Title
Cluster Algebras and Integrable Systems

Author
Williams, Harold

Publication Date
2014

Cluster Algebras and Integrable Systems

by

Harold Matthew Williams

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Nicolai Reshetikhin, Chair
Professor Lauren Williams
Professor John Littlejohn

Spring 2014

Cluster Algebras and Integrable Systems

Abstract

Cluster Algebras and Integrable Systems

by

Harold Matthew Williams

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Nicolai Reshetikhin, Chair

We present a series of results at the interface of cluster algebras and integrable systems, discussing various connections to the broader world of representation theory, geometry, and mathematical physics.

In chapter 3 we develop a rigorous theory of Poisson-Lie structures on ind-algebraic groups and treat the case of symmetrizable Kac-Moody groups within this framework. We use this as a setting for the construction of integrable systems on Hamiltonian reductions of symplectic leaves of affine Lie groups, providing generalizations of the relativistic periodic Toda chain to all affine types.

In chapter 4 we formulate and prove a precise relationship between the Chamber Ansatz of [FZ99] and the general phenomenon of duality between cluster varieties. We also extend the construction of cluster structures on double Bruhat cells of algebraic groups to the setting of symmetrizable Kac-Moody groups, in particular encompassing the examples considered in chapter 3.

In chapter 5 we realize the cluster structures associated with Q-systems as amalgamations of those on double Bruhat cells of simple algebraic groups. We use this to identify Q-system dynamics with those of a factorization mapping, thus deducing their integrability in a uniform way for various Dynkin types, and relate them to the Fomin-Zelevinsky twist automorphism. In the process we also provide cluster realizations of twisted Q-systems.

In chapter 6 we identify the Hamiltonians of the open quadratic Toda system (equivalently the conserved quantities of the Q-systems studied in chapter 5) as cluster characters, certain generating functions of Euler characteristics of quiver Grassmannians. Heuristically this means the Hamiltonians should be interpreted as generalized canonical basis elements, and we explain how such an expression is predicted by the appearance of the relevant cluster structures in supersymmetric gauge theory.

This dissertation is for my parents, David and Cathy Williams. I find it completely unjustifiable that I should have had opportunity to be raised by two such wonderful and supportive human beings. My greatest hope for this dissertation is that it makes them happy to see the product of me getting to do something I love, a privilege I owe entirely to them.

Contents

Acknowledgments

I owe a tremendous debt to Kolya Reshetikhin for his mentorship and support during the past several years. While I've learned mathematics from many, many people, most of what I've learned about how to be a mathematician is due to him. Moreover, I thank him for suggesting what would turn out to be such a fertile research direction. This dissertation would not exist without discussions with and comments from many other mathematicians, too numerous to list here. But certainly such a list would include Vladimir Fock, Lauren Williams, Philippe Di Francesco, Rinat Kedem, Bernard Leclerc, Andy Neitzke, Christof Geiss, the late Andrei Zelevinsky, Michael Gekhtman, Tomoki Nakanishi, Theo Johnson-Freyd, Pablo Solis, and Qi You. I would also like to thank Rubina Kwon for her support and above all patience during the preparation of this dissertation. While working on the results presented here I received generous financial support from NSF grants DMS-12011391, DMS-0901431, and DMS-0943745, and the Centre for Quantum Geometry of Moduli Spaces at Aarhus University.

Chapter 1

Introduction and Overview

The broad theme of this dissertation is the interplay between cluster algebras and integrable systems within the larger context of representation theory, geometry, and mathematical physics.

Cluster algebras emerged around the turn of the century as abstractions of combinatorics arising in the theory of canonical bases [FZ02]. They were quickly discovered both to possess a deep theory of their own and to arise in many unanticipated mathematical and physical contexts, from representation theory [Lec10] and total positivity [Fom10] to the geometry of moduli spaces [FG06b] and quantum field theory [CNV10].

Integrable systems on the other hand have a long history in mathematics and physics, dating back to the 19th century. An integrable system is essentially a Hamiltonian system with maximal symmetry, or more precisely a Poisson manifold with a maximal collection of Poisson-commuting functions. The position of integrable systems in modern mathematics is largely characterized by the fact that their symmetry is often an expression of some deeper underlying structure, typically geometric, representation-theoretic, or combinatorial in nature. It is from this point of view that the connection between integrable systems and cluster algebras seems most natural, since cluster algebras themselves usually reflect some larger geometric or combinatorial structure.

In chapter 2 we collect some necessary background material, mostly on Kac-Moody groups and cluster algebras. Informally, a cluster structure on a variety is an infinite family of toric charts with distinguished coordinates, and transition functions of a specific form [FZ02]. To each coordinatized chart (called a cluster) is associated a skew-symmetrizable "exchange" matrix, which encodes the transition functions (called cluster transformations) to another cluster (which we say is obtained by mutation). An explicit rule produces the new exchange matrix from the original one, so the mutation process can be iterated indefinitely, recovering the entire infinite family of clusters. The special coordinate functions on each chart are called cluster variables; the set of all cluster variables is a linearly independent subset of the coordinate ring of the variety, endowing it with an abstraction of (a subset of) a canonical basis.

Chapter 3 is concerned with the development of a rigorous theory of Poisson-Lie structures on ind-algebraic groups. In particular we treat the standard Poisson structure on a symmetrizable Kac-Moody group. We use this as a setting for the construction of integrable systems on Hamiltonian reductions of symplectic leaves of affine Lie groups, providing generalizations of the relativistic periodic Toda chain to all affine types.

The symplectic leaves of a symmetrizable Kac-Moody group are classified by its double Bruhat cells. In Chapter 4 we extend the construction of cluster structures on double Bruhat cells of algebraic groups to this setting. We also formulate and prove a precise relationship between the Chamber Ansatz of [FZ99] and the general phenomenon of duality between cluster varieties. Roughly speaking, we explain how the formula for the Chamber Ansatz is a consequence of the presence of two dual cluster structures on the simply-connected and adjoint forms of a double Bruhat cell, explaining the relationship between the approaches of [FG06a] and [BFZ05].

In chapter 5 we turn to Q-systems, certain recurrence relations arising in the representation theory of quantum loop algebras. In [Ked08; DK09] these were discovered to be expressible as sequences of cluster transformations. We prove that the relevant cluster structures are in fact amalgamations of those on Coxeter double Bruhat cells of simple algebraic groups. We use this to identify Q-system dynamics with those of factorization mappings, deducing their integrability in a uniform way for various Dynkin types, and relate them to the Fomin-Zelevinsky twist automorphism. In the process we also provide cluster realizations of twisted Q-systems.

Finally, in chapter 6 we identify the conserved quantities of the A_n Q-systems (equivalently the Hamiltonians of the open quadratic Toda system) as cluster characters, certain generating functions of Euler characteristics of quiver Grassmannians. Heuristically this means the Hamiltonians should be interpreted as generalized canonical basis elements, and we explain how such an expression is predicted by the appearance of the relevant cluster structures in supersymmetric gauge theory. In particular, these cluster structures also coincide with that those on the moduli spaces of irregular local systems associated with the Seiberg-Witten geometry of pure $\mathcal{N} = 2$ $SU(N)$ Yang-Mills theory.

The results of chapters 3 and 4 are based on [Wil13b; Wil13a], respectively.

Chapter 2

Background on Lie Theory and Cluster Algebras

In this chapter we collect the essential background material on Lie theory (especially Kac-Moody groups) and cluster algebras that will be required later. The material is mostly standard, and references are given throughout. The only minor exceptions are some statements such as Proposition 2.1.21 which are straightforward generalizations to the Kac-Moody case of known statements about simple algebraic groups.

2.1 Lie Theory and Kac-Moody Groups

Kac-Moody Algebras

We briefly recall the theory of Kac-Moody algebras [Kac94]. A generalized Cartan matrix C is an $r \times r$ integer matrix such that

1. $C_{ii} = 2$ for all $1 \leq i \leq r$
2. $C_{ij} \leq 0$ for $i \neq j$
3. $C_{ij} = 0$ if and only if $C_{ji} = 0$.

We will assume throughout that C is symmetrizable; that is, there exist positive integers $d_1, \ldots, d_r$ such that $d_i C_{ij} = d_j C_{ji}$ for all $1 \leq i, j \leq r$. To the matrix C is associated a Lie algebra $\mathfrak{g} := \mathfrak{g}(C)$. The Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ contains simple coroots $\{\alpha_1^\vee, \ldots, \alpha_r^\vee\}$, its dual contains simple roots $\{\alpha_1, \ldots, \alpha_r\}$, and these satisfy $\langle \alpha_j | \alpha_i^\vee \rangle = C_{ij}$. The dimension of $\mathfrak{h}$, which we denote throughout by $\widetilde{r}$, is equal to $2r - \operatorname{rank}(C)$.

The algebra $\mathfrak{g}$ is generated by $\mathfrak{h}$ and the Chevalley generators $\{e_1, f_1, \ldots, e_r, f_r\}$, subject to the relations

1. $[h, h'] = 0$ for all $h, h' \in \mathfrak{h}$

2. $[h, e_i] = \langle \alpha_i | h \rangle e_i$
3. $[h, f_i] = -\langle \alpha_i | h \rangle f_i$
4. $[e_i, f_i] = \alpha_i^\vee$
5. $[e_i, f_j] = ad(e_i)^{1-C_{ij}} e_j = ad(f_i)^{1-C_{ij}} f_j = 0$ for all $i \neq j$.

The roots of $\mathfrak{g}$ are the elements $\alpha \in \mathfrak{h}^*$ such that

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [h, X] = \langle \alpha | h \rangle X \text{ for all } h \in \mathfrak{h}\}$$

is nonzero. Any nonzero root is a sum of simple roots with either all positive or all negative integer coefficients, and we say it is a positive or negative root accordingly. We then have subalgebras

$$\mathfrak{n}_+ = \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha, \quad \mathfrak{n}_- = \bigoplus_{\alpha < 0} \mathfrak{g}_\alpha.$$

If $\mathfrak{g}'$ denotes the derived subalgebra of $\mathfrak{g}$ and $\mathfrak{h}' = \bigoplus_{i=1}^r \mathbb{C}\alpha_i^\vee$, then we have vector space decompositions

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \quad \mathfrak{g}' = \mathfrak{n}_- \oplus \mathfrak{h}' \oplus \mathfrak{n}_+.$$

The Weyl group W is the subgroup of $\mathrm{Aut}(\mathfrak{h}^*)$ generated by the simple reflections

$$s_i : \beta \mapsto \beta - \langle \beta | \alpha_i^\vee \rangle \alpha_i.$$

A nonzero root is said to be real if it is conjugate to a simple root under W, and imaginary otherwise. A reduced word for an element of W is an expression $w = s_{i_1} \cdots s_{i_n}$ such that n is as small as possible; the length $\ell(w)$ is then defined as the length of such a reduced word.

