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On Viazovska's modular form inequalities

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Viazovska proved that the E_8 lattice sphere packing is the densest sphere packing in 8 dimensions. Her proof relies on two inequalities between functions defined in terms of modular and quasimodular forms. We give a direct proof of these inequalities that does not rely on computer calculations.

Jacobi thetanull function | Eisenstein series | modular form | inequality | sphere packing

Significance

Viazovska's 2017 solution of the sphere packing problem in 8 dimensions used modular forms to solve a fundamental problem in geometry. A key element of Viazovska's proof consisted of proving two inequalities satisfied by certain functions defined in terms of modular forms. The inequalities were verified through computer calculations, but the approach appeared complicated and lacking in conceptual insight. This paper gives a proof of Viazovska's modular form inequalities that removes the reliance on computer calculation, thereby simplifying and clarifying Viazovska's groundbreaking work. The approach may be useful in analyzing problems where similar modular form inequalities have appeared, such as in the sphere packing problem in dimension 24 and in the study of universal optimality in energy minimization problems.

Viazovska (1) proved that the sphere packing associated with the E_8 lattice, which has a packing density of $\frac{\pi^4}{384}$, is the densest sphere packing in 8 dimensions. Her proof relied on properties of certain functions, denoted $\phi_0(z)$ and $\psi_I(z)$, which were defined in terms of classical modular and quasimodular forms: the Eisenstein series E_2, E_4, and E_6, and the Jacobi thetanull functions θ_2, θ_3, and θ_4. A key step in the proof consisted of showing that these functions satisfied a certain pair of inequalities; this was essential to verifying that a radial function defined by taking an integral transform of $\phi_0(z)$ and $\psi_I(z)$ (combined in a particular way) was the so-called magic function that had been conjectured to exist by Cohn and Elkies (2) and certifies the correct sphere packing bound.

The goal of this paper is to give a direct proof of Viazovska's inequalities. To recall the result, let z denote a complex variable taking values in the upper half plane, and denote $q = e^{\pi i z}$. Let $\sigma_\alpha(n) = \sum_{d\,|\,n} d^\alpha$ denote the divisor function. Recall the definitions of the functions E_2, E_4, E_6, θ_2, θ_3, and θ_4:

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^{2n}, \qquad \theta_2(z) = \textstyle\sum_{n=-\infty}^{\infty} q^{(n+1/2)^2},$$
$$E_4(z) = 1 + 240\textstyle\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}, \quad \theta_3(z) = \sum_{n=-\infty}^{\infty} q^{n^2},$$
$$E_6(z) = 1 - 504\textstyle\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}, \quad \theta_4(z) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}.$$

Next, set

$$\phi_0(z) = 1728\frac{(E_2(z)E_4(z) - E_6(z))^2}{E_4(z)^3 - E_6(z)^2}, \qquad [1]$$

$$\psi_I(z) = 128\left(\frac{\theta_3(z)^4 + \theta_4(z)^4}{\theta_2(z)^8} + \frac{\theta_4(z)^4 - \theta_2(z)^4}{\theta_3(z)^8}\right), \qquad [2]$$

and define functions $A(t)$, $B(t)$ of a real variable $t > 0$ by

$$A(t) = -t^2\phi_0(i/t) - \frac{36}{\pi^2}\psi_I(it),$$
$$B(t) = -t^2\phi_0(i/t) + \frac{36}{\pi^2}\psi_I(it).$$

Theorem 1 (Viazovska's modular form inequalities). *The functions $A(t)$, $B(t)$ satisfy*

$$A(t) < 0 \qquad (t > 0), \qquad [V1]$$
$$B(t) > 0 \qquad (t > 0). \qquad [V2]$$

Viazovska's original proof of Theorem 1 relied heavily on computer calculations. The proof consisted of two main steps: first, analogues of the inequalities [**V1**]–[**V2**] were verified numerically for approximating functions $A_0^{(6)}(t)$, $A_\infty^{(6)}(t)$, $B_0^{(6)}(t)$, $B_\infty^{(6)}(t)$ of $A(t)$ and $B(t)$, which were formed by truncating the asymptotic expansions of $A(t)$ and $B(t)$ near $t = 0$ and $t = \infty$; this could be done in a finite calculation. Second, rigorous

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bounds were derived that made it possible to deduce the inequalities [**V1**]–[**V2**] from the corresponding inequalities for the approximating functions.

Another pair of inequalities of similar flavor to [**V1**]–[**V2**] was proved by Cohn et al. (3) in their subsequent proof of optimality of the Leech lattice packing in 24 dimensions. Their proof used different techniques, but that proof as well remained dependent on extensive computer calculations.

Below, we give a proof of Theorem 1 that is fully human-verifiable and requires no numerical calculations beyond the elementary manipulation of a few standard mathematical constants. This helps to simplify and demystify a critical step in Viazovska's celebrated sphere packing proof.

1. Proof of [V1]

It is sufficient to prove that $\phi_0(it) > 0$ and $\psi_I(it) > 0$ for all $t > 0$. The first of these claims follows immediately from the standard identities (4, pp. 20, 21, 49)

$$E_4^3 - E_6^2 = 1728q^2 \prod_{n=1}^{\infty}(1-q^{2n})^{24}, \qquad [3]$$

$$E_2E_4 - E_6 = \frac{3}{2\pi i}\frac{dE_4}{dz} = 720\sum_{n=1}^{\infty} n\sigma_3(n)q^{2n}, \qquad [4]$$

which imply that both $E_4^3 - E_6^2$ and $E_2E_4 - E_6$ take positive real values on the positive imaginary axis.

For the claim about $\psi_I(it)$, recall Jacobi's identity $\theta_2^4 + \theta_4^4 = \theta_3^4$ (see ref. 4, p. 28), and set $\lambda(z) = \theta_2^4/\theta_3^4 = 1 - \theta_4^4/\theta_3^4$ (the modular lambda function; see ref. 4, p. 63). It is clear from these defining relations of $\lambda(z)$ that for $t > 0$, $\lambda(it)$ takes real values in $(0, 1)$. Now note that

$$\begin{aligned}\frac{1}{128}\psi_I &= \frac{\theta_3^4+\theta_4^4}{\theta_2^8} + \frac{\theta_4^4-\theta_2^4}{\theta_3^8}\\ &= \frac{1}{\theta_3^4}\cdot\frac{\theta_3^8+\theta_3^4\theta_4^4}{\theta_2^8} + \frac{1}{\theta_3^4}\cdot\frac{\theta_4^4-\theta_2^4}{\theta_3^4}\\ &= \frac{1}{\theta_3^4}\left(\frac{1}{\lambda^2} + \frac{1}{\lambda}\cdot\frac{1-\lambda}{\lambda} + (1-\lambda) - \lambda\right)\\ &= \frac{1}{\theta_3^4}\frac{(1-\lambda)(2+\lambda+2\lambda^2)}{\lambda^2}.\end{aligned}$$

Since the function $x \mapsto \frac{(1-x)(2+x+2x^2)}{x^2}$ is positive for $x \in (0, 1)$, and since $\theta_3(it)^4 > 0$ for $t > 0$, we get the claim that $\psi_I(it) > 0$. □

2. Proof of [V2]

We will make use of the standard modular transformation properties (1, pp. 996–997)

$$\theta_2(z+1)^4 = -\theta_2(z)^4, \qquad \theta_2(-1/z)^4 = -z^2\,\theta_4(z)^4, \quad [5]$$

$$\theta_3(z+1)^4 = \theta_4(z)^4, \qquad \theta_3(-1/z)^4 = -z^2\,\theta_3(z)^4, \quad [6]$$

$$\theta_4(z+1)^4 = \theta_3(z)^4, \qquad \theta_4(-1/z)^4 = -z^2\,\theta_2(z)^4, \quad [7]$$

$$E_2(z+1) = E_2(z), \qquad E_2(-1/z) = z^2E_2(z) - \frac{6iz}{\pi}, \quad [8]$$

$$E_4(z+1) = E_4(z), \qquad E_4(-1/z) = z^4E_4(z), \quad [9]$$

$$E_6(z+1) = E_6(z), \qquad E_6(-1/z) = z^6E_6(z). \quad [10]$$

Using [**8**]–[**10**], a simple calculation shows that

$$z^2\phi_0(-1/z) = 1728\Big[\frac{(E_2E_4-E_6)^2}{E_4^3-E_6^2}z^2 - \frac{12i}{\pi}\cdot\frac{E_4(E_2E_4-E_6)}{E_4^3-E_6^2}z - \frac{36}{\pi^2}\left(\frac{E_4^2}{E_4^3-E_6^2}\right)\Big].$$

(This is a slightly simplified version of equation 29 from ref. 1.) Similarly, with the help of [**5**]–[**7**], we see that

$$z^2\psi_I(-1/z) = -128\left(\frac{\theta_3^4+\theta_2^4}{\theta_4^8} + \frac{\theta_2^4-\theta_4^4}{\theta_3^8}\right).$$

We will separate the proof of [**V2**] into two parts, proving separately that

$$B(t) > 0 \text{ for } t \ge 1 \quad \text{and} \quad t^2B(1/t) > 0 \text{ for } t \ge 1,$$

that is, equivalently, that

$$\frac{\pi^2}{36}t^2\phi_0(i/t) < \psi_I(it) \qquad \text{for } t \ge 1,$$

$$\frac{\pi^2}{36}\phi_0(it) < t^2\psi_I(i/t) \qquad \text{for } t \ge 1.$$

It is convenient to clear the denominators in each of these inequalities by multiplying both sides by $E_4^3 - E_6^2$, which is also equal to $\frac{27}{4}(\theta_2\theta_3\theta_4)^8$ by a well-known identity. (4, p. 29) We therefore define

$$f(z) = \frac{1}{864}\cdot\frac{\pi^2}{36}(E_4^3-E_6^2)\phi_0(z) = \frac{\pi^2}{18}(E_2E_4-E_6)^2, \quad [11]$$

$$\begin{aligned}\widetilde{f}(z) &= -\frac{1}{864}\cdot\frac{\pi^2}{36}(E_4^3-E_6^2)z^2\phi_0(-1/z)\\ &= -\frac{\pi^2}{18}(E_2E_4-E_6)^2z^2 + \frac{2\pi i}{3}E_4(E_2E_4-E_6)z + 2E_4^2,\end{aligned} \quad [12]$$

$$\begin{aligned}g(z) &= -\frac{1}{864}(E_4^3-E_6^2)z^2\psi_I(-1/z)\\ &= \theta_2^8(\theta_3^{12}+\theta_2^4\theta_3^8+\theta_2^4\theta_4^8-\theta_4^{12}),\end{aligned} \quad [13]$$

$$\begin{aligned}\widetilde{g}(z) &= \frac{1}{864}(E_4^3-E_6^2)\psi_I(z)\\ &= \theta_4^8(\theta_3^{12}+\theta_4^4\theta_3^8+\theta_2^8\theta_4^4-\theta_2^{12}).\end{aligned} \quad [14]$$

By the above remarks, in order to deduce [**V2**], it will be sufficient to prove the following inequalities:

$$f(it) < g(it) \qquad \text{for } t \ge 1, \qquad \textbf{[V2-I]}$$

$$\widetilde{f}(it) < \widetilde{g}(it) \qquad \text{for } t \ge 1. \qquad \textbf{[V2-II]}$$

As a final bit of preparation, recall the known explicit evaluations

$$E_2(i) = \frac{3}{\pi}, \qquad E_4(i) = \frac{3\Gamma(1/4)^8}{64\pi^6}, \quad E_6(i) = 0, \qquad \textbf{[15]}$$

$$\theta_2(i) = \frac{\Gamma(1/4)}{(2\pi)^{3/4}}, \quad \theta_3(i) = \frac{\Gamma(1/4)}{\sqrt{2}\,\pi^{3/4}}, \qquad \theta_4(i) = \frac{\Gamma(1/4)}{(2\pi)^{3/4}}. \qquad \textbf{[16]}$$

Here, $\Gamma(\cdot)$ denotes the Euler gamma function. The numerical value of $\Gamma(1/4)$ is approximately 3.62561 (5). For the proof of Eq. **16**, see ref. 6, p. 325 and ref. 7, equation 2.21, p. 307. The identity $E_2(i) = 3/\pi$ is an immediate consequence of Eq. **8**. The relation $E_6(i) = 0$ is proved in ref. 8, p. 40, and the formula for $E_4(i)$ follows from Eq. **16** and the identity $E_4 = \frac{1}{2}(\theta_2^8+\theta_3^8+\theta_4^8)$, proved, e.g., in ref. 4, p. 29; see also ref. 9, p. 290.

A. Proof of [V2-I]. The functions $f(z)$, $g(z)$ have Fourier expansions

$$f(z) = 28800\pi^2 q^4 + 1036800\pi^2 q^6 + 14169600\pi^2 q^8 + \ldots =: \sum_{n=4}^{\infty} a_n q^n, \qquad \textbf{[17]}$$

$$g(z) = 20480 q^3 + 2015232 q^5 + 41656320 q^7 + \ldots =: \sum_{n=3}^{\infty} b_n q^n. \qquad \textbf{[18]}$$

The coefficients a_n in Eq. **17** are nonnegative: This is immediate from Eq. **4**. Similarly, we have $b_n \geq 0$ for all n. To see this, let $\gamma(z) = \theta_2^8\theta_3^{12} + \theta_2^{12}\theta_3^8$, and observe that, by Eqs. **5**–**7**, $g(z)$ can be represented as

$$g(z) = \gamma(z) - \gamma(z+1). \qquad \textbf{[19]}$$

The Fourier coefficients of γ are manifestly nonnegative, and, since the substitution $z \mapsto z+1$ corresponds to replacing each occurrence of q by $-q$ in the Fourier series, the relationship [**19**] means that the Fourier expansion of g consists of twice the odd terms in the Fourier expansion of γ and therefore also has nonnegative coefficients.

From the above remarks, it now follows that the function $t \mapsto e^{3\pi t} f(it) = \sum_{n=4}^{\infty} a_n e^{-\pi(n-3)t}$ is a nonincreasing function of t. Using [**15**], we then get for all $t \geq 1$ the bound

$$e^{3\pi t} f(it) \leq e^{3\pi} f(i) = e^{3\pi}\frac{\pi^2}{18}\left(\frac{3}{\pi}\frac{3\Gamma(1/4)^8}{64\pi^6} - 0\right)^2 = e^{3\pi}\frac{9\Gamma(1/4)^{16}}{8192\,\pi^{12}} \approx 13130.47. \qquad \textbf{[20]}$$

On the other hand, by Eq. **18** and the observation about the nonnegativity of the coefficients b_n, the bound $e^{3\pi t} g(it) = 20480 + \sum_{n=4}^{\infty} b_n e^{-\pi(n-3)t} \geq 20480$ holds for all $t > 0$. Combining this with [**20**] gives [**V2-I**]. □

B. Proof of [V2-II]. In a similar vein, we examine the q-series expansions of $\widetilde{f}(z), \widetilde{g}(z)$ and their properties. From Eqs. **12** and **14**, we obtain expansions of the forms

$$\widetilde{f}(z) = 2 + (480\pi i z + 960)q^2 + (-28800\pi^2 z^2 + 123840\pi i z + 123840)q^4 + (-1036800\pi^2 z^2 + 3150720\pi i z + 2100480)q^6 + \ldots =: \sum_{n=0}^{\infty} c_n(z) q^n, \qquad \textbf{[21]}$$

$$\widetilde{g}(z) = 2 + 240q^2 - 10240q^3 + 134640q^4 - 1007616q^5 + \ldots =: \sum_{n=0}^{\infty} d_n q^n. \qquad \textbf{[22]}$$

Here, [**22**] is a conventional Fourier series, whereas [**21**] is a more unusual expansion in powers of $q = e^{\pi i z}$ in which each coefficient $c_n(z)$ is itself a quadratic polynomial in z. It is convenient to renormalize these expressions, defining new functions

$$\widetilde{F}(z) = -\frac{\widetilde{f}(z)-2}{q^2} = -\sum_{n=2}^{\infty} c_n(z) q^{n-2} = (-480\pi i z - 960) + (28800\pi^2 z^2 - 123840\pi i z - 123840)q^2 + \ldots, \qquad \textbf{[23]}$$

$$\widetilde{G}(z) = -\frac{\widetilde{g}(z)-2}{q^2} = -\sum_{n=2}^{\infty} d_n q^{n-2} = -240 + 10240q - 134640q^2 + 1007616q^3 + \ldots. \qquad \textbf{[24]}$$

The inequality [**V2-II**] can now be restated as the claim that $\widetilde{G}(it) < \widetilde{F}(it)$ for all $t \geq 1$. This will follow from the combination of the following two lemmas.

Lemma 2. *$\widetilde{G}(it) \leq 288$ for all $t \geq 1$.*

Lemma 3. *$\widetilde{F}(it) \geq 468$ for all $t \geq 1$.*

The following auxiliary claim will be used in the proof of Lemma 2.

Lemma 4. *We have $(-1)^n d_n \geq 0$ for $n \geq 0$.*

Proof: By Eqs. **5**–**7**, the function $\widetilde{g}(z+1) = \sum_{n=0}^{\infty}(-1)^n d_n q^n$ can be written as

$$\widetilde{g}(z+1) = \theta_3^{12}\theta_2^8 + \theta_3^8\theta_2^{12} + \theta_3^{12}\theta_4^8 + \theta_3^8\theta_4^{12}. \qquad \textbf{[25]}$$

The claim is that the Fourier series of this function has nonnegative coefficients. This fact was proved by Slipper (10, p. 76), who deduced it from a certain identity representing the function on the right-hand side of Eq. **25** in terms of the theta series of a certain 20-dimensional lattice, known as "DualExtremal(20,2)a". Here is a self-contained proof that uses only elementary properties of the thetanull functions. Denote for convenience

$$Z = \theta_3^4, \qquad X = \theta_2^4, \qquad Y = 2Z - X.$$

Then X and Z have Fourier series with nonnegative coefficients, and, again recalling the identity $\theta_2^4+\theta_4^4=\theta_3^4$, we see that $Y=\theta_3^4+\theta_4^4=\theta_3(z)^4+\theta_3(z+1)^4$ (recall [**6**] above), so the Fourier series of Y also has nonnegative coefficients. Now, observe that $\widetilde{g}(z+1)$ can be expressed as

$$\begin{aligned}\widetilde{g}(z+1)&=Z^3X^2+Z^2X^3+Z^3(Z-X)^2+Z^2(Z-X)^3\\&=\frac{1}{16}\left(6X^5+15X^4Y+10X^3Y^2+Y^5\right),\end{aligned}$$

and therefore also has nonnegative Fourier coefficients. □

Proof of Lemma 2: Define

$$H(z)=\frac{\widetilde{G}(z)-\widetilde{G}(z+1)}{2}=\sum_{m=1}^{\infty}(-d_{2m+1})q^{2m-1}, \quad [26]$$

$$=10240q+10007616q^3+\dots. \quad [27]$$

Two crucial properties of $H(z)$ are a) the function $t\mapsto H(it)=\sum_{m=1}^{\infty}(-d_{2m+1})e^{-\pi(2m-1)t}$ is nonincreasing (each summand is nonincreasing, by Lemma 4); and b) $\widetilde{G}(it)+240\le H(it)$ for all $t>0$ (this follows from Lemma 4 together with the observation that the constant coefficient in Eq. **24** is -240). Now note that, by Eqs. **5**–**7**, **14**, and **24**, $H(z)$ can be expressed explicitly as

$$\begin{aligned}H(z)&=-\frac{1}{2}q^{-2}\Big[\theta_4^8(\theta_3^{12}+\theta_4^4\theta_3^8+\theta_2^8\theta_4^4-\theta_2^{12})-2\\&\qquad-\theta_3^8(\theta_4^{12}+\theta_3^4\theta_4^8+\theta_2^8\theta_3^4+\theta_2^{12})+2\Big]\\&=\frac{1}{2}q^{-2}\left(\theta_2^8\theta_3^{12}+\theta_2^{12}\theta_3^8+\theta_2^{12}\theta_4^8-\theta_2^8\theta_4^{12}\right)\\&=\frac{1}{2}q^{-2}\left(\theta_2^8(\theta_3^{12}-\theta_4^{12})+\theta_2^{12}(\theta_3^8+\theta_4^8)\right).\end{aligned}$$

Therefore, using the evaluations [**16**], we get that for all $t\ge 1$,

$$\begin{aligned}\widetilde{G}(it)&\le -240+H(it)\le -240+H(i)\\&=-240+\frac{e^{2\pi}}{2}\left(\frac{\Gamma(1/4)}{(2\pi)^{3/4}}\right)^{20}\left((2^{1/4})^{12}-1+(2^{1/4})^8+1\right)\\&=-240+\frac{e^{2\pi}}{2}\frac{\Gamma(1/4)^{20}}{(2\pi)^{15}}(8+4)\\&=-240+6e^{2\pi}\frac{\Gamma(1/4)^{20}}{(2\pi)^{15}}\approx 287.02,\end{aligned}$$

as claimed. □

Proof of Lemma 3: We strategically separate $\widetilde{F}(z)$ into three components, defining

$$\widetilde{F}_1(z)=-480\pi iz+(28800\pi^2z^2-123840\pi iz-123840)q^2, \quad [28]$$

$$\widetilde{F}_2(z)=\frac{\pi^2}{18}q^{-2}(E_2E_4-E_6)^2z^2-2q^{-2}(E_4^2-1)+(-28800\pi^2z^2+123840)q^2, \quad [29]$$

$$\widetilde{F}_3(z)=-\frac{2\pi i}{3}q^{-2}E_4(E_2E_4-E_6)z+(480\pi iz+123840\pi izq^2), \quad [30]$$

so that, by Eqs. **12** and **23**, we have

$$\widetilde{F}(z)=\widetilde{F}_1(z)+\widetilde{F}_2(z)+\widetilde{F}_3(z). \quad [31]$$

