}_{\lambda'}) : L_{\lambda'}] = [\Gamma_0(G/B, \mathcal{O}_{\lambda}) : L_{\lambda}]$, which proves the exact sequence. □

8.4 Cohomology groups in the block $F^{(a,b)}$ with $a = b + 3n$, $n > 1$.

For $n > 1$, we assume $b = 1$. The dominant weights close to the walls of the Weyl chamber have different arrangements in this case and they are correspondingly denoted:

$\lambda_{t_2+1} + \rho = (a + 2, 2, 1|t_2 + 1)$;
$\lambda_{t_2+\frac{1}{2}} + \rho = (a + \frac{3}{2}, \frac{3}{2}, \frac{1}{2}|t_2 + \frac{1}{2})$;
$\lambda_{t_3-\frac{1}{2}} + \rho = (a - \frac{1}{2}, \frac{3}{2}, \frac{1}{2}|t_3 - \frac{1}{2})$;
$\lambda_{t_3-1} + \rho = (a - 1, 2, 1|t_3 - 1)$;
...
$\lambda_{\frac{1}{2}} + \rho = (t_1 + \frac{1}{2}, t_2 - \frac{1}{2}, t_3 - \frac{1}{2}|\frac{1}{2})$;
$\lambda_0 + \rho = (t_1, t_2, t_3|0)$;
$\lambda_{-\frac{1}{2}} + \rho = (t_1 - \frac{1}{2}, t_2 + \frac{1}{2}, t_3 + \frac{1}{2}|\frac{1}{2})$;
...
$\lambda_{-\frac{t_3}{2}+1} + \rho = (\frac{a}{2} + \frac{3}{2}, \frac{a}{2} - \frac{1}{2}, \frac{a}{2} - \frac{3}{2}|\frac{t_3}{2} - 1)$;
$\lambda_{-\frac{t_3}{2}+\frac{1}{2}} + \rho = (\frac{a}{2} + 1, \frac{a}{2}, \frac{a}{2} - 1|\frac{t_3}{2} - \frac{1}{2})$;

$\lambda_{-\frac{t_2}{2}-\frac{1}{2}}+\rho=(\frac{a}{2}+\frac{3}{2},\frac{a}{2}+\frac{1}{2},\frac{a}{2}-\frac{1}{2}|\frac{t_2}{2}+\frac{1}{2});$
$\lambda_{-\frac{t_2}{2}-1}+\rho=(\frac{a}{2}+2,\frac{a}{2}+1,\frac{a}{2}-1|\frac{t_2}{2}+1);$
...
$\lambda_{-t_1+1}+\rho=(a,a-1,1|t_1-1);$
$\lambda_{-t_1+\frac{1}{2}}+\rho=(a+\frac{1}{2},a-\frac{1}{2},\frac{1}{2}|t_1-\frac{1}{2});$
$\lambda_{-t_1-\frac{1}{2}}+\rho=(a+\frac{3}{2},a+\frac{1}{2},\frac{1}{2}|t_1+\frac{1}{2});$
$\lambda_{-t_1-1}+\rho=(a+2,a+1,1|t_1+1).$

Lemma 8.4.1 *For all $\lambda\in F^{(a,b)}$ such that $\lambda\neq\lambda_0$, we have $\Gamma_1(G/B,\mathcal{O}_\lambda)=0$.*

Proof. For generic weights, this follows from Lemma 6.2.2. For weights close to the walls of the Weyl chamber, we compute from Lemma 5.1.5 in a similar way as for $\mathcal{F}^{(1,1)}$ in Lemma 7.1.2 or for generic weights. □

Lemma 8.4.2 *For non-generic weight $\lambda=\lambda_{t_2+1}\in F^{(a,1)}$, we have an exact sequence:*

$$0\longrightarrow L_{\lambda_{t_2+1}}\longrightarrow\Gamma_0(G/B,\mathcal{O}_{\lambda_{t_2+1}})\longrightarrow L_{\lambda_{t_2+\frac{1}{2}}}\longrightarrow 0$$

Proof. Follows from computation using Lemma 5.1.5, that $[\Gamma_0(G/B,\mathcal{O}_{\lambda_{t_2+1}}):L_{\lambda_{t_2+\frac{1}{2}}}]\leq 1$ and $[\Gamma_0(G/B,\mathcal{O}_{\lambda_{t_2+1}}):L_{\lambda_\sigma}]=0$ for $\sigma\neq\lambda_{t_2+\frac{1}{2}},\lambda_{t_2+1}$.

We also know that

$$0=sdim\Gamma_0(G/B,\mathcal{O}_{\lambda_{t_2+1}})=sdimL_{\lambda_{t_2+1}}+[\Gamma_0(G/B,\mathcal{O}_{\lambda_{t_2+1}}):L_{\lambda_{t_2+\frac{1}{2}}}]sdimL_{\lambda_{t_2+\frac{1}{2}}}.$$

Since, starting with generic weight, we know that $sdimL_{\lambda_{t_2+1}}\neq 0$, we must have that $[\Gamma_0(G/B,\mathcal{O}_{\lambda_{t_2+1}}):L_{\lambda_{t_2+\frac{1}{2}}}]\neq 0$, proving the lemma.

□

Lemma 8.4.3 *For non-generic weight $\lambda=\lambda_{-t_1-1}\in F^{(a,1)}$, we have an exact sequence:*

$$0\longrightarrow L_{\lambda_{-t_1-1}}\longrightarrow\Gamma_0(G/B,\mathcal{O}_{\lambda_{-t_1-1}})\longrightarrow L_{\lambda_{-t_1-\frac{1}{2}}}\longrightarrow 0$$

Proof. Similar to Lemma 8.4.11. □

Lemma 8.4.4 *For non-generic weight $\lambda=\lambda_{-t_1-\frac{1}{2}}\in F^{(a,1)}$, we have an exact sequence:*

$$0\longrightarrow L_{\lambda_{-t_1-\frac{1}{2}}}\longrightarrow\Gamma_0(G/B,\mathcal{O}_{\lambda_{-t_1-\frac{1}{2}}})\longrightarrow L_{\lambda_{-t_1+\frac{1}{2}}}\longrightarrow 0$$

Proof. Similar to Lemma 8.3.2. □

Lemma 8.4.5 *For non-generic weight $\lambda = \lambda_{t_2+\frac{1}{2}} \in F^{(a,1)}$, we have an exact sequence:*

$$0 \longrightarrow L_{\lambda_{t_2+\frac{1}{2}}} \longrightarrow \Gamma_0(G/B, \mathcal{O}_{\lambda_{t_2+\frac{1}{2}}}) \longrightarrow L_{\lambda_{t_3-\frac{1}{2}}} \longrightarrow 0$$

Proof. Similar to Lemma 8.3.3. □

Lemma 8.4.6 *For non-generic weight $\lambda = \lambda_c \in F^{(a,1)}$ with $c \in I_4$, we have an exact sequence:*

$$0 \longrightarrow L_{\lambda_c} \longrightarrow \Gamma_0(G/B, \mathcal{O}_{\lambda_c}) \longrightarrow L_{\lambda_c-\beta_4} \longrightarrow 0$$

where $\beta_4 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}|\frac{1}{2})$

Proof. We use Lemma 8.4.5, together with induction. □

Lemma 8.4.7 *For non-generic weight $\lambda = \lambda_c \in F^{(a,1)}$ with $c \in I_7$ such that $c \neq -\frac{t_2}{2} - \frac{1}{2}, -\frac{t_2}{2} - 1$, we have an exact sequence:*

$$0 \longrightarrow L_{\lambda_c} \longrightarrow \Gamma_0(G/B, \mathcal{O}_{\lambda_c}) \longrightarrow L_{\lambda_c-\beta_7} \longrightarrow 0$$

where $\beta_7 = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}|\frac{1}{2})$.

Proof. We use Lemma 8.4.5, together with induction. □

Lemma 8.4.8 *For non-generic weight $\lambda = \lambda_c \in F^{(a,1)}$ with $c \in I_5$, we have an exact sequence:*

$$0 \longrightarrow L_{\lambda_c} \longrightarrow \Gamma_0(G/B, \mathcal{O}_{\lambda_c}) \longrightarrow L_{\lambda_c-\beta_5} \longrightarrow 0$$

where $\beta_5 = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}|\frac{1}{2})$.

Proof. We use Lemma 8.4.6, together with induction. □

Lemma 8.4.9 *For $\lambda \in F^{(a,b)}$ such that $\lambda = \lambda_0$, we have* $\Gamma_0(G/B, \mathcal{O}_{\lambda_0}) = \Gamma_1(G/B, \mathcal{O}_{\lambda_0}) = L_{\lambda_0}$.

Proof. From Lemma 5.1.5, we have that any simple subquotient in $\Gamma_0(G/B, \mathcal{O}_{\lambda_0})$ has weight less or equal to λ_0. Thus, $\Gamma_0(G/B, \mathcal{O}_{\lambda_0}) = L_{\lambda_0}$.

Also, we have

$$sdim\,\Gamma_0(G/B, \mathcal{O}_{\lambda_0}) = sdim\,\Gamma_1(G/B, \mathcal{O}_{\lambda_0})$$

and $\Gamma_0(G/B, \mathcal{O}_{\lambda_0}) = L_{\lambda_0}$. Since from above lemma, $sdim\lambda_0 \neq 0$, we have $\Gamma_1(G/B, \mathcal{O}_{\lambda_0}) \neq 0$. From Lemma 5.1.5, we can see that any simple subquotient in $\Gamma_1(G/B, \mathcal{O}_{\lambda_0})$ has weight less or equal to λ_0. This proves the lemma. □

It remains to understand the cohomology groups for weights with $c \in I_5$, and weights λ_c with $c \neq -\frac{t_2}{2} - \frac{1}{2}, -\frac{t_2}{2} - 1 \in I_7$.

We need the following lemma first:

Lemma 8.4.10 *We have* $sdimL_{\lambda_{-\frac{t_3}{2}-2}} = -sdimL_{\lambda_{-\frac{t_3}{2}}}$.

Proof. Follows from Lemma 6.2.3 and Lemma 8.4.8 and the fact that the parity of the weight in I_6 will coincide with the sign of the superdimension. □

Lemma 8.4.11 *For* $\lambda_{-\frac{t_2}{2}-1}, \lambda_{-\frac{t_2}{2}-\frac{1}{2}} \in F^{(a,1)}$*, we have exact sequences:*

$$0 \longrightarrow L_{\lambda_{-\frac{t_2}{2}-1}} \longrightarrow \Gamma_0(G/B, \mathcal{O}_{\lambda_{-\frac{t_2}{2}-1}}) \longrightarrow L_{\lambda_{-\frac{t_2}{2}-\frac{1}{2}}} \longrightarrow 0$$

$$0 \longrightarrow L_{\lambda_{-\frac{t_2}{2}-\frac{1}{2}}} \longrightarrow \Gamma_0(G/B, \mathcal{O}_{\lambda_{-\frac{t_2}{2}-\frac{1}{2}}}) \longrightarrow L_{\lambda_{-\frac{t_3}{2}+\frac{1}{2}}} \longrightarrow 0$$

Proof. Since Lemma 5.1.5, doesn't give good description of cohomology groups in these cases, we first apply translation functor to the dominant weights $\lambda_{-\frac{t_3}{2}+1}$, $\lambda_{-\frac{t_3}{2}+\frac{1}{2}}$, $\lambda_{-\frac{t_2}{2}-\frac{1}{2}}$, $\lambda_{-\frac{t_2}{2}-1}$ twice to get dominant weights $\lambda'_{-\frac{t_3}{2}+1}$, $\lambda'_{-\frac{t_3}{2}+\frac{1}{2}}$, $\lambda'_{-\frac{t_2}{2}-\frac{1}{2}}$, $\lambda'_{-\frac{t_2}{2}-1}$ in the equivalent block $\mathcal{F}^{(a+2,3)}$.

Then we apply odd reflections with respect to odd roots $\beta, \beta', \beta'', \beta'''$ to obtain dominant weights $\lambda''_{\frac{-t_3}{2}+1}$, $\lambda''_{-\frac{t_3}{2}+\frac{1}{2}}$, $\lambda''_{-\frac{t_2}{2}-\frac{1}{2}}$, $\lambda''_{-\frac{t_2}{2}-1}$ with respect to another Borel subalgebra B''.

