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2019
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A constrained optimization problem for the Fourier transform

By

Dominique Maldague

A dissertation submitted in partial satisfaction of the

requirements for the degree of<br>Doctor of Philosophy<br>in<br>Mathematics<br>in the<br>Graduate Division<br>of the<br>University of California, Berkeley<br>Committee in charge:<br>Professor Michael Christ, Chair<br>Professor Daniel Tataru<br>Professor Chung-Pei Ma

Spring 2019

Abstract<br>A constrained optimization problem for the Fourier transform<br>by<br>Dominique Maldague<br>Doctor of Philosophy in Mathematics<br>University of California, Berkeley<br>Professor Michael Christ, Chair

Among functions $f$ majorized by indicator functions $1_{E}$, which functions have maximal ratio $\|\widehat{f}\|_{q} /|E|^{1 / p}$ ? We investigate the existence of maximizers, using a concentration compactness approach and ingredients from additive combinatorics to establish properties of maximizing sequences. For exponents $q$ sufficiently close to even integers, we exploit variational techniques and combinatorial results to identify all maximizers. This follows from establishing a sharper version of an associated inequality: if the input $(f, E)$, where $|f| \leqslant 1_{E}$, has a certain structure, then the ratio $\frac{\|\widehat{f}\|_{q}}{|E|^{1 / p}}$ is at least a certain quantitative distance from being optimal.

Dedicated to my loving partner Franco, my brother Alex, and my therapist Ava, who supported me while I was a graduate student. Also dedicated to all of the women who I am lucky to have in my family, including my mom Laurie, Sandra, Lorraine, Lucie, Isabelle, Madeleine, Chloé, Juliette, Pénélope, Miriam, Jocelyne, Andrea, and Abigail.

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## Chapter 1

## Introduction

Define the Fourier transform as $\mathcal{F}(f)(\xi)=\widehat{f}(\xi)=\int_{\mathbb{R}^{d}} e^{-2 \pi i x \cdot \xi} f(x) d x$ for a function $f: \mathbb{R}^{d} \rightarrow$ $\mathbb{C}$. The Fourier transform is a contraction from $L^{1}\left(\mathbb{R}^{d}\right)$ to $L^{\infty}\left(\mathbb{R}^{d}\right)$ and is unitary on $L^{2}\left(\mathbb{R}^{d}\right)$. Interpolation gives the Hausdorff-Young inequality $\|\hat{f}\|_{q} \leqslant\|f\|_{p}$ where $p \in(1,2), 1=\frac{1}{p}+\frac{1}{q}$. In [2], Beckner proved the sharp Hausdorff-Young inequality

$$
\begin{equation*}
\|\widehat{f}\|_{q} \leqslant \mathbf{C}_{q}^{d}\|f\|_{p} \tag{1.1}
\end{equation*}
$$

where $\mathbf{C}_{q}=p^{1 / 2 p} q^{-1 / 2 q}$. In 1990, Lieb proved that Gaussians are the only maximizers of (1.1), meaning that $\|\widehat{f}\|_{q} /\|f\|_{p}=\mathbf{C}_{q}^{d}$ if and only if $f=c \exp (-Q(x, x)+v \cdot x)$ where $Q$ is a positive definite real quadratic form, $v \in \mathbb{C}^{d}$ and $c \in \mathbb{C}$. In 2014, Christ established a sharpened Hausdorff-Young inequality by bounding $\|\widehat{f}\|_{q}-\mathbf{C}_{p}^{d}\|f\|_{p}$ by a negative multiple of an $L^{p}$ distance function of $f$ to the Gaussians.

In [16], Christ studied the existence of maximizers for the ratio $\left\|\widehat{1_{E}}\right\|_{q} /|E|^{1 / p}$ where $E \subset \mathbb{R}^{d}$ is a positive Lebesgue measure set. For $d \geqslant 1, q \in(2, \infty)$, and $p=q^{\prime}$, define

$$
\mathbf{A}_{q, d}:=\sup _{|E|<\infty} \frac{\left\|\widehat{1_{E}}\right\|_{q}}{|E|^{1 / p}}
$$

where the supremum is taken over Lebesgue measurable subsets of $\mathbb{R}^{d}$ of finite measure. Building on the work of Burchard in [7], Christ identified maximizing sets to be ellipsoids for exponents $q \geqslant 3$ sufficiently close to even integers [16]. This dissertation presents analogous results for a related inequality.

Our variant of the Hausdorff-Young inequality replaces indicator functions by bounded multiples and modifies the functional as follows. For $d \geqslant 1, q \in(2, \infty)$, and $p=q^{\prime}$, we consider the inequality

$$
\begin{equation*}
\|\widehat{f}\|_{q} \leqslant \mathbf{B}_{q, d}|E|^{1 / p} \tag{1.2}
\end{equation*}
$$

and define the quantities

$$
\begin{align*}
\Psi_{q}(E) & :=\sup _{|f|<E} \frac{\|\hat{f}\|_{q}}{\left\|1_{E}\right\|_{p}}  \tag{1.3}\\
\mathbf{B}_{q, d} & :=\sup _{E} \Psi_{q}(E) \tag{1.4}
\end{align*}
$$

where $|f|<E$ means $|f| \leqslant 1_{E}$ and the supremum is taken over all Lebesgue measurable sets $E \subset \mathbb{R}^{d}$ with positive, finite Lebesgue measures. This quantity $\mathbf{B}_{q, d}$ satisfies $\mathbf{B}_{q, d} \leqslant \mathbf{C}_{p}^{d}$ by their definitions and $\mathbf{B}_{q, d}<\mathbf{C}_{q, d}$ by [16]. The supremum (1.4) is equal to

$$
\sup _{f \in L(p, 1)} \frac{\|\widehat{f}\|_{q}}{\|f\|_{\mathcal{L}}} \quad \text { where } \quad\|f\|_{\mathcal{L}}=\inf \left\{\|a\|_{\ell^{1}}:|f|=\sum_{n} a_{n}\left|E_{n}\right|^{-1 / p} 1_{E_{n}}, a_{n}>0,\left|E_{n}\right|<\infty\right\}
$$

We prove this equivalence in Proposition 5 in $\$ 2.1$. Lorentz spaces can be defined by real interpolation between $L^{p}$ spaces. Since the quasinorm $\|\cdot\|_{\mathcal{L}}$ induces the standard topology on the Lorentz space $L(p, 1)$, this is a natural quantity to study.

Christ used continuum versions of theorems of Balog-Szemerédi and Freŭman from additive combinatorics to understand the underlying structure of functions with nearly optimal ratio $\|\widehat{f}\|_{q} /\|f\|_{p}$ in [15] and for sets $E$ with nearly optimal ratio $\left\|\widehat{1_{E}}\right\|_{q} /\left\|1_{E}\right\|_{p}$ in [16]. We use similar techniques to prove properties of maximizing sequences for (1.1). Our most complete result is Proposition 28, which describes a localization property for maximizing sequences. The full precompactness result is presented in the following conjecture.

Conjecture 1. Let $d \geqslant 1$ and $q \in(2, \infty), p=q^{\prime}$. Let $\left(E_{\nu}\right)$ be a sequence of Lebesgue measurable subsets of $\mathbb{R}^{d}$ with $\left|E_{\nu}\right| \in \mathbb{R}^{+}$and let $f_{\nu}$ be Lebesgue measurable functions on $\mathbb{R}^{d}$ satisfying $\left|f_{\nu}\right| \leqslant 1_{E_{\nu}}$. Suppose that $\lim _{\nu \rightarrow \infty}\left|E_{\nu}\right|^{-1 / p}\left\|\widehat{f}_{\nu}\right\|_{q}=\mathbf{B}_{q, d}$. Then there exists a subsequence of indices $\nu_{k}$, a Lebesgue measurable set $E \subset \mathbb{R}^{d}$ with $0<|E|<\infty$, a Lebesgue measurable function $f$ on $\mathbb{R}^{d}$ satisfying $|f| \leqslant 1_{E}$, a sequence $\left(T_{\nu}\right)$ of affine automorphisms of $\mathbb{R}^{d}$, and a sequence of vectors $v_{\nu} \in \mathbb{R}^{d}$ such that

$$
\lim _{k \rightarrow \infty}\left\|e^{-2 \pi i v_{\nu_{k}} \cdot x} f_{\nu_{k}} \circ T_{\nu_{k}}^{-1}-f\right\|_{p}=0 \quad \text { and } \quad \lim _{k \rightarrow \infty}\left|T_{\nu_{k}}\left(E_{\nu_{k}}\right) \Delta E\right|=0
$$

There is an open technical difficulty to proving the above conjecture, which is presented as Question 9 in $\$ 4$.

Proposition 2. If there is an affirmative answer to Question 9, then the claim in Conjecture 1 holds.

The conditional existence of maximizers is a direct consequence. A simplified outline of this partial argument is as follows.

1. Begin by proving basic principles of concentration compactness: "no slacking" and "cooperation" (see \$3.1).
2. If $|f| \leqslant 1_{E}$ with $|E|=1$ satisfies $\|\widehat{f}\|_{q} \geqslant \eta$ for $\eta>0$, then $f$ satisfies a related Young's convolution inequality: for appropriate $\gamma, r, s,\left\||f|^{\gamma} *|f|^{\gamma}\right\|_{r} \geqslant \eta^{s}$.
3. By continuum analogues of theorems of Balog-Szemerédi and Frĕ̆man, $|f| \leqslant 1_{E}$ with $|E|=1$ satisfying $\left\||f|^{\gamma} *|f|^{\gamma}\right\|_{r} \geqslant \eta^{s}$ must place a portion of its $L^{p}$ mass on a continuum multiprogression of controlled rank and Lebesgue measure.
4. Combine concentration compactness principles with the specific additive structure we have from the relation to Young's convolution inequality to conclude that a function $|f| \leqslant 1_{E}$ satisfying $\|\widehat{f}\|_{q} \geqslant(1-\delta) \mathbf{B}_{q, d}|E|^{1 / p}$ for small $\delta>0$ is mostly supported on a multiprogression of controlled rank and size.
5. By precomposing a near-extremizer with an affine transformation $\mathcal{T}$, we can change variables to guarantee that the continuum multiprogression is mostly contained in $\mathbb{Z}^{d} \times[-\delta, \delta]^{d}$. We must guarantee that the Jacobian of $\mathcal{T}$ is bounded below since otherwise we could trivially collapse any bounded set to a small neighborhood of the origin.
6. The Fourier transform of a function living on $\mathbb{Z}^{d} \times[-\delta, \delta]^{d}$ decomposes into a discrete and a continuous Fourier transform, and a near-extremizer for 1.2 must be a nearextremizer of each step of the decomposition. Since near-extremizers of the discrete Fourier transform must mostly be supported on a single $n \in \mathbb{Z}^{d}$, this gives extra structure. We prove that the only multiprogression structure which is favorable at each step of the decomposition is one mostly contained in a single convex set $[-\delta, \delta]$.
7. If $|f| \leqslant 1_{E}$ is a near-extremizer, then $\hat{f}|\widehat{f}|^{q-2}$ is a near-extremizer of a related dual inequality (see $\$ 2.2$ ). The above reasoning may also be carried out for this dual inequality, except for step (6), which may or may not be true in the dual setting. If step (6) holds in the dual setting, we conclude that a significant portion of the $L^{p}$ mass of $f$ and $\hat{f}$ must be localized to ellipsoids (or other convex sets) of controlled size.
8. Via a composition with an affine transformation and modulation by a character, we can assume that $f$ and $\hat{f}$ are localized (respectively) to the unit ball $\mathbb{B}$ and and ellipsoid $\mathcal{E}$ centered at the origin. We prove a reversed uncertainty bound: $|\mathcal{E}||\mathbb{B}| \leqslant C$ and furthermore $\mathcal{E} \subset C \mathbb{B}$ for an appropriate $C>0$.
9. It follows that for any sequence of function $\left|f_{\nu}\right| \leqslant 1_{E_{\nu}}$ with $\left|E_{\nu}\right| \in \mathbb{R}^{+}$and $\left\|\widehat{f}_{\nu}\right\|_{q}\left|E_{\nu}\right|^{-1 / p} \rightarrow \mathbf{B}_{q, d}$, after $\left(f_{\nu}, E_{\nu}\right)$ is renormalized to $\left(F_{\nu}, A_{\nu}\right)$ by appropriate symmetries of the inequality, we have weakly convergent subsequences of $F_{\nu}$ and $1_{A_{\nu}}$. Finally, we get $L^{p}$ convergence via a convexity argument involving the $\|\cdot\|_{\mathcal{L}}$ norm.
If we restrict the exponents $q$, then our results are much more complete. In Chapter 6, we prove $\mathbf{B}_{q, d}=\mathbf{A}_{q, d}$, identify the maximizers, and prove a quantitative stability result for the inequality

$$
\begin{equation*}
\|\widehat{f}\|_{q} \leqslant \mathbf{B}_{q, d}|E|^{1 / p} \tag{1.5}
\end{equation*}
$$

when $q$ is near an even integer $m \geqslant 4$. We define some notation in order to state our main theorem. Let $\mathfrak{E}$ denote the set of ellipsoids in $\mathbb{R}^{d}$. Let $A \Delta B$ denote the symmetric difference $(A \backslash B) \cup(B \backslash A)$. For any Lebesgue measurable subset $E \subset \mathbb{R}^{d}$ with $|E| \in(0, \infty)$, define

$$
\begin{equation*}
\operatorname{dist}(E, \mathfrak{E}):=\inf _{\mathcal{E} \in \mathfrak{E}} \frac{|\mathcal{E} \Delta E|}{|E|} \tag{1.6}
\end{equation*}
$$

where the inf is taken over all ellipsoids satisfying $|\mathcal{E}|=|E|$. Let $\mathfrak{L}$ denote the set of affine functions $L: \mathbb{R}^{d} \rightarrow \mathbb{R}$. For $e^{i g} \in L^{2}(E)$ with $g$ real-valued, define

$$
\begin{equation*}
\operatorname{dist}_{E}\left(e^{i g}, \mathfrak{L}\right):=\inf _{L \in \mathfrak{L}}\left\|e^{i g}-e^{i L}\right\|_{L^{2}(E)} \tag{1.7}
\end{equation*}
$$

Theorem 3. Let $d \geqslant 1$. For each even integer $m \in\{4,6,8, \ldots\}$ there exists $\delta(m)>0$ such that the following conclusions hold for all exponents satisfying $|q-m| \leqslant \delta(m)$. Firstly, if $E \subset \mathbb{R}^{d}$ is a Lebesgue measurable set of finite measure, and $f, g$ are real-valued functions with $0 \leqslant f \leqslant 1$, then

$$
\mathbf{B}_{q, d}=\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q} /|E|^{1 / p}
$$

if and only if $f e^{i g} 1_{E}=e^{i L} 1_{\mathcal{E}}$, for some $L \in \mathfrak{L}$ and $\mathcal{E} \in \mathfrak{E}$. Secondly, there exists $c_{q, d}>0$ such that for every set $E \subset \mathbb{R}^{d}$ with $|E|=1$, and for all real-valued functions $f, g$ with $0 \leqslant f \leqslant 1$,

$$
\begin{equation*}
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q} \leqslant \mathbf{B}_{q, d}^{q}-c_{q, d}\left[\|f-1\|_{L^{1}(E)}+\operatorname{dist}_{E}\left(e^{i g}, \mathfrak{L}\right)^{2}+\operatorname{dist}(E, \mathfrak{E})^{2}\right] \tag{1.8}
\end{equation*}
$$

This theorem refines 1.5 by bounding $\mathbf{B}_{q, d}-\|\hat{f}\|_{q} /|E|^{1 / p}$ below by a function of a distance of $(f, E)$ to the set of maximizers (or extremizers) of 1.5 ). The optimality of the $L^{1}$ norm and exponent 1 in $\|f-1\|_{L^{1}(E)}$ is proved in Lemma 50. The optimality of the $L^{2}$ norm and exponent 2 in $\operatorname{dist}_{E}\left(e^{i g}, \mathfrak{L}\right)^{2}$ from (1.8) is proved in $\$ 6.8$. The optimality of the exponent 2 of $\operatorname{dist}(E, \mathfrak{E})^{2}$ is addressed in [16].

We rely on the hypothesis that $q$ is near an even integer $m \geqslant 4$ to identify maximizers. For $q \in(2, \infty)$ not near an even integer, it is not known which sets $E$ maximize $\left\|\widehat{1_{E}}\right\|_{q} /|E|^{1 / p}$. Furthermore, if $q=2 m$ for $\mathbb{Z} \ni m \geqslant 2$, then for any $|f| \leqslant 1_{E}$ where $E$ is a Lebesgue measurable set and $|E| \in(0, \infty)$, we have the inequality

$$
\begin{equation*}
\|\widehat{f}\|_{q}^{q}=\|f * \cdots * f\|_{2}^{2} \leqslant\left\|1_{E} * \cdots * 1_{E}\right\|_{2}^{2}=\left\|\widehat{1_{E}}\right\|_{q}^{q} \tag{1.9}
\end{equation*}
$$

where the convolution products are $m$-fold. The failure of $\|\widehat{f}\|_{q} \leqslant\|\widehat{f \mid}\|_{q}$ for general $f \in L^{q^{\prime}}$ was shown for $q=3$ by Hardy and Littlewood and for all other exponents not in $\{2,4,6, \ldots\}$ by Boas [5]. Thus it is not obvious that $\mathbf{B}_{q, d}$ is no larger than $\mathbf{A}_{q, d}$. By Theorem3, ellipsoids are among maximizers for certain exponents $q$, so the following corollary is an immediate consequence.
Corollary 4. Let $d \geqslant 1$. For each even integer $m \in\{4,6,8, \ldots\}$ there exists $\delta(m)>0$ such that if $|q-m| \leqslant \delta(m)$, then $\mathbf{B}_{q, d}=\mathbf{A}_{q, d}$.

This material is based upon work supported by the National Science Foundation Graduate Research Fellowship under Grant No. DGE 1106400.

## Chapter 2

## Different versions of the inequality

### 2.1 An inequality for the Lorentz space $L(p, 1)$

There are many quasinorms which induce the same topology on $L(p, q)$ spaces. For the special case of $p>1$ and $q=1$, we will show that our extremization problem can be phrased using various quasinorms (and one norm defined by Calderón) on $L(p, 1)$. Let $\mathbf{B}_{q, d}$ be as before.

Definition 1. Let $d \geqslant 1$. Define $\|f\|_{\mathcal{L}}$ for a measurable function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ by

$$
\|f\|_{\mathcal{L}}=\inf \left\{\left\|\left(a_{n}\right)\right\|_{\ell^{1}}:|f|=\sum_{n} a_{n}\left|E_{n}\right|^{-1 / p} 1_{E_{n}}, a_{n} \geqslant 0,\left|E_{n}\right|<\infty\right\} .
$$

The following definitions 2, 3, and 4 are from Chapter V, §3 in [36].
Definition 2. Let $d \geqslant 1$. Define $f^{*}$ for $t>0$ by

$$
f^{*}(t)=\inf \{r:|\{x:|f(x)|>r\}| \leqslant t\} .
$$

Definition 3. Let $d \geqslant 1,1 \leqslant p<\infty, q$ the conjugate of $p$. Define $\|f\|_{p 1}^{*}$ for a measurable function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ by

$$
\|f\|_{p 1}^{*}=\frac{1}{p} \int_{0}^{\infty} t^{-1 / q} f^{*}(t) d t
$$

Definition 4. Let $d \geqslant 1,1 \leqslant p<\infty, q$ the conjugate of $p$. Define $\|f\|_{p 1}$ for a measurable function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ by

$$
\|f\|_{p 1}=\frac{1}{p} \int_{0}^{\infty} t^{-1 / q-1} \int_{0}^{t} f^{*}(u) d u d t .
$$

Definition 5. Let $d \geqslant 1,1 \leqslant p<\infty$. The space $L(p, 1)$ is defined as all measurable functions $f: \mathbb{R}^{d} \rightarrow \mathbb{C}$ satisfying $\|f\|_{p 1}^{*}<\infty$.

See the appendix for the relationships between $\|\cdot\|_{\mathcal{L}},\|\cdot\|_{p 1}^{*}$, and $\|\cdot\|_{p 1}$, and that they generate the same topology on $L(p, 1)$. In particular, it is proved that $\|f\|_{\mathcal{L}}=\|f\|_{p 1}^{*}$ for all measurable $f: \mathbb{R}^{d} \rightarrow \mathbb{C}$ (where one quantity is infinite if and only if the other quantity is as well).

Proposition 5. For $d \geqslant 1, q \in(2, \infty)$, and $p$ the dual exponent to $q$,

$$
\mathbf{B}_{q, d}=\sup _{f \in L^{p}} \frac{\|\hat{f}\|_{q}}{\|f\|_{\mathcal{L}}}
$$

Proof. Let $|f|=\sum_{n} a_{n}\left|E_{n}\right|^{-1 / p} 1_{E_{n}}$ where $a_{n} \geqslant 0$ and $\left|E_{n}\right|<\infty$. Then $\|f\|_{p} \leqslant \sum_{n} a_{n}$, so $\|f\|_{p} \leqslant\|f\|_{\mathcal{L}}$. By the Hausdorff-Young inequality, the constant $A_{\mathcal{L}}$ defined by

$$
\begin{equation*}
A_{\mathcal{L}}:=\sup _{f \in L^{p}} \frac{\|\widehat{f}\|_{q}}{\|f\|_{\mathcal{L}}} \tag{2.1}
\end{equation*}
$$

is finite.
We want to show that $\mathbf{B}_{q, d}:=\sup _{|E|<\infty} \sup _{|f|<1_{E}} \frac{\|\hat{f}\|_{q}}{|E|^{1 / p}}=\sup _{f \in L^{p}} \frac{\|\hat{f}\|_{q}}{\|f\|_{\mathcal{L}}}=: A_{\mathcal{L}}$.
If $|f|=\sum a_{n}\left|E_{n}\right|^{-1 / p} 1_{E_{n}}$ with $a_{n} \geqslant 0,\left|E_{n}\right|<\infty$, then

$$
\frac{\|\widehat{f}\|_{q}}{\sum\left|a_{n}\right|} \leqslant \frac{\sum\left|a_{n}\left\|\left.E_{n}\right|^{-1 / p}\right\| \widehat{\overline{1 E}_{n}} \|_{q}\right.}{\sum\left|a_{n}\right|} \leqslant \mathbf{B}_{q, d}
$$

so $A_{\mathcal{L}} \leqslant \mathbf{B}_{q, d}$.
For the other direction, since simple functions are dense in $L^{p}\left(\mathbb{R}^{d}\right)$, it suffices to consider $f=\sum a_{n} 1_{A_{n}}$ where $A_{n}$ are disjoint and $f$ is majorized by the indicator of a Lebesgue measurable set $E$ of size one. Then $\sum\left|a_{n}\left\|\left.A_{n}\right|^{1 / p} \leqslant|E|^{1 / p} \sum\left|a_{n}\right|=|E|^{1 / p}\right\| f \|_{1} \leqslant|E|^{\frac{1}{p}+1}=1\right.$. Rearranged, this means

$$
\|\widehat{f}\|_{q}=\frac{\|\widehat{f}\|_{q}}{|E|^{1 / p}} \leqslant \frac{\|\widehat{f}\|_{q}}{\sum\left|a_{n}\right|\left|A_{n}\right|^{1 / p}} \leqslant \frac{\|\widehat{f}\|_{q}}{\|f\|_{\mathcal{L}}}
$$

so $\mathbf{B}_{q, d} \leqslant A_{\mathcal{L}}$.

Lemma 6. If $f \in L(p, 1)$ satisfies $\mathbf{B}_{q, d}=\|f\|_{\mathcal{L}}^{-1}\|\widehat{f}\|_{q}$, then

$$
f=a e^{i \varphi} 1_{E}
$$

for some scalar $a \in \mathbb{R}^{+}$, Lebesgue measurable function $\varphi: \mathbb{R}^{d} \rightarrow \mathbb{R}$, and a Lebesgue measurable set $E$ of finite measure.
Proof. By Lemma 60, we also have that $\mathbf{B}_{q, d}=\left(\|f\|_{p 1}^{*}\right)^{-1}\|\widehat{f}\|_{q}$. Let $E=\{(y, s):|f(y)|>s\}$. Let $e^{i \varphi}=f /|f|$ so we can use the layer cake representation

$$
f(x)=e^{i \varphi(x)} \int_{0}^{\infty} 1_{E}(x, t) d t
$$

Then

$$
\begin{aligned}
\|\widehat{f}\|_{q} & =\left\|\left(\int_{0}^{\infty} e^{i \varphi(x)} 1_{E}(x, t) d t\right)^{\wedge}\right\|_{q}=\left\|\int_{0}^{\infty} \widehat{e^{i \varphi} 1_{E}}(\xi, t) d t\right\|_{q} \\
& \leqslant \int_{0}^{\infty}\left\|\widehat{e^{i \varphi} 1_{E}}(\xi, t)\right\|_{q} d t \\
& \leqslant \int_{0}^{\infty} \mathbf{B}_{q, d}|\{x:|f(x)|>t\}|^{1 / p} d t \\
& =\mathbf{B}_{q, d} \int_{0}^{\infty} \int_{0}^{|\{x:|f(x)|>t\}|} \frac{1}{p} u^{-1 / q} d u d t \\
& =\mathbf{B}_{q, d} \int_{0}^{|\{x:|f(x)|>0\}|} \int_{0}^{f^{*}(u)} \frac{1}{p} u^{-1 / q} d t d u=\mathbf{B}_{q, d}\|f\|_{p 1}^{*} .
\end{aligned}
$$

Since $\mathbf{B}_{q, d}=\left(\|f\|_{p 1}^{*}\right)^{-1}\|\widehat{f}\|_{q}$, the above sequence of inequalities are actually equalities. Equality in the Minkowski integral inequality implies that for a.e. $(\xi, t) \in \mathbb{R}^{d} \times \mathbb{R}^{+}$,

$$
\widehat{e^{i \varphi} 1_{E}}(\xi, t)=h(\xi) g(t)
$$

for some measurable functions $h, g$. Since $e^{i \varphi} 1_{E}(x, t) \in L^{2}$, in particular, $h$ and $\check{h}$ in $L^{2}$.

$$
1_{E}(x, t)=e^{-i \varphi(x)} \breve{h}(x) g(t) .
$$

But then for every $(x, t)$ satisfying $|f(x)|>t$, we have

$$
e^{-i \varphi(x)} \breve{h}(x) g(t)=1
$$

Suppose $|f(x)|>|f(y)|>0$. Then for all $0 \leqslant t<f(y)$,

$$
e^{-i \varphi(x)} \check{h}(x)=g(t)^{-1}=e^{i \varphi(y)} \widetilde{h(y)},
$$

which is a contradiction unless $|f(x)|$ is constant on its support. Thus $f$ takes the form $a e^{i \varphi} 1_{S}$ where $S=\operatorname{supp} f$ and $a \in \mathbb{R}^{+}$.

The conditional existence corollary to Proposition 2 in terms of Lorentz norms is
Corollary 7. Suppose that there is an affirmative answer to Question g. Let $d \geqslant 1, p \in(1,2)$, $q$ the conjugate exponent of $p$. First, we have $\frac{B_{q, d}}{q}=\sup _{0 \neq g \in L(p, 1)}\|g\|_{p 1}^{-1}\|\widehat{g}\|_{q}$. Second, if $f \in L(p, 1)$ satisfies $\frac{\mathbf{B}_{q, d}}{q}=\|f\|_{p 1}^{-1}\|\widehat{f}\|_{q}$, then

$$
f=a e^{i \varphi} 1_{E}
$$

for some scalar $a \in \mathbb{R}^{+}$, Lebesgue measurable function $\varphi: \mathbb{R}^{d} \rightarrow \mathbb{R}$, and a Lebesgue measurable set $E$ of finite measure.

See $\$ 5.4$ for the proof of the Corollary 7 .

### 2.2 The dual inequality

Recall the definition of the optimal constant $\mathbf{B}_{q, d}$

$$
\mathbf{B}_{q, d}=\sup _{0<|E|<\infty} \sup _{|f| \leqslant 1} \frac{\|\widehat{f}\|_{q}}{|E|^{1 / p}} .
$$

By exploiting $L^{p}$ duality and Plancherel's theorem, we also have the expressions:

$$
\mathbf{B}_{q, d}=\sup _{|E|<\infty} \sup _{|f| \leqslant 1_{E}} \sup _{\|g\|_{p} \leqslant 1} \frac{|\langle\widehat{f}, g\rangle|}{|E|^{1 / p}}=\sup _{|E|<\infty} \sup _{\|g\|_{p} \leqslant 1} \frac{\left\langle 1_{E},\right| \widehat{g}| \rangle \mid}{|E|^{1 / p}},
$$

the last of which motivates the following definition.
Definition 6. Let $d \geqslant 1$ and $q \in[1, \infty)$, and $p$ be the conjugate exponent to $q$. Define the norm $\|\cdot\|_{q, *}$ of a function $g \in L^{q}\left(\mathbb{R}^{d}\right)$ to be

$$
\|g\|_{q, *}=\sup _{0<|E|<\infty}|E|^{-1 / p} \int_{E}|g|
$$

where the supremum is taken over Lebesgue measurable subsets $E \subset \mathbb{R}^{d}$ of positive, finite measure.

Note that by Hölder's inequality, if $g \in L^{q}$, then $\|g\|_{q, *} \leqslant\|g\|_{q}<\infty$. Thus for $f \in L^{p}$ with $p \in(1,2)$ and $q$ the conjugate exponent,

$$
\|\widehat{f}\|_{q, *} \leqslant\|f\|_{p}
$$

is a corollary of the Hausdorff-Young inequality.
Proposition 8. For $d \geqslant 1, q \in(2, \infty)$, and $p$ the dual exponent to $q$,

$$
\mathbf{B}_{q, d}=\sup _{\|f\|_{p} \leqslant 1}\|\widehat{f}\|_{q, *} .
$$

Furthermore, if $|f| \leqslant 1_{E},|E|<\infty$ satisfies $\|\widehat{f}\|_{q} \geqslant(1-\delta) \mathbf{B}_{q, d}|E|^{1 / p}$ for some $\delta>0$, then

$$
\|\left(|\widehat{f}|^{q-2} \widehat{f}\right)^{\check{ }\left\|_{q, *} \geqslant(1-\delta)^{q} \mathbf{B}_{q, d}\right\||\widehat{f}|^{q-2} \widehat{f} \|_{p} . . . . . . .}
$$

Proof. Let $f \in L^{p}\left(\mathbb{R}^{d}\right)$. Consider a Lebesgue measurable set $E \subset \mathbb{R}^{d}$ of finite measure such that $|\widehat{f}| \neq 0$ a.e. on $E$ and write $\widehat{f}=e^{-i \varphi}|\widehat{f}|$ for a real-valued phase function $\varphi$ equal to 0 off of the support of $\hat{f}$. Using Plancherel's theorem and Hölder's inequality, we then have

$$
\begin{aligned}
|E|^{-1 / p} \int_{E}|\widehat{f}| & =|E|^{-1 / p} \int 1_{E} e^{i \varphi} \widehat{f}=|E|^{-1 / p} \int \widehat{e^{i \varphi} 1_{E}} f \\
& \leqslant|E|^{-1 / p}\left\|\widehat{e^{i \varphi} 1_{E}}\right\|_{q}\|f\|_{p} \\
& \leqslant \mathbf{B}_{q, d}\|f\|_{p}
\end{aligned}
$$

so that $\sup _{\|f\|_{p} \leq 1}\|\hat{f}\|_{q, *} \leqslant \mathbf{B}_{q, d}$.
Now suppose that for $|f| \leqslant 1_{E},|E|<\infty$, and $\delta>0$ we have $\|\widehat{f}\|_{q} \geqslant(1-\delta) \mathbf{B}_{q, d}|E|^{1 / p}$. Then $|\hat{f}|^{q-2} \hat{f} \in L^{p}$ since $\left\||\hat{f}|^{q-2} \hat{f}\right\|_{p}^{p}==\|\hat{f}\|_{q}^{q}$. Then

$$
\left.\begin{aligned}
\int_{E}\left|\left(|\hat{f}|^{q-2} \hat{f}\right)\right| & \geqslant \int|f|\left|\left(|\hat{f}|^{q-2} \hat{f}\right)\right| \\
& \geqslant\left|\int f \overline{\left(|\hat{f}|^{q-2} \hat{f}\right)}\right| \\
& =\left|\int \widehat{f}\right| \widehat{f} \mid q^{q-2} \hat{f}
\end{aligned}\left|=\int\right| \hat{f}\right|^{q} .
$$

Rearranging and using that $\left\||\widehat{f}|^{q-2} \widehat{f}\right\|_{p}=\|\widehat{f}\|_{q}^{q / p} \leqslant \mathbf{B}_{q, d}^{q / p}|E| q / p^{2}$,

$$
|E|^{-1 / p} \int_{-E}\left|\left(|\widehat{f}|^{q-2} \hat{f}\right)\right| \geqslant(1-\delta)^{q} \mathbf{B}_{q, d}^{q-q / p}\left\||\hat{f}|^{q-2} \widehat{f}\right\|_{p}=\left.(1-\delta)^{q} \mathbf{B}_{q, d}\| \| \hat{f}\right|^{q-2} \hat{f} \|_{p},
$$

proving that we can find $g \in L^{p}$ such that $\|\hat{g}\|_{q, *}\|g\|_{p}^{-1}$ is arbitrarily close to $\mathbf{B}_{q, d}$.

For $q \in(2, \infty)$ and $p$ the conjugate exponent of $q$, the inequality

$$
\begin{equation*}
\|\hat{g}\|_{q, *} \leqslant \mathbf{B}_{q, d}\|g\|_{p} . \tag{2.2}
\end{equation*}
$$

is amenable to the same analysis as our main inequality (1.2), and each lemma we prove about (1.2) will have an analogue for this dual inequality.

## Chapter 3

## Structure of near-maximizers

### 3.1 Quasi-extremal principles

We establish the quasi-extremal principles "no slacking" and "cooperation". No slacking guarantees that every part of a near-extremizer is a quasi-extremizers. Cooperation guarantees that these small parts work together in a compatible way (e.g. have nontrivial intersection of supports).

Definition 7. Let $d \geqslant 1, q \in(2, \infty)$ and $p=q^{\prime}$. A nonzero function $f$ satisfying $|f| \leqslant 1_{E} \in L^{p}$ is a $\delta$-quasi-extremizer for (1.2) if

$$
\|\widehat{f}\|_{q} \geqslant \delta \mathbf{B}_{q, d}|E|^{1 / p}
$$

By a quasi-extremizer, we mean a $\delta$-quasi-extremizer for some small $\delta>0$.

### 3.1.1 No slacking

Lemma 9 (No slacking). For any $p, q \in(1, \infty)$ there exist $c, C_{0}<\infty$ with the following property. Let $\delta>0,|E|<\infty,|f| \leqslant 1_{E}$. Suppose that

$$
\|\widehat{f}\|_{q} \geqslant(1-\delta) \mathbf{B}_{q, d}|E|^{1 / p}
$$

Suppose that $f=g+h$ where $g=1_{A} f, h=1_{B} f$, and $A \cap B=\varnothing$, and that

$$
|B| \geqslant C_{0} \delta|E|
$$

Then

$$
\|\widehat{h}\|_{q} \geqslant c \delta|E|^{1 / p}
$$

Proof. There exists $C<\infty$ such that for any $G, H \in L^{q}$,

$$
\|G+H\|_{q}^{q} \leqslant\|G\|_{q}^{q}+C\|G\|_{q}^{q-1}\|H\|_{q}+C\|H\|_{q}^{q} .
$$

Consequently,

$$
\begin{aligned}
\|\widehat{g+h}\|_{q}^{q} & \leqslant\|\widehat{g}\|_{q}^{q}+C\|\widehat{g}\|_{q}^{q-1}\|\widehat{h}\|_{q}+C\|\widehat{h}\|_{q}^{q} \\
& \leqslant \mathbf{B}_{q, d}^{q}|A|^{q / p}+C \mathbf{B}_{q, d}^{q-1}|A|^{(q-1) / p}\|\widehat{h}\|_{q}+C\|\widehat{h}\|_{q}^{q} .
\end{aligned}
$$

On the other hand, $|E|=|A|+|B|$. Without loss of generality, assume $|E|=1$, so that $|A|,|B| \leqslant 1$. Thus

$$
\begin{aligned}
(1-\delta)^{q} & \leqslant \frac{\|\widehat{f}\|_{q}^{q}}{\mathbf{B}_{q, d}^{q}|E|^{q / p}}=\frac{\|\widehat{f}\|_{q}^{q}}{\mathbf{B}_{q, d}^{q}} \\
& \leqslant \mathbf{B}_{q, d}^{-q}\left(\mathbf{B}_{q, d}^{q}|A|^{q / p}+C \mathbf{B}_{q, d}^{q-1}|A|^{(q-1) / p}\|\widehat{h}\|_{q}+C\|\widehat{h}\|_{q}^{q}\right) \\
& =|A|^{q / p}+C \mathbf{B}_{q, d}^{-1}|A|^{(q-1) / p}\|\widehat{h}\|_{q}+C \mathbf{B}_{q, \|}^{-q}\|\widehat{h}\|_{q}^{q} \\
& \leqslant(1-|B|)^{q / p}+C \mathbf{B}_{q, d}^{-1}|A|^{(q-1) / p}\|\widehat{h}\|_{q}+C \mathbf{B}_{q, d}^{-1}\|\widehat{h}\|_{q} \\
& \leqslant 1-c_{p}|B|+2 C \mathbf{B}_{q, d}^{-1}\|\widehat{h}\|_{q} .
\end{aligned}
$$

Then we have

$$
\begin{aligned}
2 C \mathbf{B}_{q, d}^{-1}\|\widehat{h}\|_{q} & \geqslant(1-\delta)^{q}-1+c_{p}|B| \\
& \geqslant 1-O(\delta)-1+|B| \\
& \geqslant|B|-O(\delta) \\
& \geqslant C_{0}^{p} \delta-O(\delta) \\
& \geqslant c \delta
\end{aligned}
$$

provided $C_{0}$ is large enough.
Lemma 10 (No slacking dual). For each $d \geqslant 1$ and $q \in(2, \infty)$ there exist $\delta_{0}, c, C_{0}<\infty$ with the following property. Let $\delta \in\left(0, \delta_{0}\right]$ and let $f=g+h$ where $f, g, h \in L^{p}\left(\mathbb{R}^{d}\right)$ and $g, h$ are disjointly supported on $A, B$ respectively. Suppose that

$$
\|\widehat{f}\|_{q, *} \geqslant(1-\delta) \mathbf{B}_{q, d}\|f\|_{p}
$$

and that

$$
\|h\|_{p} \geqslant C_{0} \delta^{1 / p}\|f\|_{p}
$$

Then

$$
\|\widehat{h}\|_{q, \infty} \geqslant c \delta\|f\|_{p}
$$

Proof. Using the hypothesis that $f$ is near-extremizing,

$$
\begin{aligned}
(1-\delta)\|f\|_{p} \mathbf{B}_{q, d} \leqslant\|\widehat{f}\|_{q, *} & \leqslant\|\widehat{g}\|_{q, *}+\|\widehat{h}\|_{q, *} \\
& \leqslant \mathbf{B}_{q, d}\|g\|_{p}+\|\widehat{h}\|_{q, *} \\
& \leqslant \mathbf{B}_{q, d}\left(\|f\|_{p}^{p}-\|h\|_{p}^{p}\right)^{1 / p}+\|\widehat{h}\|_{q, *} \\
& \leqslant \mathbf{B}_{q, d}\left(\|f\|_{p}^{p}-C_{0}^{p} \delta\|f\|_{p}^{p}\right)^{1 / p}+\|\widehat{h}\|_{q, *} .
\end{aligned}
$$

Rearranging the above inequality gives

$$
\left((1-\delta)-\left(1-C_{0}^{p} \delta\right)^{1 / p}\right) \mathbf{B}_{q, d}\|f\|_{p} \leqslant\|\hat{h}\|_{q, *}
$$

Finally, we can arrange that $\left|C_{0}^{p} \delta\right|<1$, so

$$
\begin{aligned}
1-\delta-\left(1-C_{0}^{p} \delta\right)^{1 / p} & =-\delta+\frac{1}{p} C_{0}^{p} \delta+O\left(\delta^{2}\right) \\
& =\left(C_{0}^{p} / p-1\right) \delta+O\left(\delta^{2}\right) .
\end{aligned}
$$

If $C_{0}^{p} / p-2>0$ and $\delta$ is small enough, we have the result.

### 3.1.2 Cooperation

Lemma 11. Let $p \in[1,2)$ and $q \in[2, \infty)$. There exist $c, C \in \mathbb{R}^{+}$with the following property. Let $0 \neq f \in L^{p}$ satisfy $|f| \leqslant 1_{E}$ and $\|\widehat{f}\|_{q} \geqslant(1-\delta) \mathbf{B}_{q, d}|E|^{1 / p}$. Suppose that $f=f^{\sharp}+f^{b}$ where supp $f^{\sharp}=A$ and supp $f^{b}=B$ satisfy

$$
\begin{gathered}
A \cup B=E, \quad A \cap B=\varnothing \\
\text { and } \quad \min (|A|,|B|) \geqslant \eta^{p}|E|
\end{gathered}
$$

Then

$$
\left\|\widehat{f}^{\sharp} \cdot \hat{f}^{b}\right\|_{q / 2} \geqslant\left(c \eta^{p}-C \delta\right)|E|^{2 / p} .
$$

Proof.

$$
\begin{aligned}
& \|\widehat{f}\|_{q}^{q} \leqslant \int\left(\left|\widehat{f^{\sharp}}\right|^{2}+\left|\hat{f}^{b}\right|^{2}\right)|\widehat{f}|^{q-2}+2 \int\left|\widehat{f^{\sharp}} \cdot \hat{f^{b}}\right||\widehat{f}|^{q-2} \\
& \leqslant\left(\left\|\left|\widehat{f}^{\sharp}\right|^{2}\right\|_{q / 2}+\left\|\left|\widehat{f^{\sharp}}\right|^{2}\right\|_{q / 2}\right)\left\||\hat{f}|^{q-2}\right\|_{q /(q-2)}+2\left\|\widehat{f^{\sharp}} \cdot \widehat{f}^{b}\right\|_{q / 2}\left\|\mid \widehat{f}^{q-2}\right\|_{q /(q-2)} \\
& =\left(\|\widehat{f}\|_{q}^{2}+\left\|\widehat{f^{\sharp}}\right\|_{q}^{2}\right)\|\widehat{f}\|_{q}^{q-2}+2\left\|\widehat{f^{\sharp}} \cdot \widehat{f}^{b}\right\|_{q / 2}\|\widehat{f}\|_{q}^{q-2} \\
& \leqslant\left(|A|^{2 / p}+|B|^{2 / p}\right) \mathbf{B}_{q, d}^{q}|E|^{(q-2) / p}+2\left\|\widehat{f^{\sharp}} \cdot \hat{f}^{b}\right\|_{q / 2} \mathbf{B}_{q, d}^{q-2}|E|^{(q-2) / p} .
\end{aligned}
$$

Rearranging gives

$$
\begin{aligned}
\left\|\hat{f}^{\sharp} \cdot \hat{f}^{b}\right\|_{q / 2} & \geqslant\left(2 \mathbf{B}_{q, d}^{q-2}|E|^{(q-2) / p}\right)^{-1}\left(\|\widehat{f}\|_{q}^{q}-\left(|A|^{2 / p}+|B|^{2 / p}\right) \mathbf{B}_{q, d}^{q}|E|^{(q-2) / p}\right) \\
& \geqslant\left(2 \mathbf{B}_{q, d}^{q-2}|E|^{(q-2) / p}\right)^{-1}\left((1-\delta)^{q} \mathbf{B}_{q, d}^{q}|E|^{\mid / p}-\left(|A|^{2 / p}+|B|^{2 / p}\right) \mathbf{B}_{q, d}^{q}|E|^{(q-2) / p}\right) \\
& \geqslant 2^{-1} \mathbf{B}_{q, d}^{2}\left((1-\delta)^{q}|E|^{2 / p}-|A|^{2 / p}-|B|^{2 / p}\right) .
\end{aligned}
$$

Note that since $p<2$,

$$
\left(|A|^{2 / p}+|B|^{2 / p}\right)^{p / 2} \leqslant|A|+|B| \leqslant|E|
$$

with strict inequality unless $|A|$ or $|B|$ is 0 . Without loss of generality, suppose that $|E|=1$.

We want to show there exists $c \in \mathbb{R}^{+}$such that for $\eta$ small enough and $\eta^{p} \leqslant \min (|A|,|B|)$,

$$
\frac{|A|^{2 / p}+|B|^{2 / p}}{(|A|+|B|)^{2 / p}}=|A|^{2 / p}+|B|^{2 / p} \leqslant 1-c \eta^{p}
$$

By assumption, $|A|,|B| \in\left[\eta^{p}, 1-\eta^{p}\right]$, so $|A|^{2 / p}+|B|^{2 / p} \leqslant\left(\eta^{p}\right)^{2 / p}+\left(1-\eta^{p}\right)^{2 / p}$. For all $\eta>0$ sufficiently small, there exists $c>0$ so that $\left(\eta^{p}\right)^{2 / p}+\left(1-\eta^{p}\right)^{2 / p} \leqslant 1-c \eta^{p}$.

Finally, using $|A|+|B|=|E|=1$,

$$
\begin{aligned}
\left\|\hat{f}^{\sharp} \cdot \hat{f}^{b}\right\|_{q / 2} & \geqslant 2^{-1} \mathbf{B}_{q, d}^{2}\left((1-\delta)^{q}-|A|^{2 / p}-|B|^{2 / p}\right) \\
& \geqslant 2^{-1} \mathbf{B}_{q, d}^{2}\left((1-\delta)^{q}-\left(1-c \eta^{p}\right)\right) \\
& \geqslant c \eta^{p}-C \delta .
\end{aligned}
$$

Lemma 12. For each $d \geqslant 1$ and $q \in(2, \infty)$ there exist $\delta_{0}, c, C_{0}<\infty$ with the following property. Let $\delta \in\left(0, \delta_{0}\right]$ and let $f=g+h$ where $f, g, h \in L^{p}\left(\mathbb{R}^{d}\right)$ and $g$, $h$ are disjointly supported. Let $\eta^{p} \geqslant \delta$. Suppose that the following inequalities hold.

$$
\begin{gathered}
\|\widehat{f}\|_{q, *} \geqslant(1-\delta) \mathbf{B}_{q, d}\|f\|_{p}, \\
\min \left(\|g\|_{p},\|h\|_{p}\right) \geqslant C_{0} \eta\|f\|_{p} .
\end{gathered}
$$

Then

$$
\left\||\widehat{g}|^{1 / 2}|\widehat{h}|^{1 / 2}\right\|_{q, *} \geqslant c \delta\|f\|_{p} \mathbf{B}_{q, d} .
$$

Proof. Take $E \subset \mathbb{R}^{d}$ with $|E| \in(0, \infty)$ satisfying

$$
|E|^{-1 / p} \int_{E}|\hat{f}| \geqslant(1-2 \delta) \mathbf{B}_{q, d}\|f\|_{p}
$$

By replacing $E$ with $E \cap\{\widehat{f} \neq 0\}$, we can assume that $\widehat{f}$ is nonzero on $E$. For $\lambda>0$ a large constant to be chosen later, define $E_{\lambda, g}=\{x \in E:|\widehat{g}|>\lambda|\widehat{h}|\}$ and $E_{\lambda, h}=\{x \in E:|\widehat{h}|>\lambda|\widehat{g}|\}$. Note that

$$
\begin{aligned}
\int_{E}|\widehat{f}| & =\int_{E_{\lambda, g}}|\widehat{f}|+\int_{E_{\lambda, h}}|\widehat{f}|+\int_{E \backslash\left(E_{\lambda, g} \cup E_{\lambda, h}\right.}|\widehat{f}| \\
& \leqslant(1+1 / \lambda) \int_{E_{\lambda, g}}|\widehat{g}|+(1+1 / \lambda) \int_{E_{\lambda, h}}|\widehat{h}|+\int_{E \backslash\left(E_{\lambda, g} \cup E_{\lambda, h}\right)}(|\hat{g}|+|\widehat{h}|) \\
& \leqslant(1+1 / \lambda) \int_{E_{\lambda, g}}|\widehat{g}|+(1+1 / \lambda) \int_{E_{\lambda, h}}|\widehat{h}|+\int_{E \backslash\left(E_{\lambda, g} \cup E_{\lambda, h}\right)}\left(|\widehat{g}|^{1 / 2} \lambda^{1 / 2}|\widehat{h}|^{1 / 2}+\lambda^{1 / 2}|\widehat{g}|^{1 / 2}|\widehat{h}|^{1 / 2}\right) \\
& \leqslant(1+1 / \lambda)\left(\left|E_{\lambda, g}\right|^{1 / p}\|\widehat{g}\|_{q, *}+\left|E_{\lambda, h}\right|^{1 / p}\|\widehat{h}\|_{q, *}\right)+2 \lambda^{1 / 2} \int_{E \backslash\left(E_{\lambda, g} \cup E_{\lambda, h}\right)}|\widehat{g}|^{1 / 2}|\widehat{h}|^{1 / 2} .
\end{aligned}
$$

Using our main dual inequality, we have

$$
\left|E_{\lambda, g}\right|^{1 / p}\|\widehat{g}\|_{q, *}+\left|E_{\lambda, h}\right|^{1 / p}\|\widehat{h}\|_{q, *} \leqslant\left(\left|E_{\lambda, g}\right|^{1 / p}\|g\|_{p}+\left|E_{\lambda, h}\right|^{1 / p}\|h\|_{p}\right) \mathbf{B}_{q, d}
$$

and by Hölder's inequality,

$$
\left|E_{\lambda, g}\right|^{1 / p}\|g\|_{p}+\left|E_{\lambda, h}\right|^{1 / p}\|h\|_{p} \leqslant\left(\left|E_{\lambda, g}\right|^{p / p}+\left|E_{\lambda, h}\right|^{p / p}\right)^{1 / p}\left(\|g\|_{p}^{q}+\|h\|_{p}^{q}\right)^{1 / q} \leqslant|E|^{1 / p}\left(\|g\|_{p}^{q}+\|h\|_{p}^{q}\right)^{1 / q} .
$$

Also

$$
\|g\|_{p}^{q}+\|h\|_{p}^{q} \leqslant \max \left(\|g\|_{p}^{q-p},\|h\|_{p}^{q-p}\right)\left(\|g\|_{p}^{p}+\|h\|_{p}^{p}\right)=\max \left(\|g\|_{p}^{q-p},\|h\|_{p}^{q-p}\right)\|f\|_{p}^{p}
$$

Now we use the hypothesis that $\min \left(\|g\|_{p},\|h\|_{p}\right) \geqslant C_{0} \eta\|f\|_{p}$ to say

$$
\max \left(\|g\|_{p}^{p},\|h\|_{p}^{p}\right)=\|f\|_{p}^{p}-\min \left(\|g\|_{p},\|h\|_{p}\right) \leqslant\|f\|_{p}^{p}\left(1-C_{0}^{p} \eta^{p}\right)
$$

In summary,

$$
\begin{aligned}
\left|E_{\lambda, g}\right|^{1 / p}\|\widehat{g}\|_{q, *}+\left|E_{\lambda, h}\right|^{1 / p}\|\widehat{h}\|_{q, *} & \left.\leqslant|E|^{1 / p}\left(\|f\|_{p}^{p}\right)^{1 / q}\|f\|_{p}^{(q-p) / q}\left(1-C_{0}^{p} \eta^{p}\right)^{(q-p) / q}\right) \mathbf{B}_{q, d} \\
& =|E|^{1 / p}\|f\|_{p}\left(1-C_{0}^{p} \eta^{p}\right)^{(q-p) / q} \mathbf{B}_{q, d}
\end{aligned}
$$

Putting everything together, we have

$$
\begin{aligned}
& (1-2 \delta) \mathbf{B}_{q, d}\|f\|_{p} \leqslant(1+1 / \lambda)\|f\|_{p}\left(1-C_{0}^{p} \eta^{p}\right)^{(q-p) / q} \mathbf{B}_{q, d}+\lambda^{1 / 2}|E|^{-1 / p} \int_{E \backslash\left(E_{\lambda, h} \cup E_{\lambda, h}\right)}|\widehat{g} \widehat{h}|^{1 / 2} \\
& \left(1-2 \delta-(1+1 / \lambda)\left(1-(1-p / q) C_{0}^{p} \eta^{p}+O\left(\eta^{2 p}\right)\right) \mathbf{B}_{q, d}\|f\|_{p} \leqslant \lambda^{1 / 2}|E|^{-1 / p} \int_{E \backslash\left(E_{\lambda, h} \cup E_{\lambda, h}\right)}|\widehat{g} \widehat{h}|^{1 / 2}\right. \\
& \left(-2 \delta-1 / \lambda+(1+1 / \lambda)(1-p / q) C_{0}^{p} \eta^{p}+O\left(\eta^{2 p}\right)\right) \mathbf{B}_{q, d}\|f\|_{p} \leqslant \lambda^{1 / 2}|E|^{-1 / p} \int_{E}|\widehat{g} \widehat{h}|^{1 / 2} .
\end{aligned}
$$

The desired inequality follows from choosing $\lambda=\delta^{-1}, \eta^{p} \geqslant \delta$ and $C_{0}$ large enough.

### 3.2 Multiprogression structure of quasi-extremizers

In this section, we relate quasi-extremizers for (1.2) to quasi-extremizers for Young's convolution inequality. Then we exploit the connection between Young's convolution inequality and principles of additive combinatorics which imply that quasi-extremizing functions for Young's inequality have significant support on sets with arithmetic structure. We use the following definition and notation for multiprogressions.

Definition 8. A discrete multiprogression $\mathbf{P}$ in $\mathbb{R}^{d}$ of rank $r$ is a function

$$
\mathbf{P}: \prod_{i=1}^{r}\left\{0,1, \ldots, N_{i}-1\right\} \rightarrow \mathbb{R}^{d}
$$

of the form

$$
\mathbf{P}\left(n_{1}, \ldots, n_{r}\right)=\left\{a+\sum_{i=1}^{r} n_{i} v_{i}: 0 \leqslant n_{i}<N_{i}\right\}
$$

for some $a \in \mathbb{R}^{d}$, some $v_{j} \in \mathbb{R}^{d}$, and some positive integers $N_{1}, \ldots, N_{r}$. A continuum multiprogression $P$ in $\mathbb{R}^{d}$ of rank $r$ is a function

$$
P: \prod_{i=1}^{r}\left\{0,1, \ldots, N_{i}-1\right\} \times[0,1]^{d} \rightarrow \mathbb{R}^{d}
$$

of the form

$$
\left(n_{1}, \ldots, n_{d} ; y\right) \mapsto a+\sum_{i} n_{i} v_{i}+s y
$$

where $a, v_{i} \in \mathbb{R}^{d}$ and $s \in \mathbb{R}^{+}$. The size of $P$ is defined to be

$$
\sigma(P)=s^{d} \prod_{i} N_{i}
$$

$P$ is said to be proper if this mapping is injective.
We will often identify a multiprogression with its range, and will refer to multiprogressions as if they were sets rather than functions. If $P$ is proper then the Lebesgue measure of its range equals its size. For a discussion of properties of multiprogressions, see $\S 5$ of [15].

Lemma 13 (Quasi-extremizers for Young's inequality). Let $r \in(1, \infty)$ and suppose that the exponent $t$ defined by $1+t^{-1}=2 r^{-1}$ also belongs to ( $1, \infty$ ). For each $\delta>0$, there exist $c_{\delta}, C_{\delta} \in(0, \infty)$ such that for any $|f| \leqslant 1_{E}$ with $0<|E|<\infty$ and $|E|^{2 / r} \delta \leqslant\|f * f\|_{t}$, there exist a disjoint, measurable partition $E=A \cup B$ and a proper continuum multiprogression $P$ such that

$$
\begin{gathered}
A \subset P \\
|P| \leqslant C_{\delta}|A| \\
\operatorname{rank}(P) \leqslant C_{\delta} \\
\left\|f-1_{A} f\right\|_{r} \leqslant\left(1-c_{\delta}\right)\|f\|_{r} .
\end{gathered}
$$

Proof. This lemma follows from the proof of Lemma 6.1 in [15] where we specialize to the case $f_{1}=f_{2}$ and use the relation $|E|^{2 / r} \geqslant\|f\|_{r}^{2}$.

Lemma 14. Let $d \geqslant 1$ and $p \in(1,2)$. Let $\eta>0$. Suppose that $E$ is a measurable set and $f$ is a nonzero function satisfying $|f| \leqslant 1_{E} \in L^{p}\left(\mathbb{R}^{d}\right)$ and $|E|^{1 / p} \eta \leqslant\|\widehat{f}\|_{p^{\prime}}$. If $p \leqslant 4 / 3$, $|E|^{2 / p} \eta^{2} \leqslant\||f| *|f|\|_{t}$ for $t^{-1}=2 p^{-1}-1$. If $4 / 3<p$, then there exists $\gamma=\gamma(p) \in \mathbb{R}^{+}$such that $|E|^{3 / 2} \eta^{\gamma} \leqslant\left\||f|^{4 / 3} *|f|^{4 / 3}\right\|_{2}$.

Proof. First suppose that $p \leqslant \frac{4}{3}$. Then applying Plancherel's theorem and the HausdorffYoung inequality, we have

$$
|E|^{1 / p} \eta \leqslant\|\widehat{f}\|_{p^{\prime}}=\|\widehat{f * f}\|_{p^{\prime} / 2}^{1 / 2} \leqslant\|f * f\|_{\left(p^{\prime} / 2\right)^{\prime}}^{1 / 2} \leqslant\||f| *|f|\|_{t}^{1 / 2}
$$

where $t=\frac{p^{\prime} / 2}{p^{\prime} / 2-1}=\frac{p /(p-1)}{p /(p-1)-2}=\left(2 p^{-1}-1\right)^{-1}$.
Write $f(x)=g(x) e^{i \varphi(x)}$ where $\varphi(x)$ is real-valued and $g \geqslant 0$. Note that for $\operatorname{Re} z>0$, we can define $f_{z}:=g^{z} e^{i \varphi} \in L^{p / \operatorname{Re} z}$.

Assume that $\frac{4}{3}<p$. Since $\frac{p}{2}<1<\frac{3 p}{4}$, there exists $\theta \in(0,1)$ such that $1=(1-\theta) p 2^{-1}+$ $\theta 3 p 4^{-1}$. By the Three Lines Lemma proof of the Riesz-Thorin theorem,

$$
\|\widehat{f}\|_{p^{\prime}} \leqslant \sup _{\operatorname{Re} z=p / 2}\left\|\widehat{f}_{z}\right\|_{2}^{1-\theta} \sup _{\operatorname{Re} z=3 p / 4}\left\|\widehat{f}_{z}\right\|_{(4 / 3)^{\prime}}^{\theta}=\|f\|_{p}^{(1-\theta) p 2^{-1}} \sup _{\operatorname{Re} z=3 p / 4}\left\|\widehat{f}_{z}\right\|_{(4 / 3)^{\prime}}^{\theta} .
$$

Combining this with the quasi-extremal hypothesis for $f$ gives

$$
\begin{aligned}
|E|^{1 / p} \eta & \leqslant\|f\|_{p}^{(1-\theta) p 2^{-1}} \sup _{\operatorname{Re} z=4 / 3}\left\|\widehat{f}_{z}\right\|_{(4 / 3)^{\prime}}^{\theta} \\
& \leqslant|E|^{(1-\theta) 2^{-1}} \sup _{\operatorname{Re} z=4 / 3}\left\|\widehat{f_{z} * f_{z}}\right\|_{2}^{\theta / 2} \\
& =|E|^{(1-\theta) 2^{-1}} \sup _{\operatorname{Re} z=4 / 3}\left\|f_{z} * f_{z}\right\|_{2}^{\theta / 2} \\
& \leqslant|E|^{(1-\theta) 2^{-1}}\left\||f|^{4 / 3} *|f|^{4 / 3}\right\|_{2}^{\theta / 2} .
\end{aligned}
$$

Rearranging, we can write

$$
|E|^{3 \theta / 4} \eta \leqslant\left\||f|^{4 / 3} *|f|^{4 / 3}\right\|_{2}^{\theta / 2}
$$

so $|E|^{3 / 2} \eta^{\gamma} \leqslant\left\||f|^{4 / 3} *|f|^{4 / 3}\right\|_{2}$ for some $\gamma>0$.

Proposition 15 (Structure of quasi-extremizers). Let $d \geqslant 1$, let $\Lambda \subset(1,2)$ be a compact set, and let $\eta>0$. There exist $C_{\eta}, c_{\eta} \in \mathbb{R}^{+}$with the following property for all $p \in \Lambda$. Suppose that $0 \neq f \in L^{p}\left(\mathbb{R}^{d}\right),|f| \leqslant 1_{E}$ with $|E|<\infty$, and $\|\widehat{f}\|_{q} \geqslant \eta|E|^{1 / p}$. Then there exists a multiprogression $P$ and a disjoint, measurable partition $E=A \cup B$ such that

$$
\begin{gathered}
A \subset P \\
|P| \leqslant C_{\eta}|A| \\
\operatorname{rank} P \leqslant C_{\eta} \\
\left\|f-1_{A} f\right\|_{p} \leqslant\left(1-c_{\eta}\right)\|f\|_{p} .
\end{gathered}
$$

Proof. Combine Lemmas 14 and 13 .

Lemma 16. Let $d \geqslant 1$, let $\Lambda \subset(2, \infty)$ be compact, and let $\eta \in(0,1]$. There exist $C_{\eta}, c_{\eta}>0$ with the following property for all $q \in \Lambda$. Suppose that $0 \neq f \in L^{q^{\prime}}\left(\mathbb{R}^{d}\right)$ satisfies $\|\hat{f}\|_{q, *} \geqslant$ $\eta\|f\|_{p}$. Then there exist a proper continuum multiprogression $P$ and a disjointly supported Lebesgue measurable decomposition $f=g+h$ such that

$$
\begin{array}{r}
g<P, \\
\|g\|_{p} \geqslant c_{\eta}\|f\|_{p} \\
\|g\|_{\infty}|P|^{1 / p} \leqslant C_{\eta}\|f\|_{p} \\
\operatorname{rank} P \leqslant C_{\eta} .
\end{array}
$$

Proof. This follows from Proposition 6.4 in [15] since $\|f\|_{p} \eta \leqslant\|\hat{f}\|_{q, *} \leqslant\|\hat{f}\|_{q}$.

### 3.3 Multiprogression structure of near-extremizers

The following is a restatement of Lemma 5.5 of [15], included here for the reader's convenience.

Lemma 17 (Compatibility of nonnegligibly interacting multiprogressions). Let $d \geqslant 1$. Let $\Lambda$ be a compact subset of $(1,2)$. Let $\lambda>0$ and $R<\infty$. There exists $C<\infty$, depending only $\lambda, R, d, \Lambda$, with the following property. Let $p \in \Lambda$. Let $P, Q \subset \mathbb{R}^{d}$ be nonempty proper continuum multiprogressions of ranks $\leqslant R$. Let $\varphi<P$ and $\psi<Q$ be functions that satisfy $\|\varphi\|_{\infty}|P|^{1 / p} \leqslant 1$ and $\|\psi\|_{\infty}|Q|^{1 / p} \leqslant 1$. If

$$
\|\widehat{\varphi} \widehat{\psi}\|_{q / 2} \geqslant \lambda
$$

then

$$
\begin{gathered}
\max (|P|,|Q|) \leqslant C \min (|P|,|Q|) \\
|P+Q| \leqslant C \min (|P|,|Q|) .
\end{gathered}
$$

Lemma 18. Let $d \geqslant 1$, and let $\Lambda \subset(1,2)$ be a compact set. For any $\epsilon>0$ there exist $\delta>0, N_{\epsilon}<\infty$, and $C_{\epsilon}<\infty$ with the following property for all $p \in \Lambda$. Let $|E|<\infty$ and $|f| \leqslant 1_{E}$ be such that $\|\widehat{f}\|_{q} \geqslant(1-\delta) \mathbf{B}_{q, d}|E|^{1 / p}$. Then there exist a measurable decomposition $f=g+h$, where $g=g 1_{A}, h=h 1_{B}$, and $A \cap B=\varnothing$, and continuum multiprogressions $\left\{P_{i}: 1 \leqslant i \leqslant N_{\epsilon}\right\}$ such that

$$
\begin{gathered}
|B| \leqslant \epsilon|E| \\
\sum_{i}\left|P_{i}\right| \leqslant C_{\epsilon}|E| \\
A \subset{ }_{i=1}^{N_{\epsilon}} P_{i} \\
\operatorname{rank} P_{i} \leqslant C_{\epsilon} \\
\|g\|_{p} \geqslant c_{\delta}\|f\|_{p} .
\end{gathered}
$$

Proof. We define an iterative process following the proof of Theorem 7.1 from [15]. Setting $\eta_{\delta}=1-\delta$, we may apply Proposition 15 to obtain a disjoint decomposition $E=A_{1} \cup B_{1}$ with a multiprogression $P_{1}$ satisfying

$$
\left|P_{1}\right| \leqslant C_{\eta_{\delta}}\left|A_{1}\right|, \quad \text { rank } P_{1} \leqslant C_{\eta_{\delta}}, \quad\left\|1_{A_{1}} f\right\|_{p} \geqslant c_{\eta_{\delta}}\|f\|_{p}
$$

Suppose that $\left|B_{1}\right|>\epsilon|E|$ (the case $\left|B_{1}\right| \leqslant \epsilon|E|$ will be analyzed below). By Lemma 9 with $\delta<\epsilon / C_{0}$,

$$
\left\|\widehat{1_{B_{1}} f}\right\|_{q} \geqslant \frac{c}{C_{0}} \epsilon|E|^{1 / p}
$$

where $c, C_{0}$ are as in the lemma. Define $\eta_{\epsilon}=\frac{c}{C_{0}} \epsilon$. Then we apply Proposition 15 to $1_{B_{1}} f$ to obtain a disjoint decomposition $B_{1}=A_{2} \cup B_{2}$ with the corresponding conclusions.

For the $k$-th step in the process, we halt if $\left|B_{k-1}\right| \leqslant \epsilon|E|$. If $\left|B_{k-1}\right|>\epsilon|E|$, then by Lemma 9. we have $\left\|\widehat{1_{B_{k-1}} f}\right\|_{q} \geqslant \eta_{\epsilon}|E|^{1 / p}$. Then applying Proposition 15, we get $B_{k-1}=A_{k} \cup B_{k}$ with the conclusions of the proposition.

We note that this process terminates after finitely many steps since all of the $B_{i}$ are disjoint and after $m$ steps, $|E| \geqslant\left|B_{1}\right|+\cdots+\left|B_{m}\right|>m \epsilon|E|$. Thus we may suppose we have obtained a disjoint decomposition

$$
E=A_{1} \cup \cdots \cup A_{n} \cup B_{n}
$$

where $\left|B_{i}\right|>\epsilon|E|$ for $1 \leqslant i<n$ and $\left|B_{n}\right| \leqslant \epsilon|E|$. We also have multiprogressions $P_{i}$ satisfying $\left|P_{1}\right| \leqslant C_{\eta_{\delta}}\left|A_{1}\right|$, rank $P_{1} \leqslant C_{\eta_{\delta}}$ and for $1<i \leqslant n,\left|P_{i}\right| \leqslant C_{\eta_{\epsilon}}\left|A_{i}\right|$, rank $P_{i} \leqslant C_{\eta_{\epsilon}}$. Thus

$$
\sum_{i}\left|P_{i}\right| \leqslant C_{\epsilon}|E|,
$$

$A:=\cup_{i} A_{i} \subset \cup_{i} P_{i}$, rank $P_{i} \leqslant C_{\epsilon}$, and

$$
\left\|1_{A} f\right\|_{p} \geqslant\left\|1_{A_{1}} f\right\|_{p} \geqslant c_{\delta}\|f\|_{p},
$$

as desired.
Lemma 19 (More structured decomposition). Let $d \geqslant 1$, and let $\Lambda \subset(1,2)$ be a compact set. For any $\epsilon>0$ there exist $\delta>0, N_{\epsilon}<\infty$, and $C_{\epsilon}<\infty$ with the following property for all $p \in \Lambda$. Let $|E|<\infty$ and $|f| \leqslant 1_{E}$ be such that $\|\widehat{f}\|_{q} \geqslant(1-\delta) \mathbf{B}_{q, d}|E|^{1 / p}$. Then there exist a measurable decomposition $f=g+h$, where $g=g 1_{A}, h=h 1_{B}$, and $A \cap B=\varnothing$, and $a$ continuum multiprogression $P$ such that

$$
\begin{gathered}
|B| \leqslant \epsilon|E| \\
|P| \leqslant C_{\epsilon}|E| \\
A \subset P \\
\operatorname{rank} P \leqslant C_{\epsilon} .
\end{gathered}
$$

Proof. First we define $E_{\lambda}=\{x \in E:|f(x)| \leqslant \lambda\}$. Note that by the Hausdorff-Young inequality,

$$
\left\|\widehat{1_{E_{\lambda}} f}\right\|_{q} \leqslant\left\|1_{E_{\lambda}} f\right\|_{p} \leqslant \lambda|E|^{1 / p} .
$$

Assume that $\left|E_{\lambda}\right|>\epsilon|E|$. Then by Lemma 9 ,

$$
\left\|\widehat{1_{E_{\lambda}} f}\right\|_{q} \geqslant \frac{c_{0} \epsilon}{C}|E|^{1 / p}:=\eta_{\epsilon}|E|^{1 / p} .
$$

Thus if we take $\lambda=\eta_{\epsilon} \epsilon$, we are guaranteed that $\left|E_{\lambda}\right|<\epsilon|E|$. Now without loss of generality, assume that $|f| \geqslant \eta_{\epsilon} \epsilon$ on $E$.

We define an iterative process with an outer and an inner loop. For the step 1 of the outer loop, letting $\eta_{\delta}=1-\delta$, apply Proposition 15 to get $E=A_{1} \cup B_{1}$ where $A_{1}$ is contained in a multiprogression $P_{1}$ satisfying the conclusions of the proposition. At step $N$ of the outer loop, we have a measurable decomposition

$$
f=G_{N}+H_{N}
$$

where $H_{N}=1_{B_{N}} H_{N}$ and $G_{N}=1_{A_{N}} G_{N}$, where $A_{N} \cap B_{N}=\varnothing$ and $A_{N}$ is contained in a multiprogression $P_{N}$ with $\left|P_{N}\right| \leqslant C_{\epsilon}|E|$, rank $P_{N} \leqslant C_{\epsilon}$, and $\left\|G_{N}\right\|_{p} \geqslant c_{\delta}\|f\|_{p}$. If $\left|B_{N}\right|<$ $\epsilon|E|$, then we halt. Otherwise, initiate step $(N, 1)$ of the inner loop. Since $\left|B_{N}\right| \geqslant \epsilon|E|$, by Lemma 9, $\left\|\widehat{1_{B_{N}} f}\right\|_{q} \geqslant \eta_{\epsilon}|E|^{1 / p}$. Thus we can decompose $B_{N}$ into $S_{N, 1}$ (contained in a multiprogression) and $R_{N, 1}$ using Proposition 15. The halting criterion for the ( $N, j$ ) th step is $\left|R_{N, j}\right| \leqslant \frac{1}{2} \epsilon|E|$ or $\left\|\widehat{G_{N}} \widehat{1_{S_{N, j}} f}\right\|_{q / 2} \geqslant \rho|E|^{2 / p}$. If neither is satisfied in step $(N, j)$, then $\left|R_{N, j}\right|>\frac{1}{2} \epsilon|E|$, so repeat the argument described for step ( $N, 1$ ) replacing $B_{N}$ by $R_{N, j}$. After $k$ iterations of the inner loop, we note that

$$
\left|B_{N}\right| \geqslant\left|R_{N, 1}\right|+\cdots+\left|R_{N, k}\right| \geqslant k \epsilon|E|,
$$

so the inner loop terminates in a maximum of $M_{\epsilon}$ steps.
Suppose that the inner loop terminates at step $k$ because $\left|R_{N, k}\right| \leqslant \frac{1}{2} \epsilon|E|$ but $\left\|\widehat{G_{N}} \widehat{1_{S_{N, k}} f}\right\|_{q / 2}<$ $\rho|E|^{2 / p}$. Then $\left\|\widehat{G_{N}} \widehat{1_{S_{N, j}} f}\right\|_{q / 2}<\rho|E|^{2 / p}$ for $1 \leqslant j \leqslant k$. Define $h=\sum_{j=1}^{k} 1_{S_{N, j}} f$. Note that

$$
\begin{equation*}
\left\|\widehat{G_{N}} \widehat{h}\right\|_{q / 2} \leqslant \sum_{j=1}^{k}\left\|\widehat{G_{N}} \widehat{1_{S_{N, j}}}\right\|_{q / 2}<M_{\epsilon} \rho|E|^{2 / p} \tag{3.1}
\end{equation*}
$$

However, $|\operatorname{supp} h|=\sum_{j=1}^{k}\left|S_{N, k}\right| \geqslant\left|B_{N}\right|-\left|R_{N, k}\right| \geqslant \epsilon|E|-\frac{1}{2} \epsilon|E|=\frac{\epsilon}{2}|E|$ and

$$
\left|\operatorname{supp} G_{N}\right|=\left|A_{N}\right| \geqslant\left\|G_{N}\right\|_{p}^{p} \geqslant c_{\delta}\|f\|_{p}^{p} \geqslant c_{\delta} \eta_{\epsilon} \epsilon|E|^{1 / p}
$$

where we used the assumption that $|f| \geqslant \eta_{\epsilon} \epsilon$ discussed at the beginning of the proof. Finally note that $\left\|\widehat{G_{N}+h}\right\|_{q} \geqslant\|\widehat{f}\|_{q}-\left\|\widehat{1_{R_{N, k}} f}\right\| q \geqslant\left(1-\epsilon-\epsilon^{1 / p}\right) \mathbf{B}_{q, d}|E|^{1 / p}$. Thus, choosing $\delta$ and $\rho$ small enough, (3.1) contradicts Lemma 11 .

Thus the halting criterion for the inner loop yields a function $1_{S_{N, k}} f$ such that

$$
\begin{equation*}
\left\|\widehat{G_{N}} \widehat{1_{S_{N, k}} f}\right\|_{q / 2} \geqslant \rho|E|^{2 / p} \tag{3.2}
\end{equation*}
$$

The function $1_{S_{N, k}} f$ also satisfies

$$
\begin{equation*}
\left\|1_{S_{N, k}} f\right\|_{p} \geqslant c_{\epsilon}\left\|1_{R_{N, k-1}} f\right\|_{p} \geqslant c_{\epsilon} \eta_{\epsilon} \epsilon\left|R_{N, k-1}\right|^{1 / p} \geqslant c_{\epsilon} \eta_{\epsilon} \epsilon^{1+1 / p}|E|^{1 / p} . \tag{3.3}
\end{equation*}
$$

If $Q_{N}$ is the multiprogression associated to $S_{N, k}$, then Lemma 17 (taking $\varphi=\frac{1}{C_{\epsilon \epsilon E \mid}} 1_{A_{N}} f$ and $\psi=\frac{1}{C_{\epsilon}|E|} 1_{S_{N, k}} f$, which satisfies the hypotheses for small enough $\rho$ ) implies that $\mid P_{N}+$ $Q_{N} \mid \leqslant C_{\epsilon}^{\prime} \min \left(\left|P_{N}\right|,\left|Q_{N}\right|\right)$. Thus there exists a continuum multiprogression $P_{N+1}$ of rank $\leqslant C_{\epsilon}$ containing $P_{N}$ and $Q_{N}$ and satisfying $\left|P_{N+1}\right| \leqslant C_{\epsilon}|E|$.

Set $G_{N+1}=G_{N}+1_{S_{N, k}} f$. Then $H_{N+1}:=f-G_{N+1}$ has support called $B_{N+1}$. If $\left|B_{N+1}\right| \leqslant \epsilon|E|$, then we're done. If not, proceed to outer loop step $N+2$. Note that for each outer loop step, we have

$$
\left\|G_{N+1}\right\|_{p}^{p} \geqslant\left\|G_{N}\right\|_{p}^{p}+\left\|1_{S_{N, k}} f\right\|_{p}^{p} \geqslant\left\|G_{N}\right\|_{p}^{p}+c_{\epsilon} \eta_{\epsilon} \epsilon^{p+1}|E|
$$

Thus the outer loop terminates in at most $N_{\epsilon}$ steps. Note that since the ranks of $P_{N}$ and $Q_{N}$ at most add at each step of the outer loop, the rank of the ultimate multiprogression is controlled by $M_{\epsilon}>0$.

Lemma 20. Let $d \geqslant 1$, and let $\Lambda \subset(1,2)$ be a compact set. For any $\epsilon>0$ there exist $\delta>0$, $N_{\epsilon}<\infty$, and $C_{\epsilon}<\infty$ with the following property for all $p \in \Lambda$. Let $|E|<\infty$ and $|f| \leqslant 1_{E}$ be such that $\|\widehat{f}\|_{q, *} \geqslant(1-\delta) \mathbf{B}_{q, d}\|f\|_{p}$. Then there exists a measurable decomposition $f=g+h$ where $g=1_{A} g, h=1_{B} h, A \cap B=\varnothing$, and there is a continuum multiprogression $P$ such that

$$
\begin{gathered}
\|h\|_{p} \leqslant \epsilon\|f\|_{p} \\
\|g\|_{\infty}|P|^{1 / p} \leqslant C_{\epsilon}\|f\|_{p} \\
A \subset P \\
\operatorname{rank} P \leqslant C_{\epsilon} .
\end{gathered}
$$

Proof. Using the hypothesis $\|\widehat{f}\|_{q, *} \geqslant(1-\delta) \mathbf{B}_{q, d}\|f\|_{p}$, by Lemma 16 there exists a disjoint decomposition $f=g_{1}+h_{1}$ where $g_{1}$ is supported on a multiprogression $P_{1}$ with rank $P_{1} \leqslant C_{\delta}$, $\left\|g_{1}\right\|_{p} \geqslant c_{\delta}\|f\|_{p},\left\|g_{1}\right\|_{\infty}\left|P_{1}\right|^{1 / p} \leqslant C_{\delta}\|f\|_{p}$. If $\left\|h_{1}\right\|_{p}<\epsilon\|f\|_{p}$, then we halt.

To further refine the decomposition in the case that $\left\|h_{1}\right\|_{p} \geqslant \epsilon\|f\|_{p}$, define an iterative process with input ( $g_{1}, h_{1}$ ) and output ( $g_{2}, h_{2}$ ) where $f=g_{2}+h_{2}$ and $g_{2}, h_{2}$ satisfy certain properties below. Apply Lemma 10 to conclude that $\left\|\widehat{h_{1}}\right\|_{q, *} \geqslant \eta_{\epsilon}\|f\|_{p}$ for $\eta_{\epsilon}>0$. Then apply Lemma 16 to get $h_{1}=u_{1}+v_{1}$ where $u_{1}$ is supported on a multiprogression $Q_{1}$, rank $Q_{1} \leqslant C_{\epsilon}$, $\left\|u_{1}\right\|_{\infty}\left|Q_{1}\right|^{1 / p} \leqslant C_{\epsilon}\|f\|_{p}$, and $\left\|u_{1}\right\|_{p} \geqslant c_{\epsilon}\left\|h_{1}\right\|_{p} \geqslant c_{\epsilon} \epsilon\|f\|_{p}$.

Choose $\delta$ suffciently small to ensure that $c_{\delta} \geqslant \epsilon c_{\epsilon}$. Since $\min \left(\left\|g_{1}\right\|_{p},\left\|u_{1}\right\|_{p}\right) \geqslant \epsilon c_{\epsilon}\|f\|_{p}$, by Lemma 12, $\left\|\left|\widehat{g_{1}}\right|^{1 / 2}\left|\widehat{u_{1}}\right|^{1 / 2}\right\|_{q, *} \geqslant \rho(\epsilon)\|f\|_{p} \mathbf{B}_{q, d}$ for $\rho(\epsilon)>0$. But then Lemma 17 (taking
$\varphi=\frac{1}{C_{\delta}\|f\|_{p}} g_{1}$ and $\left.\psi=\frac{1}{C_{\epsilon}\|f\|_{p}} u_{1}\right)$ implies that $\max \left(\left|P_{1}\right|,\left|Q_{1}\right|\right) \leqslant C_{\epsilon}^{\prime} \min \left(\left|P_{1}\right|,\left|Q_{1}\right|\right)$ and $\mid P_{1}+$ $Q_{1} \mid \leqslant C_{\epsilon}^{\prime} \min \left(\left|P_{1}\right|,\left|Q_{1}\right|\right)$. Thus there exists a continuum multiprogression $P_{2}$ of rank $\leqslant C_{\epsilon, \delta}$ containing $P_{1}$ and $Q_{1}$ and satisfying $\left|P_{2}\right| \leqslant C_{\epsilon}$. Define $g_{2}:=g_{1}+u_{1}$ and $h_{2}:=v_{1}$.

If $\left\|h_{2}\right\|_{p}<\epsilon\|f\|_{p}$, then halt. If $\left\|h_{2}\right\|_{p} \geqslant \epsilon\|f\|_{p}$, repeat the process described above with input $\left(g_{2}, h_{2}\right)$.

After $n$ steps of this iteration, we have a decomposition $f=g_{n}+h_{n}$ and a multiprogression $P_{n}$ of controlled size and rank containing the support of $g_{n}$ and satisfying $\left\|g_{n}\right\|_{\infty}\left|P_{n}\right|^{1 / p} \leqslant$ $C_{\epsilon}\|f\|_{p}$, and

$$
\left\|g_{n}\right\|_{p}^{p}=\left\|g_{1}\right\|_{p}^{p}+\left\|u_{1}\right\|_{p}^{p}+\cdots+\left\|u_{n-1}\right\|_{p}^{p} \geqslant\left(c_{\delta}^{p}+(n-1) c_{\epsilon}^{p} \epsilon^{p}\right)\|f\|_{p}^{p} .
$$

Thus the loop terminates in at most $n_{\epsilon}$ steps. Note that since the ranks of $P_{n}$ and $Q_{n}$ at most add at each step of the process, the rank of the ultimate multiprogression is controlled by a constant depending on $\epsilon$. Also, $\left|P_{n}\right| \leqslant\left(C_{\epsilon}^{\prime}\right)^{n-1}\left(\min \left(\left|P_{1}\right|,\left|Q_{1}\right|, \ldots,\left|Q_{n-1}\right|\right)\right.$.

Finally we note that

$$
\begin{aligned}
\left\|g_{n}\right\|_{\infty}\left|P_{n}\right|^{1 / p} & \leqslant\left(C_{\epsilon}^{\prime}\right)^{(n-1)}\left(\left\|g_{1}\right\|_{\infty}\left|P_{1}\right|^{1 / p}+\left\|u_{1}\right\|_{\infty}\left|Q_{1}\right|^{1 / p}+\cdots+\left\|u_{n-1}\right\|_{\infty}\left|Q_{n-1}\right|^{1 / p}\right) \\
& \leqslant\left(C_{\epsilon}^{\prime}\right)^{n}(n-1)\|f\|_{p} .
\end{aligned}
$$

## Chapter 4

## Exploitation of $\mathbb{Z}^{\kappa} \times \mathbb{R}^{d}$

### 4.1 Analysis of the discrete Hausdorff-Young inequality

Let $\mathbb{T}$ denote the quotient group $\mathbb{R} / \mathbb{Z}$. Extend the previous notation and define the Fourier transform $\hat{\wedge}: \mathbb{Z}^{\kappa} \times \mathbb{R}^{d} \rightarrow \mathbb{T}^{\kappa} \times \mathbb{R}^{d}$ by

$$
\widehat{f}(\theta, \xi)=\int_{\mathbb{R}^{d}} \sum_{n \in \mathbb{Z}^{k}} e^{-2 \pi i x \cdot \xi} e^{-2 \pi i n \cdot \theta} f(n, x) d x
$$

where $\theta \in \mathbb{T}^{d}$. This can be decomposed as $\mathcal{F} \circ \tilde{\mathcal{F}}$ where

$$
\begin{aligned}
\mathcal{F} g(\theta, \xi) & =\sum_{n \in \mathbb{Z}^{\kappa}} g(n, \xi) e^{-2 \pi i n \cdot \theta} \\
\tilde{\mathcal{F}} f(n, \xi) & =\int_{\mathbb{R}^{d}} f(n, x) e^{-2 \pi i x \cdot \xi} d x
\end{aligned}
$$

If we treat the operator $\mathcal{F}$ as the partial Fourier transform with respect to the first coordinate and $\tilde{F}$ the corresponding transform for the second coordinate, then we can say $\mathcal{F} \circ \tilde{\mathcal{F}}=\tilde{\mathcal{F}} \circ \mathcal{F}$ (even though the operators on the left and right are not precisely the same).

Lemma 21. Let $d, \kappa \geqslant 1$, and $p \in(1,2), q=p^{\prime}$. The optimal constant $\mathbf{A}(q, d, \kappa)$ in the inequality

$$
\begin{equation*}
\|\widehat{f}\|_{q} \leqslant \mathbf{A}(q, d, \kappa)|E|^{1 / p} \tag{4.1}
\end{equation*}
$$

where $E \subset \mathbb{Z}^{\kappa} \times \mathbb{R}^{d}$ satisfies $|E|<\infty$ and $|f| \leqslant 1_{E}$, satisfies

$$
\mathbf{A}(q, d, \kappa)=\mathbf{B}_{q, d} .
$$

The optimal constant $\mathbf{A}^{\prime}(q, d, \kappa)$ for the inequality

$$
\|\widehat{f}\|_{q, *} \leqslant \mathbf{A}^{\prime}(q, d, \kappa)\|f\|_{p}
$$

for $\mathbb{Z}^{\kappa} \times \mathbb{R}^{d}$ likewise satisfies $\mathbf{A}^{\prime}(q, d, \kappa)=\mathbf{B}_{q, d}$.

Proof of Lemma 21. We analyze the mixed $L^{p} \operatorname{norms} L_{n}^{p} L_{\xi}^{q}\left(\mathbb{Z}_{n}^{\kappa} \times \mathbb{R}_{\xi}^{d}\right)$ and $L_{\xi}^{q} L_{n}^{p}\left(\mathbb{Z}_{n}^{\kappa} \times \mathbb{R}_{\xi}^{d}\right)$, given respectively by

$$
\|g\|_{L_{n}^{p} L_{\xi}^{q}}=\left(\sum_{n}\left(\int|g(n, \xi)|^{q} d \xi\right)^{p / q}\right)^{1 / p} \quad \text { and } \quad\|g\|_{L_{\xi}^{q} L_{n}^{p}}=\left(\int\left(\sum_{n}|g(n, \xi)|^{p}\right)^{q / p} d \xi\right)^{1 / q}
$$

There are corresponding norms for $L_{\theta}^{s} L_{x}^{t}\left(\mathbb{T}_{\theta}^{\kappa} \times \mathbb{R}_{x}^{d}\right)$ and $L_{x}^{t} L_{\theta}^{s}\left(\mathbb{T}_{\theta}^{\kappa} \times \mathbb{R}_{x}^{d}\right)$. Since $q \geqslant p$, we have by Minkowski's integral inequality that

$$
\|g\|_{L_{\theta}^{q} L_{x}^{p}\left(\mathbb{T}^{\kappa} \times \mathbb{R}^{d}\right)} \leqslant\|g\|_{L_{x}^{p} L_{\theta}^{q}\left(\mathbb{T}^{\kappa} \times \mathbb{R}^{d}\right)}
$$

If $\mathfrak{F}$ denotes the Fourier transform from $\mathbb{Z}^{\kappa}$ to $\mathbb{T}^{\kappa}$ defined by

$$
\mathfrak{F} g(\theta)=\sum_{n} g(n) e^{-2 \pi i n \cdot \theta}
$$

then the optimal constant in the corresponding Hausdorff-Young inequality for $p \in(1,2)$ is 1. Thus if $|f| \leqslant 1_{E}$ for $E \subset \mathbb{Z}^{\kappa}$ and $|E|<\infty$, we have

$$
\begin{equation*}
\|\mathfrak{F} f\|_{q} \leqslant\|f\|_{p} \leqslant|E|^{1 / p} . \tag{4.2}
\end{equation*}
$$

This means that for $g \in L_{\xi}^{q} L_{n}^{p}\left(\mathbb{Z}_{n}^{\kappa} \times \mathbb{R}_{\xi}^{d}\right)$,

$$
\begin{aligned}
\|\mathcal{F} g\|_{L_{\xi}^{q} L_{\theta}^{q}} & =\left(\iint|\mathcal{F} g(\theta, \xi)|^{q} d \theta d \xi\right)^{1 / q} \\
& \leqslant\left(\int\left(\sum_{n}|g(n, \xi)|^{p}\right)^{q / p} d \xi\right)^{1 / q},
\end{aligned}
$$

so $\mathcal{F}$ is a contraction from $L_{\xi}^{q} L_{n}^{p}\left(\mathbb{R}_{\xi}^{d} \times \mathbb{Z}_{n}^{\kappa}\right)$ to $L_{\xi}^{q} L_{\theta}^{q}\left(\mathbb{R}_{\xi}^{d} \times \mathbb{T}_{\theta}^{\kappa}\right)$.
Let $|f| \leqslant 1_{E} \in L^{p}\left(\mathbb{Z}^{\kappa} \times \mathbb{R}^{d}\right)$. For $n \in \mathbb{Z}^{\kappa}$, define the subset $E_{n} \subset \mathbb{R}^{d}$ and the function $f_{n}: \mathbb{R}^{d} \rightarrow \mathbb{C}$ by

$$
\begin{array}{r}
E_{n}=\left\{x \in \mathbb{R}^{d}:(n, x) \in E\right\} \\
f_{n}(x)=f(n, x), \tag{4.4}
\end{array}
$$

noting that $f_{n} \in L^{p}\left(\mathbb{R}^{d}\right)$. Since $\left|f_{n}\right| \leqslant 1_{E_{n}}$,

$$
\begin{align*}
\|\tilde{\mathcal{F}} f\|_{L_{n}^{p} L_{\xi}^{q}} & =\left(\sum_{n}\left(\int|\tilde{\mathcal{F}} f(n, \xi)|^{q} d \xi\right)^{p / q}\right)^{1 / p} \\
& =\left(\sum_{n}\left(\int\left|\int f_{n}(x) e^{-2 \pi i x \cdot \xi} d x\right|^{q} d \xi\right)^{p / q}\right)^{1 / p} \\
& \leqslant\left(\sum_{n} \mathbf{B}_{q, d}^{p}\left|E_{n}\right|\right)^{1 / p}=\mathbf{B}_{q, d}|E|^{1 / p} \tag{4.5}
\end{align*}
$$

Combining the above inequalities yields

$$
\begin{equation*}
\|\hat{f}\|_{L^{q}\left(\mathbb{Z}^{\kappa} \times \mathbb{R}^{d}\right)}=\|\mathcal{F} \tilde{\mathcal{F}} f\|_{L_{\xi}^{q} L_{\theta}^{q}} \leqslant\|\tilde{\mathcal{F}} f\|_{L_{\xi}^{q} L_{n}^{p}} \leqslant\|\tilde{\mathcal{F}} f\|_{L_{n}^{p} L_{\xi}^{q}} \leqslant \mathbf{B}_{q, d}|E|^{1 / p} \tag{4.6}
\end{equation*}
$$

where we use 4.5) in the last inequality. Thus $\mathbf{A}(q, d, \kappa) \leqslant \mathbf{B}_{q, d}$. Now let $|f| \leqslant 1_{E} \in L^{p}\left(\mathbb{R}^{d}\right)$ be given. Define $E_{0}=\{0\} \times E$ and $f_{0}: \mathbb{Z}^{\kappa} \times \mathbb{R}^{d} \rightarrow \mathbb{C}$ by $f_{0}(n, x)=0$ for $n \neq 0$ and $f_{0}(0, x)=$ $f(x)$. Let $\tilde{\mathfrak{F}}$ denote the Fourier transform on $\mathbb{R}^{d}$ defined by $\tilde{\mathfrak{F}} g(\xi)=\int g(x) e^{-2 \pi i x \cdot \xi} d x$. Then

$$
\begin{aligned}
\|\tilde{\mathfrak{F}} f\|_{L^{q}\left(\mathbb{R}^{d}\right)} & =\left\|\mathcal{F} f_{0}(0, \cdot)\right\|_{L_{\xi}^{q}} \\
& =\|\mathcal{F} \tilde{f}\|_{L_{\xi}^{q} L_{n}^{q}} \\
& \leqslant \mathbf{A}(q, d, \kappa)\left|E_{0}\right|^{1 / p}=\mathbf{A}(q, d, \kappa)|E|^{1 / p} .
\end{aligned}
$$

This yields the reverse inequality $\mathbf{A}(q, d, \kappa) \geqslant \mathbf{B}_{q, d}$.
Now consider $\mathbf{A}^{\prime}(q, d, \kappa)$. Let $f \in L^{p}\left(\mathbb{Z}^{\kappa} \times \mathbb{R}^{d}\right)$ and let $E \subset \mathbb{Z}^{k} \times \mathbb{R}^{d}$ be a Lebesgue measurable set satisfying $|E| \in \mathbb{R}^{+}$. Writing $E_{\theta}=\{\xi:(\theta, \xi) \in E\}$,

$$
\begin{aligned}
\int_{E}|\widehat{f}| & =\int_{\mathbb{T}^{\kappa}} \int_{\mathbb{R}^{d}}|\mathcal{F} \tilde{\mathcal{F}} f(\theta, \xi)| 1_{E}(\theta, \xi) d \xi d \theta \\
& \leqslant \int_{\mathbb{T}^{\kappa}}\|\tilde{\mathcal{F}} \mathcal{F} f(\theta, \cdot)\|_{L_{\xi}^{q, *}}\left|E_{\theta}\right|^{1 / p} d \theta \\
& \leqslant \int_{\mathbb{T}^{\kappa}} \mathbf{B}_{q, d}\|\mathcal{F} f(\theta, \cdot)\|_{L_{x}^{p}}\left|E_{\theta}\right|^{1 / p} d \theta \\
& \leqslant \mathbf{B}_{q, d}\left(\int_{\mathbb{T}^{\kappa}}\|\mathcal{F} f(\theta, \cdot)\|_{L_{x}^{p}}^{q} d \theta\right)^{1 / q}\left(\int_{\mathbb{T}^{\kappa}}\left|E_{\theta}\right|^{p / p} d \theta\right)^{1 / p} \\
& \leqslant \mathbf{B}_{q, d}\|\mathcal{F} f\|_{L_{x}^{p} L_{\theta}^{q}|E|^{1 / p}} \\
& \leqslant \mathbf{B}_{q, d}\|f\|_{L^{p}}|E|^{1 / p}
\end{aligned}
$$

so $\mathbf{A}^{\prime}(q, d, \kappa) \leqslant \mathbf{B}_{q, d}$. For the reverse inequality, let $f \in L^{p}\left(\mathbb{R}^{d}\right)$ and let $E \subset \mathbb{R}^{d}$ be a Lebesgue measurable set with $|E| \in \mathbb{R}^{+}$. Let $f_{0}$ and $E_{0}$ be defined as above. Then

$$
\begin{aligned}
\int_{E}|\tilde{\mathfrak{F}} f| & =\sum_{n} \int\left|\mathcal{F} f_{0}(n, \xi)\right| 1_{E_{0}}(n, \xi) d \xi \\
& \leqslant \mathbf{A}^{\prime}(q, d, \kappa)\left|E_{0}\right|^{1 / p}\left\|f_{0}\right\|_{L^{p}\left(\mathbb{Z}^{\kappa} \times \mathbb{R}^{d}\right)} \\
& =\mathbf{A}^{\prime}(q, d, \kappa)|E|^{1 / p}\|f\|_{L^{p}\left(\mathbb{R}^{d}\right)} .
\end{aligned}
$$

In the remainder of the subsection, we prove the following two propositions concerning the structure of near-extremizers of the sharp Hausdorff-Young inequality on $\mathbb{Z}^{\kappa} \times \mathbb{R}^{d}$.

Proposition 22. Let $d, \kappa \geqslant 1$ and $q \in(2, \infty), p=q^{\prime}$. Let $\delta>0$ be small. Let $0 \neq$ $f \in L^{p}\left(\mathbb{Z}^{\kappa} \times \mathbb{R}^{d}\right),|f| \leqslant 1_{E}$ where $E \subset \mathbb{Z}^{\kappa} \times \mathbb{R}^{d}$ is Lebesgue measurable and $|E|<\infty$. If $\|\widehat{f}\|_{q} \geqslant(1-\delta) \mathbf{B}_{q, d}|E|^{1 / p}$, then there exists $m \in \mathbb{Z}^{\kappa}$ such that

$$
\begin{equation*}
\left|E_{m}\right| \geqslant\left(1-o_{\delta}(1)\right)|E| \tag{4.7}
\end{equation*}
$$

where $E_{m}$ is defined in (4.3).
The analogous proof of Proposition 22 for the dual inequality fails to go through, which is the reason our results are partial. This leads to the following question, which is left open. Our final precompactness result is conditional on a positive answer to this question.

Question 9. Let $d, \kappa \geqslant 1$ and $q \in(2, \infty), p=q^{\prime}$. Let $\delta>0$ be small. Let $0 \neq f \in L^{p}\left(\mathbb{Z}^{\kappa} \times \mathbb{R}^{d}\right)$. If $\|\hat{f}\|_{q, *} \geqslant(1-\delta) \mathbf{B}_{q, d}\|f\|_{p}$, then must there exist $m \in \mathbb{Z}^{\kappa}$ such that

$$
\begin{equation*}
\left\|f_{m}\right\|_{L^{p}\left(\mathbb{R}^{d}\right)} \geqslant\left(1-o_{\delta}(1)\right)\|f\|_{L^{p}\left(\mathbb{Z}^{\kappa} \times \mathbb{R}^{d}\right)} \tag{4.8}
\end{equation*}
$$

where $f_{m}$ is defined in (4.4)?
In the analysis of $\mathbf{A}(q, d, \kappa)$ from Lemma 21, we proved a string of inequalities in (4.6). Combining these inequalities with the assumption that $(f, E)$ are $\delta$-near extremizing yields the following lemma, which requires no further proof.

Lemma 23. Let $d, \kappa \geqslant 1$ and $q \in(2, \infty)$. Set $p=q^{\prime}$. Let $\delta>0$, let $E \subset \mathbb{Z}^{\kappa} \times \mathbb{R}^{d}$ be a Lebesgue measurable set with $|E| \in \mathbb{R}^{+}$, and let $f$ be a measurable function satisfying $|f| \leqslant 1_{E}$. If $\|\hat{f}\|_{q} \geqslant(1-\delta) \mathbf{B}_{q, d}|E|^{1 / p}$, then all of the following hold:

$$
\begin{align*}
&\|\mathcal{F} \tilde{\mathcal{F}} f\|_{L_{\xi}^{q} L_{\theta}^{q}} \geqslant(1-\delta)\|\tilde{\mathcal{F}} f\|_{L_{n}^{p} L_{\xi}^{q}}  \tag{4.9}\\
&\|\tilde{\mathcal{F}} f\|_{L_{\xi}^{q} L_{n}^{p}} \geqslant(1-\delta)\|\tilde{\mathcal{F}} f\|_{L_{n}^{p} L_{\xi}^{q}}  \tag{4.10}\\
&\|\tilde{\mathcal{F}} f\|_{L_{n}^{p} L_{\xi}^{q}} \geqslant(1-\delta) \mathbf{B}_{q, d}|E|^{1 / p} \tag{4.11}
\end{align*}
$$

The inequalities listed in Lemma 23 will be used to establish the following weak result, which is a preliminary for showing that any near extremizer of the lifted problem is mostly supported on one slice of the $\mathbb{Z}^{\kappa}$ variable.

Lemma 24. Let $E \subset \mathbb{Z}^{\kappa} \times \mathbb{R}^{d}$ and $|f| \leqslant 1_{E}$ satisfy $\|\widehat{f}\|_{q} \geqslant(1-\delta) \mathbf{B}_{q, d}|E|^{1 / p}$. There exists a disjointly supported decomposition

$$
\tilde{\mathcal{F}} f(n, \xi)=g(n, \xi)+h(n, \xi)
$$

where

$$
\|h\|_{L_{\xi}^{q} L_{n}^{p}} \leqslant o_{\delta}(1)|E|^{1 / p}
$$

and for each $\xi \in \mathbb{R}^{d}$ there exists $n(\xi) \in \mathbb{Z}^{\kappa}$ such that

$$
g(n, \xi)=0 \quad \text { for all } n \neq n(\xi)
$$

Proof of Lemma 24. This is completely analogous to the proof of Lemma 10.14 in [16].
Let $\eta=\delta^{1 / 2}$. Since $|f| \leqslant 1_{E}$, for each $n \in \mathbb{Z}^{\kappa}$ the function $\tilde{\mathcal{F}} f(n, \xi)$ is a continuous function of $\xi$. Thus $\varphi_{\xi}(n):=\tilde{\mathcal{F}} f(n, \xi)$ is well-defined for every $\xi \in \mathbb{R}^{d}$. Define

$$
\mathcal{G}=\left\{\xi \in \mathbb{R}^{d}: \varphi_{\xi} \neq 0, \quad\left\|\widehat{\varphi_{\xi}}\right\|_{L_{\theta}^{q}} \geqslant(1-\eta)\left\|\varphi_{\xi}\right\|_{L_{n}^{p}}\right\} .
$$

Here, $\hat{\cdot}$ denotes the Fourier transform for $\mathbb{Z}^{\kappa}$. Then

$$
\begin{aligned}
\|\mathcal{F} \tilde{\mathcal{F}} f\|_{L_{\theta}^{q} L_{\xi}^{q}}^{q} & =\int_{\mathbb{R}^{d} \backslash \mathcal{G}}\left\|\widehat{\varphi_{\xi}}\right\|_{L_{\theta}^{q}}^{q} d \xi+\int_{\mathcal{G}}\left\|\widehat{\varphi_{\xi}}\right\|_{L_{\theta}^{q}}^{q} d \xi \\
& \leqslant(1-\eta)^{q} \int_{\mathbb{R}^{d} \backslash \mathcal{G}}\left\|\varphi_{\xi}\right\|_{L_{n}^{p}}^{q} d \xi+\int_{\mathcal{G}}\left\|\varphi_{\xi}\right\|_{L_{n}^{p}}^{q} d \xi \\
& \leqslant \int_{\mathbb{R}^{d}}\|\tilde{\mathcal{F}} f\|_{L_{n}^{p}}^{q} d \xi-c \eta \int_{\mathbb{R}^{d} \backslash \mathcal{G}}\|\tilde{\mathcal{F}} f\|_{L_{n}^{p}}^{q} d \xi .
\end{aligned}
$$

Combining this with 4.9), we get

$$
\begin{aligned}
(1-\delta)^{q}\|\tilde{\mathcal{F}} f\|_{L_{n}^{p} L_{\xi}^{q}}^{q} & \leqslant\|\mathcal{F} \tilde{\mathcal{F}} f\|_{L_{\xi}^{q} L_{\theta}^{q}}^{q} \\
& \leqslant \int_{\mathbb{R}^{d}}\|\tilde{\mathcal{F}} f\|_{L_{n}^{p}}^{q} d \xi-c \eta \int_{\mathbb{R}^{d} \backslash \mathcal{G}}\|\tilde{\mathcal{F}} f\|_{L_{n}^{p}}^{q} d \xi .
\end{aligned}
$$

Rearranging the above inequality, obtain

$$
\begin{equation*}
\int_{\mathbb{R}^{d} \backslash \mathcal{G}}\|\tilde{\mathcal{F}} f\|_{L_{n}^{p}}^{q} d \xi \leqslant c^{\prime} \delta^{1 / 2}\|\tilde{\mathcal{F}} f\|_{L_{n}^{p} L_{\xi}^{q}}^{q} . \tag{4.12}
\end{equation*}
$$

For each $\xi \in \mathcal{G},\left\|\widehat{\varphi_{\xi}}\right\|_{L_{\theta}^{q}} \geqslant(1-\eta)\left\|\varphi_{\xi}\right\|_{L_{n}^{p}}$, so we can invoke the argument beginning in line (7) of [25] or Theorem 1.3 from [8] to get $n=n(\xi) \in \mathbb{Z}^{\kappa}$ such that

$$
\left\|\varphi_{\xi}\right\|_{L^{p}\left(\mathbb{Z}^{k} \backslash\{n(\xi)\}\right)} \leqslant o_{\eta}(1)\left\|\varphi_{\xi}\right\|_{L^{p}\left(\mathbb{Z}^{\kappa}\right)} .
$$

Define

$$
g(n, \xi)= \begin{cases}\varphi_{\xi}(n) & \text { if } n=n(\xi), \xi \in \mathcal{G} \\ 0 & \text { else. }\end{cases}
$$

Let $h(n, \xi):=\tilde{\mathcal{F}} f(n, \xi)-g(n, \xi)$. Note that $g$ satisfies the conclusions of the lemma by its definition. To bound $\|h\|_{L_{\xi}^{q} L_{n}^{p}}$, we use the definition of $g$ as well as 4.12) to get

$$
\begin{aligned}
\|h\|_{L^{q} L_{n}^{p}}^{q} & \leqslant \int_{\mathcal{G}}\|\tilde{\mathcal{F}} f-g\|_{L_{n}^{p}}^{q} d \xi+\int_{\mathbb{R}^{d} \backslash \mathcal{G}}\|\tilde{\mathcal{F}} f\|_{L_{n}^{p}}^{q} d \xi \\
& =\int_{\mathcal{G}}\left(\|\tilde{\mathcal{F}} f\|_{L^{p}\left(\mathbb{Z}^{\kappa} \backslash n(\xi)\right)}^{p}+|\tilde{\mathcal{F}} f(n(\xi), \xi)-g(n(\xi), \xi)|^{p}\right)^{q / p} d \xi+\int_{\mathbb{R}^{d} \backslash \mathcal{G}}\|\tilde{\mathcal{F}} f\|_{L_{n}^{p}}^{q} d \xi \\
& =\int_{\mathcal{G}}\left(\|\tilde{\mathcal{F}} f\|_{L^{p}}^{p} \mathbb{Z}^{\kappa} \backslash n(\xi)\right) \\
& \leqslant 0)^{q / p} d \xi+\int_{\mathbb{R}^{d} \backslash \mathcal{G}}\|\tilde{\mathcal{F}} f\|_{L_{n}^{p}}^{q} d \xi \\
& \left(o_{\eta}(1)\|\tilde{\mathcal{F}} f\|_{L^{p}\left(\mathbb{Z}^{\kappa}\right)}\right)^{q} d \xi+c^{\prime} \delta^{1 / 2}\|\tilde{\mathcal{F}} f\|_{L_{\xi}^{q} L_{n}^{p}}^{q}=o_{\delta}(1)\|\tilde{\mathcal{F}} f\|_{L_{\xi}^{q} L_{n}^{p}}^{q} .
\end{aligned}
$$

Proof of Proposition 22. Let $\tilde{\mathcal{F}} f=g+h$ as in Lemma 24 . Combining $\|h\|_{L_{\xi}^{q} L_{n}^{p}} \leqslant o_{\delta}(1)(1)|E|^{1 / p}$ with 4.11) implies $\|h\|_{L_{\xi}^{q} L_{n}^{p}} \leqslant o_{\delta}(1)\|\tilde{\mathcal{F}} f\|_{L_{n}^{p} L_{\xi}^{q}}$. Using this with 4.9) gives

$$
\|g\|_{L_{\xi}^{q} L_{n}^{p}}+\|h\|_{L_{\xi}^{q} L_{n}^{p}} \geqslant\|\mathcal{F} \tilde{\mathcal{F}} f\|_{L_{\xi}^{q} L_{\theta}^{q}} \geqslant(1-\delta)\|\tilde{\mathcal{F}} f\|_{L_{n}^{p} L_{\xi}^{q}},
$$

from which we conclude

$$
\|g\|_{L_{\xi}^{q} L_{n}^{p}} \geqslant\left(1-o_{\delta}(1)\right)\|\tilde{\mathcal{F}} f\|_{L_{n}^{p} L_{\xi}^{q}} .
$$

Noting that $\|g\|_{L_{\xi}^{q} L_{n}^{p}}^{q}=\|g\|_{L_{\xi}^{q} L_{n}^{q}}^{q}$, we further have

$$
\begin{equation*}
\|g\|_{L_{\xi}^{q} L_{n}^{q}}=\|g\|_{L_{\xi}^{q} L_{n}^{p}} \geqslant\left(1-o_{\delta}(1)\right)\|\tilde{\mathcal{F}} f\|_{L_{n}^{p} L_{\xi}^{q}} \geqslant\left(1-o_{\delta}(1)\right)\|g\|_{L_{n}^{p} L_{\xi}^{q}} . \tag{4.13}
\end{equation*}
$$

Let $M=\sup _{n}\|g(n, \cdot)\|_{L_{\xi}^{q}}^{q}($ which is finite by (4.13) ) and calculate using 4.13)

$$
\begin{aligned}
M^{\frac{q-p}{p q}}\left(\int|g(n(\xi), \xi)|^{q} d \xi\right)^{1 / q} & \geqslant\left(1-o_{\delta}(1)\right) M^{\frac{q-p}{p q}}\left(\sum_{n}\left(\int|g(n, \xi)|^{q} d \xi\right)^{p / q}\right)^{1 / p} \\
& \geqslant\left(1-o_{\delta}(1)\right)\left(\sum_{n} \int|g(n, \xi)|^{q} d \xi\right)^{1 / p} \\
& =\left(1-o_{\delta}(1)\right)\left(\int|g(n(\xi), \xi)|^{q} d \xi\right)^{1 / p}
\end{aligned}
$$

and therefore

$$
M \geqslant\left(1-o_{\delta}(1)\right)^{\frac{p q}{q-p}}\left(\int|g(n(\xi), \xi)|^{q} d \xi\right)^{\left(\frac{1}{p}-\frac{1}{q}\right)\left(\frac{p q}{q-p}\right)}=\left(1-o_{\delta}(1)\right) \int|g(n(\xi), \xi)|^{q} d \xi
$$

Thus there exists $n \in \mathbb{Z}^{\kappa}$ such that

$$
\int|g(n, \xi)|^{q} d \xi \geqslant\left(1-o_{\delta}(1)\right)\left(\|\tilde{\mathcal{F}} f\|_{L_{\xi}^{q} L_{n}^{p}}-\|h\|_{L_{\xi}^{q} L_{n}^{p}}\right)^{q} \geqslant\left(1-o_{\delta}(1)\right) \mathbf{B}_{q, d}^{q}|E|^{q / p} .
$$

Then

$$
\mathbf{B}_{q, d}^{q}\left|E_{n}\right|^{q / p} \geqslant \int|g(n, \xi)|^{q} d \xi \geqslant\left(1-o_{\delta}(1)\right) \mathbf{B}_{q, d}^{q}|E|^{q / p}
$$

so $\left|E_{n}\right|^{1 / p} \geqslant\left(1-o_{\delta}(1)\right)|E|^{1 / p}$.

### 4.2 Lifting to $\mathbb{Z}^{\kappa} \times \mathbb{R}^{d}$

Definition 10. Let $\mathcal{Q}_{d}=\left[-\frac{1}{2}, \frac{1}{2}\right]^{d}$. To any function $f: \mathbb{R}^{d} \rightarrow \mathbb{C}$, associate the function $f^{\dagger}: \mathbb{Z}^{d} \times \mathbb{R}^{d} \rightarrow \mathbb{C}$ defined by

$$
f^{\dagger}(n, x)= \begin{cases}f(n+x) & \text { if } x \in \mathcal{Q}_{d} \\ 0 & \text { if } x \notin \mathcal{Q}_{d}\end{cases}
$$

For a measurable set $E \subset \mathbb{R}^{d}$, let $E^{\dagger}$ be the set in $\mathbb{Z}^{d} \times \mathbb{R}^{d}$ defined by

$$
E^{\dagger}=\{(n, x): n+x \in E\}
$$

We abuse the notation of $\hat{.}$ in the following lemmas: if $g: \mathbb{R}^{d} \rightarrow \mathbb{C}$, then $\hat{g}(\xi)=$ $\int_{\mathbb{R}^{d}} e^{-2 \pi i x \cdot \xi} g(x) d x$ and if $g: \mathbb{Z}^{d} \times \mathbb{R}^{d} \rightarrow \mathbb{C}$, then $\widehat{g}(\theta, \xi)=\sum_{n \in \mathbb{Z}^{d}} \int_{\mathbb{R}^{d}} e^{-2 \pi i n \cdot \theta} e^{-2 \pi i x \cdot \xi} g(n, x) d x$.

Lemma 25. Let $d \geqslant 1$ and $q \in(2, \infty), p=q^{\prime}$. Let $\delta, \eta>0$ be small. Let $E \subset \mathbb{R}^{d}$ be a Lebesgue measurable set with $|E| \in \mathbb{R}^{+}$. Suppose that

$$
\text { distance }\left(x, \mathbb{Z}^{d}\right) \leqslant \eta \quad \text { for all } x \in E
$$

and that for $|f| \leqslant 1_{E}$,

$$
\|\widehat{f}\|_{L^{q}\left(\mathbb{R}^{d}\right)} \geqslant(1-\delta) \mathbf{B}_{q, d}|E|^{1 / p}
$$

Then

$$
\left\|\widehat{f}^{\dagger}\right\|_{L^{q}\left(\mathbb{T}^{d} \times \mathbb{R}^{d}\right)} \geqslant\left(1-\delta-o_{\eta}(1)\right) \mathbf{B}_{q, d}\left|E^{\dagger}\right|^{1 / p}
$$

Proof. The conclusion of Lemma 9.1 of [15] is that for some $C, \gamma \in \mathbb{R}^{+}$, we have

$$
\left|\left\|\hat{f^{\dagger}}\right\|_{L^{q}\left(\mathbb{T}^{d} \times \mathbb{R}^{d}\right)}-\|\widehat{f}\|_{L^{q}\left(\mathbb{R}^{d}\right)}\right| \leqslant C \eta^{\gamma}\|f\|_{L^{p}\left(\mathbb{R}^{d}\right)} .
$$

It follows that

$$
\begin{aligned}
\|\widehat{f}\|_{L^{q}\left(\mathbb{T}^{d} \times \mathbb{R}^{d}\right)} & \geqslant(1-\delta) \mathbf{B}_{q, d}|E|^{1 / p}-C \eta^{\gamma}\|f\|_{p} \geqslant(1-\delta) \mathbf{B}_{q, d}|E|^{1 / p}-C \eta^{\gamma}|E|^{1 / p} \\
& =\left(1-\delta-o_{\eta}(1)\right) \mathbf{B}_{q, d}\left|E^{\dagger}\right|^{1 / p}
\end{aligned}
$$

where we used that $|E|=\left|E^{\dagger}\right|$.
The following lemma is analogous to the previous lemma. Ultimately, it is necessary to establish analogous results for the norm $\|\cdot\|_{q, *}$ because we will use it to translate localization properties of near-extremizers to the Fourier transforms of near-extremizers.

Lemma 26. Let $d \geqslant 1$ and $q \in(2, \infty)$, $p=q^{\prime}$. Let $\delta, \eta>0$ be small. Let $0 \neq f \in L^{p}\left(\mathbb{R}^{d}\right)$. Suppose that

$$
f \neq 0 \Longrightarrow \text { distance }\left(x, \mathbb{Z}^{d}\right) \leqslant \eta
$$

and that

$$
\|\widehat{f}\|_{L^{q, *}\left(\mathbb{R}^{d}\right)} \geqslant(1-\delta) \mathbf{B}_{q, d}\|f\|_{p} .
$$

Then

$$
\|{\widehat{f^{\dagger}}}_{\left\|_{L^{q, *}\left(\mathbb{T}^{d} \times \mathbb{R}^{d}\right)} \geqslant\left(1-2 \delta-o_{\eta}(1)\right) \mathbf{B}_{q, d}\right\| f^{\dagger} \|_{L^{p}\left(\mathbb{Z}^{d} \times \mathbb{R}^{d}\right)} . . . . . .}
$$

Proof. Let $\xi=n(\xi)+\alpha(\xi)$ where $n(\xi) \in \mathbb{Z}^{d}$ and $\alpha(\xi) \in\left[-\frac{1}{2}, \frac{1}{2}\right]^{d}=\mathcal{Q}_{d}$. The proof of Lemma 9.1 in [15] demonstrates that

$$
\begin{equation*}
\left\|\hat{f}^{\dagger}(\theta, n(\xi)+\alpha(\xi))-\widehat{f}(n(\xi)+\theta)\right\|_{L_{\theta, \xi}^{q}} \leqslant o_{\eta}(1)\|f\|_{p} \tag{4.14}
\end{equation*}
$$

Let $E \subset \mathbb{R}^{d}$ be such that $|E|^{-1 / p} \int_{E}|\widehat{f}(\xi)| d \xi \geqslant(1-2 \delta) \mathbf{B}_{q, d}\|f\|_{L_{x}^{p}}$. Define the lifted set $\tilde{E}=\left\{(\theta, \xi) \in \mathbb{T}^{d} \times \mathbb{R}^{d}: \theta+n(\xi) \in E\right\}$. Using (4.14, we calculate

$$
\begin{aligned}
\int_{\tilde{E}}\left|\widehat{f^{\dagger}}(\theta, \xi)\right| d \theta d \xi & \geqslant \int_{\tilde{E}}|\widehat{f}(n(\xi)+\theta)| d \theta d \xi-\int_{\tilde{E}}\left|\widehat{f^{\dagger}}(\theta, n(\xi)+\alpha(\xi))-\widehat{f}(n(\xi)+\theta)\right| d \theta d \xi \\
& \geqslant \int_{E}|\widehat{f}(\xi)| d \xi-|\tilde{E}|^{1 / p}\left\|\widehat{f^{\dagger}}(\theta, n(\xi)+\alpha(\xi))-\widehat{f}(n(\xi)+\theta)\right\|_{L_{\theta, \xi}^{q}} \\
& \geqslant|\tilde{E}|^{1 / p}(1-2 \delta) \mathbf{B}_{q, d}\|f\|_{L_{x}^{p}}-|\tilde{E}|^{1 / p} o_{\eta}(1)\|f\|_{L_{x}^{p}} \\
& =|\tilde{E}|^{1 / p}\left(1-2 \delta-o_{\eta}(1)\right) \mathbf{B}_{q, d}\left\|f^{\dagger}\right\|_{L_{n, x}^{p}}
\end{aligned}
$$

Translating general near-extremizers of $(1.2)$ and $(2.2)$ to near-extremizers satisfying the hypotheses of the previous two lemmas respectively will be much easier with the following Proposition 5.2 from [15], stated here for the reader's convenience.

Proposition 27. (Approximation by $\mathbb{Z}^{d}$ ). For each $d \geqslant 1$ and $\mathbf{r} \geqslant 0$ there exists $c>0$ with the following property. Let $P$ be a continuum multiprogression in $\mathbb{R}^{d}$ of rank $\mathbf{r}$, whose Lebesgue measure satisfies $|P|=1$. Let $\delta \in\left(0, \frac{1}{2}\right]$. There exists $\mathcal{T} \in$ Aff( $(d)$ whose Jacobian determinant satisfies

$$
|\operatorname{det} J(\mathcal{T})| \geqslant c \delta^{d \mathbf{r}+d^{2}}
$$

such that

$$
\|\mathcal{T}(x)\|_{\mathbb{R}^{d} / \mathbb{Z}^{d}}<\delta \quad \text { for all } x \in P
$$

## Chapter 5

## Conditional argument for precompactness

### 5.1 Spatial localization

Theorem 28. Let $d \geqslant 1$ and $q \in(2, \infty), p=q^{\prime}$. For every $\epsilon>0$ there exists $\delta>0$ with the following property. Let $E$ be a measurable set with $|E| \in \mathbb{R}^{+}$and $|f| \leqslant 1_{E}$. If $\|\widehat{f}\|_{q} \geqslant(1-\delta) \mathbf{B}_{q, d}|E|^{1 / p}$, then there exists an ellipsoid $\mathcal{E} \subset \mathbb{R}^{d}$ satisfying

$$
\begin{align*}
|E \backslash \mathcal{E}| & \leqslant \epsilon|E|  \tag{5.1}\\
|\mathcal{E}| & \leqslant C_{\epsilon}|E| . \tag{5.2}
\end{align*}
$$

Proof. Assume that $|E|^{1 / p} \mathbf{B}_{q, d}(1-\delta) \leqslant\|\widehat{f}\|_{q}$, where $\delta$ is to be chosen below.

1. Using the structural lemma for near extremizers of (1.2), Lemma 19 with $\epsilon_{0}>0$ to be chosen later, we obtain a decomposition $E=A \cup B$ and a multiprogression $P$ satisfying

$$
\begin{gathered}
E=A \cup B, \quad A \cap B=\varnothing \\
|B| \leqslant \epsilon_{0}|E| \\
|P| \leqslant C_{\epsilon_{0}}|E|, \\
A<P \\
\operatorname{rank} P \leqslant C_{\epsilon_{0}} .
\end{gathered}
$$

2. By precomposing $f$ with an affine transformation, assume without loss of generality that $|P|=1$. Then for a fixed $\delta_{0} \in\left(0, \frac{1}{2}\right]$ to be chosen below, Proposition 5.2 in [15], otherwise known as Proposition 27 in this paper, allows us to find a $c=c(d, p)$ as well as $\mathcal{T} \in \operatorname{Aff}(d)$ such that

$$
\begin{array}{r}
|\operatorname{det} J(\mathcal{T})| \geqslant c \delta_{0}^{d C_{\epsilon_{0}}+d^{2}} \quad \text { and } \\
\|\mathcal{T}(A)\|_{\mathbb{R}^{d} / \mathbb{Z}^{d}}<\delta_{0}
\end{array}
$$

where $J(\mathcal{T})$ is the Jacobian matrix of $\mathcal{T}$.
3. Now taking $\eta_{0}=\delta_{0}$ in the hypothesis of Lemma 25, we are guaranteed that since

$$
\left\|1_{A} \widehat{f \circ \mathcal{T}^{-1}}\right\|_{q} \geqslant(1-\delta) \mathbf{B}_{q, d}\left|\mathcal{T}^{-1}(E)\right|^{1 / p}-\left\|\widehat{1_{B} f}\right\|_{q} \geqslant\left(1-\delta-o_{\epsilon_{0}}(1)\right) \mathbf{B}_{q, d}|A|^{1 / p}
$$

and $\|\mathcal{T}(A)\|_{\mathbb{R}^{d} / \mathbb{Z}^{d}}<\delta_{0}$, we have

$$
\left\|\left(1_{A} \widehat{f \circ \mathcal{T}^{-1}}\right)^{\dagger}\right\|_{L^{q}\left(\mathbb{T}^{d} \times \mathbb{R}^{d}\right)} \geqslant\left(1-\delta-o_{\epsilon_{0}}(1)-o_{\delta_{0}}(1)\right) \mathbf{B}_{q, d}\left|\mathcal{T}(A)^{\dagger}\right|^{1 / p}
$$

where $\widehat{\cdot}$ here denotes the Fourier transform on $\mathbb{Z}^{d} \times \mathbb{R}^{d}$.
4. Then Proposition 22 gives the existence of $m \in \mathbb{Z}^{d}$ such that

$$
\left|\mathcal{T}(A) \cap\left(m+[1 / 2,1 / 2)^{d}\right)\right| \geqslant\left(1-o_{\delta}(1)-o_{\epsilon_{0}}(1)-o_{\delta_{0}}(1)\right)|\mathcal{T}(A)| .
$$

5. Last, we note that the cube $Q:=m+[1 / 2,1 / 2)^{d}$ satisfies

$$
\begin{aligned}
\left|E \backslash \mathcal{T}^{-1}(Q)\right| & \leqslant|A|+|B|-\left|A \cap \mathcal{T}^{-1}(Q)\right| \\
& \leqslant|A|+\epsilon_{0}|E|-\left(1-o_{\delta}(1)-o_{\epsilon_{0}}(1)-o_{\delta_{0}}(1)\right)|A| \\
& \leqslant\left(\epsilon_{0}+o_{\delta}(1)+o_{\epsilon_{0}}(1)+o_{\delta_{0}}(1)\right)|E|
\end{aligned}
$$

Note that $\epsilon_{0}$ and $\delta_{0}$ may be chosen freely, and $\delta$ may be taken small enough after fixing an $\epsilon_{0}$ and $\delta_{0}$. Thus we may choose $\epsilon_{0}$, $\delta_{0}$, and then $\delta$ small enough so that $\left|E \backslash \mathcal{T}^{-1}(Q)\right| \leqslant \epsilon|E|$. We also note that

$$
\begin{aligned}
\left|\mathcal{T}^{-1}(Q)\right| & =|\operatorname{det} J(\mathcal{T})|^{-1}|Q| \\
& =|\operatorname{det} J(\mathcal{T})|^{-1}|P| \\
& \leqslant\left(c \delta_{0}^{d C_{\epsilon_{0}}+d^{2}}\right)^{-1} C_{\epsilon_{0}}|E| \\
& =\tilde{C}_{\epsilon}|E| .
\end{aligned}
$$

Finally, since $Q$ is comparable in size (up to dimensional constants) to the smallest ball which contains it, we are done.

Proposition 29. Suppose that there is an affirmative answer to Question 9. Let $d \geqslant 1$ and $q \in(2, \infty), p=q^{\prime}$. For every $\epsilon>0$ there exists $\delta>0$ with the following property. Let $0 \neq f \in L^{q^{\prime}}\left(\mathbb{R}^{d}\right)$ satisfy $\|\hat{f}\|_{q, *} \geqslant(1-\delta) \mathbf{B}_{q, d}\|f\|_{p}$. There exists an ellipsoid $\mathcal{E} \subset \mathbb{R}^{d}$ and $a$ decomposition $f=\phi+\psi$ such that

$$
\begin{gathered}
\|\psi\|_{q^{\prime}}<\epsilon\|f\|_{p} \\
\phi \equiv 0 \quad \text { on } \mathbb{R}^{d} \backslash \mathcal{E} \\
\|\phi\|_{\infty}|\mathcal{E}|^{1 / p} \leqslant C_{\epsilon}\|f\|_{p} .
\end{gathered}
$$

Proof. We follow an analogous argument as that in the proof of Theorem 28, replacing the near extremizer structure Lemma 19 by the analogous structure theorem for the dual problem, Lemma 20. For step (3) in the proof of Theorem 28, we use Lemma 26 in place of Lemma 25. For step (4), use an affirmative answer to Question 9 instead of Proposition 22. The conclusion is that using analogous notation as in the proof of Theorem 28 ,

$$
\left\|1_{A} f \circ \mathcal{T}^{-1} 1_{Q}\right\|_{L^{p}\left(\mathbb{R}^{d}\right)} \geqslant\left(1-o_{\delta}(1)-o_{\epsilon_{0}}(1)-o_{\delta_{0}}(1)\right)\left\|1_{A} f \circ \mathcal{T}^{-1}\right\|_{L^{p}\left(\mathbb{R}^{d}\right)}
$$

where $\epsilon_{0}$ and $\delta_{0}$ may be chosen freely, and $\delta$ may be taken small enough after fixing an $\epsilon_{0}$ and $\delta_{0}$. Let $\mathcal{E}$ be the smallest ellipsoid containing $\mathcal{T}^{-1}(Q)$ and define $\phi=1_{A \cap \mathcal{E}} f$, so $\psi=f-\varphi$. Then for small enough parameters $\delta_{0}, \epsilon_{0}$ and then $\delta$,

$$
\|\psi\|_{p}<\epsilon\|f\|_{p} \quad \text { and } \quad \varphi<1_{\mathcal{E}}
$$

By the construction, we also have that

$$
\begin{aligned}
\|\varphi\|_{\infty}|\mathcal{E}|^{1 / p} & \leqslant c_{d}\left\|1_{A} f\right\|_{\infty}\left|\mathcal{T}^{-1}(Q)\right| \\
& \leqslant c_{d} C_{\epsilon_{0}}\|f\|_{p}\left|\operatorname{det} J\left(\mathcal{T}^{-1}\right)\right| \\
& \leqslant c_{d} C_{\epsilon_{0}}\|f\|_{p}\left(c \delta_{0}^{d C_{\epsilon_{0}}+d^{2}}\right)^{-1}
\end{aligned}
$$

so we are done.

### 5.2 Frequency localization

Proposition 30. Suppose that there is an affirmative answer to Question 9. Let $d \geqslant 1$ and $q \in(2, \infty), p=q^{\prime}$. For every $\epsilon>0$ there exists $\delta>0$ with the following property. Let $E$ be a Lebesgue measurable set with $|E| \in \mathbb{R}^{+}$. Suppose that $|f| \leqslant 1_{E}$ satisfies $\|\widehat{f}\|_{q} \geqslant$ $(1-\delta) \mathbf{B}_{q, d}|E|^{1 / p}$. Then there exists an ellipsoid $\mathcal{E}^{\prime} \subset \mathbb{R}^{d}$ and a decomposition $\widehat{f}=\Phi+\Psi$ such that

$$
\begin{gathered}
\|\Psi\|_{q^{\prime}}<\epsilon\|\widehat{f}\|_{p} \\
\Phi \equiv 0 \quad \text { on } \mathbb{R}^{d} \backslash \mathcal{E}^{\prime} \\
\|\Phi\|_{\infty}\left|\mathcal{E}^{\prime}\right|^{1 / p} \leqslant C_{\epsilon}\|f\|_{p} .
\end{gathered}
$$

Proof. In the proof of Proposition 8 we showed that if $(f, E)$ is a near-extremizing pair for (1.2), then $\widehat{f}|\hat{f}|^{q-2}$ is a near-extremizer for 2.1). Thus we may apply Proposition 29 to obtain a decomposition $\widehat{f}|\widehat{f}|^{q-2}=\varphi+\psi$ and take $\Phi=\varphi|\varphi|^{(2-q) /(q-1)}$ and $\Psi=\psi|\psi|^{(2-q) /(q-1)}$ for the desired decomposition.

### 5.3 Compatibility of approximating ellipsoids

We will show that $\mathcal{E}$ and $\mathcal{E}^{\prime}$ are dual to each other, up to bounded factors and independent translations. For $s \in \mathbb{R}^{+}$and $E \subset \mathbb{R}^{d}$, we consider the dilated set $s E=\{s y: y \in E\}$.

Definition 11. The polar set $\mathcal{E}^{*}$ of a balanced, bounded, convex set with nonempty interior $\mathcal{E} \subset \mathbb{R}^{d}$ is

$$
\mathcal{E}^{*}=\{y:|\langle x, y\rangle| \leqslant 1 \text { for every } x \in \mathcal{E}\}
$$

where $\langle\cdot, \cdot\rangle$ denotes the Euclidean inner product.
Lemma 31. Suppose that there is an affirmative answer to Question 9. Let $d \geqslant 1$ and let $\Lambda \subset(1,2)$ be a compact set. There exists $\eta_{0}>0$ such that the following property holds for $0<\eta<\eta_{0}$. Let $\eta>0$. Let $p \in \Lambda$ and let $q=p^{\prime}$. Suppose $\|\widehat{f}\|_{q} \geqslant(1-\rho(\eta)) \mathbf{B}_{q, d}|E|^{1 / p}$ for a function $\rho:[0,1] \rightarrow \mathbb{R}^{+}$where $\rho(\eta) \rightarrow 0$ as $\eta \rightarrow 0$ sufficiently fast so that there exists an ellipsoid $\mathcal{E}+u$ satisfying the conclusions of Theorem 28 with $\epsilon=\eta$ and an ellipsoid $\tilde{\mathcal{E}}+v$ and disjoint decomposition $\widehat{f}=\Phi+\Psi$ satisfying the conclusions of Proposition 29 with $\epsilon=\eta$, where $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are ellipsoids centered at the origin and $u, v \in \mathbb{R}^{d}$. Then there exists a constant $C=C(d, \Lambda, \eta)$ such that

$$
\mathcal{E} \subset C \tilde{\mathcal{E}}^{*} \quad \text { and } \quad \tilde{\mathcal{E}} \subset C \mathcal{E}^{*}
$$

Proof. By constants, we mean quantities which are permitted to depend on $d, \Lambda, \eta$. By replacing $f$ and $1_{E}$ with $e^{2 \pi i x \cdot v} f(x+u)$ and $1_{E}(x+u)$ respectively, we may assume without loss of generality that $u, v=0$. By dilating $f$ and $E$ by $|\mathcal{E}|^{1 / d}$, we may further assume that $|\mathcal{E}|=1$.

First, we will prove that $|\tilde{\mathcal{E}}|=|\mathcal{E}||\tilde{\mathcal{E}}| \leqslant C$. We have assumed that

$$
\begin{equation*}
(1-\rho(\eta)) \mathbf{B}_{q, d}|E|^{1 / p} \leqslant\|\widehat{f}\|_{q}, \tag{5.3}
\end{equation*}
$$

and hence by Theorem 28 we know that

$$
|E \backslash \mathcal{E}| \leqslant \eta|E|, \quad|\mathcal{E}| \leqslant C_{\eta}|E|
$$

and by Proposition 29 that

$$
\|\Psi\|_{q} \leqslant \eta|E|^{1 / p}, \quad \Phi<\tilde{\mathcal{E}}, \quad\|\Phi\|_{\infty}|\tilde{\mathcal{E}}|^{1 / q} \leqslant C_{\eta}|E|^{1 / p}
$$

Let $S_{\alpha}=\left\{\xi:|\widehat{f}(\xi)| \geqslant \alpha|\tilde{\mathcal{E}}|^{-1 / q}\right\}$ and $\lambda_{\eta}=\left\{\xi:|\widehat{f}(\xi)| \leqslant C_{\eta}|E|^{1 / p}|\tilde{\mathcal{E}}|^{-1 / q}\right\}$. We decompose the following integral as

$$
\begin{aligned}
\int_{\mathbb{R}^{d}}|\widehat{f}|^{q} d \xi & =\int_{\tilde{\mathcal{E}}^{c}}|\widehat{f}|^{q} d \xi+\int_{\tilde{\mathcal{E}} \cap \lambda_{\eta}^{c}}|\widehat{f}|^{q} d \xi+\int_{\tilde{\mathcal{E}} \cap\left(\lambda_{\eta} \cap S_{\alpha}\right)}|\widehat{f}|^{q} d \xi+\int_{\tilde{\mathcal{E}}^{\prime} \cap\left(\lambda_{\eta} \cap S_{\alpha}^{c}\right)}|\widehat{f}|^{q} d \xi \\
& :=A+B+C+D
\end{aligned}
$$

We will bound each integral defined above. First, we use the properties of the decomposition $\widehat{f}=\Phi+\Psi$ to note that

$$
A=\int_{\tilde{\mathcal{E}}^{c}}|\Psi|^{q} d \xi \leqslant\|\Psi\|_{q}^{q} \leqslant \eta^{q}|E|^{q / p} .
$$

Next, to control $B$, we use the property that $\widehat{f}=\Phi+\Psi$ is a disjointly supported decomposition and $\|\Phi\|_{\infty} \leqslant C_{\eta}|E|^{1 / p}|\tilde{\mathcal{E}}|^{-1 / q}$, so $\Phi=0$ a.e. on $\lambda_{\eta}^{c}$. Namely,

$$
B=\int_{\tilde{\mathcal{E}} \cap \lambda_{\eta}^{c}}|\Psi|^{q} d \xi \leqslant \eta^{q}|E|^{q / p}
$$

For $C$, we use that $|\hat{f}| \leqslant C_{\eta}|E|^{1 / p}|\tilde{\mathcal{E}}|^{-1 / q}$ on $\lambda_{\eta}$ to get

$$
C \leqslant C_{\eta}^{q}|E|^{q / p}|\tilde{\mathcal{E}}|^{-1}\left|\tilde{\mathcal{E}} \cap S_{\alpha}\right| .
$$

Finally, we have for $D$ that

$$
D \leqslant\left|\tilde{\mathcal{E}} \cap \lambda_{\eta}\right| \alpha^{q}|\tilde{\mathcal{E}}|^{-1} \leqslant \alpha^{q} .
$$

Combining the upper bounds for $A, B, C, D$ with (5.3), we have

$$
\begin{aligned}
(1-\rho(\eta))^{q} \mathbf{B}_{q, d}^{q}|E|^{q^{q / p} \leqslant} \leqslant & \int_{\mathbb{R}^{d}}|f|^{q} d \xi \\
= & A+B+C+D \\
\leqslant & \eta^{q}|E|^{q / p}+\eta^{q}|E|^{q / p} \\
& \quad+C_{\eta}^{q}|E|^{q / p}|\tilde{\mathcal{E}}|^{-1}\left|\tilde{\mathcal{E}} \cap S_{\alpha}\right|+\alpha^{q}
\end{aligned}
$$

Rearranging, we get

$$
C_{\eta}^{-q}\left[(1-\rho(\eta))^{q} \mathbf{B}_{q, d}^{q}-2 \eta^{q}-\alpha^{q}|E|^{-q / p}\right]|\tilde{\mathcal{E}}| \leqslant\left|\tilde{\mathcal{E}} \cap S_{\alpha}\right|
$$

Finally, since $|\mathcal{E}|=1$ and $|\mathcal{E}| \leqslant C_{\eta}|E|$, we have

$$
C_{\eta}^{-q}\left[\left(1-o_{\eta}(1)\right)^{q} \mathbf{B}_{q, d}^{q}-2 \eta^{q}-\alpha^{q} C_{\eta}^{q / p}\right]|\tilde{\mathcal{E}}| \leqslant\left|\tilde{\mathcal{E}} \cap S_{\alpha}\right| .
$$

Choose $\alpha$ small enough so that

$$
\frac{1}{2} C_{\eta}^{-q}\left[\left(1-o_{\eta}(1)\right)^{q} \mathbf{B}_{q, d}^{q}-2 \eta^{q}\right] \leqslant C_{\eta}^{-q}\left[\left(1-o_{\eta}(1)\right)^{q} \mathbf{B}_{q, d}^{q}-2 \eta^{q}-\alpha^{q} C_{\eta}^{q / p}\right]
$$

so $\alpha$ only depends on $\eta$. Thus for $c^{\prime}=c^{\prime}(\eta)>0$ and $\alpha=\alpha(\eta)$, we can conclude that

$$
c^{\prime}|\tilde{\mathcal{E}}| \leqslant\left|\tilde{\mathcal{E}} \cap S_{\alpha}\right|
$$

Since $|f| \leqslant 1_{E}$ and $|E|<\infty, f$ is in $L^{2}$. Since $|\mathcal{E}|=1$, note that $|E|=|E \cap \mathcal{E}|+|E \backslash \mathcal{E}| \leqslant$ $1+\eta|E|$, so we can assume $|E| \leqslant 2$. Using these two observations, we have

$$
2 \geqslant|E| \geqslant\|f\|_{2}^{2}=\|\widehat{f}\|_{2}^{2} \geqslant \int_{S_{\alpha}}|\widehat{f}(\xi)|^{2} d \xi \geqslant \alpha^{2}|\tilde{\mathcal{E}}|^{-2 / q}\left|S_{\alpha}\right| \geqslant \alpha^{2} c^{\prime}|\tilde{\mathcal{E}}|^{1-2 / q}
$$

so $|\mathcal{E}||\tilde{\mathcal{E}}|=|\tilde{\mathcal{E}}| \leqslant C^{\prime}$ for $C^{\prime}=C^{\prime}(\eta)$.
Now assume via composition with an affine transformation that $\tilde{\mathcal{E}}=\mathbb{B}$ and that $\mathcal{E}=\{x$ : $\left.\sum_{j=1}^{d} s_{j}^{-2} x_{j}^{2} \leqslant 1\right\}$. We wish to show that $\mathbb{B} \subset C \mathcal{E}^{*}$ and that $\mathcal{E} \subset C \mathbb{B}$, where $C$ is permitted to depend on $\eta$. We know from the earlier discussion that $|\mathcal{E}||\mathbb{B}| \leqslant C_{\eta}$. Since $|\mathcal{E}|=c_{d} \prod_{j=1}^{d} s_{j}$, it remains to show that the smallest $s_{i}$, say $s_{1}$, is bounded below. Using the same notation as earlier, we note that

$$
\left\|\partial_{\xi_{1}} \widehat{\mathcal{E}_{\mathcal{E}}}\right\|_{q} \leqslant 2 \pi \mathbf{B}_{q, d}\left\|x_{1} 1_{\mathcal{E}} f\right\|_{p} \leqslant 2 \pi \mathbf{B}_{q, d} s_{1}|E|^{1 / p}
$$

and that

$$
\begin{aligned}
\left\|\widehat{1_{\mathcal{E}} f}\right\|_{L^{q}(\mathbb{B})} & \geqslant\|\widehat{f}\|_{L^{q}(\mathbb{B})}-\left\|\widehat{1_{E \backslash \mathcal{E}}}\right\|_{q} \\
& \geqslant\|\hat{f}\|_{L^{q}\left(\mathbb{R}^{d}\right)}-\|\Psi\|_{q}-\mathbf{B}_{q, d}|E \backslash \mathcal{E}|^{1 / p} \\
& \geqslant(1-\rho(\eta)) \mathbf{B}_{q, d}|E|^{1 / p}-\eta|E|^{1 / p}-\mathbf{B}_{q, d} \eta^{1 / p}|E|^{1 / p} \\
& \geqslant\left(1-o_{\eta}(1)\right) \mathbf{B}_{q, d}|E|^{1 / p} .
\end{aligned}
$$

We also have

$$
\||E|^{-1 / p} \widehat{\overline{1 \mathcal{E}} \|_{q}} \leqslant \mathbf{B}_{q, d}
$$

Thus we are in the situation where there are functions $h$ satisfying $\left\|\partial_{\xi_{1}} h\right\|_{L^{q}\left(\mathbb{R}^{d}\right)} \leqslant 2 \pi \mathbf{B}_{q, d} s_{1}(h)$ for a positive quantity $s_{1}$ associated to each $h$ and $\|h\|_{L^{q}(\mathbb{B})}>\frac{1}{2} \mathbf{B}_{q, d}>0$. If there are functions $h=|E|^{-1 / p} \widehat{1_{\mathcal{E}} f}$ fitting the above regime and for which $s_{1}(h) \rightarrow 0$, then $\|h\|_{L^{q}\left(\mathbb{R}^{d}\right)} \rightarrow \infty$. Since we have the uniform upper bound $\left\||E|^{-1 / p} \widehat{1_{\mathcal{E}} f}\right\|_{q} \leqslant \mathbf{B}_{q, d}$, there must be a positive lower bound depending on $\eta$ for the values of $s_{1}$, which completes the proof.

### 5.4 Precompactness

We restate Proposition 2 for the reader's convenience.
Proposition 2 Suppose that there is an affirmative answer to Question 9. Let $d \geqslant 1$ and $q \in(2, \infty), p=q^{\prime}$. Let $\left(E_{\nu}\right)$ be a sequence of Lebesgue measurable subsets of $\mathbb{R}^{d}$ with $\left|E_{\nu}\right| \in \mathbb{R}^{+}$and let $f_{\nu}$ be Lebesgue measurable functions on $\mathbb{R}^{d}$ satisfying $\left|f_{\nu}\right| \leqslant 1_{E_{\nu}}$. Suppose that $\lim _{\nu \rightarrow \infty}\left|E_{\nu}\right|^{-1 / p}\left\|\widehat{f}_{\nu}\right\|_{q}=\mathbf{B}_{q, d}$. Then there exists a subsequence of indices $\nu_{k}$, a Lebesgue measurable set $E \subset \mathbb{R}^{d}$ with $0<|E|<\infty$, a Lebesgue measurable function $f$ on $\mathbb{R}^{d}$ satisfying $|f| \leqslant 1_{E}$, a sequence $\left(T_{\nu}\right)$ of affine automorphisms of $\mathbb{R}^{d}$, and a sequence of vectors $v_{\nu} \in \mathbb{R}^{d}$ such that

$$
\lim _{k \rightarrow \infty}\left\|e^{-2 \pi i v_{\nu_{k}} \cdot x} f_{\nu_{k}} \circ T_{\nu_{k}}^{-1}-f\right\|_{p}=0 \quad \text { and } \quad \lim _{k \rightarrow \infty}\left|T_{\nu_{k}}\left(E_{\nu_{k}}\right) \Delta E\right|=0
$$

In order to prove Proposition 2, we first prove the following lemma.

Lemma 32. Suppose that there is an affirmative answer to Question 9. Let $d \geqslant 1$ and $q \in(2, \infty), p=q^{\prime}$. Let $\left(E_{\nu}\right)$ be a sequence of Lebesgue measurable subsets of $\mathbb{R}^{d}$ with $\left|E_{\nu}\right| \in \mathbb{R}^{+}$. Let $f_{\nu}$ be Lebesgue measurable functions satisfying $\left|f_{\nu}\right| \leqslant 1_{E_{\nu}}$. Suppose that $\lim _{\nu \rightarrow \infty}\left|E_{\nu}\right|^{-1 / p}\left\|\hat{f}_{\nu}\right\|_{q}=\mathbf{B}_{q, d}$. Then there exists a sequence of elements $T_{\nu} \in$ Aff( $(d)$ and vectors $v_{\nu} \in \mathbb{R}^{d}$ such that $\left|T_{\nu}\left(E_{\nu}\right)\right|$ is uniformly bounded and the sequence of functions $\left(\widehat{g_{\nu}}\right)$ where $g_{\nu}=e^{-2 \pi i v_{\nu} \cdot x} f_{\nu} \circ T_{\nu}^{-1}$ is precompact in $L^{q}\left(\mathbb{R}^{d}\right)$.

Proof of Lemma 32. Let $f_{\nu}$ and $E_{\nu}$ satisfy the hypotheses. Let $\epsilon_{0}=\min \left(\frac{1}{4}, \eta_{0}\right)$ where $\eta_{0}$ is the threshold from Lemma 31. For each sufficiently large $\nu$, (1) there exists an ellipsoid $\mathcal{E}_{\nu}$ satisfying the conclusions of Theorem 28 with $\epsilon=\epsilon_{0}$ and (2) there exists an ellipsoid $\mathcal{F}_{\nu}$ and disjointly supported decomposition $f_{\nu}=\Phi_{\nu}+\Psi_{\nu}$ satisfying the conclusions of Proposition 30 with $\epsilon=\epsilon_{0}$.

Let $u_{\nu}, v_{\nu} \in \mathbb{R}^{d}$ be the centers of the $\mathcal{E}_{\nu}$ and $\mathcal{F}_{\nu}$ respectively. By replacing $f_{\nu}$ by $e^{-2 \pi i v_{\nu} \cdot x} f_{\nu}\left(x+u_{\nu}\right)$ and $1_{E_{\nu}}$ by $1_{E_{\nu}-u_{n}}$, we may reduce to the case $u_{\nu}=v_{\nu}=0$. By composing $f_{n}$ and $1_{E_{\nu}}$ with an element of the general linear group on $\mathbb{R}^{d}$, we may reduce to the case in which $\mathcal{E}_{n}$ is the unit ball $\mathbb{B}$ of $\mathbb{R}^{d}$. Continue to denote these modified functions by $f_{\nu}$ and $1_{E_{\nu}}$.

For each $\epsilon>0$, there exists $N<\infty$ such that for each $\nu \geqslant N$, Theorem 28 associates to $\left(f_{\nu}, E_{\nu}\right)$ an ellipsoid $\mathcal{E}_{\nu, \epsilon}$ and Proposition 30 associates to $\left(f_{\nu}, E_{\nu}\right)$ an ellipsoid $\mathcal{F}_{\nu, \epsilon}$ and a disjointly supported decomposition $\widehat{f}_{\nu}=\Phi_{\nu, \epsilon}+\Psi_{\nu, \epsilon}$.

Symmetries of the inequality have been exploited to normalize so that $\mathcal{E}_{\nu}=\mathbb{B}$, so by Lemma 31, $\mathcal{F}_{n}$ are balls centered at the origin with radii comparable to 1 . We claim that this ensures corresponding normalizations for $\mathcal{E}_{\nu, \epsilon}, \mathcal{F}_{\nu, \epsilon} ; \epsilon$-dependent symmetries are not needed.

According to Theorem 28,

$$
\begin{array}{r}
\left|E_{\nu} \backslash \mathbb{B}\right| \leqslant \epsilon_{0}\left|E_{\nu}\right| \quad \text { and } \quad|\mathbb{B}| \leqslant C_{0}\left|E_{\nu}\right| \\
\left|E_{\nu} \backslash \mathcal{E}_{\nu, \epsilon}\right| \leqslant \epsilon\left|E_{\nu}\right| \quad \text { and } \quad\left|\mathcal{E}_{\nu, \epsilon}\right| \leqslant C_{\epsilon}\left|E_{\nu}\right|,
\end{array}
$$

provided that $\nu$ is sufficiently large and $\epsilon$ is sufficiently small. This implies that

$$
\begin{aligned}
\frac{3}{4} C_{0}^{-1}|\mathbb{B}| \leqslant\left(1-\epsilon_{0}\right)\left|E_{\nu}\right| & \leqslant\left|\mathbb{B} \cap E_{\nu}\right|=\left|\left(\mathbb{B} \cap E_{\nu}\right) \backslash \mathcal{E}_{\nu, \epsilon}\right|+\left|\mathbb{B} \cap E_{\nu} \cap \mathcal{E}_{\nu, \epsilon}\right| \\
& \leqslant \epsilon\left|E_{\nu}\right|+\left|\mathbb{B} \cap \mathcal{E}_{\nu, \epsilon}\right| \leqslant \epsilon \frac{4}{3}|\mathbb{B}|+\left|\mathbb{B} \cap \mathcal{E}_{\nu, \epsilon}\right|,
\end{aligned}
$$

so there is a $c>0$ such that $\left|\mathbb{B} \cap \mathcal{E}_{\nu, \epsilon}\right| \geqslant c$ where $c$ is independent of $\epsilon$ and $\nu$. This lower bound combined with the upper bound $\left|\mathcal{E}_{\nu, \epsilon}\right| \leqslant C_{\epsilon}\left|E_{\nu}\right| \leqslant C_{\epsilon} \frac{4}{3}|\mathbb{B}|$ implies the $\mathcal{E}_{\nu, \epsilon}$ are contained in a ball centered at 0 with radius depending only on $\epsilon$.

By Proposition 30, for sufficiently large $\nu$ and sufficiently small $\epsilon$,

$$
\begin{aligned}
& \left\|\Phi_{\nu}-\hat{f}_{\nu}\right\|_{q} \leqslant \epsilon_{0}\left\|\hat{f}_{\nu}\right\|_{q} \quad \text { and } \quad\left\|\Phi_{\nu, \epsilon}-\hat{f}_{\nu}\right\|_{q} \leqslant \epsilon\left\|\hat{f}_{\nu}\right\|_{q}, \\
& \left\|\Phi_{\nu}\right\|_{\infty} \leqslant C_{0}\left\|f_{\nu}\right\|_{p} \quad \text { and } \quad\left\|\Phi_{\nu, \epsilon}\right\|_{\infty}\left|\mathcal{E}_{\nu, \epsilon}\right|^{1 / q} \leqslant C_{\epsilon}\left\|f_{\nu}\right\|_{p} .
\end{aligned}
$$

For each $\xi \in \mathbb{R}^{d}$, each of $\Phi_{\nu}(\xi), \Phi_{\nu, \epsilon}(\xi)$ is equal either to $\widehat{f}_{\nu}(\xi)$, or to 0 . From these inequalities and this fact, along with the support relations $\Phi_{\nu}<\mathbb{B}$ and $\Phi_{\nu, \epsilon}<\mathcal{F}_{\nu, \epsilon}$, it follows that

$$
\begin{aligned}
\left\|\Phi_{\nu, \epsilon}\right\|_{q} & =\left\|1_{\mathcal{F}_{\nu, \epsilon}} \widehat{f}_{\nu}\right\|_{q} \geqslant(1-\epsilon)\left\|\widehat{f}_{\nu}\right\|_{q} \\
\left\|1_{\mathcal{F}_{\nu, \epsilon} \mid C_{0} \mathbb{B}} \widehat{f}_{\nu}\right\|_{q} & \leqslant\left\|\Phi_{\nu}-\Phi_{\nu, \epsilon}\right\|_{q} \leqslant\left(\epsilon_{0}+\epsilon\right)\left\|\widehat{f}_{\nu}\right\|_{q}
\end{aligned}
$$

where $\mathcal{F}_{\nu} \subset C_{0} \mathbb{B}$. Thus $\left\|1_{\mathcal{F}_{\nu, \epsilon} \cap C_{0} \mathbb{B}} \widehat{f}_{\nu}\right\|_{q} \geqslant\left(1-\epsilon_{0}-2 \epsilon\right)\left\|\widehat{f}_{\nu}\right\|_{q}$. Combined with the inequalities

$$
\begin{gathered}
\left\|\hat{f}_{\nu}\right\|_{q} \geqslant \frac{1}{2} \mathbf{B}_{q, d}\left|E_{\nu}\right|^{1 / p} \geqslant \frac{1}{2} \mathbf{B}_{q, d} C_{0}^{-1 / p}|\mathbb{B}|^{1 / p} \text { and } \\
\left\|1_{\mathcal{F}_{\nu, \epsilon} \cap C_{0} \mathbb{B}} \hat{f}_{\nu}\right\|_{q} \leqslant\left|\mathcal{F}_{\nu, \epsilon} \cap C_{0} \mathbb{B}\right|^{1 / q}\left\|\widehat{f}_{\nu}\right\|_{\infty} \leqslant\left|\mathcal{F}_{\nu, \epsilon} \cap C_{0} \mathbb{B}\right|^{1 / q}\left|E_{\nu}\right| \leqslant\left|\mathcal{F}_{\nu, \epsilon} \cap C_{0} \mathbb{B}\right|^{1 / q} \frac{4}{3}|\mathbb{B}|,
\end{gathered}
$$

we conclude $\left|\mathcal{F}_{\nu, \epsilon} \cap C_{0} \mathbb{B}\right| \geqslant c$ where $c>0$ is independent of $\nu, \epsilon$.
Another consequence of the inequalities from Proposition 30 is that $\left\|\Phi_{\nu, \epsilon}\right\|_{\infty} \geqslant c\left\|\Phi_{\nu}\right\|_{\infty}$, where $c>0$ is independent of $\nu, \epsilon$. Indeed,

$$
\begin{aligned}
\left(1-\epsilon_{0}\right)\left\|\widehat{f}_{\nu}\right\|_{q} & \leqslant\left\|\Phi_{\nu}\right\|_{q}=\left\|\Phi_{\nu}\right\|_{L^{q}\left(C_{0} \mathbb{B}\right)} \leqslant\left\|\Phi_{\nu}-\Phi_{\nu, \epsilon}\right\|_{q}+\left\|\Phi_{\nu, \epsilon}\right\|_{L^{q}\left(C_{0} \mathbb{B}\right)} \\
& \leqslant\left(\epsilon_{0}+\epsilon\right)\left\|\widehat{f}_{\nu}\right\|_{q}+\left.\left|\Phi_{\nu, \epsilon} \|_{\infty}\right| C_{0} \mathbb{B}\right|^{1 / q}
\end{aligned}
$$

so for a constant $c>0$ independent of $\nu, \epsilon$,

$$
\left\|\Phi_{\nu, \epsilon}\right\|_{\infty} \geqslant c\left\|\widehat{f}_{\nu}\right\|_{q} \geqslant c \frac{1}{2} \mathbf{B}_{q, d}\left|E_{\nu}\right|^{1 / p} \geqslant c \frac{1}{2} \mathbf{B}_{q, d}\left\|f_{\nu}\right\|_{p}
$$

Then $\left|\mathcal{F}_{\nu, \epsilon}\right|^{1 / q} \leqslant C_{\epsilon}\left\|f_{\nu}\right\|_{p}\left\|\Phi_{\nu, \epsilon}\right\|_{\infty}^{-1} \leqslant C_{\epsilon} \frac{2}{c \mathbf{B}_{q, d}}$. The uniform lower bound on $\left|\mathcal{F}_{\nu, \epsilon} \cap C_{0} \mathbb{B}\right|$ and the $\epsilon$-dependent upper bound on $\left|\mathcal{F}_{\nu, \epsilon}\right|$ imply that $\mathcal{F}_{\nu, \epsilon} \subset C_{\epsilon} \mathbb{B}$ for sufficiently large $\nu$.

Now note the uniform bound

$$
\left\|\widehat{1_{C_{0} \mathbb{B}} f_{\nu}}\right\|_{q} \leqslant \mathbf{B}_{q, d}\left|E_{\nu}\right|^{1 / q} \leqslant \mathbf{B}_{q, d} \frac{4}{3}|\mathbb{B}|^{1 / p} .
$$

By the Hausdorff-Young inequality,

$$
\left\|\nabla \widehat{1_{C_{0} \mathbb{B}} f_{\nu}}\right\|_{q} \leqslant\left\||x| 1_{\mathcal{E}_{\nu, \epsilon},}\right\|_{p} \leqslant C_{\epsilon}
$$

since $|x| 1_{\mathcal{E}_{\nu, \epsilon}}$ is bounded by the diameter of $\mathcal{E}_{\nu, \epsilon}$ and the volumes $\left|\mathcal{E}_{\nu, \epsilon}\right|$ are bounded above uniformly in $\nu$. Thus by Rellich's theorem, on any fixed bounded subset of $\mathbb{R}^{d}$, we can find an $L^{q}$ convergent subsequence of $\left(\widehat{1_{\mathcal{E}_{\nu, \epsilon}} f_{\nu}}\right)$. Since this is true for each $\epsilon,\left\|\widehat{f}_{\nu}\right\|_{q}$ is bounded uni-
 in $L^{q}\left(\mathbb{R}^{d}\right)$ on any fixed bounded subset of $\mathbb{R}^{d}$. Since $\mathcal{F}_{\nu, \epsilon}$ is contained in a ball independent of $\nu$ for each fixed $\epsilon$, and since $\int_{\xi \notin \mathcal{F} \nu, \epsilon}\left|\widehat{f}_{\nu}(\xi)\right|^{q} d \xi \rightarrow 0$ as $\epsilon \rightarrow 0$, the sequence $\left(\widehat{f}_{\nu}\right)$ is precompact in $L^{q}\left(\mathbb{R}^{d}\right)$.

Proof of Proposition 2. From Lemma 32, we can assume that the sequence $\left(\widehat{f}_{\nu}\right)$ is convergent in $L^{q}\left(\mathbb{R}^{d}\right)$ and that the supports $E_{\nu}$ satisfy $\left|E_{\nu}\right| \leqslant \frac{4}{3}|\mathbb{B}|$. By passing to a subsequence, we may also assume that $\lim _{\nu \rightarrow \infty}\left|E_{\nu}\right|=a$ where $0<a<\infty$ and by precomposing $f_{\nu}$ and $1_{E_{\nu}}$ by affine transformations, we may assume that $\left|E_{\nu}\right|=1$ for all $\nu$. Then

$$
\lim _{\nu \rightarrow \infty}\left\|\widehat{f}_{\nu}\right\|_{q}=\mathbf{B}_{q, d}
$$

Since $\left\|f_{\nu}\right\|_{2} \leqslant\left|E_{\nu}\right|^{1 / 2}=1$ for all $\nu$, by the Banach-Alaoglu theorem, there is a weak-* convergent subsequence (which we just denote $\left(f_{\nu}\right)$ ) to a limit $f \in L^{2}$. Note that since weak-* convergence of $\left(f_{\nu}\right)$ to $f$ implies convergence as tempered distributions, it must be that $\left(\hat{f}_{\nu}\right)$ converge to $\hat{f}$ as tempered distributions. Since $\left(\hat{f}_{\nu}\right)$ is a convergent sequence in $L^{q}$, it must therefore be true that $\widehat{f}_{\nu} \rightarrow \hat{f}$ strongly in $L^{q}$.

We claim that

$$
\lim _{\nu, \mu \rightarrow \infty} \frac{1}{2}\left\|f_{\nu}+f_{\mu}\right\|_{\mathcal{L}}=1
$$

Indeed, $\left\|f_{\nu}+f_{\mu}\right\|_{\mathcal{L}} \leqslant\left|E_{\nu}\right|^{1 / p}+\left|E_{\mu}\right|^{1 / p}=2$, so

$$
\mathbf{B}_{q, d}=\lim _{\nu, \mu \rightarrow \infty}\left\|\frac{1}{2} \widehat{f}_{\nu}+\frac{1}{2} \widehat{f}_{\mu}\right\|_{q} \leqslant \lim _{\nu, \mu \rightarrow \infty} \frac{\left\|\frac{1}{2} \widehat{f}_{\nu}+\frac{1}{2} \widehat{f}_{\mu}\right\|_{q}}{\left\|\frac{1}{2} f_{\nu}+\frac{1}{2} f_{\mu}\right\|_{\mathcal{L}}} \leqslant \mathbf{B}_{q, d}
$$

where we used Proposition 5 in the final inequality. Also observe that

$$
\lim _{\nu, \mu \rightarrow \infty}\left\|\frac{1}{2} f_{\nu}+\frac{1}{2} f_{\mu}\right\|_{\mathcal{L}} \leqslant\left\|\frac{1}{2} 1_{E_{\nu}}+\frac{1}{2} 1_{E_{\mu}}\right\|_{\mathcal{L}} \leqslant 1
$$

so $\lim _{\nu, \mu \rightarrow \infty}\left\|\frac{1}{2} 1_{E_{\nu}}+\frac{1}{2} 1_{E_{\mu}}\right\|_{\mathcal{L}}=1$. By Lemma 59, since $\frac{1}{2} 1_{E_{\nu}}+\frac{1}{2} 1_{E_{\mu}}=1_{E_{\nu} \cap E_{\mu}}+\frac{1}{2} 1_{E_{\nu} \Delta E_{\mu}}$,

$$
\left\|\frac{1}{2} 1_{E_{\nu}}+\frac{1}{2} 1_{E_{\mu}}\right\|_{\mathcal{L}}=\frac{1}{2}\left|E_{\nu} \cap E_{\mu}\right|^{1 / p}+\frac{1}{2}\left|E_{\nu} \cup E_{\mu}\right|^{1 / 2} .
$$

Let $\delta_{\nu, \mu}>0$ be defined by $\left|E_{\nu} \cap E_{\mu}\right|=1-\delta_{\nu \mu}$, so $\left|E_{\nu} \cup E_{\mu}\right|=1+\delta_{\nu \mu}$. Since there exists $c>0$ so $(1-\delta)^{1 / p}+(1+\delta)^{1 / p} \leqslant 2-c \delta^{2}$ for $|\delta| \leqslant 1$, conclude that $\lim _{\nu, \mu \rightarrow \infty}\left|E_{\nu} \cap E_{\mu}\right|=1$. It follows that $\left|E_{\nu} \Delta E_{\mu}\right| \rightarrow 0$, so there exists a Lebesgue measurable set $E \subset \mathbb{R}^{d}$ such that $\left|E_{\nu} \Delta E\right| \rightarrow 0$.

Note that for each $0<\eta<1$,

$$
\begin{aligned}
\frac{1}{2}\left\|f_{\nu}+f_{\mu}\right\|_{\mathcal{L}} & \leqslant\left\|1_{\left\{\frac{1}{2}\left|f_{\nu}+f_{\mu}\right|>1-\eta\right\}}+(1-\eta) 1_{\left\{0<\frac{1}{2}\left|f_{\nu}+f_{\mu}\right| \leqslant 1-\eta\right\}}\right\|_{\mathcal{L}} \\
& =\eta\left|\left\{\frac{1}{2}\left|f_{\nu}+f_{\mu}\right|>1-\eta\right\}\right|^{1 / p}+(1-\eta)\left|\left\{0<\frac{1}{2}\left|f_{\nu}+f_{\mu}\right|\right\}\right|^{1 / p} \\
& \leqslant \eta\left|\left\{\frac{1}{2}\left|f_{\nu}+f_{\mu}\right|>1-\eta\right\}\right|^{1 / p}+(1-\eta)\left|E_{\nu} \cup E_{\mu}\right|^{1 / p} .
\end{aligned}
$$

Since $\lim _{\nu, \mu \rightarrow \infty}\left|E_{\nu} \cup E_{\mu}\right|=\lim _{\nu, \mu \rightarrow \infty} \frac{1}{2}\left\|f_{\nu}+f_{\mu}\right\|_{\mathcal{L}}=1$, we conclude that

$$
\lim _{\nu, \mu \rightarrow \infty}\left|\left\{x: \frac{1}{2}\left|f_{\nu}(x)+f_{\mu}(x)\right|>1-\eta\right\}\right|=1 .
$$

It follows that $\lim _{\nu, \mu \rightarrow \infty}\left\|f_{\nu}+f_{\mu}\right\|_{2}=2$, and so $\lim _{\nu, \mu \rightarrow \infty}\left\|f_{\nu}-f_{\mu}\right\|_{p}=0$ since $\left\|f_{\nu}-f_{\mu}\right\|_{p} \leqslant 2\left\|f_{\nu}-f_{\mu}\right\|_{2}$ and by the parallelogram law,

$$
\left\|f_{\nu}-f_{\mu}\right\|_{2}^{2}+\left\|f_{\nu}+f_{\mu}\right\|_{2}^{2}=2\left(\left\|f_{\nu}\right\|_{2}^{2}+\left\|f_{\mu}\right\|_{2}^{2}\right) .
$$

Letting $\nu, \mu \rightarrow \infty$ gives the result.

Corollary 33. Suppose that there is an affirmative answer to Question 9. Let $d \geqslant 1$ and $q \in(2, \infty), p=q^{\prime}$. There exist a measurable function $f$ and a measurable subset $E$ of $\mathbb{R}^{d}$ with $|f| \leqslant 1_{E}$ such that

$$
\mathbf{B}_{q, d}=\frac{\|\hat{f}\|_{q}}{|E|^{1 / p}}=\frac{\|\widehat{f}\|_{q}}{\|f\|_{\mathcal{L}}} .
$$

Proof of Corollary 33. By the proof of Proposition 2, there exist a sequence of Lebesgue measurable subsets $E_{\nu}$ of $\mathbb{R}^{d}$, functions $f_{\nu}$ satisfying $\left|f_{\nu}\right| \leqslant 1_{E_{\nu}}$ and $f, 1_{E} \in L^{p}\left(\mathbb{R}^{d}\right)$ with $|f|=$ $1_{E}$ which satisfy $\lim _{\nu \rightarrow \infty}\left|E_{\nu}\right|^{-1 / p}\left\|\widehat{f}_{\nu}\right\|_{q}=\mathbf{B}_{q, d}, \lim _{\nu \rightarrow \infty}\left\|f_{\nu}-f\right\|_{p}=0$, and $\lim _{\nu \rightarrow \infty}\left|E_{\nu} \Delta E\right|=0$. It follows immediately that

$$
\frac{\|\widehat{f}\|_{q}}{|E|^{1 / p}}=\frac{\|\widehat{f}\|_{q}}{\|f\|_{\mathcal{L}}}=\mathbf{B}_{q, d}
$$

Proof of Corollary 7. By Lemma 60, Lemma 6, and the inequality $\|g\|_{p 1} \leqslant q\|g\|_{p 1}^{*}$ for all $g \in L(p, 1)$ that

$$
\sup _{\substack{g \in L(p, 1) \\ g \neq 0}} \frac{\|\hat{g}\|_{q}}{\|g\|_{p 1}} \geqslant \frac{\mathbf{B}_{q, d}}{q}
$$

For the upper bound, we have a similar argument to the proof of Lemma 6. Let $0 \neq$ $g \in L(p, 1)$. Let $E=\{(y, s):|g(y)|>s\}$. Let $|g| e^{i \varphi}=g$ so we can use the layer cake representation

$$
g(x)=e^{i \varphi(x)} \int_{0}^{\infty} 1_{E}(x, s) d s .
$$

Then

$$
\begin{aligned}
\|\widehat{g}\|_{q} & =\left\|\left(\int_{0}^{\infty} e^{i \varphi(x)} 1_{E}(x, s) d s\right)^{\wedge}\right\|\left\|_{q}=\right\| \int_{0}^{\infty} \widehat{e^{i \varphi} 1_{E}}(\xi, s) d s \|_{q} \\
& \leqslant \int_{0}^{\infty}\left\|\widehat{e^{i \varphi} 1_{E}}(\xi, s)\right\|_{q} d s \\
& \leqslant \int_{0}^{\infty} \mathbf{B}_{q, d}|\{x:|g(x)|>s\}|^{1 / p} d s \\
& =\frac{\mathbf{B}_{q, d}}{q} \int_{0}^{\infty}\left\|1_{\{x:|g(x)|>s\}}\right\|_{p 1} d s \\
& =\frac{\mathbf{B}_{q, d}}{q} \int_{0}^{\infty} \int_{0}^{\infty} t^{-1 / q-1} \int_{0}^{t} 1_{\{x:|g(x)|>s\}}^{*}(u) d u d t d s \\
& \leqslant \frac{\mathbf{B}_{q, d}}{q} \int_{0}^{\infty} t^{-1 / q-1} \int_{0}^{t} g^{*}(u) d u d t=\frac{\mathbf{B}_{q, d}}{q}\|g\|_{p 1},
\end{aligned}
$$

so

$$
\sup _{\substack{g \in L(p, 1) \\ g \neq 0}} \frac{\|\widehat{g}\|_{q}}{\|g\|_{p 1}}=\frac{\mathbf{B}_{q, d}}{q}
$$

Now if $0 \neq f \in L(p, 1)$ satisfies $\frac{\|\hat{f}\|_{q}}{\|f\|_{p 1}}=\frac{\mathbf{B}_{q, d}}{q}$, then by repeating the previous analysis, the above inequalities are equalities. Equality in the Minkowski integral inequality implies that for a.e. $(\xi, s) \in \mathbb{R}^{d} \times \mathbb{R}^{+}$,

$$
\widehat{e^{i \varphi} 1_{E}}(\xi, s)=h(\xi) g(s)
$$

for some measurable functions $h, g$. Since $e^{i \varphi} 1_{E}(x, t) \in L^{2}$, in particular, $h$ and $\check{h}$ in $L^{2}$.

$$
1_{E}(x, s)=e^{-i \varphi(x)} \breve{h}(x) g(s) .
$$

But then for every $(x, s)$ satisfying $|f(x)|>s$, we have

$$
e^{-i \varphi(x)} \breve{h}(x) g(t)=1 .
$$

Suppose $|f(x)|>|f(y)|>0$. Then for all $0 \leqslant s<f(y)$,

$$
e^{-i \varphi(x)} \breve{h}(x)=g(s)^{-1}=e^{i \varphi(y)} \widetilde{h(y)}
$$

which is a contradiction unless $|f(x)|$ is constant on its support. Thus $f$ takes the form $a e^{i \varphi} 1_{S}$ where $S \subset \mathbb{R}^{d}$ is a Lebesgue measurable subset and $a \in \mathbb{R}^{+}$.

For the existence of such an extremizer, by the proof of Proposition 2, there exist a sequence of Lebesgue measurable subsets $E_{\nu}$ of $\mathbb{R}^{d}$, functions $f_{\nu}$ satisfying $\left|f_{\nu}\right| \leqslant 1_{E_{\nu}}$, and $f, 1_{E} \in L^{p}\left(\mathbb{R}^{d}\right)$ with $|f|=1_{E}$ which satisfy $\lim _{\nu \rightarrow \infty}\left|E_{\nu}\right|^{-1 / p}\left\|\hat{f}_{\nu}\right\|_{q}=\mathbf{B}_{q, d}, \lim _{\nu \rightarrow \infty}\left\|f_{\nu}-f\right\|_{p}=0$, and $\lim _{\nu \rightarrow \infty}\left|E_{\nu} \Delta E\right|=0$. Thus there exists $f \in L(p, 1)$ satisfying

$$
\frac{\|\widehat{f}\|_{q}}{q|E|^{1 / p}}=\frac{\|\widehat{f}\|_{q}}{\|f\|_{p 1}}=\frac{\mathbf{B}_{q, d}}{q}
$$

## Chapter 6

## Sharpened inequalities for exponents $q \geqslant 4$ close to even integers

### 6.1 Background for sharpened inequalities and a general approach

The existence and identification of maximizers has been studied for many inequalities in analysis. Talenti determined the maximizers for the Sobolev inequality $\|f\|_{p^{*}} \leqslant B(n)\|\nabla f\|_{p}$ for $\mathbb{R}^{n}$ [37]. Maximizers for the Hardy-Littlewood-Sobolev inequality were determined by Lieb [29] and with an alternative method by Frank and Lieb [24]. Extremizing cases for the isoperimetric inequalities and the Brunn-Minkowski inequality have also been studied.

There is extensive work in sharpened inequalities of the same general form as 1.8 . Bianchi and Egnell [4 proved a sharpened Sobolev inequality for the special case $p=2$, with general $p$ treated by Cianchi, Fusco, Maggi, and Pratelli [17]. Chen, Frank, and Weth established a quantitative sharpened fractional Sobolev inequality in 9]. Fusco, Maggi, and Pratelli [23] obtained a strengthened isoperimetric inequality. Figalli, Maggi, and Pratelli [18] used mass transportation techniques to sharpen isoperimetric inequalities and the BrunnMinkowski inequality for convex sets.

Let $\mathbb{B}$ denote the $d$-dimensional unit ball. In this paper, the proof of the quantitative stability result (1.8) implies that $\mathbf{B}_{q, d}=\left\|\widehat{1_{\mathbb{B}}}\right\|_{q} /|\mathbb{B}|^{1 / p}$, so maximizers exist. In addition, the inequality 1.8 implies that the only maximizers are functions equivalent to $1_{\mathbb{B}}$ acted on by symmetries of the inequality. The quantitative stability result determining the extremizers is analogous to the argument of Christ in [16].

A general approach outlined by Bianchi and Egnell [4] to prove stability results like (1.8) for $4 \leqslant q \in 2 \mathbb{N}$ is as follows. It is immediate from 1.9 that $\mathbf{A}_{q, d}=\mathbf{B}_{q, d}$. Since for $|f| \leqslant 1_{E}$,

$$
\frac{\|\widehat{f}\|_{q}}{|E|^{1 / p}}=\frac{\left\|\widehat{e^{2 L} f \circ} \varphi\right\|_{q}}{\left|\varphi^{-1}(E)\right|^{1 / p}}
$$

for all affine transformations $\varphi: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ and $L \in \mathfrak{L}$, we can also say that functions of the
form $e^{i L} 1_{\mathcal{E}}$ where $L \in \mathfrak{L}$ and $\mathcal{E} \in \mathfrak{E}$ are among the extremizers for 1.5 . Establishing (1.8) would then show that they are the only extremizers. Christ's Theorem 34, stated below, plus the inequality in (1.9) will provide the starting point for our proof of (1.8).
Theorem 34. [16] Let $d \geqslant 1$. For each even integer $m \in\{4,6,8, \ldots\}$ there exists $\delta(m)>0$ such that the following three conclusions hold for all exponents satisfying $|q-m| \leqslant \delta(m)$. Let $p$ be the conjugate exponent to $q$. Firstly,

$$
\mathbf{A}_{q, d}=\left\|\widehat{\hat{1}_{E}}\right\|_{q} /|E|^{1 / p} \quad \text { for any } E \in \mathfrak{E} .
$$

Secondly, ellipsoids are the only extremizers; for any Lebesgue measurable set $E \subset \mathbb{R}^{d}$, with $0<|E|<\infty, \Phi_{q}(E)=\mathbf{A}_{q, d}$ if and only if $E$ is an ellipsoid. Thirdly, there exists $\tilde{c}_{q, d}>0$ such that for every set $E \subset \mathbb{R}^{d}$ with $|E|=1$,

$$
\begin{equation*}
\left\|\widehat{1_{E}}\right\|_{q}^{q} \leqslant \mathbf{A}_{q, d}^{q}-\tilde{c}_{q, d} \operatorname{dist}(E, \mathfrak{E})^{2} . \tag{6.1}
\end{equation*}
$$

Definition 12. For $\delta$ a small positive constant, we say that $|f| \leqslant 1_{E}$ is a $\delta$ near extremizer of $\sqrt{1.5}$, or just a near extremizer, if $(1-\delta) \mathbf{B}_{q, d}^{q}|E|^{q / p} \leqslant\|\widehat{f}\|_{q}^{q}$.

If $|E|=1, f, g$ are real-valued functions with $0 \leqslant f \leqslant 1$, and $f e^{i g} 1_{E}$ is NOT a $\delta$ near extremizer of (1.5), then

$$
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q} \leqslant \mathbf{B}_{q, d}^{q}(1-\delta) \leqslant \mathbf{B}_{q, d}^{q}-\frac{\delta}{9} \mathbf{B}_{q, d}^{q}\left[\|f-1\|_{L^{1}(E)}+\operatorname{dist}_{E}\left(e^{i g}, \mathfrak{L}\right)^{2}+\operatorname{dist}(E, \mathfrak{E})^{2}\right]
$$

since $\|f-1\|_{L^{1}(E)} \leqslant 1, \operatorname{dist}_{E}\left(e^{i g}, \mathfrak{L}\right)^{2} \leqslant 4$, and $\operatorname{dist}(E, \mathfrak{E}) \leqslant 4$. Thus in the case that $f e^{i g} 1_{E}$ is not a $\delta$ near extremizer, 1.8 is trivially satisfied with $c_{q, d}=\frac{\delta}{9} \mathbf{B}_{q, d}^{q}$.

Now assume that $f e^{i g} 1_{E}$ is a $\delta$ near extremizer. From (1.9) and (6.1) for $4 \leqslant q \in 2 \mathbb{N}$, we can immediately say that

$$
\tilde{c}_{q, d} \operatorname{dist}(E, \mathfrak{E})^{2} \leqslant \delta \mathbf{B}_{q, d} .
$$

By precomposing $f e^{i g} 1_{E}$ with an appropriate affine transformation, we can assume that $|E \Delta \mathbb{B}|^{2}$ is bounded by a constant multiple of $\delta$. We work more to prove that $f$ must be close to 1 and $e^{i g}$ close to $e^{i L}$ for some $L \in \mathfrak{L}$ in $\S 6.3$.

In the case that $f e^{i g-i L} 1_{E}$ is close to $1_{\mathbb{B}}$, we will be able to control the error in a Taylor expansion of $\left\|f \widehat{e^{i g-i L}} 1_{E}\right\|_{q}^{q}$ about $\left\|\widehat{\mathbb{1}_{\mathbb{B}}}\right\|_{q}^{q}$ which is developed in $\S 6.4$. To simplify the Taylor expansion analysis, we treat the special case of $E=\mathbb{B}$ for near-even integer exponents $q$ in $\$ 6.6$.

In 6.3.1, we generalize the previous discussion to $3 \leqslant q$ near even integers using the equicontinuity of the functional

$$
q \mapsto\|\widehat{f}\|_{q}
$$

on $q \in(2, \infty)$ where $|f| \leqslant 1_{E}$, $E$ a Lebesgue measurable set with $|E|<\infty$. Finally, for real valued functions $f$ and $g$ with $0 \leqslant f \leqslant 1$ and a Lebesgue measurable set $E \subset \mathbb{R}^{d}$ of finite measure, we prove (1.8) for near extremizers in three cases: (1) majority modulus $f$ variation, (2) majority support $E$ variation, and (3) mostly frequency $g$ variation, which we address in Proposition 49, Proposition 54, and Proposition 55 respectively.

### 6.2 Equicontinuity of $q \mapsto\left\|\widehat{f 1_{E}}\right\|_{q}$ for $|E|=1$.

The following equicontinuity result for the optimal constant $\mathbf{B}_{q, d}$ as a function of $q$ will be used to make a perturbative argument generalizing bounds for one exponent to nearby exponents.

Lemma 35. Let $d \geqslant 1$ and $r \in(2, \infty)$. As $f 1_{E}$ varies over all subsets satisfying $|E|=1$ and functions satisfying $|f| \leqslant 1$, the functions $q \mapsto\left\|\widehat{f 1_{E}}\right\|_{q}$ form an equicontinuous family of functions of $q$ on any compact subset of $(2, \infty)$.

Proof. This follows from the proof of Lemma 3.1 from [16] with $f 1_{E}$ in place of $1_{E}$.
An immediate consequence of the equicontinuity lemma is the following corollary.
Corollary 36. For each mapping $d \geqslant 1$, the mapping $(2, \infty) \ni q \mapsto \mathbf{B}_{q, d} \in R^{+}$is continuous.
The following corollary and lemma will be used in $\S 6.3 .1$ to outline the strategy of the proof of Theorem 3.

Corollary 37. Let $d \geqslant 1$ and $\bar{q} \geqslant 4$ be an even integer with conjugate exponent $\bar{p}$. Let $\delta>0$, $E$ be a Lebesgue measurable subset of $\mathbb{R}^{d}$, and $f: \mathbb{R}^{d} \rightarrow \mathbb{C}$ satisfy $|f| \leqslant 1$. Let $q>2$ with conjugate exponent $p$. If

$$
\left\|\widehat{f 1_{E}}\right\|_{q}^{q} /|E|^{q / p} \geqslant \mathbf{B}_{q, d}^{q}-\delta,
$$

then

$$
\left\|\widehat{f 1_{E}}\right\|_{\bar{q}}^{\bar{q}} /|E|^{\bar{q} / \bar{p}} \geqslant \mathbf{B}_{\bar{q}, d}^{\bar{q}}-o_{q-\bar{q}}(1)-\delta
$$

where $o_{q-\bar{q}}(1)$ is a function which tends to zero as $|q-\bar{q}|$ goes to zero.
Proof. Since $\Psi_{q}$ is invariant under dilations, it suffices to consider when $|E|=1$. Then the conclusion follows from the preceding Lemma 35 and Corollary 36 .

The purpose of the following lemma is to confirm that Theorem 3 is trivial unless $\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q}$ is close to $\mathbf{B}_{q, d}^{q}$.

Lemma 38. Let $d \geqslant 1$ and $q \geqslant 2$ with conjugate exponent $p$. Let $0<\delta<1$, let $E \subset \mathbb{R}^{d}$ be a Lebesgue measurable set with $|E|=1$ and let $f, g$ be real-valued functions with $0 \leqslant f \leqslant 1$. If

$$
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q} \leqslant \mathbf{B}_{q, d}^{q}-\delta
$$

then

$$
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q} \leqslant \mathbf{B}_{q, d}^{q}-\frac{\delta}{6}\left[\|f-1\|_{L^{1}(E)}+\inf _{L \in \mathfrak{L}}\left\|e^{i g}-e^{i L}\right\|_{L^{2}(E)}^{2}+\operatorname{dist}(E, \mathfrak{E})^{2}\right]
$$

Proof. It suffices to note the following inequalities.

$$
\begin{aligned}
\|f-1\|_{L^{1}(E)} & \leqslant 2|E| \leqslant 2 \\
\inf _{L \in \mathfrak{L}}\left\|e^{i g}-e^{i L}\right\|_{L^{2}(E)} & \leqslant\left\|e^{i g}-1\right\|_{L^{2}(E)} \leqslant 2|E|^{1 / 2} \leqslant 2 \\
\operatorname{dist}(E, \mathfrak{E}) & \leqslant \frac{|E \Delta \lambda \mathbb{B}|}{|E|} \leqslant 2
\end{aligned}
$$

where we define $\lambda$ by $|\lambda \mathbb{B}|=1$, where $\mathbb{B}$ denotes the unit ball in $\mathbb{R}^{d}$.

### 6.3 Structure of near-extremizers of the form $f e^{i g} 1_{E}$ for $q=2 m$.

In this section, let $f$ be a real valued function with $0 \leqslant f(x) \leqslant 1$ a.e., let $g$ be a real valued function, and let $E \subset \mathbb{R}^{d}$ be a Lebesgue measurable set. Recall that for even integers $q$, we know that $\left\|\widehat{1_{\mathbb{B}}}\right\|_{q} /|\mathbb{B}|^{1 / p}=\mathbf{B}_{q, d}$. We carefully unpackage the structure of near-extremizers of (1.5) of the form $f e^{i g} 1_{E}$ for even $q$. By proving that (possibly after composition with an affine function) $e^{i g}$ must be close to a multiple of a character and that $\|f-1\|_{1}$ and $|E \Delta \mathbb{B}|$ must be small, we guarantee that a Taylor expansion of $\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q}$ about $\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}$ will have an error that we can control (see $\$ 6.4$ ).

Since $q$ is even, we can write $\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q}$ as an $m$-fold convolution product using Plancherel's theorem:

$$
\begin{align*}
& \left\|\widehat{f e^{i g} 1_{E}}\right\|_{2 m}^{2 m}=\int_{E^{2 m-1}} f\left(x_{1}\right) \cdots f\left(x_{m}\right) f\left(y_{2}\right) \cdots f\left(y_{m}\right) f(L(x, y)) \times  \tag{6.2}\\
& \cos \left(g\left(x_{1}\right)+\cdots+g\left(x_{m}\right)-g\left(y_{2}\right)-\cdots-g\left(y_{m}\right)-g(L(x, y))\right) 1_{E}(L(x, y)) d x d y
\end{align*}
$$

where $x=\left(x_{1}, \ldots, x_{m}\right) \in \mathbb{R}^{m d}, y=\left(y_{2}, \ldots, y_{m}\right) \in \mathbb{R}^{(m-1) d}$, and $L(x, y)=x_{1}+\cdots+x_{m}-$ $y_{2}-\cdots y_{m}$. From this expression, it is clear that

$$
\begin{align*}
& \left\|\widehat{f e^{i g} 1_{E}}\right\|_{q} \leqslant\left\|\widehat{1_{E}}\right\|_{q}  \tag{6.3}\\
& \left\|\widehat{f e^{i g} 1_{E}}\right\|_{q} \leqslant\left\|\widehat{f 1_{E}}\right\|_{q}  \tag{6.4}\\
& \left\|\widehat{f e^{i g} 1_{E}}\right\|_{q} \leqslant\left\|\overrightarrow{e^{i g} f 1_{E}}\right\|_{q} . \tag{6.5}
\end{align*}
$$

If $(1-\delta) \mathbf{B}_{q, d}|E|^{1 / p} \leqslant\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}$, then by (6.3),

$$
(1-\delta) \mathbf{A}_{q, d}|E|^{1 / p} \leqslant\left\|\widehat{1_{E}}\right\|_{q}
$$

where $\mathbf{A}_{q, d}=\sup _{E} \frac{\|\widehat{1 E}\|_{q}}{|E|^{1 / p}}$ and equals $\mathbf{B}_{q, d}$ since $q$ is even. By Christ's Theorem 2.6 in [16], conclude that

$$
\left|T^{-1}(E) \Delta \mathbb{B}\right| \leqslant 2 \operatorname{dist}(E, \mathfrak{E}) \leqslant O\left(\delta^{1 / 2}\right)
$$

where $T \in \operatorname{Aff}\left(\mathbb{R}^{d}\right)$ is an affine automorphism of $\mathbb{R}^{d}$ and the big-O depends on dimension and is uniform for $q$ in a compact subset of $(3, \infty)$.

Replacing our near-extremizer $f e^{i g} 1_{E}$ by $f \circ T e^{i g \circ T} 1_{E} \circ T$, we may assume that $|E \Delta \mathbb{B}| \leqslant$ $O\left(\delta^{1 / 2}\right)$.

Define a measurable function $f_{0}: \mathbb{R}^{d} \rightarrow[0,1]$ by $f_{0}=f 1_{E \cap \mathbb{B}}+1_{\mathbb{B} \backslash E}$. Note that

$$
\begin{align*}
\left\|\widehat{f 1_{E}}\right\|_{q} & \leqslant\left\|\widehat{f_{0} 1_{\mathbb{B}}}\right\|_{q}+\left\|\widehat{f 1_{E}}-\widehat{f_{0} 1_{\mathbb{B}}}\right\|_{q} \leqslant\left\|\widehat{f_{0} 1_{\mathbb{B}}}\right\|_{q}+\left\|f 1_{E \backslash \mathbb{B}}-1_{\mathbb{B} \backslash E}\right\|_{p} \\
& \leqslant\left\|\widehat{f_{0} 1_{\mathbb{B}}}\right\|_{q}+|E \Delta \mathbb{B}|^{1 / p} \leqslant\left\|\widehat{f_{0} 1_{\mathbb{B}}}\right\|_{q}+O\left(\delta^{1 / 2 p}\right) . \tag{6.6}
\end{align*}
$$

In the following lemma, we consider $\left\|\widehat{f_{0} 1_{\mathbb{B}}}\right\|_{q}$.
Lemma 39. Let $d \geqslant 1$ and let $q \geqslant 4$. Suppose $0 \leqslant f \leqslant 1_{\mathbb{B}}$. Then

$$
\left\|\widehat{f 1_{\mathbb{B}}}\right\|_{q}^{q} \leqslant\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-c\|f-1\|_{L^{1}(\mathbb{B})}
$$

for $c=\inf _{\mathbb{B}} K_{q}>0$ where $K_{q}$ is the $(q-1)$-fold convolution product $1_{\mathbb{B}} * \cdots * 1_{\mathbb{B}}$.
Proof. Letting $q=2 m$, we have

$$
\begin{aligned}
\left\|\widehat{f 1_{\mathbb{B}}}\right\|_{2 m}^{2 m}= & \left\|\widehat{1_{\mathbb{B}}}\right\|_{2 m}^{2 m}+\left\|\widehat{f 1_{\mathbb{B}}}\right\|_{2 m}^{2 m}-\left\|\widehat{1_{\mathbb{B}}}\right\|_{2 m}^{2 m} \\
& =\left\|\hat{1_{\mathbb{B}}}\right\|_{2 m}^{2 m}+\left\langle f 1_{\mathbb{B}} * \cdots * f 1_{\mathbb{B}}, f 1_{\mathbb{B}} * \cdots * f 1_{\mathbb{B}}\right\rangle-\left\langle 1_{\mathbb{B}} * \cdots * 1_{\mathbb{B}}, 1_{\mathbb{B}} * \cdots * 1_{\mathbb{B}}\right\rangle \\
& \leqslant\left\|\widehat{1_{\mathbb{B}}}\right\|_{2 m}^{2 m}+\left\langle f 1_{\mathbb{B}} * 1_{\mathbb{B}} \cdots * 1_{\mathbb{B}}, 1_{\mathbb{B}} * \cdots * 1_{\mathbb{B}}\right\rangle-\left\langle 1_{\mathbb{B}} * \cdots * 1_{\mathbb{B}}, 1_{\mathbb{B}} * \cdots * 1_{\mathbb{B}}\right\rangle \\
& =\left\|\widehat{1_{\mathbb{B}}}\right\|_{2 m}^{2 m}+\left\langle(f-1) 1_{\mathbb{B}}, K_{q}\right\rangle \\
& \leqslant\left\|\widehat{1_{\mathbb{B}}}\right\|_{2 m}^{2 m}-c\|f-1\|_{L^{1}(\mathbb{B})}
\end{aligned}
$$

where each convolution product has $m$ factors and we used that $K_{q}>0$ on $\mathbb{B}$.
Combine (6.6) with Lemma 39 to reason that if $f e^{i g} 1_{E}$ is a near-extremizer and $|E \Delta \mathbb{B}| \leqslant$ $O\left(\delta^{1 / 2}\right)$, then

$$
\|f-1\|_{L^{1}(E)} \leqslant\left\|f_{0}-1\right\|_{L^{1}(\mathbb{B})}+2|E \Delta \mathbb{B}| \leqslant O\left(\delta^{1 / 2 p}\right)
$$

Now define $g_{0}: \mathbb{R}^{d} \rightarrow \mathbb{R}$ by $g_{0}=1_{E \cap \mathbb{B}} g$. Then

$$
\begin{aligned}
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q} & \leqslant\left\|\widehat{e^{i g_{0} 1_{\mathbb{B}}}}\right\|_{q}+\left\|e^{i g} 1_{E}-e^{i g_{0}} 1_{\mathbb{B}}\right\|_{p}+\left\|(f-1) 1_{E}\right\|_{p} \\
& \leqslant\left\|e^{i g_{0} 1_{\mathbb{B}}}\right\|_{q}+|E \Delta \mathbb{B}|^{1 / p}+\left\|(f-1) 1_{E}\right\|_{1}^{1 / p} \\
& =\left\|\widehat{e^{i g_{0}} 1_{\mathbb{B}}}\right\|_{q}+O\left(\delta^{1 / 2 p^{2}}\right) .
\end{aligned}
$$

Thus it remains to understand the case in which $e^{i g_{0}} 1_{\mathbb{B}}$ is a near extremizer. The naive approach used to understand $f 1_{\mathbb{B}}$ in Lemma 39 breaks down when considering $e^{i g} 1_{\mathbb{B}}$ since an expression with $g$ appears within the argument of the cosine in 6.2 . One tool we have at our disposal is the following Proposition 8.2 of Christ from [14], stated here for the reader's convenience.

Lemma 40. For each dimension $d \geqslant 1$ there exists a constant $K<\infty$ with the following property. Let $B \subset \mathbb{R}^{d}$ be a ball centered at the origin with positive radius, and let $\eta \in\left(0, \frac{1}{2}\right]$, and let $\delta>0$ be sufficiently small. For $j \in\{1,2,3\}$, let $f_{j}: 2 B \rightarrow \mathbb{C}$ be Lebesgue measurable functions that vanish only on sets of Lebesgue measure zero. Suppose that

$$
\left|\left\{(x, y) \in B^{2}:\left|f_{1}(x) f_{2}(y) f_{3}(x+y)^{-1}-1\right|>\eta\right\}\right|<\delta|B|^{2} .
$$

Then for each index $j$ there exists an affine function $L_{j} n: \mathbb{R}^{d} \rightarrow \mathbb{C}$ such that

$$
\left|\left\{x \in B:\left|f_{j}(x) e^{-L_{j}(x)}-1\right|>K \eta^{1 / K}\right\}\right| \leqslant K \delta|B| .
$$

We will use this lemma in order to obtain structure for $g$, as described in the following proposition.

Proposition 41. Let $d \geqslant 1$ and let $q \geqslant 4$ be an even integer with conjugate exponent $p$. There exist positive constants $\tilde{K}, \delta_{0}>0$, depending only on $d$, with the following property. Suppose that $(1-\delta) \mathbf{B}_{q, d}|\mathbb{B}|^{1 / p} \leqslant\left\|e^{i g} 1_{\mathbb{B}}\right\|_{q}$ for $\delta \leqslant \delta_{0}$ and $g$ real-valued. Then there exists an affine function $L_{1}: \mathbb{R}^{d} \rightarrow \mathbb{R}$ such that

$$
\int_{\mathbb{B}}\left|e^{i L_{1}(x)-i g(x)}-1\right| d x \leqslant \tilde{K} \delta^{1 /(8 \tilde{K})}
$$

Proof. Let $q=2 m$. We use the expression

$$
\widehat{e^{i g} 1_{\mathbb{B}}} \|_{q}^{q}=\int_{\mathbb{B}^{q-1}} \cos \left(g\left(x_{1}\right)+\cdots+g\left(x_{m}\right)-g\left(y_{2}\right)-\cdots-g\left(y_{m}\right)-g(L(x, y))\right) 1_{\mathbb{B}}(L(x, y)) d x d y
$$

where $x=\left(x_{1}, \ldots, x_{m}\right) \in \mathbb{R}^{m d}, y=\left(y_{2}, \ldots, y_{m}\right) \in \mathbb{R}^{(m-1) d}$, and $L(x, y)=x_{1}+\cdots+x_{m}-$ $y_{2}-\cdots y_{m}$. Let $A(x, y)=g\left(x_{1}\right)+\cdots+g\left(x_{m}\right)-g\left(y_{2}\right)-\cdots-g\left(y_{m}\right)-g(L(x, y))$. Since $e^{i g} 1_{\mathbb{B}}$ is a near-extremizer, we have

$$
\begin{aligned}
(1-\delta) \mathbf{B}_{q, d}|\mathbb{B}|^{1 / p} & \leqslant\left\|\widehat{e^{i g} 1_{\mathbb{B}}}\right\|_{q}^{q}=\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+\left\|\widehat{e^{i g} 1_{\mathbb{B}}}\right\|_{q}^{q}-\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q} \\
& =\mathbf{B}_{q, d}-\int_{\mathbb{B} q-1}|\cos (A(x, y))-1| 1_{\mathbb{B}}(L(x, y)) d x d y
\end{aligned}
$$

so $\int_{\mathbb{B}^{q-1}}|\cos (A(x, y))-1| 1_{\mathbb{B}}(L(x, y)) d x d y \leqslant \mathbf{B}_{q, d}|\mathbb{B}|^{1 / p} \delta$. We use this in the following:

$$
\begin{aligned}
\int_{\mathbb{B}^{q-1}}\left|e^{i A(x, y)}-1\right| 1_{\mathbb{B}}(L(x, y)) d x d y & \left.=\int_{\mathbb{B}^{q-1}}(\cos (A(x, y))-1)^{2}+(\sin (A(x, y)))^{2}\right)^{1 / 2} 1_{\mathbb{B}}(L(x, y)) d x d y \\
& =\int_{\mathbb{B}^{q-1}} \sqrt{2}|\cos (A(x, y))-1|^{1 / 2} 1_{\mathbb{B}}(L(x, y)) d x d y \\
& \leqslant \sqrt{2}\left|\mathbb{B}^{q-1}\right|^{1 / 2}\left(\int_{\mathbb{B}^{q-1}}|\cos (A(x, y))-1| 1_{\mathbb{B}}(L(x, y)) d x d y\right)^{1 / 2} \\
& \leqslant \sqrt{2}|\mathbb{B}|^{(q-1) / 2}\left(\mathbf{B}_{q, d}|\mathbb{B}|^{1 / p} \delta\right)^{1 / 2}
\end{aligned}
$$

Set

$$
\begin{equation*}
\tilde{\delta}=\int_{\mathbb{B}^{q-1}}\left|e^{i A(x, y)}-1\right| 1_{\mathbb{B}}(L(x, y)) d x d y \leqslant C \delta^{1 / 2} \tag{6.7}
\end{equation*}
$$

Note that sometimes we will abuse the notation $L: L(x, y)$ where $x \in \mathbb{R}^{m d}, y \in \mathbb{R}^{(m-1) d}$ and $L\left(x_{1}+x_{2}, x^{\prime}, y\right)$ where $x_{1}, x_{2} \in \mathbb{R}^{d}, x^{\prime} \in \mathbb{R}^{(m-2) d}, y \in \mathbb{R}^{(m-1) d}$ mean the same thing. Define the function $\alpha$ by

$$
\begin{equation*}
\alpha\left(x_{1}+x_{2}, x^{\prime}, y\right):=g\left(x_{3}\right)+\cdots+g\left(x_{m}\right)-g\left(y_{2}\right)-\cdots-g\left(y_{m}\right)-g\left(L\left(x_{1}+x_{2}, x^{\prime}, y\right)\right) \tag{6.8}
\end{equation*}
$$

so that $A(x, y)=g\left(x_{1}\right)+g\left(x_{2}\right)+\alpha\left(x_{1}+x_{2}, x^{\prime}, y\right)$ for $L(x, y) \in \mathbb{B}$. Define the set $S_{\tilde{\delta}} \subset \mathbb{R}^{(q-3) d}$ to be

$$
S_{\tilde{\delta}}:=\left\{\left(x^{\prime}, y\right) \in \mathbb{B}^{q-3}: \int_{\mathbb{B}^{2}}\left|e^{i\left(g\left(x_{1}\right)+g\left(x_{2}\right)+\alpha\left(x_{1}+x_{2}, x^{\prime}, y\right)\right)}-1\right| 1_{\mathbb{B}}(L(x, y)) d x_{1} d x_{2}>\tilde{\delta}^{1 / 2}\right\} .
$$

Using Chebyshev's inequality in (6.7), we know that $\left|S_{\tilde{\delta}}\right| \leqslant \tilde{\delta}^{1 / 2}$. Fix an $\left(x^{\prime}, y\right) \in \mathbb{B}^{q-3} \backslash S_{\tilde{\delta}}$ such that $\left|\left(x^{\prime}, y\right)\right|<2 \inf \left\{\left|\left(w^{\prime}, z\right)\right|:\left(w^{\prime}, z\right) \in \mathbb{B}^{q-3} \backslash S_{\tilde{\delta}}\right\}$. Since $\left|S_{\tilde{\delta}}\right| \leqslant \tilde{\delta}^{1 / 2}$, there must be some positive intersection between $\mathbb{B}^{q-3} \backslash S_{\tilde{\delta}}$ and the ball in $\mathbb{R}^{(q-3) d}$ centered at the origin of radius $c_{(q-3) d}^{-1 /(q-3) d)}(2 \tilde{\delta})^{1 /(2(q-3) d)}$, where $c_{(q-3) d}$ is the volume of the $(q-3) d$-dimensional unit ball. Thus

$$
\left|\left(x^{\prime}, y\right)\right| \leqslant 2 c_{(q-3) d}^{-1 /((q-3) d}(2 \tilde{\delta})^{1 /(2(q-3) d)}=: b(q, d) \tilde{\delta}^{1 /(2(q-3) d)} .
$$

For our fixed $\left(x^{\prime}, y\right)$, let $a$ denote

$$
\begin{equation*}
a:=x_{3}^{\prime}+\cdots x_{m}^{\prime}-y_{2}-\cdots-y_{m} . \tag{6.9}
\end{equation*}
$$

Note that

$$
\begin{aligned}
|a| & \leqslant\left|x_{3}^{\prime}\right|+\cdots+\left|x_{m}^{\prime}\right|+\left|y_{2}\right|+\cdots+\left|y_{m}\right| \\
& \leqslant(q-3)^{1 / 2}\left(\left|x_{3}^{\prime}\right|^{2}+\cdots+\left|x_{m}^{\prime}\right|^{2}+\left|y_{2}\right|^{2}+\cdots+\left|y_{m}\right|^{2}\right)^{1 / 2} \\
& =(q-3)^{1 / 2}\left|\left(x^{\prime}, y\right)\right|^{1 / 2}<(q-3)^{1 / 2} b(q, d) \tilde{\delta}^{1 /(2(q-3) d)} .
\end{aligned}
$$

Since $\left(x^{\prime}, y\right) \notin S_{\tilde{\delta}}$, we apply Chebyshev's inequality again to get

$$
\begin{equation*}
\left|\left\{\left(x_{1}, x_{2}\right) \in \mathbb{B}^{2}:\left|e^{i\left(g\left(x_{1}\right)+g\left(x_{2}\right)+\alpha\left(x_{1}+x_{2}, x^{\prime}, y\right)\right)}-1\right| 1_{\mathbb{B}}(L(x, y))>\tilde{\delta}^{1 / 4}\right\}\right|<\tilde{\delta}^{1 / 4} . \tag{6.10}
\end{equation*}
$$

In order to satisfy the hypotheses of Lemma 40, we need to eliminate the indicator function $1_{\mathbb{B}}(L(x, y))$ from the set. We accomplish this by shrinking the size of the ball. Indeed let $r:=\frac{1}{2}(1-|a|)$. It is now clear why we chose $\left(x^{\prime}, y\right)$ with nearly minimal modulus: to make $r$ reasonably close to $1 / 2$. Then

$$
\begin{aligned}
& \left|\left\{\left(x_{1}, x_{2}\right) \in B(r)^{2}:\left|e^{i\left(g\left(x_{1}\right)+g\left(x_{2}\right)+\alpha\left(x_{1}+x_{2}, x^{\prime}, y\right)\right)}-1\right|>\tilde{\delta}^{1 / 4}\right\}\right| \\
& \quad \leqslant\left|\left\{\left(x_{1}, x_{2}\right) \in \mathbb{B}^{2}:\left|e^{i\left(g\left(x_{1}\right)+g\left(x_{2}\right)+\alpha\left(x_{1}+x_{2}, x^{\prime}, y\right)\right)}-1\right| 1_{\mathbb{B}}(L(x, y))>\tilde{\delta}^{1 / 4}\right\}\right| \leqslant \tilde{\delta}^{1 / 4}
\end{aligned}
$$

where $B(r)$ denotes the $d$-dimensional ball of radius $r$ centered at the origin. Thus the hypotheses of Lemma 40 are satisfied. The lemma guarantees the existence of an affine function $L_{0}: \mathbb{R}^{d} \rightarrow \mathbb{C}$ such that

$$
\begin{equation*}
\left|\left\{x \in B(r):\left|e^{i g(x)} e^{-L_{0}(x)}-1\right|>K \tilde{\delta}^{1 /(4 K)}\right\}\right| \leqslant K \tilde{\delta}^{1 / 4} r^{-d} \tag{6.11}
\end{equation*}
$$

where $K>0$ depends only on the dimension. Note that there exists a constant $c>0$ such that $\left|e^{i g(x)} e^{-L_{0}(x)}-1\right| \geqslant c\left|e^{i g(x)} e^{-i \operatorname{Im} L_{0}(x)}-1\right|$ for all $x \in \mathbb{B}$. Then for $K^{\prime}=K / c$, it follows from (6.11) that

$$
\left|\left\{x \in B(r):\left|e^{i g(x)-i \operatorname{Im} L_{0}(x)}-1\right|>K^{\prime} \tilde{\delta}^{1 /(4 K)}\right\}\right| \leqslant K \tilde{\delta}^{1 / 4} r^{-d}
$$

Since we know that $e^{i g} \approx e^{i \operatorname{Im} L_{0}}$ on the majority of $B(r)$, we can use the set whose measure is bounded in (6.10) to make conclusions about $\alpha$ on the set $B(r)+B(r)$ (which by the definition of $\alpha$ gives us information about $g$ on $B(2 r)$ ). More precisely, we have that

$$
\begin{align*}
\mid\left\{\left(x_{1}, x_{2}\right) \in\right. & \left.B(r)^{2}:\left|e^{i \operatorname{Im} L_{0}\left(x_{1}+x_{2}\right)+i \alpha\left(x_{1}+x_{2}, x^{\prime}, y\right)}-1\right|>4 K^{\prime} \tilde{\delta}^{1 /(4 K)}\right\} \mid \leqslant \\
& \left|\left\{\left(x_{1}, x_{2}\right) \in B(r)^{2}:\left|e^{i \operatorname{Im} L_{0}\left(x_{1}+x_{2}\right)-i g\left(x_{1}\right)-i g\left(x_{2}\right)}-1\right|>2 K^{\prime} \tilde{\delta}^{1 /(4 K)}\right\}\right| \\
& +\left|\left\{\left(x_{1}, x_{2}\right) \in B(r)^{2}:\left|e^{i\left(g\left(x_{1}\right)+g\left(x_{2}\right)+\alpha\left(x_{1}+x_{2}, x^{\prime}, y\right)\right)}-1\right|>2 K^{\prime} \tilde{\delta}^{1 /(4 K)}\right\}\right| \\
\leqslant & 2\left|\left\{\left(x_{1}, x_{2}\right) \in B(r)^{2}:\left|e^{i \operatorname{Im} L_{0}\left(x_{1}\right)-i g\left(x_{1}\right)}-1\right|>K^{\prime} \tilde{\delta}^{1 /(4 K)}\right\}\right|+\tilde{\delta}^{1 / 4} \\
\leqslant & 2 c_{d} r^{d} K \tilde{\delta}^{1 / 4} r^{-d}+\tilde{\delta}^{1 / 4}\left(2 c_{d} K+1\right) \tilde{\delta}^{1 / 4} \tag{6.12}
\end{align*}
$$

where $c_{d}$ is the volume of the $d$-dimensional unit ball.
Recalling the definition of $\alpha$ from (6.8) and $a$ from (6.9) we write for $x_{1}, x_{2} \in B(r)$

$$
\begin{aligned}
e^{i \operatorname{Im} L_{0}\left(x_{1}+x_{2}\right)+i \alpha\left(x_{1}+x_{2}, x^{\prime}, y\right)} & =e^{i \operatorname{Im} L_{0}\left(x_{1}+x_{2}\right)+i g\left(x_{3}\right)+\cdots+i g\left(x_{m}\right)-i g\left(y_{2}\right)-\cdots-i g\left(y_{m}\right)-i g\left(L\left(x_{1}+x_{2}, x^{\prime}, y\right)\right)} \\
& =e^{i \operatorname{Im} L_{0}\left(x_{1}+x_{2}\right)+i g\left(x_{3}\right)+\cdots+i g\left(x_{m}\right)-i g\left(y_{2}\right)-\cdots-i g\left(y_{m}\right)-i g\left(x_{1}+x_{2}+a\right)}
\end{aligned}
$$

Combining this expression with the bound from line (6.12), we have actually shown that for $L_{1}: \mathbb{R}^{d} \rightarrow \mathbb{R}$ the affine function defined by $L_{1}(w)=\operatorname{Im} L_{0}(w)-\operatorname{Im} L_{0}(a)+i g\left(x_{3}\right)+\cdots+$ $i g\left(x_{m}\right)-i g\left(y_{2}\right)-\cdots-i g\left(y_{m}\right)$,

$$
\begin{equation*}
\left|\left\{\left(x_{1}, x_{2}\right) \in B(r)^{2}:\left|e^{i L_{1}\left(x_{1}+x_{2}+a\right)-i g\left(x_{1}+x_{2}+a\right)}-1\right|>4 K^{\prime} \tilde{\delta}^{1 / 4}\right\}\right| \leqslant\left(2 c_{d} K+1\right) \tilde{\delta}^{1 / 4} \tag{6.13}
\end{equation*}
$$

Let $A$ denote the set whose measure is bounded above in 6.13). Let $E \subset \mathbb{R}^{d}$ denote the set

$$
E:=\left\{x \in B(2 r):\left|e^{L_{1}(w+a)-i g(w+a)}-1\right|>4 K^{\prime} \tilde{\delta}^{1 / 4}\right\} .
$$

Writing line (6.13) in reverse order, we relate $A$ to $E$ as follows.

$$
\begin{aligned}
\left(2 c_{d} K+1\right) \tilde{\delta}^{1 / 4} & \geqslant \iint_{B(r)^{2}} 1_{A}\left(x_{1}, x_{2}\right) d x_{1} d x_{2} \\
& =\iint_{B(r)^{2}} 1_{E}\left(x_{1}+x_{2}\right) d x_{1} d x_{2} \\
& =\left\langle 1_{B(r)} * 1_{B(r)}, 1_{E}\right\rangle \\
& \geqslant\left(\inf \left\{1_{B(r)} * 1_{B(r)}(x): x \in B(1-\eta)\right\}\right)|E \cap B(1-\eta)|
\end{aligned}
$$

Note that there is a dimensional constant $a_{d}>0$ such that $1_{\mathbb{B}}(r) * 1_{\mathbb{B}}(r)(x)>a_{d} \eta$ if $\eta<1$ and $x \in B(1-\eta)$. Thus if we pick $\eta=\tilde{\delta}^{1 / 8}$,

$$
\left|E \cap B\left(1-\tilde{\delta}^{1 / 8}\right)\right| \leqslant a_{d}^{-1}\left(2 c_{d} K+1\right) \tilde{\delta}^{1 / 8}
$$

Putting everything together, we can now bound the $L^{1}$ norm of $\left|e^{i L_{1}-i g}-1\right|$ :

$$
\begin{aligned}
\int_{\mathbb{B}}\left|e^{i L_{1}(x)-i g(x)}-1\right| d x= & \int_{E \cap B\left(1-\tilde{\delta}^{1 / 8}\right)}\left|e^{i L_{1}(x)-i g(x)}-1\right| d x+\int_{E \backslash B\left(1-\tilde{\delta}^{1 / 8}\right)}\left|e^{L_{1}(x)-i g(x)}-1\right| d x \\
& +\int_{\mathbb{B} \backslash E}\left|e^{L_{1}(x)-i g(x)}-1\right| d x \\
\leqslant & {\left[a_{d}^{-1}\left(2 c_{d} K+1\right) \tilde{\delta}^{1 / 8}+c_{d}(d+1) \tilde{\delta}^{1 / 8}\right](2)+c_{d} 4 K^{\prime} \tilde{\delta}^{1 /(4 K)} }
\end{aligned}
$$

Finally, we consider maximizers of the form $e^{i g} 1_{\mathbb{B}}$.
Lemma 42. Let $d \geqslant 1$. Suppose that $q \geqslant 4$ is an even integer and that $\left\|\widehat{e^{i g} 1_{\mathbb{B}}}\right\|_{q}^{q}=\mathbf{B}_{q, d}$. Then there exists an affine function $L: \mathbb{R}^{d} \rightarrow \mathbb{R}$ such that $e^{i g}=e^{i L}$ on $\mathbb{B}$.

This lemma follows from a standard argument. Using the expression from (6.2), the equality $\left\|\widehat{e^{i g} 1_{\mathbb{B}}}\right\|_{q}=\mathbf{B}_{q, d}=\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}$ leads to a functional equation. By taking smooth approximations (which still satisfy the functional equation) and using derivatives, we find that $e^{i g}$ has the desired form.

### 6.3.1 Proof strategy for Theorem 3.

Fix a dimension $d \geqslant 1$ and an even integer $\bar{q} \geqslant 4$. By Lemma 38, to prove Theorem 3, it is sufficient to find $\delta>0$ and $\rho>0$ so that if $|q-\bar{q}|<\rho$ and if

$$
\begin{equation*}
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q} \geqslant \mathbf{B}_{q, d}^{q}-\delta \tag{6.14}
\end{equation*}
$$

where $f, g$ are real valued with $0 \leqslant f \leqslant 1$ and $E \subset \mathbb{R}^{d}$ is a Lebesgue measurable subset with $|E|=1$, then

$$
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q} \leqslant \mathbf{B}_{q, d}^{q}-c_{q, d}\left[\|f-1\|_{L^{1}(E)}+\operatorname{dist}_{E}\left(e^{i g}, \mathfrak{L}\right)^{2}+\operatorname{dist}(E, \mathfrak{E})^{2}\right] .
$$

By Corollary 37, the hypothesis $\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q} \geqslant \mathbf{B}_{q, d}^{q}-\delta$ implies that

$$
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{\bar{q}}^{\bar{q}} \geqslant \mathbf{B}_{\bar{q}, d}^{\bar{q}}-o_{q-\bar{q}}(1)-\delta .
$$

If $\delta, \rho$ are sufficiently small, then by the previous section, there exists an affine automorphism $T: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ so that $\left|T^{-1}(E)\right|=|\mathbb{B}|$ and $\left|T^{-1}(E) \Delta \mathbb{B}\right| \leqslant 2 \operatorname{dist}(E, \mathfrak{E}) \leqslant$ $C_{\bar{q}, d}\left(o_{q-\bar{q}}(1)+\delta^{1 / 2}\right)$. Let $\bar{p}$ be the conjugate exponent to $\bar{q}$.

Define $f^{\prime}: \mathbb{B} \rightarrow[0,1]$ by $f \circ T$ on $T^{-1}(E) \cap \mathbb{B}$ and by 1 on $\mathbb{B} \backslash T^{-1}(\mathbb{B})$. By Lemma 39,

$$
\begin{aligned}
\left\|\widehat{\mathbb{1}_{\mathbb{B}}}\right\|_{\bar{q}}^{\bar{q}}-c_{\bar{q}}\left\|f^{\prime}-1\right\|_{L^{1}(\mathbb{B})} & \geqslant\left\|\widehat{f^{\prime} 1_{\mathbb{B}}}\right\|_{\bar{q}}^{\bar{q}} \\
& \geqslant|\mathbb{B}|^{\mid \bar{q} / \bar{p}} \mathbf{B}_{\bar{q}, d}^{\bar{q}}-o_{q-\bar{q}}(1)-o_{\delta}(1) \\
& =\left\|\widehat{1_{\mathbb{B}}}\right\|_{\bar{q}}^{\bar{q}}-o_{q-\bar{q}}(1)-o_{\delta}(1) .
\end{aligned}
$$

Thus $\|f \circ T-1\|_{L^{1}\left(T^{-1}(E)\right)} \leqslant\left\|f^{\prime}-1\right\|_{L^{1}(\mathbb{B})}+\|f \circ T-1\|_{L^{1}\left(T^{-1}(E) \backslash \mathbb{B}\right)} \leqslant o_{q-\bar{q}}(1)+o_{\delta}(1)$.
Define $g^{\prime}: \mathbb{B} \rightarrow \mathbb{R}$ by $g \circ T$ on $T^{-1}(E) \cap \mathbb{B}$ and 0 on $\mathbb{B} \backslash T^{-1}(E)$. Since $\|\left(f \circ T e^{i g \circ T} 1_{E} \circ\right.$ $T)^{\wedge}\left\|_{\bar{q}} /\left|T^{-1}(E)\right|^{1 / \bar{p}}=\right\|\left(f e^{i g} 1_{E}\right)^{\wedge} \|_{\bar{q}}$,

$$
\begin{aligned}
\left\|\widehat{e^{i g^{\prime}} 1_{\mathbb{B}}}\right\|_{\bar{q}} \geqslant & \left\|\left(f \circ T e^{i g \circ T} 1_{E} \circ T\right)^{\wedge}\right\|_{\bar{q}}-\left\|f \circ T e^{i g \circ T} 1_{E} \circ T-e^{i g^{\prime}} 1_{\mathbb{B}}\right\|_{\bar{p}} \\
\geqslant & \left|T^{-1}(E)\right|^{1 / \bar{p}} \mathbf{B}_{\bar{q}, d}-o_{q-\bar{q}}(1)-o_{\delta}(1) \\
& -\|f \circ T-1\|_{L^{1}(E)}^{1 / \bar{p}}-\left|T^{-1}(E) \Delta \mathbb{B}\right|^{1 / \bar{p}} \\
= & |\mathbb{B}|^{1 / \bar{p}} \mathbf{B}_{\bar{q}, d}-o_{q-\bar{q}}(1)-o_{\delta}(1) .
\end{aligned}
$$

Then Proposition 41 applies, so there exists a real-valued affine function $L: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ so that

$$
\left\|e^{i g^{\prime}}-e^{i L}\right\|_{L^{1}(\mathbb{B})} \leqslant o_{q-\bar{q}}(1)+o_{\delta}(1)
$$

If for a.e. $x \in T^{-1}(\mathbb{B})$ we choose a representative of the equivalence class $[g \circ T(x)-L(x)] \in$ $\mathbb{R} /(2 \pi)$ with values in some range $[-M, M]$, then

$$
\|g \circ T-L\|_{L^{2}\left(T^{-1}(E)\right)}^{2} \leqslant M^{2}\left|T^{-1}(E) \backslash \mathbb{B}\right|+M\left\|e^{i g^{\prime}}-e^{i L}\right\|_{L^{1}(\mathbb{B})} \leqslant M^{2}\left(o_{q-\bar{q}}(1)+o_{\delta}(1)\right) .
$$

We will prove in Propositions 49, 54, and 55 that

$$
\left\|\left(f \circ T e^{i(g \circ T-L)} 1_{E} \circ T\right)^{\wedge}\right\|_{q}^{q} \leqslant\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-c_{q, d}\left[\|f \circ T-1\|_{L^{1}\left(T^{-1}(E)\right)}+\operatorname{dist}_{T^{-1}(E)}\left(e^{i g \circ T}, \mathfrak{L}\right)^{2}+\left|T^{-1}(E) \Delta \mathbb{B}\right|^{2}\right]
$$

Since $\left\|\left(f \circ T e^{i(g \circ T-L)} 1_{E} \circ T\right)^{\hat{}}\right\|_{q} /|\mathbb{B}|^{1 / p}=\left\|\left(f e^{i g} 1_{E}\right)^{\hat{}}\right\|_{q}$, it follows that

$$
\begin{aligned}
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q} & \leqslant \mathbf{B}_{q, d}^{q}-|\mathbb{B}|^{-1 / p} c_{q, d}\left[\|f \circ T-1\|_{L^{1}\left(T^{-1}(E)\right)}+\operatorname{dist}_{T^{-1}(E)}\left(e^{i g \circ T}, \mathfrak{L}\right)^{2}+\left|T^{-1}(E) \Delta \mathbb{B}\right|^{2}\right] \\
& \leqslant \mathbf{B}_{q, d}^{q}-|\mathbb{B}|^{1 / q} c_{q, d}\left[\|f-1\|_{L^{1}(E)}+\operatorname{dist}_{E}\left(e^{i g}, \mathfrak{L}\right)^{2}+\operatorname{dist}(E, \mathfrak{E})^{2}\right]
\end{aligned}
$$

where we used that $|\mathbb{B}|=\left|T^{-1}(E)\right|=\left|\operatorname{det} T^{-1}\right||E|=\left|\operatorname{det} T^{-1}\right|$.

### 6.4 A Taylor expansion representation of $\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q}$.

Assuming that $|f| \leqslant 1_{E}$ is close to $1_{\mathbb{B}}$ in the appropriate sense, we can find a good representation of $\|\widehat{f}\|_{q}^{q}$ using a Taylor expansion about $\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}$. This is analogous to Lemma 3.4 in [16]. The functions in the following definition arise in the Taylor expansion.

Definition 13. For $d \geqslant 1$ and $q \in(3, \infty)$, we define the functions $K_{q}$ and $L_{q}$ on $\mathbb{R}^{d}$ by

$$
\begin{array}{r}
\widehat{K_{q}}=\left|\widehat{1_{\mathbb{B}}}\right|^{q-2} \widehat{1_{\mathbb{B}}} \\
\widehat{L_{q}}=\left|\widehat{1_{\mathbb{B}}}\right|^{q-2} . \tag{6.16}
\end{array}
$$

The basic properties of $K_{q}$ and $L_{q}$ are discussed in 6.4 below. For any function $f$ on $\mathbb{R}^{d}$, we let $\tilde{f}$ be the function $f(x)=f(-x)$.

Lemma 43. Let $d \geqslant 1$ and $q \in(3, \infty)$ with conjugate exponent $q^{\prime}$. Let $E \subset \mathbb{R}^{d}$ and $|f| \leqslant 1_{E}$. Set $h=f 1_{E}-1_{\mathbb{B}}$. For sufficiently small $\|h\|_{q^{\prime}}$,

$$
\begin{aligned}
\|\widehat{f}\|_{q}^{q}= & \left\|\widehat{\mathbb{1}_{\mathbb{B}}}\right\|_{q}^{q}+q\left\langle K_{q}, \operatorname{Re} h\right\rangle-\frac{1}{4} q(q-2)\left\langle\operatorname{Im} h * \operatorname{Im} h, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle\operatorname{Im} h, \operatorname{Im} h * L_{q}\right\rangle \\
& +\frac{1}{4} q(q-2)\left\langle\operatorname{Re} h * \operatorname{Re} h, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle\operatorname{Re} h * \operatorname{Re} \tilde{h}, L_{q}\right\rangle+O\left(\|h\|_{q^{\prime}}^{3}\right) .
\end{aligned}
$$

If $q$ belongs to a compact subset of $(3, \infty)$, the constant implicit in the notation $O(\cdot)$ may be taken to be independent of $q$.

Proof. Using the Taylor expansion

$$
|1+t|^{q}=1+q \operatorname{Re} t+\frac{1}{2} q(q-1)(\operatorname{Re} t)^{2}+\frac{1}{2} q(\operatorname{Im} t)^{2}+O\left(|t|^{3}+|t|^{q}\right)
$$

valid for $q \in(3, \infty)$ and the fact that $\widehat{1_{\mathbb{B}}}$ is real-valued, we have

$$
\begin{aligned}
\left|\widehat{1_{\mathbb{B}}}+\widehat{h}\right|^{q} & =\left|\widehat{1_{\mathbb{B}}}\right|^{q}+q(\operatorname{Re} \hat{h}) \widehat{1_{\mathbb{B}}}\left|\widehat{1_{\mathbb{B}}}\right|^{q-2}+\frac{1}{2} q(q-1)(\operatorname{Re} \widehat{h})^{2}\left|\widehat{\mathbb{1}_{\mathbb{B}}}\right|^{q-2} \\
& +\frac{1}{2} q(\operatorname{Im} \hat{h})^{2}\left|\widehat{\overline{1}_{\mathbb{B}}}\right|^{q-2}+O\left(|\hat{h}|^{3}\left|\widehat{1_{\mathbb{B}}}\right|^{q-3}\right)+O\left(|\widehat{h}|^{q}\right)
\end{aligned}
$$

Next we integrate over $\mathbb{R}^{d}$ to obtain

$$
\begin{aligned}
\|\widehat{f}\|_{q}^{q}= & \left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+q\left\langle K_{q}, \operatorname{Re} h\right\rangle+\frac{1}{2} q(q-1)\left\langle(\operatorname{Re} \widehat{h})^{2}, \widehat{L_{q}}\right\rangle \\
& +\frac{1}{2} q\left\langle(\operatorname{Im} \widehat{h})^{2}, \widehat{L_{q}}\right\rangle+O\left(\|h\|_{q^{\prime}}^{3}+\|h\|_{q^{\prime}}^{q}\right) \\
= & \left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+q\left\langle K_{q}, \operatorname{Re} h\right\rangle+\frac{1}{8} q(q-1)\left\langle(\widehat{h}+\overline{\widehat{h}})^{2}, \widehat{L_{q}}\right\rangle \\
& -\frac{1}{8} q\left\langle(\widehat{h}-\overline{\widehat{h}})^{2}, \widehat{L_{q}}\right\rangle+O\left(\|h\|_{q^{\prime}}^{3}\right)
\end{aligned}
$$

Using Plancherel's theorem, recalling that the $\simeq$ notation denotes the reflected function, using the relation $\tilde{L_{q}}=L_{q}$, and exploiting the equality $\left\langle f_{1} * f_{2}, f_{3}\right\rangle=\left\langle f_{1}, \tilde{f}_{2} * f_{4}\right\rangle$ for real-valued functions $f_{i}$, we further compute

$$
\begin{aligned}
\|\widehat{f}\|_{q}^{q}= & \left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+q\left\langle K_{q}, \operatorname{Re} h\right\rangle+\frac{1}{8} q(q-1)\left\langle(h+\overline{\tilde{h}}) *(h+\overline{\tilde{h}}), L_{q}\right\rangle \\
& -\frac{1}{8} q\left\langle(h-\tilde{\tilde{h}}) *(h-\overline{\tilde{h}}), L_{q}\right\rangle+O\left(\|h\|_{q^{\prime}}^{3}\right) .
\end{aligned}
$$

By expanding $h$ and $\tilde{h}$ into real and imaginary parts and reorganizing, obtain

$$
\begin{aligned}
\|\widehat{f}\|_{q}^{q}= & \left\|\hat{1}_{\mathbb{B}}\right\|_{q}^{q}+q\left\langle K_{q}, \operatorname{Re} h\right\rangle-\frac{1}{4} q(q-2)\left\langle\operatorname{Im} h * \operatorname{Im} h, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle\operatorname{Im} h, \operatorname{Im} h * L_{q}\right\rangle \\
& +\frac{1}{4} q(q-2)\left\langle\operatorname{Re} h * \operatorname{Re} h, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle\operatorname{Re} h * \operatorname{Re} \tilde{h}, L_{q}\right\rangle+O\left(\|h\|_{q^{\prime}}^{3}\right) .
\end{aligned}
$$

### 6.4.1 The functions $K_{q}$ and $L_{q}$.

Let $K_{q}$ and $L_{q}$ be the functions defined in (6.15) and 6.16). We state the facts proved in in [16] about $K_{q}$ and $L_{q}$ that we will need for our analysis of the Taylor expansion. See $\S 3.3$ and $\S 3.4$ from [16] for detailed explanations.

As is well known, $\widehat{1_{\mathbb{B}}}$ is a radially symmetric real-valued real analytic function which satisfies

$$
\begin{equation*}
\left|\widehat{1_{\mathbb{B}}}(\xi)\right|+\left|\nabla \widehat{1_{\mathbb{B}}}(\xi)\right| \leqslant C_{d}(1+|\xi|)^{-(d+1) / 2} . \tag{6.17}
\end{equation*}
$$

The following are consequences of (6.17):
Lemma 44. Let $d \geqslant 1$ and $q \in(3, \infty)$. The functions $K_{q}, L_{q}$ are real-valued, radially symmetric, bounded and Hölder continuous of some positive order. Moreover, $K_{q}(x) \rightarrow 0$ as $|x| \rightarrow \infty$ and likewise for $L_{q}(x)$. The function $K_{q}$ is continuously differentiable, and $x \cdot \nabla K_{q}$ is likewise real-valued, radially symmetric, and Hölder continuous of some positive order. These conclusions hold uniformly for $q$ in any compact subset of $(3, \infty)$.

Lemma 45. For each $d \geqslant 1, K_{q}, L_{q}$, and $x \cdot \nabla_{x} K_{q}$ depend continuously on $q \in(3, \infty)$. This holds in the sense that for each compact subset $\Lambda \subset(3, \infty)$, the mappings $q \mapsto K_{q}$ and $q \mapsto L_{q}$ are continuous from $\Lambda$ to the space of continuous functions on $\mathbb{R}^{d}$ that tend to zero at infinity. Moreover, there exists $\rho>0$ such that this mapping from $\Lambda$ to the space of bounded Hölder continuous functions of order $\rho$ on any bounded subset of $\mathbb{R}^{d}$ is continuous. The two conclusions also hold for $q \mapsto x \cdot \nabla_{x} K_{q}$.

The following lemma is an immediate consequence of the boundedness of $L_{q}$.
Lemma 46. Let $d \geqslant 1$ and $q \in(3, \infty)$. Let $h_{1}, h_{2} \in L^{1}\left(\mathbb{R}^{d}\right)$. Then

$$
\left\langle h_{1} * L_{q}, h_{2}\right\rangle=O\left(\left\|h_{1}\right\|_{1}\left\|h_{2}\right\|_{1}\right) .
$$

Lemma 47. For each $d \geqslant 1$ and each even integer $m \geqslant 4$ there exists $\eta=\eta(d, m)>0$ such that whenever $|q-m|<\eta$, there exists $c>0$ such that whenever $|y| \leqslant 1 \leqslant|x| \leqslant 2$,

$$
\begin{equation*}
K_{q}(y) \geqslant K_{q}(x)+c| | x|-|y|| \tag{6.18}
\end{equation*}
$$

Also, $\inf _{x \in 2 \mathbb{B}} K_{q}(x)>0$ and

$$
\begin{equation*}
\min _{|x| \leqslant 1-\delta} K_{q}(x)>\max _{|x| \geqslant 1+\delta} K_{q}(x) \quad \text { for all } \delta>0 . \tag{6.19}
\end{equation*}
$$

Proof. From the proof of Lemma 3.10 in [16], if $m$ is an even integer greater than 3 and $|y|=1$, then the map $t \mapsto K_{m}(t y)$ has strictly negative derivative for all $t \in(0, m-1)$. The inequalities 6.18) and (6.19) are a direct result. Since $\inf _{x \in 2 \mathbb{B}} K_{q}(x)=\inf _{x \in 2 \mathbb{B}} 1_{\mathbb{B}} * \cdots * 1_{\mathbb{B}}(x)$ is obviously positive when $q$ is even, the same holds for near $q$ by the continuity of $q \mapsto K_{q}$.

Remark 14. We will use the fact that $\inf _{x \in 2 \mathbb{B}} K_{q}(x)$ is positive for $q$ near even integers extensively throughout the paper.

### 6.4.2 A more detailed Taylor expansion in terms of the support, frequency, and modulus.

In order to better understand the effects of specific variations of $1_{\mathbb{B}}$, we consider $f e^{i g} 1_{E}$ where $0 \leqslant f \leqslant 1, g$ is real valued, and $|E| \in \mathbb{R}^{+}$.

Lemma 48. Let $d \geqslant 1$ and $q \in(3, \infty)$. Let $E \subset \mathbb{R}^{d}$. Suppose that $|E \Delta \mathbb{B}|,\|g\|_{L^{2}(E)}$, and $\|f-1\|_{L^{1}(E)}$ are sufficiently small. Then

$$
\begin{aligned}
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q}= & \left\|\widehat{1_{E}}\right\|_{q}^{q}+q\left\langle K_{q}, f \cos g 1_{E \backslash \mathbb{B}}-1_{E \backslash \mathbb{B}}\right\rangle-\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+\left\|\widehat{f^{\prime} e^{i g^{\prime}} 1_{\mathbb{B}}}\right\|_{q}^{q} \\
& +O\left(\left(\|g\|_{2}^{2}+\|f-1\|_{L^{1}(E)}\right)|E \Delta \mathbb{B}|^{1 / 2}\right)+O\left(\|g\|_{L^{2}(E)}^{3}+\|f-1\|_{L^{1}(E)}^{2}+|E \Delta \mathbb{B}|^{3 / q^{\prime}}\right)
\end{aligned}
$$

where $g^{\prime}=g$ on $E \cap \mathbb{B}, g^{\prime}=0$ on $\mathbb{R}^{d} \backslash(E \cap \mathbb{B})$ and $f^{\prime}=f$ on $E \cap \mathbb{B}, f^{\prime}=1$ on $\mathbb{B} \backslash E, f^{\prime}=0$ on $\mathbb{R}^{d} \backslash \mathbb{B}$.

Proof. As in Lemma 43 , set $h=f e^{i g} 1_{E}-1_{\mathbb{B}}$. Using the expression for $\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q}$ from Lemma

43, we replace some of the terms with $h$ by $h-1_{E}+1_{E}$ and expand to obtain

$$
\begin{align*}
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q}= & \left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+q\left\langle K_{q}, f \cos g 1_{E}-1_{E}\right\rangle+q\left\langle K_{q}, 1_{E}-1_{\mathbb{B}}\right\rangle \\
& -\frac{1}{4} q(q-2)\left\langle f \sin g 1_{E} * f \sin g 1_{E}, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle f \sin g 1_{E}, f \sin g 1_{E} * L_{q}\right\rangle \\
& +\frac{1}{4} q(q-2)\left\langle\left(1_{E}-1_{\mathbb{B}}\right) *\left(1_{E}-1_{\mathbb{B}}\right), L_{q}\right\rangle \\
& +\frac{1}{4} q^{2}\left\langle\left(1_{E}-1_{\mathbb{B}}\right) *\left(1_{\tilde{E}}-1_{\mathbb{B}}\right), L_{q}\right\rangle \\
& +O\left(\left\|f \cos g 1_{E}-1_{E}\right\|_{1}\left\|f \cos g 1_{E}-1_{\mathbb{B}}\right\|_{1}+\left\|f \cos g 1_{E}-1_{E}\right\|_{1}^{2}\right)+O\left(\|h\|_{q^{\prime}}^{3}\right) \\
= & \left\|\widehat{1_{E}}\right\|_{q}^{q}+q\left\langle K_{q}, f \cos g 1_{E}-1_{E}\right\rangle  \tag{6.20}\\
& -\frac{1}{4} q(q-2)\left\langle f \sin g 1_{E} * f \sin g 1_{E}, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle f \sin g 1_{E}, f \sin g 1_{E} * L_{q}\right\rangle \\
& +O\left(\left\|f \cos g 1_{E}-1_{E}\right\|_{1}\left\|f \cos g 1_{E}-1_{\mathbb{B}}\right\|_{1}\right) \\
& +O\left(\left\|f e^{i g} 1_{E}-1_{E}\right\|_{q^{\prime}}^{3}+\left\|1_{E}-1_{\mathbb{B}}\right\|_{q^{\prime}}^{3}\right) .
\end{align*}
$$

where we used that Lemma 43 gives

$$
\begin{aligned}
\left\|\widehat{1_{E}}\right\|_{q}^{q} & =\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+q\left\langle K_{q}, 1_{E}-1_{\mathbb{B}}\right\rangle+\frac{1}{4} q(q-2)\left\langle\left(1_{E}-1_{\mathbb{B}}\right) *\left(1_{E}-1_{\mathbb{B}}\right), L_{q}\right\rangle \\
& +\frac{1}{4} q^{2}\left\langle\left(1_{E}-1_{\mathbb{B}}\right),\left(1_{E}-1_{\mathbb{B}}\right) * L_{q}\right\rangle+O\left(|E \Delta \mathbb{B}|^{3 / q^{\prime}}\right) .
\end{aligned}
$$

Lemma $\sqrt[43]{ }$ also gives the following Taylor expansion for $\| \widehat{f^{\prime} e^{i g^{\prime}} 1_{E} \|_{q}^{q}}$.

$$
\begin{aligned}
\left\|\widehat{f^{\prime} e^{i g^{\prime}}} 1_{\mathbb{B}}\right\|_{q}^{q} & =\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+q\left\langle K_{q},\left(f^{\prime} \cos g^{\prime}-1\right) 1_{\mathbb{B}}\right\rangle \\
& -\frac{1}{4} q(q-2)\left\langle f^{\prime} \sin g^{\prime} 1_{\mathbb{B}} * f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}, f^{\prime} \sin g^{\prime} 1_{\mathbb{B}} * L_{q}\right\rangle \\
& +\frac{1}{4} q(q-2)\left\langle\left(f^{\prime} \cos g^{\prime}-1\right) 1_{\mathbb{B}} *\left(f^{\prime} \cos g^{\prime}-1\right) 1_{\mathbb{B}}, L_{q}\right\rangle \\
& +\frac{1}{4} q^{2}\left\langle\left(f^{\prime} \cos g^{\prime}-1\right) 1_{\mathbb{B}},\left(f^{\prime} \cos g^{\prime}-1\right) 1_{\mathbb{B}} * L_{q}\right\rangle+O\left(\left\|f^{\prime} e^{i g^{\prime}}-1\right\|_{L^{q^{\prime}(\mathbb{B})}}^{3}\right) \\
& =\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+q\left\langle K_{q},\left(f^{\prime} \cos g^{\prime}-1\right) 1_{\mathbb{B}}\right\rangle \\
& -\frac{1}{4} q(q-2)\left\langle f^{\prime} \sin g^{\prime} 1_{\mathbb{B}} * f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}, f^{\prime} \sin g^{\prime} 1_{\mathbb{B}} * L_{q}\right\rangle \\
& +O\left(\left\|f^{\prime} \cos g^{\prime}-1\right\|_{L^{1}(\mathbb{B})}^{2}\right)+O\left(\left\|f^{\prime} e^{i g^{\prime}}-1\right\|_{L^{\prime}(\mathbb{B})}^{3}\right) .
\end{aligned}
$$

Using that $\left\|\left(f^{\prime} e^{i g^{\prime}}-1\right) 1_{\mathbb{B}}\right\|_{q^{\prime}}=\left\|\left(f e^{i g}-1\right) 1_{E \cap \mathbb{B}}\right\|_{q^{\prime}} \leqslant\left\|\left(f e^{i g}-1\right) 1_{E}\right\|_{q^{\prime}}$ and that $\| f^{\prime} \cos g^{\prime}-$ $1\left\|_{L^{1}(\mathbb{B})}^{2}=\right\| f \cos g-1\left\|_{L^{1}(E \cap \mathbb{B})}^{2} \leqslant\right\| f \cos g 1_{E}-1_{E}\left\|_{1}\right\| f \cos g 1_{E}-1_{\mathbb{B}} \|_{1}$, we will extract the
expression for $\| \widehat{f^{\prime} e^{i g^{\prime}} 1_{E} \|_{q}^{q} \text { above from some of the terms from } 6 \text {.20). First write some terms }}$ from (6.20) in terms of $g^{\prime}$ :

$$
\begin{aligned}
& \left\langle K_{q}, f \cos g 1_{E}-1_{E}\right\rangle=\left\langle K_{q}, f \cos g 1_{E}-1_{E}-f^{\prime} \cos g^{\prime} 1_{\mathbb{B}}+1_{\mathbb{B}}\right\rangle+\left\langle K_{q}, f^{\prime} \cos g^{\prime} 1_{\mathbb{B}}-1_{\mathbb{B}}\right\rangle \\
& \quad=\left\langle K_{q}, f \cos g 1_{E \backslash \mathbb{B}}-1_{E \backslash \mathbb{B}}\right\rangle+\left\langle K_{q}, f^{\prime} \cos g^{\prime} 1_{\mathbb{B}}-1_{\mathbb{B}}\right\rangle, \\
& \left\langle f \sin g 1_{E} * f \sin g 1_{E}, L_{q}\right\rangle \\
& =\left\langle\left(f \sin g 1_{E}-f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}+f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}\right) *\left(f \sin g 1_{E}-f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}+f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}\right), L_{q}\right\rangle \\
& \quad=\left\langle f^{\prime} \sin g^{\prime} 1_{\mathbb{B}} * f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}, L_{q}\right\rangle+O\left(\left\|f \sin g 1_{E}-f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}\right\|_{1}\left\|f \sin g 1_{E}\right\|_{1}\right) \\
& \quad+O\left(\left\|f \sin g 1_{E}-f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}\right\|_{1}^{2}\right) \\
& =\left\langle f^{\prime} \sin g^{\prime} 1_{\mathbb{B}} * f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}, L_{q}\right\rangle+O\left(\|g\|_{2}^{2}|E \Delta \mathbb{B}|^{1 / 2}\right) \\
& \left\langle f \sin g 1_{E}, f \sin g 1_{E} * L_{q}\right\rangle \\
& =\left\langle\left(f \sin g 1_{E}-f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}+f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}\right),\left(f \sin g 1_{E}-f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}+f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}\right) * L_{q}\right\rangle \\
& \quad=\left\langle f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}, f^{\prime} \sin g^{\prime} 1_{\mathbb{B}} * L_{q}\right\rangle+O\left(\left\|f \sin g 1_{E}-f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}\right\|_{1}\left\|f \sin g 1_{E}\right\|_{1}\right. \\
& \quad+O\left(\left\|f \sin g 1_{E}-f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}\right\|_{1}^{2}\right) \\
& \quad=\left\langle f^{\prime} \sin g^{\prime} 1_{\mathbb{B}}, f^{\prime} \sin g^{\prime} 1_{\mathbb{B}} * L_{q}\right\rangle+O\left(\|g\|_{2}^{2}|E \Delta \mathbb{B}|^{1 / 2}\right)
\end{aligned}
$$



$$
\begin{aligned}
q\left\langle K_{q}, f\right. & \left.\cos g 1_{E}-1_{E}\right\rangle-\frac{1}{4} q(q-2)\left\langle f \sin g 1_{E} * f \sin g 1_{E}, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle f \sin g 1_{E}, f \sin g 1_{E} * L_{q}\right\rangle \\
= & q\left\langle K_{q}, f \cos g 1_{E \backslash \mathbb{B}}-1_{E \backslash \mathbb{B}}\right\rangle-\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+\left\|\widehat{f^{\prime} e^{i g^{\prime}} 1_{\mathbb{B}}}\right\|_{q}^{q} \\
& +O\left(\|g\|_{2}^{2}|E \Delta \mathbb{B}|^{1 / 2}\right)+O\left(\left\|f \cos g 1_{E}-1_{E}\right\|_{1}\left\|f \cos g 1_{E}-1_{\mathbb{B}}\right\|_{1}\right)+O\left(\left\|\left(f e^{i g}-1\right) 1_{E}\right\|_{q^{\prime}}^{3}\right) .
\end{aligned}
$$

Inputting this into 6.20) gives

$$
\begin{aligned}
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q}= & \left\|\widehat{1_{E}}\right\|_{q}^{q}+q\left\langle K_{q}, f \cos g 1_{E \backslash \mathbb{B}}-1_{E \backslash \mathbb{B}}\right\rangle-\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+\left\|\widehat{f^{\prime} e^{i g^{\prime}} 1_{\mathbb{B}}}\right\|_{q}^{q} \\
& +O\left(\|g\|_{2}^{2}|E \Delta \mathbb{B}|^{1 / 2}\right)+O\left(\left\|f \cos g 1_{E}-1_{E}\right\|_{1}\left\|f \cos g 1_{E}-1_{\mathbb{B}}\right\|_{1}\right) \\
& +O\left(\left\|f e^{i g} 1_{E}-1_{E}\right\|_{q^{\prime}}^{3}+\left\|1_{E}-1_{\mathbb{B}}\right\|_{q^{\prime}}^{3}\right)
\end{aligned}
$$

so it remains to understand the big-O terms. Note that

$$
\left\|f e^{i g} 1_{E}-1_{E}\right\|_{q^{\prime}} \leqslant\|f-1\|_{L^{1}(E)}^{1 / q^{\prime}}+\|g\|_{L^{2}(E)}(2|E|)^{\left(2-q^{\prime}\right) /\left(2 q^{\prime}\right)} .
$$

We also have that $\left\|f \cos g 1_{E}-1_{E}\right\|_{1} \leqslant\|g\|_{L^{2}(E)}^{2}+\|f-1\|_{L^{1}(E)}$ and

$$
\left\|f \cos g 1_{E}-1_{\mathbb{B}}\right\|_{1} \leqslant\|g\|_{L^{2}(E)}^{2}+\|f-1\|_{L^{1}(E)}+|E \Delta \mathbb{B}|
$$

Thus, noting that $3 / q^{\prime}>2$, we can simplify the big-O terms to

$$
O\left(\left(\|g\|_{2}^{2}+\|f-1\|_{L^{1}(E)}\right)|E \Delta \mathbb{B}|^{1 / 2}\right)+O\left(\|g\|_{L^{2}(E)}^{3}+\|f-1\|_{L^{1}(E)}^{2}+|E \Delta \mathbb{B}|^{3 / q^{\prime}}\right)
$$

### 6.5 Mostly modulus variation

By mostly modulus variation, we mean $\|f-1\|_{L^{1}(E)}^{1 / 2} \geqslant \max \left(M N|E \Delta \mathbb{B}|, N\|g\|_{L^{2}(E)}\right)$. The parameters $M$ and $N$ are consistent with the other two cases described in $\$ 6.7$ and $\$ 6.8$. Let $K_{q}, L_{q}$ be the functions defined in (6.15) and (6.16) in the following discussion. We use our most basic Taylor expansion from Lemma 43 to understand this case.

Proposition 49. Let $d \geqslant 1$ and let $\bar{q} \geqslant 4$ be an even integer. There exist $M, N \in \mathbb{R}^{+}, \delta_{0}>0$, and $\rho>0$ all depending on $\bar{q}$ and $d$ such that the following holds. Let $q \in(3, \infty), E \subset \mathbb{R}^{d}$ be a Lebesgue measurable set with $|E| \leqslant|\mathbb{B}|, 0 \leqslant f \leqslant 1$, and $g$ be real valued. Suppose that $\|f-1\|_{L^{1}(\mathbb{B})} \leqslant \delta_{0},\|g\|_{L^{2}(E)} \leqslant \delta_{0},|E \Delta \mathbb{B}| \leqslant \delta_{0}$, and $|q-\bar{q}| \leqslant \rho$. If

$$
\|f-1\|_{L^{1}(E)}^{1 / 2} \geqslant \max \left(M N|E \Delta \mathbb{B}|, N\|g\|_{L^{2}(E)}\right)
$$

then

$$
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q} \leqslant\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-c_{q, d}\|f-1\|_{L^{1}(E)}
$$

for a constant $c_{q, d}>0$ depending only on the exponent $q$ and on the dimension.
Proof. We begin with the Taylor expansion for $\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q}$ from Lemma 43 .

$$
\begin{align*}
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q}= & \left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+q\left\langle K_{q}, f \cos g 1_{E}-1_{\mathbb{B}}\right\rangle \\
& -\frac{1}{4} q(q-2)\left\langle f \sin g 1_{E} * f \sin g 1_{E}, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle f \sin g 1_{E}, f \sin g 1_{E} * L_{q}\right\rangle \\
& +\frac{1}{4} q(q-2)\left\langle\left(f \cos g 1_{E}-1_{\mathbb{B}}\right) *\left(f \cos g 1_{E}-1_{\mathbb{B}}\right), L_{q}\right\rangle \\
& +\frac{1}{4} q^{2}\left\langle\left(f \cos g 1_{E}-1_{\mathbb{B}}\right) *\left(\tilde{f} \cos \tilde{g} 1_{\tilde{E}}-1_{\mathbb{B}}\right), L_{q}\right\rangle \\
& +O\left(\left\|f e^{i g} 1_{E}-1_{\mathbb{B}}\right\|_{q^{\prime}}^{3}\right) \\
= & \left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+q\left\langle K_{q}, f \cos g 1_{E}-1_{\mathbb{B}}\right\rangle+O\left(\|\sin g\|_{L^{1}(E)}^{2}+\left\|f \cos g 1_{E}-1_{\mathbb{B}}\right\|_{1}^{2}+\left\|f e^{i g} 1_{E}-1_{\mathbb{B}}\right\|_{q^{\prime}}^{3}\right) . \tag{6.21}
\end{align*}
$$

Analyze the inner product term in (6.21). Let $G_{+}=\{x \in E: \cos g(x) \geqslant 0\}, G_{-}=\{x \in$ $E: \cos g(x)<0\}, K_{+}=\left\{x \in \mathbb{R}^{d}: K_{q}(x) \geqslant 0\right\}$, and $K_{-}=\left\{x \in \mathbb{R}^{d}: K_{q}(x)<0\right\}$. The term $\left\langle K_{q}, f \cos g 1_{E}-1_{\mathbb{B}}\right\rangle$ may be handled as follows:

$$
\begin{aligned}
\left\langle K_{q}, f \cos g 1_{E}-1_{\mathbb{B}}\right\rangle & =\left\langle K_{q}, f \cos g\left(1_{E \cap G_{+} \cap K_{+}}+1_{E \cap G_{+} \cap K_{-}}+1_{E \cap G_{-} \cap K_{+}}+1_{E \cap G_{-} \cap K_{-}}\right)-1_{\mathbb{B}}\right\rangle \\
& \leqslant\left\langle K_{q}, f \cos g 1_{E \cap G_{+} \cap K_{+}}-1_{\mathbb{B}}\right\rangle+\left\langle K_{q}, f \cos g 1_{E \cap G_{-} \cap K_{-}}\right\rangle \\
& \leqslant\left\langle K_{q}, f 1_{E \cap G_{+} \cap K_{+}}-1_{\mathbb{B}}\right\rangle+C\left|G_{-}\right|
\end{aligned}
$$

where $C=C(q, d)>0$ is a constant. Note that on $G_{-}$, we must have $|g| \geqslant \pi / 2$, so $\left|G_{-}\right| \leqslant \frac{4}{\pi^{2}}\|g\|_{L^{2}(E)}^{2} \leqslant \frac{4}{\pi^{2} N^{2}}\|f-1\|_{L^{1}(E)}$. Now consider $\left\langle K_{q}, f 1_{E \cap G_{+} \cap K_{+}}-1_{\mathbb{B}}\right\rangle$ :

$$
\begin{aligned}
& \left\langle K_{q}, f 1_{\left(E \cap G_{+} \cap K_{+}\right) \cap \mathbb{B}}-1_{\mathbb{B} \cap E}\right\rangle+\left\langle K_{q}, f 1_{\left(E \cap G_{+} \cap K_{+}\right) \backslash \mathbb{B}}-1_{\mathbb{B} \backslash E}\right\rangle \\
& \quad \leqslant\left\langle K_{q}, f 1_{E \cap \mathbb{B}}-1_{\mathbb{B} \cap E}\right\rangle+\left\langle K_{q}, f 1_{\left(E \cap K_{+}\right) \backslash \mathbb{B}}-1_{\mathbb{B} \backslash E}\right\rangle \\
& \leqslant-\inf _{\mathbb{B}} K_{q} \cdot\|f-1\|_{L^{1}(E \cap \mathbb{B})}+\sup _{|x|>1} K_{q}(x) \cdot\left\|f 1_{E \backslash \mathbb{B}}\right\|_{1}-\inf _{\mathbb{B}} K_{q} \cdot|\mathbb{B} \backslash E| .
\end{aligned}
$$

By Lemma 47, we know that $\inf _{\mathbb{B}} K_{q}=\sup _{|x|>1} K_{q}(x)=\left.K_{q}\right|_{\partial \mathbb{B}}$. We assumed that $|E| \leqslant|\mathbb{B}|$ (so $-|\mathbb{B} \backslash E| \leqslant-|E \backslash \mathbb{B}|$ ) in the hypotheses. Using these two observations, we further simplify and bound the above.

$$
\begin{aligned}
\sup _{|x|>1} K_{q}(x) \cdot\left\|f 1_{E \backslash \mathbb{B}}\right\|_{1}-\inf _{\mathbb{B}} K_{q} \cdot|\mathbb{B} \backslash E| & \leqslant-\left.K_{q}\right|_{\partial \mathbb{B}}\left(|E \backslash \mathbb{B}|-\left\|f 1_{E \backslash \mathbb{B}}\right\|_{1}\right) \\
& =-\left.K_{q}\right|_{\partial \mathbb{B}}\|f-1\|_{L^{1}(E \backslash \mathbb{B})} .
\end{aligned}
$$

In summary, we have shown that

$$
\left\langle K_{q}, f \cos g 1_{E}-1_{\mathbb{B}}\right\rangle \leqslant-\inf _{\mathbb{B}} K_{q} \cdot\|f-1\|_{L^{1}(E)}+C \frac{1}{N^{2}}\|f-1\|_{L^{1}(E)}
$$

Now analyze the error term in 6.21). The error $O\left(\left\|f \cos g 1_{E}-1_{\mathbb{B}}\right\|_{1}^{2}\right)$ can be replaced by $O\left(\|f-1\|_{L^{1}(E)}^{2}+|E \Delta \mathbb{B}|^{2}\right)$ because

$$
\begin{aligned}
\left\|f \cos g 1_{E}-1_{\mathbb{B}}\right\|_{1} & \leqslant\left\|f \cos g 1_{E}-\cos g 1_{E}\right\|_{1}+\left\|\cos g 1_{E}-1_{E}\right\|_{1}+|E \Delta \mathbb{B}| \\
& \leqslant\|f-1\|_{L^{1}(E)}+\|g\|_{L^{2}(E)}^{2}+|E \Delta \mathbb{B}|
\end{aligned}
$$

and similarly, $O\left(\left\|f e^{i g} 1_{E}-1_{\mathbb{B}}\right\|_{q^{\prime}}^{3}\right)$ can be replaced by $O\left(\|f-1\|_{L^{1}(E)}^{3 /\left(2 q^{\prime}\right)}\right)$ since

$$
\begin{aligned}
\left\|f e^{i g} 1_{E}-1_{\mathbb{B}}\right\|_{q^{\prime}} & \leqslant\left\|f e^{i g} 1_{E}-e^{i g} 1_{E}\right\|_{q^{\prime}}+\left\|e^{i g} 1_{E}-1_{E}\right\|_{q^{\prime}}+\left\|1_{E}-1_{\mathbb{B}}\right\|_{q^{\prime}} \\
& \leqslant\|f-1\|_{L^{1}(E)}+\|g\|_{L^{2}(E)}|E|^{\left(q^{\prime}-2\right) /\left(q^{\prime}\right)^{2}}+|E \Delta \mathbb{B}|^{1 / q^{\prime}}
\end{aligned}
$$

Also note that since $|\sin x| \leqslant|x|,\|\sin g\|_{1}^{2} \leqslant\|g\|_{1}^{2} \leqslant|\mathbb{B}|\|g\|_{L^{2}(E)}^{2} \leqslant \frac{|\mathbb{B}|}{N^{2}}\|f-1\|_{L^{1}(E)}$.
Returning to (6.21), we now have the following bound.

$$
\begin{aligned}
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q}= & \left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+q\left\langle K_{q}, f \cos g 1_{E}-1_{\mathbb{B}}\right\rangle+O\left(\|\sin g\|_{L^{1}(E)}^{2}+\left\|f \cos g 1_{E}-1_{\mathbb{B}}\right\|_{1}^{2}+\left\|f e^{i g} 1_{E}-1_{\mathbb{B}}\right\|_{q^{\prime}}^{3}\right) \\
\leqslant & \left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-q \inf _{\mathbb{B}} K_{q} \cdot\|f-1\|_{L^{1}(E)}+C_{1}\left(\frac{1}{N^{2}}+\frac{1}{(M N)^{2}}\right)\|f-1\|_{L^{1}(E)} \\
& +O\left(\|f-1\|_{L^{1}(E)}^{3 /\left(2 q^{\prime}\right)}\right)
\end{aligned}
$$

Thus for $M, N$ large enough and $\delta_{0}$ small enough, we have the desired conclusion.

The exponent 1 and the $L^{1}(E)$ norm of $\|f-1\|_{L^{1}(E)}$ in the previous proposition are optimal in the following sense.
Lemma 50. Let $d \geqslant 1, p \geqslant 1, N>0$, and $\bar{q} \geqslant 4$ be an even integer. There exists $\rho=\rho(\bar{q}, d)>0$ such that the following holds. If for some $q>3$ satisfying $|q-\bar{q}|<\rho$, there exists $c_{q, d}>0$ such that for any function $f: \mathbb{R}^{d} \rightarrow[0,1]$,

$$
\begin{equation*}
\left\|\widehat{f 1_{\mathbb{B}}}\right\|_{q}^{q} \leqslant\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-c_{\bar{q}, d}\|f-1\|_{L^{p}(\mathbb{B})}^{N} \tag{6.22}
\end{equation*}
$$

then $N \geqslant p$.
Note that for a function $f$ satisfying $|f-1| \leqslant 1$ and a subset $E$ with $|E|=1$, if $N \geqslant p$,

$$
\|f-1\|_{L^{p}(E)}^{N} \leqslant\|f-1\|_{L^{N}(E)}^{N} \leqslant\|f-1\|_{L^{1}(E)}
$$

so Theorem 3 is stronger than if $\|f-1\|_{L^{1}(E)}$ were replaced by $\|f-1\|_{L^{p}(E)}^{N}$.
Proof. For each $n \in \mathbb{N}$, let $f_{n}$ be the indicator function of the ball

$$
B_{n}:=\left\{x \in \mathbb{R}^{d}:|x|<1-1 / n\right\} .
$$

Use Lemma 43 with $h_{n}=f_{n} 1_{\mathbb{B}}-1_{\mathbb{B}}$ :

$$
\begin{aligned}
\left\|\widehat{f_{n} 1_{\mathbb{B}}}\right\|_{q}^{q}= & \left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+q\left\langle K_{q},\left(f_{n}-1\right) 1_{\mathbb{B}}\right\rangle+\frac{1}{4} q(q-2)\left\langle h_{n} * h_{n}, L_{q}\right\rangle \\
& +\frac{1}{4} q^{2}\left\langle h_{n} * \tilde{h_{n}}, L_{q}\right\rangle+O\left(\left\|h_{n}\right\|_{q^{\prime}}^{3}+\left\|h_{n}\right\|_{q^{\prime}}^{q}\right)
\end{aligned}
$$

where $q^{\prime}$ is the conjugate exponent of $q$. This combined with the hypothesis (6.22) leads to

$$
\begin{gathered}
\tilde{c}_{\bar{q}, d}\left(1-(1-1 / n)^{d}\right)^{N / p}+O\left(\left(1-(1-1 / n)^{d}\right)^{3 / q^{\prime}}\right)=c_{\bar{q}, d}\left\|f_{n}-1\right\|_{L^{p}(\mathbb{B})}^{N}+O\left(\left\|h_{n}\right\|_{q^{\prime}}^{3}\right) \\
\leqslant q\left\langle K_{q},\left(1-f_{n}\right) 1_{\mathbb{B}}\right\rangle \leqslant C\left(1-(1-1 / n)^{d}\right) .
\end{gathered}
$$

Let $n$ tend to infinity to conclude that $N / p \geqslant 1$.

### 6.6 The special case $E=\mathbb{B}$ for $q$ near an even integer.

Let $K_{q}, L_{q}$ be the functions defined in (6.15) and 6.16). In order to treat the remaining cases of mostly support variation (in $\$ 6.7$ ) and mostly frequency variation (in $\$ 6.8$ ) of our near-extremizer $f e^{i g} 1_{E}$ with $0 \leqslant f \leqslant 1, g$ real-valued, begin with Lemma 48, Let $g^{\prime}=g$ on $E \cap \mathbb{B}$ and $g^{\prime}=0$ on $B \backslash E$ and let $f^{\prime}=f$ on $E \cap \mathbb{B}$ and $f^{\prime}=1$ on $B \backslash E$. Recall the statement from Lemma 48:

$$
\begin{aligned}
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q}= & \left\|\widehat{1_{E}}\right\|_{q}^{q}+q\left\langle K_{q}, f \cos g 1_{E \backslash \mathbb{B}}-1_{E \backslash \mathbb{B}}\right\rangle-\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+\left\|\widehat{f^{\prime} e^{i g^{\prime}} 1_{\mathbb{B}}}\right\|_{q}^{q} \\
& +O\left(\left(\|g\|_{2}^{2}+\|f-1\|_{L^{1}(E)}\right)|E \Delta \mathbb{B}|^{1 / 2}\right)+O\left(\|g\|_{L^{2}(E)}^{3}+\|f-1\|_{L^{1}(E)}^{2}+|E \Delta \mathbb{B}|^{3 / q^{\prime}}\right) .
\end{aligned}
$$

In this section, we work to understand the term $\left\|\widehat{f^{\prime} e^{i g^{\prime}} 1_{\mathbb{B}}}\right\|_{q}^{q}$ above.

### 6.6.1 A new Taylor expansion for $\left\|\widehat{f e^{i g} 1_{\mathbb{B}}}\right\|_{q}^{q}$ when $\|f-1\|_{1},\|g\|_{2} \ll 1$.

The structural information we have obtained from Lemma 39 and Proposition 41 guarantees that if $f e^{i g} 1_{\mathbb{B}}$ is a near-extremizer, then, after possibly replacing $g$ by an affine translation of $g, f e^{i g} 1_{\mathbb{B}}$ is reasonably close to $1_{\mathbb{B}}$. Thus a Taylor expansion of $\left\|\widehat{f e^{i g} 1_{\mathbb{B}}}\right\|_{q}^{q}$ about $\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}$ will have an error that we can control. Furthermore, since we know that $\left|e^{i g}-1\right|$ is small on the majority of $\mathbb{B}$, we can expand the $\sin (g)$ and $\cos (g)$ that appear in Lemma 43 using Taylor series approximations. We split up the set $\mathbb{B}$ into a subset where the frequency $|g|$ is small and the remainder set where the frequency is not small in order to use polynomials to approximate the trigonometric terms in the next lemma. Define

$$
\mathbb{B}_{g}^{\epsilon}:=\{x \in \mathbb{B}:|g(x)|>\epsilon\}
$$

for $0<\epsilon<\pi / 2$ and $A_{g}:=\{x \in \mathbb{B}: \cos g(x) \geqslant 0\}$. Note that in the following lemma, we do not require a specific equivalence representative of $g(x) \in \mathbb{R} /(2 \pi)$ for $x \in \mathbb{B}$.

Lemma 51. Let $d \geqslant 1$ and let $q \geqslant 4$ be an even integer. Let $f, g$ be real valued functions on $\mathbb{B}$ with $0 \leqslant f \leqslant 1$. There exists $\delta_{0}>0$, depending on $q$ and on $d$, such that if $\|f-1\|_{L^{1}(\mathbb{B})} \leqslant \delta_{0}$ and $\|g\|_{L^{2}(\mathbb{B})} \leqslant \delta_{0}$, then

$$
\begin{aligned}
\left\|\widehat{f e^{i g} 1_{\mathbb{B}}}\right\|_{q}^{q} & \leqslant\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-q \inf _{\mathbb{B}} K_{q} \cdot\left(\|\cos g-1\|_{L^{1}\left(A_{g} \cap \mathbb{B}_{g}^{\epsilon}\right)}+\left|\mathbb{B}_{g}^{\epsilon} \backslash A_{g}\right|\right) \\
& -\frac{q}{2}\left\langle K_{q}, g^{2} 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}\right\rangle-\frac{1}{4} q(q-2)\left\langle g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}} * g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}, g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}} * L_{q}\right\rangle \\
& +\epsilon^{2} O\left(\|g\|_{L^{2}\left(\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}\right)}^{2}+O\left(\left|\mathbb{B}_{g}^{\epsilon}\right|^{1 / 2}\|g\|_{L^{2}(\mathbb{B})}^{2}\right)+O\left(\|f-1\|_{L^{1}(\mathbb{B})}\|g\|_{L^{2}(\mathbb{B})}\right)\right. \\
& +O\left(\|f-1\|_{L^{1}(\mathbb{B})}^{2}+\|g\|_{L^{2}(\mathbb{B})}^{3}\right)
\end{aligned}
$$

where $\mathbb{B}_{g}^{\epsilon}=\{x \in \mathbb{B}:|g(x)|>\epsilon\}$ for any $0<\epsilon<\pi / 2$ and $A_{g}=\{x \in \mathbb{B}: \cos g(x) \geqslant 0\}$.
Proof. We use the Taylor expansion from in Lemma 43.

$$
\begin{align*}
\left\|\widehat{f e^{i g} 1_{\mathbb{B}}}\right\|_{q}^{q} & =\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+q\left\langle K_{q}, f \cos g 1_{\mathbb{B}}-1_{\mathbb{B}}\right\rangle-\frac{1}{4} q(q-2)\left\langle f \sin g 1_{\mathbb{B}} * f \sin g 1_{\mathbb{B}}, L_{q}\right\rangle  \tag{6.23}\\
& +\frac{1}{4} q^{2}\left\langle f \sin g 1_{\mathbb{B}}, f \sin g 1_{\mathbb{B}} * L_{q}\right\rangle+\frac{1}{4} q(q-2)\left\langle\left(f \cos g 1_{\mathbb{B}}-1_{\mathbb{B}}\right) *\left(f \cos g 1_{\mathbb{B}}-1_{\mathbb{B}}\right), L_{q}\right\rangle \\
& +\frac{1}{4} q^{2}\left\langle\left(f \cos g 1_{\mathbb{B}}-1_{\mathbb{B}}\right) *\left(\tilde{f} \cos \tilde{g} 1_{\mathbb{B}}-1_{\mathbb{B}}\right), L_{q}\right\rangle+O\left(\left\|f e^{i g}-1\right\|_{L^{q^{\prime}(\mathbb{B})}}^{3}\right)
\end{align*}
$$

Bound the terms with two cosines by $O\left(\|f \cos g-1\|_{L^{1}(\mathbb{B})}^{2}\right)$. Since

$$
\begin{aligned}
\|f \cos g-1\|_{L^{1}(\mathbb{B})} & \leqslant\|f-1\|_{L^{1}(\mathbb{B})}+\|\cos g-1\|_{L^{1}(\mathbb{B})} \\
& \leqslant\|f-1\|_{L^{1}(\mathbb{B})}+\|g\|_{L^{2}(\mathbb{B})}^{2},
\end{aligned}
$$

we can replace $O\left(\|f \cos g-1\|_{L^{1}(\mathbb{B})}^{2}\right)$ by $O\left(\|f-1\|_{L^{1}(\mathbb{B})}^{2}+\|g\|_{L^{2}(\mathbb{B})}^{4}\right)$.

For the terms with sine, first we eliminate the $f$ factor.

$$
\begin{aligned}
\left\langle f \sin g 1_{\mathbb{B}}, f \sin g 1_{\mathbb{B}} * L_{q}\right\rangle & =\left\langle(f-1+1) \sin g 1_{\mathbb{B}},(f-1+1) \sin g 1_{\mathbb{B}} * L_{q}\right\rangle \\
& =\left\langle\sin g 1_{\mathbb{B}}, \sin g 1_{\mathbb{B}} * L_{q}\right\rangle+O\left(\|(f-1) \sin g\|_{L^{1}(\mathbb{B})}\|f \sin g\|_{L^{1}(\mathbb{B})}\right) \\
& \leqslant\left\langle\sin g 1_{\mathbb{B}}, \sin g 1_{\mathbb{B}} * L_{q}\right\rangle+O\left(\|f-1\|_{L^{1}(\mathbb{B})}\|g\|_{L^{2}(\mathbb{B})}\right) .
\end{aligned}
$$

Next, split the ball into $\mathbb{B}_{g}^{\epsilon}$ and $\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}$.

$$
\begin{aligned}
\left\langle\sin g 1_{\mathbb{B}}, \sin g 1_{\mathbb{B}} * L_{q}\right\rangle & =\left\langle\left(\sin g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}+\sin g 1_{\mathbb{B}_{g}^{\epsilon}}\right),\left(\sin g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}+\sin g 1_{\mathbb{B}_{g}^{\epsilon}}\right) * L_{q}\right\rangle \\
& \leqslant\left\langle\sin g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}, \sin g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}} * L_{q}\right\rangle+O\left(\left\|\sin g 1_{\mathbb{B}_{g}^{\epsilon}}\right\|_{1}\left\|\sin g 1_{\mathbb{B}}\right\|_{1}\right) \\
& \leqslant\left\langle\sin g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}, \sin g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}} * L_{q}\right\rangle+O\left(\left|\mathbb{B}_{g}^{\epsilon}\right|^{1 / 2}\|g\|_{L^{2}(\mathbb{B})}\right) .
\end{aligned}
$$

Together, we have

$$
\begin{aligned}
-\frac{1}{4} q(q-2)\langle f & \left.\sin g 1_{\mathbb{B}} * f \sin g 1_{\mathbb{B}}, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle f \sin g 1_{\mathbb{B}}, f \sin g 1_{\mathbb{B}^{*}} * L_{q}\right\rangle \\
\leqslant & -\frac{1}{4} q(q-2)\left\langle\sin g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}} * \sin g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle\sin g 1_{\mathbb{B} \backslash \mathbb{B}_{g}}, \sin g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}} * L_{q}\right\rangle \\
& O\left(\left|\mathbb{B}_{g}^{\epsilon}\right|^{1 / 2}\|g\|_{L^{2}(\mathbb{B})}^{2}\right)+O\left(\|f-1\|_{L^{1}(\mathbb{B})}\|g\|_{L^{2}(\mathbb{B})}\right)
\end{aligned}
$$

Finally, for the term with one cosine, recall $A_{g}=\{x \in \mathbb{B}: \cos g(x) \geqslant 0\}$. On the set $\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}=\{x \in \mathbb{B}:|g(x)| \leqslant \epsilon\}$ where $\epsilon<\pi / 2$, we also have $\cos g>0$, so $\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon} \subset A_{g}$. Calculate

$$
\begin{aligned}
\left\langle K_{q}, f \cos g 1_{\mathbb{B}}-1_{\mathbb{B}}\right\rangle \leqslant & \left\langle K_{q}, f \cos g 1_{A_{g}}-1_{\mathbb{B}}\right\rangle \\
\leqslant & \left\langle K_{q}, \cos g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}-1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}\right\rangle+\left\langle K_{q}, \cos g 1_{A_{g} \cap \mathbb{B}_{g}^{\epsilon}}-1_{A_{g} \cap \mathbb{B}_{g}^{\epsilon}}\right\rangle \\
& -\inf _{\mathbb{B}} K_{q} \cdot\left|\mathbb{B}_{g}^{\epsilon} \backslash A_{g}\right| \\
\leqslant & \left\langle K_{q}, \cos g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}-1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}\right\rangle-\inf _{\mathbb{B}} K_{q} \cdot\left(\|\cos g-1\|_{L^{1}\left(A_{g} \cap \mathbb{B}_{g}^{\epsilon}\right)}+\left|\mathbb{B}_{g}^{\epsilon} \backslash A_{g}\right|\right) .
\end{aligned}
$$

Using the above analysis in 6.23, we have

$$
\begin{align*}
\left\|\widehat{f e^{i g} 1_{\mathbb{B}}}\right\|_{q}^{q} \leqslant & \left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+q\left\langle K_{q}, \cos g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}-1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}\right\rangle-q \inf _{\mathbb{B}} K_{q} \cdot\left(\|\cos g-1\|_{L^{1}\left(A \cap \mathbb{B}_{g}^{\epsilon}\right)}+\left|\mathbb{B}_{g}^{\epsilon} \backslash A\right|\right)  \tag{6.24}\\
& -\frac{1}{4} q(q-2)\left\langle\sin g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}} * \sin g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle\sin g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}, \sin g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}} * L_{q}\right\rangle \\
& O\left(\left|\mathbb{B}_{g}^{\epsilon}\right|^{1 / 2}\|g\|_{L^{2}(\mathbb{B})}^{2}\right)+O\left(\|f-1\|_{L^{1}(\mathbb{B})}\|g\|_{L^{2}(\mathbb{B})}\right) \\
& +O\left(\|f-1\|_{L^{1}(\mathbb{B})}^{2}+\|g\|_{L^{2}(\mathbb{B})}^{4}\right)+O\left(\left\|f e^{i g}-1\right\|_{L^{q^{\prime}(\mathbb{B})}}^{3}\right) .
\end{align*}
$$

Approximate the remaining trigonometric functions by the following Taylor expansions for $t \in \mathbb{R}$ :

$$
\sin t=t+O\left(t^{3}\right) \quad \text { and } \quad \cos t=1-\frac{t^{2}}{2}+O\left(t^{4}\right)
$$

This combined with the definition of $\mathbb{B}_{g}^{\epsilon}$ gives

$$
\begin{aligned}
& q\left\langle K_{q}, \cos g 1_{B \backslash \mathbb{B}_{g}^{\epsilon}}-1_{B \backslash \mathbb{B}_{g}^{\epsilon}}\right\rangle-\frac{1}{4} q(q-2)\left\langle\sin g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}} * \sin g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}, L_{q}\right\rangle \\
&+\frac{1}{4} q^{2}\left\langle\sin g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}, \sin g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}} * L_{q}\right\rangle \\
&= q\left\langle K_{q},\left(1-g^{2} / 2+O\left(g^{4}\right)\right) 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}-1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}\right\rangle \\
&-\frac{1}{4} q(q-2)\left\langle\left(g+O\left(g^{3}\right)\right) 1_{\mathbb{B}_{\mathbb{B}} \backslash \mathbb{B}_{g}^{\epsilon}} *\left(g+O\left(g^{3}\right)\right) 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}, L_{q}\right\rangle \\
&+\frac{1}{4} q^{2}\left\langle\left(g+O\left(g^{3}\right)\right) 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}},\left(g+O\left(g^{3}\right)\right) 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}} * L_{q}\right\rangle \\
& \leqslant-\frac{q}{2}\left\langle K_{q}, g^{2} 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}\right\rangle+\epsilon^{2} O\left(\|g\|_{L^{2}\left(\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}\right)}^{2}\right)-\frac{1}{4} q(q-2)\left\langle g 1_{\mathbb{B}^{\prime} \backslash \mathbb{B}_{g}^{\epsilon}} * g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}, L_{q}\right\rangle \\
&+\frac{1}{4} q^{2}\left\langle g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}, g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}} * L_{q}\right\rangle+O\left(\|g\|_{L^{2}(\mathbb{B})}^{3}\right)
\end{aligned}
$$

Finally, we note that since $q^{\prime}<3 / 2$ and $\left\|f e^{i g}-1\right\|_{L^{q^{\prime}}(\mathbb{B})} \leqslant\|f-1\|_{L^{q^{\prime}}(\mathbb{B})}+\|g\|_{L^{2}(\mathbb{B})}(2|\mathbb{B}|)^{\left(2-q^{\prime}\right) / 2 q^{\prime}}$, the error terms may be combined to

$$
O\left(\|f-1\|_{L^{2}(\mathbb{B})}^{2}+\|g\|_{L^{2}(\mathbb{B})}^{3}+\|g\|_{L^{2}(\mathbb{B})}^{4}+\left\|f e^{i g}-1\right\|_{L^{q^{\prime}}(\mathbb{B})}^{3}\right) \leqslant O\left(\|f-1\|_{L^{2}(\mathbb{B})}^{2}+\|g\|_{L^{2}(\mathbb{B})}^{3}\right) .
$$

### 6.6.2 Connection with a spectral problem.

In the previous section, for $f e^{i g} 1_{\mathbb{B}}$ with $\|f-1\|_{L^{1}(\mathbb{B})}$ and $\|g\|_{L^{2}(\mathbb{B})}$ small, we expressed $\left\|\widehat{f e^{i g} 1_{\mathbb{B}}}\right\|_{q}^{q}$ as $\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}$ plus a quadratic form in $g 1_{\mathbb{B} \backslash \mathbb{B}_{g}}$ and a small error. In this section, we analyze a spectral problem concerning that quadratic form when $q \geqslant 4$ is an even integer in order to obtain a more descriptive upper bound for $\left\|\widehat{f e^{i g} 1_{\mathbb{B}}}\right\|_{q}^{q}$.