We now make the following elementary observations:

(a) The function $t\mapsto\widetilde{F}_1(it)$ is monotone increasing on $[1,\infty)$.

Proof: Assume that $t\ge 1$. A trivial calculation gives that

$$\begin{aligned}\frac{d}{dt}\left(\widetilde{F}_1(it)\right)&=480\pi e^{-2\pi t}\left(e^{2\pi t}+120\pi^2t^2-636\pi t+774\right)\\&\ge 480\pi e^{-2\pi t}\left(e^{2\pi}+120\pi^2t^2-636\pi t+774\right).\end{aligned}$$

The last expression is of the form $e^{-2\pi t}$ times a quadratic polynomial in t, which, it is easy to check, is positive on the real line. Thus, we have shown that $\widetilde{F}_1'(t)>0$ for $t\ge 1$, which proves the claim.

(b) The function $t\mapsto\widetilde{F}_2(it)$ is monotone increasing on $[1,\infty)$.

Proof: Let $(\alpha_n)_{n=2}^{\infty}$ and $(\beta_n)_{n=1}^{\infty}$ be the coefficients in the Fourier series

$$(E_2E_4-E_6)^2=\sum_{n=2}^{\infty}\alpha_nq^{2n},\quad E_4^2-1=\sum_{n=1}^{\infty}\beta_nq^{2n}.$$

Clearly, $\alpha_n\ge 0$ [Eq. **4**] and $\beta_n\ge 0$ for all n. One can also easily check that $\alpha_2=518400$ and $\beta_2=61920$. Then, on inspection of Eq. **29**, we see that

$$\widetilde{F}_2(it)=-2\beta_1+\sum_{n=3}^{\infty}\left(-\frac{\pi^2}{18}\alpha_nt^2-2\beta_n\right)e^{-\pi(2n-2)t}.$$

The summand associated with $n=2$ is precisely canceled out by the term $(-28800\pi^2z^2+123840)q^2$ in Eq. **29**. Now for each $n\ge 3$, the nth summand in this series is easily seen to be an increasing function of t for $t\ge\frac{1}{(n-1)\pi}$, so in particular for $t\ge 1$. Thus, $t\mapsto\widetilde{F}_2(it)$ is also increasing for $t\ge 1$.

(c) $\widetilde{F}_3(it)\ge 0$ for all $t>0$.

Proof: Let $(\delta_n)_{n=1}^{\infty}$ be the coefficients in the Fourier series $E_4(E_2E_4-E_6)=\sum_{n=1}^{\infty}\delta_nq^{2n}$. Then $\delta_n\ge 0$ for all n, and we have $\delta_1=720$ and $\delta_2=185760$. Referring to Eq. **30**, we then see that

$$\widetilde{F}_3(it)=\frac{2\pi t}{3}\sum_{n=3}^{\infty}\delta_ne^{-\pi(2n-2)t}\ge 0,$$

since the summands associated with $n=1,2$ are canceled by the term $(480\pi iz+123840\pi izq^2)$ in Eq. **30**.

Finally, combining [**31**] with the observations (a)–(c) above, we get that for $t\ge 1$,

$$\begin{aligned}\widetilde{F}(it)&\ge\widetilde{F}_1(it)+\widetilde{F}_2(it)\ge\widetilde{F}_1(i)+\widetilde{F}_2(i)\\&=480\pi+123840e^{-2\pi}\\&\quad+e^{2\pi}\Big(-\frac{\pi^2}{18}(E_2(i)E_4(i)-E_6(i))^2-2(E_4(i)^2-1)\Big)\\&=480\pi+123840e^{-2\pi}+e^{2\pi}\left(2-\frac{45\Gamma(1/4)^{16}}{8192\pi^{12}}\right)\\&\approx 468.39,\end{aligned}$$

as claimed. □

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