We have:
$\lambda_{-\frac{t_3}{2}+1} + \rho = (\frac{a}{2} + \frac{3}{2}, \frac{a}{2} - \frac{1}{2}, \frac{a}{2} - \frac{3}{2} | \frac{t_3}{2} - 1)$;
$\lambda_{-\frac{t_3}{2}+\frac{1}{2}} + \rho = (\frac{a}{2} + 1, \frac{a}{2}, \frac{a}{2} - 1 | \frac{t_3}{2} - \frac{1}{2})$;

$\lambda_{-\frac{t_2}{2}-\frac{1}{2}} + \rho = (\frac{a}{2}+\frac{3}{2}, \frac{a}{2}+\frac{1}{2}, \frac{a}{2}-\frac{1}{2} | \frac{t_2}{2}+\frac{1}{2});$
$\lambda_{-\frac{t_2}{2}-1} + \rho = (\frac{a}{2}+2, \frac{a}{2}+1, \frac{a}{2}-1 | \frac{t_2}{2}+1).$

After applying the translation functor twice, we have:

$\lambda'_{-\frac{t_3}{2}+1} + \rho = (\frac{a}{2}+2, \frac{a}{2}+1, \frac{a}{2}-2 | \frac{t_3}{2}-\frac{1}{2});$
$\lambda'_{-\frac{t_3}{2}+\frac{1}{2}} + \rho = (\frac{a}{2}+2, \frac{a}{2}+1, \frac{a}{2}-1 | \frac{t_3}{2}+\frac{1}{2});$
$\lambda'_{-\frac{t_2}{2}-\frac{1}{2}} + \rho = (\frac{a}{2}+\frac{5}{2}, \frac{a}{2}+\frac{1}{2}, \frac{a}{2}-\frac{1}{2} | \frac{t_3}{2}+1);$
$\lambda'_{-\frac{t_2}{2}-1} + \rho = (\frac{a}{2}+\frac{7}{2}, \frac{a}{2}+\frac{1}{2}, \frac{a}{2}-\frac{1}{2} | \frac{t_3}{2}+2).$

After applying odd reflections we have:

$\lambda''_{-\frac{t_3}{2}+1} + \rho'' = (\frac{a}{2}+\frac{3}{2}, \frac{a}{2}+\frac{3}{2}, \frac{a}{2}-\frac{3}{2} | \frac{t_3}{2});$
$\lambda''_{-\frac{t_3}{2}+\frac{1}{2}} + \rho'' = (\frac{a}{2}+2, \frac{a}{2}+1, \frac{a}{2}-1 | \frac{t_3}{2}+\frac{1}{2});$
$\lambda''_{-\frac{t_2}{2}-\frac{1}{2}} + \rho'' = (\frac{a}{2}+\frac{5}{2}, \frac{a}{2}+\frac{1}{2}, \frac{a}{2}-\frac{1}{2} | \frac{t_3}{2}+1);$
$\lambda''_{-\frac{t_2}{2}-1} + \rho'' = (\frac{a}{2}+\frac{7}{2}, \frac{a}{2}+\frac{1}{2}, \frac{a}{2}-\frac{1}{2} | \frac{t_3}{2}+2).$

From Lemma 3.2.2, the positive odd roots with respect to the new Borel B'' are all the odd roots with first coordinate $\frac{1}{2}$.

Now computation using Lemma 5.1.5 with respect to B'', implies:

$[\Gamma_0(G/B'', \mathcal{O}_{\lambda''_{-\frac{t_3}{2}-2}}) : L_{\lambda''_{-\frac{t_3}{2}-1}}] \leq 2$ and

$[\Gamma_0(G/B'', \mathcal{O}_{\lambda''_{-\frac{t_3}{2}-2}}) : L_{\lambda''_{-\frac{t_3}{2}-\frac{1}{2}}}] \leq 1$ and

$[\Gamma_0(G/B'', \mathcal{O}_{\lambda''_{-\frac{t_3}{2}-2}}) : L_{\lambda''_{-\frac{t_3}{2}}}] = 0.$

Also,
$[\Gamma_0(G/B'', \mathcal{O}_{\lambda''_{-\frac{t_3}{2}-1}}) : L_{\lambda''_{-\frac{t_3}{2}-\frac{1}{2}}}] = 1$ and

$[\Gamma_0(G/B'', \mathcal{O}_{\lambda''_{-\frac{t_3}{2}-1}}) : L_{\lambda''_{-\frac{t_3}{2}}}] = 0.$

Also,
$[\Gamma_0(G/B'', \mathcal{O}_{\lambda''_{-\frac{t_3}{2}-\frac{1}{2}}}) : L_{\lambda''_{-\frac{t_3}{2}}}] = 1.$

All other multiplicities are zero.

Since the odd reflections with respect to the weight $\lambda_{-\frac{t_3}{2}-2}$ are typical, by Lemma 8.3.8, we have the first equality below. And, since T is an equivalence, by Lemma 5.2.4 we have the second equality. From Lemma 5.1.5 with respect to Borel B, we have the equality to 0.

$$[\Gamma_0(G/B'', \mathcal{O}_{\lambda''_{-\frac{t_3}{2}-2}}) : L_{\lambda''_{-\frac{t_3}{2}-1}}] = [\Gamma_0(G/B, \mathcal{O}_{\lambda'_{-\frac{t_3}{2}-2}}) : L_{\lambda'_{-\frac{t_3}{2}-1}}] =$$
$$= [\Gamma_0(G/B, \mathcal{O}_{\lambda_{-\frac{t_3}{2}-2}}) : L_{\lambda_{-\frac{t_3}{2}-1}}] \leq 1.$$

We also have $0 = sdim\Gamma_0(G/B'', \mathcal{O}_{\lambda''_{-\frac{t_3}{2}-\frac{1}{2}}}) = sdimL_{\lambda''_{-\frac{t_3}{2}-\frac{1}{2}}} + [\Gamma_0(G/B'', \mathcal{O}_{\lambda''_{-\frac{t_3}{2}-\frac{1}{2}}}) : L_{\lambda''_{-\frac{t_3}{2}}}] sdimL_{\lambda''_{-\frac{t_3}{2}}} = sdimL_{\lambda''_{-\frac{t_3}{2}-\frac{1}{2}}} + sdimL_{\lambda''_{-\frac{t_3}{2}}}$, implying

$$sdimL_{\lambda''_{-\frac{t_3}{2}-\frac{1}{2}}} = -sdimL_{\lambda''_{-\frac{t_3}{2}}}.$$

Similarly, we get:

$$sdimL_{\lambda''_{-\frac{t_3}{2}-\frac{1}{2}}} = -sdimL_{\lambda''_{-\frac{t_3}{2}-1}}.$$

We have:

$$0 = sdim\Gamma_0(G/B'', \mathcal{O}_{\lambda''_{-\frac{t_3}{2}-2}}) =$$
$$= sdimL_{\lambda''_{-\frac{t_3}{2}-2}} + [\Gamma_0(G/B'', \mathcal{O}_{\lambda''_{-\frac{t_3}{2}-2}}) : L_{\lambda''_{-\frac{t_3}{2}-1}}] sdimL_{\lambda''_{-\frac{t_3}{2}-1}} +$$
$$+ [\Gamma_0(G/B'', \mathcal{O}_{\lambda''_{-\frac{t_3}{2}-2}}) : L_{\lambda''_{-\frac{t_3}{2}-\frac{1}{2}}}] sdimL_{\lambda''_{-\frac{t_3}{2}-\frac{1}{2}}}.$$

From above $[\Gamma_0(G/B'', \mathcal{O}_{\lambda''_{-\frac{t_3}{2}-2}}) : L_{\lambda''_{-\frac{t_3}{2}-1}}] \leq 1$ and $[\Gamma_0(G/B'', \mathcal{O}_{\lambda''_{-\frac{t_3}{2}-2}}) : L_{\lambda''_{-\frac{t_3}{2}-\frac{1}{2}}}] \leq 1$.

Since $sdimL_{\lambda_{-\frac{t_3}{2}-2}} = -sdimL_{\lambda_{-\frac{t_3}{2}}}$ from Lemma 8.4.10, we must have

$[\Gamma_0(G/B'', \mathcal{O}_{\lambda''_{-\frac{t_3}{2}-2}}) : L_{\lambda''_{-\frac{t_3}{2}-1}}] = 1$ and $[\Gamma_0(G/B'', \mathcal{O}_{\lambda''_{-\frac{t_3}{2}-2}}) : L_{\lambda''_{-\frac{t_3}{2}-\frac{1}{2}}}] = 0.$

Again using the fact that the odd reflections were typical with respect to $\lambda_{-\frac{t_3}{2}-2}$ and Lemma 5.2.4, we have:

$$[\Gamma_0(G/B, \mathcal{O}_{\lambda_{-\frac{t_3}{2}-2}}) : L_{\lambda_{-\frac{t_3}{2}-1}}] = 1 \text{ and } [\Gamma_0(G/B, \mathcal{O}_{\lambda_{-\frac{t_3}{2}-2}}) : L_{\lambda_{-\frac{t_3}{2}-\frac{1}{2}}}] = 0.$$

Similarly, we obtain the second exact sequence.

□

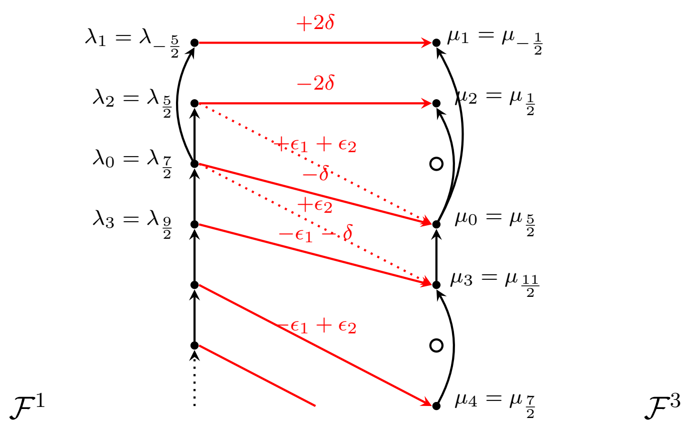
Chapter 9

Equivalence of blocks in $G(3)$

9.1 Equivalence of blocks $\mathcal{F}^1$ and $\mathcal{F}^3$

Let $\mathfrak{g} = G(3)$. We prove the equivalence of the blocks $\mathcal{F}^1$ and $\mathcal{F}^3$ as the first step of mathematical induction of proving the equivalence of the blocks $\mathcal{F}^a$ and $\mathcal{F}^{a+2}$. We follow similar argument as for the symmetric blocks of $F(4)$.

The following is the picture of the translator functor from block $\mathcal{F}^1$ to $\mathcal{F}^3$. It is defined by $T(L_\lambda) = (L_\lambda \otimes \mathfrak{g})^3$. The non-filled circles represent the acyclic weights in the block occurring on the walls of the Weyl chamber. The filled circles represent dominant weights in the block. The horizontal arrows are maps $\lambda \mapsto \lambda + \gamma$, with $\gamma \in \Delta$ is the root above the arrow. In this section, we will show that the solid arrows represent the maps $L_\lambda \mapsto T(L_\lambda)$.



In the above picture we have: $\lambda_1+\rho = (2,3|-\frac{5}{2})$; $\lambda_2+\rho = (2,3|\frac{5}{2})$; $\lambda_0+\rho = (3,4|\frac{7}{2})$;

$\lambda_3 + \rho = (4, 5|\frac{9}{2})$; $\mu_1 + \rho = (2, 3| - \frac{1}{2})$; $\mu_2 + \rho = (2, 3|\frac{1}{2})$; $\mu_0 + \rho = (3, 4|\frac{5}{2})$; $\mu_3 + \rho = (3, 5|\frac{7}{2})$.

Note that the indices are distinct from the index c describing λ, they are described in the picture.

Lemma 9.1.1 *Any dominant weight $\lambda \in F^1$ with $\lambda \neq \lambda_1$ and λ_2 can be obtained from λ_0 by adding root $\beta = (1, 1|1)$ finitely many times.*

Proof. From Theorem 4.4.7, since $a = 0$, we have $J_2, J_3 = \emptyset$. If $c \neq \pm\frac{5}{2}$, which correspond to $\lambda_c \neq \lambda_1$ and λ_2, we have $\lambda_c = \lambda_0 + (c - \frac{7}{2})\beta$, where $\beta = (1, 1|1)$. □

Lemma 9.1.2 *For a dominant weight $\lambda \in F^1$ with $\lambda \neq \lambda_i$ for $i = 1, 2$, we have $\Gamma_i(G/B, \mathcal{O}_\lambda) = 0$ for $i > 0$.*

Proof. Assume $\lambda \neq \lambda_i$ for $i = 1, 2$ and $\Gamma_i(G/B, \mathcal{O}_\lambda) \neq 0$ for $i > 0$. Then there is $\mu \in F^1$ dominant weight such that L_μ occurs in $\Gamma_i(G/B, \mathcal{O}_\lambda)$ with non-zero multiplicity.

For $\lambda \neq \lambda_s$ for $s = 1, 2$, we have by Lemma 9.1.1, $\lambda + \rho = \lambda_0 + \rho + n\beta = (3 + n, 4 + n|\frac{7}{2} + n)$.

By Lemma 5.1.5, we know that $\mu + \rho = w(\lambda + \rho) - \sum_{\alpha \in I} \alpha$ for $w \in W$ of length i, with $I \subset \Delta_1^+$. The last coordinate of $\mu + \rho$ is in

$$[\frac{7}{2} + n - 5, \frac{7}{2} + n] \cap (\frac{1}{2}\mathbb{Z}_{\geq 7} \cup \pm\frac{5}{2}).$$

Assume $n = 0$. Then the last coordinate of $\mu + \rho$ is $\frac{5}{2}$ or $\pm\frac{7}{2}$. By computation there are only three possibilities $\mu = \lambda_i$ with $i = 0, 2$ and in each case $w = id$. This implies $\Gamma_i(G/B, \mathcal{O}_{\lambda_0}) = 0$ for $i > 0$.

Assume $n \geq 1$. Then the last coordinate of $\mu + \rho$ is in

$$[\frac{7}{2} + n - 5, \frac{7}{2} + n] \cap (\frac{1}{2}\mathbb{Z}_{\geq 5}).$$

By computation only $w = id$ is possible. Thus, $\Gamma_i(G/B, \mathcal{O}_\lambda) = 0$ for $i > 0$. □

Lemma 9.1.3 *For a dominant weight $\lambda \in F^1$ with $\lambda \neq \lambda_i$ for $i = 1, 2$, we have $[\Gamma_0(G/B, \mathcal{O}_\lambda) : L_\mu] = 0$ for $\mu \neq \lambda$ and $\mu \neq \lambda - \alpha$ for any $\alpha \in \Delta_{\bar{1}}$.*

Proof. Similarly to the previous lemma, by Lemma 9.1.1, $\lambda + \rho = \lambda_0 + \rho + n\beta = (3 + n, 4 + n|\frac{7}{2} + n)$.

Assume $n = 0$. Then the last coordinate of $\mu + \rho$ is $\frac{5}{2}$ or $\pm\frac{7}{2}$. By computation there are only three possibilities $\mu = \lambda_i$ with $i = 0, 1, 2$ and in each case $w = id$. This implies $[\Gamma_0(G/B, \mathcal{O}_{\lambda_0}), L_\mu] = 0$ for $\mu \neq \lambda_0 - \alpha$.

Assume $n \geq 1$. Then the last coordinate of $\mu + \rho$ is in

$$[\frac{7}{2} + n - 5, \frac{7}{2} + n] \cap (\frac{1}{2}\mathbb{Z}_{\geq 5}).$$

By computation only $w = id$ is possible when $\mu + \rho$ has last coordinate equal the last coordinate of $\lambda + \rho$ minus 1 or $\mu = \lambda$, in each case there is a unique set I. Thus, $[\Gamma_0(G/B, \mathcal{O}_\lambda) : L_\mu] = 0$ for $\mu \neq \lambda$ and $\mu \neq \lambda - \alpha$ for any $\alpha \in \Delta_{\bar{1}}$. □

Lemma 9.1.4 *For a dominant weight $\lambda \in F^1$, we have $sdim L_\lambda = \pm 2$ if $\lambda \neq \lambda_i$ for $i = 1, 2$.*

Proof. We prove this by induction starting with a generic weight $\lambda \in F^1$. We have $sdim L_\lambda = \pm 2$ by computation from generic formula for superdimension. The weights in F^1 can be obtained successively from λ by subtracting odd root β from Lemma 9.1.1.

By Lemma 5.1.7 and Lemma 9.1.2, we have

$$0 = sdim\Gamma_0(G/B, \mathcal{O}_\lambda) = sdim L_\lambda + [\Gamma_0(G/B, \mathcal{O}_\lambda) : L_{\lambda-\alpha}] sdim L_{\lambda-\alpha}.$$

Since $sdim L_\lambda = \pm 2$ and $[\Gamma_0(G/B, \mathcal{O}_{\lambda-\alpha}) : L_{\lambda-\alpha}] \leq 1$ from proof of previous lemma, we must have $[\Gamma_0(G/B, \mathcal{O}_\lambda) : L_{\lambda-\alpha}] = 1$ and $sdim L_{\lambda-\alpha} = \mp 2$. By induction, this way from generic weight we obtain L_{λ_0}. Thus, $sdim L_{\lambda_0} = \pm 2$. □

Lemma 9.1.5 *We have $\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) = L_{\lambda_1}$.*

Proof. From Lemma 5.1.5, we have $[\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) : L_\sigma] = 0$ for $\sigma \neq \lambda_1$. We know $[\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) : L_{\lambda_1}] = 1$ from Lemma 5.1.4. □

Lemma 9.1.6 *We have $\Gamma_1(G/B, \mathcal{O}_{\lambda_1}) = L_{\lambda_2}$.*

Proof. We have

$$0 = sdim\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) - sdim\Gamma_1(G/B, \mathcal{O}_{\lambda_1})$$

and

$$sdim\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) = sdimL_{\lambda_1} = 1,$$

since L_{λ_1} is the trivial module. This implies that $sdim\Gamma_1(G/B, \mathcal{O}_{\lambda_1}) = 1$. Hence, $\Gamma_1(G/B, \mathcal{O}_{\lambda_1}) = L_{\lambda_1}$ or $\Gamma_1(G/B, \mathcal{O}_{\lambda_1}) = L_{\lambda_2}$. This is true since $\Gamma_1(G/B, \mathcal{O}_{\lambda_1}) : L_\sigma] = 0$.

We have

$$ch\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) - ch\Gamma_1(G/B, \mathcal{O}_{\lambda_1}) = \frac{D_1 e^\rho}{D_0} \sum_{w \in W} sgn(w) e^{w(\lambda_1 + \rho)}.$$

The expression on the right is not zero, since the lowest degree term in the numerator is not zero by computation. This implies $\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) \neq \Gamma_1(G/B, \mathcal{O}_{\lambda_1})$. Thus, $\Gamma_1(G/B, \mathcal{O}_{\lambda_1}) = L_{\lambda_2}$.

□

Lemma 9.1.7 *We have* $sdimL_{\lambda_1} = sdimL_{\lambda_2} = 1$.

Proof. This follows from previous two lemmas and since

$$sdim\Gamma_0(G/B, \mathcal{O}_{\lambda_1}) = sdim\Gamma_1(G/B, \mathcal{O}_{\lambda_1}).$$

□

Lemma 9.1.8 *The cohomology group* $\Gamma_0(G/B, \mathcal{O}_{\lambda_0})$ *has a filtration with quotients* L_{λ_0}, L_{λ_1}, *and* L_{λ_2}.*We know that* L_{λ_0} *is a quotient of* $\Gamma_0(G/B, \mathcal{O}_{\lambda_0})$. *The kernel of that quotient has a filtration with subquotients* L_{λ_1}, L_{λ_2}.

Proof. From previous lemmas, we have $sdimL_{\lambda_0} = \pm 2$, $sdimL_{\lambda_1} = sdimL_{\lambda_2} = 1$. We also know from Lemma 5.1.5, $[\Gamma_0(G/B, \mathcal{O}_{\lambda_0}) : L_\sigma] = 0$, unless $\sigma = \lambda_i$ with $i = 0, 1, 2$. From Lemma 5.1.5, we have $[\Gamma_0(G/B, \mathcal{O}_{\lambda_0}) : L_{\lambda_0}] = 1$, $[\Gamma_0(G/B, \mathcal{O}_{\lambda_0}) : L_{\lambda_1}] \leq 1$, $[\Gamma_0(G/B, \mathcal{O}_{\lambda_0}) : L_{\lambda_2}] \leq 1$.

We have

$$0 = sdim\Gamma_0(G/B, \mathcal{O}_{\lambda_0}) = sdimL_{\lambda_0} + [\Gamma_0(G/B, \mathcal{O}_{\lambda_0}) : L_{\lambda_1}]sdimL_{\lambda_1} +$$

$$+[\Gamma_0(G/B,\mathcal{O}_{\lambda_0}):L_{\lambda_2}]sdimL_{\lambda_2}.$$

This implies that $[\Gamma_0(G/B,\mathcal{O}_{\lambda_0}):L_{\lambda_1}]=[\Gamma_0(G/B,\mathcal{O}_{\lambda_0}):L_{\lambda_2}]=1$, and $sdimL_{\lambda_0}=-2$. □

Lemma 9.1.9 *We have* $\Gamma_0(G/B,\mathcal{O}_{\lambda_2})=L_{\lambda_2}$ *and* $\Gamma_1(G/B,\mathcal{O}_{\lambda_2})=L_{\lambda_1}$.

Proof. From Lemma 5.1.5, we have $[\Gamma_0(G/B,\mathcal{O}_{\lambda_2}):L_\sigma]=0$ for $\sigma\neq\lambda_i$ with $i=1,2$. We know $[\Gamma_0(G/B,\mathcal{O}_{\lambda_2}):L_{\lambda_2}]=1$ from Lemma 5.1.4. We need to show $[\Gamma_0(G/B,\mathcal{O}_{\lambda_2}):L_{\lambda_1}]=0$.

From Lemma 5.1.9, since $\lambda_2=w(\lambda_1+\rho)-\rho$, with w reflection with respect to root δ, we have

$$ch\Gamma_0(G/B,\mathcal{O}_{\lambda_1})-ch\Gamma_1(G/B,\mathcal{O}_{\lambda_1})=-ch\Gamma_0(G/B,\mathcal{O}_{\lambda_2})+ch\Gamma_1(G/B,\mathcal{O}_{\lambda_2}).$$

From Lemma 9.1.5, we have $\Gamma_0(G/B,\mathcal{O}_{\lambda_1})=L_{\lambda_1}$. From Lemma 9.1.6, we have $\Gamma_1(G/B,\mathcal{O}_{\lambda_1})=L_{\lambda_2}$. From Lemma 5.1.5, we know that $[\Gamma_1(G/B,\mathcal{O}_{\lambda_2}):L_{\lambda_2}]=0$. We also know that $[\Gamma_0(G/B,\mathcal{O}_{\lambda_2}):L_{\lambda_2}]=1$. The above equation gives that

$$[\Gamma_1(G/B,\mathcal{O}_{\lambda_2}):L_{\lambda_1}]-[\Gamma_0(G/B,\mathcal{O}_{\lambda_2}):L_{\lambda_1}]=1.$$

We show that $\Gamma_1(G/B,\mathcal{O}_{\lambda_2})=L_{\lambda_1}$, which together with previous equality implies $[\Gamma_0(G/B,\mathcal{O}_{\lambda_2}):L_{\lambda_1}]=0$ and proves the lemma.

Consider the typical weight μ, with $\mu+\rho=(3,4|\frac{5}{2})$. The module $(L_\mu\otimes\mathfrak{g})^1$ has a filtration with quotients $\mathcal{O}_\lambda$ with $\lambda=\lambda_i$ with $i=0,2$. As $\lambda_2<\lambda_0$, we have an exact sequence:

$$0\to\mathcal{O}_{\lambda_0}\to(\mathcal{O}_\mu\otimes\mathfrak{g})^{\Phi^{-1}(\chi)}\to\mathcal{O}_{\lambda_2}\to 0.$$

Applying Lemma 5.1.1, gives the following long exact sequence (add details):

$$0\to\Gamma_1(G/B,\mathcal{O}_{\lambda_2})\to\Gamma_0(G/B,\mathcal{O}_{\lambda_0})\to(L_\mu\otimes\mathfrak{g})^\chi\to\Gamma_0(G/B,\mathcal{O}_{\lambda_2})\to 0.$$

From previous lemma, we have $[\Gamma_0(G/B,\mathcal{O}_{\lambda_0}):L_{\lambda_1}]=1$. From the long exact sequence we have $[\Gamma_1(G/B,\mathcal{O}_{\lambda_2}):L_{\lambda_1}]\leq[\Gamma_0(G/B,\mathcal{O}_{\lambda_0}):L_{\lambda_1}]=1$. Since $sdim\Gamma_1(G/B,\mathcal{O}_{\lambda_2})=sdim\Gamma_0(G/B,\mathcal{O}_{\lambda_2})\neq 0$, we have $[\Gamma_1(G/B,\mathcal{O}_{\lambda_2}):L_{\lambda_1}]\neq 0$. This proves the lemma. □

Lemma 9.1.10 *We have $T(L_{\lambda_i}) = L_{\mu_i}$, for all $i \neq 2, 0$.*

Proof. By definition, $T(L_{\lambda_i}) = (L_{\lambda_i} \otimes \mathfrak{g})^3$. For every λ_i with $i \neq 2, 0$, there is a unique dominant weight in block $\mathcal{F}^3$ of the form $\lambda_i + \gamma$ with $\gamma \in \Delta$. Thus, the lemma follows from Lemma 5.2.7. □

Lemma 9.1.11 *We have $T(L_{\lambda_0}) = L_{\mu_0}$.*

Proof. By definition, $T(L_{\lambda_0}) = (L_{\lambda_0} \otimes \mathfrak{g})^3$. The only dominant weights in the block $\mathcal{F}^3$ of the form $\lambda_0 + \gamma$ with $\gamma \in \Delta$ are μ_3 and μ_0.

The module L_{λ_3} is a quotient of $\Gamma_0(G/B, \mathcal{O}_{\lambda_3})$ from Lemma 5.1.4. We obtain the following exact sequence from Lemma 9.1.3:

$$0 \to L_{\lambda_0} \to \Gamma_0(G/B, \mathcal{O}_{\lambda_3}) \to L_{\lambda_3} \to 0.$$

Since T is an exact functor, we get the following exact sequence:

$$0 \to T(L_{\lambda_0}) \to T(\Gamma_0(G/B, \mathcal{O}_{\lambda_3})) \to T(L_{\lambda_3}) \to 0.$$

We have $T(L_{\lambda_3}) = L_{\mu_3}$, from Lemma 9.1.10. By lemma Lemma 5.2.1 and Lemma 5.2.2, we have

$$T(\Gamma_0(G/B, \mathcal{O}_{\lambda_3})) = \Gamma_0(G/B, T(\mathcal{O}_{\lambda_3})) = \Gamma_0(G/B, \mathcal{O}_{\mu_3}).$$

The later module has a unique quotient L_{μ_3}. Therefore, $T(L_{\lambda_0})$ has no simple subquotient L_{μ_3}, which proves the statement.

□

Lemma 9.1.12 *We have $T(L_{\lambda_2}) = L_{\mu_2}$.*

Proof. By definition of translation functor $T(L_{\lambda_2}) = (L_{\lambda_2} \otimes \mathfrak{g})^3$. The only dominant weights in the block $\mathcal{F}^3$ of the form $\lambda_2 + \gamma$ with $\gamma \in \Delta$ are μ_2 and μ_0.

We know that L_{λ_0} is a quotient of $\Gamma_0(G/B, \mathcal{O}_{\lambda_0})$ from Lemma 9.1.11. The kernel of that quotient has a filtration with subquotients L_{λ_1}, L_{λ_2}. We have the following exact sequence from Lemma 9.1.8:

$$0 \to S \to \Gamma_0(G/B, \mathcal{O}_{\lambda_0}) \to L_{\lambda_0} \to 0.$$

Since T is an exact functor, we get the following exact sequence:

$$0 \to T(S) \to T(\Gamma_0(G/B, \mathcal{O}_{\lambda_0})) \to T(L_{\lambda_0}) \to 0.$$

From Lemma 9.1.11, we have $T(L_{\lambda_0}) = L_{\mu_0}$. The kernel $T(S)$ of that quotient has a filtration with subquotients $T(L_{\lambda_1})$, $T(L_{\lambda_2})$. By lemma Lemma 5.2.1 and Lemma 5.2.2, we have $T(\Gamma_0(G/B, \mathcal{O}_{\lambda_0})) = \Gamma_0(G/B, T(\mathcal{O}_{\lambda_0})) = \Gamma_0(G/B, \mathcal{O}_{\mu_0})$. The later module has a unique quotient L_{μ_0}. Therefore, it follows from the exact sequence that $T(S)$ has no simple subquotient L_{μ_0}. Thus, $T(L_{\lambda_2})$ has no simple subquotient L_{μ_0}. This proves the lemma.

□

Corollary 9.1.13 *For any $\lambda \in F^1$, the module $T(L_\lambda) \in F^3$ is irreducible of highest weight $\lambda + \alpha$ for some $\alpha \in \Delta$. Conversely, any irreducible module in F^3 is obtained this way.*

Proof. For any $\lambda \in F^1$, with $\lambda \neq \lambda_2$, there is a unique $\alpha \in \Delta$ with weight $\lambda + \alpha \in F^3$ dominant. Thus, $T(L_\lambda)$ is an irreducible with highest weight $\lambda + \alpha$. From previous lemma, the corollary follows.

□

Theorem 9.1.14 *The blocks $\mathcal{F}^1$ and $\mathcal{F}^3$ are equivalent as categories.*

Proof. From previous corollary, for each $\lambda_i \in F^1$, $T(L_{\lambda_i})$ is a simple module in $\mathcal{F}^3$, we denote $L_{\mu_i} = T(L_{\lambda_i})$ the simple module with highest weight $\mu_i \in F^3$. We show that $T^*(L_{\mu_i}) = L_{\lambda_i}$ for each $\mu_i \in F^3$ and T is an equivalence of the categories $\mathcal{F}^1$ and $\mathcal{F}^3$.

For all $\mu \neq \mu_0, \mu_3$, we have a unique $\gamma \in \Delta$, such that $\mu + \gamma \in F^1$ is dominant. For $\mu = \mu_0$ or μ_3, there are two possible $\gamma \in \Delta$ such that $\mu + \gamma \in F^1$ as its shown in the picture above. In these cases, $\gamma = \delta$ or $-\epsilon_1 - \epsilon_2$ such that $\mu_0 + \gamma = \lambda_0$ or λ_2. Similarly, $\gamma = \epsilon_1 + \delta$ or $-\epsilon_2$ such that $\mu_3 + \gamma = \lambda_3$ or λ_0.

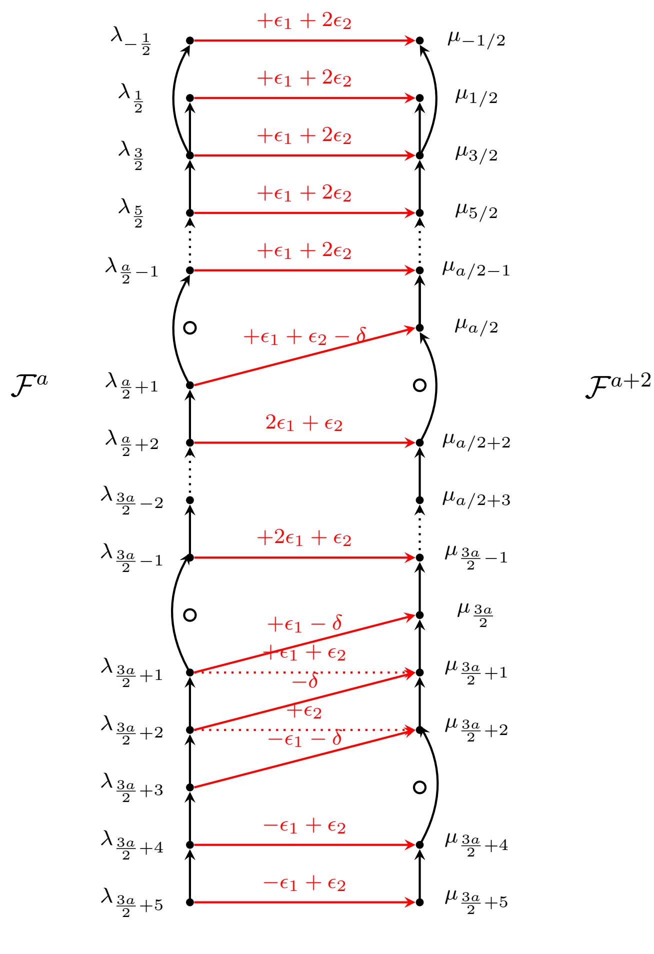
The theorem follows from Theorem 5.2.8.

□

9.2 Equivalence of blocks $\mathcal{F}^a$ and $\mathcal{F}^{a+2}$

Let again $\mathfrak{g} = G(3)$. This section is the inductive step of the proof of equivalence of the blocks of $G(3)$. We prove that all blocks are equivalent and find all cohomology groups. This section is similar to the symmetric blocks of $F(4)$.

The following is the picture of translator functor from block $\mathcal{F}^a$ to $\mathcal{F}^{a+2}$. It is defined by $T(L_\lambda) = (L_\lambda \otimes \mathfrak{g})^{a+2}$. The non-filled circles represent the acyclic weights in the block occurring on the walls of the Weyl chamber. The filled circles represent dominant weights in the block. The horizontal arrows are maps $\lambda \mapsto \lambda + \gamma$, where $\gamma \in \Delta$ is the root above the arrow. In this section, we will show that the solid arrows represent the maps $L_\lambda \mapsto T(L_\lambda)$.



Lemma 9.2.1 *For $\lambda \in F^a$, let T be an equivalence of categories $\mathcal{F}^a$ and $\mathcal{F}^{a+2}$ and $T(L_\lambda) = L'_\lambda$, then that $\Gamma_i(G/B, \mathcal{O}_{\lambda'})$ has a subqotients $L_{\lambda'_t}$ with $[\Gamma_i(G/B, \mathcal{O}_{\lambda'}) : L_{\lambda'_t}] = [\Gamma_i(G/B, \mathcal{O}_\lambda) : L_{\lambda_t}]$.*

Proof. Assume $i = 0$. Then $\Gamma_0(G/B, \mathcal{O}_{\lambda'}) = T(\Gamma_0(G/B, \mathcal{O}_\lambda))$ from Lemma 5.2.4.

Assume $i > 0$. For $\lambda \neq \lambda_t$ with $t = 1, 2$, we have $\Gamma_i(G/B, \mathcal{O}_\lambda) = 0$ from Lemma 5.1.5. For $t = 1, 2$, we know $\Gamma_0(G/B, \mathcal{O}_{\lambda_t}) = L_{\lambda_t}$ since $[\Gamma_0(G/B, \mathcal{O}_{\lambda_t}) : L_\xi] \neq 0$ for $\xi < \lambda_t$. And there are no dominant weights $\xi < \lambda_t$ in F^a for $t = 1$. For $t = 2$, we have only one such weight, namely λ_1. But, $[\Gamma_0(G/B, \mathcal{O}_{\lambda_2}) : L_{\lambda_1}] = 0$ by Lemma 5.1.5.

We know

$$sdim\Gamma_1(G/B, \mathcal{O}_{\lambda_t}) = sdim\Gamma_0(G/B, \mathcal{O}_{\lambda_t}) = sdimL_{\lambda_t}.$$

And, we know for $s \neq 1, 2$, $sdimL_{\lambda_s} > sdimL_{\lambda_1}$. This implies $\Gamma_1(G/B, \mathcal{O}_{\lambda_t}) = L_{\lambda_s}$ for $s = 1, 2$.