Definition 15. Define $T_{q}: L^{2}(\mathbb{B}) \rightarrow L^{2}(\mathbb{B})$ to be the linear operator which is the composition of multiplication by $K_{q}^{-1 / 2}$, followed by convolution with $L_{q}$, followed by multiplication by $K_{q}^{-1 / 2}$. That is, for $h \in L^{2}(\mathbb{B})$,

$$
\left.h \stackrel{T_{q}}{\longmapsto} K_{q}^{-1 / 2}\left(K_{q}^{-1 / 2} h 1_{\mathbb{B}} * L_{q}\right)\right|_{\mathbb{B}} .
$$

Observe that $T_{q}$ is bounded on $L^{2}(\mathbb{B})$ since $K_{q}$ is bounded above and below by positive quantities on $\mathbb{B}$, so

$$
\left\|K_{q}^{-1 / 2}\left(K^{-1 / 2} h * L_{q}\right)\right\|_{L^{2}(\mathbb{B})} \leqslant\left\|K_{q}^{-1 / 2}\right\|_{L^{\infty}(\mathbb{B})}^{2}\left\|L_{q}\right\|_{L^{1}(2 \mathbb{B})}\|h\|_{L^{2}(\mathbb{B})} .
$$

Using that $\widehat{1_{\mathbb{B}}}$ is a real-valued function that satisfies $\| \widehat{1_{\mathbb{B}}}(\xi) \mid \leqslant C_{d}(1+|\xi|)^{-(d+1) / 2}$ and that $\widehat{L_{q}}=\left|\widehat{1_{\mathbb{B}}}\right|^{q-2}$, we will show that $T_{q}$ is a compact operator. Since $K_{q}$ is bounded above and below by positive constants on $2 \mathbb{B}$, multiplication by $K_{q}^{-1 / 2}$ defines a bounded operator on $L^{2}(\mathbb{B})$ and therefore it suffices to show that convolution with $L_{q}$ is compact. But convolution with any continuous function, followed by restriction to the ball, defines a compact operator, so $T_{q}$ is compact.

Finally, since $K_{q}$ is real and $L_{q}$ is symmetric,

$$
\left\langle T_{q} h, f\right\rangle=\left\langle K_{q}^{-1 / 2} h * L_{q}, K_{q}^{-1 / 2} h\right\rangle=\left\langle K_{q}^{-1 / 2} h, L_{q} * K_{q}^{-1 / 2} h\right\rangle=\left\langle h, T_{q} f\right\rangle,
$$

so $T_{q}$ is self-adjoint. Let $Q_{q}$ be the quadratic form on $L^{2}(\mathbb{B})$ defined by

$$
Q_{q}(f, h)=\left\langle f, T_{q} h\right\rangle
$$

where $f, h \in L^{2}(\mathbb{B})$. As above, $\tilde{f}(x)=f(-x)$.
Definition 16. Let $d \geqslant 1$ and $q \geqslant 4$ be an even integer. Let $\mathcal{H}$ denote the subspace of $L^{2}(\mathbb{B})$ of functions of the form $K_{q}^{1 / 2}(x)(\alpha \cdot x+b)$ where $\alpha \in \mathbb{R}^{d}$ and $b \in \mathbb{R}$. Let $P_{\mathcal{H}}: L^{2}(\mathbb{B}) \rightarrow L^{2}(\mathbb{B})$ denote the orthogonal projection onto $\mathcal{H}$.

Lemma 52. Let $d \geqslant 1$. Let $q \geqslant 4$ be an even integer. Then there exists $c>0$ depending on the dimension and $q$ such that

$$
\begin{equation*}
-\frac{q}{2}\|h\|_{L^{2}(\mathbb{B})}^{2}-\frac{1}{4} q(q-2) Q_{q}(h, \tilde{h})+\frac{1}{4} q^{2} Q_{q}(h, h) \leqslant-c\left\|\left(I-P_{\mathcal{H}}\right) h\right\|_{L^{2}(\mathbb{B})}^{2} \tag{6.25}
\end{equation*}
$$

for every real-valued $h \in L^{2}(\mathbb{B})$.
Proof. Since $T_{q}$ is a compact, self-adjoint linear operator, we can write $L^{2}(\mathbb{B})$ as a direct sum of eigenspaces. For a fixed eigenvalue, we can further orthogonally decompose the corresponding eigenspace into even eigenfunctions and odd eigenfunctions since $K_{q}=\tilde{K}_{q}$ and if $T_{q} \varphi=\lambda \varphi, 1 / 2(\varphi+\tilde{\varphi})+1 / 2(\varphi-\tilde{\varphi})$ is a unique representation of $\varphi$ as a sum of an even eigenvector with eigenvalue $\lambda$ and an odd eigenvector with eigenvalue $\lambda$. Since $T_{q}$ can be regarded as an operator on real-valued functions in $L^{2}(\mathbb{B})$, we can assume that the eigenfunctions are real-valued. Thus we can expand $h$ as say $h=\sum_{n=0}^{\infty} h_{n}$, where the $h_{n}$ are pairwise orthogonal eigenfunctions of $T_{q}$, either even or odd, real-valued, and associated with eigenvalues $\lambda_{n}$. The spectrum is real and $\lambda_{n} \rightarrow 0$ as $n \rightarrow \infty$. Assume that $\left|\lambda_{n}\right|$ is nonincreasing and calculate

$$
\begin{align*}
-\frac{q}{2}\|h\|_{L^{2}(\mathbb{B})}^{2}- & \frac{1}{4} q(q-2) Q_{q}(h, \tilde{h})+\frac{1}{4} q^{2} Q_{q}(h, h) \\
= & \sum_{n \leqslant N}\left(-\frac{q}{2}\left\|h_{n}\right\|_{L^{2}(\mathbb{B})}^{2}-\frac{1}{4} q(q-2) \lambda_{n}\left\langle h_{n}, \tilde{h_{n}}\right\rangle+\frac{1}{4} q^{2} \lambda_{n}\left\|h_{n}\right\|_{2}\right) \\
& +\sum_{n>N}\left(-\frac{q}{2}\left\|h_{n}\right\|_{L^{2}(\mathbb{B})}^{2}-\frac{1}{4} q(q-2) \lambda_{n}\left\langle h_{n}, \tilde{h}_{n}\right\rangle+\frac{1}{4} q^{2} \lambda_{n}\left\|h_{n}\right\|_{2}^{2}\right) \\
\leqslant & \sum_{n \leqslant N}\left(-\frac{q}{2}\left\|h_{n}\right\|_{L^{2}(\mathbb{B})}^{2}-\frac{1}{4} q(q-2) \lambda_{n}\left\langle h_{n}, \tilde{h_{n}}\right\rangle+\frac{1}{4} q^{2} \lambda_{n}\left\|h_{n}\right\|_{2}^{2}\right)  \tag{6.26}\\
& +\left(-\frac{q}{2}+\frac{1}{2} q(q-1)\left|\lambda_{N}\right|\right) \sum_{n>N}\left\|h_{n}\right\|_{L^{2}(\mathbb{B})}^{2}
\end{align*}
$$

We return to this expression (6.26) after understanding the case for a single eigenfunction. Fix an eigenfunction $\varphi$ of $T_{q}$ with eigenvalue $\lambda$. We analyze

$$
\begin{equation*}
-\frac{q}{2}\|\varphi\|_{L^{2}(\mathbb{B})}^{2}-\frac{1}{4} q(q-2) Q_{q}(\varphi, \tilde{\varphi})+\frac{1}{4} q^{2} Q_{q}(\varphi, \varphi) . \tag{6.27}
\end{equation*}
$$

Note that since $K_{q}$ and $L_{q}$ are even functions, $T_{q} \tilde{\varphi}=\lambda \tilde{\varphi}$ as well. If $\lambda=0$, then 6.25 for $h=\varphi$ is trivial since

$$
\begin{aligned}
-\frac{q}{2}\|\varphi\|_{L^{2}(\mathbb{B})}^{2} & -\frac{1}{4} q(q-2) Q_{q}(\varphi, \tilde{\varphi})+\frac{1}{4} q^{2} Q_{q}(\varphi, \varphi)=-\frac{q}{2}\|\varphi\|_{L^{2}(\mathbb{B})}^{2} \\
& \leqslant-\frac{q}{2}\left\|\left(I-P_{\mathcal{H}}\right) \varphi\right\|_{L^{2}(\mathbb{B})}^{2} .
\end{aligned}
$$

Thus we can assume that $\lambda \neq 0$. In this case,

$$
\left|\lambda \left\|\varphi ( x ) \left|=\left|K_{q}^{-1 / 2}(x)\left(K^{-1 / 2} \varphi * L_{q}\right)(x)\right| \leqslant\left\|K_{q}^{-1 / 2}\right\|_{L^{\infty}(\mathbb{B})}^{2}\|\varphi\|_{L^{2}(\mathbb{B})}\left\|L_{q}\right\|_{L^{2}(\mathbb{B})},\right.\right.\right.
$$

so $\|\varphi\|_{L^{\infty}(\mathbb{B})}$ is finite. Following [15] and [10], 6.27) is analyzed for an eigenfunction $\varphi$ by considering a Taylor expansion of $\left\|\widehat{e^{t \psi \psi}}\right\|_{q}^{q}$ where $\psi=K_{q}^{-1 / 2} \varphi$ and $t \in \mathbb{R}$ is an auxiliary parameter. Choose $t>0$ sufficiently small so that $\cos t \psi \geqslant 0$ on $\mathbb{B}$ and the hypotheses of Lemma 51 are satisfied with $f=1$ and $g=t \psi$. Executing the proof of Lemma 51 without expanding the term $\left\langle K_{q},(f \cos (g)-1) 1_{\mathbb{B}}\right\rangle$ yields the following equality:

$$
\begin{aligned}
\left\|\widehat{e^{i t \psi} 1_{\mathbb{B}}}\right\|_{q}^{q} & =\|{\hat{1_{\mathbb{B}}} \|_{q}^{q}+q\left\langle K_{q},(\cos (t \psi)-1) 1_{\mathbb{B}}\right\rangle-\frac{1}{4} q(q-2)\left\langle t \psi 1_{\mathbb{B} \backslash \mathbb{B}_{t \psi}^{\epsilon}} * t \psi 1_{\mathbb{B} \backslash \mathbb{B}_{t \psi}^{\epsilon}}, L_{q}\right\rangle}+\frac{1}{4} q^{2}\left\langle t \psi 1_{\mathbb{B} \backslash \mathbb{B}_{t \psi}^{\epsilon}}, t \psi 1_{\mathbb{B} \backslash \mathbb{B}_{t \psi}^{\epsilon}} * L_{q}\right\rangle+O\left(\left|\mathbb{B}_{t \psi}^{\epsilon}\right|^{1 / 2} t^{2}\|\psi\|_{L^{2}(\mathbb{B})}^{2}+t^{3}\|\psi\|_{L^{2}(\mathbb{B})}^{3}\right) .
\end{aligned}
$$

where $\epsilon \in(0, \pi / 2)$ and $\mathbb{B}_{t \psi}^{\epsilon}=\left\{x \in \mathbb{B}: t\left|K^{-1 / 2}(x) \varphi(x)\right|>\epsilon\right\}$. Since $\|\varphi\|_{L^{\infty}(\mathbb{B})}<\infty$, for $t<\epsilon\left(\left\|K^{-1 / 2} \varphi\right\|_{L^{\infty}(\mathbb{B})}\right)^{-1}$, the set $\mathbb{B}_{t \psi}^{\epsilon}$ is empty. Thus the statement we have from the proof of Lemma 51 for $t<\epsilon\left(\left\|K^{-1 / 2} \varphi\right\|_{L^{\infty}(\mathbb{B})}\right)^{-1}$ is

$$
\begin{aligned}
\left\|\widehat{e^{i t \psi} 1_{\mathbb{B}}}\right\|_{q}^{q} & =\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-q\left\langle K_{q},(\cos (t \psi)-1) 1_{\mathbb{B}}\right\rangle-\frac{1}{4} q(q-2)\left\langle t \psi 1_{\mathbb{B}} * t \psi 1_{\mathbb{B}}, L_{q}\right\rangle \\
& +\frac{1}{4} q^{2}\left\langle t \psi 1_{\mathbb{B}}, t \psi 1_{\mathbb{B}} * L_{q}\right\rangle+O\left(t^{3}\|\psi\|_{L^{2}(\mathbb{B})}^{3}\right) .
\end{aligned}
$$

Now if we expand the cosine, we have the equality

$$
\begin{align*}
\left\|\widehat{e^{i t \psi} 1_{\mathbb{B}}}\right\|_{q}^{q} & =\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-\frac{q}{2}\left\langle K_{q},(t \psi)^{2} 1_{\mathbb{B}}\right\rangle+t^{4} O\left(\|\psi\|_{L^{\infty}(\mathbb{B})}^{2}\|\psi\|_{L^{2}(\mathbb{B})}^{2}\right)-\frac{1}{4} q(q-2)\left\langle t \psi 1_{\mathbb{B}} * t \psi 1_{\mathbb{B}}, L_{q}\right\rangle \\
& +\frac{1}{4} q^{2}\left\langle t \psi 1_{\mathbb{B}}, t \psi 1_{\mathbb{B}} * L_{q}\right\rangle+O\left(t^{3}\|\psi\|_{L^{2}(\mathbb{B})}^{3}\right) \\
& =\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-t^{2}\left[\frac{q}{2}\left\langle K_{q},(\psi)^{2} 1_{\mathbb{B}}\right\rangle-\frac{1}{4} q(q-2)\left\langle\psi 1_{\mathbb{B}} * \psi 1_{\mathbb{B}}, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle\psi 1_{\mathbb{B}}, \psi 1_{\mathbb{B}} * L_{q}\right\rangle\right]+O_{\varphi}\left(t^{3}\right) \\
& =\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-t^{2}\left[\frac{q}{2}\|\varphi\|_{L^{2}(\mathbb{B})}^{2}-\frac{1}{4} q(q-2) Q_{q}(\varphi, \tilde{\varphi})+\frac{1}{4} q^{2} Q_{q}(\varphi, \varphi)\right]+O_{\varphi}\left(t^{3}\right) \\
& =\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-t^{2}\left[\frac{q}{2}\|\varphi\|_{L^{2}(\mathbb{B})}^{2}-\frac{\lambda}{4} q(q-2)\langle\varphi, \tilde{\varphi}\rangle+\frac{\lambda}{4} q^{2}\|\varphi\|_{L^{2}(\mathbb{B})}^{2}\right]+O_{\varphi}\left(t^{3}\right) \tag{6.28}
\end{align*}
$$

where the final big- $O_{\varphi}$ depends on the dimension, the exponent $q$, and on the $L^{\infty}$ and $L^{2}$ norms of $\varphi$. Note that we used that $|\psi|$ is bounded above and below by a constant (depending on $q$ ) multiple of $\varphi$ on $\mathbb{B}$. Since $q$ is an even integer, $\left\|\widehat{e^{i t \psi} 1_{\mathbb{B}}}\right\|_{q}^{q} \leqslant\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}$ and so

$$
\begin{equation*}
\frac{q}{2}\|\varphi\|_{L^{2}(\mathbb{B})}^{2}-\frac{\lambda}{4} q(q-2)\langle\varphi, \tilde{\varphi}\rangle+\frac{\lambda}{4} q^{2}\|\varphi\|_{L^{2}(\mathbb{B})}^{2} \geqslant 0 \tag{6.29}
\end{equation*}
$$

from (6.28). Expressing $q=2 m$, we can also write

$$
\left\|e^{i t \psi} 1_{\mathbb{B}}\right\|_{q}^{q}=\operatorname{Re} \int_{\mathbb{B} q-1} e^{i t\left(\psi\left(x_{1}\right)+\cdots+\psi\left(x_{m}\right)-\psi\left(y_{2}\right)-\cdots-\psi\left(y_{m}\right)-\psi(L(x, y))\right)} 1_{\mathbb{B}}(L(x, y)) d x d y
$$

where $x=\left(x_{1}, \ldots, x_{m}\right), y=\left(y_{2}, \ldots, y_{m}\right)$, and $L(x, y)=x_{1}+\cdots+x_{m}-y_{2}-\cdots y_{m}$. Let $\alpha(x, y)=\psi\left(x_{3}\right)+\cdots+\psi\left(x_{m}\right)-\psi\left(y_{2}\right)-\cdots-\psi\left(y_{m}\right)-\psi(L(x, y))$. Then for all sufficiently small $t$, since $\cos (\theta)-1 \leqslant-\frac{\theta^{2}}{4}$ for $|\theta| \leqslant \theta / 2$,

$$
\begin{align*}
& \left\|\widehat{e^{i t \psi} 1_{\mathbb{B}}}\right\|_{q}^{q}=\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+\widehat{e^{i t \psi} 1_{\mathbb{B}}}\left\|_{q}^{q}-\right\| \widehat{\mathbb{1}_{\mathbb{B}}} \|_{q}^{q} \\
& =\left\|\widehat{\mathbb{1}_{\mathbb{B}}}\right\|_{q}^{q}-\int_{B^{q-1}}\left|\cos \left(t\left(\psi\left(x_{1}\right)+\psi\left(x_{2}\right)+\alpha(x, y)\right)\right)-1\right| 1_{\mathbb{B}}(L(x, y)) d x d y \\
& \leqslant\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-\frac{t^{2}}{4} \int_{B^{q-1}}\left(\psi\left(x_{1}\right)+\psi\left(x_{2}\right)+\alpha(x, y)\right)^{2} 1_{\mathbb{B}}(L(x, y)) d x d y \tag{6.30}
\end{align*}
$$

Combining (6.28) with 6.30 gives

$$
\begin{aligned}
\frac{t^{2}}{4} \int_{B^{q-1}}\left(\psi\left(x_{1}\right)+\psi\left(x_{2}\right)+\right. & \alpha(x, y))^{2} 1_{\mathbb{B}}(L(x, y)) d x d y \\
& \leqslant t^{2}\left[\frac{q}{2}\|\varphi\|_{L^{2}(\mathbb{B})}^{2}-\frac{\lambda}{4} q(q-2)\langle\varphi, \tilde{\varphi}\rangle+\frac{\lambda}{4} q^{2}\|\varphi\|_{L^{2}(\mathbb{B})}^{2}\right]+O_{\varphi}\left(t^{3}\right)
\end{aligned}
$$

for all sufficiently small $t>0$. If the coefficient of $t^{2}$ on the right hand side is 0 , then

$$
\int_{B^{q-1}}\left(\psi\left(x_{1}\right)+\psi\left(x_{2}\right)+\alpha(x, y)\right)^{2} 1_{\mathbb{B}}(L(x, y)) d x d y=0
$$

which means that

$$
\left\|\widehat{e^{i t \psi} 1_{\mathbb{B}}}\right\|_{q}^{q}=\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-\int_{B^{q-1}}\left|\cos \left(t\left(\psi\left(x_{1}\right)+\psi\left(x_{2}\right)+\alpha(x, y)\right)\right)-1\right| 1_{\mathbb{B}}(L(x, y)) d x d y=\left\|\hat{1}_{\mathbb{B}}\right\|_{q}^{q}
$$

By Lemma 42, $e^{i t \psi}=e^{i(\alpha \cdot x+b)}$ for some $\alpha \in \mathbb{R}^{d}$ and $b \in \mathbb{R}$. Thus the inequality 6.29) is strict unless $e^{i K^{-1 / 2} \varphi}$ takes the form $e^{i(\alpha \cdot x+b)}$. Using that $\lambda$ is nonzero, we have for each $x \in \mathbb{B}$ the expression

$$
\varphi(x)=\lambda^{-1} K_{q}^{-1 / 2}(x)\left(K_{q}^{-1 / 2} \varphi * L_{q}\right)(x)
$$

Since $K_{q}^{-1 / 2}$ is continuous and $L_{q} \in L^{2}\left(\mathbb{R}^{d}\right), \varphi$ is continuous on $\mathbb{B}$. Note that $e^{i K_{q}^{-1 / 2} \varphi}=$ $e^{i(\alpha \cdot x+b)}$ implies that $K_{q}^{-1 / 2} \varphi(x)=\alpha \cdot x+b+f(x)$ for some function $f: \mathbb{B} \rightarrow 2 \pi \mathbb{Z}$. The only continuous such function is constant, so $\varphi(x)=K_{q}^{1 / 2}\left(\alpha \cdot x+b^{\prime}\right)$ for $b^{\prime}=b+2 \pi n$ for some $n \in \mathbb{Z}$. Conclude that the inequality (6.29) is strict unless $\varphi \in \mathcal{H}$, where $\mathcal{H}$ was defined in Definition 16.

Finally, we use this in (6.26) and conclude that there exists $c>0$ so that

$$
\begin{aligned}
-\frac{q}{2}\|h\|_{L^{2}(\mathbb{B})}^{2}- & \frac{1}{4} q(q-2) Q_{q}(h, \tilde{h})+\frac{1}{4} q^{2} Q_{q}(h, h) \\
& \leqslant-c \sum_{\substack{n \leqslant N \\
h_{n} \notin \mathcal{H}}}\left\|h_{n}\right\|_{L^{2}(\mathbb{B})}^{2}-c \sum_{n>N}\left\|h_{n}\right\|_{L^{2}(\mathbb{B})}^{2} \\
& \leqslant-c\left\|\left(I-P_{\mathcal{H}}\right) h\right\|_{L^{2}(\mathbb{B})}^{2} .
\end{aligned}
$$

### 6.6.3 Conclusion of the spectral analysis for $q$ near an even integer and $E=\mathbb{B}$

Let $K_{q}, L_{q}$ be the functions defined in (6.15) and (6.16). Use the frequency Taylor expansion from Lemma 51 and compare the main terms with $K_{q}$ and $L_{q}$ to analogous terms with $K_{\bar{q}}$ and $L_{\bar{q}}$ where $\bar{q}$ is the closest even integer. Then make use of the spectral analysis in Lemma 52 to obtain the following theorem. We will use the following theorem in the proof of Proposition 54 in $\$ 6.7$ and the proof of Proposition 55 in $\$ 6.8$.

Theorem 53. Let $d \geqslant 1$ and let $\bar{q} \geqslant 4$ be an even integer. There exist $\delta_{0}, \rho,>0$ all depending on the dimension and $\bar{q}$ as well as $c_{q, d}>0$ so that the following holds. Let $q \in(3, \infty), E \subset \mathbb{R}^{d}$ be a Lebesgue measurable set with $|E| \leqslant|\mathbb{B}|, 0 \leqslant f \leqslant 1$, and $g$ be real valued. If $|q-\bar{q}|<\rho$, $\|f-1\|_{L^{1}(\mathbb{B})} \leqslant \delta_{0},\|g\|_{L^{2}(\mathbb{B})} \leqslant \delta_{0}$, and $|g| \leqslant \frac{5 \pi}{4}$, then

$$
\begin{aligned}
\left\|\widehat{f e^{i g} 1_{\mathbb{B}}}\right\|_{q}^{q} \leqslant & \left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-c_{q, d}\left\|\left(I-P_{\mathcal{H}}\right) K_{\bar{q}}^{1 / 2} g\right\|_{L^{2}(\mathbb{B})}^{2}+o_{q-\bar{q}}(1)\|g\|_{L^{2}(\mathbb{B})}^{2} \\
& +O\left(\|f-1\|_{L^{1}(\mathbb{B})}\|g\|_{L^{2}(\mathbb{B})}\right)+O\left(\|f-1\|_{L^{1}(\mathbb{B})}^{2}+\|g\|_{L^{2}(\mathbb{B})}^{5 / 2} .\right.
\end{aligned}
$$

Proof. The function $f e^{i g} 1_{\mathbb{B}}$ satisfies the hypotheses of Lemma 51. We have the expansion

$$
\begin{align*}
& \left\|\widehat{f e^{i g} 1_{\mathbb{B}}}\right\|_{q}^{q} \leqslant\left\|\widehat{\mathbb{1}_{\mathbb{B}}}\right\|_{q}^{q}-q \inf _{\mathbb{B}} K_{q} \cdot\left(\|\cos g-1\|_{L^{1}\left(A_{g} \cap \mathbb{B}_{g}^{\epsilon}\right)}+\left|\mathbb{B}_{g}^{\epsilon} \backslash A_{g}\right|\right)  \tag{6.31}\\
& -\frac{q}{2}\left\langle K_{q}, g^{2} 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\mathbb{E}}}\right\rangle-\frac{1}{4} q(q-2)\left\langle g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{e}} * g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{s}}, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{s}}, g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{s}} * L_{q}\right\rangle \\
& +\epsilon^{2} O\left(\|g\|_{L^{2}\left(\mathbb{B} \mathbb{B}_{g}^{(\mathbb{E}}\right)}^{2}\right)+O\left(\left\|\left.\mathbb{B}_{g}^{\epsilon}\right|^{1 / 2}\right\| g \|_{L^{2}(\mathbb{B})}^{2}\right)+O\left(\|f-1\|_{L^{1}(\mathbb{B})}\|g\|_{L^{2}(\mathbb{B})}\right) \\
& +O\left(\|f-1\|_{L^{1}(\mathbb{B})}^{2}+\|g\|_{L^{2}(\mathbb{B})}^{3}\right)
\end{align*}
$$

We analyze the three main terms in the expansion:

$$
\begin{align*}
& -\frac{q}{2}\left\langle K_{q}, g^{2} 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{s}}\right\rangle-\frac{1}{4} q(q-2)\left\langle g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{s}} * g 1_{\mathbb{B} \backslash \mathbb{B}_{g}}, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{s}}, g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{c}} * L_{q}\right\rangle \\
& =-\frac{q}{2}\left\langle K_{q}-K_{\bar{q}}+K_{\bar{q}}, g^{2} 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{c}}\right\rangle-\frac{1}{4} q(q-2)\left\langle g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}} * g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\varepsilon}}, L_{q}-L_{\bar{q}}+L_{\bar{q}}\right\rangle \\
& +\frac{1}{4} q^{2}\left\langle g 1_{\mathbb{B} \backslash \mathbb{R}_{g}^{c}}, g 1_{\mathbb{B} \backslash \mathbb{R}_{g}^{\mathscr{E}}} *\left(L_{q}-L_{\bar{q}}+L_{\bar{q}}\right\rangle\right\rangle \\
& \leqslant-c_{\bar{q}, d}\left\|\left(I-P_{\mathcal{H}}\right) K_{\bar{q}}^{1 / 2} g 1_{\mathbb{B} \mid \mathbb{M} \boldsymbol{E}}\right\|_{L^{2}(\mathbb{B})}^{2}+o_{q-\bar{q}}(1)\|g\|_{L^{2}\left(\mathbb{B} \backslash \mathbb{\mathbb { B }}{ }_{\xi}\right)}^{2} \tag{6.32}
\end{align*}
$$

where we use Lemma 52 and Lemma 45 in (6.32). Using this in (6.31) gives

$$
\begin{align*}
\left\|\widehat{f e^{i g} 1_{\mathbb{B}}}\right\|_{q}^{q} & \leqslant\left\|\widehat{\mathbb{1}_{\mathbb{B}}}\right\|_{q}^{q}-q \inf _{\mathbb{B}} K_{q} \cdot\left(\|\cos g-1\|_{L^{1}\left(A_{g} \cap \mathbb{B}_{g}^{\epsilon}\right)}+\left|\mathbb{B}_{g}^{\epsilon} \backslash A_{g}\right|\right)  \tag{6.33}\\
& -c_{\bar{q}, d}\left\|\left(I-P_{\mathcal{H}}\right) K_{\bar{q}}^{1 / 2} g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}\right\|_{L^{2}(\mathbb{B})}^{2}+o_{q-\bar{q}}(1)\|g\|_{L^{2}\left(\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}\right)}^{2} \\
& +\epsilon^{2} O\left(\|g\|_{L^{2}\left(\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}\right)}^{2}\right)+O\left(\left|\mathbb{B}_{g}^{\epsilon}\right|^{1 / 2}\|g\|_{L^{2}(\mathbb{B})}^{2}\right)+O\left(\|f-1\|_{L^{1}(\mathbb{B})}\|g\|_{L^{2}(\mathbb{B})}\right) \\
& +O\left(\|f-1\|_{L^{1}(\mathbb{B})}^{2}+\|g\|_{L^{2}(\mathbb{B})}^{3}\right) .
\end{align*}
$$

Since $|g| \leqslant \frac{5 \pi}{4}$, we can combine the $\left\|\left(I-P_{\mathcal{H}}\right) K_{\bar{q}}^{1 / 2} g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{e}}\right\|_{L^{2}(\mathbb{B})}^{2}$ above with the other negative term above as follows. Choose $c_{0}>0$ so that $1-\cos \theta \geqslant c_{0} \theta^{2}$ for $|\theta| \leqslant \frac{5 \pi}{4}$. Then

$$
\begin{align*}
\|\cos g-1\|_{L^{1}\left(A_{g} \cap \mathbb{B}_{g}^{\epsilon}\right)}+ & \left|\mathbb{B}_{g}^{\epsilon} \backslash A_{g}\right|+\left\|\left(I-P_{\mathcal{H}}\right) K_{\bar{q}}^{1 / 2} g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}\right\|_{L^{2}(\mathbb{B})}^{2} \geqslant c_{0}\|g\|_{L^{2}\left(A_{g} \cap \mathbb{B}_{g}^{\epsilon}\right)}^{2} \\
& +\frac{16}{25 \pi^{2}}\|g\|_{L^{2}\left(\mathbb{B}_{g}^{\epsilon} \backslash A_{g}\right)}^{2}+\left\|\left(I-P_{\mathcal{H}}\right) K_{\bar{q}}^{1 / 2} g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}\right\|_{L^{2}(\mathbb{B})}^{2} \\
\geqslant & c_{0}\left\|K_{\bar{q}}\right\|_{L^{\infty}(\mathbb{B})}^{-1}\left\|K_{\bar{q}}^{1 / 2} g\right\|_{L^{2}\left(A_{g} \cap \mathbb{B}_{g}^{\epsilon}\right)}^{2}+\frac{16}{25 \pi^{2}}\left\|K_{\bar{q}}\right\|_{L^{\infty}(\mathbb{B})}^{1}\left\|K_{\bar{q}}^{1 / 2} g\right\|_{L^{2}\left(\mathbb{B}_{g} \backslash A_{g}\right)}^{2} \\
& \quad+\left\|\left(I-P_{\mathcal{H}}\right) K_{\bar{q}}^{1 / 2} g 1_{\mathbb{B} \backslash \mathbb{B}_{g} \epsilon}\right\|_{L^{2}(\mathbb{B})}^{2} \\
\geqslant & C_{0}\left\|K_{\bar{q}}^{1 / 2} g 1_{\mathbb{B}_{g}^{\epsilon}}+\left(I-P_{\mathcal{H}}\right) K_{\bar{q}}^{1 / 2} g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}\right\|_{L^{2}(\mathbb{B})}^{2} \\
= & C_{0}\left\|K_{\bar{q}}^{1 / 2} g-P_{\mathcal{H}}\left(K_{\bar{q}}^{1 / 2} g 1_{\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}}\right)\right\|_{L^{2}(\mathbb{B})}^{2} \geqslant C_{0}\left\|K_{\bar{q}}^{1 / 2} g-P_{\mathcal{H}}\left(K_{\bar{q}}^{1 / 2} g\right)\right\|_{L^{2}(\mathbb{B})}^{2} \tag{6.34}
\end{align*}
$$

for an appropriate constant $C_{0}>0$. So we have for another constant $c>0$

$$
\begin{align*}
\left\|\widehat{f e^{i g} 1_{\mathbb{B}}}\right\|_{q}^{q} & \leqslant\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-c\left\|\left(I-P_{\mathcal{H}}\right) K_{\bar{q}}^{1 / 2} g\right\|_{L^{2}(\mathbb{B})}^{2}  \tag{6.35}\\
& +o_{q-\bar{q}}(1)\|g\|_{L^{2}\left(\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}\right)}^{2}+\epsilon^{2} O\left(\|g\|_{L^{2}\left(\mathbb{B} \backslash \mathbb{B}_{g}^{\epsilon}\right)}^{2}\right) \\
& +O\left(\left|\mathbb{B}_{g}^{\epsilon}\right|^{1 / 2}\|g\|_{L^{2}(\mathbb{B})}^{2}\right)+O\left(\|f-1\|_{L^{1}(\mathbb{B})}\|g\|_{L^{2}(\mathbb{B})}\right) \\
& +O\left(\|f-1\|_{L^{1}(\mathbb{B})}^{2}+\|g\|_{L^{2}(\mathbb{B})}^{3}\right) .
\end{align*}
$$

Use the bound $\left|\mathbb{B}_{g}^{\epsilon}\right|^{1 / 2} \leqslant \epsilon^{-1}\|g\|_{L^{2}(\mathbb{B})}$ and choose $\epsilon=\|g\|_{L^{2}(\mathbb{B})}^{1 / 2}$ to simplify the above to

$$
\begin{align*}
\left\|\widehat{f e^{i g} 1_{\mathbb{B}}}\right\|_{q}^{q} \leqslant & \left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-c\left\|\left(I-P_{\mathcal{H}}\right) K_{\bar{q}}^{1 / 2} g\right\|_{L^{2}(\mathbb{B})}^{2}+o_{q-\bar{q}}(1)\|g\|_{L^{2}(\mathbb{B})}^{2}  \tag{6.36}\\
& +O\left(\|f-1\|_{L^{1}(\mathbb{B})}\|g\|_{L^{2}(\mathbb{B})}\right)+O\left(\|f-1\|_{L^{1}(\mathbb{B})}^{2}+\|g\|_{L^{2}(\mathbb{B})}^{5 / 2}\right) .
\end{align*}
$$

### 6.7 Mostly support variation

By mostly support variation, we mean $M N|E \Delta \mathbb{B}| \geqslant \max \left(N\|g\|_{L^{2}(E)},\|f-1\|_{L^{1}(E)}^{1 / 2}\right)$. The parameters $M$ and $N$ are consistent with the other two cases described in $\$ 6.5$ and $\$ 6.8$. Let $K_{q}, L_{q}$ be the functions defined in (6.15) and (6.16). We employ the more detailed Taylor expansion from Lemma 48 in this section.

Proposition 54. Let $d \geqslant 1$ and let $\bar{q} \geqslant 4$ be an even integer and $M, N \in \mathbb{R}^{+}$. There exists $\delta_{0}=\delta_{0}(\bar{q}, d, M, N)>0$ and $\rho=\rho\left(\delta_{0}, \bar{q}, M, N\right)>0$ such that the following holds. Let $q \in(3, \infty), E \subset \mathbb{R}^{d}$ be a Lebesgue measurable set with $|E| \leqslant|\mathbb{B}|, 0 \leqslant f \leqslant 1$, and $g$ be real valued. Suppose that $\|f-1\|_{L^{1}(\mathbb{B})} \leqslant \delta_{0},\|g\|_{L^{2}(E)} \leqslant \delta_{0},|E \Delta \mathbb{B}| \leqslant 2 \operatorname{dist}(E, \mathfrak{E}) \leqslant \delta_{0}$, and $|q-\bar{q}| \leqslant \rho$. If

$$
M N|E \Delta \mathbb{B}| \geqslant \max \left(N\|g\|_{L^{2}(E)},\|f-1\|_{L^{1}(E)}^{1 / 2}\right)
$$

then

$$
\left\|\widehat{f e^{i g} 1_{\mathbb{B}}}\right\|_{q}^{q} \leqslant\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-c_{q, d} \operatorname{dist}(E, \mathfrak{E})^{2}
$$

for a constant $c_{q, d}>0$ depending only on the exponent $q$ and on the dimension.
Proof. We begin with the expression from Lemma 48, in which the terms with $f$ and $g$ are separated from terms with just the support $E$. Recall that $g^{\prime}=g$ on $E \cap \mathbb{B}$ and $g^{\prime}=0$ on $B \backslash E$ and that $f^{\prime}=f$ on $E \cap \mathbb{B}$ and $f^{\prime}=1$ on $B \backslash E$. We have

$$
\begin{align*}
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q}= & \left\|\widehat{1_{E}}\right\|_{q}^{q}+q\left\langle K_{q}, f \cos g 1_{E \backslash \mathbb{B}}-1_{E \backslash \mathbb{B}}\right\rangle-\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+\left\|\widehat{f^{\prime} e^{i g^{\prime}} 1_{\mathbb{B}}}\right\|_{q}^{q}  \tag{6.37}\\
& +O\left(\left(\|g\|_{2}^{2}+\|f-1\|_{L^{1}(E)}\right)|E \Delta \mathbb{B}|^{1 / 2}\right)+O\left(\|g\|_{L^{2}(E)}^{3}+\|f-1\|_{L^{1}(E)}^{2}+|E \Delta \mathbb{B}|^{3 / q^{\prime}}\right) .
\end{align*}
$$

We use Christ's Theorem 2.6 from [16] to bound $\left\|\widehat{1_{E}}\right\|_{q}$ :

$$
\left\|\widehat{1_{E}}\right\|_{q}^{q} \leqslant\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-c_{q, d}|E \Delta \mathbb{B}|^{2}
$$

By Theorem 53, we control $\left\|\widehat{f^{\prime} e^{i g^{\prime}} 1_{\mathbb{B}}}\right\|_{q}$ as follows.

$$
\begin{aligned}
\left\|\widehat{f^{\prime} e^{i g^{\prime}} 1_{\mathbb{B}}}\right\|_{q}^{q} \leqslant & \left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-c_{q, d}\left\|\left(I-P_{\mathcal{H}}\right) K_{\bar{q}}^{1 / 2} g^{\prime}\right\|_{L^{2}(\mathbb{B})}^{2}+o_{q-\bar{q}}(1)\left\|g^{\prime}\right\|_{L^{2}(\mathbb{B})}^{2} \\
& \left.+O\left(\left\|f^{\prime}-1\right\|_{L^{1}(\mathbb{B})}\right)\left\|g^{\prime}\right\|_{L^{2}(\mathbb{B})}\right)+O\left(\left\|f^{\prime}-1\right\|_{L^{1}(\mathbb{B})}^{2}+\left\|g^{\prime}\right\|_{L^{2}(\mathbb{B})}^{5 / 2}\right) \\
\leqslant & \left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+0+o_{q-\bar{q}}(1) M^{2}|E \Delta \mathbb{B}|^{2} \\
& +O_{M, N}\left(|E \Delta \mathbb{B}|^{3}\right)+O_{M, N}\left(\|\left. E \Delta \mathbb{B}\right|^{4}+|E \Delta \mathbb{B}|^{5 / 2}\right) \\
= & \left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+o_{q-\bar{q}}(1) M^{2}|E \Delta \mathbb{B}|^{2}+O_{M, N}\left(|E \Delta \mathbb{B}|^{5 / 2}\right) .
\end{aligned}
$$

Finally, to bound the inner product term from (6.37), using that $K_{\bar{q}} \geqslant 0$, calculate

$$
\begin{aligned}
\left\langle K_{q}, f \cos g 1_{E \backslash \mathbb{B}}-1_{E \backslash \mathbb{B}}\right\rangle & =\left\langle K_{q}-K_{\bar{q}}+K_{\bar{q}}, f \cos g 1_{E \backslash \mathbb{B}}-1_{E \backslash \mathbb{B}}\right\rangle \\
& \leqslant o_{q-\bar{q}}(1)\|f \cos g-1\|_{L^{1}(E \backslash \mathbb{B}}+\left\langle K_{\bar{q}},(f \cos g-1) 1_{E \backslash \mathbb{B}}\right\rangle \\
& \leqslant o_{q-\bar{q}}(1)\left(\|f-1\|_{L^{1}(E \backslash \mathbb{B})}+\|\cos g-1\|_{L^{1}(E \backslash \mathbb{B}}\right)+0 \\
& \leqslant o_{q-\bar{q}}(1)\left(M^{2} N^{2}|E \Delta \mathbb{B}|^{2}+M^{2}|E \Delta \mathbb{B}|^{2}\right) .
\end{aligned}
$$

Using the above bounds in (6.37) gives

$$
\begin{aligned}
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q} \leqslant & \left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-c_{q, d}|E \Delta \mathbb{B}|^{2}+M^{2} N^{2} o_{q-\bar{q}}(1)|E \Delta \mathbb{B}|^{2} \\
& +O_{M, N}\left(|E \Delta \mathbb{B}|^{5 / 2}+|E \Delta \mathbb{B}|^{3 / q^{\prime}}\right)
\end{aligned}
$$

If $\delta_{0}, \rho$ are chosen sufficiently small, then we have the desired result.

### 6.8 Mostly frequency variation

By mostly frequency variation, we mean $\max \left(\|f-1\|_{1}^{1 / 2}, M N|E \Delta \mathbb{B}|\right) \leqslant N\|g\|_{L^{2}(E)}$. The parameters $M$ and $N$ are consistent with the other two cases described in 86.5 and $\S 6.7$. Let $K_{q}, L_{q}$ be the functions defined in (6.15) and 6.16). As in $\$ 6.7$, we employ Lemma 48 to analyze the contributions from the frequency $g$.

Proposition 55. Let $d \geqslant 1$ and let $\bar{q}>3$ be an even integer and $N \in \mathbb{R}^{+}$. There exist $\delta_{0}=\delta_{0}(\bar{q}, d)>0, \rho\left(\delta_{0}, \bar{q}, N\right)>0$, and $M=M(\bar{q}, \rho) \in \mathbb{N}$, such that the following holds. Let $q \in(3, \infty), E \subset \mathbb{R}^{d}$ be a Lebesgue measurable set with $|E| \leqslant|\mathbb{B}|, 0 \leqslant f \leqslant 1$, and $-\pi \leqslant g \leqslant \pi$. Suppose that $\|f-1\|_{L^{1}(\mathbb{B})} \leqslant \delta_{0},\|g\|_{L^{2}(E)} \leqslant \delta_{0},\left\|e^{i g}-1\right\|_{L^{2}(E \cap \mathbb{B})} \leqslant 2 \inf _{\mathbb{R} \text { - affine }} \| e^{i(g-L)}-$ $1 \|_{L^{2}(E \cap \mathbb{B})},|E \Delta \mathbb{B}| \leqslant \delta_{0},|E|=|\mathbb{B}|$, and $|q-\bar{q}| \leqslant \rho\left(\delta_{0}, \bar{q}\right)$. If

$$
\max \left(\|f-1\|_{1}^{1 / 2}, M N|E \Delta \mathbb{B}|\right) \leqslant N\|g\|_{L^{2}(E)}
$$

then

$$
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q} \leqslant\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-c_{q, d} \inf _{\substack{\text { Laffine } \\ \mathbb{R} \text {-valued }}}\left\|e^{i(g-L)}-1\right\|_{L^{2}(E)}
$$

for a constant $c_{q, d}>0$ depending only on the exponent $q$ and on the dimension.
For use in the subsequent proof of Proposition 55, we state a version of Lemma 4.1 from [16] with the special case $\eta=1$, noting that $q_{d}$ in the statement may be taken to be equal to 3 .

Lemma 56. [16] Let $d \geqslant 1$ and $\bar{q} \geqslant 4$ be an even integer. There exists $\delta_{0}=\delta_{0}(\bar{q})>0$ and $c, C, \rho, \alpha \in \mathbb{R}^{+}$with the following property. Let $E \subset \mathbb{R}^{d}$ be a Lebesgue measurable set satisfying $|E|=|\mathbb{B}|$. If $|q-\bar{q}|<\rho$ and $|E \Delta \mathbb{B}| \leqslant \delta_{0}$, then

$$
\left\|\widehat{1_{E}}\right\|_{q}^{q} \leqslant\left\|\widehat{1_{E \cap 2 \mathbb{B}}}\right\|_{q}^{q}-c|E \backslash 2 \mathbb{B}|+C|E \Delta \mathbb{B}| \cdot|E \backslash 2 \mathbb{B}|+C|E \Delta \mathbb{B}|^{2+\alpha} .
$$

Proof. (of Proposition 55)
Use the expression from Lemma 48 in which the terms with $f$ and $g$ are separated from the terms with only the support $E$. Majorize the big-O terms with $\|f-1\|_{L^{1}(E)}$ or $|E \Delta \mathbb{B}|$ by terms with $\|g\|_{L^{2}(E)}$.

$$
\begin{gather*}
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q}=\left\|\widehat{1_{E}}\right\|_{q}^{q}+q\left\langle K_{q}, f \cos g 1_{E \backslash \mathbb{B}}-1_{E \backslash \mathbb{B}}\right\rangle-\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+\left\|\widehat{f^{\prime} e^{i g^{\prime}} 1_{\mathbb{B}}}\right\|_{q}^{q}  \tag{6.38}\\
+O_{N}\left(\|g\|_{L^{2}(E)}^{5 / 2}+\|g\|_{L^{2}(E)}^{3 / q^{\prime}}\right)
\end{gather*}
$$

where $f^{\prime}=f$ on $E \cap \mathbb{B}$ and $f^{\prime}=1$ on $\mathbb{B} \backslash E$ and $g^{\prime}=g$ on $E \cap \mathbb{B}$ and $g^{\prime}=0$ on $\mathbb{B} \backslash E$. We further analyze $\left\|\widehat{1_{E}}\right\|_{q}^{q}$ and $\left\langle K_{q}, f \cos g 1_{E \backslash \mathbb{B}}-1_{E \backslash \mathbb{B}}\right\rangle$.

Use Lemma 56 to extract $-|E \backslash 2 \mathbb{B}|$ from $\left\|\widehat{1_{E}}\right\|_{q}^{q}$ :

$$
\begin{aligned}
\left\|\widehat{1_{E}}\right\|_{q}^{q} & \leqslant\left\|\widehat{1_{E \cap 2 \mathbb{B}}}\right\|_{q}^{q}-c|E \backslash 2 \mathbb{B}|+C|E \Delta \mathbb{B}| \cdot|E \backslash 2 \mathbb{B}|+C|E \Delta \mathbb{B}|^{2+\alpha} \\
& \leqslant\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-c|E \backslash 2 \mathbb{B}|+\frac{C}{M^{2}}\|g\|_{2}^{2}+O\left(\|g\|_{2}^{2+\alpha}\right) \\
& \leqslant\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-c / 4\left\|e^{i g}-1\right\|_{L^{2}(E \backslash 2 \mathbb{B})}^{2}+\frac{C}{M^{2}}\|g\|_{2}^{2}+O\left(\|g\|_{2}^{2+\alpha}\right) .
\end{aligned}
$$

As above, let $\bar{q}$ denote the nearest even integer to $q$. Next bound the term $\left\langle K_{q}, f \cos g 1_{E \backslash \mathbb{B}}-\right.$ $\left.1_{E \backslash \mathbb{B}}\right\rangle$ above by a negative multiple of $\left\|e^{i g}-1\right\|_{L^{2}((E \cap 2 \mathbb{B}) \backslash \mathbb{B}}^{2}$ plus an error term. Let $A_{g}=\{x \in$ $E \backslash \mathbb{B}: \cos g \geqslant 0\}$.

$$
\begin{align*}
\left\langle K_{q}-K_{\bar{q}}+K_{\bar{q}},\right. & \left.f \cos g 1_{E \backslash \mathbb{B}}-1_{E \backslash \mathbb{B}}\right\rangle=o_{q-\bar{q}}(1)\|f \cos g-1\|_{L^{1}(E \backslash \mathbb{B})}+\left\langle K_{\bar{q}}, f \cos g 1_{E \backslash \mathbb{B}}-1_{E \backslash \mathbb{B}}\right\rangle \\
& \leqslant o_{q-\bar{q}}(1)\|f \cos g-1\|_{L^{1}(E \backslash \mathbb{B})}+\left\langle K_{\bar{q}}, f \cos g 1_{A_{g}}-1_{E \backslash \mathbb{B}}\right\rangle \\
& \leqslant o_{q-\bar{q}}(1)\left(\|f-1\|_{L^{1}(E)}+\|g\|_{L^{2}(E)}^{2}\right)+\left\langle K_{\bar{q}}, \cos g 1_{A_{g}}-1_{E \backslash \mathbb{B}}\right\rangle \\
& \leqslant o_{q-\bar{q}}(1)\left(N^{2}+1\right)\|g\|_{L^{2}(E \backslash \mathbb{B})}^{2}+\left\langle K_{\bar{q}}, f \cos g 1_{A_{g}}-1_{A_{g}}\right\rangle-\left\langle K_{\bar{q}}, 1_{(E \backslash \mathbb{B}) \backslash A_{g}}\right\rangle \\
& \leqslant o_{q-\bar{q}}(1)\left(N^{2}+1\right)\|g\|_{L^{2}(E \backslash \mathbb{B})}^{2}-\inf _{2 \mathbb{B}} K_{\bar{q}} \cdot\left(\|\cos g-1\|_{L^{1}\left(A_{g} \cap 2 \mathbb{B}\right)}+\left|(E \cap 2 \mathbb{B}) \backslash\left(\mathbb{B} \cup A_{g}\right)\right|\right) \\
& \leqslant o_{q-\bar{q}}(1)\left(N^{2}+1\right)\|g\|_{L^{2}(E \backslash \mathbb{B})}^{2}-\inf _{2 \mathbb{B}} K_{\bar{q}} \cdot\left\|e^{i g}-1\right\|_{L^{2}((E \cap 2 \mathbb{B}) \backslash \mathbb{B})}^{2} \tag{6.39}
\end{align*}
$$

where we used that $K_{\bar{q}} \geqslant 0$ everywhere.
By Theorem 53, since $g$ is real-valued with $|g| \leqslant \pi$,

$$
\begin{aligned}
\left\|\widehat{f^{\prime} e^{i g^{\prime}} 1_{\mathbb{B}}}\right\|_{q}^{q} \leqslant & \left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-c_{q, d}\left\|\left(I-P_{\mathcal{H}}\right) K_{\bar{q}}^{1 / 2} g^{\prime}\right\|_{L^{2}(\mathbb{B})}^{2}+o_{q-\bar{q}}(1)\left\|g^{\prime}\right\|_{L^{2}(\mathbb{B})}^{2} \\
& +O\left(\left\|f^{\prime}-1\right\|_{L^{1}(\mathbb{B})}\left\|g^{\prime}\right\|_{L^{2}(\mathbb{B})}\right)+O\left(\left\|f^{\prime}-1\right\|_{L^{1}(\mathbb{B})}^{2}+\left\|g^{\prime}\right\|_{L^{2}(\mathbb{B})}^{5 / 2}\right) \\
\leqslant & \left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-c_{q, d}\left\|\left(I-P_{\mathcal{H}}\right) K_{\bar{q}}^{1 / 2} g 1_{E \cap \mathbb{B}}\right\|_{L^{2}(\mathbb{B})}^{2}+o_{q-\bar{q}}(1)\|g\|_{L^{2}(E)}^{2} \\
& +O_{N}\left(\|g\|_{L^{2}(E)}^{5 / 2}\right) .
\end{aligned}
$$

Recalling the definition of $\mathcal{H}$ in Definition 16 and the hypotheses about $g$, note that

$$
\begin{aligned}
\left\|\left(I-P_{\mathcal{H}}\right) K_{\bar{q}}^{1 / 2} g 1_{E \cap \mathbb{B}}\right\|_{L^{2}(\mathbb{B})} & \geqslant \inf _{\mathbb{B}} K_{\bar{q}}^{1 / 2} \cdot\left\|g-K_{\bar{q}}^{-1 / 2} P_{\mathcal{H}}\left(K_{\bar{q}}^{1 / 2} g 1_{E \cap \mathbb{B}}\right)\right\|_{L^{2}(E \cap \mathbb{B})} \\
& \geqslant \inf _{\mathbb{B}} K_{\bar{q}}^{1 / 2} \cdot\left\|e^{i g}-e^{i K_{\bar{q}}^{-1 / 2} P_{\mathcal{H}}\left(K_{\bar{q}}^{1 / 2} g 1_{E \cap \mathbb{B}}\right)}\right\|_{L^{2}(E \cap \mathbb{B})} \\
& \geqslant \inf _{\mathbb{B}} K_{\bar{q}}^{1 / 2} \cdot \inf _{\substack{L \text { affine } \\
\text { valued }}}\left\|e^{i(g-L)}-1\right\|_{L^{2}(E \cap \mathbb{B})} \\
& \geqslant \frac{1}{2} \inf _{\mathbb{B}} K_{\bar{q}}^{1 / 2} \cdot\left\|e^{i g}-1\right\|_{L^{2}(E \cap \mathbb{B})} .
\end{aligned}
$$

Combining the above analysis yields

$$
\begin{align*}
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q} & \leqslant\left\|\widehat{\mathbb{B}_{\mathbb{B}}}\right\|_{q}^{q}-c / 4\left\|e^{i g}-1\right\|_{L^{2}(E \backslash 2 \mathbb{B})}^{2}+\frac{C}{M^{2}}\|g\|_{L^{2}(E)}^{2}+o_{q-\bar{q}}(1)\left(N^{2}+1\right)\|g\|_{L^{2}(E \backslash \mathbb{B})}^{2} \\
& -\inf _{2 \mathbb{B}} K_{\bar{q}} \cdot\left\|e^{i g}-1\right\|_{L^{2}((E \cap 2 \mathbb{B}) \backslash \mathbb{B})}^{2} \\
& -\frac{c_{q, d}}{4} \inf _{\mathbb{B}} K_{\bar{q}} \cdot\left\|e^{i g}-1\right\|_{L^{2}(E \cap \mathbb{B})}^{2}+o_{q-\bar{q}}(1)\|g\|_{L^{2}(E)}^{2}+O_{N}\left(\|g\|_{L^{2}(E)}^{2+\epsilon}\right) \\
& =\left\|\widehat{\mathbb{1}_{\mathbb{B}}}\right\|_{q}^{q}-\tilde{c}\left\|e^{i g}-1\right\|_{L^{2}(E)}^{2}  \tag{6.40}\\
& +\left(\frac{C}{M^{2}}+o_{q-\bar{q}}(1) N^{2}\right)\|g\|_{L^{2}(E)}^{2}+O_{N}\left(\|g\|_{L^{2}(E)}^{2+\epsilon}\right)
\end{align*}
$$

where $2+\epsilon=\min \left(2+\alpha, 5 / 2,3 / q^{\prime}\right)$ and $\tilde{c}>0$ depends on $\bar{q}$ and $d$. Since $|g| \leqslant \pi,\left|e^{i g}-1\right|^{2} \geqslant$ $\pi^{-2} g^{2}$ almost everywhere on $E$. Thus for $\delta_{0}$ and $\rho$ sufficiently small depending on $N$ and $M$ small enough depending on $\bar{q}$,

$$
\left\|\widehat{f e^{i g} 1_{E}}\right\|_{q}^{q}=\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-\frac{\tilde{c}}{2}\left\|e^{i g}-1\right\|_{L^{2}(E)}^{2}
$$

which proves the proposition.

### 6.8.1 Optimality of the $L^{2}$ norm and the exponent 2

We show that the exponent 2 and the $L^{2}$ norm in the term $\inf _{\mathbb{R} L \text { affine }}\left\|e^{i(g-L)}-1\right\|_{L^{2}(E)}^{2}$ from Theorem 3 are optimal in Lemma 58. First we prove a technical sublemma.

Notation 17. Let $d \geqslant 1$ and let $\mathbb{B}=\left\{x \in \mathbb{R}^{d}:|x| \leqslant 1\right\}$. Let the projection $R(f)$ of a realvalued function $f$ be defined for each $x$ in the domain of $f$ by $R(f)(x) \equiv f(x) \bmod (2 \pi)$ and $R(f)(x) \in[-\pi, \pi)$.

Sublemma 57. Let $d \geqslant 1, p \geqslant 1$. There exists $\epsilon>0$ such that the following holds. If $g: \mathbb{R}^{d} \rightarrow \mathbb{R}$ satisfies $\|g\|_{\infty} \leqslant \epsilon$, then

$$
\inf _{\substack{L \text { affine } \\ \mathbb{R} \text {-valued }}}\left\|e^{i(g-L)}-1\right\|_{L^{p}(\mathbb{B})} \geqslant \frac{1}{2} \inf _{\underset{\mathbb{R} \text { affine }}{ }\|g-L\|_{L^{p}(\mathbb{B})} .} \| g \text { valued }
$$

Proof. Using the notation $R$ above and that $\left|e^{i \theta}-1\right| \geqslant \frac{1}{2} \theta$ for all $\theta \in[-\pi, \pi)$, note that

$$
\begin{aligned}
\inf _{\substack{L \text { affine } \\
\mathbb{R}^{\text {valued }}}}\left\|e^{i(g-L)}-1\right\|_{L^{p}(\mathbb{B})} & =\inf _{\substack{\mathbb{R}_{\text {a fanine }} \text {-valued }}}\left\|e^{i R(g-L)}-1\right\|_{L^{p}(\mathbb{B})} \\
& \geqslant \frac{1}{2} \inf _{\mathbb{R}^{L} \text { affine }}\|R(g-L)\|_{L^{p}(\mathbb{B})} .
\end{aligned}
$$

By the definition of $R$,

$$
\inf _{\substack{L \text { affine } \\ \mathbb{R}^{L} \text {-valued }}}\|R(g-L)\|_{L^{p}(\mathbb{B})} \leqslant \inf _{\substack{L \text { affine } \\ \mathbb{R}^{-v a l u e d}}}\|g-L\|_{L^{p}(\mathbb{B})} .
$$

For the reverse inequality, it suffices to note that

$$
\inf _{\substack{L \text { affine } \\ \mathbb{R} \text {-valued }}}\|g-L\|_{L^{p}(\mathbb{B})}=\inf _{L: \mathbb{B} \rightarrow \mathbb{R} \text { affine }}^{|L| \leqslant 3} \mid\|g-L\|_{L^{p}(\mathbb{B})}
$$

since for $|L| \leqslant 3$ and $\epsilon<0.1, R(g-L)=g-L$. Indeed, suppose for $L_{0}(x)=x \cdot \alpha+b$ where $\alpha \in \mathbb{R}^{d}$ and $b \in \mathbb{R}$ that

$$
\begin{equation*}
\left\|g-L_{0}\right\|_{L^{p}(\mathbb{B})} \leqslant 2 \inf _{\substack{L \text { affine } \\ \mathbb{R}^{- \text {valued }}}}\|g-L\|_{L^{p}(\mathbb{B})} \leqslant 2 \epsilon|\mathbb{B}|^{1 / p} \tag{6.41}
\end{equation*}
$$

Suppose for $x \in \mathbb{B}$ that $\left|L_{0}(x)\right| \geqslant 3$. Since $|g| \leqslant \epsilon$, there exists some $y \in \mathbb{B}$ such that $\left|L_{0}(y)\right| \leqslant 1$. Then

$$
2 \leqslant\left|L_{0}(x)-L_{0}(y)\right|=|\alpha \cdot(x-y)| \leqslant 2|\alpha|,
$$

so on the set $S:=\{z \in \mathbb{B}:|z-x|<1\},\left|L_{0}(z)\right| \geqslant 2$. Thus

$$
\left\|g-L_{0}\right\|_{L^{p}(\mathbb{B})} \geqslant\left\|g-L_{0}\right\|_{L^{p}(S)} \geqslant(2-\epsilon)|S|^{1 / p} \geqslant|S|^{1 / p} .
$$

Since $|S| \geqslant\left|\mathbb{B} \cap\left(\mathbb{B}+e_{1}\right)\right|$ where $e_{1}=(1,0, \ldots, 0) \in \mathbb{R}^{d}$, if $\epsilon<\min \left(\frac{\left|\mathbb{B} \cap\left(\mathbb{B}+e_{1}\right)\right|^{1 / p}}{2|\mathbb{B}|^{1 / p}}, 0.1\right)$, this contradicts (6.41).

Lemma 58. Let $d \geqslant 1, p \geqslant 1, N>0$, and $\bar{q} \geqslant 4$ an even integer. There exists $\rho=\rho(\bar{q}, d)>$ 0 such that the following holds. If for some $q>3$ satisfying $|q-\bar{q}|<\rho$, there exists $c_{q, d}>0$ such that for any function $g: \mathbb{R}^{d} \rightarrow \mathbb{R}$,

$$
\begin{equation*}
\left\|\widehat{e^{i g} 1_{\mathbb{B}}}\right\|_{q}^{q} \leqslant\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}-c_{\bar{q}, d} \inf _{\substack{\text { Laffine } \\ \mathbb{R} \text {-valued }}}\left\|e^{i(g-L)}-1\right\|_{L^{p}(\mathbb{B})}^{N}, \tag{6.42}
\end{equation*}
$$

then $N \geqslant 2$ and $N \geqslant p$.
Note that this lemma proves optimality of exponent 2 and the $L^{2}$ norm in the term $\inf _{\mathbb{R}^{L} \text {-affine }}\left\|e^{i(g-L)}-1\right\|_{L^{2}(E)}^{2}$ from Theorem 3 because if $N \geqslant 2$ and $N \geqslant p$, for any real-valued function $g$, real-valued affine function $L$, and subset $E$ with $|E|=1$,

$$
\left\|e^{i(g-L)}-1\right\|_{L^{p}(E)}^{N} \leqslant\left\|e^{i(g-L)}-1\right\|_{L^{N}(E)}^{N} \leqslant 2^{N-2}\left\|e^{i(g-L)}-1\right\|_{L^{2}(\mathbb{B})}^{2}
$$

Proof. Consider the family of real-valued functions $\epsilon g_{n}$ with small parameter $\epsilon>0$ and $n \in \mathbb{N}$, where $g_{n}$ is the indicator function of the annulus

$$
A_{n}:=\left\{x \in \mathbb{R}^{d}: 1-1 / n<|x|<1\right\} .
$$

Choose $\epsilon>0$ small enough so Lemma 43 applies. Then if $q^{\prime}$ is the conjugate exponent to $q$,

$$
\begin{align*}
& \left\|\widehat{e^{i \epsilon g_{n}} 1_{\mathbb{B}}}\right\|_{q}^{q}=\left\|\widehat{1_{\mathbb{B}}}\right\|_{q}^{q}+q\left\langle K_{q},\left(\cos \left(\epsilon g_{n}\right)-1\right) 1_{\mathbb{B}}\right\rangle  \tag{6.43}\\
& \quad-\frac{1}{4} q(q-2)\left\langle\sin \left(\epsilon g_{n}\right) 1_{\mathbb{B}} * \sin \left(\epsilon g_{n}\right) 1_{\mathbb{B}}, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle\sin \left(\epsilon g_{n}\right) 1_{\mathbb{B}}, \sin \left(\epsilon g_{n}\right) 1_{\mathbb{B}} * L_{q}\right\rangle \\
& \quad+O\left(\left\|\cos \left(\epsilon g_{n}\right)-1\right\|_{L^{1}(\mathbb{B})}^{2}\right)+O\left(\left\|e^{i \epsilon g_{n}}-1\right\|_{L^{q^{\prime}}(\mathbb{B})}^{3}\right)
\end{align*}
$$

Since $2|\cos (\theta)-1|=\left|e^{i \theta}-1\right|^{2}$ for $\theta \in \mathbb{R},\left\|\cos \left(\epsilon g_{n}\right)-1\right\|_{L^{1}(\mathbb{B})}^{2} \leqslant\left\|e^{i \epsilon g_{n}}-1\right\|_{L^{2}(\mathbb{B})}^{4} \leqslant \epsilon^{4}\left\|g_{n}\right\|_{L^{2}(\mathbb{B})}^{4}$. Since $1<q^{\prime}<2$, by Hölder's inequality, $\left\|e^{i \epsilon g_{n}}-1\right\|_{L^{q^{\prime}}(\mathbb{B})} \leqslant\left\|e^{i \epsilon g_{n}}-1\right\|_{L^{2}(\mathbb{B})}|\mathbb{B}|^{\left(2-q^{\prime}\right) /\left(2 q^{\prime}\right)}$. Thus we can majorize the big-O terms by $O\left(\epsilon^{3}\left\|g_{n}\right\|_{L^{2}(\mathbb{B})}^{2}\right)$. Combining (6.43) with our hypothesis (6.42) and rearranging, we have for a constant $C_{q, d}>0$

$$
\begin{aligned}
c_{q, d} \inf _{\mathbb{R}_{\text {Lafine }} \text { valued }}\left\|e^{i\left(\epsilon g_{n}-L\right)}-1\right\|_{L^{p}(\mathbb{B})}^{N}+O\left(\epsilon^{3}\left\|g_{n}\right\|_{2}^{3}\right) & \leqslant q\left\langle K_{q},\left(\cos \left(\epsilon g_{n}\right)-1\right) 1_{\mathbb{B}}\right\rangle \\
& -\frac{1}{4} q(q-2)\left\langle\sin \left(\epsilon g_{n}\right) 1_{\mathbb{B}} * \sin \left(\epsilon g_{n}\right) 1_{\mathbb{B}}, L_{q}\right\rangle+\frac{1}{4} q^{2}\left\langle\sin \left(\epsilon g_{n}\right) 1_{\mathbb{B}}, \sin \left(\epsilon g_{n}\right) 1_{\mathbb{B}} * L_{q}\right\rangle \\
& \leqslant C_{q, d} \epsilon^{2}\left\|g_{n}\right\|_{L^{2}(\mathbb{B})}^{2}
\end{aligned}
$$

For $\epsilon>0$ small enough, Sublemma 57 gives

$$
\begin{aligned}
\frac{c_{q, d}}{2} \inf _{\substack{L \\
\mathbb{R}^{- \text {affine }} \text { valued }}}\left\|\epsilon g_{n}-L\right\|_{L^{p}(\mathbb{B})}^{N}+O\left(\epsilon^{3}\left\|g_{n}\right\|_{2}^{3}\right) & \leqslant c_{q, d} \inf _{\substack{L \text { affine } \\
\mathbb{R}^{L}-\text { valued }}}\left\|e^{i\left(\epsilon g_{n}-L\right)}-1\right\|_{L^{p}(\mathbb{B})}^{N}+O\left(\epsilon^{3}\left\|g_{n}\right\|_{2}^{3}\right) \\
& \leqslant C_{q, d} \epsilon^{2}\left\|g_{n}\right\|_{L^{2}(\mathbb{B})}^{2} .
\end{aligned}
$$

Divide by $\epsilon^{2}$ to obtain

$$
\begin{equation*}
\frac{c_{q, d}}{2} \epsilon^{N-2} \inf _{\substack{\mathbb{R}_{\text {affine }} \text {-valued }}}\left\|g_{n}-L\right\|_{L^{p}(\mathbb{B})}^{N}+O\left(\epsilon\left\|g_{n}\right\|_{2}^{3}\right) \leqslant C_{q, d}\left\|g_{n}\right\|_{L^{2}(\mathbb{B})}^{2} \tag{6.44}
\end{equation*}
$$

Since $g_{n}$ is not an affine function, letting $\epsilon \rightarrow 0$ implies that $N-2 \geqslant 0$.
Note that

$$
\left\|g_{n}\right\|_{2}^{2}=c\left(1-(1-1 / n)^{d}\right)
$$

for a dimensional constant $c>0$. Choose real valued affine functions $L_{n}$ so that

$$
\frac{1}{2}\left\|g_{n}-L_{n}\right\|_{L^{p}(\mathbb{B})} \leqslant \inf _{\substack{L \text { affine } \\ \mathbb{R} \text {-valued }}}\left\|g_{n}-L\right\|_{L^{p}(\mathbb{B})}
$$

Using these expressions in (6.44) gives

$$
\frac{c_{q, d}}{2^{N+1}} \epsilon^{N-2}\left\|g_{n}-L_{n}\right\|_{L^{p}(\mathbb{B})}^{N}+O\left(\epsilon\left(1-(1-1 / n)^{d}\right)^{3 / 2}\right) \leqslant C_{q, d} c\left(1-(1-1 / n)^{d}\right)
$$

Since

$$
\frac{1}{2}\left\|g_{n}-L_{n}\right\|_{L^{p}(\mathbb{B})} \leqslant \inf _{\substack{L \text { affine } \\ \mathbb{R}^{- \text {valued }}}}\left\|g_{n}-L\right\|_{L^{p}(\mathbb{B})} \leqslant\left\|g_{n}\right\|_{p}=c^{1 / p}\left(1-(1-1 / n)^{d}\right)^{1 / p}
$$

we have $\left\|g_{n}-L_{n}\right\|_{L^{p}\left(\mathbb{B} \backslash A_{n}\right)}=\left\|L_{n}\right\|_{L^{p}\left(\mathbb{B} \backslash A_{n}\right)}=o_{n}(1)$ and $\left|g_{n}(x)-L_{n}(x)\right| \geqslant 1-o_{n}(1)$ for $x \in A_{n}$. Thus

$$
\begin{gathered}
\frac{c_{q, d}}{2^{N+1}} \epsilon^{N-2}\left(1-o_{n}(1)\right)^{N / p} c^{N / p}\left(1-(1-1 / n)^{d}\right)^{N / p}+O\left(\epsilon\left(1-(1-1 / n)^{d}\right)^{3 / 2}\right) \leqslant \\
\quad \frac{c_{q, d}}{4} \epsilon^{N-2}\left\|g_{n}-L_{n}\right\|_{L^{p}\left(A_{n}\right)}^{N}+O\left(\epsilon\left(1-(1-1 / n)^{d}\right)^{3 / 2}\right) \\
\leqslant c C_{q, d}\left(1-(1-1 / n)^{d}\right)
\end{gathered}
$$

which after dividing by $\left(1-(1-1 / n)^{d}\right)$ yields

$$
\frac{c_{q, d}}{2^{N+1}} \epsilon^{N-2}\left(1-o_{n}(1)\right)^{N / p} c^{N / p}\left(1-(1-1 / n)^{d}\right)^{N / p-1}+O\left(\epsilon\left(1-(1-1 / n)^{d}\right)^{1 / 2}\right) \leqslant c C_{q, d}
$$

This inequality holds for arbitrarily large $n$, so $N / p \geqslant 1$.

## Chapter 7

## Appendix: The Lorentz space $L(p, 1)$

We relate the three quasinorms on $L(p, 1)$ defined in $\$ 2.1$. In the following lemma, we prove a formula for $\|s\|_{\mathcal{L}}$ where $s$ is a nonnegative simple function.
Lemma 59. Let $d \geqslant 1$. Let $s=\sum_{n=1}^{N} a_{n} 1_{A_{n}}$ where the $A_{n}$ are pairwise disjoint and of finite Lebesgue measure and $0<a_{1}<\cdots<a_{N}$. Let $a_{0}=0$ and let $B_{n}=\cup_{k=n}^{N} A_{k}$ for $n=1, \ldots, N$. Then

$$
\|s\|_{\mathcal{L}}=\sum_{n=1}^{N}\left(a_{n}-a_{n-1}\right)\left|B_{n}\right|^{1 / p}
$$

Proof. First we prove for any $k \geqslant 1$ that when $c_{0}=0<c_{1}<c_{2}<\cdots<c_{k}$ and $C_{j}=\cup_{i=j}^{k} E_{i}$ for measurable sets $E_{i} \subset \mathbb{R}^{d}$ of finite measure,

$$
\begin{equation*}
\sum_{j=1}^{k}\left(c_{j}-c_{j-1}\right)\left|C_{j}\right|^{1 / p} \leqslant \sum_{j=1}^{k} c_{j}\left|E_{j}\right|^{1 / p} \tag{7.1}
\end{equation*}
$$

If $k=1$, then clearly $\sum_{j=1}^{k}\left(c_{j}^{n}-c_{j-1}^{n}\right)\left|C_{j}^{n}\right|^{1 / p}=c_{1}\left|C_{1}\right|^{1 / p}=\sum_{j=1}^{1} c_{j}\left|E_{j}\right|^{1 / p}$. Suppose for $k \geqslant 1$ that when $c_{1}<c_{2}<\cdots<c_{k}$ and $C_{j}=\cup_{i=j}^{k} E_{i}$ for measurable sets $E_{i} \subset \mathbb{R}^{d}$ of finite measure,

$$
\sum_{j=1}^{k}\left(c_{j}-c_{j-1}\right)\left|C_{j}\right|^{1 / p} \leqslant \sum_{j=1}^{k} c_{j}\left|E_{j}\right|^{1 / p}
$$

Then

$$
\begin{aligned}
\sum_{j=1}^{k+1}\left(c_{j}-c_{j-1}\right)\left|C_{j}^{n}\right|^{1 / p} & =c_{1}\left|C_{1}\right|^{1 / p}+\left(c_{2}-c_{1}\right)\left|C_{2}\right|^{1 / p}+\cdots+\left(c_{k+1}-c_{k}\right)\left|C_{k+1}\right|^{1 / p} \\
& \leqslant c_{1}\left(\left|E_{1}\right|^{1 / p}+\left|C_{2}\right|^{1 / p}\right)+\left(c_{2}-c_{1}\right)\left|C_{2}\right|^{1 / p}+\cdots+\left(c_{k+1}-c_{k}\right)\left|C_{k+1}\right|^{1 / p} \\
& =c_{1}\left|E_{1}\right|^{1 / p}+\left(c_{2}-c_{0}\right)\left|C_{2}\right|^{1 / p}+\cdots+\left(c_{k+1}-c_{k}\right)\left|C_{k+1}\right|^{1 / p} \\
& \leqslant c_{1}\left|E_{1}\right|^{1 / p}+\sum_{j=2}^{k+1} c_{j}\left|E_{j}\right|^{1 / p}=\sum_{j=1}^{k+1} c_{j}\left|E_{j}\right|^{1 / p}
\end{aligned}
$$

so (7.1) is proved.
Next we prove the lemma inductively, where notation is as in the statement of the lemma. If $N=1$, suppose $a_{1} 1_{A_{1}}=\sum_{j=1}^{\infty} b_{j} 1_{S_{j}}$ where $b_{j} \geqslant 0,\left|S_{j}\right|<\infty$. Then

$$
\begin{aligned}
a_{1}\left|A_{1}\right|^{1 / p}=\left\|a_{1} 1_{A_{1}}\right\|_{p} & =\left\|\sum_{j=1}^{N} b_{j} 1_{S_{j}}+\sum_{j=N+1}^{\infty} b_{j} 1_{S_{j}}\right\|_{p} \\
& \leqslant \sum_{j=1}^{\infty} b_{j}\left|S_{j}\right|^{1 / p}+\lim _{N \rightarrow \infty}\left\|\sum_{j=N+1}^{\infty} b_{j} 1_{S_{j}}\right\|_{p}
\end{aligned}
$$

where $\lim _{N \rightarrow \infty}\left\|\sum_{j=N+1}^{\infty} b_{j} 1_{S_{j}}\right\|_{p}=0$ by Lebesgue's dominated convergence theorem.
Now suppose that the lemma holds for $N-1 \geqslant 1$. Consider

$$
\sum_{n=1}^{N} a_{n} 1_{A_{n}}=\sum_{j=1}^{\infty} b_{j} 1_{S_{j}}
$$

where $b_{j-1} \geqslant b_{j} \geqslant 0, S_{j} \subset \cup_{n=1}^{N} A_{n}$, the $S_{j}$ are distinct, and $\left|S_{j}\right|>0$. From (7.1), we have for each $M>0$ that

$$
\sum_{j=1}^{M}\left(b_{j}-b_{j+1}\right)\left|\cup_{k=1}^{j} S_{k}\right|^{1 / p} \leqslant \sum_{j=1}^{M} b_{j}\left|S_{j}\right|^{1 / p}
$$

Letting $M \rightarrow \infty$ and noting that $\sum_{j=1}^{\infty} b_{j} 1_{S_{j}}=\sum_{j=1}^{\infty}\left(b_{j}-b_{j+1}\right) 1_{\cup_{k=1}^{j} S_{k}}$, we can assume that $S_{1} \subset S_{2} \subset \cdots$ and $b_{j} \geqslant 0$ but are not necessarily decreasing. Since the simple function $s$ achieves its $L^{\infty}$ norm on $A_{N}$, and the series takes its maximum on $S_{1}$, we must have $S_{1}=A_{1}$ and

$$
a_{N}=\sum_{j=1}^{\infty} b_{j},
$$

so $\sum_{k=j}^{\infty} b_{k} \rightarrow 0$ as $j \rightarrow \infty$. The simple function $s$ achieves its minimum (on a set of positive measure) in $\cup_{n=1}^{N} A_{n}$ on $A_{1}$, but the series takes the values of $\sum_{k=j}^{\infty} b_{k}$ on positive measure sets in $\cup_{n=1}^{N} A_{n}$, so there is no minimum unless $\sum_{k=j}^{\infty} b_{k}$ is zero for large enough $j$. Thus we may write

$$
\sum_{n=1}^{N}\left(a_{n}-a_{n-1}\right) 1_{B_{n}}=\sum_{j=1}^{M} b_{j} 1_{S_{j}}
$$

where $S_{j} \subset S_{j+1}$ and $b_{j}>0$. We note $B_{1}=S_{M}$ and $a_{1}=b_{M}$. Then invoking the inductive hypothesis, we have

$$
\sum_{n=2}^{N}\left(a_{n}-a_{n-1}\right)\left|B_{n}\right|^{1 / p}+a_{1}\left|B_{1}\right|^{1 / p} \leqslant \sum_{j=1}^{M-1} b_{j}\left|S_{j}\right|^{1 / p}+b_{M}\left|S_{M}\right|^{1 / p}
$$

as desired.

For all $f \in L(p, 1)$,

$$
\begin{equation*}
\|f\|_{p 1}^{*} \leqslant\|f\|_{p 1} \leqslant \frac{p}{p-1}\|f\|_{p 1}^{*}, \tag{7.2}
\end{equation*}
$$

which is proved in Chapter V, $\S 3$ in [36]. From the nonincreasing property of $f^{*}$, it is clear that $\frac{1}{t} \int_{0}^{t} f^{*}(u) d u \geqslant f^{*}(t)$ for $t>0$, which implies that $\|f\|_{p 1} \geqslant\|f\|_{p 1}^{*}$. This combined with (7.2) implies that $\|f\|_{p 1}^{*}$ is finite if and only if $\|f\|_{p 1}$ is finite.

Lemma 60. Let $d \geqslant 1$. Let $p>1$ and let $q$ be the conjugate exponent to $p$. For all measurable functions $f: \mathbb{R}^{d} \rightarrow \mathbb{C}$ with $\|f\|_{\mathcal{L}}<\infty$ and $\|f\|_{p 1}^{*}<\infty,\|f\|_{\mathcal{L}}=\|f\|_{p 1}^{*}$.

Proof. First we show the equivalence for nonnegative simple functions. Write $s=\sum_{n=1}^{N} a_{n} 1_{A_{n}}$ where the $A_{n}$ are pairwise disjoint and $0<a_{1}<\cdots<a_{N}$. Let $a_{0}=0$ and let $B_{n}=\cup_{k=n}^{N} A_{k}$ for $n=1, \ldots, N$, and let $\left|B_{N+1}\right|=0$.

Calculate

$$
\begin{aligned}
\|s\|_{p 1}^{*} & =\frac{1}{p} \int_{0}^{\infty} t^{-1 / q} s^{*}(t) d t=\frac{1}{p} \sum_{n=0}^{N-1} \int_{\left|B_{N-n+1}\right|}^{\left|B_{N-n}\right|} t^{-1 / q} \inf \{r:|\{x:|s(x)|>r\}| \leqslant t\} d t \\
& =\frac{1}{p} \sum_{n=0}^{N-1} \int_{\left|B_{N-n+1}\right|}^{\left|B_{N-n}\right|} t^{-1 / q} \inf \left\{r:|\{x:|s(x)|>r\}| \leqslant\left|B_{N-n+1}\right|\right\} d t \\
& =\frac{1}{p} \sum_{n=0}^{N-1} a_{N-n} \int_{\left|B_{N-n+1}\right|}^{\left|B_{N-n}\right|} t^{-1 / q} d t \\
& =\sum_{n=0}^{N-1} a_{N-n}\left(\left|B_{N-n}\right|^{1 / p}-\left|B_{N-n+1}\right|^{1 / p}\right) \\
& =\sum_{n=0}^{N-1} a_{N-n}\left|B_{N-n}\right|^{1 / p}-\sum_{n=0}^{N-1} a_{N-n}\left|B_{N-n+1}\right|^{1 / p} \\
& =\sum_{n=1}^{N} a_{n}\left|B_{n}\right|^{1 / p}-\sum_{n=1}^{N} a_{n-1}\left|B_{n}\right|^{1 / p}=\sum_{n=1}^{N}\left(a_{n}-a_{n-1}\right)\left|B_{n}\right|^{1 / p} .
\end{aligned}
$$

Thus by Lemma 59, we have $\|s\|_{\mathcal{L}}=\|s\|_{p 1}^{*}$ for all nonnegative simple functions.
Next, consider a $f \in L(p, 1)$ with finite support $A$ and $L^{\infty}$ norm $M>0$. From the definition of $\|\cdot\|_{\mathcal{L}}$ and Lemma 59, we can choose nonnegative simple functions $|f|-1 / n \leqslant$ $s_{n} \leqslant|f|$ such that $\lim _{n \rightarrow \infty} s_{n}(x)=|f(x)|$ for a.e. $x \in \mathbb{R}^{d}$ and

$$
\|f\|_{\mathcal{L}}=\lim _{n \rightarrow \infty}\left\|s_{n}\right\|_{\mathcal{L}}=\lim _{n \rightarrow \infty}\left\|s_{n}\right\|_{p 1}^{*} .
$$

Note that

$$
\frac{1}{p} \int_{0}^{\infty} t^{-1 / q} s_{n}^{*}(t) d t=\frac{1}{p} \int_{0}^{|A|} t^{-1 / q} s_{n}^{*}(t) d t
$$

Since $\left(|f|-1 / n 1_{A}\right)^{*} \leqslant s_{n}^{*} \leqslant|f|^{*}$, we have the upper bound

$$
\frac{1}{p} \int_{0}^{|A|} t^{-1 / q} s_{n}^{*}(t) d t \leqslant \frac{1}{p} \int_{0}^{|A|} t^{-1 / q} f^{*}(t) d t
$$

and the lower bound

$$
\begin{aligned}
\frac{1}{p} \int_{0}^{|A|} t^{-1 / q} \inf \left\{r:\left|\left\{x: s_{n}(x)>r\right\}\right| \leqslant t\right\} d t & \geqslant \frac{1}{p} \int_{0}^{|A|} t^{-1 / q} \inf \{r:|\{x:|f(x)|>r+1 / n\}| \leqslant t\} d t \\
& =\frac{1}{p} \int_{0}^{|A|} t^{-1 / q} \inf \{r:|\{x:|f(x)|>r\}| \leqslant t\} d t-p|A|^{1 / p} \frac{1}{n}
\end{aligned}
$$

Thus by the squeeze theorem, we have that $\lim _{n \rightarrow \infty}\left\|s_{n}\right\|_{p 1}^{*}=\|f\|_{p 1}^{*}$.
For general $f \in L(p, 1)$, define $f_{n}=f 1_{\{1 / n \leqslant|f| \leqslant n\}}$. We argue that $\|f\|_{\mathcal{L}}=\lim _{n \rightarrow \infty}\left\|f_{n}\right\|_{\mathcal{L}}$.
If $\left|f_{n}\right|=\sum_{n} a_{n} 1_{A_{n}},|f| 1_{\{|f|>n\}}=\sum_{m} b_{m} 1_{B_{m}}$, and $|f| 1_{\{|f|<1 / n\}}=\sum_{k} c_{k} 1_{C_{k}}$, then $|f|=$ $\sum_{n} a_{n} 1_{A_{n}}+\sum_{m} b_{m} 1_{B_{m}}+\sum_{k} c_{k} 1_{C_{k}}$ and so

$$
\|f\|_{\mathcal{L}} \leqslant\left\|f_{n}\right\|_{\mathcal{L}}+\left\||f| 1_{\{|f|>n\}}\right\|_{\mathcal{L}}+\left\||f| 1_{|f|<1 / n\}}\right\|_{\mathcal{L}}
$$

Since if $|f|=\sum_{m} e_{m} 1_{E_{m}}, e_{m}>0$ with $\sum_{m} e_{m}\left|E_{m}\right|^{1 / p}<\infty$ then $\left|f_{n}\right|=\sum_{m} e_{m} 1_{E_{m} \cap\{1 / n \leqslant|f| \leqslant n\}}$ with

$$
\sum_{m} e_{m}\left|E_{m} \cap\{1 / n \leqslant|f| \leqslant n\}\right|^{1 / p} \leqslant \sum_{m} e_{m}\left|E_{m}\right|^{1 / p}<\infty
$$

we also have $\left\|f_{n}\right\|_{\mathcal{L}} \leqslant\|f\|_{\mathcal{L}}$. To show that $\|f\|_{\mathcal{L}}=\lim _{n \rightarrow \infty}\left\|f_{n}\right\|_{\mathcal{L}}$, it suffices to show that $\lim _{n \rightarrow \infty}\left\|f 1_{\{|f|>n\}}\right\|_{\mathcal{L}}=\lim _{n \rightarrow \infty}\left\|f 1_{\{|f|>1 / n\}}\right\|_{\mathcal{L}}=0$.

If $|f|=\sum_{m} e_{m} 1_{E_{m}}, e_{m}>0$ with $\sum_{m} e_{m}\left|E_{m}\right|^{1 / p}<\infty$, then

$$
\limsup _{n \rightarrow \infty}\left\||f| 1_{\{|f|>n\}}\right\|_{\mathcal{L}} \leqslant \limsup _{n \rightarrow \infty} \sum_{m} e_{m}\left|E_{m} \cap\{|f|>n\}\right|^{1 / p}=0
$$

where we used the monotone convergence theorem in the last line. Similarly, we have that

$$
\limsup _{n \rightarrow \infty}\left\|f 1_{\{|f|<1 / n\}}\right\|_{\mathcal{L}} \leqslant \limsup _{n \rightarrow \infty} \sum_{m} e_{m}\left|E_{m} \cap\{|f|<1 / n\}\right|^{1 / p} .
$$

Since $\sum_{m} e_{m}\left|E_{m} \cap\{|f|<1 / n\}\right|^{1 / p}$ is a decreasing sequence in $n$,

$$
\limsup _{n \rightarrow \infty} \sum_{m} e_{m}\left|E_{m} \cap\{|f|<1 / n\}\right|^{1 / p}=\inf _{n} \sum_{m} e_{m}\left|E_{m} \cap\{|f|<1 / n\}\right|^{1 / p} .
$$

We also have for each $M>1$

$$
\begin{aligned}
\inf _{n} \sum_{m} e_{m}\left|E_{m} \cap\{|f|<1 / n\}\right|^{1 / p} & \leqslant \inf _{n} \sum_{m \leqslant M} e_{m}\left|E_{m} \cap\{|f|<1 / n\}\right|^{1 / p}+\sum_{m>M} e_{m}\left|E_{m}\right|^{1 / p} \\
& =\sum_{m \leqslant M} e_{m}\left|E_{m} \cap\{|f|=0\}\right|^{1 / p}+\sum_{m>M} e_{m}\left|E_{m}\right|^{1 / p} \\
& =\sum_{m>M} e_{m}\left|E_{m}\right|^{1 / p} .
\end{aligned}
$$

Letting $M$ go to infinity, we have $\lim _{n \rightarrow \infty}\left\|f 1_{\{|f|<1 / n\}}\right\|_{\mathcal{L}}=0$. Conclude that $\|f\|_{\mathcal{L}}=\lim _{n \rightarrow \infty}\left\|f_{n}\right\|_{\mathcal{L}}=$ $\lim _{n \rightarrow \infty}\left\|f_{n}\right\|_{p 1}^{*}$.

Finally, we need to show that $\lim _{n \rightarrow \infty}\left\|f_{n}\right\|_{p 1}^{*}=\|f\|_{p 1}^{*}$. Since $\left\|f_{n}\right\|_{p 1}^{*} \leqslant\|f\|_{p 1}^{*}$ for each $n$ and $\lim _{M \rightarrow \infty} \int_{M}^{\infty} t^{-1 / q} f_{n}^{*}(t) d t \leqslant \lim _{M \rightarrow \infty} \int_{M}^{\infty} t^{-1 / q} f^{*}(t) d t=0$, it suffices to show that for each $M>0$,

$$
\lim _{n \rightarrow \infty} \int_{0}^{M} t^{-1 / q} f_{n}^{*}(t) d t \geqslant \int_{0}^{M} t^{-1 / q} f^{*}(t) d t
$$

We note that

$$
\begin{aligned}
\int_{0}^{M} t^{-1 / q} f_{n}^{*}(t) d t & =\int_{0}^{M} t^{-1 / q} \inf \left\{r:\left|\left\{x:\left|f_{n}(x)\right|>r\right\}\right| \leqslant t\right\} d t \\
& \geqslant \int_{0}^{M} t^{-1 / q} \inf \left\{r:\left|\left\{x:|f(x)| 1_{\{|f| \leqslant n\}}>r\right\}\right| \leqslant t\right\} d t-p M^{1 / p} \frac{1}{n} \\
& =\geqslant \int_{0}^{M} t^{-1 / q} f^{*}(t+|\{x:|f(x)|>n\}|) d t-p M^{1 / p} \frac{1}{n}
\end{aligned}
$$

Since $\lim _{n \rightarrow \infty}|\{x:|f(x)|>n\}|=0$ and $f^{*}$ is a.e. continuous, by the Lebesgue dominated convergence theorem,

$$
\lim _{n \rightarrow \infty} \int_{0}^{M} t^{-1 / q} f_{n}^{*}(t) d t \geqslant \int_{0}^{M} t^{-1 / q} f^{*}(t) d t
$$

Corollary 61. Let $d \geqslant 1$. Let $p>1$ and let $q$ be the conjugate exponent to $p$. For all measurable functions $f: \mathbb{R}^{d} \rightarrow \mathbb{C},\|f\|_{\mathcal{L}}<\infty$ if and only if $\|f\|_{p 1}^{*}<\infty$.

Proof. Suppose that $\|f\|_{\mathcal{L}}<\infty$. We showed in the proof of Lemma 60 that $\|f\|_{\mathcal{L}}=$ $\lim _{n \rightarrow \infty}\left\|f_{n}\right\|_{\mathcal{L}}$ where $\left|f_{n}\right|$ are bounded with finite support and monotonically increasing a.e. to $|f|$. We also showed that for those $f_{n}, \lim _{n \rightarrow \infty}\left\|f_{n}\right\|_{p 1}^{*}=\|f\|_{p 1}^{*}$, so $\|f\|_{p 1}^{*}<\infty$.

Next, suppose $\|f\|_{p 1}^{*}<\infty$. By definition of $\|f\|_{\mathcal{L}}$ (regardless of whether this quantity is finite or infinite), there exist simple functions $0 \leqslant s_{n} \leqslant|f|$ such that $\|f\|_{\mathcal{L}}=\lim _{n \rightarrow \infty}\left\|s_{n}\right\|_{\mathcal{L}}$. But we showed in the proof of Lemma 60 that $\left\|s_{n}\right\|_{\mathcal{L}}=\left\|s_{n}\right\|_{p 1}^{*}$ for each $n$. Since $\left\|s_{n}\right\|_{p 1}^{*} \leqslant\|f\|_{p 1}^{*}$ for all $n$, we must have $\|f\|_{\mathcal{L}}<\infty$.

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