We fix a complex algebraic torus H with Lie algebra $\mathfrak{h}$, which in the following section will be the Cartan subgroup of the group associated with $\mathfrak{g}$. The integral weight lattice $P := \mathrm{Hom}(H, \mathbb{C}^*)$ can be regarded as a sublattice of $\mathfrak{h}^*$, with

$$\langle \omega | \alpha_i^\vee \rangle \in \mathbb{Z}$$

for all $\omega \in P$ and all simple coroots $\alpha_i^\vee$. We fix once and for all a basis $\{\omega_1, \ldots, \omega_{\widetilde{r}}\}$ of P, the *fundamental weights*, such that

$$\langle \omega_j | \alpha_i^\vee \rangle = \delta_{i,j}, \quad 1 \leq i \leq r, \quad 1 \leq j \leq \widetilde{r}.$$

The choice of fundamental weights lets us uniquely define C_{ij} for $r \leq i \leq \widetilde{r}$ by the requirement that

$$\alpha_j = \sum_{1 \leq i \leq \widetilde{r}} C_{ij} \omega_i. \tag{2.1.1}$$

Given $a \in H$, we will denote the value of the character $\lambda \in P$ at a as a^λ. Conversely, given $t \in \mathbb{C}^*$ and a cocharacter $\lambda^\vee \in \mathrm{Hom}(\mathbb{C}^*, H)$, we write $t^{\lambda^\vee}$ for the corresponding element of H. Having fixed the basis $\omega_1, \ldots, \omega_{\widetilde{r}}$ of P, we have a corresponding dual basis of the cocharacter lattice $\mathrm{Hom}(\mathbb{C}^*, H)$. We denote its elements by $\alpha_1^\vee, \ldots, \alpha_{\widetilde{r}}^\vee$, since for $i < r$ these are just the coroots of G.

The set of dominant weights is $P_+ := \{\lambda \in P : \langle \lambda | \alpha_i^\vee \rangle \geq 0 \text{ for all } 1 \leq i \leq r\}$. For each $\lambda \in P_+$ there is an irreducible $\mathfrak{g}$-representation $L(\lambda)$ with highest weight λ, unique up to isomorphism. The representation $L(\lambda)$ is the direct sum of finite-dimensional $\mathfrak{h}$-weight spaces, and its graded dual $L(\lambda)^\vee$ is an irreducible lowest-weight representation.

Let σ be the involution of $\mathfrak{g}$ determined by

$$\sigma(h) = -h \text{ for all } h \in H, \quad \sigma(e_i) = -f_i, \quad \sigma(f_i) = -e_i, \tag{2.1.2}$$

and let $\rho_\lambda : \mathfrak{g} \to \operatorname{End} L(\lambda)$ be the map defining the action of $\mathfrak{g}$ on $L(\lambda)$. Then there is a $\mathfrak{g}$-module isomorphism between $L(\lambda)^\vee$ and the representation whose underlying vector space is $L(\lambda)$ and whose $\mathfrak{g}$-action is given by $\rho_\lambda \circ \sigma$. In particular this isomorphism yields a nondegenerate symmetric bilinear form

$$L(\lambda) \otimes L(\lambda) \cong L(\lambda)^\vee \otimes L(\lambda) \to \mathbb{C}.$$

We say $\mathfrak{g}(C)$ is of finite type if C is positive definite, and affine type if C is positive semidefinite. In the former case it is a finite-dimensional semisimple Lie algebra, while in the latter it admits an alternative description in terms of loop algebras.

More precisely, let $\mathfrak{g}(C)$ be a semisimple Lie algebra with Cartan matrix C. Its loop algebra $L\mathfrak{g} := \mathfrak{g}(C) \otimes \mathbb{C}[z^{\pm 1}]$ has a universal central extension $\widetilde{L\mathfrak{g}} := \mathbb{C}c \oplus L\mathfrak{g}$ with bracket

$$[Xz^m + Ac, Yz^n + Bc] = [X, Y]z^{m+n} + \delta_{m+n,0}\langle X, Y \rangle c.$$

The action of $\frac{d}{dz}$ on $L\mathfrak{g}$ by derivations extends to an action on $\widetilde{L\mathfrak{g}}$, so we have the semidirect product $\widehat{L\mathfrak{g}} := \mathbb{C}\frac{d}{dz} \ltimes \widetilde{L\mathfrak{g}}$. There is an extended Cartan matrix $\widetilde{C}$ such that $\widehat{L\mathfrak{g}} \cong \mathfrak{g}(\widetilde{C})$ and $\widetilde{L\mathfrak{g}} \cong \mathfrak{g}'(\widetilde{C})$. To form $\widetilde{C}$ we adjoin an extra row and column to C by setting

$$C_{0,0} = 2, \quad C_{k,0} = -\theta(\alpha_k^\vee), \quad \text{and} \quad C_{0,i} = -\alpha_i(\alpha_\theta^\vee).$$

Here $\theta = \sum_{i=1}^r \theta_i \alpha_i$ is the highest root of $\mathfrak{g}(C)$, and we will always normalize the form on $\mathfrak{g}(C)$ so that $\langle \theta, \theta \rangle = 2$ (to simplify later formulas we will also use the convention $\theta_0 = 1$). Note that we index the simple roots of a general Kac-Moody algebra by $\{1, \ldots, r\}$, while we index affine simple roots by $\{0, \ldots, r\}$. Every affine Kac-Moody algebra is either of the form $\widehat{L\mathfrak{g}}$ or a twisted version thereof; for simplicity we will always take "affine" to mean "untwisted affine" unless explicitly stated.

Kac-Moody Groups and Double Bruhat Cells

To a generalized Cartan matrix C we may also associate a group G, which is a simply-connected complex algebraic group when C is positive-definite [KP83a; Kum02]. In general

G is an ind-algebraic group, and shares many important properties with the simple algebraic groups, in particular a Bruhat decomposition and generalized Gaussian factorization.

For each real root α, G contains a one-parameter subgroup $x_\alpha(t)$, and G is generated by these together with the Cartan subgroup H (for simple roots, we will write $x_{\pm i}(t) := x_{\pm\alpha_i}(t)$). We denote the subgroups generated by the positive and negative real root subgroups by N_+ and N_-, respectively, and we also have the positive and negative Borel subgroups $B_\pm := H \ltimes N_\pm$.

For each $1 \leq i \leq r$ there is a unique embedding $\varphi_i : SL_2 \to G$ such that

$$\varphi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = t^{\alpha_i^\vee}, \quad \varphi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = x_i(t), \quad \varphi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = x_{-i}(t).$$

The Weyl group W is isomorphic with $N_G(H)/H$, where $N_G(H)$ is the normalizer of H in G. The simple reflections s_i have representatives in G of the form

$$\overline{s_i} = x_i(-1)x_{-i}(1)x_i(-1) = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.1.3}$$

$$\overline{\overline{s_i}} = x_i(1)x_{-i}(-1)x_i(1) = \varphi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.1.4}$$

In particular, for any $w \in W$ we have well-defined representatives

$$\overline{w} = \overline{s_{i_1}} \cdots \overline{s_{i_n}}, \quad \overline{\overline{w}} = \overline{\overline{s_{i_1}}} \cdots \overline{\overline{s_{i_n}}},$$

where $s_{i_1} \cdots s_{i_n}$ is any reduced word for w.

Recall that an ind-variety X is the union of an increasing sequence of finite-dimensional varieties X_n whose inclusions $X_n \hookrightarrow X_{n+1}$ are closed embeddings [Sha81]. We say a map $X \xrightarrow{\phi} Y$ of ind-varieties is regular if for all $i \in \mathbb{N}$ there exists an $n(i)$ such that $\phi(X_i) \subset Y_{n(i)}$ and the restrictions $X_i \xrightarrow{\phi|_{X_i}} Y_{n(i)}$ are regular. If the X_n are affine, the coordinate ring of X is

$$\mathbb{C}[X] = \varprojlim \mathbb{C}[X_n],$$

topologized as an inverse limit of discrete vector spaces; regular maps of affine ind-varieties induce continuous homomorphisms between their coordinate rings. We can also form products of ind-varieties in the obvious way.

Definition 2.1.5. An ind-algebraic group (or ind-group) X is an ind-variety with a regular group operation $X \times X \to X$. □

To define the ind-group structure on G, consider the integrable $\mathfrak{g}$-representation

$$V = \bigoplus_{i=1}^{\widetilde{r}} (L(\omega_i) \oplus L(\omega_i)^\vee).$$

The group G acts on integrable highest weight representations of $\mathfrak{g}$ and their restricted duals, hence on V. If v_i and $v_i^\vee$ are the highest and lowest weight vectors of $L(\omega_i)$ and $L(\omega_i)^\vee$, respectively, the map $g \mapsto g \cdot \sum_{i=1}^{\widetilde{r}}(v_i + v_i^\vee)$ embeds G injectively into V. We may filter V by finite direct sums of its weight spaces, and the intersections of G with these are closed subvarieties that define an ind-group structure on G [Kum02, p. 7.4.14]. The subgroups H, $N_\pm$, and $B_\pm$ are then closed subgroups.

Proposition 2.1.6. ([Kum02, pp. 6.5.8, 7.4.11]) *The multiplication map $N_- \times H \times N_+ \to G$ is a biregular isomorphism onto an open subvariety G_0. Thus for any $g \in G_0$ we may write*

$$g = [g]_-[g]_0[g]_+$$

for some unique $[g]_\pm \in N_\pm$ and $[g]_0 \in H$. Moreover, the maps

$$G_0 \to N_\pm \text{ (resp. } H), \quad g \mapsto [g]_\pm \text{ (resp. } [g]_0)$$

are regular.

Proposition 2.1.7. ([GLS11, p. 7.2]) *We have*

$$G_0 = \{x \in G | \Delta^{\omega_j}(x) \neq 0 \text{ for all } 1 \leq j \leq \widetilde{r}\},$$

where the Δ^{ω_j} are the principal minors of Definition 2.1.19.

Proposition 2.1.8. ([Kum02, p. 7.4.2]) *The group G has positive and negative Bruhat decompositions*

$$G = \bigsqcup_{w \in W} B_+ \dot{w} B_+ = \bigsqcup_{w \in W} B_- \dot{w} B_-,$$

where $\dot{w}$ is any representative of w in G.

In particular, G is a disjoint union of the *double Bruhat cells*

$$G^{u,v} := B_+ \dot{u} B_+ \cap B_- \dot{v} B_-.$$

To obtain a more explicit description of the double Bruhat cells, we introduce the $\ell(w)$-dimensional unipotent subgroups

$$N_+(w) := N_+ \cap \dot{w} N_- \dot{w}^{-1}, \quad N_-(w) := N_- \cap \dot{w}^{-1} N_+ \dot{w}$$

associated to any $w \in W$. These have complementary infinite-dimensional subgroups

$$N'_+(w) := N_+ \cap \dot{w} N_+ \dot{w}^{-1}, \quad N'_-(w) := N_- \cap \dot{w}^{-1} N_- \dot{w}.$$

Proposition 2.1.9. ([Kum02, p. 6.1.3]) *For any $w \in W$, the multiplication maps*

$$N_\pm(w) \times N'_\pm(w) \to N_\pm$$

are biregular isomorphisms.

The Bruhat decomposition then admits the following refinement:

Corollary 2.1.10. *The natural maps*

$$N_+(w) \to N_+(w)\dot{w}B_+/B_+, \quad N_-(w) \to B_-\backslash B_-\dot{w}N_-(w)$$

are biregular isomorphisms. In particular, the Bruhat cells can be written as

$$B_+\dot{w}B_+ = N_+(w)\dot{w}B_+, \quad B_-\dot{w}B_- = B_-\dot{w}N_-(w).$$

Corollary 2.1.11. *For any $x \in B_+\dot{w}B_+$, we have $\dot{w}^{-1}x \in G_0$. Then*

$$\pi_+(x) := \dot{w}[\dot{w}^{-1}x]_-\dot{w}^{-1} \in N_+(w)$$

and $x = \pi_+(x)\dot{w}b_+$ for some $b_+ \in B_+$. Similarly, if $x \in B_-\dot{w}B_-$, then $x\dot{w}^{-1} \in G_0$,

$$\pi_-(x) := \dot{w}^{-1}[x\dot{w}^{-1}]_+\dot{w} \in N_-(w),$$

and $x = b_-\dot{w}\pi_-(x)$ for some $b_- \in B_-$.