We have

$$ch\Gamma_0(G/B, \mathcal{O}_{\lambda_t}) - ch\Gamma_1(G/B, \mathcal{O}_{\lambda_t}) = \frac{D_1 e^\rho}{D_0} \sum_{w \in W} sgn(w) e^{w(\lambda_t + \rho)}.$$

The expression on the right is not zero, since one can compute that the lowest degree term in the numerator is not zero.

Hence, $ch\Gamma_1(G/B, \mathcal{O}_{\lambda_t}) \neq ch\Gamma_0(G/B, \mathcal{O}_{\lambda_t})$ implying $\Gamma_1(G/B, \mathcal{O}_{\lambda_t}) = L_{\lambda_s}$ with $s \neq t$. This proves the lemma.

□

Lemma 9.2.2 *Let $\lambda \in F^a$ be dominant, then there is unique $\gamma \in \Delta$ such that $\lambda + \gamma \in \mathcal{F}^{a+1}$ is dominant, unless $\lambda = \lambda_c$ with $c = \frac{3}{2}a + 1$ or $\frac{3}{2}a + 2$. See the diagram above.*

Proof. We have $\lambda_{\frac{3}{2}a+1} + \rho = (a + 1, 2a + 1, \frac{3}{2}a + 1)$ or $\lambda_{\frac{3}{2}a+2} + \rho = (a + 2, 2a + 2, \frac{3}{2}a + 2)$.

For every $c \geq -\frac{1}{2}$, there is at most one dominant $\lambda \in F^a$, with $\lambda + \rho = (b_1, b_2|c)$. Assume $\gamma \in \Delta$ is such that $\lambda + \gamma \in F^{a+2}$, then $\lambda + \rho + \gamma$ must have last coordinate $c \pm 1$, or c. Thus in generic cases, the last coordinate of $\lambda + \gamma + \rho$ and $\lambda + \rho$ are in the same interval J_i. The exceptional cases occur around walls of the Weyl chamber, when $c = \frac{a}{2} + 1$, $\frac{3a}{2} + 1$, $\frac{3a}{2} + 2$, $\frac{3a}{2} + 3$. And only for the cases $\frac{3a}{2} + 1$, $\frac{3a}{2} + 2$, there are two possible γ.

We show that the last coordinates of $\lambda+\gamma+\rho$ and $\lambda+\rho$ are the same in generic cases, and thus, there is at most one such γ, proving the uniqueness.

Note that for generic λ, $(\lambda+\rho,\alpha)=0$ and $(\lambda+\gamma+\rho,\alpha)=0$ are true for the same $\alpha\in\Delta_{\bar{1}}^{+}$ (see Remark 4.4.8 above). That means $(\gamma,\alpha)=0$, and this is impossible for $\gamma=2\delta$, if γ is odd then this implies $\gamma=\pm\alpha$, which is impossible since then λ and $\lambda+\gamma$ are in the same block. While when $\gamma\neq 2\delta$ is even the statement is clear.

The existence in generic cases: for $c\in I_i$, there is a corresponding root γ. □

Lemma 9.2.3 *We have $T(L_{\lambda_i})=L_{\lambda_i+\gamma}$, for all $i\neq\frac{3}{2}a+1$ or $\frac{3}{2}a+2$ and for the unique $\gamma\in\Delta$ in the previous lemma.*

Proof. By definition, $T(L_{\lambda_i})=(L_{\lambda_i}\otimes\mathfrak{g})^{a+2}$. For every $i\neq\frac{3}{2}a+1$ or $\frac{3}{2}a+2$, there is a unique dominant weight μ_i in the block F^{a+2} of the form $\lambda_i+\gamma$ with $\gamma\in\Delta$ as its shown in the picture above. Thus, the lemma follows from Lemma 5.2.7. □

Lemma 9.2.4 *Assume for each $\lambda\in F^a$, $T(L_\lambda)$ is a simple module in $\mathcal{F}^{a+2}$, denoted $L_{\lambda'}=T(L_\lambda)$. Then categories $\mathcal{F}^a$ and $\mathcal{F}^{a+2}$ are equivalent.*

Proof. We show that T defined by $T(L_\lambda)=(L_\lambda\otimes\mathfrak{g})^{a+2}$ is an equivalence of categories $\mathcal{F}^a$ and $\mathcal{F}^{a+2}$.

By hypothesis, for each $\lambda_i\in F^a$, $T(L_{\lambda_i})$ is a simple module in $\mathcal{F}^{a+2}$, we denote $L_{\mu_i}=T(L_{\lambda_i})$ the simple module with highest weight $\mu_i\in F^{a+2}$. We show that $T^*(L_{\mu_i})=L_{\lambda_i}$ for each $\mu_i\in F^{a+2}$.

For all $\mu\neq\mu_{\frac{3a}{2}+1},\mu_{\frac{3a}{2}+2}$, we have a unique $\gamma\in\Delta$, such that $\mu+\gamma\in F^a$.

For $\mu=\mu_{\frac{3a}{2}+1},\mu_{\frac{3a}{2}+2}$, there are two possible $\gamma\in\Delta$ such that $\mu+\gamma\in F^a$ as its shown in the picture above.
Here we have, $\gamma=-\epsilon_1-\epsilon_2$ and δ such that $\mu_{\frac{3a}{2}+1}+\gamma=\lambda_{\frac{3a}{2}+1}$ and $\lambda_{\frac{3a}{2}+2}$. Similarly, we have $\gamma=-\epsilon_2$ and $\epsilon_1+\delta$ such that $\mu_{\frac{3a}{2}+2}+\gamma=\lambda_{\frac{3a}{2}+2}$ and $\lambda_{\frac{3a}{2}+3}$.

From this, the statement follows from Theorem 5.2.8.

□

Lemma 9.2.5 *Let $\mathfrak{g} = G(3)$ and $\lambda \in F^a$ such that $\lambda + \rho = (a+2, 2a+2|\frac{3}{2}a+2)$. Let $\alpha = -\delta$. Then $T(L_\lambda) = L_{\lambda-\alpha}$.*

Proof. We will assume that blocks $\mathcal{F}^c$ for $c \leq a$ are all equivalent. Then using this assumption we will prove the lemma. This lemma together with the next lemma implies the equivalence of $\mathcal{F}^a$ and $\mathcal{F}^{a+2}$. Thus, we use a complicated induction in a similar for the case of $F(4)$.

From our assumption and Lemma 9.2.1, we obtain all cohomology groups for $\mathcal{F}^a$, since we know them for $\mathcal{F}^1$ from the previous section.

By definition, we have $\lambda = \lambda_{\frac{3}{2}a+2}$, $\lambda - \alpha = \mu_{\frac{3}{2}a+1}$, and $T(L_{\lambda_{\frac{3}{2}a+2}}) = (L_{\lambda_{\frac{3}{2}a+2}} \otimes \mathfrak{g})^{a+2}$.

The only dominant weights in the block $\mathcal{F}^{a+2}$ of the form $\lambda_{\frac{3}{2}a+2} + \gamma$ with $\gamma \in \Delta$ are $\mu_{\frac{3}{2}a+2}$ and $\mu_{\frac{3}{2}a+1}$.

We know that $L_{\lambda_{\frac{3}{2}a+3}}$ is a quotient of $\Gamma_0(G/B, \mathcal{O}_{\lambda_{\frac{3}{2}a+3}})$ from Lemma 5.1.4. We have the following exact sequence from our inductive assumption and from Lemma 9.1.3:

$$0 \to L_{\lambda_{\frac{3}{2}a+2}} \to \Gamma_0(G/B, \mathcal{O}_{\lambda_{\frac{3}{2}a+3}}) \to L_{\lambda_{\frac{3}{2}a+3}} \to 0.$$

Since T is an exact functor, we have another exact sequence:

$$0 \to T(L_{\lambda_{\frac{3}{2}a+2}}) \to T(\Gamma_0(G/B, \mathcal{O}_{\lambda_{\frac{3}{2}a+3}})) \to T(L_{\lambda_{\frac{3}{2}a+3}}) \to 0.$$

From Lemma 9.2.3, we have $T(L_{\lambda_{\frac{3}{2}a+3}}) = L_{\mu_{\frac{3}{2}a+2}}$. By lemma Lemma 5.2.1 and Lemma 5.2.2, we have

$$T(\Gamma_0(G/B, \mathcal{O}_{\lambda_{\frac{3}{2}a+3}})) = \Gamma_0(G/B, T(\mathcal{O}_{\lambda_{\frac{3}{2}a+3}})) = \Gamma_0(G/B, \mathcal{O}_{\mu_{\frac{3}{2}a+2}}).$$

The module $\Gamma_0(G/B, \mathcal{O}_{\mu_{\frac{3}{2}a+2}})$ has a unique quotient $L_{\mu_{\frac{3}{2}a+2}}$. From the last exact sequence, $T(L_{\lambda_{\frac{3}{2}a+2}})$ has no simple subquotient $L_{\mu_{\frac{3}{2}a+2}}$. This proves the lemma.

□

Lemma 9.2.6 *Let $\mathfrak{g} = G(3)$ and $\lambda \in F^a$ such that $\lambda + \rho = (a+1, 2a+1|\frac{3}{2}a+1)$. If $a > 1$, let $\alpha = \epsilon_1 - \delta$. Then $T(L_\lambda) = L_{\lambda-\alpha}$.*

Proof. We will again assume that blocks $\mathcal{F}^c$ for $c \leq a$ are all equivalent and using this assumption we will prove the lemma. This lemma together with the previous one, will prove the equivalence of $\mathcal{F}^a$ and $\mathcal{F}^{a+2}$. Thus, we use a complicated induction in a.

From our assumption and Lemma 9.2.1, we obtain all cohomology groups for $\mathcal{F}^a$, since we know them for $\mathcal{F}^1$ from the previous section.

By definition, $\lambda = \lambda_{\frac{3}{2}a+1}$, $\lambda - \alpha = \mu_{\frac{3}{2}a}$, and $T(L_{\lambda_{\frac{3}{2}a+1}}) = (L_{\lambda_{\frac{3}{2}a+1}} \otimes \mathfrak{g})^{a+2}$. The only dominant weights with central character corresponding to block $\mathcal{F}^{a+2}$ of the form $\lambda_{\frac{3}{2}a+1} + \gamma$ with $\gamma \in \Delta$ are $\mu_{\frac{3}{2}a+1}$ and $\mu_{\frac{3}{2}a}$.

We know that $L_{\lambda_{\frac{3}{2}a+2}}$ is a quotient of $\Gamma_0(G/B, \mathcal{O}_{\lambda_{\frac{3}{2}a+2}})$ from Lemma 5.1.4. We have the following exact sequence:

$$0 \to L_{\lambda_{\frac{3}{2}a+1}} \to \Gamma_0(G/B, \mathcal{O}_{\lambda_{\frac{3}{2}a+2}}) \to L_{\lambda_{\frac{3}{2}a+2}} \to 0.$$

Since T is an exact functor, we have:

$$0 \to T(L_{\lambda_{\frac{3}{2}a+1}}) \to T(\Gamma_0(G/B, \mathcal{O}_{\lambda_{\frac{3}{2}a+2}})) \to T(L_{\lambda_{\frac{3}{2}a+2}}) \to 0.$$

From Lemma 9.2.3, we have $T(L_{\lambda_{\frac{3}{2}a+2}}) = L_{\mu_{\frac{3}{2}a+1}}$. By lemma Lemma 5.2.1 and Lemma 5.2.2, we have

$$T(\Gamma_0(G/B, \mathcal{O}_{\lambda_{\frac{3}{2}a+2}})) = \Gamma_0(G/B, T(\mathcal{O}_{\lambda_{\frac{3}{2}a+2}})) = \Gamma_0(G/B, \mathcal{O}_{\mu_{\frac{3}{2}a+1}}).$$

The module $\Gamma_0(G/B, \mathcal{O}_{\mu_{\frac{3}{2}a+1}})$ has a unique quotient $L_{\mu_{\frac{3}{2}a+1}}$. The last exact sequence implies that $T(L_{\lambda_{\frac{3}{2}a+1}})$ has no simple subquotient $L_{\mu_{\frac{3}{2}a+1}}$. This proves the lemma. □

Lemma 9.2.7 *The categories $\mathcal{F}^a$ and $\mathcal{F}^{a+2}$ are equivalent for all $a \geq 1$.*

Proof. This follows from Theorem 5.2.8 together with Lemma 9.2.3, Lemma 9.2.6, and Lemma 9.2.6. □

Chapter 10

Characters and superdimension

The following lemma summarizes some results from sections 5-8 on the multiplicities of simple modules L_μ in the cohomology groups $\Gamma_i(G/B, \mathcal{O}_\lambda) = H^i(G/B, \mathcal{O}_\lambda^*)^*$. It is used to prove some of the main results in this thesis. Recall that $\lambda_0, \lambda_1, \lambda_2$ are the special weights defined above.

Lemma 10.0.8 *For all simple modules $L_\lambda \in \mathcal{F}^{(a,b)}$ (or $\mathcal{F}^a$) such that $\lambda \neq \lambda_0, \lambda_1, \lambda_2$, there is a unique dominant weight $\mu \in F^{(a,b)}$ (or F^a) with $\mu = \lambda - \sum_{i=1}^n \alpha_i$ with $\alpha_i \in \Delta_1^+$ and $n \in \{1, 2, 3, 4\}$ such that we have an exact sequence:*

$$0 \longrightarrow L_\lambda \longrightarrow \Gamma_0(G/B, \mathcal{O}_\lambda) \longrightarrow L_\mu \longrightarrow 0$$

We also have $\Gamma_i(G/B, \mathcal{O}_\lambda) = 0$ for $i > 0$.

10.1 Superdimension formulae

We denote $s(\lambda) := p(\lambda)$ if $\lambda = \lambda_c$ with $c \in I_i$ or J_i with $i = 1, 3, 6, 8$. And $s(\lambda) := p(\lambda) + 1$ if $\lambda = \lambda_c$ with $c \in I_i$ or J_i with $i = 2, 4, 5, 7$.

Theorem 10.1.1 *Let $\mathfrak{g} = F(4)$. Let $\lambda \in F^{(a,b)}$ and $\mu + \rho_l = a\omega_1 + b\omega_2$. If $\lambda \neq \lambda_1,\ \lambda_2$, the following superdimension formula holds:*

$$sdim\, L_\lambda = (-1)^{s(\lambda)} 2 dim\, L_\mu(\mathfrak{g}_x). \tag{10.1}$$

For the special weights, we have:

$$sdim\, L_{\lambda_1} = sdim\, L_{\lambda_2} = dim\, L_\mu(\mathfrak{g}_x). \tag{10.2}$$

Proof. For generic weight, the theorem follows from Theorem 6.1.4.

For other cases, if $\lambda \neq \lambda_0, \lambda_1, \lambda_2$ we have

$$0 = sdim H^0(G/B, \mathcal{O}_\lambda^*)^* = sdim L_\lambda + [H^0(G/B, \mathcal{O}_\lambda^*)^* : L_\mu] sdim L_\mu,$$

where μ is the unique dominant weight in Lemma 10.0.8. This gives

$$sdim L_\lambda = -sdim L_\mu.$$

From Lemma 10.0.8, we have $\mu = \lambda - \sum_{i=1}^n \alpha_i$ with $\alpha_i \in \Delta_{\bar{1}}^+$ and $n \in \{1, 2, 3, 4\}$.

Thus, if n is even, we have $p(\mu) = p(\lambda)$, thus in those cases the sign changes. This occurs each time the last coordinates of μ and λ belong to adjacent intervals. Thus, the theorem follows. □

Theorem 10.1.2 *Let $\mathfrak{g} = G(3)$. Let $\lambda \in F^a$ and $\mu + \rho_l = a\omega_1$. If $\lambda \neq \lambda_1$, λ_2, the following superdimension formula holds:*

$$sdim\, L_\lambda = (-1)^{s(\lambda)} 2 dim\, L_\mu(\mathfrak{g}_x). \tag{10.3}$$

For the special weights, we have:

$$sdim\, L_{\lambda_1} = sdim\, L_{\lambda_2} = dim\, L_\mu(\mathfrak{g}_x). \tag{10.4}$$

Proof. Similar to the proof for $F(4)$. □

10.2 Kac-Wakimoto conjecture

A root α is called *isotropic* if $(\alpha, \alpha) = 0$. The *degree of atypicality* of the weight λ the maximal number of mutually orthogonal linearly independent isotropic roots α such that $(\lambda + \rho, \alpha) = 0$. The *defect* of $\mathfrak{g}$ is the maximal number of linearly independent mutually orthogonal isotropic roots. The above theorem proves the following conjecture by Kac-Wakimoto Conjecture for $\mathfrak{g} = F(4)$ and $G(3)$, see [14].

Theorem 10.2.1 *The superdimension of a simple module of highest weight λ is nonzero if and only if the degree of atypicality of the weight is equal to the defect of the Lie superalgebra.*

Proof. Follows from Theorem 10.1.1 and Theorem 10.1.2. □

10.3 Character formulae

In this section, we prove a Weyl character type formula for the dominant atypical weights.

Lemma 10.3.1 *For a dominant weight λ and corresponding μ and n from Lemma 10.0.8, there is a unique $\sigma \in W$ such that $\lambda + \rho - \sigma(\mu + \rho) = n\alpha$ for $\alpha \in \Delta_{\bar{1}}$ satisfying $(\lambda + \rho, \alpha) = 0$. Also, sign$\sigma = (-1)^{n-1}$.*

If $\beta \in \Delta_{\bar{1}}$ is such that $(\mu + \rho, \beta) = 0$, then $\sigma(\beta) = \alpha$.

Proof. This follows from Lemma 5.1.5. □

Theorem 10.3.2 *For a dominant weight $\lambda \neq \lambda_1, \lambda_2$, let $\alpha \in \Delta_{\bar{1}}$ be such that $(\lambda + \rho, \alpha) = 0$. Then*

$$chL_\lambda = \frac{D_1 \cdot e^\rho}{D_0} \cdot \sum_{w \in W} sign(w) \cdot w(\frac{e^{\lambda+\rho}}{(1 + e^{-\alpha})}). \tag{10.5}$$

For $\lambda = \lambda_i$ with $i = 1, 2$, we have the following similar formula:

$$chL_\lambda = \frac{D_1 \cdot e^\rho}{2D_0} \cdot \sum_{w \in W} sign(w) \cdot w(\frac{e^{\lambda+\rho}(2 + e^{-\alpha})}{(1 + e^{-\alpha})}). \tag{10.6}$$

Proof. Let μ be dominant weight, then it corresponds to some λ and n in Lemma 10.0.8 such that we have:

$$0 \longrightarrow L_\lambda \longrightarrow \Gamma_0(G/B, \mathcal{O}_\lambda) \longrightarrow L_\mu \longrightarrow 0 .$$

It follows that $ch(\Gamma_0(G/B, \mathcal{O}_\lambda)) = ch(L_\lambda) + ch(L_\mu)$.

Assume the formula is true for λ. We show that this together with Lemma 6.1.3, proves the formula for μ. Since we can obtain each dominant weight from generic one by similar correspondence, from Lemma 6.1.2 the formula follows for all dominant weights.

We have from Lemma 10.3.1:

$$\frac{e^{\lambda+\rho}}{1 + e^{-\alpha}} + (-1)^{n-1}\sigma(\frac{e^{\mu+\rho}}{1 + e^{-\beta}}) = \frac{e^{\lambda+\rho}}{1 + e^{-\alpha}} + (-1)^{n-1}(\frac{e^{\lambda+\rho-n\alpha}}{1 + e^{-\alpha}}) =$$

$$= \frac{e^{\lambda+\rho}(1+(-1)^{n-1}e^{-n\alpha})}{1+e^{-\alpha}} = e^{\lambda+\rho}(1+\sum_{i=1}^{n-1}(-1)^i e^{-i\alpha}).$$

Using the above equation, we have:

$$ch(L_\mu) = ch(\Gamma_0(G/B,\mathcal{O}_\lambda)) - ch(L_\lambda) = \frac{D_1 \cdot e^\rho}{D_0} \cdot \sum_{w\in W} signw \cdot w(e^{\lambda+\rho} - \frac{e^{\lambda+\rho}}{(1+e^{-\alpha})}) =$$

$$= \frac{D_1 \cdot e^\rho}{D_0} \cdot \sum_{w\in W} sign(w\sigma)\cdot(w\sigma)(\frac{e^{\mu+\rho}}{(1+e^{-\beta})} + \sum_{i=1}^{n}(-1)^i e^{\lambda+\rho-i\alpha}) =$$

$$= \frac{D_1 \cdot e^\rho}{D_0} \cdot \sum_{w\in W} sign(w)\cdot w(\frac{e^{\mu+\rho}}{(1+e^{-\beta})}) + \frac{D_1 \cdot e^\rho}{D_0} \cdot \sum_{w\in W} sign(w)\cdot w(\sum_{i=1}^{n}(-1)^i e^{\lambda+\rho-i\alpha}).$$

The second summand is zero as the weights $\lambda - i\alpha$ are acyclic. Thus we get the required formula.

Similarly, for $\mu = \lambda_1, \lambda_2$, we have

$$ch(L_{\lambda_1}) + ch(L_{\lambda_2}) = ch(\Gamma_0(G/B,\mathcal{O}_{\lambda_0})) - ch(L_{\lambda_0}) =$$

$$= \frac{D_1 \cdot e^\rho}{D_0} \cdot \sum_{w\in W} signw \cdot w(e^{\lambda_0+\rho} - \frac{e^{\lambda_0+\rho}}{(1+e^{-\alpha_0})}) =$$

$$= \frac{D_1 \cdot e^\rho}{D_0} \cdot \sum_{w\in W} sign(w\sigma)\cdot(w\sigma)(\frac{e^{\lambda_1+\rho}}{(1+e^{-\alpha_1})} + \sum_{i=1}^{n}(-1)^i e^{\lambda_0+\rho-i\alpha_0}) =$$

$$= \frac{D_1 \cdot e^\rho}{D_0} \cdot \sum_{w\in W} sign(w)\cdot w(\frac{e^{\mu+\rho}}{(1+e^{-\beta})}) + \frac{D_1 \cdot e^\rho}{D_0} \cdot \sum_{w\in W} sign(w)\cdot w(\sum_{i=1}^{n}(-1)^i e^{\lambda+\rho-i\alpha}).$$

The second summand is zero as the weights $\lambda + \rho - i\alpha$ are acyclic. Thus we get the required formula for the sum $ch(L_{\lambda_1}) + ch(L_{\lambda_2})$.

On the other hand, we have

$$ch(L_{\lambda_1}) - ch(L_{\lambda_2}) = ch(\Gamma_0(G/B,\mathcal{O}_{\lambda_1})) - ch(\Gamma_1(G/B,\mathcal{O}_{\lambda_1})) =$$

$$= \frac{D_1 \cdot e^{\rho}}{D_0} \cdot \sum_{w \in W} sign(w) \cdot w(e^{\lambda_1 + \rho}).$$

Adding both equations above, we get:

$$ch(L_{\lambda_1}) = \frac{D_1 \cdot e^{\rho}}{2D_0} \cdot \sum_{w \in W} sign(w) \cdot w(\frac{e^{\lambda_1 + \rho}(2 + e^{-\alpha_1})}{(1 + e^{-\alpha_1})}).$$

The same proof works for the weight λ_2. □

Chapter 11

Indecomposable modules

11.1 Notions from category theory

The following results from category theory have been taken from [6]. A small Abelian $\mathbb{C}$-linear category $\mathcal{A}$ is called *nice* if morphism spaces are finite-dimensional, if every object in $\mathcal{A}$ has a finite composition series, and if $\mathcal{A}$ contains enough projectiles. Fitting's lemma holds for nice categories:

Lemma 11.1.1 *(Fitting) Let $\mathcal{A}$ be a nice category. Then*

(i) The endomorphism ring of any indecomposable object is finite dimensional and local.

(ii) Any object satisfies Krull-Schmidt theorem.

(iii) Any indecomposable projective object has a unique simple quotient.

(iv) Any object has a unique up to isomorphism projective cover.

(v) For any object M, the number of isomorphism classes of indecomposable projective objects P such that $Hom_{\mathcal{A}}(P, M) \neq 0$ is finite.

Let X^+ denote the set of all isomorphism classes of simple objects in $\mathcal{A}$, then there is a natural bijection between X^+ and the set of isomorphism classes in indecomposable projective modules. For $\lambda \in X^+$, we let $S(\lambda)$ denote the corresponding simple object and $P(\lambda)$ the projective cover.

By a *quiver* we mean a directed graph. Given a quiver with vertex set X^+, we can define a $\mathbb{C}$-linear category $\mathbb{C}Q$. Its objects are vertices of Q, the space of morphisms $Hom_{\mathbb{C}Q}(\lambda, \mu)$ between two vertices is the space of formal linear combinations of paths from λ to μ, with the composition of morphisms linearly extending the concatenations of paths.

By a *representation of a quiver* Q, we mean a finite dimension X^+-graded vector space $V = \oplus_{\lambda \in X^+} V_\lambda$ and linear maps $\phi|_V : V_\lambda \to V_\mu$ corresponding to each arrow $\phi : \lambda \to \mu$ of the quiver. We get linear maps $Hom_{\mathbb{C}Q}(\lambda, \mu) \to Hom_{\mathbb{C}}(V_\lambda, V_\mu)$, which are compatible with composition. By a *morphism of representations* we mean a morphisms of X^+-graded spaces that commute with the action of all arrows. Representations of Q form an abelian category denoted Q-mod.

Let $\mathcal{A}$ be a nice category, then *Ext-quiver* is the quiver Q, which has vertex set the set X^+ of isomorphism classes of simple objects and the number of arrows from vertex λ to the vertex μ is

$$d_{\lambda,\mu} = dim Ext^1_{\mathcal{A}}(S(\lambda), S(\mu)).$$

Since $\mathcal{A}$ contains enough projective objects, $Ext^1_{\mathcal{A}}(M, N)$ is well defined and finite dimensional vector space for any objects M and N of $\mathcal{A}$.

For two vertices λ and $\mu \in X^+$, $rad(P(\mu), P(\lambda))$ is defined to be the set of all noninvertible morphisms from $P(\mu)$ to $P(\lambda)$. Then

$$rad(P(\mu), P(\lambda)) = Hom_{\mathcal{A}}(P(\mu), radP(\lambda)).$$

$rad^n(P(\mu), P(\lambda))$ is the subspace of $rad(P(\mu), P(\lambda))$ consisting of sums of products of n noninvertible maps between projectives.

Lemma 11.1.2 *([4]) There is a canonical isomorphism*

$$Ext^1_{\mathcal{A}}(S(\lambda), S(\mu)) \cong Hom_{\mathcal{A}}(P(\mu), radP(\lambda)/rad^2P(\lambda))^*.$$

Say we have $d_{\lambda\mu}$ arrows from λ to μ, denoted by $(\phi^i_{\lambda\mu})_{i=1,\dots,d_{\lambda\mu}}$. Let $\mathcal{R}_{\lambda\mu}$ be a bijection from$(\phi^i_{\lambda\mu})_{i=1,\dots,d_{\lambda\mu}}$ to a set of $d_{\lambda\mu}$ morphisms in $rad(P(\mu), P(\lambda))$, such that the bijection is onto modulo $rad^2(P(\mu), P(\lambda))$.

Lemma 11.1.3 *([4]) There is a unique well defined family of linear maps*

$$\bar{\mathcal{R}}_{\lambda\mu} : Hom_{\mathbb{C}Q}(\lambda, \mu) \to Hom_{\mathcal{A}}(P(\lambda), P(\mu)),$$

such that $\mathcal{R}_{\lambda\mu}(\phi^i_{\lambda\mu}) = \bar{\mathcal{R}}_{\lambda\mu}(\phi^i_{\lambda\mu})$ *and compatible with composition.*

The map $R : (\lambda, \mu) \to Ker\bar{\mathcal{R}}_{\lambda\mu}$ is a system of relations on Q. If we let $\mathcal{G}$ to be spectroid of $\mathcal{A}$, then the categories $\mathbb{C}Q/R$ and $\mathcal{G}^{op}$ are equivalent. Here, spectroid of $\mathcal{A}$ is the full subcategory consisting of indecomposable projective modules.

Theorem 11.1.4 *(The Quiver Theorem, [4]) Let $\mathcal{A}$ be a nice category and Q its Ext-quiver, and R relations above. There exists an equivalence of categories*

$$e : \mathcal{A} \to Q/R$$

such that $e(M) \cong \oplus_{\lambda \in X^+} Hom_{\mathcal{A}}(P(\lambda), M)$ as graded vector spaces.

11.2 Quivers

The following lemma shows that $\mathcal{C}$ is a nice category.

Lemma 11.2.1 *([8]) (i) The category $\mathcal{C}$ contains enough projective modules.*

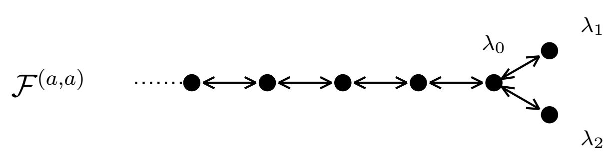
(ii) Projective and injective modules coinside in $\mathcal{C}$.

(iii) For any λ, $\mu \in X^+$, we have: $Ext^1(L_\lambda, L_\mu) \cong Ext^1(L_\mu, L_\lambda)$.

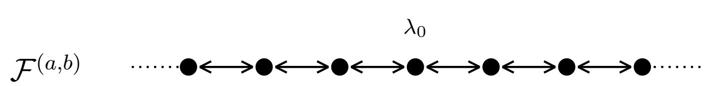
A *quiver diagram* is a directed graph that has vertices the irreducible representations of $\mathfrak{g}$, and the number of arrows from vertex λ to the vertex μ is $dimExt^1_{\mathcal{A}}(L_\lambda, L_\mu)$.

Theorem 11.2.2 *Let $\mathfrak{g} = F(4)$.*

(1)For the symmetric block $\mathcal{F}^{(a,a)}$, we have the following quiver diagram, which is of type D_∞:

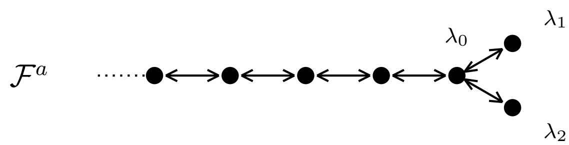


(2) For the non-symmetric block $\mathcal{F}^{(a,b)}$, we have the following quiver diagram, which is of type A_∞:



Let $\mathfrak{g} = G(3)$.

(3) For the block $\mathcal{F}^a$*, we have the following quiver diagram, which is of type* D_∞*:*



Proof. For $\mu, \mu' \neq \lambda_1, \lambda_2$, assume μ and μ' are the adjacent vertices of the quiver with $\mu > \mu'$. From Lemma 10.0.8 and Lemma 5.1.5, we have

$$dim\, Ext^1(L_\mu, L_{\mu'}) = [\Gamma_0(G/B, \mathcal{O}_\mu) : L_{\mu'}].$$

Since the category $\mathcal{C}$ is a contravariant, from Lemma 11.2.1, we also have

$$Ext^1(L_{\mu'}, L_\mu) = Ext^1(L_\mu, L_{\mu'}).$$

□

11.3 Projective modules

Lemma 11.3.1 *Let* $\mathfrak{g} = F(4)$ *(or* $G(3)$*). Then the projective indecomposable modules in the block* $\mathcal{F}^{(a,b)}$ *(or* $\mathcal{F}^a$*) have the following radical layer structure:*

If $\lambda_i \in F^{(a,b)}$ *or* $\lambda_i \in F^{(a,a)}$ *(or* F^a*) with* $i = 0, 1, 2$*. Then* P_{λ_i} *has a radical layer structure:*

$$L_{\lambda_i}$$
$$L_{\lambda_{i-1}} \oplus L_{\lambda_{i+1}}$$
$$L_{\lambda_i}$$

where λ_{i-1} *and* λ_{i+1} *are the adjacent vertices of* λ_i *in the quiver.*

If $\lambda_i \in F^{(a,a)}$ *with* $i = 1, 2$*. Then* P_{λ_i} *has a radical layer structure:*

$$L_{\lambda_i}$$
$$L_{\lambda_0}$$

$$L_{\lambda_i}$$

For $\lambda_0 \in F^{(a,a)}$ *(or* F^a*),* P_{λ_0} *has a radical layer structure:*

$$L_{\lambda_0}$$
$$L_{\lambda_1} \oplus L_{\lambda_2} \oplus L_{\lambda_3}$$
$$L_{\lambda_0}$$

Proof. For the top radical layer structure, we have:

$$P_\lambda / rad\, P_\lambda = soc\, P_\lambda \cong L_\lambda,$$

since projective morphisms in $\mathcal{C}$ are injective and have a simple socle (see [24]).

Since $rad\, P_{\lambda_i} / rad^2\, P_{\lambda_i}$ is the direct sum of simple modules which have a non-split extension by L_{λ_i}, for the middle radical layer structure, we have:

$$rad\, P_{\lambda_i} / rad^2\, P_{\lambda_i} \cong L_{\lambda_{i-1}} \oplus L_{\lambda_{i+1}}.$$

Also, since from Theorem 11.2.2, we have:

$Ext^1(L_{\lambda_i}, L_{\lambda_{i-1}}) \neq 0$, $Ext^1(L_{\lambda_i}, L_{\lambda_{i+1}}) \neq 0$, and $Ext^1(L_{\lambda_i}, L_\sigma) \neq 0$ for $\sigma \neq \lambda_{i-1}, \lambda_{i+1}$.

Similarly, we obtain the middle layer for the special weights.

By BGG reciprocity from [8], we have

$$[P_\lambda : L_\mu] = \sum_\nu [P_\lambda : \varepsilon_\nu] \cdot [\varepsilon_\nu : L_\mu] = \sum_\nu [\varepsilon_\nu : L_\lambda] \cdot [\varepsilon_\nu : L_\mu].$$

Thus,

$$[P_\lambda : L_\mu] = \begin{cases} 2 & \text{if } \mu = \lambda; \\ 1 & \text{if } \mu \text{ is adjacent to } \lambda. \end{cases}$$

This implies that there are only three radical layers. Therefore, for the bottom radical layer structure, we have:

$$rad^2\, P_\lambda \cong L_\lambda.$$

□

11.4 Germoni's conjecture and the indecomposable modules

The following theorem together with results in [9] for other Lie superalgebras proves a conjecture by J. Germoni (Theorem 11.4.2).

Theorem 11.4.1 *The blocks of atypicality 1 are tame.*

Proof. Follows from Theorem 11.2.2 and Lemma 11.3.1. □

Theorem 11.4.2 *Let $\mathfrak{g}$ be a basic classical Lie superalgebra. Then all tame blocks are of atypicality less or equal 1.*

Proof. Follows from Theorem 11.4.1, since all the blocks for $F(4)$ and $G(3)$ are of atypicality less or equal 1. Also, it follows from [9] for other Lie superalgebras. □

For $\mathcal{F}^{(a,b)}$ and for $\mathcal{F}^{(a,a)}$, $\mathcal{F}^{a}$ if $l \geq 3$, we let d_l^+ denote the arrow from vertex with weight λ_l to the adjacent vertex λ_l' on the left in the quiver. And let d_l^- denote the arrow in the opposite direction.

These arrows correspond to the irreducible morphisms $D^{\pm}_{\lambda_l}$ from $P_{\lambda_l'}$ to P_{λ_l}.

For $\mathcal{F}^{(a,a)}$, $\mathcal{F}^{a}$, also let d_0^+ denote the arrow from vertex λ_0 to λ_3 and d_0^- the arrow in the opposite direction. Similarly, for $i = 1, 2$, denote by d_i^+ the arrow from vertex λ_1 to λ_0 and d_i^- the arrow in the opposite direction.

The following theorem together with Theorem 11.1.4 gives a description of the indecomposable modules.

Theorem 11.4.3 *The quivers A_∞ and D_∞ are the ext-quiver for atypical blocks $\mathcal{F}^{(a,b)}$ and $\mathcal{F}^{(a,a)}$ of $F(4)$ and the quiver D_∞ is the ext-quiver for atypical block $\mathcal{F}^{a}$ of $G(3)$ with the following relations:*

For $\mathcal{F}^{(a,b)}$, we have:

$$d^+d^- + d^-d^+ = (d^+)^2 = (d^-)^2 = 0 \text{ ,where } d^{\pm} = \sum_{l\in\mathbb{Z}} d_l^{\pm}$$

For $\mathcal{F}^{(a,a)}$ or $\mathcal{F}^{a}$ we have the following relations:

$$d_l^- d_{l+1}^- = d_{l+1}^+ d_l^+ = 0, \text{ for } l \geq 3$$

$$d_1^- d_2^+ = d_2^- d_1^+ = d_0^+ d_2^+ = d_2^- d_0^- = d_0^- d_3^- = d_3^+ d_0^+ = d_1^- d_0^- = d_0^+ d_1^+ = 0$$

$$d_l^- d_l^+ = d_{l+1}^+ d_{l+1}^- \text{ for } l \geq 3$$

$$d_1^+ d_1^- = d_2^+ d_2^- = d_0^- d_0^+.$$

Proof. The above relations follow by computations in [6] or [7], since the radical filtrations of projectives are the same. Using Lemma 11.1.2, Theorem 11.2.2, and Lemma 11.3.1, we obtain the statement.

□

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Appendix A

The following computations were done using Maple 16.01 program.

A.1 Program for computing the superdimension for generic weights for $F(4)$

as:=[-1/2,-1/2,-1/2, 1/2], [-1/2,-1/2, 1/2, 1/2], [-1/2, 1/2,-1/2, 1/2], [1/2,-1/2,-1/2, 1/2], [-1/2, 1/2, 1/2, 1/2], [1/2,-1/2, 1/2, 1/2], [1/2, 1/2,-1/2, 1/2];

for i from 1 to 7 do for j from i+1 to 7 do lprint(as[i]+as[j]); od; od;

for i from 1 to 7 do for j from i+1 to 7 do for k from j+1 to 7 do lprint(as[i]+as[j]+as[k]); od; od; od;

for i from 1 to 7 do for j from i+1 to 7 do for k from j+1 to 7 do for l from k+1 to 7 do lprint(as[i]+as[j]+as[k]+as[l]); od; od; od; od;

for i from 1 to 7 do for j from i+1 to 7 do for k from j+1 to 7 do for l from k+1 to 7 do for m from l+1 to 7 do lprint(as[i]+as[j]+as[k]+as[l]+as[m]); od; od; od; od; od;

for i from 1 to 7 do for j from i+1 to 7 do for k from j+1 to 7 do for l from k+1 to 7 do for m from l+1 to 7 do for n from m+1 to 7 do lprint(as[i]+as[j]+as[k]+as[l]+as[m]+as[n]); od; od; od; od; od; od;

for i from 1 to 7 do for j from i+1 to 7 do for k from j+1 to 7 do for l from k+1 to 7 do for m from l+1 to 7 do for n from m+1 to 7 do for o from n+1 to 7 do lprint(as[i]+as[j]+as[k]+as[l]+as[m]+as[n]+as[o]); od; od; od; od; od; od; od;

a[0]:=[0,0,0,0];

t:=0; for i from 0 to 7 do tot[i]:=0; end do; for c from 1 to 1 do tot[0]:=tot[0] + expand((1/90)*(2*(w-a[0][c][4])+4)*(x-a[0][c][1])*(y-a[0][c][2])*(z-a[0][c][3])*(x-a[0][c][1]+y-a[0][c][2])*(x-a[0][c][1]+z-a[0][c][3])*(y-a[0][c][2]+z-a[0][c][3])*(x-a[0][c][1]-y+a[0][c][2])*(x-a[0][c][1]-z+a[0][c][3])*(y-a[0][c][2]-z+a[0][c][3])); end do;

for j from 1 to 7 do tot[1]:=tot[1] + expand((1/90)*(2*(w-a[1][j][4])+4)*(x-a[1][j][1])*(y-a[1][j][2])*(z-a[1][j][3])*(x-a[1][j][1]+y-a[1][j][2])*(x-a[1][j][1]+z-a[1][j][3])*(y-a[1][j][2]+z-a[1][j][3])*(x-a[1][j][1]-y+a[1][j][2])*(x-a[1][j][1]-z+a[1][j][3])*(y-a[1][j][2]-z+a[1][j][3])); end do;

for k from 1 to 21 do tot[2]:=tot[2] + expand((1/90)*(2*(w-a[2][k][4])+4)*(x-a[2][k][1])*(y-a[2][k][2])*(z-a[2][k][3])*(x-a[2][k][1]+y-a[2][k][2])*(x-a[2][k][1]+z-a[2][k][3])*(y-a[2][k][2]+z-a[2][k][3])*(x-a[2][k][1]-y+a[2][k][2])*(x-a[2][k][1]-z+a[2][k][3])*(y-a[2][k][2]-z+a[2][k][3])); end do;

for l from 1 to 35 do tot[3]:=tot[3] + expand((1/90)*(2*(w-a[3][l][4])+4)*(x-a[3][l][1])*(y-a[3][l][2])*(z-a[3][l][3])*(x-a[3][l][1]+y-a[3][l][2])*(x-a[3][l][1]+z-a[3][l][3])*(y-a[3][l][2]+z-a[3][l][3])*(x-a[3][l][1]-y+a[3][l][2])*(x-a[3][l][1]-z+a[3][l][3])*(y-a[3][l][2]-z+a[3][l][3])); end do;

for m from 1 to 35 do tot[4]:=tot[4] + expand((1/90)*(2*(w-a[4][m][4])+4)*(x-a[4][m][1])*(y-a[4][m][2])*(z-a[4][m][3])*(x-a[4][m][1]+y-a[4][m][2])*(x-a[4][m][1]+z-a[4][m][3])*(y-a[4][m][2]+z-a[4][m][3])*(x-a[4][m][1]-y+a[4][m][2])*(x-a[4][m][1]-z+a[4][m][3])*(y-a[4][m][2]-z+a[4][m][3])); end do;

for p from 1 to 21 do tot[5]:=tot[5] + expand((1/90)*(2*(w-a[5][p][4])+4)*(x-a[5][p][1])*(y-a[5][p][2])*(z-a[5][p][3])*(x-a[5][p][1]+y-a[5][p][2])*(x-a[5][p][1]+z-a[5][p][3])*(y-a[5][p][2]+z-a[5][p][3])*(x-a[5][p][1]-y+a[5][p][2])*(x-a[5][p][1]-z+a[5][p][3])*(y-a[5][p][2]-z+a[5][p][3])); end do;

for r from 1 to 7 do tot[6]:=tot[6] + expand((1/90)*(2*(w-a[6][r][4])+4)*(x-a[6][r][1])*(y-a[6][r][2])*(z-a[6][r][3])*(x-a[6][r][1]+y-a[6][r][2])*(x-a[6][r][1]+z-a[6][r][3])*(y-a[6][r][2]+z-a[6][r][3])*(x-a[6][r][1]-y+a[6][r][2])*(x-a[6][r][1]-z+a[6][r][3])*(y-a[6][r][2]-z+a[6][r][3])); end do;

for s from 1 to 1 do tot[7]:=tot[7] + expand((1/90)*(2*(w-a[7][s][4])+4)*(x-a[7][s][1])*(y-a[7][s][2])*(z-a[7][s][3])*(x-a[7][s][1]+y-a[7][s][2])*(x-a[7][s][1]+z-a[7][s][3])*(y-a[7][s][2]+z-a[7][s][3])*(x-a[7][s][1]-y+a[7][s][2])*(x-a[7][s][1]-z+a[7][s][3])*(y-a[7][s][2]-z+a[7][s][3])); end do;

t:=tot[0]-tot[1]+tot[2]-tot[3]+tot[4]-tot[5]+tot[6]-tot[7];

A.2 Program for computing the superdimension for generic weights for $G(3)$

as:=[0,0,1], [1,0,1], [-1,0,1], [0,1,1], [0,-1,1], [-1,-1,1];

for i from 1 to 6 do for j from i+1 to 6 do lprint(as[i]+as[j]); od; od;

for i from 1 to 6 do for j from i+1 to 6 do for k from j+1 to 6 do lprint(as[i]+as[j]+as[k]); od; od; od;

for i from 1 to 6 do for j from i+1 to 6 do for k from j+1 to 6 do for l from k+1 to 6 do lprint(as[i]+as[j]+as[k]+as[l]); od; od; od; od;

for i from 1 to 6 do for j from i+1 to 6 do for k from j+1 to 6 do for l from k+1 to 6 do for m from l+1 to 6 do lprint(as[i]+as[j]+as[k]+as[l]+as[m]); od; od; od; od; od;

for i from 1 to 6 do for j from i+1 to 6 do for k from j+1 to 6 do for l from k+1 to 6 do for m from l+1 to 6 do for n from m+1 to 6 do lprint(as[i]+as[j]+as[k]+as[l]+as[m]+as[n]); od; od; od; od; od; od;

a[0]:=[[0,0,0]];

t:=0; for i from 0 to 6 do tot[i]:=0; end do; for c from 1 to 1 do tot[0]:=tot[0] + expand((1/240)*(2*w-2*a[0][c][3]+7)*(x-a[0][c][1])*(y-a[0][c][2])*(x-a[0][c][1]+y-a[0][c][2])*(-x+a[0][c][1]+y-a[0][c][2])*(2*x-2*a[0][c][1]-y+a[0][c][2])*(-x+a[0][c][1]+2*y-2*a[0][c][2])); end do;

for j from 1 to 6 do tot[1]:=tot[1] + expand((1/240)*(2*w-2*a[1][j][3]+7)*(x-a[1][j][1])*(y-a[1][j][2])*(x-a[1][j][1]+y-a[1][j][2])*(-x+a[1][j][1]+y-a[1][j][2])*(2*x-2*a[1][j][1]-y+a[1][j][2])*(-x+a[1][j][1]+2*y-2*a[1][j][2])); end do;

for k from 1 to 15 do tot[2]:=tot[2] + expand((1/240)*(2*w-2*a[2][k][3]+7)*(x-a[2][k][1])*(y-a[2][k][2])*(x-a[2][k][1]+y-a[2][k][2])*(-x+a[2][k][1]+y-a[2][k][2])*(2*x-2*a[2][k][1]-y+a[2][k][2])*(-x+a[2][k][1]+2*y-2*a[2][k][2])); end do;

for l from 1 to 20 do tot[3]:=tot[3] + expand((1/240)*(2*w-2*a[3][l][3]+7)*(x-a[3][l][1])*(y-a[3][l][2])*(x-a[3][l][1]+y-a[3][l][2])*(-x+a[3][l][1]+y-a[3][l][2])*(2*x-2*a[3][l][1]-y+a[3][l][2])*(-x+a[3][l][1]+2*y-2*a[3][l][2])); end do;

for m from 1 to 15 do tot[4]:=tot[4] + expand((1/240)*(2*w-2*a[4][m][3]+7)*(x-a[4][m][1])*(y-a[4][m][2])*(x-a[4][m][1]+y-a[4][m][2])*(-x+a[4][m][1]+y-a[4][m][2])*(2*x-2*a[4][m][1]-y+a[4][m][2])*(-x+a[4][m][1]+2*y-2*a[4][m][2])); end do;

for p from 1 to 6 do tot[5]:=tot[5] + expand((1/240)*(2*w-2*a[5][p][3]+7)*(x-a[5][p][1])*(y-a[5][p][2])*(x-a[5][p][1]+y-a[5][p][2])*(-x+a[5][p][1]+y-a[5][p][2])*(2*x-2*a[5][p][1]-y+a[5][p][2])*(-x+a[5][p][1]+2*y-2*a[5][p][2])); end do;

for r from 1 to 1 do tot[6]:=tot[6] + expand((1/240)*(2*w-2*a[6][r][3]+7)*(x-a[6][r][1])*(y-a[6][r][2])*(x-a[6][r][1]+y-a[6][r][2])*(-x+a[6][r][1]+y-a[6][r][2])*(2*x-2*a[6][r][1]-y+a[6][r][2])*(-x+a[6][r][1]+2*y-2*a[6][r][2])); end do;

t:=tot[0]-tot[1]+tot[2]-tot[3]+tot[4]-tot[5]+tot[6];