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ABSTRACT

The role of analyticity in established theories of macroscopic and atomic phenomena is briefly reviewed, with emphasis on the difficulty of using analyticity here as a starting principle. Contrast is made with nuclear phenomena where experiments suggest the absence of arbitrary parameters and where analyticity of the S matrix seems attractive as an *a priori* theoretical postulate. The notions of maximal analyticity of first and second degree are discussed.

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I. INTRODUCTION

It is a commonplace observation that in one form or another almost all physical theories have involved analytic functions. This circumstance is so widespread that physicists tend to forget the exceptional status of analytic functions in mathematics. Fermi used to say: "When in doubt expand in a power series." This statement reflects the belief, shared by most of us I'm sure, that natural laws are likely to depend analytically on any physical parameter which can be continuously varied.

Why have we developed such a prejudice? I have asked many colleagues this question, and the only consistent answer I get is that all successful theories of the past have contained this feature. Some degree of continuity in the dependence on physically continuous parameters is required by *a priori* common-sense considerations, but the extraordinary smoothness of analytic functions is not demanded by sheer logic. Neither, of course, is the related fact that past theories have been expressible through differential equations. An intuitively plausible theorem of Poincaré provides a formal connection between these two questions. Stated roughly: If the coefficients of a differential equation depend analytically on some quantity, then the solution of the equation will be analytic. Poincaré tells us, in other words, that theories based on differential equations preserve and propagate any analyticity we insert into the coefficients.

Why, on the other hand, should the coefficients be analytic? In more or less complete dynamical theories such as that of classical electrodynamics they often turn out to be so, and one might argue that it would be uneconomical for nature to have arranged for the propagation of a property that doesn't exist. In less specific general frameworks, however, such as that of nonrelativistic quantum mechanics, it is perfectly possible to consider coefficients that are not analytic. At the 1964 Washington meeting of the American Physical Society, Wigner told an interesting story. He said that shortly after the invention of quantum mechanics he asked Von Neumann if it were not strange that the formalism failed to require analyticity. Von Neumann replied that analytic functions constituted such a restricted and special class of functions that there was no reason why they should be the only ones admitted into physics. Wigner concluded this story with the remark that, Von Neumann notwithstanding, the subsequent 35 years have shown that in a deep sense physics _is_ based on analytic functions.

It is pointless to seek a logical origin for this circumstance. Physical theory cannot be based on logic; it is always a matter of guesswork based on observation of nature. One cannot, for example, argue that it is logical for classical mechanics to be expressible through second-order differential equations. This simply is the scheme that works. With regard to analyticity there is another question which is to the point: Does analyticity in itself constitute a suitable _a priori_ principle for the formulation of theories?

In the past the answer has been negative, perhaps because of the importance of variable parameters. All theories have tolerated a variety of solutions corresponding to different choices of certain constants. Classical Maxwell theory, for example, permits particles of any charge and mass. A central issue has been to state this arbitrariness in a precise fashion, and the framework of differential or integral equations has proved appropriate and economical. Conceivably one might formulate these same theories in a manner that emphasizes analyticity at the beginning, but in so doing the characterization of the arbitrariness in the theory is likely to become awkward and unesthetic. For example, the rule for the nature and the location of singularities that are arbitrary, as opposed to those that are determined, may not appear particularly simple and beautiful.

During the past ten years, nevertheless, a feeling among many theoretical physicists has been growing that the description of natural phenomena on the subatomic level may be facilitated if analyticity is employed as a primary rather than a derived concept. It has not yet proved possible in this area to formulate specific and experimentally successful differential or integral equations, while at the same time strong experimental evidence in favor of analyticity has developed. Furthermore there is a growing suspicion that a successful theory of strong interactions will not tolerate arbitrary parameters; perhaps no arbitrariness of any kind is possible, all nuclear particles enjoying an equivalent status. The concept of global analyticity may furnish the appropriate framework for a theory according to which no singularities are arbitrary but all are determined by general principles.

II. ANALYTICITY AND FIELD THEORY

Historically the recognition of analyticity in the theory of subatomic particles developed as much from studies of field theory as from experimental observations.[1] One often hears it said, in fact, that dispersion relations for scattering amplitudes were "derived" from field theory.[2] There is truth in this statement but it can be misleading. It is conceivable that scattering amplitudes for massive nuclear particles should have the analyticity properties commonly ascribed to them, while at the same time the association of fields with these particles is meaningless. In other words one does not know how to construct fields purely in terms of analytic scattering amplitudes. The field concept involves a larger set of functions than is achieved by the analytic continuation of the scattering matrix,[3] and the existence of this larger domain does not necessarily follow from that of the smaller. The point I am trying to make is that, if in the future it should develop that a field theory of nuclear particles is impossible, the conjectured analyticity properties of the nuclear scattering matrix still may survive. There is ample precedent in the history of physics for special aspects of an unsuccessful theory to appear again in a quite different theory that does succeed.

To avoid misunderstanding, let me emphasize that the foregoing remarks do not apply to the form factors often used in the analysis of weak or electromagnetic interactions. The existence of such form factors is closely related to the existence of the corresponding electromagnetic and leptonic fields. Form factors are not scattering amplitudes, however, the distinction being important for the present discussion. There is no experimental evidence to support the existence of form factors for strong interactions.

III. MAXIMAL ANALYTICITY OF THE FIRST DEGREE

What are the analytic properties of the S matrix that have been suggested by theories based on differential equations? Many of these properties were noticed during the thirties and forties in nonrelativistic quantum theory, but they received general acceptance only during the last decade when, against the background of field theory, they were seen not to conflict with relativistic requirements. Ironically these properties have been formulated only in the absence of zero-mass particles or when the interaction with zero-mass particles is treated as a small perturbation. The scattering matrix has not yet even been defined in the general case, where the infrared problem must be faced. I say "ironically" because the best established dynamical theories are those associated with zero-mass particles--the photon and the graviton. In spite of this unsatisfactory situation, it is worth considering here the standard analyticity properties of the S matrix in the absence of infrared complications. The latter are tied up with the theory of measurement, with which no one seems prepared as yet to deal.

Field theory suggests that all the singularities of the scattering matrix are determined by a knowledge of the particles that exist. The primary singularities are poles--associated with individual particles--and branch points--associated with the thresholds of channels containing two or more particles. A pole position in the energy complex plane is precisely at the mass of the corresponding particle, while a branch point occurs at the sum of masses of the particles making up the channel in question. Both stable and unstable particles are conjectured to follow this rule, unstable particles being assigned complex masses. The nature of a particular threshold branch point is made precise by a formula giving the change in a scattering amplitude when a single circuit is made around the branch point. This discontinuity formula may be described as an analytic continuation of the unitarity condition.

It is conjectured that <u>all</u> other singularities of the scattering matrix follow from the discontinuity formulas, once the particle poles and threshold branch points are given. This conjecture I shall refer to as maximal analyticity of the first degree. It represents the combined observations of so many theorists over so many years that it is impossible to present a fair list of credits. I learned only recently, for example, that it was Kramers who first became aware of the particle-pole association. Certainly it was his work and that of Kronig that paved the way for the analytic continuation of the S matrix.[4] Threshold branch points were recognized as such early in nuclear physics by Wigner.[5]

It is easier to pinpoint certain of the dynamical deductions made within the last decade on the basis of S-matrix analyticity. Gell-Mann and Goldberger noticed that continuation from positive to negative energies (often called "crossing") corresponds to the existence of antiparticles,[6] and Mandelstam showed how the Yukawa force is a consequence of crossing when one considers simultaneously singularities in angle and energy.[7] Low and I showed that the discontinuity formula has dynamical content similar to that of an equation of motion, being capable of generating poles that correspond to composite particles.[8] I shall come back to dynamical questions later; for the moment, let me recall some of the experimental verifications of first-degree analyticity.

To begin with, there is the widespread success of the Breit-Wigner formula approximating a scattering amplitude by one or a few unstable particle poles near the physical region. Next, the detailed nature of branch points associated with two-particle thresholds has been checked with high accuracy through the so-called effective range formulas. Both of these verifications imply that there are no strong singularities near the physical region that are not envisaged by maximal analyticity. This latter inference receives further support from the rate of convergence in partial-wave representations of the angular dependence of experimental amplitudes. Mathematically a Legendre polynomial expansion converges at a rate determined by the nearest singularity. The experimentally observed convergence rates are always in qualitative agreement with the nearest singularity predicted by maximal analyticity. In a few cases the agreement has been shown to be quantitative.

Perhaps the most impressive verification of first-degree analyticity has been through the dispersion relations for forward pion-nucleon scattering, written down by Goldberger.[9] By a fortunate chance the total discontinuity across all predicted cuts in this case can be experimentally determined and only a single pole appears. Measurement of the scattering amplitude then not only verifies the absence of additional singularities on the so-called physical sheet but leads to an accurate determination of the pole residue. This same residue appears with poles in nucleon-nucleon scattering--where an independent experimental determination leads to the same value.

The combined weight of nuclear reaction measurements over a 30-year span leaves little doubt that the singularities in the neighborhood of the physical region correspond to maximal analyticity of the first degree. Furthermore if singularities appear at distant points in the complex plane in conflict with this conjecture, experiments show them to be relatively weak in their influence on the physical region.

IV. SECOND-DEGREE ANALYTICITY

At this point it must be stated that even with maximal analyticity the discontinuity formulas predict an enormously complicated collection of singularities in complex regions distant from the physical region. The elucidation of this structure is one of the key problems of strong-interaction physics; it has not yet been shown even in principle that the predicted singularity distribution is unique and self-consistent. The following phenomenon, in particular, causes concern: Take an elastic scattering amplitude and assume a single pole in momentum transfer, corresponding to the existence of a particle with appropriate quantum numbers. Then investigate what other singularities result as a consequence of the formula for the discontinuity around the elastic threshold. An infinite sequence of branch points in momentum transfer develops without inconsistency, but the sum over this infinite set turns out to generate poles in energy, perhaps an infinite number of poles. If the residue of the original pole in momentum transfer is not enormous, most of the new poles in energy will be far from the physical region, but with a positive starting residue of moderate strength a few energy poles easily can come close enough so as to be identifiable as particles. It is natural to refer to such particles as composites of the original two-body system. At this point one evidently must begin again, studying the singularities generated by the composite particles. If the total number of poles somehow could be stabilized, it could be made plausible through Feynman graphs that the other singularities form a self-consistent set, but there is no known way to control the number of poles. They multiply like rabbits.

An important experimental comment should be made at this point. If one examines the known poles of the strong-interaction S matrix--that is, the nuclear particles that have been discovered--it turns out that any one of them might have its origin in the mechanism just described. It is found, in other words, that whenever an energy pole appears there is a pole (or poles) in momentum transfer of a magnitude and sign such that it (they) might have generated the energy pole. One can easily imagine combinations of poles that would not fit this pattern (leptons, in fact, do not), but the observed nuclear family appears to do so. In other words, all the strongly interacting particles seem to be composite structures. Using Gell-Mann's phrase, we are dealing with what appears to be a nuclear democracy.

At this point the conjecture of second-degree analyticity begins to take shape. Roughly speaking it is that there are no "unnecessary" poles; all are consequences of the mechanism just described and cannot be eliminated. This conjecture is not motivated by field theory--only by experiment. It is supported in a negative sense by the failure of theorists to have found consistent field equations that would give one or a few particle poles an aristocratic (elementary) status. This failure, by the way, encourages the suspicion that the assumption of second-degree analyticity is redundant. That is to say, the combination of first-degree analyticity and unitarity may be impossible except for a democratic collection of poles.

A more precise form of the second-degree analyticity conjecture employs the notion of analytic continuation in angular momentum. Theoretical analysis shows that all composite-particle poles generated by the mechanism described above group themselves into families, the members of each being connected by an analytic interpolation in angular momentum of the type noticed by Regge in nonrelativistic quantum theory.[10] Furthermore, studies of perturbation expansions of field theory make it appear likely that the introduction of arbitrary elementary particles necessarily leads to poles in the S matrix that are not of the Regge type.[11] One is thus led to the following reformulation of second-degree analyticity: All poles are Regge poles.[12]

V. THE BOOTSTRAP CONJECTURE

Suppose that, excluding the photon and the leptons, all poles in fact are of the Regge type, corresponding to composite particles. What free conditions (i.e., variable parameters) remain in the theory? It is hard to find any chink into which parameters can be inserted if we forego our traditional prerogative of arbitrarily assigning a few pole positions and residues. In fact, from a purely theoretical standpoint it looks unlikelv that any S matrix at all can be found that satisfies our requirements. One would probably dismiss the whole idea as ridiculous if we did not see experimental indications that somehow nature has been able to solve this puzzle. It is not now and never will be feasible experimentally to examine the entire strong-interaction S matrix, but as emphasized above, the individual local regions that have been measured tend to support the conjecture of full analyticity--first and second degree. The possibility must be faced, therefore, in the description of strong interactions that for the first time in the history of physics we shall be dealing with a dynamical theory in which there are no variable parameters. Perhaps the only unitary and fully analytic S matrix not equal to the unit matrix is the strong-interaction S matrix we observe in nature.

Let me summarize this discussion of strong interactions with a chart showing four possible theoretical situations currently receiving attention:

A. There are variable parameters conveniently expressed through field theory. These parameters are perhaps awkward to introduce through analyticity. First-degree analyticity is nonetheless present and useful. Continuation off the mass shell is meaningful.

1. A few non-Regge poles are present representing elementary particles.

2. All poles are of Regge type. (This possibility is unlikely if perturbation expansions are a reliable guide.)

B. There are no variable parameters. All poles are Regge poles.

1. Off-mass-shell continuation and the field concept are meaningful without equations of motion. The physical content of field theory and analyticity are equivalent.

2. Off-mass-shell continuation and fields are meaningless. Analyticity remains valid.

In all four situations analyticity plays a useful role, but only in B.2 does it have a crucial role. In B.1, currently favored by the most sophisticated theorists, analyticity may turn out to be more convenient than the field concept as a starting point.

VI. ELECTROMAGNETISM AND WEAK INTERACTIONS

Almost all theories of electromagnetism and weak interactions fall into the category A.1,where the role of analyticity is least interesting from a fundamental point of view. As a practical matter, of course, as already mentioned, it is useful to analytically continue the form factors that describe the coupling of nuclear particles to weak and electromagnetic currents. With conserved currents it turns out to be possible by this route to calculate the form factors from a knowledge of the strong-interaction S matrix. Methods based on analytic continuation also have been developed to calculate the electromagnetic mass shifts of nuclear particles. All calculations in this area are of a perturbation nature, however, and of a basically straightforward character. The underlying field theory is assumed to be secure in the weak coupling limit.

Many physicists are disturbed by the suggestion that strong-interaction theory should fall into the category B.1 or even B.2 while electromagnetism is solidly entrenched in A.1. Yukawa's original proposal about the nuclear force, after all, was based on a presumed analogy between strong and electromagnetic interactions.[13] A similar analogy may be said to exist with respect to gravitation, however, and few of us look to general relativity as a source of inspiration in constructing nuclear theory. After all, the most remarkable aspect of the history of physics is that approximate laws of good accuracy have been found for separate ranges of phenomena. At no time, in fact, have we ever been in possession of a comprehensive dynamical law, and there is little reason to believe that the immediate future of physics will differ from the past in this respect. Laws will tend to become more comprehensive, but at any given time progress requires the identification of those areas of nature that can meaningfully be approximated as separate. Strong interactions appear to constitute such a subdivision.

Since the word "philosophy" appears in the title of this conference I feel less embarrassment than usual in emphasizing a possible reason for the difference in status between electromagnetism and strong interactions. Concerning weak interactions I have nothing to suggest. The point about electromagnetism is that it constitutes the tool with which we perform the measurements on which physics is based. Perhaps gravitation can in principle serve the same purpose, but a long-range force is essential. Furthermore, as Bohr so often emphasized, the existence of classical systems that can observe each other depends on the smallness of the fine structure constant. Thus the zero-mass photon, with its weak coupling to matter, plays a role in physics that cannot be filled by any of the strongly interacting particles.

It is unfashionable at present to inject measurement considerations into physical theory because this subject has suffered such an extended period of sterility. It is my belief, however, that the photon properties are interlocked with the theory of measurement, perhaps even with the meaning of space and time, and will never be explained purely by dynamical considerations. In contrast the parameters of strong interactions, having no connection with the measurement process, have a chance of being determined by dynamics. In this dynamics analyticity will play a key role, perhaps the central role.

FOOTNOTES AND REFERENCES

† This work was performed under the auspices of the U. S. Atomic Energy Commission.

1. For a review of this history see the articles by M. L. Goldberger and S. Mandelstam in "La Theorie Quantique des Champs," in The Proceedings of the 12th Solvay Congress (Interscience Publishers, New York, 1961), p. 179 and 209.
2. The term "dispersion relations" refers to the Cauchy formulae expressing an analytic function, with pole and branch point singularities only, in terms of the residues of its poles and the discontinuities across its cuts.
3. Often one characterizes the difference by saying that scattering amplitudes are "on the mass shell," while the existence of fields requires a meaning for continuations to arbitrary complex masses.
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