Proposition 2.1.12. *The map*

$$G^{u,v} \to N_+(u) \times N_-(v) \times H, \quad x \mapsto (\pi_+(x), \pi_-(x), [\overline{u}^{-1}x]_0)$$

provides an isomorphism of $G^{u,v}$ with the open set

$$\{(n_+, n_-, h) | \overline{v}n_-n_+^{-1}\overline{u}^{-1} \in G_0\} \subset N_+(u) \times N_-(v) \times H.$$

In particular, $G^{u,v}$ is a rational affine variety of dimension $\ell(u) + \ell(v) + \widetilde{r}$.

Proof. By an elementary calculation one checks that

$$(n_+, n_-, h) \mapsto n_+\overline{u}h[\overline{v}n_-n_+^{-1}\overline{u}^{-1}]_+$$

provides the inverse map. By Proposition 2.1.7 the given open set is the nonvanishing locus of the pullback of $\prod_{1\leq j\leq \widetilde{r}} \Delta^{\omega_j} \in \mathbb{C}[G]$ along the regular map

$$(n_+, n_-, h) \mapsto \overline{v}n_-n_+^{-1}\overline{u}^{-1}.$$

The last statement then follows since $N_+(u) \times N_-(v) \times H$ is an open subvariety of $\mathbb{A}^{\ell(u)+\ell(v)+\widetilde{r}}$. ∎

For each simple root α, G' has a corresponding SL_2 subgroup G_α generated by $x_{\pm\alpha}(t)$. In Theorem 3.2.10 we will use the following observation:

Proposition 2.1.13. *G' is generated by the simple root SL_2 subgroups G_α.*[1]

[1]Since G' is infinite-dimensional it does not suffice to observe that the Lie algebras of the G_α together generate $\mathfrak{g}$. For example, the Lie algebra of $N_+ \subset \widetilde{LSL_2}$ is generated by the two simple positive root spaces, yet N_+ is not generated by *any* proper subcollection of the 1-parameter positive root subgroups [KP83a].

Proof. It suffices to show that the real root 1-parameter subgroups lie in the subgroup generated by the G_α, since these generate G'. By definition a real root β is one of the form $w(\alpha)$ for some simple root α and $w \in W$. Then we can write the subgroup $x_\beta(t)$ as $\dot{w}x_\alpha(t)\dot{w}^{-1}$ for any representative $\dot{w}$ of w in G'. But by eq. (2.1.3) this can be written in terms of simple root 1-parameter subgroups. □

Remark 2.1.14. We could also consider a completed version of the Kac-Moody group G, as in [Kum02, p. 6.1.16]. In the affine case, this corresponds to using the formal loop group rather than the polynomial loop group. However, only the smaller group G has a double Bruhat decomposition, since the completed group does not have a Bruhat decomposition with respect to B_-. Furthermore, the formal loop group does not admit evaluation representations, so it is not the right object to consider in the context of the integrable systems constructed in Section 3.4. □

Affine Kac-Moody Groups

In affine type, Kac-Moody groups admit an alternative description as central extensions of loop groups. Let C be a finite type Cartan matrix and G the corresponding simply connected complex algebraic group with Lie algebra $\mathfrak{g}$. To avoid conflating this group with the associated infinite-dimensional group, we will generally use $\mathcal{G}$ rather than G to denote the Kac-Moody group of the extended matrix $\widetilde{C}$ (likewise $\mathcal{U}_\pm$ and $\mathcal{B}_\pm$ will denote the unipotent and Borel subgroups of $\mathcal{G}$). If $LG := G(\mathbb{C}[z^{\pm 1}])$ is the group of regular maps from $\mathbb{C}^*$ to G, there is a universal central extension

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \widetilde{LG} \longrightarrow LG \longrightarrow 1$$

and an isomorphism $\mathcal{G}' \cong \widetilde{LG}$. The rotation action of $\mathbb{C}^*$ on LG extends to $\widetilde{LG}$, and $\mathcal{G}$ is isomorphic with the semidirect product $\mathbb{C}^* \ltimes \widetilde{LG}$ [Kum02, p. 13.2.9].

The central extension splits canonically over the subgroups $G(\mathbb{C}[z])$ and $G(\mathbb{C}[z^{-1}])$ of LG, so we have $\mathbb{C}^* \times G(\mathbb{C}[z]), \mathbb{C}^* \times G(\mathbb{C}[z^{-1}]) \subset \widehat{LG}$. Evaluation at $z = 0$ gives a homomorphism $\mathbb{C}^* \times G(\mathbb{C}[z]) \to G$, and $\mathcal{B}_+$ is the preimage of the positive Borel subgroup of G. Similarly $\mathcal{B}_- \subset \mathbb{C}^* \times G(\mathbb{C}[z^{-1}])$ is the preimage of the negative Borel subgroup of G under evaluation at $z = \infty$ [Kum02, p. 13.2.2]. The Cartan subgroup $\widetilde{H}$ of $\widetilde{LG}$ splits as the product of the center of $\widetilde{LG}$ and the Cartan subgroup H of G, embedded as constant maps (we write the Cartan subgroup of an affine Kac-Moody group as $\widetilde{H}$ to distinguish it from the Cartan subgroup of G).

A faithful n-dimensional G-representation yields a closed embedding $G \hookrightarrow \mathrm{Mat}_{n\times n}$, hence an inclusion $LG \hookrightarrow \mathrm{Mat}_{n\times n} \otimes \mathbb{C}[z^{\pm 1}]$. The subsets

$$LG_m := \left\{ A(z) = \sum_{k=-m}^{m} A_{ij}^k z^k : A(z) \in LG \right\} \subset \mathrm{Mat}_{n\times n} \otimes \mathbb{C}[z^{\pm 1}]$$

are affine varieties, and the natural maps $LG_m \hookrightarrow LG_{m+1}$ are closed embeddings. This defines an ind-variety structure on LG, which is independent of the choice of representation.

It is clear that under this ind-variety structure the evaluation maps $LG \to G$ are regular; the same cannot be said of the ind-variety structure LG inherits as a Kac-Moody group. Our discussion of double Bruhat cells is based on the latter structure, but for integrable systems we will consider functions pulled back along evaluation maps. Thus to ensure these yield regular functions on double Bruhat cells we must verify the compatibility of the two ind-variety structures. This is essentially well-known, but for convenience we include a proof. We use LG_{pol} to refer to LG with the ind-variety structure described in this section, and LG_{KM} to refer to the ind-variety structure described in Section 2.1.

Proposition 2.1.15. *The ind-variety structures LG_{pol} and LG_{KM} are equivalent. That is, the identity map is a biregular isomorphism between them.*

Proof. We first show that the induced structures $(\mathcal{U}_\pm)_{pol}$ and $(\mathcal{U}_\pm)_{KM}$ are equivalent (note that $\mathcal{U}_\pm$ is manifestly a closed subgroup of LG_{pol}). If $w_\circ$ is the longest element of the Weyl group of G, $\mathcal{U}'_-(w_\circ)$ and $\mathcal{U}_-(w_\circ)$ are closed subgroups of LG_{pol}, and Proposition 2.1.9 is clearly true for $(\mathcal{U}_\pm)_{pol}$. Thus showing the claim for $\mathcal{U}_\pm$ reduces to showing it for $\mathcal{U}'_\pm(w_\circ)$.

We now invoke the corresponding theorem about the affine Grassmannian $X := LG/G(\mathbb{C}[z]) = \widetilde{LG}/\mathcal{P}$, where $\mathcal{P} \subset \widetilde{LG}$ is the parabolic subgroup corresponding to the subset $\{\alpha_1, \dots, \alpha_r\} \subset \{\alpha_0, \dots, \alpha_r\}$ of simple affine roots. Like LG, X has two equivalent but a priori distinct ind-variety structures [Kum02, p. 13.2.18]. First, it is a disjoint union of Schubert cells $X_w = \mathcal{B}_+ \dot{w} \mathcal{P}/\mathcal{P}$, and is filtered by finite-dimensional projective varieties

$$X_n = \bigcup_{\ell(w) \leq n} X_w.$$

Alternatively, X can be written as an increasing union of closed subvarieties of finite-dimensional Grassmannians. We refer the reader to [Kum02, p. 13.2.15] for the precise construction, noting only that it is clear that LG_{pol} acts regularly on X. In particular, $\mathcal{U}'_-(w_\circ)_{pol}$ acts faithfully on the dense open subset of $\widetilde{LG}_0/\mathcal{P}$, and $\mathcal{U}'_-(w_\circ)_{pol} \cong \mathcal{G}_0/\mathcal{P} \cong \mathcal{U}'_-(w_\circ)_{KM}$. The claim for $\mathcal{U}_+$ follows similarly.

In particular, the two ind-variety structures on $\mathcal{U}_- \times H \times \mathcal{U}_+$ coincide. By Proposition 2.1.6 this is isomorphic with an open subset $LG_0 \subset LG_{KM}$. But it is clear that LG_0 is open in LG_{pol}, and that Proposition 2.1.6 holds for LG_{pol}. Thus the two ind-variety structures on $\widetilde{LG}_0$ are equivalent, and since the translates of $\widetilde{LG}_0$ form an open cover of LG the proposition follows.

□

Remark 2.1.16. All but finitely many of the varieties used in either definition of the ind-variety structure are singular, and unavoidably so: in [FGT08] it was shown that X and LG cannot be written locally as an increasing union of smooth subvarieties. Thus LG is not a complex manifold, even though we have the following property: for any $g \in LG$ the canonical map

$$\varprojlim \mathrm{Sym}^*(m_i(g)/m_i(g)^2) \to \varprojlim \bigoplus_{n=0}^{\infty} m_i(g)^n/m_i(g)^{n+1}$$

is an isomorphism, where $m_i(g) \subset \mathbb{C}[LG_i]$ is the vanishing ideal of g [Kum02, p. 4.3.7]. □

Strongly Regular Functions and Generalized Minors

When G is infinite-dimensional, there are several natural algebras of functions one may consider on it. Being an ind-variety, G is the increasing union of finite-dimensional varieties, and the inverse limit of their coordinate rings is a complete topological algebra of functions on G. For our purposes it is more practical to consider a proper subalgebra of this, the ring of strongly regular functions.

Given a dominant integral weight $\lambda \in P_+$ we have an irreducible highest-weight $\mathfrak{g}$-module $L(\lambda)$ and its graded dual $L(\lambda)^\vee$, both of which integrate to representations of G. Recall from Section 2.1 that $L(\lambda)$ is equipped with a nondegenerate bilinear form. For each $v_1, v_2 \in L(\lambda)$, we use this to define a function on G by taking

$$g \mapsto \langle v_1 | g \cdot v_2 \rangle.$$

We regard this as a matrix coefficient of the image of g in End $L(\lambda)$.

Definition 2.1.17. ([KP83b]) The algebra of *strongly regular* functions, which we will denote simply by $\mathbb{C}[G]$, is the algebra generated by all such matrix coefficients of irreducible highest-weight representations. □

Proposition 2.1.18. ([KP83b, Theorem 1]) *The algebra $\mathbb{C}[G]$ is closed under the $G \times G$ action*

$$((g_1, g_2) \cdot f)(g) = f(g_1^{-1} g g_2).$$

Furthermore, as $G \times G$-modules there is an isomorphism

$$\mathbb{C}[G] \cong \bigoplus_{\lambda \in P_+} (L(\lambda)^\vee \otimes L(\lambda)).$$

Definition 2.1.19. Given a fundamental weight ω_i and a pair $w, w' \in W$, the *generalized minor* $\Delta^{\omega_i}_{w,w'}$ is the matrix coefficient

$$g \mapsto \langle \overline{w} v_{\omega_i} | g \overline{w'} v_{\omega_i} \rangle,$$

where v_{ω_i} is a highest-weight vector of $L(\omega_i)$. The *principal minor* $\Delta^{\omega_i} := \Delta^{\omega_i}_{e,e}$ is characterized by the fact that on the dense open set G_0,

$$\Delta^{\omega_i} : g = [g]_-[g]_0[g]_+ \mapsto [g]_0^{\omega_i}.$$

The other minors can then be expressed in terms of Δ^{ω_i} by

$$\Delta^{\omega_i}_{w,w'}(g) = \Delta^{\omega_i}(\overline{w}^{-1} g \overline{w'}).$$

□

Proposition 2.1.20. *The algebra $\mathbb{C}[G]$ is a unique factorization domain in which the generalized minors are prime. Two minors $\Delta^{\omega_j}_{u,v}$ and $\Delta^{\omega_i}_{u',v'}$ are relatively prime unless $u\omega_j = u'\omega_i$ and $v\omega_j = v'\omega_i$.*

Proof. That $\mathbb{C}[G]$ is a unique factorization domain is Theorem 3 in [KP83b], and the fact that the principal minors are prime is contained in the proof thereof. Since an arbitrary generalized minor only differs from a principal minor by an automorphism of $\mathbb{C}[G]$, it is also prime.

If $u\omega_j = u'\omega_i$ and $v\omega_j = v'\omega_i$, it is clear from Definition 2.1.19 that the generalized minors $\Delta^{\omega_j}_{u,v}$ and $\Delta^{\omega_i}_{u',v'}$ differ by a scalar multiple. On the other hand, if $u\omega_j \neq u'\omega_i$ or $v\omega_j \neq v'\omega_i$, it is clear from the decomposition in Proposition 3.2.12 that $\Delta^{\omega_j}_{u,v}$ and $\Delta^{\omega_i}_{u',v'}$ are linearly independent. But the only units of $\mathbb{C}[G]$ are the constant functions [KP83b, p. 2.1c], so the proposition follows. □

The identity established in the next proposition plays a key role in the cluster algebras constructed on double Bruhat cells, providing the prototypical example of an exchange relation. It is a direct generalization of [FZ99, p. 1.17], which in turn generalizes several classical determinantal identities. The proof below follows that in [FZ99, p. 1.17], though when the Cartan matrix does not have full rank and $r < \widetilde{r} = \dim(H)$ it is important to use eq. (2.1.1) in interpreting the right-hand side of the identity.

Proposition 2.1.21. *Suppose $u, v \in W$ satisfy $\ell(us_i) > \ell(u)$ and $\ell(vs_i) > \ell(v)$ for some $1 \leq i \leq r$. Then*

$$\Delta^{\omega_i}_{u,v}\Delta^{\omega_i}_{us_i,vs_i} = \Delta^{\omega_i}_{us_i,v}\Delta^{\omega_i}_{u,vs_i} + \prod_{\substack{1\leq k\leq \widetilde{r}\\ k\neq i}} (\Delta^{\omega_k}_{u,v})^{-C_{ki}}.$$

Proof. It suffices to consider $u = v = e$. In the case of arbitrary u, v, showing both sides are equal when evaluated at some $x \in G$ is then equivalent to showing both sides take the same value at $\overline{u}^{-1}x\overline{v}$ in the identity case.

Let

$$f_1 = \Delta^{\omega_i}_{e,e}\Delta^{\omega_i}_{s_i,s_i} - \Delta^{\omega_i}_{s_i,e}\Delta^{\omega_i}_{e,s_i}, \quad f_2 = \prod_{\substack{1\leq k\leq \widetilde{r}\\ k\neq i}} (\Delta^{\omega_k}_{e,e})^{-C_{ki}}.$$

We claim that f_1 and f_2 satisfy the following conditions, where we consider $\mathbb{C}[G]$ as a $G \times G$ representation as in Proposition 3.2.12:

1. They are invariant under $N_- \times N_+$.

2. They have weight $(\alpha_i - 2\omega_i, 2\omega_i - \alpha_i)$.

3. They both evaluate to 1 at the identity.

These conditions uniquely determine a function on the dense subset G_0, hence on all of G, so together imply the proposition.

The fact that f_2 satisfies the given conditions is essentially immediate; for (2) we must recall the definition of C_{ij} for $r \leq j \leq \widetilde{r}$ in eq. (2.1.1). Likewise conditions (2) and (3) hold straightforwardly for f_1.

We claim then that f_1 is invariant under right translations by N_+. Clearly it is invariant under right translation by $x_j(t)$ for $j \neq i$ and $t \in \mathbb{C}$, so we need only show that it is invariant under right translations by $x_i(t)$.

It is immediate that $\Delta^{\omega_i}_{e,e}(xx_i(t)) = \Delta^{\omega_i}_{e,e}(x)$ and $\Delta^{\omega_i}_{s,e}(xx_i(t)) = \Delta^{\omega_i}_{s,e}(x)$. We claim further that

$$\Delta^{\omega_i}_{e,s_i}(xx_i(t)) = \Delta^{\omega_i}_{e,s_i}(x) + t\Delta^{\omega_i}_{e,e}(x), \tag{2.1.22}$$

$$\Delta^{\omega_i}_{s_i,s_i}(xx_i(t)) = \Delta^{\omega_i}_{s_i,s_i}(x) + t\Delta^{\omega_i}_{s_i,e}(x). \tag{2.1.23}$$

To see this, first note that for a highest-weight vector v_{ω_i} of $L(\omega_i)$ we have

$$x_i(t)\overline{s_i} \cdot v_{\omega_i} = \overline{s_i} \cdot v_{\omega_i} + tv_{\omega_i}. \tag{2.1.24}$$

This is a simple computation in SL_2 representation theory; when we decompose $L(\omega_i)$ as a $\varphi_i(SL_2)$-representation, v_{ω_i} generates a copy of the standard SL_2-representation. But now eqs. (2.1.22) and (2.1.23) follow immediately in light of Definition 2.1.19, and we conclude that

$$\begin{aligned} f_1(xx_i(t)) &= \Delta^{\omega_i}_{e,e}(x)(\Delta^{\omega_i}_{s_i,s_i}(x) + t\Delta^{\omega_i}_{s_i,e}(x)) - \Delta^{\omega_i}_{s_i,e}(x)(\Delta^{\omega_i}_{e,s_i}(x) + t\Delta^{\omega_i}_{e,e}(x)) \\ &= f_1(x). \end{aligned}$$

One easily checks that $f_1(x) = f_1(\sigma(x^{-1}))$, where σ is the automorphism of G induced from eq. (2.1.2). From this the right N_+-invariance of f_1 implies its left N_--invariance, hence condition (1) indeed holds for f_1. □

2.2 Cluster Algebras

In this section we fix some basic definitions and facts concerning cluster algebras and $\mathcal{X}$-coordinates. More extensive references include [FZ07; FG09; GHK13]. The only nonstandard item is our discussion of amalgamation: while this is usually understood as a gluing operation between seeds [FG06a], we will require self-amalgamations of individual indecomposable seeds.

Definition 2.2.1. (Seeds) A seed Σ consists of:

1. An index set $I = I_f \sqcup I_u$ with a decomposition into frozen and unfrozen indices.

2. An $I \times I$ exchange matrix B with $B_{ij} \in \mathbb{Z}$ unless $i, j \in I_f$.

3. Skew-symmetrizers $d_i \in \mathbb{Z}_{>0}$ such that $B_{ij}d_j = B_{ji}d_i$. □

Definition 2.2.2. (Mutation) For any unfrozen index k the mutation of Σ at k is the seed $\mu_k(\Sigma)$ defined as follows. It has the same index set, frozen and unfrozen subsets, and skew-symmetrizers as Σ. Its exchange matrix $\mu_k(B)$ is given by

$$\mu_k(B)_{ij} = \begin{cases} -B_{ij} & i = k \text{ or } j = k \\ B_{ij} + \frac{1}{2}(|B_{ik}|B_{kj} + B_{ik}|B_{kj}|) & i, j \neq k. \end{cases} \tag{2.2.3}$$ □

Two seeds Σ and Σ' are said to be mutation equivalent if they are related by a finite sequence of mutations. Note that the term seed is often taken to include the additional data of an identification of the corresponding cluster variables with a transcendence basis of a fixed function field.

Definition 2.2.4. (Cluster Variables and $\mathcal{X}$-coordinates) To a seed Σ we associate two Laurent polynomial rings $\mathbb{C}[A_i^{\pm1}]$ and $\mathbb{C}[X_i^{\pm1}]$, whose generators are indexed by I and referred to as cluster variables and $\mathcal{X}$-coordinates, respectively. These are the coordinate rings of two algebraic tori, denoted by $\mathcal{A}_\Sigma$ and $\mathcal{X}_\Sigma$. There is a canonical map $p_\Sigma : \mathcal{A}_\Sigma \to \mathcal{X}_\Sigma$ defined by $p_\Sigma^* X_i = \prod_{j \in I} A_j^{B_{ij}}$. The torus $\mathcal{X}_\Sigma$ has a canonical Poisson structure given by

$$\{X_i, X_j\} = B_{ij} d_j X_i X_j.$$ □

While working over the complex numbers is sufficient for our purposes, it is not essential. Also, what we refer to as $\mathcal{X}$-coordinates are often called Y-variables elsewhere in the literature.

Remark 2.2.5. The tori $\mathcal{A}_\Sigma$ and $\mathcal{X}_\Sigma$ are dual in following sense: the ring $\mathbb{C}[X_i^{\pm1}]$ should be identified with the group ring of the free abelian group $\mathbb{Z}I$ generated by I, and (when B is skew-symmetric) $\mathbb{C}[A_i^{\pm1}]$ should be identified with the group ring of its dual lattice $(\mathbb{Z}I)^*$. In particular, the exchange matrix endows $\mathbb{Z}I$ with a skew-symmetric form, which is the origin of the map p_Σ and the Poisson structure on $\mathcal{X}_\Sigma$. □

Definition 2.2.6. (Cluster Transformations) To each mutation μ_k of seeds is associated a pair of rational maps between the corresponding tori, called cluster transformations and also denoted by μ_k. These satisfy

$$\begin{tikzcd} \mathcal{A}_\Sigma \arrow[r, dashed, "\mu_k"] \arrow[d, "p_\Sigma"] & \mathcal{A}_{\Sigma'} \arrow[d, "p_{\Sigma'}"] \\ \mathcal{X}_\Sigma \arrow[r, dashed, "\mu_k"] & \mathcal{X}_{\Sigma'} \end{tikzcd},$$

where $\Sigma' = \mu_k(\Sigma)$, and are defined explicitly by[2]

$$\mu_k^*(A_i') = \begin{cases} A_i & i \neq k \\ A_k^{-1}\left(\prod_{B_{kj}>0} A_j^{B_{kj}} + \prod_{B_{kj}<0} A_j^{-B_{kj}}\right) & i = k \end{cases} \tag{2.2.7}$$

and

$$\mu_k^*(X_i') = \begin{cases} X_i X_k^{[B_{ik}]_+}(1+X_k)^{-B_{ik}} & i \neq k \\ X_k^{-1} & i = k, \end{cases} \tag{2.2.8}$$

where $[B_{ik}]_+ := \max(0, B_{ik})$. □

The new cluster variables A_i' could also be defined directly as elements of the function field $\mathbb{C}(\mathcal{A}_\Sigma)$, omitting specific mention of the torus $\mathcal{A}_\Sigma'$.

Definition 2.2.9. (Cluster Algebras and $\mathcal{X}$-varieties) The $\mathcal{A}$- and $\mathcal{X}$-spaces $\mathcal{A}_{|\Sigma|}$ and $\mathcal{X}_{|\Sigma|}$ are the schemes obtained from gluing together along cluster transformations all such tori of seeds mutation equivalent to an initial seed Σ. The map p_Σ extends to a map $p_{|\Sigma|}\colon \mathcal{A}_{|\Sigma|} \to \mathcal{X}_{|\Sigma|}$, and the Poisson structure on $\mathcal{X}_\Sigma$ extends to one on $\mathcal{X}_{|\Sigma|}$. The upper cluster algebra $\overline{\mathcal{A}}(\Sigma)$ is the algebra of regular functions on $\mathcal{A}_{|\Sigma|}$, or equivalently

$$\overline{\mathcal{A}}(\Sigma) := \mathbb{C}[\mathcal{A}_{|\Sigma|}] = \bigcap_{\Sigma' \sim \Sigma} \mathbb{C}[\mathcal{A}_{\Sigma'}] \subset \mathbb{C}(\mathcal{A}_{|\Sigma|}).$$

The cluster algebra $\mathcal{A}(\Sigma)$ is the subalgebra of the function field $\mathbb{C}(\mathcal{A}_{|\Sigma|})$ generated by the collection of all cluster variables of seeds mutation equivalent to Σ. □

Although in general the $\mathcal{A}$- and $\mathcal{X}$-spaces associated with a seed can be defined over $\mathbb{Z}$, we will only consider the associated complex schemes in the remainder of the paper. In fact, since the expressions in eqs. (2.2.7) and (2.2.8) are subtraction-free, one can consider the associated $\mathbb{P}$-points of these spaces for any semifield $\mathbb{P}$. This leads in particular to the notion of the positive real part of these spaces, but this will not play a direct role in the present work.

A key property of cluster algebras is the Laurent phenomenon, summarized in the following proposition.

Proposition 2.2.10. ([FZ02, p. 3.1]) *For any seed Σ the cluster algebra $\mathcal{A}(\Sigma)$ is contained in the upper cluster algebra $\overline{\mathcal{A}}(\Sigma)$. In other words, the cluster variables of any seed are Laurent polynomials in the cluster variables of any seed mutation equivalent to it.*

A generic seed is mutation equivalent to infinitely many other seeds. However, the following proposition guarantees that in favorable circumstances an upper cluster algebra is already determined by a finite number of them.

[2]Note that our exchange matrix conventions are transpose to those of, for example, [FZ07].

Proposition 2.2.11. ([BFZ05, p. 1.9]) *Let Σ be a seed such that the submatrix of B formed by its unfrozen rows has full rank. Then*

$$\overline{\mathcal{A}}(\Sigma) = \mathbb{C}[\mathcal{A}_\Sigma] \cap \bigcap_{k \in I_u} \mathbb{C}[\mathcal{A}_{\mu_k(\Sigma)}].$$

In other words, the upper cluster algebra $\overline{\mathcal{A}}(\Sigma)$ only depends on Σ and the seeds obtained from it by a single mutation.

For seeds with frozen variables, the map $p_{|\Sigma|}\colon \mathcal{A}_{|\Sigma|} \to \mathcal{X}_{|\Sigma|}$ admits a family of modifications depending on an $I_f \times I_f$ matrix. This fact is crucial for the quantization of cluster algebras, and in the present context we will it is also essential for understanding the cluster structures associated with double Bruhat cells as in Proposition 4.2.28.

Proposition 2.2.12. *Let M be an $I \times I$ matrix such that $M_{ij} = 0$ unless both i and j are frozen. Let Σ be any seed such that $\widetilde{B} = B + M$ is an integer matrix, and let $p_M : \mathcal{A}_\Sigma \to \mathcal{X}_\Sigma$ be the regular map defined by*

$$p_M^*(X_i) = \prod_{j \in I} A_j^{\widetilde{B}_{ij}}.$$

Then p_M extends to a regular map $p_M : \mathcal{A}_{|\Sigma|} \to \mathcal{X}_{|\Sigma|}$.[3]

Proof. First observe that if Σ' is any seed mutation equivalent to Σ, its exchange matrix B' again has the property that $B' + M$ has integer entries. This follows from the fact that the mutation rules eq. (2.2.3) can only change the exchange matrix entries by integer values. In particular, the formula in the statement of the proposition yields a regular map $p'_M : \mathcal{A}_{\Sigma'} \to \mathcal{X}_{\Sigma'}$ when we replace B by B'.

To check that these descend to a map $\mathcal{A}_{|\Sigma|} \to \mathcal{X}_{|\Sigma|}$, we must verify that they commute with the cluster transformations. That is, if Σ' is obtained from Σ by mutation at k, we want to show that that there is a commutative diagram

$$\begin{tikzcd}
\mathcal{A}_\Sigma \arrow[r, dashed, "\mu_k"] \arrow[d, "p_M"] & \mathcal{A}_{\Sigma'} \arrow[d, "p'_M"] \\
\mathcal{X}_\Sigma \arrow[r, dashed, "\mu_k"] & \mathcal{X}_{\Sigma'}
\end{tikzcd}$$

[3] A special case of this is proved in [GSV03, Lemma 1.3].

Note that $p_M^*(X_i) = p^*(X_i) \prod_{j \in I_f} A_j^{M_{ij}}$. If $i \neq k$, we have

$$\begin{aligned}(\mu_k \circ p_M)^*(X_i') &= \mu_k^*\big(p(X_i') \prod_{j \in I_f} (A_j')^{M_{ij}}\big) \\ &= (\mu_k \circ p)^*(X_i') \prod_{j \in I_f} A_j^{M_{ij}}\end{aligned}$$

and

$$\begin{aligned}(p_M \circ \mu_k)^*(X_i') &= p_M^*\big(X_i X_k^{[B_{ik}]_+}(1+X_k)^{-B_{ik}}\big) \\ &= (p \circ \mu_k)^*(X_i') \prod_{j \in I_0} A_j^{M_{ij}},\end{aligned}$$

and the equality of these follows from their equality in the $M = 0$ case. On the other hand, since $p^*(X_k) = p_M^*(X_k)$, it follows trivially that $(\mu_k \circ p_M)^*(X_k') = (p_M \circ \mu_k)^*(X_k')$, and the proposition follows. $\square$

Definition 2.2.13. (σ-periods) Let $\widehat{\mu} = \mu_{i_1} \circ \cdots \circ \mu_{i_k}$ be a sequence of mutations of a seed Σ and σ a permutation of I such that

$$\widehat{\mu}(B)_{ij} = B_{\sigma(i)\sigma(j)}.$$

In other words, $\widehat{\mu}(\Sigma)$ and Σ are isomorphic after relabeling by σ. Then we say $\widehat{\mu}$ is a σ-period of Σ, or that $\widehat{\mu}$ is a mutation-periodic sequence when σ and Σ are understood. To such a mutation-periodic sequence is associated a pair of rational automorphisms of the tori $\mathcal{A}_\Sigma$ and $\mathcal{X}_\Sigma$, denoted by $\widehat{\mu}_\sigma$, which we refer to as cluster automorphisms and which are intertwined by the map p_Σ. More formally, these are defined by

$$\widehat{\mu}_\sigma^*(A_i) = (\mu_{i_1} \circ \cdots \circ \mu_{i_k})^*(A_{\sigma^{-1}(i)}), \quad \widehat{\mu}_\sigma^*(X_i) = (\mu_{i_1} \circ \cdots \circ \mu_{i_k})^*(X_{\sigma^{-1}(i)}). \qquad \square$$

Definition 2.2.14. (Amalgamation) If Σ, $\widetilde{\Sigma}$ are seeds and $\pi\colon I \twoheadrightarrow \widetilde{I}$ a surjection of their index sets, we say $\widetilde{\Sigma}$ is the amalgamation of Σ along π if

1. For all distinct $i, j \in I$, $\pi(i) = \pi(j)$ implies $i, j \in I_f$ and $B_{ij} = 0$.

2. For all $k, \ell \in \widetilde{I}$,
$$\widetilde{B}_{k\ell} = \sum_{\substack{i,j:\pi(i)=k,\\ \pi(j)=\ell}} B_{ij}.$$

3. $\pi(I_u) \subset \widetilde{I}_u$.

4. $d_i = d_{\pi(i)}$ for all $i \in I$.

To such an amalgamation of seeds is associated an amalgamation map $\pi\colon \mathcal{X}_\Sigma \twoheadrightarrow \mathcal{X}_{\widetilde{\Sigma}}$, which is Poisson and defined by

$$\pi^*(\widetilde{X}_j) = \prod_{i:\pi(i)=j} X_i. \qquad \square$$

In particular, an amalgamation $\widetilde{\Sigma}$ of Σ can be associated with any bijection $\varphi : I_1 \xrightarrow{\sim} I_2$ between disjoint subsets of I_f such that $B_{i,\varphi(i)} = 0$ and $d_i = d_{\varphi(i)}$ for all $i \in I_1$. We set $\widetilde{I} = I \smallsetminus I_1$, $\widetilde{I}_u = I_u$, $\widetilde{I}_f = I_f \smallsetminus I_1$, defining the map $\pi : I \twoheadrightarrow \widetilde{I}$ as the identity on $I \smallsetminus I_1$ and φ on I_1. The exchange matrix $\widetilde{B}$ is then uniquely determined by the hypotheses of Definition 2.2.14.

Remark 2.2.15. In the spirit of Remark 2.2.5, amalgamation should be understood as deriving from an inclusion of lattices $\mathbb{Z}\widetilde{I} \subset \mathbb{Z}I$, where for each $i \in \widetilde{I}$ we identify the generator e_i of $\mathbb{Z}\widetilde{I}$ with the element $\sum_{\pi(j)=i} e_j$ of $\mathbb{Z}I$. $\square$

Definition 2.2.14 is somewhat flexible about the relation between frozen and unfrozen subsets of I and $\widetilde{I}$, and in typical situations we may have $\pi(i)$ be unfrozen though i is frozen. It is also typically the case that Σ is a direct sum of two other seeds Σ_1 and Σ_2 (for the obvious notion of direct sum), and the map φ identifies some frozen indices of Σ_1 with frozen indices of Σ_2. However, our examples require the more general notion given here.

A crucial feature of amalgamations is that under certain mild conditions they commute with cluster transformations:

Proposition 2.2.16. *Suppose $\widetilde{\Sigma}$ is the amalgamation of Σ along $\pi : I \twoheadrightarrow \widetilde{I}$, and that π also satisfies the hypotheses of Definition 2.2.14 with respect to $\mu_k(\Sigma)$ and $\mu_k(\widetilde{\Sigma})$ for some unfrozen index k. Then $\mu_k(\widetilde{\Sigma})$ is also the amalgamation of $\mu_k(\Sigma)$ along π, and the respective amalgamation maps and cluster transformations commute:*

$$\begin{tikzcd}
\mathcal{X}_\Sigma \arrow[r, dashed, "\mu_k"] \arrow[d, two heads, "\pi"] & \mathcal{X}_{\Sigma'} \arrow[d, two heads, "\pi"] \\
\mathcal{X}_{\widetilde{\Sigma}} \arrow[r, dashed, "\mu_k"] & \mathcal{X}_{\widetilde{\Sigma}'}.
\end{tikzcd}$$

Proof. For each $i \in \widetilde{I}$, we must check that $(\pi \circ \mu_k)^* X'_i = (\mu_k \circ \pi)^* X'_i$. This is clear for $i = k$, while for $i \neq k$ we have

$$(\pi \circ \mu_k)^* X'_i = \prod_{\pi(j)=i} (X_j X_k^{[B_{jk}]_+} (1 + X_k)^{-B_{jk}})$$
$$(\mu_k \circ \pi)^* X'_i = (\prod_{\pi(j)=i} X_j) X_k^{[B_{ik}]_+} (1 + X_k)^{-B_{ik}}.$$

Since $B_{ik} = \sum_{\pi(j)=i} B_{jk}$ by assumption, the result follows if

$$\sum_{\pi(j)=i} [B_{jk}]_+ = [\sum_{\pi(j)=i} B_{jk}]_+.$$

This in turn holds if B_{jk} and $B_{\ell k}$ are of the same sign whenever $\pi(j) = \pi(k) = i$. But if B_{jk} and $B_{\ell k}$ were of opposite signs for some such j, ℓ, $B'_{j\ell}$ would be nonzero, contradicting our hypothesis about π. $\square$

When frozen variables of two distinct seeds are glued together by an amalgamation, the assumption that π satisfies the needed hypotheses with respect to the mutated seeds always holds. However, when Σ is not a direct sum this need not be the case. For example, if B is the adjacency matrix of the quiver

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet}$$

then we can form an amalgamation by gluing the vertices 1 and 3 together. However, after mutating the original quiver at vertex 2, we will have $B'_{13} \neq 0$, hence this is no longer an admissible amalgamation.

Chapter 3

Infinite-dimensional Poisson-Lie Theory and Affine Integrable Systems

3.1 Introduction

The goals of this chapter are to set up a rigorous, working theory of Poisson-Lie structures on ind-algebraic groups, treat the case of symmetrizable Kac-Moody groups within this framework, and use this as a setting for the construction of integrable systems on symplectic leaves of affine Lie groups.

The development of Poisson-Lie theory, that is, of Poisson structures compatible with a group operation, accompanied the discovery of quantum groups in the context of quantum integrable systems [Dri88]. The resulting subject witnessed a rich interplay between Poisson geometry, the representation theory of quantum algebras, and exact solvability of statistical and quantum systems. Though Poisson brackets on loop groups are often related to more interesting physical models than those on finite-dimensional Lie groups, in practice they are dealt with less rigorously as well. The literature on Poisson-Lie theory contains many treatments of the foundations of the finite-dimensional case [KS96; CP94; RSTS94], generally referred to without comment when infinite-dimensional examples are treated in applications. While this is satisfactory for performing computations relevant to any given model, it is not from the perspective of setting up a complete mathematical theory.

The sort of infinite-dimensional groups for which we aim to fill this gap are ind-algebraic groups, geometrically the increasing unions of finite-dimensional algebraic varieties. These include in particular the groups associated with Kac-Moody algebras of arbitrary type and groups of algebraic loops into a simple Lie group. For these Kac-Moody groups we also generalize the classification of their symplectic leaves by double Bruhat cells, well-known in finite type.

Theorem. (3.2.7, 3.2.10, 3.2.13, 3.3.3) *The completed coordinate ring of a symmetrizable Kac-Moody group G is a topological Poisson algebra. Its symplectic leaves are classified by the double Bruhat cells of G, which are smooth, finite-dimensional Poisson subvarieties.*

We note that although the essential features of the finite-type case carry over completely to the general case, there are fundamental geometric differences that demand careful consideration. In particular, vector fields on ind-varieties can not in general be integrated, making even the existence of symplectic leaves a nontrivial fact. Moreover, affine Kac-Moody groups, our main examples, are known to be everywhere singular [FGT08], a pathology obviously quite foreign to the finite-dimensional case and which indicates the care needed when passing to infinite dimensions.

After developing these foundations, we describe a class of completely integrable Hamiltonian systems generalizing the relativistic periodic Toda lattice, introduced in [Rui90]. We identify the phase space of this particular system with a double Bruhat cell of the $A_n^{(1)}$ affine Kac-Moody group, and its Hamiltonians with restrictions of invariant functions. This refines the well-known observation that it admits a Lax form which is Hamiltonian with respect to the Poisson-Lie bracket induced by the trigonometric r-matrix [Sur91]. A larger family of systems can then be obtained by transporting the construction to other double Bruhat cells and other groups. On a general double Bruhat cell the invariant functions will not necessarily restrict to a maximal set of Poisson-commuting functions, but we show that a sufficient condition for this is that the cell correspond to a pair of Coxeter elements in the affine Weyl group. This construction generalizes that of [Hof+00], which treated semisimple algebraic groups and where the term Coxeter-Toda lattice was introduced for the resulting systems.

Theorem. (3.4.6) *For an affine Kac-Moody group G and a Coxeter element c of the affine Weyl group, the conjugation quotient $G^{c,c}/H$ is equipped with a canonical integrable system, a generalized relativistic periodic Toda lattice.*

3.2 Ind-Groups and Poisson-Lie Theory

This section is devoted to foundational results on the Poisson-Lie theory of ind-algebraic groups, and Kac-Moody groups in particular. Recall that a Poisson-Lie group is a Lie group equipped with a Poisson structure such that the group operation $G \times G \to G$ is a Poisson map; we refer to [KS96; CP94; RSTS94] for a detailed exposition in the finite-dimensional case.

Standard Poisson-Lie Structure on SL_2

We briefly review the standard Poisson structure on SL_2; this is both a model for the general case, and essential for the explicit computations we will perform in Section 3.4. The Lie algebra $\mathfrak{sl}_2$ has generators

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and an invariant form unique up to fixing the scalar $d := \frac{2}{(H,H)}$. If $\Omega_d \in \mathfrak{g} \otimes \mathfrak{g}$ is the corresponding Casimir, we write $\Omega_d = \Omega_{+-} + \Omega_0 + \Omega_{-+}$, where $\Omega_0 \in \mathfrak{h} \otimes \mathfrak{h}, \Omega_{+-} \in \mathfrak{n}_+ \otimes \mathfrak{n}_-$, and $\Omega_{-+} \in \mathfrak{n}_- \otimes \mathfrak{n}_+$. We have the standard quasitriangular r-matrix is

$$r = \Omega_0 + 2\Omega_{+-} = d(\frac{1}{2}H \otimes H + 2X \otimes Y). \tag{3.2.1}$$

That is, r is a solution of the classical Yang-Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0,$$

and its symmetric part is adjoint invariant [CP94, p. 2.1.11].

Trivializing the tangent bundle by right translations, we define a Poisson bivector whose value at $g \in SL_2$ is $\mathrm{Ad}_g(r) - r$. The resulting tensor is skew-symmetric since the symmetric part of r is invariant, and its compatibility with the group structure is immediate by construction. Moreover, the Yang-Baxter equation implies the Jacobi identity for the corresponding Poisson bracket [KS96, p. 4.2].

Given the parametrization

$$SL_2 = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : AD - BC = 1 \right\},$$

the Poisson brackets of the coordinate functions are

$$\{B, A\} = dAB, \quad \{B, D\} = -dBD, \quad \{B, C\} = 0,$$
$$\{C, A\} = dAC, \quad \{C, D\} = -dCD, \quad \{D, A\} = 2dBC.$$

To notate the dependence of the bracket on d, we denote the corresponding Poisson algebraic group by $SL_2^{(d)}$.

Poisson Ind-Varieties

Our treatment of infinite-dimensional Poisson-Lie theory is based on the following definition; for simplicity all ind-varieties are tacitly taken to be affine unless stated otherwise.

Definition 3.2.2. A *Poisson ind-variety* is an ind-variety X with a Poisson bracket on $\mathbb{C}[X]$, continuous as a map $\mathbb{C}[X] \otimes \mathbb{C}[X] \to \mathbb{C}[X]$. A *Poisson map* is a regular map of ind-varieties which intertwines the Poisson brackets on their coordinate rings. □

Whenever $V = \varprojlim V_i$ and $W = \varprojlim W_i$ are inverse limits of (discrete) vector spaces, we have the completed tensor product $V \widehat{\otimes} W := \varprojlim V_i \otimes W_i$. For example, if X and Y are ind-varieties, $\mathbb{C}[X] \widehat{\otimes} \mathbb{C}[Y]$ is just the coordinate ring of $X \times Y$. $V \otimes W$ sits in $V \widehat{\otimes} W$ as a dense subspace with respect to its inverse limit topology, and whenever we refer to a topology on $V \otimes W$ (as in the preceding definition) we mean its subspace topology.

Remark 3.2.3. The role of the inverse limit topology on V is to restrict our attention to operations that can be defined through the V_i. A linear map $\phi : V \to W$ is continuous if and only if for each i and all $k \gg 0$ there are linear maps $\phi_{ki} : V_k \to W_i$ which commute with each other, the maps defining the inverse systems, and ϕ in the obvious ways (note that for each i, ϕ_{ki} is defined for k sufficiently large, but how large k must be depends on i). In other words, taking the inverse limit is a full and faithful functor from the category of pro-vector spaces indexed by $\mathbb{N}$ to the category of topological vector spaces. This allows us to go back and forth between topological statements about V and purely algebraic statements about the V_i. In particular, we have the following useful observation:

Lemma 3.2.4. *Let $\phi : V \to A$ and $\psi : W \to B$ be continuous linear maps between inverse limits of discrete vector spaces (indexed by $\mathbb{N}$). Then $\phi \otimes \psi$ extends continuously to a map $\phi \widehat{\otimes} \psi : V \widehat{\otimes} W \to A \widehat{\otimes} B$ of completed tensor products.*

Proof. Since ϕ and ψ are continuous, they are determined by collections of maps $\{\phi_{ki} : V_k \to A_i \mid k \gg 0\}$ and $\{\psi_{ki} : W_k \to B_i \mid k \gg 0\}$ as above. But then for each i we have linear maps $\phi_{ki} \otimes \psi_{ki} : V_k \otimes W_k \to A_i \otimes B_i$ for k sufficiently large. These readily satisfy the necessary compatibility requirements, hence yield a continuous linear map $\phi \widehat{\otimes} \psi : V \widehat{\otimes} W \to A \widehat{\otimes} B$. □

□

Proposition 3.2.5. *For any Poisson ind-varieties X and Y, $X \times Y$ has a canonical Poisson structure.*

Proof. The bracket on $\mathbb{C}[X] \otimes \mathbb{C}[Y] \subset \mathbb{C}[X \times Y]$ may be given by the usual formula $\{f \otimes \phi, g \otimes \psi\}_{X \times Y} := \{f, g\}_X \otimes \phi\psi + fg \otimes \{\phi, \psi\}_Y$. The fact that this extends to all of $\mathbb{C}[X \times Y]$ follows from Lemma 3.2.4 and the continuity of the brackets on X and Y. □

Definition 3.2.6. A *Poisson Ind-Group* is an ind-algebraic group G which is a Poisson ind-variety and whose group operation $G \times G \to G$ is Poisson. □

As in the case of SL_2, it will be convenient to define Poisson brackets implicitly by providing a bivector field. However, the groups we are interested in need not be inductive limits of smooth varieties (see Remark 2.1.16), so we must be careful in discussing their tangent bundles. The following proposition guarantees that nonetheless the trivialized tangent bundle behaves as expected.

Proposition 3.2.7. *Let $\mathcal{G}$ be an ind-group and $\mathfrak{g}$ its Lie algebra. There is a bijection between continuous n-derivations of $\mathbb{C}[\mathcal{G}]$ and regular maps $\mathcal{G} \to \bigwedge^n \mathfrak{g}$ (by n-derivation we mean a skew-symmetric map $\mathbb{C}[\mathcal{G}] \widehat{\otimes} \ldots \widehat{\otimes} \mathbb{C}[\mathcal{G}] \to \mathbb{C}[\mathcal{G}]$ which is a derivation in each position). Given a map $K : \mathcal{G} \to \bigwedge^n \mathfrak{g}$, the corresponding n-derivation $\widetilde{K}$ takes the functions $f_1, \ldots, f_n \in \mathbb{C}[\mathcal{G}]$ to the function*

$$\widetilde{K}(f_1, \ldots, f_n) : g \mapsto \langle K(g) | d_e \ell_g^* f_1 \wedge \cdots \wedge d_e \ell_g^* f_n \rangle.$$

Proof. We prove the case $n = 1$, the higher rank case not being substantively different. We first show that the regularity of K ensures that the stated formula takes regular functions to regular functions, and that this assignment is continuous. Note that $\mathfrak{g}$ is an ind-variety via its filtration by the $T_e\mathcal{G}_i$, and that there is a correspondence between regular maps $K : \mathcal{G} \to \mathfrak{g}$ and continuous linear maps $K^* : \mathfrak{g}^* \to \mathbb{C}[\mathcal{G}]$. Thus given K we have a continuous linear endomorphism of $\mathbb{C}[\mathcal{G}]$ given by

$$\widetilde{K} := m \circ (1\widehat{\otimes}K^*) \circ (1\widehat{\otimes}d_e) \circ \Delta.$$

Here $\Delta : \mathbb{C}[\mathcal{G}] \to \mathbb{C}[\mathcal{G}]\widehat{\otimes}\mathbb{C}[\mathcal{G}]$ is the coproduct on $\mathbb{C}[\mathcal{G}]$ and m is the extension of the multiplication map to $\mathbb{C}[\mathcal{G}]\widehat{\otimes}\mathbb{C}[\mathcal{G}]$. We have implicitly used Lemma 3.2.4 and the fact that d_e is continuous. This composition recovers the formula stated in the proposition when evaluated on a function $f \in \mathbb{C}[\mathcal{G}]$, and in particular expresses it as a manifestly continuous map from $\mathbb{C}[\mathcal{G}]$ to itself.

Conversely, given a continuous derivation $\widetilde{K}$ of $\mathbb{C}[\mathcal{G}]$, we consider the map $K^* : \mathbb{C}[\mathcal{G}] \to \mathbb{C}[\mathcal{G}]$ given by

$$K^* := m \circ (S\widehat{\otimes}\widetilde{K}) \circ \Delta,$$

where S is the antipode of $\mathbb{C}[\mathcal{G}]$. If $m_e \subset \mathbb{C}[\mathcal{G}]$ is the maximal ideal of the identity, we let the reader check that K^* annihilates m_e^2, hence descends to a continuous linear map $K^* : \mathfrak{g}^* = m_e/m_e^2 \to \mathbb{C}[\mathcal{G}]$. As observed earlier, this data is equivalent to a regular map $K : \mathcal{G} \to \mathfrak{g}$. Furthermore, from the defining property of the antipode it follows that this construction and the one above are inverse to each other. □

In particular, a Poisson structure on an ind-group $\mathcal{G}$ is determined by a Poisson bivecter $\pi : \mathcal{G} \to \bigwedge^2 \mathfrak{g}$. Restating the compatibility of the bracket on $\mathcal{G}$ with the group operation in terms of π we obtain the following definition.

Definition 3.2.8. A polyvector field $K : \mathcal{G} \to \bigwedge^n \mathfrak{g}$ is multiplicative if $K(gh) = \mathrm{Ad}_{h^{-1}}K(g) + K(h)$. □

Remark 3.2.9. The derivative $d_eK : \mathfrak{g} \to \bigwedge^n \mathfrak{g}$ of a multiplicative polyvector field is a 1-cocycle of $\mathfrak{g}$ with values in $\bigwedge^n \mathfrak{g}$. If π is a Poisson bivector, then $d_e\pi$ is a Lie cobracket which makes $\mathfrak{g}$ a Lie bialgebra. The dual of $d_e\pi$ is a continuous Lie bracket on $\mathfrak{g}^*$, which is the essentially the Poisson bracket on $\mathbb{C}[\mathcal{G}]$. That is, the maximal ideal of the identity $m_e \subset \mathbb{C}[\mathcal{G}]$ is a Lie subalgebra and $m_e^2 \subset m_e$ an ideal, hence there is an induced Lie bracket on $\mathfrak{g}^*$. We will not need this observation, except in Section 3.3 where we describe an explicit alternative description of the bracket on $\mathfrak{g}^*$ in the Kac-Moody case. □

The Standard Poisson-Lie Structure of a Kac-Moody Group

We now define the standard Poisson-Lie structure on a symmetrizable Kac-Moody group $\mathcal{G}$. The construction follows the same lines as for SL_2 (or any semisimple Lie group), but the general case presents certain technical problems absent when considering finite-dimensional groups.

The invariant form on $\mathfrak{g}$ lets us identify it $\mathcal{G}$-equivariantly with a dense subspace of $\mathfrak{g}^*$, hence $\mathfrak{g}^*\widehat{\otimes}\mathfrak{g}^*$ may be viewed as a completion of $\mathfrak{g}\otimes\mathfrak{g}$. We denote this by $\mathfrak{g}\widehat{\otimes}\mathfrak{g}$, and in particular there is an element Ω of $\mathfrak{g}\widehat{\otimes}\mathfrak{g}$ associated with the invariant form on $\mathfrak{g}$. We write Ω as $\Omega_{+-}+\Omega_0+\Omega_{-+}$, where $\Omega_0\in\mathfrak{h}\otimes\mathfrak{h}$, $\Omega_{+-}\in\mathfrak{n}_+\widehat{\otimes}\mathfrak{n}_-$, and $\Omega_{-+}\in\mathfrak{n}_-\widehat{\otimes}\mathfrak{n}_+$. Then $r=\Omega_0+2\Omega_{+-}$ is a pseudoquasitriangular r-matrix [Dri88, Section 4]; that is, r satisfies the classical Yang-Baxter equation and has adjoint-invariant symmetric part, but cannot be written as a sum of finitely many simple tensors.

As in the finite-dimensional case, we want to define a Poisson bivector $\pi:\mathcal{G}\to\bigwedge^2\mathfrak{g}$ by $\pi(g)=\mathrm{Ad}_g(r)-r$. Now, however, r is not an element of $\mathfrak{g}\otimes\mathfrak{g}$ but rather a completion thereof, so we must specifically prove that $\pi(g)$ is actually an element of $\bigwedge^2\mathfrak{g}$.

Theorem 3.2.10. *The map $g\mapsto\mathrm{Ad}_g(r)-r$ defines a bivector field $\pi:\mathcal{G}\to\bigwedge^2\mathfrak{g}$.*

Proof. First we check that $\mathrm{Ad}_g(r)-r\in\mathfrak{g}\otimes\mathfrak{g}$ for all $g\in\mathcal{G}$. We begin with the case where g lies in the SL_2 subgroup G_α for some simple root α. First decompose $\mathfrak{g}$ as a direct sum of G_α-subrepresentations corresponding to α-root strings. That is, let

$$\mathfrak{g}_{[\beta]}=\bigoplus_{n\in\mathbb{Z}}\mathfrak{g}_{\beta+n\alpha},\quad \mathfrak{g}=\bigoplus_{[\beta]\in\mathcal{Q}/\mathbb{Z}\alpha}\mathfrak{g}_{[\beta]},$$

where $\mathcal{Q}$ is the root lattice of $\mathcal{G}$. Since α is simple, for any $[\beta]$ we have either $\mathfrak{g}_{[\beta]}\subset\mathfrak{n}_+$, $\mathfrak{g}_{[\beta]}\subset\mathfrak{n}_-$, or $\beta\in\mathbb{Z}\alpha$. Furthermore, the invariant form on $\mathfrak{g}$ restricts to a nondegenerate G_α-invariant pairing between $\mathfrak{g}_{[\beta]}$ and $\mathfrak{g}_{[-\beta]}$.

Now we can rewrite the r-matrix as

$$r=r_\alpha+\sum_{\substack{[\beta]\in\mathcal{Q}/\mathbb{Z}\alpha\\ \beta>0}}r_{[\beta]}.$$

Here $r_{[\beta]}$ is the element of $\mathfrak{g}_{[\beta]}\otimes\mathfrak{g}_{[-\beta]}$ representing their G_α-invariant pairing and $r_\alpha\in\mathfrak{g}_{[\alpha]}\otimes\mathfrak{g}_{[\alpha]}$. In particular, since $r_{[\beta]}$ is G_α-invariant, $\mathrm{Ad}_g(r_{[\beta]})=r_{[\beta]}$ and

$$\mathrm{Ad}_g(r)-r=\mathrm{Ad}_g(r_{[\alpha]})-r_{[\alpha]}.$$

The right hand side is manifestly finite-rank, hence $\mathrm{Ad}_g(r)-r\in\mathfrak{g}\otimes\mathfrak{g}$ for $g\in G_\alpha$.

It is then straightforward to see that $\mathrm{Ad}_g(r)-r\in\mathfrak{g}\otimes\mathfrak{g}$ whenever g is a product of elements from simple root subgroups, and by Proposition 2.1.13 any $g\in\mathcal{G}'$ is of this form. Moreover, since r lies in the zero weight space of $\mathfrak{g}\widehat{\otimes}\mathfrak{g}$ it is fixed by the Cartan subgroup H. Since $\mathcal{G}$ is generated by H and $\mathcal{G}'$, it follows that $\mathrm{Ad}_g(r)-r\in\mathfrak{g}\otimes\mathfrak{g}$ for any $g\in\mathcal{G}$. We have $\mathrm{Ad}_g(r)-r\in\bigwedge^2\mathfrak{g}\subset\mathfrak{g}\otimes\mathfrak{g}$ because the symmetric part of r is adjoint invariant. Finally, the fact that π is regular follows from the fact that the adjoint action of $\mathcal{G}$ on $\bigwedge^2\mathfrak{g}$ is regular. $\square$

By Proposition 3.2.7, π defines a continuous skew-symmetric bracket on $\mathbb{C}[\mathcal{G}]$ satisfying the Leibniz rule. That this bracket satisfies the Jacobi identity is a consequence of the fact that r is a solution of the classical Yang-Baxter equation. To make this precise for a general Kac-Moody group we must first introduce a certain dense subalgebra of $\mathbb{C}[\mathcal{G}]$.

Recall the embedding

$$\mathcal{G} \hookrightarrow V = \bigoplus_{i=1}^{\dim(H)} (L(\omega_i) \oplus L(\omega_i)^\vee)$$

used to define the ind-variety structure on $\mathcal{G}$. The weight grading of V expresses it as a direct sum $V = \bigoplus_{\alpha \in \mathcal{Q}} V_\alpha$ of finite-dimensional subspaces.

Definition 3.2.11. The algebra of strongly regular functions on V is the symmetric algebra of its graded dual,

$$\mathbb{C}[V]_{\text{s.r.}} = \mathrm{Sym}^*(\bigoplus_{\alpha \in \mathcal{Q}} V_\alpha^*).$$

The algebra $\mathbb{C}[\mathcal{G}]_{\text{s.r.}}$ of strongly regular functions on $\mathcal{G}$ is the image of $\mathbb{C}[V]_{\text{s.r.}}$ in $\mathbb{C}[\mathcal{G}]$ under the restriction map.[1] □

Proposition 3.2.12. *$\mathbb{C}[\mathcal{G}]_{\text{s.r.}}$ is a dense subalgebra of $\mathbb{C}[\mathcal{G}]$. For any $f \in \mathbb{C}[\mathcal{G}]_{\text{s.r.}}$ and $g \in \mathcal{G}$, $\ell_g^*(f)$ is again strongly regular, and the differential $d_e f$ lies in the graded dual $\mathfrak{g}^\vee := \bigoplus_{\alpha \in \mathcal{Q}} \mathfrak{g}_\alpha^* \subset \mathfrak{g}^*$.*

Proof. The first and last statements are immediate. That $\ell_g^*(f)$ is strongly regular follows from the fact that the coadjoint action of $\mathcal{G}$ on the algebraic dual $\mathfrak{g}^*$ preserves the graded dual of $\mathfrak{g}$. □

Proposition 3.2.13. *The bracket on $\mathbb{C}[\mathcal{G}]$ defined by the bivector $\pi(g) = \mathrm{Ad}_g(r) - r$ satisfies the Jacobi identity.*

Proof. We recall the proof when $\mathcal{G}$ is a semisimple algebraic group [KS96], and then explain the necessary adjustments in the general case. First, we write the bracket as a difference of the two brackets $\{,\}_1$ and $\{,\}_2$ defined by the bivectors $\pi_1(g) = \mathrm{Ad}_g(r)$ and $\pi_2(g) = r$. Now consider separately the expressions

$$\{\phi, \{\psi, \xi\}_i\}_i + \{\psi, \{\xi, \phi\}_i\}_i + \{\xi, \{\phi, \psi\}_i\}_i$$

for $i \in \{1, 2\}$ and $\phi, \psi \in \mathbb{C}[\mathcal{G}]$. On writing these out explicitly in terms of r one sees that half of the terms vanish by the Yang-Baxter equation, while the remaining terms are the same for both $\{,\}_1$ and $\{,\}_2$. Thus they cancel when we take the difference of $\{,\}_1$ and $\{,\}_2$, yielding the Jacobi identity for the original bracket.

[1] Our use of the term "strongly regular" differs from that in section 2 of [KP83b], but is consistent with Section 4 of loc. cited.

When $\mathcal{G}$ is infinite-dimensional, this argument fails since π_1 and π_2 are not finite-rank bivectors in the sense of Proposition 3.2.7. However, in light of Proposition 3.2.12, they do define biderivations $\{,\}_1$ and $\{,\}_2$ on the algebra of strongly regular functions on $\mathcal{G}$. Moreover, the Yang-Baxter equation implies the Jacobi identity for the bracket on $\mathbb{C}[\mathcal{G}]_{\text{s.r.}}$ by an identical computation as in the finite-dimensional case. But since $\mathbb{C}[\mathcal{G}]_{\text{s.r.}}$ is dense in $\mathbb{C}[\mathcal{G}]$ and the bracket is continuous, the proposition follows. □

We call the resulting Poisson structure on $\mathcal{G}$ the standard Poisson structure. It is essentially characterized by the following proposition.

Proposition 3.2.14. *$\mathcal{G}'$ and H are Poisson subgroups of $\mathcal{G}$, the latter with the trivial Poisson structure. For any simple root α, G_α is a Poisson subgroup isomorphic with $SL_2^{(d_\alpha)}$.*

Proof. We know that only the skew-symmetric part of r, which lies in $\mathfrak{n}_+\widehat{\otimes}\mathfrak{n}_-\oplus\mathfrak{n}_-\widehat{\otimes}\mathfrak{n}_+ \subset \mathfrak{g}'\widehat{\otimes}\mathfrak{g}'$, contributes to the Poisson bivector, proving the claim for $\mathcal{G}'$. The statement about H follows from the observation that r lies in the zero weight space of $\mathfrak{g}\widehat{\otimes}\mathfrak{g}$, hence $Ad_h(r) - r = 0$ for any $h \in H$.

In the proof of Theorem 3.2.10 we found that for $g \in G_\alpha$, $\pi(g) = \mathrm{Ad}_g(r_{[\alpha]}) - r_{[\alpha]}$, where $r_{[\alpha]}$ is the component of r in the Lie algebra of G_α. But from the definition of r and eq. (3.2.1), it is clear that $r_{[\alpha]}$ is precisely the r-matrix of $SL_2^{(d_\alpha)}$, and the proposition follows. □

Proposition 3.2.15. ([RSTS94, p. 12.24]) *If $\phi, \psi \in \mathbb{C}[\mathcal{G}]$ are invariant under conjugation, then*

$$\{\phi, \psi\} = 0.$$

Proof. At any $g \in \mathcal{G}$ we check that

$$\begin{aligned}
\{\phi, \psi\}(g) &= \langle \mathrm{Ad}_g(r) - r | d\phi \wedge d\psi \rangle \\
&= \langle r | \mathrm{Ad}_g^*(d\phi \wedge d\psi) - d\phi \wedge d\psi \rangle \\
&= 0,
\end{aligned}$$

since $\mathrm{Ad}_g^*(d\phi \wedge d\psi) = d\phi \wedge d\psi$ by assumption. □

3.3 Symplectic Leaves of Kac-Moody Groups and the Double Bruhat Decomposition

In this section we show that the double Bruhat cells of a symmetrizable Kac-Moody group $\mathcal{G}$ are Poisson subvarieties, and in particular obtain a decomposition of $\mathcal{G}$ into symplectic leaves. Recall that the symplectic leaves of a finite-dimensional Poisson manifold are the orbits of its piecewise Hamiltonian flows, have canonical symplectic structures, and define a generalized foliation of $\mathcal{G}$. The existence of symplectic leaves in $\mathcal{G}$ is nontrivial, since a vector field on a general ind-variety need not have integral curves even if the ind-variety is smooth.

We will obtain an explicit characterization of the symplectic leaves of $\mathcal{G}$ in Theorem 3.3.3, but first we offer an elementary proof of their existence. We will use Propositions 2.1.12 and 3.3.12 from Section 4.2, but their proofs do not rely on the results of this section.

Proposition 3.3.1. *The double Bruhat cells $\mathcal{G}^{u,v}$ are Poisson subvarieties of $\mathcal{G}$.*

Proof. In Proposition 3.3.12 we construct dominant Poisson map $\phi_{\mathbf{i}}$ from a Poisson variety to $\mathcal{G}^{u,v}$. It follows that the closure of $\mathcal{G}^{u,v}$ in $\mathcal{G}$ is a Poisson subvariety: the kernel of $\phi_{\mathbf{i}}^*$ in $\mathbb{C}[\mathcal{G}]$ is an open Poisson ideal, hence the closure of $\mathcal{G}^{u,v}$ is the (maximal) spectrum of the Poisson algebra $\mathbb{C}[\mathcal{G}]/\mathrm{ker}\phi_{\mathbf{i}}^*$. The closure of $\mathcal{G}^{u,v}$ is can be explicitly written as

$$\bigcup_{u' \leq u, v' \leq v} \mathcal{G}^{u',v'},$$

and in particular $\mathcal{G}^{u,v}$ is the complement of a divisor in its closure. But such an open subset of an affine Poisson variety inherits a canonical Poisson structure [Van01, p. 2.35]. □

Corollary 3.3.2. *The group $\mathcal{G}$ is the disjoint union of finite-dimensional symplectic leaves.*

Proof. Follows from Proposition 3.3.1 and the fact that double Bruhat cells are smooth and finite-dimensional (Proposition 2.1.12). □

We can get a more precise description of the symplectic leaves of $\mathcal{G}$ by introducing the dual group $\mathcal{G}^\vee$ and the double group $\mathcal{D}$. These are ind-groups defined by

$$\mathcal{G}^\vee := \{(b_-, b_+) \in \mathcal{B}_- \times \mathcal{B}_+ \mid [b_-]_0 = [b_+]_0^{-1}\}, \quad \mathcal{D} := \mathcal{G} \times \mathcal{G}.$$

The dual group $\mathcal{G}^\vee$ sits inside $\mathcal{D}$ in the obvious way, and we view $\mathcal{G}$ as a subgroup of $\mathcal{D}$ via its diagonal embedding.

Theorem 3.3.3. *The symplectic leaves of a symmetrizable Kac-Moody group $\mathcal{G}$ are the connected components of its intersections with the double cosets of $\mathcal{G}^\vee$ in $\mathcal{D}$.*

The proof of this theorem proceeds in several steps, closely following [LW90] in the finite-dimensional case. The idea of the proof remains the same, but we indicate how some arguments must be rephrased or altered to remain valid in the current setting. In particular, one does not expect a priori to have such a theorem for arbitrary Poisson ind-groups, as at several points we must appeal to particular properties of Kac-Moody groups and their standard Poisson structure.

First note that the Lie algebra of $\mathcal{G}^\vee$ is

$$\mathfrak{g}^\vee = \{(X_-, X_+) \in \mathfrak{b}_- \oplus \mathfrak{b}_+ \mid [X_-]_0 = -[X_+]_0\},$$

where $[X_\pm]_0$ denotes the component of $X_\pm$ in $\mathfrak{h}$. The Lie algebra $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ of $\mathcal{D}$ is then the direct sum of $\mathfrak{g}^\vee$ and $\mathfrak{g}$, the latter embedded diagonally. Moreover, $\mathfrak{g}^\vee$ and $\mathfrak{g}$ are maximal isotropic subalgebras under the nondegenerate invariant form

$$\langle (X_1, Y_1), (X_2, Y_2) \rangle = \langle X_1, X_2 \rangle - \langle Y_1, Y_2 \rangle.$$

In particular, this form identifies $\mathfrak{g}^\vee$ with the graded dual of $\mathfrak{g}$, justifying its notation.[2]

Given this identification, the bracket on $\mathfrak{d}$ can be rewritten in terms of the coadjoint actions of $\mathfrak{g}$ and $\mathfrak{g}^\vee$ on each other. That is, if $X_1, X_2 \in \mathfrak{g}$ and $Y_1, Y_2 \in \mathfrak{g}^\vee$, then

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2] + ad^*_{Y_1} X_2 - ad^*_{Y_2} X_1, [Y_1, