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Undergraduate Students' Definitional Practices in Mathematics

By

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Abstract

Mathematical Definitional Practices of Undergraduate Students

by

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Mathematical definitions are central to learning and doing mathematics. Research has uncovered significant differences between how mathematicians and non-mathematicians construct, reason about, and refine mathematical definitions. Various strands of research provide insight into the development of definitional practices, yet an integrated approach to addressing foundational questions of development is lacking. Further, existing approaches often fail to account for the conventional nature of mathematical definitions and the social nature of defining practices. In the first paper of this dissertation, I present a sociocultural framework in which mathematical definitions are treated as cultural forms that emerge to serve specialized functions in collective practices. Building upon Saxe's (2012) treatment of the development of mathematical ideas, I describe the development of definitional practices as constituted through processes of micro-, onto-, and socio-genesis of definitional form-function relations.

In the second and third papers of this dissertation, I present two empirical studies that utilize the framework described in the first paper to examine similarities and differences in undergraduate mathematics and humanities students' approaches to defining mathematical terms related to rational number. Study 1 examines participants' conceptualizations of the function and forms of mathematical definitions. Study 2 examines participants' process of defining, with a focus on the formulation and iterative refinement of definitions for three foundational mathematical ideas – fractions, multiplication, and division. Results provide evidence of two distinct definitional practices. Mathematics students tended to define for communication and proof, as they constructed and frequently revised definitions aimed at setting a standard for how a word form could be used across a disciplinary community. Humanities students tended to define for learners, constructing definitions aimed at advancing understanding and providing procedural instructions, often defining by example. Findings imply that while college mathematics majors are being enculturated into mathematical practices of defining for a disciplinary audience, work needs to be done to provide pathways for more students to engage in mathematically authentic practices of defining.

Dedication

To Sheedvash and Bijan, with eternal gratitude.

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Acknowledgements

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1 Introduction

Mathematical definitions are central to learning and doing mathematics (Borasi, 1992; Lakatos, 1976; Poincaré, 2001). Standards such as the Common Core (Common Core State Standards Initiative, 2010) require that students "understand and use stated…definitions…in constructing arguments" and be able to "make explicit use of definitions." In recent years, scholars in mathematics education have conducted research that sheds light on students', teachers', and mathematicians' definitional activity. This body of scholarship examines how individuals and communities reason about mathematical definitions (Van Dormolen & Zaslavsky, 2003; Winicki-Landman & Leikin 2000A, 2000B; Zaslavsky & Shir, 2005), use mathematical definitions in proof, problem solving, classification, and concept formation (Edwards & Ward, 2004; Morgan, 2005; Wawro et al, 2011), and construct mathematical definitions (Dahl, 2017; Herbst et al, 2005; Kobiela & Lehrer, 2015; Martín-Molina et al, 2018; Zandieh & Rasmussen, 2010). This work indicates that disciplinary forms of defining differ significantly from students' naive everyday ways of defining. This dissertation contributes to this body of scholarship by examining differences between the definitional activity of undergraduate mathematics and humanities students.

In the first paper of my dissertation, I present a sociocultural framework within which mathematical definitions are viewed as cultural forms that emerge to serve various functions in collective practices as a useful tool for understanding the development of definitional practices. While research shows that disciplinary forms of defining differ from the definitional activity of students at various levels, there is little consensus in mathematics education research on how to characterize the development of definitional practices. The framework I present, building on Saxe's cultural-developmental treatment of form function relations (2012), aims to address key questions related to the development of mathematical definitional practices – What develops? And how does it develop? – in a way that integrates disparate lines of scholarship on mathematical definitions and defining. I describe the *what* of development as definitional forms and the functions they serve and the *how* as processes of micro-, onto-, and socio-genesis of definitional form-function relations.

The second and third papers of this dissertation draw on the theoretical framework presented in the first paper to compare the definitional activity of undergraduate mathematics and humanities students, two populations which are similar in many respects, but which differ in their level of enculturation into disciplinary mathematical practices.

The second paper investigates students' conceptualizations of both the functions served by mathematical definitions and the form that mathematical definitions should take. First, I show that while mathematics students tended to conceptualize mathematical definitions as serving to facilitate proof and communication across a community, humanities students tended to focus on the role of definitions in finding answers to routine exercises. Both participant cohorts identified the role definitions play in helping learners to understand new mathematical ideas. Second, I demonstrate differences between how participant cohorts view the form of mathematical definitions. While mathematics students preferred definitions that were precise, unambiguous, and comprehensively specified a set of instances, humanities students pointed to the need for definitions to outline the procedures for solving a problem.

Finally, the third paper of this dissertation examines students' definitional activity as they participate in an activity designed to provide opportunities for engagement in the formulation and iterative refinement of definitions in light of problematizing cases. This work provides insight into how the definitional thinking uncovered in the second paper is enacted in activity. Analyses showed, on the one hand, differences in the form of mathematical definitions that students generated. While humanities students tended to include illustrative examples within the body of their definitions, mathematics students were more likely to use specialized mathematical language in their definitions. Analyses also showed differences in participants' refinement of definitions over time. Mathematics students were more likely than humanities students both to refine their definitions to account for problematizing cases and to spontaneously propose their own problematizing cases in light of which to reconsider their definitions.

Findings from both empirical papers provide evidence of two distinct definitional practices. Mathematics students tended to define and redefine for communication and proof across a disciplinary community. Humanities students tended to define for learners, providing procedural instructions and defining by example. Findings imply that while mathematics majors are being enculturated into practices of defining for a disciplinary audience, work needs to be done to support more students to engage in mathematical definitional practices.

2 The Development of Mathematical Definitional Practices: Towards a sociocultural framework

Mathematical definitions are central to learning and doing mathematics (Borasi, 1992; Lakatos, 1976; Poincaré, 2001). Standards such as the Common Core (Common Core State Standards Initiative, 2010) require that students "understand and use stated…definitions…in constructing arguments" and be able to "make explicit use of definitions." In recent years, scholars in mathematics education have conducted research that sheds light on students', teachers', and mathematicians' definitional activity. This body of scholarship examines how individuals and communities reason about mathematical definitions (Van Dormolen & Zaslavsky, 2003; Winicki-Landman & Leikin 2000A, 2000B; Zaslavsky & Shir, 2005), use mathematical definitions in proof, problem solving, classification, and concept formation (Edwards & Ward, 2004; Morgan, 2005; Wawro et al, 2011), and construct mathematical definitions (Dahl, 2017; Herbst et al, 2005; Kobiela & Lehrer, 2015; Martín-Molina et al, 2018; Zandieh & Rasmussen, 2010). This work indicates that disciplinary forms of defining differ significantly from students' naive everyday ways of defining. Yet there is little consensus in mathematics education research on how to understand the development of definitional activity.

In this chapter, building on Saxe's cultural-developmental treatment of form function relations (2012), I propose a sociocultural framework within which mathematical definitions are viewed as cultural forms that emerge to serve various functions in collective practices as a useful tool for understanding the development of definitional practices. The framework aims to provide insight into key questions related to the development of mathematical definitional practices – What develops? And how does it develop? – in a way that integrates somewhat disparate lines of scholarship on mathematical definitions and defining.

2.1 Introducing the Framework

I introduce the proposed framework with an example of a conversation between three 10th grade students. The conversation takes place mid-way through an hour–long semi-structured task-based interview in which Tommy, Eli, and Rex engaged in an activity designed to support their co-construction of and reasoning about mathematical definitions. At this point in the interview, Eli and Rex have jointly constructed a definition for rectangle. They have been asked to compare their definition, which specifies four right angles (Def. 1 below), to another definition specifying three right angles (Def. 2 below), when Tommy joins the conversation.

DEF 1. A rectangle is a quadrilateral with four right angles.

DEF 2. A rectangle is a quadrilateral with three right angles.

1 *Tommy*: The, the thing is, I really think you have to say four right angles.
2 *Rex*: Why?
3 *Tommy*: Because you have to prove everything, right? It's very important to be specific.
4 *Rex*: But can you make one with only three?
5 *Eli*: No…it has to have *at least* three right angles.
6 *Tommy:* No, yeah, and you have to say *at least.*
7 *Eli:* I think it *is* better to have, to say four, but for this it would be more fun to, like, mess with people.
8 …

9 *Rex:* Yeah, but you can't make it with only three.
10 *Tommy:* Exactly! So there's no point in saying three.
11 *Rex*: So why would… Then there's no point in saying four.
12 *Tommy*: Ok, I feel like this is like…going in circles.
13 *Rex*: Yeah, because they're both equal.

In this episode, students grapple with the unique nature of mathematical definitions – as worked up and highly specialized conventions tailored to serve both mathematical and communicative functions. Each of the three interlocutors identifies a different function to be served by the definition – proof (Tommy, line 3), construction (Rex, line 4), and "messing with people" (Eli, line 7). Their preferences regarding definitional features accordingly differ, depending on the function that the definition is taken up to serve. Tommy argues for specificity of the four-right-angles definition (line 3), Rex argues for the minimality of the three-right-angles definition (line 11), and Eli identifies affordances of both definitions (line 7). When the context-dependent and conventional nature of definitions goes unacknowledged, the conversation "goes in circles" (Tommy, line 12).

Ultimately, the group acknowledges the fact that the definitions are "equal" (Rex, line 13). Indeed, the two definitions are equivalent in the sense that they specify the same set of mathematical objects. A quadrilateral with three right angles must, necessarily, have a fourth right angle. However, the definitions are not equivalent in terms of the functions they afford. Definition 1 might best serve a descriptive function – it describes the rectangle in a way that can be easily visualized. Definition 2, on the other hand, might best facilitate proof, specifying minimal criteria that must be satisfied to show, say, that an object is a rectangle. The friends are trying to find the "best" definition, when in fact, what is required is communal agreement on the adoption of a reasonable convention, on the one hand, and of the functions that the definition is to serve, on the other.

Seen through this lens, the development of definitional practices entails shifts in the form and function of mathematical definitions. This development is constituted through microgenetic, ontogenetic, and sociogenetic processes (Saxe, 2012). The microgenesis of definitional form-function relations takes place, for example, in Tommy's moment-to-moment activity as he structures the four-right-angles definition for rectangle to serve functions of proof. Ontogenetic processes take place as Eli's thinking shifts, for example, from thinking about definitions as serving to "mess with people" to, perhaps years hence, seeing them as valuable tools to serve problem solving and communication. Ontogenetic shifts further take place as Eli's geometry content knowledge expands and deepens. The sociogenesis of definitional form-function relations may take place over the course of a conversation such as the one described above, as interlocutors take up and revise definitional forms to serve different functions, or over the course of centuries, as in the case of the definition for dimension, which was revised repeatedly by the mathematical community to account for later insights (I further describe this historical episode later in this chapter).

In this chapter, I review the broad landscape of literature on mathematical definitions and defining and consider how this body of work bears on questions of the development of definitional practices in an integrated way.

2.2 Motivating the Framework

There is a broad landscape of literature on the mathematical definitional activity of both mathematicians and non-mathematicians. This work provides insight into how individuals formulate, reason about, negotiate, revise, and use mathematical definitions both in the classroom and in disciplinary practice. While this body of work bears on the central questions of development – what develops and how, in definitional practices? – it does not do so in an integrated way. Further, none of the current approaches take into account three key aspects of definitional practices – aspects which I outline below and which motivate the adoption of the form-function framework I propose here.

First, mathematical definitions are highly worked up and refined social conventions. By virtue of their conventionality, their ontological status is of a different order than other mathematical constructs such as lemmas, theorems, or corollaries. While, for example, the Pythagorean theorem, relating the side lengths of a right triangle, can be proven (and, in fact, necessitates proof); definitions, such as "a *right triangle* is a triangle that has one ninety degree angle," are consciously constructed meaning-relations that are 'taken-as-shared' in communities of interlocutors (Yackel & Cobb, 1996). Yet students in primary and secondary classrooms often see mathematical definitions as true and absolute, static and context-independent statements (Edwards & Ward, 2004; Foster & de Villiers, 2016; Herbst et al., 2005). Confusion among students often arises, as in the case of Tommy, Eli, and Rex, when definitions are taken as theorems that can be proven or disproven, when what is required is the adoption of a convention – an agreement about what a mathematical term or notation will mean (Rabin et al., 2013).

Further, mathematical definitions have no inherent or fixed communicative or problem solving functions. They take on different forms to serve specialized functions, as illustrated in the opening episode of this chapter. Indeed, mathematicians often select one of a number of equivalent definitions to serve different functions (Ouvrier-Buffet, 2006). They are often explicit about the fact that they are making a conscious choice of definition (Morgan, 2005). Morgan (2005, p. 108) illustrates this with an excerpt from a mathematics research paper ("we give a somewhat non-standard definition of the Hecke algebra…"), elaborating that "the modification *somewhat non standard* implies that definitions are not unique but at the same time that there exist privileged definitions that are generally acknowledged/valued by the community" (Morgan, 2005, p. 109). Borasi (1992) provides the example of alternative definitions for the circle. A metric definition defines the circle as the set of points equidistant from a center, while an analytic definition defines the circle in terms of an equation ($(x-a)^2 + (y-b)^2 = r^2$). While the two definitions are "equivalent" in the sense that they specify the same set of mathematical objects, each definition is better used to serve different functions.

Finally, much of mathematical defining – processes of formulating, reasoning about, and negotiating definitions – takes place within collective practices, recurring activities governed by social norms, involving the engagement of multiple participants (Saxe, 2012). Defining takes place as community members negotiate the meaning of terms in order to develop taken-as-shared (Cobb & Yackel, 1996) definitions. The social nature of the defining process has been described by Lakatos (1976) in his seminal account of the historical development of the definition of polyhedra. Lakatos illustrates the way in which a community jointly constructs, negotiates, and reconstructs definitions. In one excerpt (Lakatos, 1976, p. 15), a student proposes a definition, "A polyhedron is a surface consisting of a system of polygons." Another student describes an instance that satisfies that definition but should not be called a polyhedron, proposing "Take two tetrahedral which have an edge in common. Or, take two tetrahedral which have a vertex in common…" The original student adjusts his definition, stating " I admire your perverted imagination, but of course I did not mean that any system of polygons is a polyhedron. By polyhedron I meant…", and so on. This results in a collective iterative process of refining and testing definitions. As interlocutors negotiate meanings for terms to arrive at shared definitions, they engage in collective practices of defining.

Thus, mathematical definitions are conventions that are used as instruments of communication and problem solving within the collective practices of a community. This motivates a framework within which definitions are cultural forms that that serve mathematical and communicative functions within collective practices of defining.

2.3 The Development of Definitional Practices

In this section, I elaborate a framework that provides an integrative approach to understanding the development of definitional practices, with a focus on questions of what develops and how.

The proposed framework extends a body of research that investigates individuals' use of mathematical representational forms to serve reasoning and communicative functions as they participate in the collective practices of a community, whether it be in school (Saxe & Farid, 2021; Saxe, de Kirby, Kang, et al., 2015; Saxe & Sussman, 2019) or out-of-school (Saxe, 1991; Saxe & Gearhart, 1990; Saxe, 2012). This body of work describes the ways in which, as individuals engage in collective practices, they make use of cultural forms, such as a counting system, to serve various cognitive functions, such as quantifying monetary value. Both the forms themselves and the functions they serve shift over time. Similarly, as doers of mathematics engage in problem solving, the definitions they use serve various cognitive functions – as axioms for proof, as criteria for classification, as support for concept formation, and the like. The development of definitional practices involves shifts in the form and function of mathematical definitions. This development is constituted by micro-, onto-, and socio-genetic processes.

In this section, I draw on various strands of research on mathematical definitions and defining, each of which emerged somewhat separately, to provide an integrated treatment within which I regard the *what* of development as definitional forms and the functions they serve and the *how* as processes of micro-, onto-, and socio-genesis of definitional form-function relations.

2.3.1 Microgenesis of definitional form-function relations

An individual, in his moment-to-moment mathematical activity, might construct, alter, reproduce, or use definitions in service of various functions, depending on the problem context. Tommy in the opening episode of this chapter, for example, structured the four-right angles definition for rectangle to serve proving functions ("I really think you have to say four right angles… Because you have to prove everything, right?"). In this process, he highlighted a feature of the definitional form (specificity) as important. Later, in an effort to resolve problems of communication with other interlocutors, he modified the definitional form to a mathematically equivalent, but significantly different form ("you have to say *at least* [three right angles]"). In this microgenetic process, Tommy draws on a number of resources, including his intuitive ideas (or concept image, described below) and his sensorimotor experiences (Abrahamson, 2021).

There are various literatures that bear on the microgenesis of definitional forms and the functions they serve in collective practices. One body of work addresses the interplay between mathematical definitions and the resources individuals draw on in their microgenetic activity, or what researchers refer to as concept definitions and concept images (Tall & Vinner, 1981).

Another set of scholars describe the criteria that individuals deploy in formulating or evaluating definitional forms. I will describe and draw on these literatures in my treatment of the microgenesis of definitional form-function relations.

2.3.1.1 Concept Image and Concept Definition

The constructs of concept image and concept definition (Tall & Vinner, 1981) have served as important analytic tools in investigations of individuals' definitional activity (Bingolbali & Monaghan, 2008; Cantoral, 1989; Dahl, 2017; Edwards, 1997; Giraldo & Calvalho, 2006; Moore, 1994; Przenioslo, 2004; Tsamir et al, 2015; Ulusoy, 2020; Vinner & Dreyfus, 1989; Vinner, 1983; Wawro et al, 2011; Zandieh, 2000). From my perspective, this literature provides some insight into microgenetic processes as individuals work to formulate, revise, and reason about mathematical definitions.

The *concept image* for a mathematical object is generally taken to be an individual's intuitive ideas about the object - the collection of his mental pictures of the object as well as his ideas about associated properties (Vinner, 1983). The concept image may include verbal, visual, vocal, and other sensory associations (Vinner, 2011). For example, the prototype (Rosch, 1973) of a triangle – the classic image that comes to a child's mind – is that of an equilateral triangle with its base parallel to the bottom edge of a sheet of paper. The concept image is subject to change as an individual gains experience and encounters additional instances of the object. Further, particular aspects of a concept image may be evoked in response to some situations and not others (Vinner, 1991).

The *concept definition* for a mathematical object is "a form of words used to specify that concept" (Tall & Vinner, 1981), often taken to be a mathematically normative "formal" definition. This definition may well conflict with aspects of an individual's concept image. For example, young children who know that a triangle is defined to be a shape with three straight sides still often do not classify "upside down" triangles or long, thin triangles with obtuse angles as triangles due to conflicts with their concept image (Sfard, 2007). Further, the concept image may contain elements that are excluded from the definition due to minimality requirements of the latter. For example, the property that the sum of interior angles of a triangle sum to 180 degrees may well be an aspect of one's concept image, but not the concept definition.

The microgenesis of definitional form-function relations involves employing resources, among which is the concept image, to address problems in moment-to-moment activity. Research shows that individuals draw on their concept image in formulating definitions to varying extents. For example, students often reason from their concept image even though they "know", or at least can state, the definition of the concept (Dahl, 2017; Edwards & Ward, 2004; Foster & de Villiers, 2016; Moore, 1994; Vinner & Dreyfus, 1989). Further, even in the absence of any concept image (for example, when encountering a novel mathematical object), students still do not refer to the concept definition, often resorting to the concept image of a distinct but peripherally related idea (Edwards & Ward, 2004; Moore, 1994).

The dialectic between concept image and concept definition could be conceptualized as a search for equilibrium (Piaget, 1976). In the opening episode of this chapter, the students consider definitions in light of their concept image of rectangle. In this process, the definition is evaluated to account for potential counterexamples (Rex: "but you can't make [a rectangle] with only three [right angles].") As Rex considers the three-right-angles definition, he attempts to equilibrate the definition with the set of objects that he calls rectangles. The definition serves as a classificatory scheme. As an individual encounters new instances of the mathematical object, these are assimilated into his existing definitional scheme. When the individual comes into contact with an anomalous mathematical object, he alters his definition to accommodate for the anomaly. Through an iterative process of assimilation and accommodation, the definition is reproduced and altered until it reaches a state of relative equilibrium, in which the set of what one would consider to be that construct is exactly the set of objects that satisfy the proposed definition for the construct. As this process takes place, the schemata are multiplied and differentiated by their progressive accommodation to the diversities of reality.

Scholarship drawing on the constructs of concept image and concept definition to examine the definitional activity of students, teachers, and mathematicians generally does so in one of three ways. First, by cataloging participants' concept images and the ways in which they differ from standard definitions. This work examines participants' concept images of particular mathematical ideas and contrasts those with either the corresponding mathematically normative definitions or with participants' own definitions. These studies catalog students' and teachers' concept images for geometric shapes (Tsamir et al, 2015; Ulusoy, 2020), differential equations (Rasmussen, 2001), functions (Vinner & Dreyfus, 1989; Wilson, 1994), subspace (Wawro et al, 2011), derivative (Bingolbali & Monaghan, 2008; Zandieh, 2000), vector spaces and linear transformations (Sierpinska, 2000) and the like. A second strand of this work examines the ways in which participants use (or do not use) their concept image or concept definition in reasoning and problem solving (Dahl, 2017; Edwards & Ward, 2004; Foster & de Villiers, 2016; Moore, 1994; Vinner & Dreyfus, 1989). A third strand investigates how concept images and concept definitions are created or co-created within learning processes (Zandieh & Rasmussen, 2010; Bingolbali & Monaghan, 2008.)

The constructs of concept image and concept definition, while useful, are inadequate for describing the development of definitional practices in its totality. The framework emerged from a cognitive individualist perspective of learning and is still often used as such, with a few notable exceptions (Bingolbali & Monaghan, 2008; Nardi, 2006; Zandieh & Rasmussen, 2010) which are described later in this chapter. What often remains understudied are collective processes of the emergence, alteration, and communication of mathematical definitions.

2.3.1.2 Definitional Criteria

In understanding how definitional forms are generated and evaluated in microgenetic activity, scholars have specified criteria to be applied to mathematical definitions in different contexts. This body of work refers to *criteria* (Van Dormolen & Zaslavsky, 2003), desirable *features* (Zaslavsky & Shir, 2005), *characteristics* (Winicki-Landman & Leikin, 2000), or *requirements* (Borasi, 1992) for definitions, often framing sophisticated defining activities as those that result in definitions that satisfy these criteria.

This work provides insights into the features of definitional forms in a particular context – that of formal mathematics. Borasi (1992) lists a number of "commonly accepted requirements of definitions" (p. 17) in the discipline. Van Dormolen and Zaslavsky (2003) distinguish necessary criteria, which are logical necessities for the proper functioning of a deductive system, and preferred criteria, which are cultural conventions. Necessary criteria include the *criterion of hierarchy* which demands that any new concept must be described as a special case of a more general concept, the *criterion of existence* which demands that at least one instance of the newly defined concept must exist, the *criterion of equivalence* which states that multiple definitions for a given concept must be equivalent, and the *criterion of axiomatization* which implies that a definition fits into and is part of a deductive system. On the other hand, preferred criteria for mathematical definitions, as generally agreed upon by a community of mathematicians, include the *criterion of minimality* which states that no more properties of a concept be specified than necessary, the *criterion of elegance,* which is subjective and difficult to define, and the *criterion of degenerations,* which states that the definition should exclude degenerations, or instances that do not conform to our intuitive ideas of the concept.

In addition to criteria that apply to definitions in formal mathematical contexts, scholars also identify criteria for definitional forms that serve pedagogical functions. Winicki-Landman and Leikin (2000) refer these as *didactic characteristics of a definition*, while referring to features of definitional forms that serve disciplinary functions as *mathematical characteristics of a definition*. Van Dormolen & Zaslavsky (2003) support the idea that definitions take on different forms to serve different functions, distinguishing between formal mathematical definitions and "example-based descriptions" used in pedagogical contexts.

In their microgenetic activity, students and teachers attend to definitional criteria to varying extents. For example, in their study of 12th grade students' definitional activity, Zaslavsky & Shir (2005) found that participants were attentive to the need for definitions to be minimal, but did not appreciate the existence of equivalent definitions or the arbitrariness of the choice of definition. Codes et al (2020) found that prospective secondary teachers attended to the criteria of hierarchy and minimality while they did not see the need for the criteria of existence or arbitrariness. Based on findings such as these, research advocates the need for students and teachers to be assisted to attend to definitional criteria (Winicki-Landman & Leikin, 2000A). A second strand of research addresses the ways in which students and teachers might be sensitized to definitional criteria (Winicki-Landman & Leikin, 2000B).

A limitation of the focus on definitional criteria is the implication that what makes for a "good" definition is universal, either across all formal mathematical contexts or across all pedagogical contexts. Recall the conversation between Tommy, Rex, and Eli. Each individual took the definition for rectangle to serve different functions – proof, description, and "messing with people." As such, there was no universal criteria for a "good" definition without acknowledging the specialized functions to be served by the definition. Even within formal mathematical settings, alternative definitions for a given concept are selectively chosen to serve specialized functions (Morgan, 2005). Taken at face value, discussions of definitional criteria neglect the problem solving or communicative context within which definitions are used and the functions they serve within those contexts.

2.3.1.3 Mathematical definitions as mediating tools

In their microgenetic activity, individuals structure mathematical definitions to serve different functions. Ouvrier-Buffet (2006) developed a model of the defining process through interviews with mathematicians, finding that mathematicians modify definitions for a variety of reasons: "to have a better understanding of a concept or a problem, to simplify, to generalize, to explore different linked frames or connected fields than the first one, to communicate" (p. 2216-2217).

Vygotsky's construct of reverse action (1986) serves as a useful lens through which to view the ways in which definitions are drawn on to serve problem solving and communicative functions. Consider a student engaged in the classification of geometric objects. In order to determine whether an object is a triangle, the student might refer to a definition of triangle, as a planar figure with three straight sides. Then the student will extract properties from the definition to check, asking herself *is this object planar? Does it have three sides? Are all three sides straight?* If and only if she can answer each of these questions in the affirmative will she classify her object as a triangle.

According to Vygotsky (1986), signs used in problem solving act as auxiliary stimuli. An individual, encountering a task, or stimulus, has a direct response. In the case of our student, she encounters a shape and has a direct response. Based on her concept image of triangles, she has an idea of whether the shape is or is not a triangle. However, she might not directly react to the stimulus, relying instead on a definition to regulate her response. In Vygotskian terms, she introduces a sign to serve as an intermediate link between the stimulus and the response. This intermediate link is a second order stimulus (sign). She is actively engaged in establishing this link, drawing the definition into her problem-solving activities. Furthermore, the definition possesses the characteristic of reverse action in that it operates on her, not on the environment. In other words, she uses the definition to inhibit her direct response and regulate whether or not she might call the shape a triangle. In this way, the definition is the indirect means by which she completes her task. The instrumentality of mathematical definitions in problem solving and communication motivates is central to my treatment of definitions as cultural forms.

2.3.1.4 Summary

In the microgenesis of definitional form-function relations, individuals draw on resources such as their concept images and their ideas about definitional criteria to structure definitional forms to serve specialized functions.

2.3.2 Ontogenesis of definitional form-function relations

Over the course of ontogenetic development, both concept images and concept definitions shift. Wilson (1994), for example, examines a preservice teacher's evolving concept image and concept definition of function over the course of a 10-week mathematics education course. While the teacher's initial attempts to define function referred to the symbolism of functions ("A function is something written f of x equals something that contains x's"), she later developed a concept image of functions as relationships between variables and gained flexibility in relating formal definitions to her concept image. De Villiers (1998) examines the evolution of 10^{th} grade students' definitional activity as they engage in a teaching experiment designed to teach them to define, showing that students who participated in the intervention were able to construct more "correct" and minimal definitions for both known and novel geometric concepts. Bingolbali and Monaghan (2008) describe the evolution of concept images in terms of Vygotsky's (1986) complexes, a "phase on the way to concept formation."

Zandieh and Rasmussen (2010) characterize the ontogenetic development of mathematical definitional activity as the creation of greater and richer connections between students' concept image and concept definition. The authors trace a trajectory of development as shifts in the ways in which individuals use and create concept images and concept definitions, showing that as students develop their definitional activity, they are able to make connections between concept images and definitions with more facility. Vinner (1991) outlined three possible outcomes when an individual is introduced to a concept definition that conflicts with his existing concept image. (1) The concept image might be adjusted to align with the newly introduced definition, (2) The concept image might remain unchanged, while the student's memory of the teacher's definition would be forgotten or distorted, or (3) The concept image and concept definition both remain unchanged, while any conflicts between them are ignored. In the last case, the student would draw on the concept image in some problem solving situations (for example, when classifying objects) and would draw on the concept definition in other cases (for example, when asked to invoke the definition). The ontogenetic development of definitional practices would then entail a shift from the second and third approaches towards the first.

A significant development in ontogenetic processes lies in the extent to which and the ways in which mathematical definitions are used to serve cognitive and communicative functions. These ontogenetic developments can be viewed as analogous to the development of thought and language as described by Vygotsky (1986). In his treatment, words in early development are auxiliary to problem solving. As words and thought come into contact, speech accompanies problem solving. Over time, speech becomes integral to problem solving.

In the case of mathematical definitions, definitions in early life are not instrumental in communication and problem solving. Recall Sfard's (2007) example of a young child who, knowing the definition of a triangle, refused to classify a long, thin triangle as a triangle. Like Vygotsky's paint chips, for young children, definitions are auxiliary to the problem at hand. Research shows that even undergraduate students often do not rely on definitions to classify. Edwards (2004) found that even when undergraduate students know and understand a mathematical definition, they do not use it to justify or support classification.

As students develop as doers of mathematics, they begin to reference definitions in problem solving. In the wooden blocks experiment, the nonsense words on the bottom of the blocks transition from being unrelated to the problem at hand to becoming features of the objects. Similarly, at this stage, definitions become features of mathematical objects, but are not yet tools for problem solving.

Later in development, mathematical definitions are taken up as mediating tools for problem solving, serving the functions of reverse action (Vygotsky, 1986) in the same way that speech ultimately begins to serve as a tool for problem solving. Ultimately, for mathematicians, definitions serve a number of functions – to introduce the objects of a theory (Borasi, 1992; Mariotti & Fischbein, 1997; Pimm, 1993; Michener, 1978; Zaslavsky & Shir, 2005), to facilitate concept formation (Kobiela & Lehrer, 2015; Vinner, 1991; Wilson, 1990), to support communication by establishing uniformity in the meaning and use of terms (Borasi, 1992; Lakatos, 1976), and to establish foundations for proof and problem solving (Moore, 1994; Weber, 2002). For example, to prove that a quadrilateral with three right angles must have a fourth right angle, one must first refer to the definitions of right angles and quadrilaterals. In all proving activities, definitions are essential instruments that provide a basis for the construction of arguments.

Evidence of these ontogenetic developments are provided in the two studies described in subsequent chapters of this dissertation. Findings from these studies indicate that as students become enculturated into the discipline of mathematics, they engage in definitional practices in new ways. Definitions shift from serving to provide the algorithms and formulas to solve routine problems towards serving to facilitate communication and proof among a community by setting standards of usage for terms. Further, definitional forms constructed by students shift, both in terms of the language used (from less mathematical terminology towards more) and in terms of the contents of the definition (from containing illustrative examples towards minimality). Further, results show that the role of definitions shift over time, as they shift from being irrelevant to classification towards taking on a central role as criteria for classification.

2.3.3 Sociogenesis of definitional form-function relations

The sociogenesis of definitional forms and the functions they serve constitutes a process in which they are reproduced and altered in a community over time through the microgenetic activities of multiple individuals. This may take place in classroom community over the course of a semester or within a community of mathematicians over the course of centuries. Recall Lakatos' description of the historical development of the definition for polyhedron, described in section 2.2 of this dissertation, as an example of the ways in which definitional forms in the mathematical community are co-constructed as solutions to emergent collective problems.

Mathematical defining has received little attention from scholars who view it as a collective practice, despite the "social turn" in mathematics education (Lerman, 2000). Notable exceptions include Rasmussen & Zandieh (2005), who characterized mathematical learning as participating in mathematical practices such as symbolizing, algorithmatizing, and defining, as well as Kobiela and Lehrer (2015), who characterized defining as a participation framework governed by normative expectations about appropriate forms of participation (Goffman, 1981). These scholars see defining as accomplished in interaction, as community members work to negotiate the norms that govern their joint activity. Nachlieli and Tabach (2012) drew on a framework of mathematics as discourse to examine the definitional activity of a group of prospective mathematics teachers studying functions, describing the ways in which participants gradually came to take up the definition as the ultimate criterion for identifying examples of functions. Morgan (2005) drew on a similar framework to compare the social norms governing definitional practice in the field of mathematics research with school mathematics. By conducting a textual analysis of mathematical definitions in academic research journals and school textbooks, Morgan found stark differences between the two - while academic texts showed definitions to be constructed and used in creative and purposeful ways, the school textbooks presented a one-to-one word-concept relationship.

As an illustration of the sociogenesis of mathematical definitions in the history of formal mathematics, I describe the emergence and alteration of the definition of *dimension*. Up to and through most of the 19th century, mathematicians used dimension in only a vague sense. "A configuration was said to be *n*-dimensional if the least number of real parameters needed to describe its points, in some unspecified way, was *n"* (Hurewicz & Wallman, 1948, p. 4). Given this broad definition, a line was 1-dimensional because each point on a line could be denoted by a single number, while a plane was 2-dimensional because two coordinates were needed to identify a specific point. In the last part of the 19th century, several discoveries showed this definition to be untenable. One of these was Cantor's one-to-one correspondence between the line and the place, in which each point on the line was associated to a unique point on the plane such that every point on the plane was associated, in turn, with a unique point on the line. This implied, in some sense, that there were the same (infinite) number of points on a line as there were on a plane. Given the prevalent definition for dimension, the plane would be two-dimensional! There was a need for a definition of dimension that was topologically invariant (would not change if the space was stretched or contracted in particular ways). This issue was not dealt with satisfactorily until mathematicians found a topological invariant of spaces, and formulated a new definition for dimension on that basis. The new dimension concept was an important milestone in geometry, not only because it dealt with the issue noted here, but also because of the generality of objects to which it could be applied. The new definition could be applied not only to geometric objects, but also to new mathematical objects that were being developed. As new problems of communication and problem solving arose, new definitions of dimension emerged. These new definitions built on prior definitions in the sense that objects that were classified as 2-dimensional under the prior definition would still be called 2-dimensional.

At the same time, the new definition included features suited to the new problem context. As such, the form was modified to satisfy the needs of collective problem solving.

Though the example presented above is drawn from formal mathematics, analogous processes occur in pedagogical contexts, often as teachers strive to enculturate students into disciplinary practices of defining (Ball, 1993; De Villiers, 1998; Herbst et al, 2005; Kobiela & Lehrer, 2015; Mariotti & Fischbein, 1997). Consider Eli's proposal of the "*at least* three right angles" definition for rectangle in the opening episode of this chapter. This is an alternative to both existing definitions on the table. Tommy immediately takes up Eli's proposed definition, stating "No, yeah, and you have to say *at least*." In this way, the definition is reproduced in the conversation between interlocutors.

2.4 Conclusion

In this chapter, presented an integrated treatment of issues of development in mathematical practices of defining. In considering the question *What develops?* I considered mathematical definitions as cultural forms and the specialized mathematical and communicative functions that they serve in collective practices. In response to the question *How does it develop?* I described the development of definitional forms and functions as constituted through processes of micro-, onto-, and sociogenesis of form-function relations. In doing so, I drew on disparate strands of research on mathematical definitions and defining, which provide insights into questions of development but which fail to account for the conventional, function-dependent nature of mathematical definitions as well as the social nature of defining practices. These strands of prior work include but are not limited to research on relations between concept image and concept definition, scholarship focusing on definitional criteria that are to be satisfied in disciplinary and pedagogical contexts, and a small body of work taking a sociocultural approach to definitional practices. I utilize the sociocultural framework presented here to frame the empirical work I present in subsequent chapters of this dissertation.

3 A comparative analysis of undergraduates' conceptions of definitional forms and functions

3.1 Introduction

The study presented in this chapter draws on the form-function framework presented above as an analytic tool to investigate how the definitional thinking of undergraduates who are being enculturated into disciplinary mathematics (mathematics majors) differs from the definitional thinking of non-mathematics students (humanities majors). To this end, I present a comparative analysis of two groups of undergraduates – mathematics majors ($n=12$) and humanities majors ($n=12$) – engaged in task-based semi-structured interviews. Analyses provide insight into participants' conceptualizations of the functions served by mathematical definitions and the forms mathematical definitions should take.

3.2 Functions and Forms of Mathematical Definitions

I draw on a view of mathematical definitions as cultural forms that serve varying functions in collective practices. Ongoing processes of defining entail shifts in both the function and the form of these definitions over time.

3.2.1 Functions of mathematical definitions across contexts

Definitions serve several important functions in mathematical practices. First, mathematical definitions provide participants in a mathematical community a means for communicating about mathematical ideas by ensuring that they are referring to the same set of instances when using a term (Borasi, 1992). Indeed, formal communications between professional mathematicians often begin with a statement of the pertinent definition(s), regardless of interlocutors' presumed understanding of the concept (Edwards & Ward, 2008). Even with elementary level mathematical content, definitions support communication. Consider, for example, the well-known debate about whether a square is a rectangle. The issue can be resolved by ensuring that interlocutors are referring to the same set of shapes when using the term *rectangle.* This can be accomplished by agreeing on a definition for rectangles – either as quadrilaterals with four right angles or as quadrilaterals with four right angles and at least two sides of unequal length.

Definitions in non-mathematical contexts also serve to ensure that interlocutors have a common referent. Consider, for example, the necessity of a definition for two friends who disagree about whether a tomato is a fruit or a vegetable or, more seriously, the divisive consequences of the lack of a standard definition for liberty in modern American discourse.

Yet mathematical definitions play a role in disciplinary practice that goes beyond the function they serve in non-mathematical contexts. Mathematical definitions serve as the basis for deduction of properties of the concept being defined. Further, all known properties of the concept can be deduced from the definition (Morgan, 2005). In contrast to the definition of tomatoes, to which one need not, and in fact cannot, refer in order to deduce properties of redness or juiciness, mathematical definitions must determine all known properties of the concept. By setting a standard referent for the term being defined, mathematical definitions not only facilitate communication by ensuring that interlocutors are referring to the same general category of instances, but also provide a basis on which to assess the validity of all statements that are made about those instances.

A related function mathematical definitions serve in disciplinary practice is as foundations for proof (Lakatos, 1976; Moore, 1994; Weber, 2002). For example, to prove the simple statement that a square is a rectangle, one begins with a definition of rectangles (a rectangle is a quadrilateral with four right angles) and shows that any square satisfies each of those specifications – first, it is a quadrilateral and, second, it has four right angles.

In many pedagogical contexts, mathematical definitions often serve a different set of functions – to develop conceptual understanding of a novel concept and to provide the skills needed to solve routine exercises. Mathematical definitions in K-12 schooling are often presented in the context of teaching new content and serve to facilitate concept formation (Kobiela & Lehrer, 2015; Vinner, 1991; Wilson, 1990). In contrast to students in college level mathematics classes who are expected to use definitions as bases for deduction, students in elementary and secondary mathematics are provided with definitions that aim at helping them understand what, say, a fraction is or how to, for example, multiply two numbers together. In accordance with the functions they are meant to serve, these definitions take on different forms than the definitions used in college-level and professional mathematics. I address these differences in form in the next section.

In this section, I described a few of the functions mathematical definitions serve in disciplinary practice and in the classroom. This list is far from exhaustive. For example, in disciplinary practice, mathematical definitions also serve to introduce new objects into the field of inquiry, often as generalizations of some known mathematical object (Borasi, 1992; Martín-Molina et al., 2018; Zaslavsky & Shir, 2005). Importantly, mathematical definitions are also used to classify – distinguishing between what is or is not, say, a rectangle.

3.2.2 Forms of mathematical definitions across contexts

In order to fulfill the varying functions outlined above, mathematical definitions can take on different forms. At a most basic level, we can distinguish between formal mathematical definitions and informal example-based descriptions (Van Dormolen & Zaslavsky, 2003).

Formal mathematical definitions used in professional mathematics and in higher level college mathematics courses are expected to fulfill certain criteria (Edwards & Ward, 2008; Van Dormolen & Zaslavsky, 2003), of which I will provide two examples.

The criterion of hierarchy stipulates that each new concept be defined as a special case of a more general defined concept. For example, *rectangle* is defined as a type of *quadrilateral*, which is in turn defined as a special case of, perhaps, *a planar figure*. By contrast, an informal definition used in an elementary school classroom might define *rectangle* as a *shape*, with *shape* having not been previously defined.

The criterion of minimality stipulates that no more properties of a concept be specified in the definition than is required. In general, this suggests that mathematical definitions should not provide, say, example cases, or multiple interpretations of a concept. Taken to the extreme, this criterion would suggest, for example, that the definition of *rectangle* as *a quadrilateral with three right angles* is preferable to its definition as *a quadrilateral with four right angles.* Indeed, the two definitions are logically equivalent, while the former technically has fewer stipulations.

These criteria do not generally apply to definitions used in mainstream pedagogical contexts. In order to facilitate conceptual understanding, teachers in K-12 classrooms often use informal definitions that take the form of example-based descriptions. These definitions outline the general features of the concept, perhaps in more than one way, and provide illustrative exemplars (Van Dormolen & Zaslavsky, 2003). These example-based descriptions do not satisfy the criterion of hierarchy or that of minimality. Thus while these definitions have clear advantages for learning new content, they do not make for "good" definitions from a formal mathematical perspective.

The type of statement that can be called a "mathematical definition" varies across K-12 and professional mathematical contexts (Morgan, 2005). These definitions serve different functions and take on different forms. We thus expect the functions and forms of what different groups of students take to be "mathematical definitions" to differ in significant ways.

3.2.3 Shifts in definitional form-function relations

Defining entails shifts in the forms and functions of mathematical definitions, as new insights and problematizing cases are generated. The work of defining never ends, as exemplified in the case of the mathematical community's definition of *dimension,* which shifted repeatedly over the course of centuries (Eisenbud, 1995).

I anticipate differences in how undergraduate mathematics and humanities majors treat definitional forms and their functions. In K-12 education, students may be engaged with an approach to definitions that differ substantially from those associated with the definitional practice in the discipline of mathematics, a K-12 approach that I outline below. As mathematics majors participate in a new set of practices in college, they may be supported along a new line of development more characteristic of mathematicians' definitional practices, practices in which valued definitional forms may shift and take on new functions. By contrast, humanities majors carry forward the mathematical form-function world of the K-12 classroom. While they might not participate in the same line of development as the mathematics students, they develop along other lines more keyed to the humanities, taking definitional forms to serve a different set of functions.

In K-12 mathematics education, the process of defining is opaque, as students are rarely if ever offered opportunities to define or to witness defining processes. Rather, definitions are provided to students as final forms with fixed functions. Further, the forms and functions of mathematical definitions are rarely addressed in an explicit way (Wilson, 1990; Zaslavsky & Shir, 2005). Instead, defining in the mathematics classroom draw from everyday practices of defining. Indeed, students come to the classroom with a vast amount of experience with definitions in an everyday context, and thus also with an idea of what it means to define. They encounter new words in their natural language and rely on context clues to ascertain their meaning. When they are unable to do so, they might look to a dictionary for a definition. They explain the meaning of words to each other, creating their own quasi-definitions. In these non-mathematical contexts, defining a word means explaining its meaning to the extent that the audience can use it in ways that are understandable to others. In this sense, explanation and definition are indistinguishable. The criteria often applied to mathematical definitions in a professional context, say, of succinctness and non-redundancy, are not useful. Definitions are not stipulated (to dictate usage) as much as they are extracted (formed by the ways in which a word is already used) (Edwards & Ward, 2004). Thus, whereas everyday practices of defining, carried into the primary and secondary mathematics classroom, serve as a foothold that students can leverage as they learn to work with mathematical definitions, they also present some difficulties in assimilating professional functions of these vital mathematical forms.

At the college level, definitional forms take on new functions, a process that supports defining activities (the alteration of definitional forms). College mathematics students engage in, or at least are exposed to, new mathematical practices within which definitions serve the mathematical functions described previously -- comprehensively specifying how a word form is used in communication and providing a basis for proofs, to name two. At the same time, mathematical definitions continue to play a role in helping students to develop a more robust understanding of a concept.

Undergraduate mathematics education is a pivotal moment in which students, often for the first time, work with definitions as formal entities that serve new functions in their activity and, at times, engage in the actual construction of mathematical definitions (Bills & Tall, 1998; Edwards & Ward, 2008; Moore, 1994). A primary goal of the two studies reported below is to understand the emerging development of mathematical definitional practices among students, with a focus on shifts in definitional forms and specialized functions for those forms. To this end, I selected undergraduate mathematics majors as an important population to study, a population who I assumed were at the early stages of shifting their understanding of forms and functions of definitions in mathematics. Further, to determine whether, in fact, this population was developing specialized practices of defining, I compared mathematics majors with humanities majors, a same-aged and in many respects similar population, but one that was not engaged with definitional practices in the discipline of mathematics.

3.3 Methods

3.3.1 Participants and setting

Two groups of participants took part in this study: undergraduate mathematics majors (n=12) and undergraduate humanities majors (n=12).[1] Participants were recruited through an online portal open to undergraduate students at a highly ranked public university in California. All participants stated that they had come across mathematical definitions in the past. Yet most (100% of humanities majors and 75% of mathematics majors) had never created a mathematical definition.

Each participant took part in an hour long one-on-one semi-structured interview. Interviews were video and audio recorded using two audio–video cameras, one positioned to capture the participant's facial expressions and gestures, and the other to capture his dynamic inscriptional behavior. Participants were provided with paper and a writing utensil and encouraged to make liberal use of these tools as needed. Data sources included each participant's written work and the audio–video recordings of each interview.

3.3.2 Procedures

[1] Students majoring in mathematics, applied mathematics, and computer science were assigned to the mathematics-majors group. Students majoring in non-STEM disciplines were assigned to the humanities group.

The interviews were semi-structured (Ginsburg, 1997), consisting of two background questions, two questions about the functions and forms of mathematical definitions, and clarifying follow up questions to each.

Background questions:

Q1. Have you encountered any mathematical definitions? (If yes: Can you give me an example of one?)

Q2. Have you ever created a mathematical definition?

Form-function questions:

Q3. What are mathematical definitions usually used for?

Q4. What makes for a good mathematical definition? (Or: What are some features of a good mathematical definition?)

3.3.3 Analytic Approach

To analyze the interview data, I coded participants' responses for the functions they took mathematical definitions to serve and for preferred features of definitional forms. After transcribing responses, I conducted a round of in-vivo coding. I then selected and recategorized codes that appeared useful and used them to conduct a second and third round of coding. I determined inter-rater reliability for the final set of codes by coding the full data set independently from a research assistant, reaching 91.2% agreement. Differences were resolved in discussion to arrive at full agreement in the application of codes.

3.4 Results: Definitional functions

In this section, I first describe three functions of mathematical definitions identified by participants. Subsequently, I analyze similarities and differences in functions across cohorts.

3.4.1 Functions of mathematical definitions according to participants

Participants identified three functions served by mathematical definitions in response to the question "What are mathematical definitions usually used for?" These included (a) facilitating communication and proof by setting a standard among a community of mathematicians, (b) helping one to find answers to routine exercises, often numerical problems with a predefined answer, and (c) advance learners' understanding of novel ideas in a teaching/learning context.

a. Facilitate communication and proof by setting a standard.

Previously, I noted that in the discipline of mathematics, a principal function of definitions is to facilitate communication among disciplinary participants. This is done by ensuring that interlocutors are referring to the same set of instances when using a term. Relatedly, mathematical definitions serve to ensure that there is a standard basis for deduction of properties of the concept. If a participant mentioned any one of these ideas related to this definitional function – communication among mathematicians, setting a standard, or supporting deduction – I coded their response as facilitating communication and proof by setting a standard.

Each of the following responses to the question "What are mathematical definitions usually used for?" exemplifies one of the related ideas associated with this function: communication, standard setting, and proof.

> Denise: [Definitions are] a universal way to communicate with others about problems.
>
> Suzy: To set a standard for everyone who needs to be in the context of using this word. And to have the same understanding that they agree on.
>
> Yoko: I think they [definitions] are used to prove something and after proving, we can use it to solve problems."

Denise mentions communication among a community of mathematicians, while Suzy refers to the specific role mathematical definitions play in setting a standard for usage of a term. Lastly, Yoko refers to the related idea of using the specifications outlined in a definition as bases for deduction. Each of these responses was coded as referencing the overarching definitional function being discussed here.

The three ideas—communication, standard setting, and proof— are interrelated. The case of a mathematics participant, Alvin, illustrates how these ideas may work together as a coherent whole. During his interview, Alvin first referred to the role of mathematical definitions in setting a standard that allows for proof.

> Alvin: It's to basically outline a set of standard[s] for what you can or can't use when you're trying to deduce something. Cause maybe like something's defined this way versus that way. It outlines you can use this information, because that's what we call it or whatever.

In his statement, Alvin refers to elements of the definition as "information" one can use when "trying to deduce something," implying that the definition not only allows us to discriminate between instances and non-instances of the concept by checking whether a potential candidate satisfies each specification outlined in the definition, but that each specification provides a potential starting point for deducing or proving further properties of the concept.

When asked for clarification about "standards," Alvin elucidates,

> Alvin: It's because without them, it would be harder to deduce certain conclusions from information you would otherwise be given when you're trying to solve a problem or something like that.

Here, he refers to the role of mathematical definitions in problem solving – when "you're trying to solve a problem." In a problem-solving context, one uses mathematical definitions in lieu of "information you would otherwise be given," implying again that the specifications in the definition provide a basis for constructing a deductive argument (idea (3)).

When later asked who these standards are for, Alvin refers to the community of mathematicians—rather than, say, learners—as the audience of a mathematical definition,

> Alvin: It's for everyone who works in the field. They should understand it's this way, not that way. Makes collaboration a little easier.

Here, Alvin makes clear that the audience of the definition is professional mathematicians – "everyone who works in the field." Further, definitions are not meant for use by an individual, but a collaborative community communicating about ideas – "makes collaboration a little easier" (idea (1) mentioned above). It bears noting that in this statement Alvin also refers to understanding as a function of definitions ("they should understand…"), but he is clearly not referring to advancing students' understanding in a pedagogical context, a definitional function we will discuss shortly.

b. Find answers to routine exercises.

Participants identified another function of mathematical definitions: Mathematical definitions serve to support finding answers to routine exercises. Participants referred specifically to problems that had predetermined numerical or binary answers and problems that called for an algorithmic solution procedure. In my analyses, I consider these to be "routine exercises." These problems, "organized to provide practice on a particular mathematical technique that, typically, has just been demonstrated to the student," are common in mainstream mathematics classrooms (Schoenfeld, 1994). Consider the following participant responses to the question "What are mathematical definitions usually used for?"

> Mai: I see them [mathematical definitions] as rules to help you solve the problem.
>
> Courtney: For figuring out numerical problems.

In her statement, Mai refers to definitions as "rules" to help one solve a problem, implying that the problem required the application of some technique or algorithm. Similarly Courtney's statement refers explicitly to "numerical problems." Each of these responses was coded as referencing the definitional function *finding answers to routine exercises.*

c. Advance learners' understandings of novel ideas.

Participants also mentioned the role mathematical definitions play in advancing learners' understanding of novel ideas. I saw a few indicators of this in the data. First, participants often described the audience of definitions to be learners or novices, rather than, say, professional mathematicians. Second, they described the context within which mathematical definitions are used to be teaching-learning contexts, rather than, say, the disciplinary practices of a community of mathematicians. Third, they described the objects of defining to be novel, whereas in disciplinary mathematics, definitions for both familiar and novel ideas are commonplace.

Consider the response of Jenny, a humanities student, when asked about the function of mathematical definitions.

> Jenny: Teaching people who haven't heard about the concept before and getting them to understand what it is in a quick way.

Jenny implies that the audience of the definition are learners in a teaching/learning context and that the concepts being defined are unfamiliar to the audience of the definition ("people who haven't heard about the concept before").

A similar response was given by mathematics students, Suzy (who we mentioned previously as also referencing the communicative function of definitions for mathematicians) and Kara.

> Suzy: And also used to teach a new concept to people who don't know it yet.

> Kara: I would say that they're used to start building off new concepts and in a new lesson. I think math definitions are used to teach us new concepts and then with these definitions we can start using them for different concepts and start building up our mathematical skills by applying these definitions.

Kara refers to the content being defined as "new" and the context within which mathematical definitions are used as that of a classroom with "lesson[s]". She further indicates that mathematical definitions are used to "teach." Kara's statement, as well as the other statements described above, were all coded as referencing the broad definitional function *advance learners' understandings of novel ideas.*

There is coherence in participants' treatments of multiple definitional functions. Nearly one third of participants identified two definitional functions in their responses to the interview question "What are mathematical definitions usually used for?". In those cases, they often described multiple definitional functions in a coherent way. Consider the humanities student, Aubrey, who identified both *finding answers to routine exercises* and *advance learners' understanding of novel ideas* as important functions of mathematical definitions.

> Aubrey: To help people understand, not in the visual way, but in the conceptual way, trying to formulate it and, say, into a word problem, and translate it into an equation… For division it's like, say there's like twelve pieces of candy for three people. You would figure out how to divide that to three people, and twelve divided by three is four.

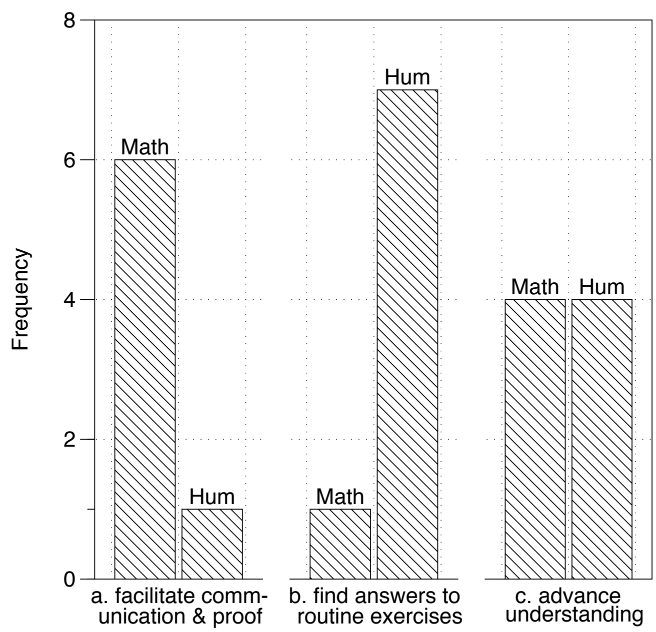
To Aubrey, mathematical definitions first "help people understand…in the conceptual way." Further, Aubrey states that definitions involve framing the concept as "a word problem" and translating it "into an equation." She references the activity of solving routine word problems, perhaps by plugging into an equation provided by the definition. Thus, her response was coded as referencing two definitional functions.

3.4.2 Similarities and differences between cohorts

To analyze for similarities and differences in the ways mathematics and humanities participants conceptualized the function of mathematical definitions, I created a dependent variable as follows. I assigned each student response to the three function codes described above, then summed the number of students in each cohort who were assigned each code. I analyzed student responses as a function of population group (mathematics vs. humanities majors).

Figure 1 displays the number of mathematics majors (Math) and humanities majors (Hum) who cited each of three functions for mathematical definitions in their responses to the interview question "What are mathematical definitions usually used for?" The figure shows that all three functions – a. facilitate communication and proof by setting a standard of usage across a community of mathematicians; b. find answers to routine exercises; and c. advance learners' understandings of novel ideas – were mentioned by at least one student in each cohort.

Figure 1
Definitional functions cited by mathematics and humanities participants



The leftmost column in Figure 1 shows that significantly more mathematics participants referred to communicative and proving functions than did humanities participants. A chi-square test of independence showed that the difference between cohorts was statistically significant, $\chi^2(1, N=24) = 5.0, p<.05$.

The central column in Figure 1 shows that humanities participants were more likely to refer to the role mathematical definitions play in helping them to find answers to routine exercises. A chi-square test of independence showed that the difference between cohorts was statistically significant, $\chi^2(1, N=24) = 6.75, p<.01$.

The rightmost column in Figure 1 shows that both participant cohorts referenced the role mathematical definitions play in helping learners to advance their understanding of novel ideas, in equal numbers. A third of the participants in each cohort cited this definitional function.

3.5 Results: Definitional forms

Having examined the functions that participants took mathematical definitions to serve, I now present analyses of the forms they preferred mathematical definitions to take. I begin by outlining three features of definitional forms that participants took to be desirable. Subsequently, I analyze similarities and differences across cohorts.

3.5.1 Forms of mathematical definitions according to participants

Participants identified three features of mathematical definitional forms in response to the question "What makes for a good mathematical definition?" These included that mathematical definitions should (a) comprehensively, precisely, and unambiguously specify all conditions for an instance to be classified as the term being defined, (b) be readable or understandable, and (c) outline procedures for solving problems.

a. Comprehensive, precise, and unambiguous

As noted previously, mathematical definitions in disciplinary practice are expected to comprehensively, precisely, and unambiguously specify how a word is used. Since a primary function of mathematical definitions is to ensure that interlocutors are referring to the same set of instances when using a term, mathematical definitions must allow disciplinary participants to consistently and unambiguously distinguish between instances and non-instances of the concept. This classification is done simply by checking whether a candidate instance satisfies all of the specifications outlined in the definition. In other words, the definition must comprehensively specify all necessary and sufficient conditions for a candidate instance to be classified as a valid instance. It must do so consistently, in the sense any mathematician using that definition would come to the same conclusion about whether a potential instance constitutes an instance of the concept, and unambiguously, in the sense that there would be no ambiguities regarding this choice.

Participant responses to "What makes for a good mathematical definition?" referred to comprehensive definitions that delimit a precise boundary between what counts as, say, a fraction, and what does not. These participants rejected the looseness and ambiguity that characterize colloquial definitional forms. Indeed, they mentioned that definitions should "account for all possibilities," be "precise and unambiguous," have "no loose ends" and "no ambiguities," be "exact" with "no room for misunderstanding," and "hold true regardless of scenario." Responses such as these were coded as citing comprehensiveness, precision, and unambiguity.

b. Procedural

Many participants' interviews revealed a preferred form of mathematical definitions: Definitional forms should instruct the reader on procedures, perhaps containing formulae or concrete examples. Participants stated, for example, that good mathematical definitions "outline specific theorems possibly, or when to use those definitions and what they apply to" (Kelly), that they are "probably something that has examples, for example, addition is adding numbers and subtraction is subtracting it" (Aubrey), or that they "contain a formula and an in-depth explanation of the formula" (Layla). Responses such as these were coded as citing the procedural nature of definitional forms.

c. Readable or understandable

A third feature of definitional forms participants cited was the readability or understandability of definitions. Some participants specifically mentioned that mathematical definitions should avoid mathematical terms and symbols. In this way, participants identified the implied audience of definitions to be novices, rather than mathematicians. As discussed previously, the implied audience is an important indicator of the function of mathematical definitions.

Consider the response of a mathematics student, Kara, who referred to both the general readability of definitions and the way in which mathematical terms and symbols should be used.

> Kara: It should be pretty clear-cut and in a readable format. So this way when a student first looks at the math definition, they wouldn't be overwhelmed with symbols and terms. It should be pretty straight-forward.

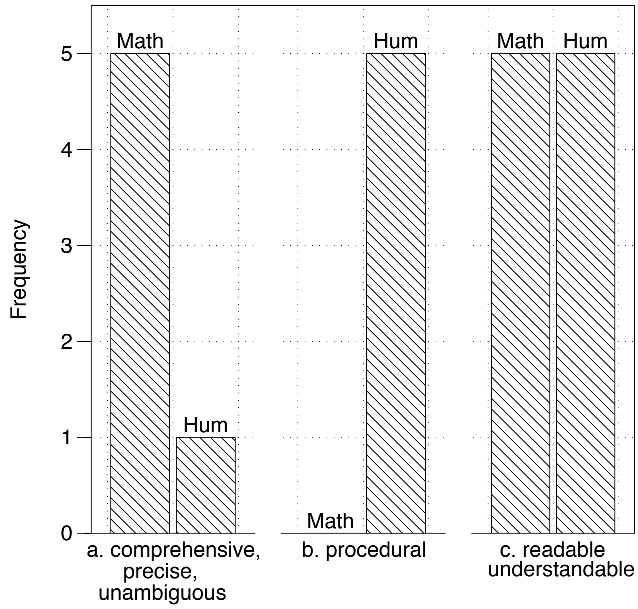
Here, Kara refers to students rather than professional mathematicians as the audience of mathematical definitions. Further, she is concerned lest students be "overwhelmed" by the symbols and terms contained in the definition. Responses such as Kara's were coded as citing the readability or understandability of definitions.

3.5.2 Similarities and differences between cohorts

To analyze for similarities and differences in the preferred definitional forms of mathematics and humanities participants, I created a dependent variable by first assigning each student response to the three definitional form codes described above, summing the number of students in each cohort who were assigned each code, and then analyzing student responses as a function of population group (mathematics vs. humanities majors).

Figure 2 displays the number of participants in each cohort (Math and Hum) that cited each of three features of mathematical definitions – that mathematical definitions should be (a) comprehensive, precise, and unambiguous, (b) procedural, or (c) readable or understandable – in their responses to the question "What makes for a good mathematical definition?"

Figure 2
Feature preferences for definitional forms cited by mathematics and humanities students



a. Comprehensive, precise, and unambiguous

The leftmost column of Figure 2 shows that nearly half of the mathematics participants (five out of 12) cited comprehensiveness, precision, and unambiguousness as preferred features of mathematical definitions. By contrast, only one humanities student did so. A chi-square test of independence showed that the difference between cohorts was statistically significant, $\chi^2(1, N=24) = 3.56, p<.10$.

This result is to be expected, given that the *forms* participants prefer mathematical definitions to take are related to the *functions* they conceptualize mathematical definitions to serve. In the previous section, we showed that mathematics participants take the primary function of mathematical definitions to be setting a standard to facilitate communication and proof across a community of mathematicians. To serve this function, definitions must comprehensively specify the set of objects that can be called, say, a fraction.

Previously, I described the response of Alvin, a mathematics participant, stating that mathematical definitions serve to *facilitate communication and proof for a community of mathematicians*. When asked about the desired features of mathematical definitions, Alvin states that the definition must be precise in its language, leaving no room for ambiguity.

> Alvin: First, there can't be any ambiguities, it has to account for all possibilities, whatever the context of the definition might be. It can't be based… it has to make sense logically, so it can't be based on a contradiction… a contradictory statement or something like that, I guess.

When Alvin states that the definition must "account for all possibilities, whatever the context of the definition might be," he implies that the definition must comprehensively specify what is contained within and what is excluded from the set of objects being defined. Alvin's statement about the form of mathematical definitions is consistent with his previous statement about the function of mathematical definitions. Both are typical of mathematics participants in general.

b. ***Procedural***

The central column of **Error! Reference source not found.** shows that nearly half of humanities participants (five out of 12) stated that mathematical definitions should outline algorithmic procedures, either by providing formulae or example cases. By contrast, no mathematics students did so. A chi-square test of independence showed that the difference between cohorts was statistically significant, $\chi^2(1, N=24) = 6.32, p<.05$.

Again, this result is to be expected, given the prior finding that humanities participants tend to take mathematical definitions to serve functions related to finding answers to routine exercises. Indeed, definitional forms that outline procedures, perhaps by providing formulae and algorithms or by describing illustrative examples, are well suited to serve this definitional function. The forms that humanities participants prefer mathematical definitions to take are thus coherent with the functions they take definitions to serve.

Recall the case of Aubrey, a humanities participant, who stated that a function of definitions is to provide support in *finding answers to routine exercises*, perhaps by "formulat[ing] [an idea]…into a word problem, and translat[ing] it into an equation." Aubrey's subsequent response about the form of mathematical definitions was coherent with her statement about the function of mathematical definitions. She stated that a good definition was "something that has examples…it needs more than a word, a conceptual sentence. It needs examples, numbers, and stuff like that." Her response was coded as preferring definitions that were *procedural.* A definitional form that is procedural in this way is well suited to help one find answers to routine exercises.

c. ***Readable or understandable***

The rightmost column of Figure 2 shows that mathematics and humanities students both stated that mathematical definitions should be readable or understandable, in equal measure (five out of 12 students).

Given prior results about differences between who participants take to be the implicit audience of mathematical definitions, I expect differences in what "readability" looks like for different participant cohorts. Indeed, a definition that is readable to mathematicians would take on a different form from one that is readable to elementary students. As I will show in Study 2, "readability" is enacted differently in the actual definitions participants construct.

3.6 Discussion

The goal of this study was to uncover undergraduate students' conceptualizations of the form and function of mathematical definitions. I found marked differences between the role mathematical definitions play for mathematics and humanities undergraduates, as well as differences in the corresponding forms definitions should take in order to serve those functions. Possible reasons for, as well as implications of, these differences are further discussed in the conclusion of this dissertation.

4 A comparative analysis of undergraduates' defining activity: Generating and refining mathematical definitions in light of counterexamples

In chapter 3, I examined participants' conceptualizations of the functions and forms of mathematical definitions by considering their explicit statements. I turn now to their enacted practices of defining. In this chapter, I examine the definitions that mathematics and humanities students generate for familiar mathematical ideas, as well as the way they modify or refine those definitions when presented with problematic counterexamples.

4.1 Introduction

In Study 2, those same students were engaged with an extended interview in which they defined three elementary mathematical ideas, *fractions*, *multiplication*, and *division*; for each, they were provided the opportunity to re-define their definitional forms when presented with a series of counterexamples. The iterative activity of defining and re-defining as they considered counterexamples to their posited definitions provided further insight into each individual's conceptualizations of both definitional forms and their functions. Together, the studies point to differences in students' epistemic definitional practices, patterned sets of actions by which members of a community propose, justify, evaluate, and legitimize knowledge claims (Cetina, 1999; Kelly, 2008). Mathematics students tended to conceptualize mathematical definitions as serving to facilitate communication and proof for an audience of mathematicians. Further, they engaged in the key mathematical practice of monster-barring (Lakatos, 1976), in which mathematical definitions are iteratively refined to account for "monsters," that is, examples that problematize the definition. By contrast, the humanities students refined their definitions less readily, focusing instead on the role of definitions in supporting novices to learn content and find answers to routine exercises.

Findings reveal that while the college mathematics majors are being enculturated into mathematical practices of defining for a disciplinary audience, college humanities majors conceptualize and formulate definitions differently. In order to provide opportunities for these students to succeed in tertiary mathematics, work needs to be done engage secondary and even elementary students in defining practices.

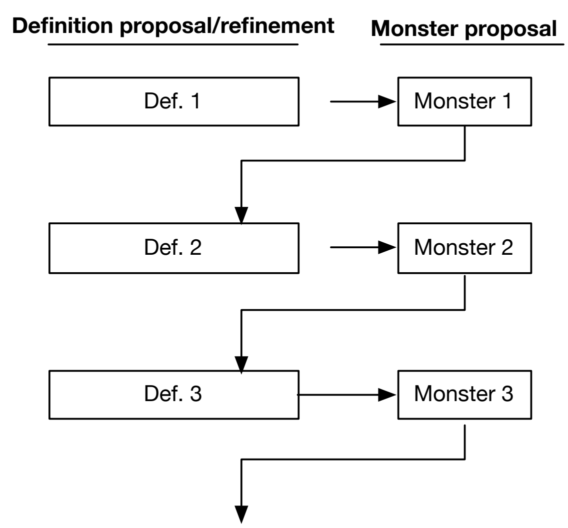
4.2 Monster-Barring

Mathematical practices (Schoenfeld, 2020), sometimes called processes *(Mathematics, 2000)*, mathematical habits of mind (Cuoco et al., 1996), or productive patterns of mathematical thinking (Schoenfeld, 2016), are recurrent forms of activity enacted by a community of mathematicians. I take definitional practices to be disciplinary forms of formulating, revising, reasoning about, and using mathematical definitions in activity. Disciplinary practices of defining are not monolithic, but rather consist of complex and interconnected strands of subpractices, referred to as aspects of definitional practice (Kobiela & Lehrer, 2015), phases in the defining process (Martín-Molina et al., 2018), or moments of work (Ouvrier-Buffet, 2006). One of these subpractices, that of monster-barring (Lakatos, 1976), is central to disciplinary defining practices and generally overlooked in K-12 curricula and standards.

In his seminal work in the philosophy of mathematics, Lakatos (1976) identifies monster-barring as a key mathematical practice for communities of mathematicians. For Lakatos, monster-barring is a definitional practice that entails the iterative refinement of mathematical definitions to account for "monsters" – examples that problematize a definition, either by satisfying the constraints of a definition that is not intended to capture them or by not satisfying the constraints of a definition that is intended to capture them. Central to the idea of monster-barring is the epistemological status of the mathematical definition as a consciously set-up convention that can undergo modifications and refinements over time, rather than as a static truism.

Figure 3 displays a simplified illustration of the monster-barring process. A definition (Def. 1) for the concept at hand is proposed. For example, *A rectangle is a shape with four sides*. An example is encountered that problematizes that definition, say a trapezoid (Monster 1). The trapezoid satisfies the properties laid out in the definition, but is not what one would classify as a rectangle. Thus the definition is refined, leading to Def. 2: *A rectangle is a shape with four sides and four right angles.* Another counterexample is proposed, say a shape with curved sides (Monster 2), and so on.

Figure 3
Illustration of monster-barring processes in which a definition is proposed (Def. 1), a monster case is encountered that problematizes the definition (Monster 1), the definition is revised accordingly (Def. 2), and so on. The iterative process continues until the set of known monster cases is exhausted.



To create learning conditions for studying the varying ways that students engage in monster-barring activity, I iteratively developed and implemented an instructional intervention that provides opportunities for students to propose definitions, reason about counterexamples, and refine their definitions as they consider those counterexamples.

4.3 A Focus on Fraction, Multiplication, and Division

I selected three familiar but challenging ideas – fraction, multiplication, and division – to engage students with defining. This choice was an important one. Indeed, Zaslavsky & Shir (2005) showed that the selection of objects of definitional activity makes a difference both in the criteria that participants apply to the evaluation of definitions and the types of reasoning in which they engage when considering alternative definitions. I applied two criteria in selecting objects of defining.

First, I aimed to extend prior work on definitional practices to new domains of mathematical content. Research into how students define dates as far back as Fawcett's (1938) implementation of an extended geometry course that provided meaningful experiences for students to define and refine their definitions for geometric concepts. Like Fawcett's study, much of the research on student engagement in defining has largely focused on geometric concepts such as triangles (Zandieh & Rasmussen, 2010), polyhedra (Lehrer & Curtis, 2000), straightness (Ouvrier-Buffet, 2006), and increasing functions (Zaslavsky & Shir, 2005). There is less research looking at students engaged in defining non-geometric concepts. Notable exceptions include Saxe et al. (2015), a study of a classroom learning about fractions, in which definitions for concepts such as the unit interval or equivalent fractions played a central role in student learning and Leikin & Winicki-Landman (2001), a study in which in-service mathematics teachers discussed the relationships between alternative definitions for the Fibonacci sequence. A few studies go even further, providing insights into how students generate their own definitions for non-geometric concepts. Ball (1993) offers an illuminating description of a student who, while learning about even and odd numbers, proposes a definition for a new type of number. Saxe et al. (2010) describe the emergence of agreements (or quasi-definitions) regarding number line principles and conventions in a tutorial "communication game." In this study, I extend a body of research that is largely focused on defining geometric concepts to the realm of non-geometric concepts.

As a second criteria for selection of content, I looked for mathematical objects with which both student groups (undergraduate mathematics and humanities students) would be familiar. At the same time, polysemous concepts would serve as a rich context within which to engage students in monster-barring activity. Fractions, for example, can be conceptualized as parts of a whole (3 parts out of 4), as the ratio or relative magnitude between two numbers (a ratio of 3 to 4), as the quotient of a division operation (3÷4), as an operator that acts on other quantities (3/4 of *x*), or as a measure (3/4 of a unit). Each of these fraction subconstructs (Behr et al., 1983) may serve as fertile ground for the emergence of a student-generated definition. The multiplicity of interpretations offers ample room for students who are familiar with the concepts under consideration to revise their definitions.

4.4 Methods

4.4.1 Participants and Setting

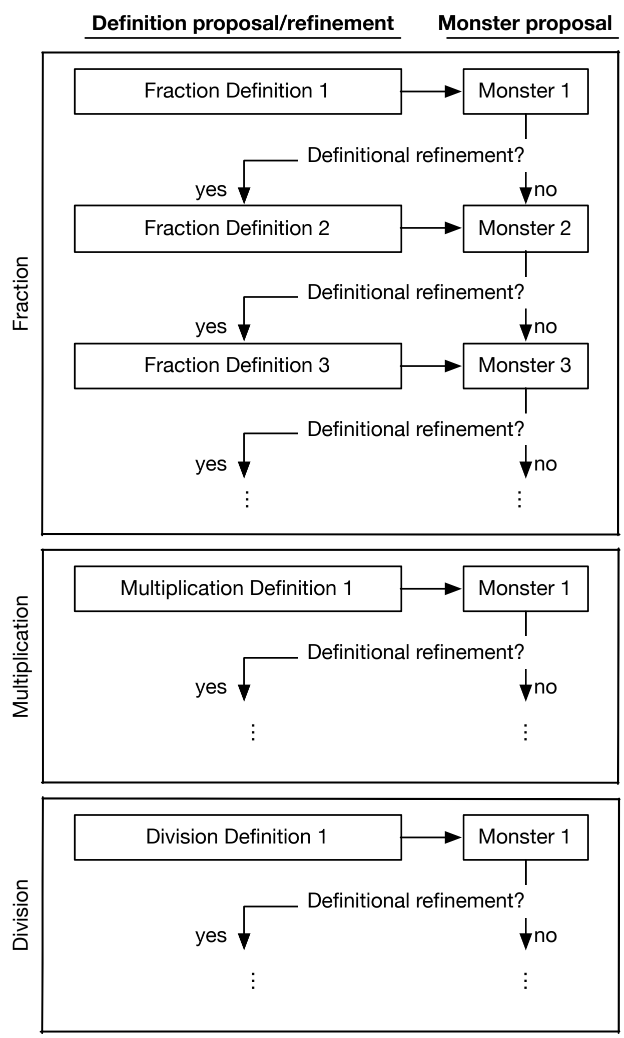
The same 24 participants from Study 1 subsequently took part in Study 2, during the same session and in the same location. General contextual information and data-collection procedures are outlined in the description of Study 1. Data sources included each participant's written work and audiovisual recordings of each interview.

4.4.2 Procedures

Each monster-barring interview consisted of three iterations, as participants engaged in formulating and iteratively refining definitions for *fraction*, then *multiplication*, and finally *division*. The iterative process of definition formulation and refinement for the three focal concepts is displayed in the three panels of Figure 4. In each round of the activity, I first asked the participant to propose a definition for the idea at hand, say fractions (Figure 4, Fraction Definition 1). Based on that definition, I proposed a counterexample that problematized their definition (Figure 4, Fraction Monster 1), asking the interviewee to explain their definition in light of the example. In response, the participant either revised their definition to account for the counterexample or explained the example without reference to the definition on the table. In either case, after some discussion, a new counterexample was proposed (Figure 4, Fraction Monster 2), often by the interviewer, but sometimes by the interviewee. Thus continued the cycle until both interlocutors exhausted their store of counterexamples and the participant was satisfied with their definition.

Figure 4

Sequence of activity in each interview, consisting of three iterations of the monster-barring activity – defining fraction, multiplication, and division.



The interview protocol was the same across participants, with variation in the proposal of monsters, which were selected based on the participant-generated definition at hand. In previous iterations of the project, I generated an inventory of prototypical student-generated definitions and developed, for each, a standard counterexample to problematize that definition. This list of definitions and associated counterexamples (see Appendix 1) was used to standardize monster-proposals in the Study 2 interviews. Data collected from the interviews confirmed that the number of monster proposals remained the same across cohorts.

As an example, a common definition students proposed for fraction was *parts of a whole*. In these cases, I proposed $\frac{4}{3}$ as an example to consider. This example would provoke consideration of the fact that the magnitude of a fraction might exceed one whole. If a student defined division as *partitioning into a number of equal parts*, I proposed $1 \div \frac{1}{2}$ to highlight the limitations of a partitive model of division, in which the dividend represents a quantity that is partitioned into a number of equally sized parts represented by the divisor. Considering this counterexample, the participant might revise their definition in terms of a quotative model of division – one in which the dividend represents a quantity that is partitioned, the divisor represents the size of each part, and the quotient is the number of resulting parts.

Note that I refer to all examples proposed over the course of the interviews as counterexamples or monsters, whether they are genuine instances, non-instances, or of ambiguous status. From a strictly Lakatosian perspective, monsters are specifically cases that must be excluded from the scope of a proposed definition. Because I left the classification of proposed cases as instances or non-instances to the participants in my interviews, I do not distinguish between the two in my analyses. In fact, the determination is often a matter of convention (for example, whether $1\frac{2}{3}$ is a fraction).

4.4.3 Analytic Approach

To understand the ways in which participants participated or did not participate in monster-barring practices, I analyzed features of the definitions that they generated, the frequency with which they refined those definitions, and the frequency with which they proposed monster cases to problematize their own definitions.

To analyze the definitions that students formulated, I first compiled a list of all the definitions that were proposed by participants. These included definitions for all three focal concepts – fraction, multiplication, and division. They also included all iterations of participants' definitions, meaning the first definition that was proposed for a concept, the revised definition after accounting for a counterexample, the revised definition after that, and so on. This resulted in 159 definitions, or 6.6 definitions per participant. To illuminate the type of definition participants formulated (example-based description or formal mathematical definition), a research assistant and I coded the full list of definitions for whether the body of each definition contained an illustrative example. To understand the implicit audience for whom the definitions were being formulated, we coded each definition for the type of language that was used, distinguishing between definitions that made use of mathematical terms (such as *reciprocal, numerator, denominator,* and *decimal*) and symbols (such as $\div$, b^{-1}, $\mathbb{Z}$, $\neq$, and x) and those that did not. The relevant coding scheme is outlined in Table A2. I determined inter-rater reliability by jointly coding 20% of the data corpus independently from a research assistant and then coding the full corpus independently, with 98.4% agreement.

As a second line of analysis, I tracked the frequency with which participants in each cohort (a) refined their definitions and (b) proposed monster cases. To do so, we used video analysis software (MaxQDA) to mark the video recording of each interview for when the participant either (a) made a written proposal or refinement of their definition for one of the focal concepts (fraction, multiplication, division) or (b) brought up an instance or non-instance of the focal idea under discussion in light of which to discuss or refine their definition.

4.5 Results: Participant-generated definitions

As described previously, mathematical definitions used in different contexts serve differing functions and, accordingly, take on different forms. Broadly, we may distinguish between example-based descriptions serving to advance learner's understanding of an idea and formal mathematical definitions serving to facilitate disciplinary communication. While informal definitions draw on illustrative examples, formal definitions adhere to the criterion of minimality. Further, while informal definitions draw on everyday language, formal definitions make use of disciplinary terminology – words and symbols that have specified meanings within the discipline. Analyses of the definitional forms that participants formulated over the course of the defining activity focused on these two key features – the inclusion of illustrative examples and the use of disciplinary terminology.

4.5.1 Definitions containing illustrative examples

I analyzed participant-generated definitions for whether the body of the definition contained illustrative examples. I found that while nearly half of the humanities participants formulated definitions that included example cases, only one mathematics student did so (see Figure 5). A chi-square test of independence did not show a significant statistical difference between the two groups, $\chi^2(1, N=24) = 3.6, p=.06$.

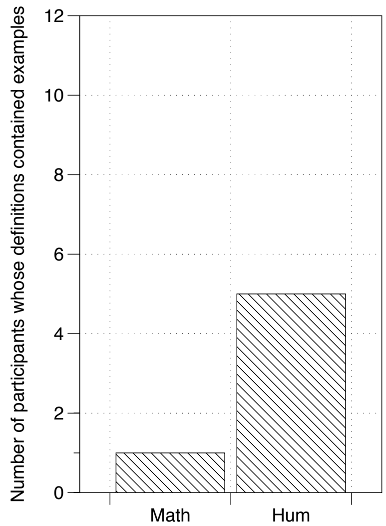
Participant-generated definitions were coded as including example cases if they contained any reference to specific numerical instances of the idea under definition. For example, the following definition, proposed by a humanities student Jimmy, refers to the case of the fraction $\frac{3}{2}$.

Def 1. A fraction is another way to represent # without decimal points. Instead $\frac{3}{2}$ it's 1.5. If you divide 3 by 2 you get 1.5.

In addition to the description of fractions as "another way to represent" decimal numbers, the definition contained the illustrative case of $\frac{3}{2}$ as a fraction representation of the decimal 1.5.

Figure 5

The number of mathematics and humanities participants who formulated definitions that contained illustrative examples

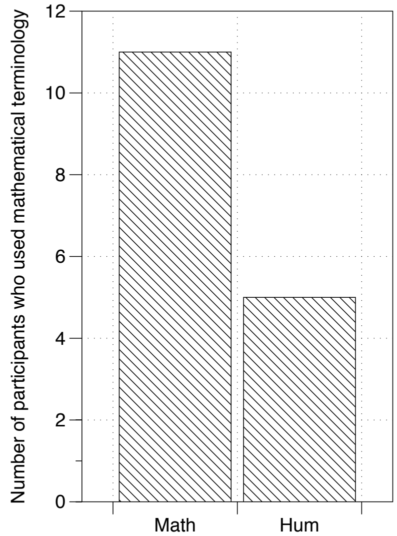


4.5.2 Definitions using mathematical terminology

My analyses of participant-generated definitions distinguished between definitions that employed mathematical words such as *reciprocal, numerator, denominator,* and *decimal*, and symbols such as $\div$, $\mathbf{b^{-1}}$, $\mathbb{Z}$, $\neq$, and $\boldsymbol{x}$, and those that did not. I found that while nearly all mathematics participants made use of mathematical terminology in their definitions, less than half of the humanities participants did so (see Figure 6). A chi-square test of independence showed the difference between the two groups to be statistically significant, $\chi^2(1, N=24) = 6.8$, $p<.05$.

Figure 6

The number of mathematics and humanities participants who formulated definitions that made use of mathematical terminology



Consider the following contrasting definitions for division proposed by a mathematics student, Alvin, and a humanities student, Andrew, respectively.

Def 2. Take two numbers a, b. Division, denoted by $a \div b, \frac{a}{b}$, or $\frac{a}{b}$, is defined as taking a and multiplying it by its reciprocal, inverse, b^{-1}, or $\frac{1}{b}$. Essentially, $a \div b$, $a \times \frac{1}{b}$ or $a \times b^{-1}$. Note: $b \neq 0$

Def 3. How to find the presence of a number in another number.

Alvin's definition both makes use of symbols that are largely mathematical, like ÷, ×, and ≠, and word forms like *reciprocal* and *inverse*. Further, Alvin makes use of a distinctly mathematical practice of using variables (in this case, *a* and *b*) to illustrate ideas. By contrast, Andrew's definition did not contain any of the predetermined markers and thus was coded as containing no disciplinary words or symbols.

Though the results of Study 1 showed that both participant cohorts voiced their preference for mathematical definitions that were readable, the findings presented here show that they enacted this preference differently. Differences in the enactment of "readability" may be due to differences in participants' own mathematical literacy as well as their projected audience for their definitions.

4.5.3 Illustrative cases of participant-constructed definitions: Contrasts between a mathematics and a humanities participant

In this section, I present the cases of two students to illustrate the ways in which the participant groups differed in the definitions they constructed and to connect these differences with the findings from Study 1 regarding each participant's conceptualizations of the function and forms of mathematical definitions.

I chose AC, a mathematics participant, and Aubrey, a humanities participant, to represent their cohorts because for each of the codes applied in both Studies 1 and 2, they received the same code as most of their cohort. In Study 1, I found that AC conceptualized mathematical definitions as *supporting communication and proof among mathematicians*. Further, he voiced a preference for mathematical definitions that were *comprehensive* and *readable*. In the present study, I found that he used mathematical terminology in the definitions he constructed. I found that Aubrey conceptualized mathematical definitions as *advancing learners' understanding* and *finding answers to routine exercises*. Further, she voiced a preference for definitions that were *procedural* and *readable*.

My analysis revealed that AC, who took the primary function of definitions to be facilitating proof for an audience of mathematicians, formulated definitions that contained complex disciplinary words and symbols, did not include illustrative examples, and were not meant to assist the reader to find answers to routine exercises. By contrast, Aubrey formulated a descriptive definition that provided multiple interpretations of the concept, described illustrative examples, and avoided disciplinary words and symbols.

Consider the following definition for division generated by AC during the defining activity.

Def. AC1. Take two numbers a, b. Division, denoted by $a \div b$, $\frac{a}{b}$, or $\frac{a}{b}$, is defined as taking a and multiplying it by its [*sic*] reciprocal, inverse, b^{-1}, or $\frac{1}{b}$. Essentially, $a \div b$, $a \times \frac{1}{b}$ or $a \times b^{-1}$. Note: $b \neq 0$.

AC makes ample usage of mathematical symbols (e.g., b^{-1}, $\neq$) and terminology (e.g., reciprocal, inverse). It seems clear that, in contrast to other participants' definitions, the audience of his definition is not school-age children. His concern is not for helping the reader find the quotient to a division problem like $3 \div 2$, as evidenced by the lack of narrative and examples. The note addended to his definition ($b \neq 0$) implies that he is concerned with excluding problematic cases, while the inclusion of multiple representations of division – $a \div b$, $\frac{a}{b}$, $\frac{a}{b}$, $a \times \frac{1}{b}$, and $a \times b^{-1}$– implies that he is concerned with comprehensively covering all cases of division.

By contrast, the following is a definition for division generated by Aubrey during the defining activity.

Def. Aubrey1. <u>Division</u>

- Dividing something (or a number) into groups / Opposite of multiplication, Divide instead of multiplying
- (12 Divided by 6) → You would figure out how to divide 12 candies to the 6 pple [*sic*]
- Dividing a number say 12 ~~by 6~~

This is the only definition for division that Aubrey proposes. She writes it down at the beginning of the division portion of the interview and does not change it throughout the interview. Within this display, we see both necessary elements of definitions that Aubrey had identified during the pre-interview. First, she notes down a general description of the idea being defined, or what she might call a "a conceptual sentence" – 'Dividing something (or a number) into groups / Opposite of multiplication, Divide instead of multiplying.' She follows this with the second necessary element of a definition: "examples and numbers and stuff like that." The second bullet point within her definition consists of an example of division, $12 \div 6$, with an explanation of how to divide, "you would figure out how to divide 12 candies to the 6 [people]."

Interestingly, Aubrey presents two different models of division in the conceptual portion of her definition: the partitive model, in which a fixed quantity is partitioned into a set number of groups, and the inverse multiplication model, in which the quotient of division is conceptualized as the missing term in a multiplicative equation (Fischbein et al., 1985). This is in line with her definitional goal of advancing understanding. In contrast to a definer whose goal for the definition is, say, to form a basis for proof, minimality is not a concern for Aubrey's definition.

Presenting as many conceptualizations of division as possible can only provide the definition-reader with more opportunities to understand.

The contrast between AC's and Aubrey's defining activity, each representative of their cohort, is summarized in Table 1. While AC described mathematical definitions that set a standard to facilitate proof and communication across a community of mathematicians, Aubrey described definitions that support learners' understanding and help them to solve routine exercises in a teaching–learning context. While AC constructed definitions that used disciplinary words and symbols to comprehensively and precisely delimit the boundaries of what counts as, in this case, division, Aubrey stated that mathematical definitions should explain and thus contain examples.

Table 1

Definitional Form–Function Relations Constructed by Typical Mathematics and Humanities Participants

		Mathematics Participant, AC	Humanities Participant, AF
Study 1	Function of definition	Communication and proof *"outline a set of standards for what you can or can't use when you're trying to deduce something."*	To advance understanding; to get answers to routine problems *"to help people understand…, say into a word problem, and translate it into an equation."*
	Audience of definition	Community of mathematicians *"It's for everyone who works in the field."* A definition *"makes collaboration a little easier."*	Individual learners Purpose of definitions is *"to help people understand…"* A definition *"needs more than a conceptual sentence, it needs examples and stuff like that."*
	Desired features of definition	Comprehensive, precise, and unambiguous *"There can't be any ambiguities, [the definition] has to account for all possibilities, whatever the context of the definition might be."*	Include examples *"A good definition would probably be something that has examples....For division, you have twelve pieces of candy between three people. Twelve divided by three is four."*
Study 2	Features of participant-generated definitions	Used disciplinary words and symbols; excluded monster cases b^{-1}, $\neq$, reciprocal, inverse	Included examples; outlined procedures *"(12 Divided by 6) → You would figure out how to divide 12 candies to the 6 pple* [sic]*"*

4.6 Results: Participants' definitional refinements

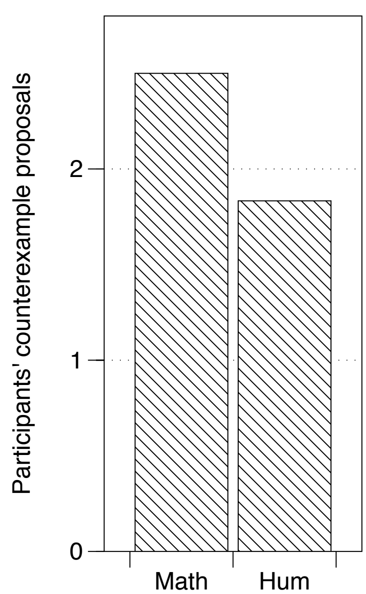
To understand participants' monster-barring activity, I considered the number of counterexamples presented to each participant and the number of times he refined his definitions in response to those monsters.

4.6.1 Counterexample presentations

Data collected from the interviews confirmed that participants in the two cohorts encountered a similar number of counterexamples. Indeed, the average number of counterexamples encountered by each mathematics participant was 11.82 (*SD* = 1.23), while the average number of counterexamples encountered by each humanities participant was 11.67 (*SD* = 3.04).

A major difference across the two cohorts was the source of these counterexamples. Counterexamples arose in two ways in the interviews. Most counterexamples were presented by the interviewer as part of the interview protocol. Additional counterexamples were spontaneously proposed by participants themselves over the course of discussions. Figure 7 gives information about this latter type of counterexample. The figure shows that the number of counterexamples spontaneously proposed by the average mathematics participant (*M*=2.5, *SD*=1.83) was greater than the number proposed by the average humanities participant (*M*=1.8, *SD*=2.04). While not statistically significant, the difference between groups may suggest that the mathematics students were more consciously concerned with monsters and with reflecting on the limitations of their definitions when presented with monsters.

Figure 7
The average number of counterexamples proposed by each mathematics and humanities participant



4.6.2 Definitional refinements

I expected the mathematics participants' definitions to undergo more refinements than humanities participants' for two reasons. First, mathematics participants had more exposure to disciplinary practices of defining in which definitions undergo iterative refinements. Second, I expected mathematics participants to have deeper understandings of the ideas being defined and, relatedly, a multiplicity of interpretations for each concept. Each such interpretation (for example, of fractions as parts of a whole, as ratio, or as quotient) was a resource for formulating an alternative definition for each idea.

Figure 8 displays the average number of definitional refinements made by each mathematics and humanities participant. This includes the cumulative number of definitions across the three focal concepts and excludes the preliminary definitions proposed for each concept. For example, if a student proposed a definition for fraction and refined it twice, then proposed a definition for multiplication and refined it once, then proposed a definition for division and did not refine it, this would be counted as three definitional refinements.

Figure 8

The average number of definitional refinements made by each mathematics and humanities student

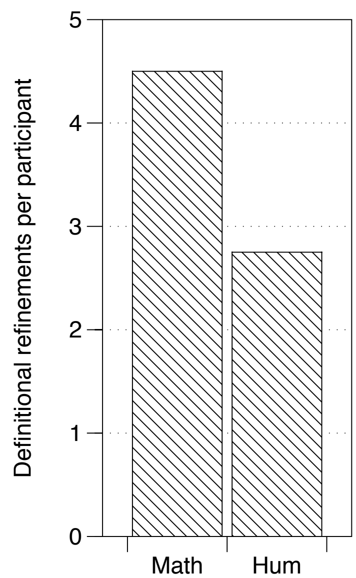


Figure 8 shows that mathematics participants refined their definitions significantly more often (M=4.50, SD=2.70) than humanities participants (M=2.75, SD = 1.60), $t(11) = -1.94, p < .05$ (one-tailed.) On average, mathematics participants refined their definitions 61% more often despite the fact that each group was faced with a similar number of counterexamples (see Figure 7). Indeed, comparing the number of monsters proposed to the number of definitional refinements, we see that mathematics participants refined their definitions once for every 2.89 monsters they encountered, while humanities participants refined their definitions once for every 4.38 monsters.

4.6.3 Illustrative cases of monster-barring activity over time: Contrasts between a mathematics and a humanities participant

To further illumine the monster-barring activity of participants over time, I examined the cases of two students. To highlight differences between the mathematics and humanities cohorts, I chose cases in which differences in the number of definitional refinements was extreme. Lee, a mathematics student, very frequently refined his definitions over the course of the activity (with 16 refinements). Aubrey, a humanities student, only refined her definitions twice.

Figure 9 consists of timelines displaying the monster-barring moves of Lee (top) and Aubrey (bottom) across the duration of each interview. The three panels represent the three activity segments of each interview – defining fractions, defining multiplication, and defining division. The upper timeline within each panel displays definitions proposed (represented by bars) and monsters encountered (represented by dots) by each participant over the course of the activity.

Figure 9

Timelines showing the monster-barring moves of two paradigmatic students

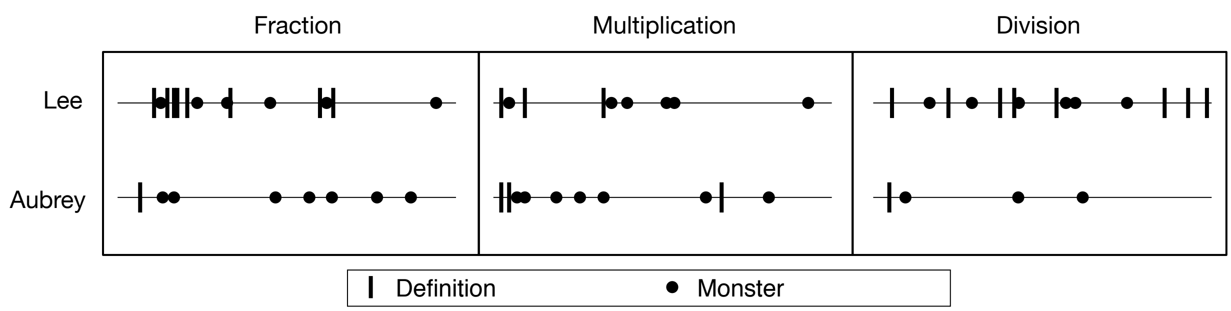


Figure 9 shows that each interviewee encountered similar numbers of monsters over the course of the interview (18 for Lee and 17 for Aubrey). Further, Figure 9 shows that many of the monsters that arose in Lee's interview (represented by dots on the upper timeline) were followed soon after by a revision of his definition (represented by bars on the upper timeline). By contrast, all but one of the monsters Aubrey encountered (represented by dots on the lower timeline) did not lead to definitional change (see the lack of bars on the lower timeline).

Indeed, Lee's definitions underwent refinements often, resulting in 19 distinct definitions. In some cases, he revised a definition to account for a monster case that was proposed by the interviewer. For example, when he was presented with the counterexample $\frac{\pi}{3}$, he revised his definition for fractions to specify that the numerator and denominator must be rational. In other cases, he revised a definition in response to a counterexample he himself had brought forward. For example, in considering his definition for fractions, he wondered aloud about the case of non-integers (this is represented as a monster/dot on Lee's timeline in Figure 9.) Ultimately, Lee decided that the numerator and denominator must be integers and revised his definition accordingly. Occasionally, Lee revised his definition unprompted, simply to clarify (represented as a bar that is not preceded by a dot on Lee's timeline in Figure 9.) For example, having written a version of his definition for fractions (Def 4 below), he asked the interviewer, "Should I write it more clearly?" and proceeded to amend his definition (resulting in Def 5 below).

Def 4. Fraction is a number [that] shows how much parts/numerator you have from a pool.

Def 5. …denotes a relative amount when how much you have and is out there can be represented by an integer.

In each of these cases, Lee readily refined his definitions.

By contrast, Aubrey more or less maintained the same definition throughout the course of each defining activity, resulting in a total of 5 distinct definitions. For example, her initial definition for fractions was as follows.

Def 6. A mathematical function that is not a whole number but part/proportion of a whole number.

She was then presented with seven monster cases, including an improper fraction ($\frac{5}{3}$), a mixed number ($1\frac{2}{3}$), an integer (4), an undefined number ($\frac{4}{0}$), and an irrational fraction ($\frac{\pi}{2}$). While she classified some of these as instances of fractions and some as non-instances, none of them prompted a revision of her definition.

4.7 Discussion

The goal of this study was to uncover similarities and differences in mathematics students' and humanities students' definitional practices, providing insights into how they formulated definitions, proposed problematizing cases, and revised their definitions in light of these cases. I found that the two groups of students were engaged in distinct practices. Evidence of this can be found both in terms of the form of definitions participants formulated, and in how they revised and problematized their definitions.

First, mathematics students tended to formulate definitions seemingly aimed at a more disciplinary audience, employing specialized mathematical terminology and symbols in their definitions. Meanwhile, humanities students formulated definitions that seemed to be aimed at more pedagogical functions, utilizing illustrative examples within the body of their definitions.

Second, mathematics students were more likely to revise their definitions and to spontaneously propose problematizing cases as they reasoned about their definitions. This suggests, corroborating findings from the study outlined in the previous chapter, that they conceptualized the definitions as needing to comprehensively specify a set of instances, thereby excluding non-instances.

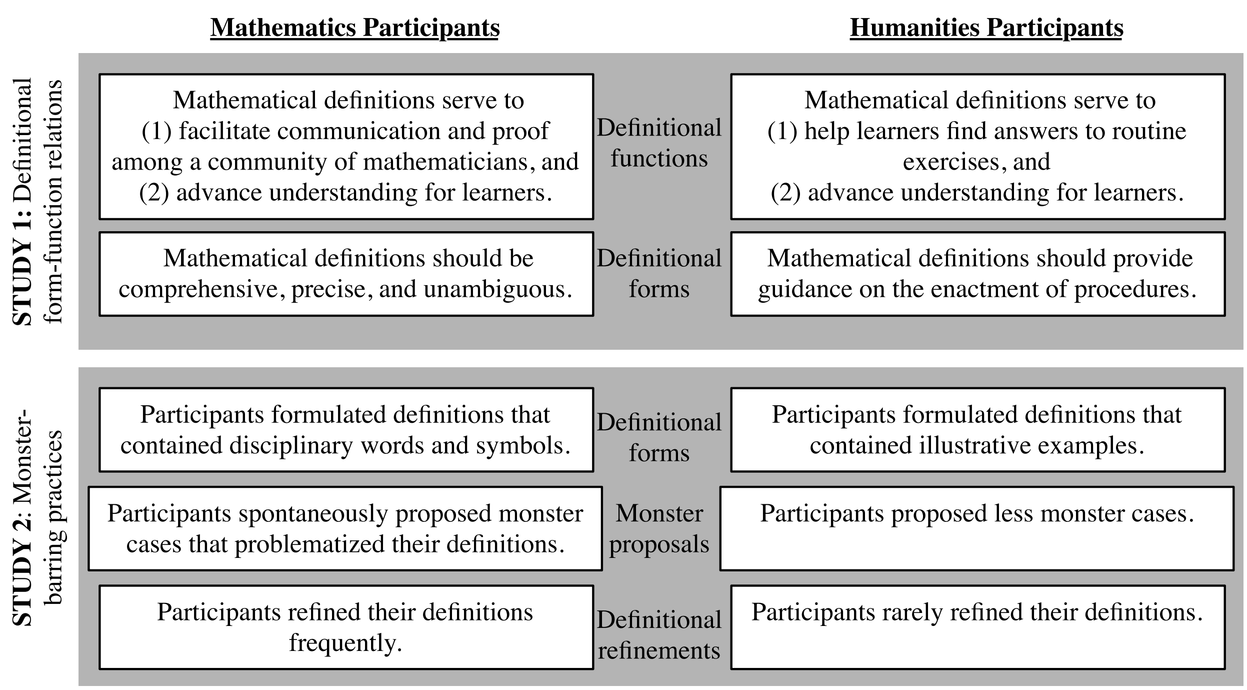
5 Discussion

This dissertation set out to investigate similarities and differences in the definitional practices of undergraduate mathematics and humanities students, with a focus on the iterative construction and refinement of definitions as they considered counterexamples. The second chapter of the dissertation provided a theoretical framework and analytic tools for carrying out this investigation. Findings from the empirical studies presented in the third and fourth chapters showed that students were engaged in two different epistemic practices, both of which utilize what we loosely call definitions. Further, these divergent practices are most plausibly a consequence of the pedagogical contexts within which students had been exposed to mathematical definitions. While this statement may be unsurprising, my findings provide a description of how school mathematics and college mathematics lead to two different epistemic practices of defining.

Evidence of two different epistemic practices of defining is shown in Figure 10, which summarizes differences between mathematics participants (left-hand column) and humanities participants (right-hand column) in their stated conceptualizations of the function and form of mathematical definitions in response to direct questioning (Study 1, upper panel) and in their enacted activity as they engaged in the formulation and iterative refinement of definitions (Study 2, lower panel).

Figure 10

Differences between mathematics participants (left) and humanities participants (right) in their conceptualizations of definitional form-function relations (Study 1) and in their monster-barring practices (Study 2)



5.1 Humanities students

In K-12 mathematics education, mathematical definitions are often presented in the context of teaching new content and developing associated mathematical skills. Definitions are provided to students, while the features and functions of mathematical definitions are rarely addressed in an explicit way. Further, the process of defining is often occluded. Without opportunities to define or witness defining processes, students come to see mathematical definitions as ready-made truth statements to be learned and then used as guidance for solving routine exercises. In my data, I encounter evidence that humanities students formulate definitions based on a conception of mathematical definitions as fully formed entities whose function is to help learners solve routine exercises.

Evidence of this is seen, first, in the findings of Study 1 regarding humanities students' conceptualizations of definitional form-function relations. Unlike mathematics participants, humanities participants tended to conceptualize mathematical definitions as forms that help learners solve routine exercises, often in a classroom context. They accordingly preferred mathematical definitions that were procedural, that is, provided instructions on how to enact procedures, perhaps containing example cases within the definition. Humanities students also tended to mention the pedagogical function of mathematical definitions – that mathematical definitions help to advance a learners understanding of the concept being defined. Relatedly, they described readability as a preferred feature of mathematical definitions, in the sense that a reader unfamiliar with the idea would understand the words and symbols used in the definition.

Findings from Study 2 showed that humanities students enacted these preferences in their defining activity by often constructing definitions that: (1) contained illustrative example cases, furthering their ability to help readers solve procedural problems; and (2) avoided discipline-specific words and symbols, supporting their readability.

A further source of evidence for divergent definitional epistemic practices was provided in Study 2. Analyses showed that humanities students were less likely to consider and modify their definitions when presented with counterexamples that problematized them. Instead, encountering an instance of a mathematical object that did not conform to their definition was often not seen as a problem. Indeed, if a definition aims to simply provide guidance for solving problems, ambiguities abound and the boundary between what is and what is not, say, a fraction can be hazy.

5.2 Mathematics students

At the college level, mathematical definitions continue to play a role in helping students to develop a more robust understanding of a given idea and aid in solving routine problems. Yet, in college level mathematics courses, students engage in (or at least are exposed to) new forms of mathematical practice within which mathematical definitions are used to solve other types of problem – problems of communication and proof. In this context, mathematical definitions set a standard that allows for coherent communication across a community of mathematicians. In my data, I encounter evidence of mathematics students' engagement in definitional practices based on a conception of mathematical definitions as specifying usage of a term, whose function is to create a standard for communication across a community.

Evidence for this is seen, first, in the findings of Study 1 regarding participants' stated conceptualizations of definitional form-function relations. Like humanities students, mathematics participants took definitions to serve to advance learner's understanding of mathematical ideas. Yet, mathematics participants also tended to conceptualize definitions as serving to facilitate communication and proof by setting a standard across a community of mathematicians. Relatedly, they tended to prefer mathematical definitions that comprehensively, precisely, and unambiguously specified the sufficient and necessary requirements for a potential candidate to be classified as an instance of the concept.

Analyses of the definitions that mathematics participants constructed (Study 2) confirmed that they were formulating definitions for a disciplinary audience. Indeed, these students often included disciplinary terminology in their definitions that would not have been accessible to novices.

Finally, mathematics students' monster-barring moves provided added evidence that they were engaged in a fundamentally different type of practice than were their peers in the humanities. We found that, to reduce ambiguities brought to light by monster cases, mathematics students refined their definitions often. With each definitional refinement, mathematics students worked towards a definition that could comprehensively and precisely specify how a word form (fraction, multiplication, or division) could be used.

5.3 Concluding remarks

This dissertation presents a preliminary study of different definitional form-function relations constitutive of two distinct epistemic practices of defining enacted by undergraduate students. Another study with quantitative analyses of a large sample size would be needed to evaluate for additional statistical significance of similar results. Further, participants in these studies were students at a highly ranked university, implying that they had a strong high school education. Studies of college students from other settings would be needed to provide generalizability to these results.

These preliminary findings imply that while college mathematics majors are being enculturated into mathematical practices of defining for a disciplinary audience, humanities majors are not. To open pathways for a more diverse body of students to pursue STEM fields, work needs to be done to provide opportunities for secondary and even elementary students to participate in disciplinary defining practices and, consequently, to obtain a richer understanding of what it means to do mathematics. To support student engagement in defining as a disciplinary practice, teachers might invite students to formulate definitions, even for elementary mathematical ideas, and problematize those definitions, perhaps by deploying a toolkit of counterexamples. This suggests that in addition to the standard definitions for foundational mathematical ideas presented in K-12 curricula, pre-service teachers might be well served by including monsters cases and the ability to deploy them effectively to problematize alternative definitions as part of their pedagogical content knowledge – the "special amalgam of content and pedagogy" needed for teaching a subject (Shulman, 1987).

6 Conclusion

This dissertation investigates how undergraduate students engage in mathematical practices of defining – formulating, reasoning about, and refining definitions. Although a rich literature examines the ways in which individuals define mathematical terms or evaluate alternative definitions, few studies consider how mathematical definitions are created and continually refined over time as a result of a collective discourse (see e.g., Lakatos, 1976). To explore the potential complexities of collective discourse around mathematical definitions at the undergraduate level, my dissertation compares how mathematics and humanities students define elementary mathematical ideas such as fractions, multiplication, and division.

Drawing on a sociocultural paradigm presented in the second chapter of the dissertation, my work takes mathematical definitions to be shifting cultural forms that serve varying functions in collective activity.

In the empirical study outlined in the third chapter, I examine how participants think about the function of mathematical definitions and, relatedly, what features might characterize a mathematical definition. Data from a series of task-based semi-structured interviews revealed that mathematics majors view the primary function of definitions to be setting a standard for communication and proof across a disciplinary community. By contrast, humanities majors tend to see definitions as serving individual learners, often by providing procedural instructions for solving routine exercises. Relatedly, the two groups revealed different preferences for the features of mathematical definitions, for example, whether definitions should contain illustrative examples.

The fourth chapter of my dissertation examines how undergraduate students in mathematics and the humanities may differentially engage in the process of defining elementary mathematical objects. Using a design-based research approach, I developed a pedagogical intervention to engage students in a Lakatosian iterative refinement of definitions in light of problematizing counterexamples. I find evidence of two distinct approaches to defining mathematical objects. While mathematics students tend to frequently refine their definitions, humanities students keep their definitions relatively static. Further, mathematics students produce definitions that use relatively complex mathematical language and symbols, suggesting that they do not primarily consider pedagogy and accessibility for new learners. By contrast, humanities students tended to formulate definitions as procedural instructions (for example, defining multiplication by describing the steps for multiplying fractions) or by example (for example, defining multiplication by presenting the case of $2 \times 3 = 6$.)

Taken together, the dissertation research makes both theoretical and pedagogical contributions to the field of educational research. With regards to theory, it recasts mathematical definitions as cultural forms that shift through time to serve collective functions. Pedagogically, the research informs the design of an approach to engaging students in formulating, reasoning about, and revising definitions.

In a next project in this area, I plan to investigate the ways in which definitional practices – formulating, reasoning about, and refining mathematical definitions – can be leveraged to support students' understandings of the core mathematical ideas, with a focus on how this work supports minoritized students' mathematical identities. In the context of a classroom teaching experiment, I plan to engage precalculus students in a series of activities to collectively posit and refine definitions related to functions. Mixed methods analyses of classroom interactions aim to illuminate the micro-processes by which conceptual understanding and disciplinary practices mutually support each other in student activity. A second line of analysis aims to uncover the effects of engagement in collaborative, iterative, and authentic processes of defining around foundational mathematical ideas. In particular, I am interested in implications of these activities for students' ideas about what mathematics is and who gets to do it. This work will provide insight into productive approaches to teaching a critical and foundational mathematical domain in the algebra-to-calculus transition. Further, the research will illustrate the coherence between mathematical content and practice goals — goals that are emphasized but often treated as distinct in instructional standards and educational research.

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Appendix: Anticipated Definitions and Associated Monsters

To standardize my proposal of monsters during the defining activities of each interview, I developed and utilized a list of anticipated student-generated definitions and associated counterexamples for the three focal concepts — fraction, multiplication, and division. This list was developed and implemented iteratively over the course of a series of pilot interviews.

The list of anticipated student-generated definitions was generated by analyzing for prototypical definitions that came up through observation and analysis of students' defining activity as they engaged in defining and refining their definitions for the three focal concepts.

The list of associated monsters was developed to best problematize each prototypical definition by surfacing discrepancies between the set of instances specified by each definition and students' intuitive ideas of the concept as applied to different cases. For example, $\frac{2}{0}$ was proposed to students who defined fractions as "one number over another" to surface the fact that the denominator of a fraction must be nonzero. While some of these "monster cases" are non-instances of the concept ($\frac{2}{0}$ is not a legitimate fraction) that prompt the definer to constrain their definition, some of them are genuine instances ($\frac{5}{3}$ is a fraction) that prompt the definer to broaden their definition. Other monster cases are of ambiguous status (are mixed numbers like $1\frac{2}{3}$ fractions?). In these cases, it was left to the interviewee to set up a definitional convention that will either exclude or include them in the set of instances that can be called, say, a *fraction*.

This list of anticipated definitions and associated monsters is not exhaustive. In cases where students proposed definitions that we did not anticipate, the interviewer spontaneously generated a monster to problematize it.

Table 2

Anticipated student-generated definitions and associated problematizing monster cases

Concept	Anticipated Student-Generated Definition	Monster Case
Fraction	Part of a whole	$\frac{5}{3}$
	Made up of a numerator over a denominator	$\frac{\pi}{3}$
	Made up of an integer over another	$\frac{2}{0}$
	Made up of an integer over a non-zero integer	$1\frac{2}{3}$
	A decimal number	$\pi = 3.14159\ldots$
	The result of dividing two whole numbers	$1\ (3 \div 3)$
Multiplication	Adding the left-hand number to itself repeatedly, a number of times specified by the right-hand number.	$\frac{1}{2} \times 3$

	Adding the right-hand number to itself repeatedly, a number of times specified by the left-hand number.	$3 \times \frac{1}{2}$
	Adding a number to itself repeatedly. OR Stretching a number by a factor represented by another number.	$\frac{1}{2} \times \frac{1}{4}$
Division	Partitioning one quantity into certain number of parts	$6 \div 3$
	Partitioning *x* into *y* parts	A figure with unequal parts
	Partitioning *x* into *y* equal parts	$1 \div \frac{1}{2}$
	How many *y's* fit into *x*	$\frac{1}{2} \div 1$
	How much of *y* fits into *x*	$1 \div \frac{1}{2}$
	What number, when multiplied by *y*, is equal to *x*	none