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ABSTRACT

We show that an approximation to the static crossing matrix leads to a soluble model for the p_{11} and p_{33} πN scattering amplitudes in which the parameters are related to forces due to particle exchanges in the t-channel. Reasonable values of these forces give the nucleon bound state and the Δ resonance in good agreement with experiment. We investigate further properties of the model.

I. INTRODUCTION

Much of our intuition in strong-interaction physics has come from the success of the static model[1] in explaining low energy pion-nucleon scattering. Especially important is the reciprocal bootstrap relationship between the N and the Δ first pointed out by Chew[2] and cited by him as evidence for the view that pion-nucleon dynamics (and, indeed, all of strong-interaction dynamics) is determined by self-consistency requirements, with no free parameters allowed. This view is extremely attractive; but since a πN bootstrap calculation typically requires a cutoff,[3] it has been difficult in practice to avoid introducing arbitrary parameters. It is usually assumed that a cutoff is needed because the effect of inelastic channels and related high-energy behavior has been mistreated (if not completely ignored).

Hendry and Stech[4] argued that, although high-energy behavior would certainly be critical in a complete calculation, it should be possible to do a self-consistent calculation of low-energy behavior with a knowledge of only nearby singularities. They represented the nearby cuts due to t-channel $(N\bar{N} \to \pi\pi)$ exchanges by a pair of conjugate poles, and showed that the position and residue of these poles could be adjusted to ensure convergence of their dispersion integrals without a high-energy cutoff. In their calculation, the t-channel singularities gave rise to an effective cutoff function which was, in principle, not arbitrary but determined by $N\bar{N} \to \pi\pi$ scattering; the resulting πN scattering amplitudes contained the N and Δ and satisfied the static crossing relations "to a high degree", but not exactly.

We would like to find, in closed form, a crossing-symmetric, unitary set of scattering amplitudes in which the only parameters introduced are related to t-channel scattering, as in the calculation of Hendry and Stech. Since the amplitudes f_{13} and f_{31} (we use the notation $f_{2I,2J}$ for the p-wave amplitudes, which are the only ones we consider) are known to be small, and since we cannot handle the problem of all four coupled p-wave amplitudes, we consider scattering only in the p_{11} and p_{33} states. Recently, Schwarz[5] noticed that the assumption $f_{13} \equiv f_{31} \equiv 0$ is inconsistent with the static crossing relation if f_{11} and f_{33} both satisfy elastic unitarity, but that a suitably modified crossing matrix[6] leads to a self-consistent soluble model, which, as emphasized by Schwarz, "satisfies an approximate crossing relation exactly instead of satisfying an exact one approximately".

In Section II below we examine the static crossing relation between the direct channel (s-channel) and the crossed πN channel (u-channel) and discuss the approximate crossing matrix. We then introduce a modified N/D representation[9,10] of the scattering amplitude in which t-channel singularities (expressing the forces due to exchange of particles in the t-channel) are contained in the N function, while the D function carries both the s- and u-channel cuts (including the nucleon and Δ exchange forces). An especially interesting feature of our model is that only the t-channel forces have to be specified at the beginning of the calculation; the s- and u-channel forces are generated self-consistently by elastic unitarity and the s-u crossing relation. Thus the nucleon exchange force is automatically taken into account when the nucleon is bound, and the force due to Δ exchange is included when the Δ resonance is formed. Also, since our model satisfies s-channel and u-channel unitarity simultaneously, the finite width of the Δ is correctly accounted for in the Δ exchange force.

It is probable that the t-channel forces are also determined by a self-consistency requirement arising from t-channel unitarity, but it is impossible to include this effect in our model. We thus use a simple four-parameter approximation for these forces. The parameters are fixed in Section III by requiring that the Δ position and width and the bound nucleon mass and coupling constant, which are all dynamically determined in our model, agree as closely as possible with the values obtained from experiment. In Section IV we show that the resulting values of the parameters correspond to t-channel forces that do approximate the forces expected from the exchange of known particles in the t-channel.

Section V is a discussion of the behavior of the scattering amplitude off the real axis, while Section VI is concerned with Levinson's Theorem and asymptotic behavior. In Section VII we consider the superconvergence relations[11] that are satisfied by our amplitude. It is shown that the t-channel forces make important contributions to these sum rules, thus overcoming some of the difficulties that arise from assuming saturation of these relations by the s- and u-channel singularities.[12] Our results are summarized and discussed in Section VIII.

II. THE MODEL

The analytic structure of πN partial wave amplitudes is by now well known;[13] in the static limit this structure simplifies considerably. The amplitudes are analytic in the ω plane cut as shown in Figure 1, where $\omega = W-M$, W being the total energy in the πN center of mass system and M the nucleon mass (we take the pion mass as the energy unit).

The discontinuity across the right-hand cut, from $+1$ to $+\infty$, is given by the unitarity relation,

$$\text{Im}\, f_{ij}(\omega + i\epsilon) = q^3 |f_{ij}(\omega)|^2 , \quad \omega > 1 , \tag{2.1}$$

where $q = (\omega^2 - 1)^{\frac{1}{2}}$ is the pion momentum in the static limit. The choice of phase-space factor in Eq. (2.1) fixes the normalization of the amplitudes; they are related to the phase shifts δ_{ij} by

$$f_{ij}(\omega) = q^{-3}\, e^{i\delta_{ij}(\omega)} \sin\delta_{ij}(\omega) , \tag{2.2}$$

in the s-channel physical region (the upper edge of the right hand cut).

The discontinuity across the left-hand cut, from $-\infty$ to -1, is related to the amplitudes of the crossed πN reaction (the u-channel). The most advantageous feature of the static model is that the crossing relations for the partial waves are particularly simple; the u-channel to s-channel crossing matrix does not mix different angular momenta. The crossing relation for p-waves is[8]

$$\begin{pmatrix} f_{11}(-\omega) \\ f_{13}(-\omega) \\ f_{31}(-\omega) \\ f_{33}(-\omega) \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 1 & -4 & -4 & 16 \\ -2 & -1 & 8 & 4 \\ -2 & 8 & -1 & 4 \\ 4 & 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} f_{11}(\omega) \\ f_{13}(\omega) \\ f_{31}(\omega) \\ f_{33}(\omega) \end{pmatrix} . \tag{2.3}$$

Since the problem of finding four unitary amplitudes obeying (2.3) is intractable, we must make some approximations. The most obvious approach would be to set $f_{13}(\omega) \equiv f_{31}(\omega) \equiv 0$, since these amplitudes are known experimentally to be small. Then we would have

$$\begin{pmatrix} f_{11}(-\omega) \\ f_{33}(-\omega) \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 1 & 16 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} f_{11}(\omega) \\ f_{33}(\omega) \end{pmatrix} , \tag{2.4}$$

but this cannot possibly be right, since applying the crossing relation (2.4) twice gives $f_{11}(\omega) = 2f_{33}(\omega)$. This relation, aside from its blatant disagreement with experiment, is inconsistent with unitarity. This is the conflict noticed by Schwarz,[5] who proposed instead the ansatz[6]

$$\begin{pmatrix} f_{11}(-\omega) \\ f_{33}(-\omega) \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f_{11}(\omega) \\ f_{33}(\omega) \end{pmatrix}, \tag{2.5}$$

which does not violate unitarity, and is somehow close to Eq. (2.4).

The sense in which (2.4) and (2.5) are close can be seen by considering an N/D static model in which the left hand cut is represented by a finite number of poles and the D function is assumed to be linear.[2,14] With these approximations the occurrence of bound states and ratios of coupling constants depend only on the eigenvalues and eigenvectors of the crossing matrix. Now for the crossing matrix in either (2.4) or (2.5) one eigenvector, (2, 1), has eigenvalue unity and the other, (2, - 1), has an eigenvalue far from unity (the latter eigenvalues are -7/9 and -1, respectively); thus both matrices lead to the same predictions in this simple model.

Since the crossing relation (2.5) is consistent with unitarity and close to the true crossing relation, and will be seen to lead to a soluble model, we shall use it in the following. Before we go on, though, it should be pointed out that there is a conflict between (2.3) and (2.5). The two equations together imply that $f_{13}(\omega)$ and $f_{31}(\omega)$ are odd functions, and that

$$4f_{13}(\omega) + 4f_{31}(\omega) - f_{11}(\omega) + 2f_{33}(\omega) = 0.$$

This last relation must lead to a violation of unitarity near the Δ resonance, where $\text{Im}\, f_{33} > \text{Im}\, f_{11}$, so we are forced to admit that our model, although self-consistent, cannot be reconciled with the full static crossing symmetry (2.3).

Scattering amplitudes f_{11} and f_{33} obeying the crossing relation (2.5) can be expressed in terms of a single analytic function by

$$f_{11}(\omega) = 2f_{33}(-\omega) \equiv f(\omega) . \tag{2.6}$$

Then on the upper edge of the right hand cut (Figure 1) we have

$$f(\omega) = q^{-3} e^{i\delta_{11}(\omega)} \sin\delta_{11}(\omega) , \tag{2.7}$$

while on the lower edge of the left hand cut (the u-channel physical region)

$$f(\omega) = 2q^{-3} e^{i\delta_{33}(|\omega|)} \sin\delta_{33}(|\omega|) . \tag{2.8}$$

Equations (2.7) and (2.8) together with the real analyticity of $f(\omega)$ completely determine the discontinuity of the inverse function across the cuts on the real axis:

$$\text{Im } f^{-1}(\omega + i\epsilon) = -\, q^3 = -\,(\omega^2 - 1)^{3/2} , \quad \omega > 1 , \tag{2.9a}$$

$$\text{Im } f^{-1}(\omega + i\epsilon) = \frac{1}{2}\, q^3 = \frac{1}{2}(\omega^2 - 1)^{3/2} , \quad \omega < -1 . \tag{2.9b}$$

This makes it convenient to express $f(\omega)$ as the quotient of two real analytic functions

$$f(\omega) = N(\omega)/D(\omega) , \tag{2.10}$$

where $D(\omega)$ carries both the left hand and the right hand cuts and $N(\omega)$ carries the singularities off the real axis. Thus we have

$$\text{Im } D(\omega + i\epsilon) = -\, q^3 N(\omega) , \; \omega > 1 , \tag{2.11a}$$

$$\text{Im } D(\omega + i\epsilon) = \frac{1}{2}\, q^3 N(\omega) , \quad \omega < -1 . \tag{2.11b}$$

The N/D method[9,10] is a well known method of constructing unitary scattering amplitudes, but the D function usually has only the right hand cut. That the N/D method can be modified in the above manner is due to the similarity between the discontinuity relations (2.9a) and (2.9b), which both follow directly from unitarity only because of the remarkably simple crossing properties of our model.

We shall make the bootstrap assumption that the D function contains all the s- and u-channel bound states and resonances as zeros of its real part. Thus neither the nucleon nor the Δ has to be put into the N function in our model; they are bootstrapped simultaneously. We also assume that the D function has no poles; we do not allow CDD[10,15] poles in our calculation. A D function with these properties can be given in terms of the usual N/D decomposition of the scattering amplitude. Consider, for example, the (physical) case where there is a nucleon pole at the origin and no $I = J = 3/2$ bound state. In this case

$$f_{11}(\omega) = N_{11}(\omega)/D_{11}(\omega) \ , \tag{2.12a}$$

$$f_{33}(\omega) = N_{33}(\omega)/D_{33}(\omega) \ , \tag{2.12b}$$

where, apart from trivial multiplicative constants,

$$D_{11}(\omega) = \omega \ \mathcal{D}_{11}(\omega) \ , \tag{2.13a}$$

$$D_{33}(\omega) = \mathcal{D}_{33}(\omega) \ , \tag{2.13b}$$

and the $\mathcal{D}_{ij}(\omega)$ are the Omnès functions,[10,16]

$$\mathcal{D}_{ij}(\omega) \equiv \exp\left[-\frac{\omega}{\pi}\int_1^{\infty} \frac{\delta_{ij}(x)\,dx}{x(x-\omega)}\right] . \tag{2.14}$$

Now it is easy to see that the D function in our model is given by

$$D(\omega) = D_{11}(\omega)\, D_{33}(-\omega) \ . \tag{2.15}$$

We shall further assume that $D(\omega)$ obeys a once-subtracted dispersion relation, which is equivalent to

$$\omega^{-1} D(\omega) \xrightarrow[\omega \to \infty]{} 0 \ . \tag{2.16}$$

This is a stronger condition than the usual assumption that D_{11} and D_{33} obey once-subtracted dispersion relations, as can be seen from Eq. (2.15); we shall return to this point in Section VI.

We now must consider the cuts from $+ i\tau$ to $+ i\infty$ and from $- i\tau$ to $- i\infty$, which we have agreed to put into the N function. These are the static limit of the circular cut[13] in the fully relativistic partial wave amplitude and come from t-channel exchanges; the mass, m, of the system exchanged being related to τ by

$$\tau = \left(\frac{1}{4} m^2 - 1\right)^{\frac{1}{2}} \ . \tag{2.17}$$

Two-pion exchange thus leads to a cut along the entire imaginary axis, but we assume that this contribution can be approximated by the ρ, $\sigma(I = J = 0$ dipion enhancement), and perhaps other resonances, so that τ is finite. Taking m as the ρ mass, for example, leads to $\tau \simeq 2.7$.

We shall go even further, and approximate these cuts by a few pairs of poles at conjugate positions on the imaginary axis, following Hendry and Stech.[4] The number of poles needed depends on the asymptotic behavior given by Eqs. (2.2) and (2.16) and the observation that $\sin\delta_{ij}(\omega)$ is bounded for positive real ω, which give

$$\omega^2 N(\omega) = \left(\omega^3 f(\omega)\right) \left(\omega^{-1} D(\omega)\right) \xrightarrow[\omega \to +\infty]{} 0 . \qquad (2.18)$$

Since we are assuming $N(\omega)$ to be a rational function, the limit must hold for $\omega \to \infty$ in any direction, which implies that we need at least two pairs of poles. In this case $N(\omega)$ is given by

$$N(\omega) = \frac{\lambda + (\lambda - \mu)\, \omega/\omega_{33}}{(\omega^2 + \alpha^2)(\omega^2 + \beta^2)} , \qquad (2.19)$$

which depends on four real parameters. The form of the numerator of (2.19) is chosen for later convenience; ω_{33} is the position of the Δ resonance,[17]

$$\omega_{33} = 2.17 . \qquad (2.20)$$

We could, of course, give $N(\omega)$ additional poles; but a proliferation of free parameters is exactly what we want to avoid. The interesting feature of our model is that it naturally involves four parameters; there are no *a priori* arguments for determining any of them. These parameters are supposed to be related to t-channel forces, though, rather than being arbitrary. We shall see if this is the case by solving the model, which is determined by Eqs. (2.11) and (2.19), together with the assumptions made about the D function.

III. THE SOLUTION

Equation (2.19) shows that if $\lambda \neq \mu$ the N function has a zero at ω_0 given by

$$\omega_0 = \frac{\lambda\, \omega_{33}}{\mu - \lambda} . \tag{3.1}$$

In this case we can perform the subtraction in the dispersion relation for $D(\omega)$ at ω_0, where $\operatorname{Im} D(\omega_0) = 0$ and $\operatorname{Re} D(\omega_0) = \nu$, say. This gives

$$D(\omega) = \nu + \left[\lambda + (\lambda - \mu)\omega/\omega_{33} \right] \left[\frac{1}{2\pi} \int_{-\infty}^{-1} \frac{(x^2 - 1)^{3/2}\, dx}{(x^2 + \alpha^2)(x^2 + \beta^2)(x - \omega)} \right.$$
$$\left. - \frac{1}{\pi} \int_1^{\infty} \frac{(x^2 - 1)^{3/2}\, dx}{(x^2 + \alpha^2)(x^2 + \beta^2)(x - \omega)} \right] . \tag{3.2}$$

In the case that $\lambda = \mu$ we can write an unsubtracted dispersion relation for $D(\omega)$, but we are then free to add a real constant ν to the D function to fix its normalization at infinity. We thus arrive at Eq. (3.2) in this case also.

Now we must consider two possibilities. If $\nu = 0$ the N and D functions have a coincident zero at ω_0 (or at infinity if $\lambda = \mu$). This gives the "extinct bound state" solution of Atkinson and Halpern,[18] in which the dependence of the amplitude on λ and μ drops out completely. The resulting amplitude cannot have a pole at the origin for the nucleon, so we ignore this solution.

If ν is not zero we can set it equal to unity by choosing the normalization of $N(\omega)$ and $D(\omega)$ appropriately. Then, except in the degenerate case $\lambda = \mu = 0$, which gives $D(\omega) \equiv 1$ and $N(\omega) \equiv f(\omega) \equiv 0$, we have

$$f^{-1}(\omega) = \frac{D(\omega)}{N(\omega)} = (\omega^2 + \alpha^2)(\omega^2 + \beta^2)\left[\frac{1}{\lambda + (\lambda - \mu)\omega/\omega_{33}} + \frac{1}{2\pi}\int_{-\infty}^{-1} \frac{(x^2 - 1)^{3/2}\,dx}{(x^2 + \alpha^2)(x^2 + \beta^2)(x - \omega)} - \frac{1}{\pi}\int_{1}^{\infty} \frac{(x^2 - 1)^{3/2}\,dx}{(x^2 + \alpha^2)(x^2 + \beta^2)(x - \omega)}\right] \tag{3.3a}$$

$$= (\omega^2 + \alpha^2)(\omega^2 + \beta^2)\left[\frac{1}{\lambda + (\lambda - \mu)\omega/\omega_{33}} + \frac{(\alpha^2 + 1)^{3/2}}{(\beta^2 - \alpha^2)(\alpha^2 + \omega^2)}\left(\frac{3}{4} + \frac{\omega}{2\pi\alpha}\sinh^{-1}\alpha\right) - \frac{(\beta^2 + 1)^{3/2}}{(\beta^2 - \alpha^2)(\beta^2 + \omega^2)}\left(\frac{3}{4} + \frac{\omega}{2\pi\beta}\sinh^{-1}\beta\right)\right] - (\omega^2 - 1)^2 \times \left[\phi(\omega) + \frac{1}{2}\phi(-\omega)\right], \tag{3.3b}$$

where

$$\phi(\omega) \equiv \frac{1}{\pi} \int_1^\infty (x^2 - 1)^{-\frac{1}{2}} \frac{dx}{x - \omega} \quad . \tag{3.3c}$$

For real or pure imaginary argument, $\phi(\omega)$ is given by

$$\phi(x \pm i\epsilon) = (x^2 - 1)^{-\frac{1}{2}} \left\{ - \frac{1}{\pi} \log\left[x + (x^2 - 1)^{\frac{1}{2}}\right] \pm i \right\}, \quad x > 1 \tag{3.4a}$$

$$= (x^2 - 1)^{-\frac{1}{2}} \frac{1}{\pi} \log\left[- x + (x^2 - 1)^{\frac{1}{2}}\right] \, , \quad x < - 1 \tag{3.4b}$$

$$= (1 - x^2)^{-\frac{1}{2}} \left(\frac{1}{2} + \frac{1}{\pi} \sin^{-1} x \right), \quad - 1 < x < 1 \tag{3.4c}$$

$$\phi(\pm ix) = (x^2 + 1)^{-\frac{1}{2}} \left(\frac{1}{2} \pm \frac{i}{\pi} \sinh^{-1} x \right) . \tag{3.4d}$$

If $f(\omega)$ is to correspond to the physical amplitudes f_{11} and f_{33}, it must have a pole at the origin and a resonance at $- \omega_{33}$. Thus λ must be chosen to give $D(0) = 0$ and μ to give $\mathrm{Re}\, D(- \omega_{33}) = 0$, or from (3.3) and (3.4)

$$\frac{1}{\lambda} = \frac{3}{4} \frac{\alpha^2\left[(\beta^2 + 1)^{3/2} - 1\right] - \beta^2\left[(\alpha^2 + 1)^{3/2} - 1\right]}{\alpha^2 \beta^2 (\beta^2 - \alpha^2)} \quad , \tag{3.5a}$$

$$\frac{1}{\mu} = - \frac{(\alpha^2 + 1)^{3/2}}{(\beta^2 - \alpha^2)(\alpha^2 + \omega_{33}^2)} \left(\frac{3}{4} - \frac{\omega_{33}}{2\pi\alpha} \sinh^{-1}\alpha\right)$$

$$+ \frac{(\beta^2 + 1)^{3/2}}{(\beta^2 - \alpha^2)(\beta^2 + \omega_{33}^2)} \left(\frac{3}{4} - \frac{\omega_{33}}{2\pi\beta} \sinh^{-1}\beta\right)$$

$$+ \frac{(\omega_{33}^2 - 1)^{3/2}}{(\alpha^2 + \omega_{33}^2)(\beta^2 + \omega_{33}^2)} \frac{1}{2\pi} \log\left[\omega_{33} + (\omega_{33}^2 - 1)^{\frac{1}{2}}\right]. \qquad (3.5b)$$

Equations (3.5) can be satisfied for finite λ and μ only if the expressions on the right hand side are non-zero; these expressions are in fact positive definite for any real α and β. The positiveness of λ and μ tells us that $N(\omega)$ is positive for $-\omega_{33} \leqslant \omega \leqslant 0$ and that, if $\lambda \neq \mu$, $\omega_0 f'(\omega_0) < 0$. Thus our model cannot possibly fit the experimental p_{11} phase shift, which increases through zero near the $N\pi\pi$ threshold.[5,19]

We can hope, though, that our model will fit the experimentally determined nucleon pole residue and Δ resonance width.[17,20]

$$\gamma_{11} = 0.243 \pm 0.006, \quad \gamma_{33} = 0.12 \pm 0.01 . \qquad (3.6)$$

We thus fix λ and μ by Eqs. (3.5) and compare the above values with the predictions of our model,

$$\gamma_{11}^{-1} = - \frac{d}{d\omega}\left[f^{-1}(\omega)\right]_0 , \qquad (3.7a)$$

$$\gamma_{33}^{-1} = -\frac{d}{d\omega}\left[\mathrm{Re}\ f_{33}^{-1}(\omega)\right]_{\omega_{33}} = 2\frac{d}{d\omega}\left[\mathrm{Re}\ f^{-1}(\omega)\right]_{-\omega_{33}}, \tag{3.7b}$$

for a range of values of α and β.

The values of γ_{11}^{-1} and γ_{33}^{-1} given by Eqs. (3.7) lie between the two curves shown in Figure 2, the upper curve giving the points for $\beta = \alpha$ and the lower curve for $\beta \gg \alpha$; increasing α moves the points along either curve to increasing values. The experimental values (3.6) are indicated by the circle with accompanying error bars. Our best fit, corresponding to $\alpha = 3.8$ and $\beta \gg \alpha$, is shown by the small square; it gives

$$\gamma_{11} = 0.2353\ , \quad \gamma_{33} = 0.136\ . \tag{3.8}$$

For $\beta \gg \alpha$, the amplitude becomes completely independent of β in the low-energy region, $\omega \lesssim \alpha$. This result follows because, in this limit, Eqs. (3.5) give, to order $1/\beta$,

$$\frac{1}{\lambda} = \frac{1}{\beta^2}\left[\frac{3}{4}\ \beta - \lambda'\right], \tag{3.9a}$$

$$\frac{1}{\mu} = \frac{1}{\beta^2}\left[\frac{3}{4}\beta - \frac{\omega_{33}}{2\pi}\sinh^{-1}\beta - \mu'\right], \tag{3.9b}$$

where

$$\lambda' = \frac{3}{4\alpha^2}\left[(\alpha^2+1)^{3/2} - 1\right] \tag{3.10a}$$

$$\mu' = \frac{1}{\alpha^2 + \omega_{33}^2} \left\{ (\alpha^2 + 1)^{3/2} \left(\frac{3}{4} - \frac{\omega_{33}}{2\pi\alpha} \sinh^{-1}\alpha \right) - \frac{(\omega_{33}^2 - 1)^{3/2}}{2\pi} \log\left[\omega_{33} + (\omega_{33}^2 - 1)^{\frac{1}{2}}\right] \right\}. \quad (3.10b)$$

Then for small ω, i.e., to order $\omega\beta^{-1}(\sinh^{-1}\beta)^2$, the inverse amplitude is approximately equal to

$$f_0^{-1}(\omega) \equiv -(\omega^2 + \alpha^2)\left[\lambda' + (\lambda' - \mu')\omega/\omega_{33}\right] + (\alpha^2 + 1)^{3/2} \times \left[\frac{3}{4} + \frac{\omega}{2\pi\alpha} \sinh^{-1}\alpha\right] - (\omega^2 - 1)^2\left[\phi(\omega) + \frac{1}{2}\phi(-\omega)\right]. \quad (3.11)$$

We shall take β large enough so that (3.11) is a good approximation to (3.3b) for $|\omega| \lesssim 5$, giving an effective three-parameter fit to the low-energy amplitudes. The resulting phase shifts, as given by (2.7) and (2.8) are compared to the 0 - 700 MeV values of Roper, Wright and Feld[19] in Figures 3 and 4. The variable used in the pion lab kinetic energy, T, which is related to ω by

$$2\,M\,T \doteq (\omega - 1)(\omega + 2M + 1) \quad . \qquad (3.12)$$

In particular, $\omega_{33} = 2.17$ corresponds to $T_{33} = 200$ MeV. The

p_{33} phase shift does not rise steeply enough above the Δ, but it does tend asymptotically to π rather than to zero. The p_{11} phase shift, as noted before, is not even qualitatively correct; it decreases through $-\pi$ at $\omega_0 \simeq \frac{3\pi}{2}\,\beta(\sinh^{-1}\beta)^{-1}$ and eventually goes to -2π at infinite energy. This somewhat surprising asymptotic behavior will be considered further in Section VI.

IV. THE t-CHANNEL FORCES

We have now explicitly constructed an amplitude with poles at $\pm i\alpha$ and $\pm i\beta$ that are supposed to represent the forces due to the exchange of particles in the t-channel. For the values of the parameters found in the previous section, this model gives a good fit to the low-energy p_{33} and p_{11} (the nucleon pole!) amplitudes. Let us now look more closely at the input forces.

The poles at $\pm i\beta$, which we shall call the "far poles", have residues given by

$$\mathrm{Res}\left[f(\omega)\ ,\ \pm i\beta\right] = -\frac{1}{\beta^2}\left(\frac{1}{2\pi} \mp \frac{3i}{4}\right)^{-1}, \tag{4.1}$$

to order $(\beta^2 \sinh^{-1}\beta)^{-1}$. The "near poles" at $\pm i\alpha$ have residues

$$\mathrm{Res}\left[f(\omega),\ \pm i\alpha\right] = \left\{\frac{1}{2\pi\alpha}\left[4\pi\alpha^3(\lambda' - \mu')/\omega_{33} - \alpha(\alpha^2 + 1) - (2\alpha^2 - 1)(\alpha^2 + 1)^{\frac{1}{2}} \sinh^{-1}\alpha\right] \pm \frac{i}{4}\left[9\alpha\,(\alpha^2 + 1)^{\frac{1}{2}} - 8\alpha\lambda'\right]\right\}^{-1} \tag{4.2a}$$

$$= (6.732 \pm 10.042\ i)^{-1} = 0.0461 \mp 0.0687\ i\ . \tag{4.2b}$$

These expressions follow, after some algebra, from Eqs. (3.3), (3.4), and (3.11); the numerical values are for $\alpha \doteq 3.8$ with

λ' and μ' given by Eqs. (3.10).

A function obeying a dispersion relation with no singularities other than the near poles is trivially given by

$$B(\omega) = \frac{0.522 + 0.0922\,\omega}{\omega^2 + \alpha^2} . \tag{4.3}$$

This "Born term", together with a similar one for the far poles would be the input in a full N/D calculation in our model. For purposes of physical interpretation, it is convenient to think of our four parameters as expressing the positions of the near and far poles, α and β, and the residues of the near poles. The residues of the far poles are then not independent, but are determined by the requirement that the amplitude have the correct behavior both at threshold and asymptotically. Thus it is likely that the near poles represent the first Born approximations to the exchanges of low-mass particles in the t-channel, while the far poles approximate the part of the double spectral function arising from the iteration of this lowest order potential in the manner proposed by Mandelstam.[10,21]

The fact that the best agreement with experiment is for $\beta \gg \alpha$ may be connected with the above interpretation; it also agrees with other models of low-energy πN scattering, which generally depend on only three parameters. In the usual reciprocal bootstrap calculation,[2] for instance, the position and residue of the nucleon pole and a cutoff parameter are needed to calculate the p_{33} amplitude, while a cutoff and the Δ position and width suffice to determine the p_{11} amplitude. Strong-coupling theories[22] for p-wave scattering also have three parameters in our sense--the mass and coupling constant of the nucleon and a cutoff.

We shall now investigate the function $B(\omega)$ given by Eq. (4.3), to check the interpretation that it represents the contribution of low mass exchanges to the t-channel force. We first note that the position of the singularities at $\pm 3.8i$ is reasonable; we can compare this to the end points of the cuts due to ρ exchange at $\pm 2.7i$ and to the position of the poles used by Hendry and Stech,[4] $\pm 5i$. These values correspond to the mass of the exchanged particles, i.e., to the range of the forces.

As a measure of the strength of the input forces, we shall take the value of the Born term at threshold.[3] The forces are then found to be attractive (positive Born term), with strength in the p_{11} and p_{33} channels of

$$B(1) = 0.040 \quad \text{and} \tag{4.4a}$$

$$\tfrac{1}{2} B(-1) = 0.014 , \tag{4.4b}$$

respectively. Donnachie, Hamilton and Lea[23] have calculated the

Born terms for πN scattering in some detail, finding the threshold values in the p_{11} and p_{33} channels of the combined force due to ρ and σ exchange to be 0.055 and 0.03, respectively. Our value is very close to theirs in the channel with the quantum numbers of the nucleon, and too small by a factor of two in the Δ channel.

The input forces needed in our model to "predict" the N and Δ are thus seen to be attractive and to have a strength and range of the correct order of magnitude. If these forces are varied, the position and width of the output N and Δ will change. To see this let us fix the range of the forces and vary their strength. This is done by fixing α and β and varying λ and μ in Eq. (3.3) or equivalently λ' and μ' in Eq. (3.11); we shall furthermore vary these parameters together in such a way that $B(\omega)$ is just multiplied by a real constant, its functional form being unaltered. The low-energy approximation to the resulting amplitude is given by

$$f_\kappa(\omega) = \left[f_0^{-1}(\omega) + \kappa g(\omega)\right]^{-1} , \qquad (4.5)$$

where $f_0^{-1}(\omega)$ is given by (3.11) with λ' and μ' obeying Eqs. (3.10), κ is a real parameter, and

$$g(\omega) = \alpha^{-2} (\omega^2 + \alpha^2) (19.081 - 3.366\, \omega) . \qquad (4.6)$$

The above assertion is verified by noting that

$$\text{Res}\left[f_\kappa(\omega),\ \pm i\alpha\right] = (1 + \kappa)^{-1}\ \text{Res}\left[f(\omega),\ \pm i\alpha\right]\ ; \qquad (4.7)$$

thus for $\kappa = 1/9$, $f_\kappa(\omega)$ is the amplitude with 10% smaller Born terms, while $\kappa = -1/11$ gives an amplitude with the input force increased by 10%. The functions $\text{Re}\ f_\kappa^{-1}(\omega) = \text{Re}\ f_0^{-1}(\omega) + \kappa g(\omega)$ for $\kappa = 0,\ 1/9,$ and $-1/11$ are plotted in Figure 5. When $\kappa = 1/9$, the nucleon position and residue are

$$\omega_{11} = 0.39\ ,\quad \gamma_{11} = 0.16\ , \qquad (4.8a)$$

while the Δ position and width change to

$$\omega_{33} = 3.37\ ,\quad \gamma_{33} = 0.124\ . \qquad (4.8b)$$

For $\kappa = -1/11$, we have

$$\omega_{11} = -0.65\ ,\qquad \gamma_{11} = 0.70\ , \qquad (4.9a)$$

$$\omega_{33} = 1.42\ ,\qquad \gamma_{33} = 0.36\ . \qquad (4.9b)$$

As κ is decreased to $-1/10$, the Δ becomes a bound state; if κ is made still more negative, the N and Δ come together and f_κ^{-1} no longer has zeros on the real axis. We shall see in the next section that the zeros have moved off the real axis to conjugate positions in the complex plane; our model has developed bound states of complex "mass". Conversely, if κ is increased sufficiently, ω_{11} becomes greater than unity; i.e., the "nucleon" changes from a bound state to a resonance; as would be expected.

Of course our model is inconsistent if $\omega_{11} \neq 0$, since this means that the "external" nucleon and the "internal" nucleon have different masses; but it is still instructive to see the results of varying the parameters. A 10% change in the strength of the t-channel forces causes large changes in the N and Δ position and width, as can be seen by comparing Eqs. (3.8), (4.8) and (4.9). The reason for this behavior lies in the well known reciprocal bootstrap mechanism;[2] the nucleon furnishes almost all the force needed to form the Δ, and <u>vice versa</u>. Thus a small increase in the t-channel forces decreases the mass and increases the coupling constant of both the N and the Δ, which results in a corresponding increase in the dominant s- and u-channel forces; a decrease in the input forces acts in just the opposite manner. The output (N and Δ masses) varies with the input (t-channel forces) by a large amplification factor which is due to the positive feedback character of the reciprocal bootstrap mechanism.

V. BEHAVIOR OF THE AMPLITUDE OFF THE REAL AXIS

In this section we want to study the behavior off the real axis of any reasonable scattering function satisfying the crossing relation (2.6), whether or not it has an N/D decomposition. By a reasonable amplitude we mean one that is a real analytic function with no essential singularity at infinity and with cuts only from $+1$ to $+\infty$ and from -1 to $-\infty$, on which it obeys the unitarity Eq. (2.9). These assumptions imply that

$$f^{-1}(\omega) = R(\omega) - (\omega^2 - 1)^2 \left[\phi(\omega) + \frac{1}{2}\phi(-\omega)\right] , \qquad (5.1a)$$

where $R(\omega)$ is a rational function and $\phi(\omega)$ is given by Eq. (3.3c). Then asymptotically

$$f^{-1}(\omega) \xrightarrow[\omega \to \infty]{} \frac{\omega^3}{2\pi} \log \omega + C\omega^{3+r} , \qquad (5.1b)$$

for some non-negative integer r, where C is real and non-zero.

We shall also insist that a reasonable $f(\omega)$ not differ too greatly from the physical amplitude in its behavior on the real axis between the s and u cuts. Specifically, $f(\omega)$ is assumed to have no zeros in $[-1, 1]$ and to fall into one of the following three classes:

(i) $f(\omega)$ has a pole with negative residue at ω_{11} in $(-1, 1)$, corresponding to the nucleon. It may or may not have a pole with positive residue at ω_{33} in $(-1, \omega_{11})$, corresponding to a bound Δ; but it has no other poles in $(-1, 1)$;

(ii) $f(\omega)$ is positive definite in $(-1, 1)$; i.e., the input forces are too weak to bind either the N or the Δ;

(iii) $f(\omega)$ is negative definite in $(-1, 1)$; this amplitude will be shown to have "bound state" poles off the real axis arising from input forces that are too attractive.

Now say that $f(\omega)$ has zeros and poles outside of $[-1, 1]$ as follows:

poles at the conjugate points z; z_i^*, $\text{Im}\, z_i > 0$; $i = 1, \cdots, p$;
zeros at the conjugate points ω_i, ω_i^*, $\text{Im}\, \omega_i > 0$; $i = 1, \cdots, m$;
zeros on the real axis at x_i with $x_i f'(x_i) < 0$; $i = 1, \cdots, n$;
any number[24] of zeros on the real axis with $xf'(x) > 0$.

Then we construct the following function

$$h(\omega) \equiv \prod_{i=1}^{p} (\omega - z_i)(\omega - z_i^*) \left[\prod_{j=1}^{m} (\omega - \omega_j)(\omega - \omega_j^*) \prod_{k=1}^{n} (\omega - x_k)^2 \right]^{-1}$$

$$\times (\omega - x_0)^{\eta} f(\omega) , \tag{5.2a}$$

where

$$x_0 = \omega_{11} \quad \text{for case i}$$

$$= 0 \quad \text{for cases ii and iii ,} \tag{5.2b}$$

and

$$\eta = +1 \quad \text{for cases i and ii}$$

$$= -1 \quad \text{for case iii .} \tag{5.2c}$$

This function is analytic in the upper half plane and has neither poles nor zeros off the real axis. On the real axis it has no poles or zeros other than simple poles with negative residue and simple zeros with positive slope. In addition, the limit of $\mathrm{Im}\, h(\omega)$ as the real axis is approached from above is non-negative. Then, by a theorem due to Symanzik,[25] which is outlined in the Appendix to this paper,

$$\mathrm{Im}\, h(\omega) \geqslant 0 \quad \text{for} \quad \mathrm{Im}\, \omega \geqslant 0 \; . \tag{5.3}$$

Functions with the property (5.3) are called Herglotz functions; such functions have been used by many authors[15,26] to study dispersion relations. The only result of the theory of these functions[27] that we need is

$$\omega^{-1}\, h(\omega) \xrightarrow[\omega \to \infty]{} C' \geqslant 0 \tag{5.4a}$$

for $\epsilon \leqslant \text{Arg}\,\omega \leqslant \Pi - \epsilon$, $\epsilon > 0$. It is clear that if $h(\omega)$ is a Herglotz function, $-h^{-1}(\omega)$ is also. Thus we also have

$$-\omega^{-1}\, h^{-1}(\omega) \xrightarrow[\omega \to \infty]{} C'' \geqslant 0 \,. \tag{5.4b}$$

Equations (5.2) and (5.4) imply that

$$2(p - m - n) + \eta - r = 2 \text{ for } r \text{ odd}, \; C < 0 \,, \tag{5.5a}$$

$$= 3 \text{ for } r \text{ even}, \tag{5.5b}$$

$$= 4 \text{ for } r \text{ odd}, \; C > 0 \,. \tag{5.5c}$$

The most obvious consequence of this relation is that $p \geqslant 1$; any reasonable amplitude must have at least one pair of poles off the real axis. An example of a reasonable scattering amplitude with only one pair of poles is f_0, defined by Eq. (3.11), which is approximately equal to the N/D solution (3.3) with $\beta \gg \alpha$ for low energies.

Let us now look at the N/D solution. It has no zeros off the real axis, so we take $m = 0$ in Eq. (5.5). Equation (3.3) shows that for $\lambda \neq \mu$ we have $n = 1$, $r = 0$, while for $\lambda = \mu$,

$n = 0$, $r = 1$, and $C > 0$. Both possibilities give

$$p = \frac{1}{2}(5 - \eta) . \tag{5.6}$$

Since the N function (2.19) has two pairs of poles off the real axis, the D function must have no zeros off the real axis in cases i and ii and one pair of zeros at conjugate points in case iii. These zeros of D are the bound states with complex "mass" referred to in the previous section. The way they develop as the input forces are increased can be seen by observing the motion as κ is lowered past $-1/10$ of the zeros of

$$f_\kappa^{-1}(\omega) \simeq -3.32\,(\omega + 0.915)^2 + 2.333 + 23.26\,\kappa$$

$$\text{for}\quad \omega \simeq -0.915,\ \kappa \simeq -0.10. \tag{5.7}$$

VI. ASYMPTOTIC BEHAVIOR

The asymptotic behavior of the best theoretical phase shifts (Figures 3 and 4) was briefly mentioned at the end of Section III. In general, any N/D amplitude with a nucleon pole at the origin and no other poles or zeros in $[-1, 1]$ that corresponds to a solution of Eq. (3.3) with $\lambda < \mu$ will have

$$\delta_{11}(\omega) \xrightarrow[\omega \to \infty]{} -2\pi + 2\pi (\log \omega)^{-1} , \tag{6.1a}$$

$$\delta_{33}(\omega) \xrightarrow[\omega \to \infty]{} \pi - \pi (\log \omega)^{-1} . \tag{6.1b}$$

Then the D functions without CDD poles, defined by Eqs. (2.13), (2.14) and (2.15) behave asymptotically like

$$|D_{11}(\omega)| \underset{\omega \to \infty}{\sim} |\omega|^{-1} (\log|\omega|)^2 , \tag{6.2a}$$

$$|D_{33}(\omega)| \underset{\omega \to \infty}{\sim} |\omega| (\log|\omega|)^{-1} , \tag{6.2b}$$

$$|D(\omega)| \underset{\omega \to \infty}{\sim} \log|\omega| . \tag{6.2c}$$

These three D functions all obey once-subtracted dispersion relations, so no CDD poles are needed.

The above phase shifts do not satisfy the usual form of Levinson's Theorem,[10,28] which states that

$$\delta(\omega) \xrightarrow[\omega \to \infty]{} \pi (n_B - n_A) , \tag{6.3a}$$

where $\delta(\omega)$ is the phase shift, normalized to zero at threshold, of an amplitude with n_A bound states and n_B CDD poles. They do satisfy the "weak Levinson's Theorem",[29]

$$\delta(\omega) \xrightarrow[\omega \to \infty]{} \pi (n_B - n_A + r) , \quad r \leq 1 , \tag{6.3b}$$

which is merely a statement that the denominator function of the amplitude obeys a once-subtracted dispersion relation. The reason that (6.3a) holds in potential scattering but not in this case is that the D functions of potential theory tend to constants asymptotically, while those given by Eqs. (6.2) certainly do not.

We could save Levinson's Theorem by adding a CDD pole to D_{33} and a zero to D_{11} in such a way that they cancel out of Eq. (2.15), leaving D unchanged. Since there is no reason to do this, the CDD pole not being needed for convergence of the dispersion relations, we shall not consider the possibility further.

Although our phase shifts do not satisfy Levinson's Theorem (6.3a), they do satisfy the Levinson-like identity

$$\delta_{11}(\omega) + \delta_{33}(\omega) \xrightarrow[\omega \to \infty]{} \pi\,(n_B' - n_A') \ , \qquad (6.4)$$

where n_A' is the total number of bound states in the p_{11} and p_{33} amplitudes (one in this case) and n_B' is the total number of CDD poles (zero in this case). Thus in our model it is possible to get a rising p_{33} phase shift without adding CDD poles, but only at the expense of a falling δ_{11}.

For values α and β other than those giving the closest agreement with experiment, Eqs. (3.5) may give values of λ and μ such that $\lambda \geqslant \mu$. The phase shifts given by Eq. (3.3) with these values of the parameters have the usual asymptotic behavior,

$$\delta_{11}(\omega) \xrightarrow[\omega \to \infty]{} -\pi \ , \quad \delta_{33}(\omega) \xrightarrow[\omega \to \infty]{} 0 \ , \qquad (6.5)$$

which satisfies both (6.3a) and (6.4) with no CDD poles. Correspondingly, the functions D_{11}, D_{33}, and D all obey once-subtracted dispersion relations.

The above discussion of asymptotics can be extended to any reasonable scattering amplitude, in the sense of the preceding section, if we keep track of the bound states correctly, excepting the "complex bound state" case iii. This case could be handled only by putting one "bound state" zero into D_{11} and the other into D_{33}, which would spoil the real analyticity of these functions. For all the other cases, the N/D solution yields phase shifts satisfying Eq. (6.4).

Let us finally consider the asymptotics of $f_0(\omega)$, defined by Eqs. (3.10) and (3.11) with $\alpha = 3.8$, which differs from our N/D solution only in its asymptotic behavior. For this amplitude,

$$\delta_{11}(\omega) \xrightarrow[\omega \to \infty]{} -\pi + 2\pi\,(\log\omega)^{-1} , \tag{6.6a}$$

$$\delta_{33}(\omega) \xrightarrow[\omega \to \infty]{} \pi - \pi\,(\log\omega)^{-1} . \tag{6.6b}$$

We can use Eqs. (2.13) and (2.14) to define $D_{11}(\omega)$ and $D_{33}(\omega)$, which have asymptotic behavior

$$|D_{11}(\omega)| \underset{\omega \to \infty}{\sim} (\log|\omega|)^2 , \tag{6.7a}$$

$$|D_{33}(\omega)| \underset{\omega \to \infty}{\sim} |\omega|\,(\log|\omega|)^{-1} . \tag{6.7b}$$

Thus the representations

$$f_{011}(\omega) \equiv f_0(\omega) = N_{11}(\omega)/D_{11}(\omega) \quad \text{and} \quad f_{033}(\omega) \equiv \tfrac{1}{2}\, f_0(-\omega) = N_{33}(\omega)/D_{33}(\omega) ,$$

both lead to N/D equations of the usual kind, with an unsubtracted dispersion relation for the N function and a once-subtracted dispersion relation for the D function. However, the D function given by Eq. (2.15), which has both the right hand and left hand cuts and no CDD poles, behaves asymptotically like $|\omega| \log |\omega|$ and so needs two subtractions. Thus $f_0(\omega)$ does not obey N/D equations of the type introduced in Section II, which illustrates the comment made after Eq. (2.16). This behavior is consistent with the argument in Section II that an amplitude with the N/D decomposition (2.10) must have at least two pairs of poles off the real axis, since $f_0(\omega)$ has only one such pair of poles. The above discussion also serves to show that the convergence of N/D equations may be critically dependent on asymptotic behavior that does not affect the solution in the low energy region.

VII. SUPERCONVERGENCE RELATIONS

Since the model considered in this paper gives phase shifts that tend asymptotically to integral multiples of π, the amplitude has the asymptotic behavior

$$\omega^3 f(\omega) \xrightarrow[\omega \to \infty]{} 0 . \tag{7.1}$$

This leads to the three superconvergence relations[11]

$$\sum_i \gamma_i + \frac{1}{\pi} \int_{-\infty}^{-1} \operatorname{Im} f(x - i\epsilon)\, dx - \frac{1}{\pi} \int_1^{\infty} \operatorname{Im} f(x + i\epsilon)\, dx = 0 , \tag{7.2a}$$

$$\sum_i \omega_i \gamma_i + \frac{1}{\pi} \int_{-\infty}^{-1} x \operatorname{Im} f(x - i\epsilon)\, dx - \frac{1}{\pi} \int_1^{\infty} x \operatorname{Im} f(x + i\epsilon)\, dx = 0 , \tag{7.2b}$$

$$\sum_i \omega_i^2 \gamma_i + \frac{1}{\pi} \int_{-\infty}^{-1} x^2 \operatorname{Im} f(x - i\epsilon)\, dx - \frac{1}{\pi} \int_1^{\infty} x^2 \operatorname{Im} f(x + i\epsilon)\, dx = 0 , \tag{7.2c}$$

where the sums run over all the poles of $f(\omega)$ at ω_i with residues γ_i.

That such superconvergence relations exist in the static model has been known for quite some time.[12] Assuming that (7.2a) is saturated by the nucleon and a zero-width Δ resonance leads to the well known prediction[2,12]

$$\gamma_{11} = 2\gamma_{33} . \tag{7.3}$$

The nucleon pole does not contribute to the second sum rule (7.2b). Assuming that only the cuts contribute to this relation leads to a contradiction since from Eqs. (2.7) and (2.8)

$$\operatorname{Im} f(x - i\epsilon) = 2q^{-3} \sin^2 \delta_{33}(|\omega|) , \quad x < -1 , \tag{7.4a}$$

$$\operatorname{Im} f(x + i\epsilon) = q^{-3} \sin^2 \delta_{11}(\omega) , \quad x > 1 , \tag{7.4b}$$

and so

$$\frac{1}{\pi}\int_{-\infty}^{-1} x \operatorname{Im} f(x - i\epsilon)\, dx - \frac{1}{\pi}\int_{1}^{\infty} x \operatorname{Im} f(x + i\epsilon)\, dx$$

$$= -\frac{1}{\pi}\int_{1}^{\infty} xq^{-3}\left[2 \sin^2 \delta_{33}(x) + \sin^2 \delta_{11}(x)\right] dx . \tag{7.4c}$$

The left hand side of the sum rule is negative definite and is large, owing to the contribution of the Δ. It has been conjectured[30] that this difficulty is related to the static approximation, and can be surmounted by using relativistic kinematics.

We would like to suggest that the contradiction results from the neglect of the t-channel singularities, can be resolved within the static model. In fact, given an amplitude satisfying (7.1) the relations (7.2) are just mathematical identities. We shall investigate the contributions of various singularities to the sum rules for our best fit solution, given by Eqs. (3.3) and (3.5) with $\beta \gg \alpha = 3.8$. For this amplitude, Eqs. (7.2a) and (7.2b) give

$$- 0.2353 + 0.0922 + 0.1554 - 0.0122 = 0, \text{ and} \tag{7.5a}$$

$$0 + 0.522 - 0.452 - 0.071 = 0 , \tag{7.5b}$$

respectively. The contributions are listed in the following order: nucleon pole, near poles (at $\pm i\alpha$), left hand cut, and right hand cut. The far poles at $\pm i\beta$ give a negligible contribution to these two sum rules, but do contribute to (7.2c). Thus the third sum rule depends on distant singularities; this is also seen in the extremely slow convergence of the integrals (the integrands go like $x^{-1}(\log x)^{-2}$ for large x). For this reason we shall only consider the first two sum rules. Note that the contribution of the near poles to these sum rules is by no means negligible relative to the cut contributions.

The left hand cut contribution to (7.5a) and (7.5b) is the same as would be contributed by a zero-width resonance in the p_{33} amplitude with position and width

$$\omega_{33}' = 2.91 , \quad \gamma_{33}' = 0.0777 . \tag{7.6}$$

These values are to be compared with those given by Eqs. (2.20) and (3.8),

$$\omega_{33} = 2.17 , \quad \gamma_{33} = 0.136 ,$$

which would contribute 0.272 and 0.59 to the first and second sum rules, respectively, in the zero-width approximation. These comparisons cast some doubt on the zero-width approximation, which is generally used in sum rule calculations.

The right hand cut contribution to (7.5a) and (7.5b) is equivalent to a zero width resonance in the p_{11} channel with

$$\omega'_{11} = 5.8 \quad \text{and} \quad \gamma'_{11} = 0.0122 , \tag{7.7}$$

but we know from the exact solution that δ_{11} is decreasing and approximately equal to -45° at ω'_{11}. Although no one would say that the sum rules predict a resonance with parameters given by (7.7), Eq. (7.6) could be used to determine the position and width of the Δ. Such a use of partial wave superconvergence relations has been advocated as a dynamical method;[30] our model shows that this procedure is inaccurate in the static limit. Using relativistic kinematics will probably not improve matters, because the sum rules tend to emphasize distant portions of the left hand cuts, which are then not given directly in terms of physical amplitudes, and which are not likely to be well approximated by the first order Born approximation.

VIII. CONCLUSIONS

We have constructed a model for pion-nucleon scattering that can represent the nucleon pole and Δ resonance rather well using quite reasonable t-channel forces. The behavior of our p_{11} amplitude in the physical region, though, does not agree with experiment. In particular, we cannot predict the zero of this amplitude near the $N\pi\pi$ threshold, a common failure of one-channel calculations, as was emphasized by Schwarz. This defect could be remedied in one of two ways; we could put either a CDD pole into our D function or an extra pair of poles into the N function, which could then have a zero in the low energy region. If fact, if the extra pair of poles in the second alternative represented short range forces, being far from the physical region, the resulting solution would differ from one with a CDD pole only in its asymptotic behavior. This situation is similar to that discussed in Section VI.

Although we could fit the p_{11} phase shift better in our one-channel model, the result would be suspect because it would involve many free parameters. This is especially true since Ball, Shaw and Wong[31] have shown that a simple two-channel ($N\pi$ and $N\sigma$) model can fit the p_{11} data quite well with only four parameters. These authors conclude that f_{11} would have a resonance rather than a nucleon bound state if the coupling between the channels were turned off, and that therefore a one-channel calculation of the nucleon mass is unreliable. In our model the forces in the $N\pi$ channel are sufficient to give a bound nucleon, but the considerations of Section IV show that the output nucleon parameters are extremely sensitive to the exact value of these forces. Thus the question of whether the nucleon bound state is formed almost entirely in the $N\pi$ channel or whether other channels (e.g., $N\sigma$) are of roughly equal importance, cannot be settled at this time in a model-independent way.

The principal result of our work is to show that, in a soluble model, elastic unitarity in the s- and u-channels and the s-u crossing relation are such powerful constraints that they completely determine the amplitude for scattering in the s- and u-channels once the t-channel singularities are specified. The strong s- and u-channel forces are determined self-consistently, while the relatively weak t-channel forces turn out to be very important. We believe that if t-channel unitarity and full crossing symmetry could be satisfied, all the forces would be determined self-consistently, and one could no longer make any clear separation between input and output at all.

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APPENDIX

The theorem used in Section V is Symanzik's[25] adaptation of the Phragmén-Lindelöf Theorem.[32] We shall outline the proof, introducing the assumptions as they are needed.

(1) Let $h(z)$ be a function holomorphic in the upper half plane.

Now define $w(z) \equiv \exp\left[ih(z)\right]$, which will also be holomorphic in the upper half plane.

(2) Let $h(z)$ have no poles on the real axis other than simple poles with negative real residue and let $\operatorname{Im} h(z)$ approach non-negative values as the real axis is approached from above.

Then $\overline{\lim_{z \to \text{real}}} |w(z)| \leq 1$ everywhere on the real axis and the Phragmén-Lindelöf Theorem allows only two cases:

(a) $|w(z)| \leq 1$ everywhere in the upper half plane; i.e., $\operatorname{Im} h(z) \geq 0$ for $\operatorname{Im} z \geq 0$, or

(b) $\overline{\lim_{r \to \infty}} \log M(r)/r > 0$ where $M(r) \equiv \max_{|z| = r} |w(z)|$; i.e., $\operatorname{Im} h(z) \to -\infty$ as $|z| \to \infty$ in some direction in the upper half plane.

Now we make the following two assumptions:

(3) $h(z)$ has no zeros in the upper half plane,

(4) $h(z)$ has no zeros on the real axis except simple zeros at which $h'(x)$ is real and positive.

Then $-h^{-1}(z)$ satisfies assumptions 1) and 2) and so the Phragmén-Lindelöf Theorem gives the result; either

(a') $\text{Im}\left[-h^{-1}(z)\right] \geqslant 0$ for $\text{Im}\, z \geqslant 0$, or

(b') $\text{Im}\left[-h^{-1}(z)\right] \rightarrow -\infty$ as $|z| \rightarrow \infty$ in some direction in the upper half plane.

Now we only need one more assumption,

(5) $h(z)$ does not have an essential singularity at infinity, to show that the only consistent result is that (a) and (a') hold. Thus the five assumptions are sufficient to prove that $h(z)$ is a Herglotz function.

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FIGURE CAPTIONS

Fig. 1. Analytic structure of the partial wave amplitudes in the static limit, showing the various cuts and the nucleon pole at $\omega = 0$.

Fig. 2. The inverse of the nucleon pole residue plotted against the inverse of the Δ resonance width; the values obtainable in our model fall between the two curves. Our best fit to the experimental values, which are given by the circle with accompanying error bars, is shown by the small square.

Fig. 3. The phase shift δ_{33} as a function of pion lab kinetic energy, T, according to Roper et al. (———) and our model (- - - - -).

Fig. 4. The phase shift δ_{11} as a function of pion lab kinetic energy, T, according to Roper et al. (———) and our model (- - - - -).

Fig. 5 Re $f_\kappa^{-1}(\omega)$ from Eq. (4.5) plotted as a function of ω for $\kappa = 0$, 1/9, and - 1/11 .

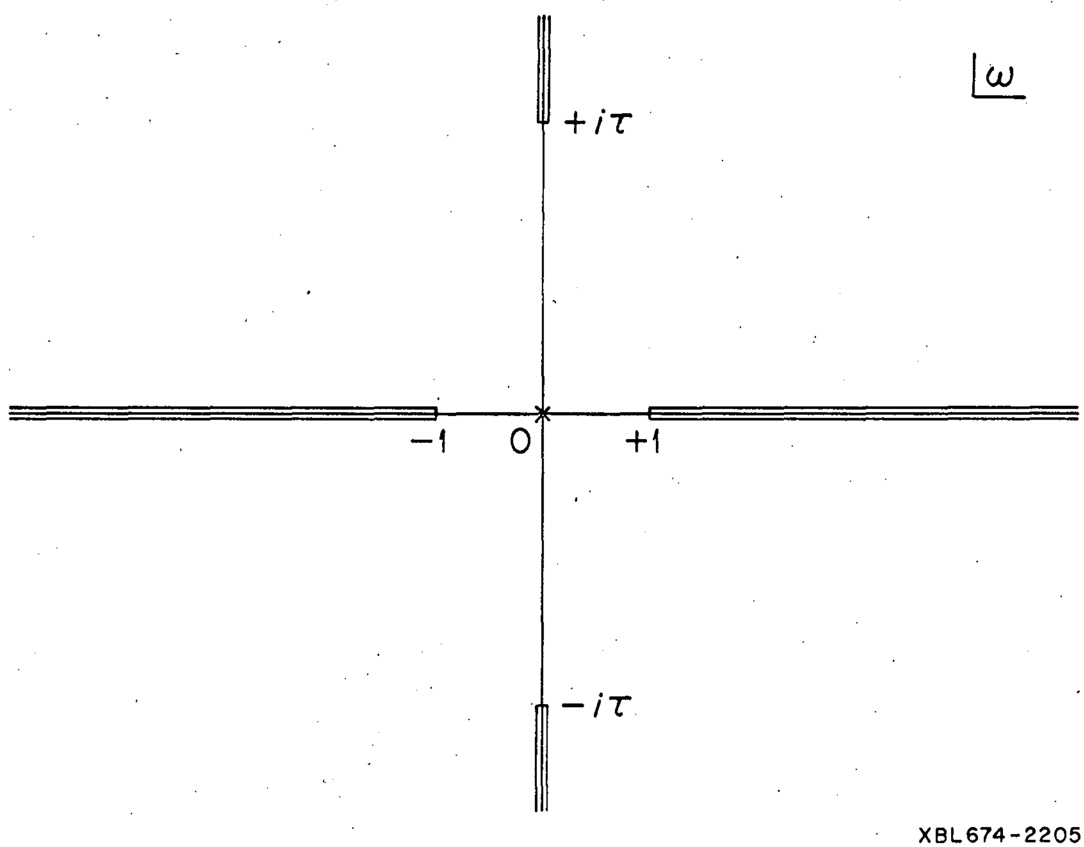

Fig. 1.

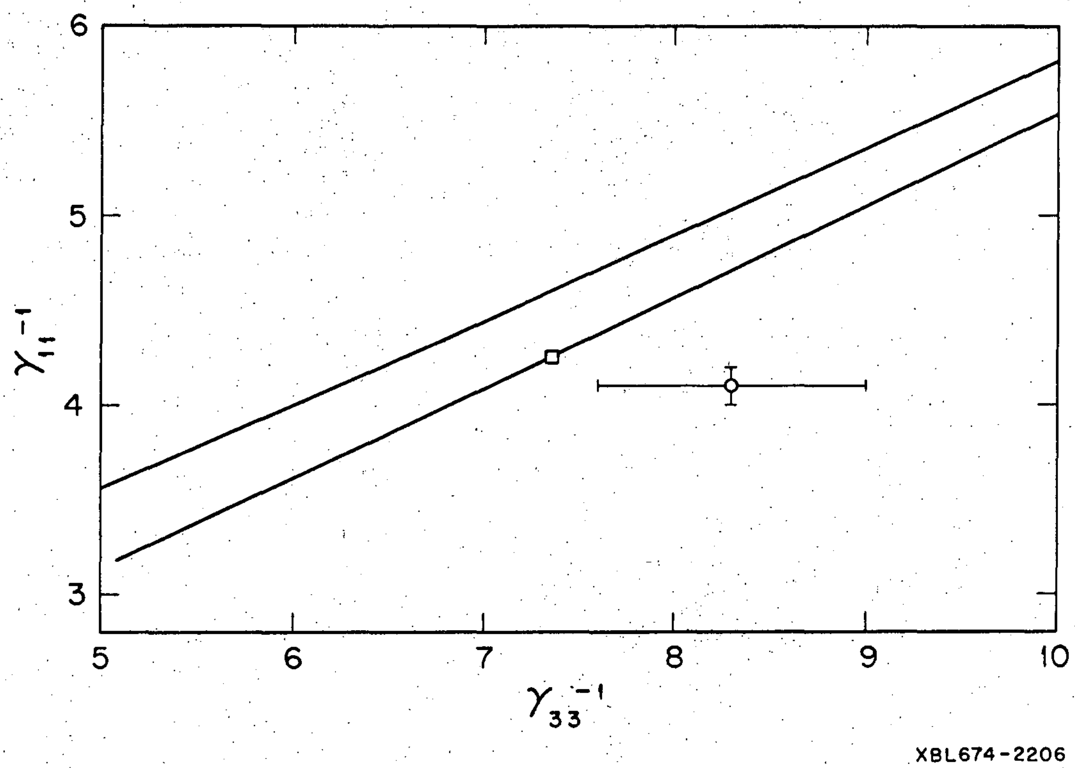

Fig. 2.

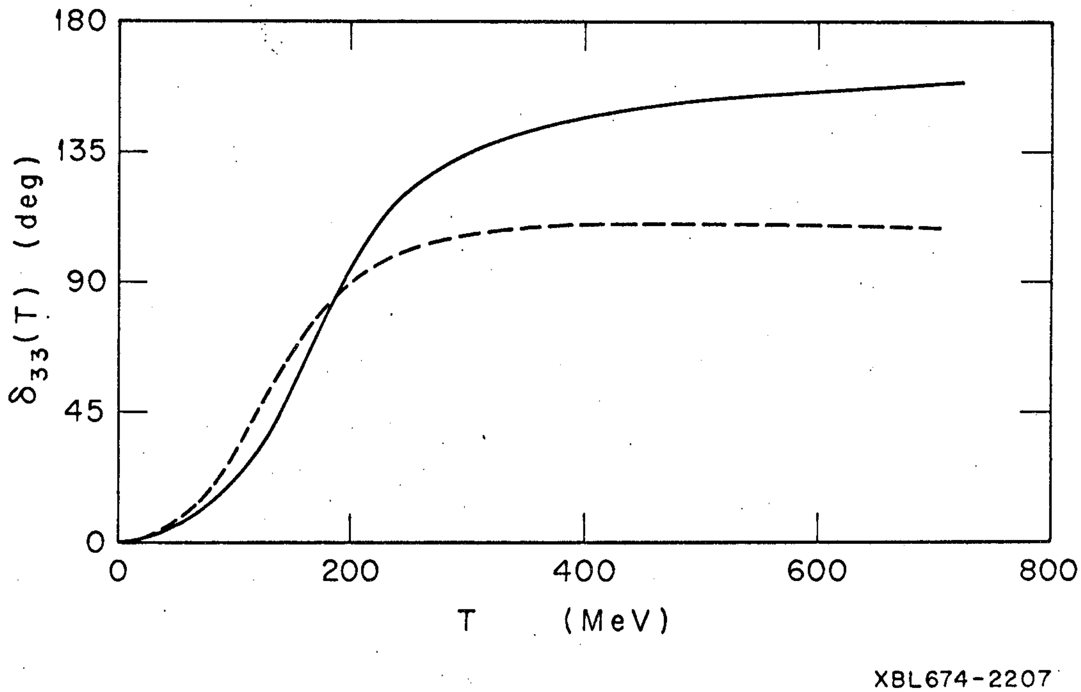

Fig. 3.

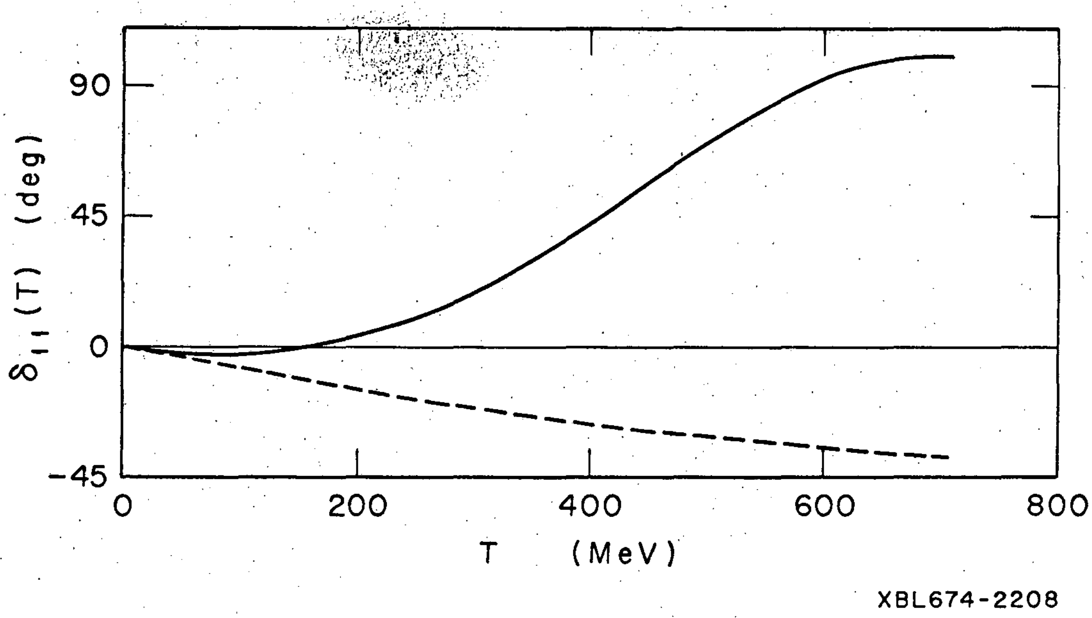

Fig. 4.

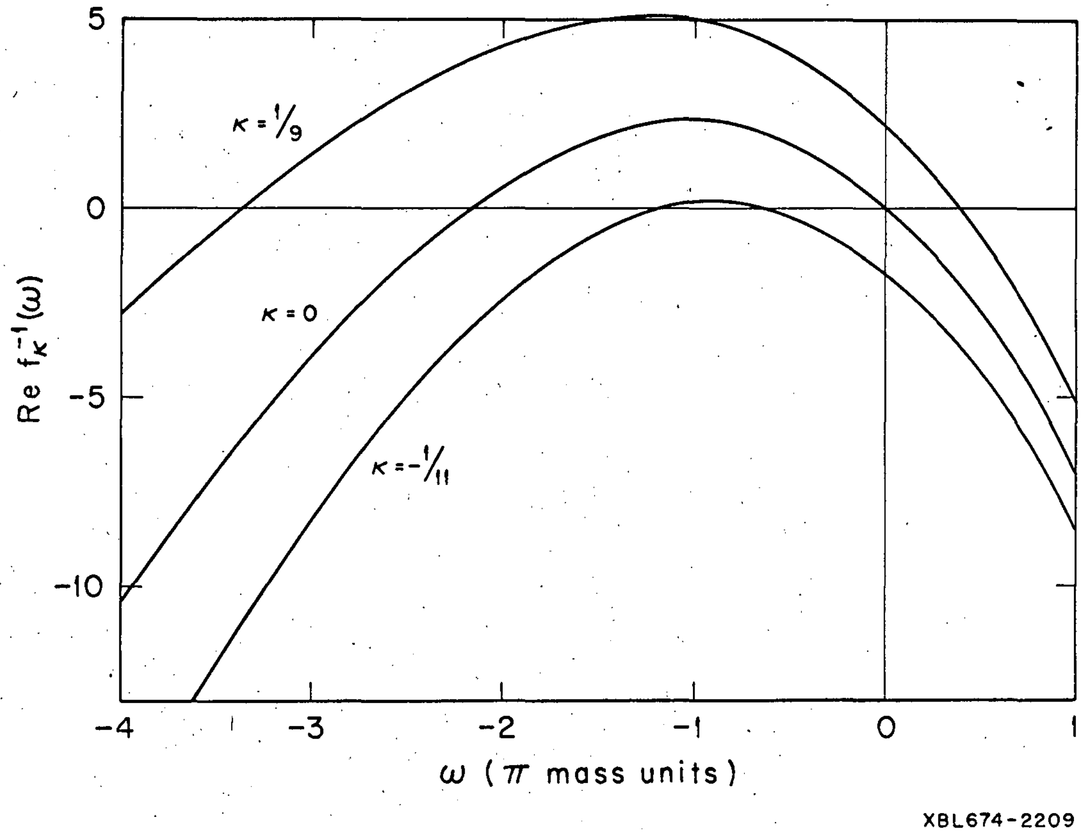
5
0
-5
-10
Re $f_\kappa^{-1}(\omega)$
$\kappa = 1/9$
$\kappa = 0$
$\kappa = -1/11$
-4
-3
-2
-1
0
1
ω (π mass units)
XBL674-2209

Fig. 5.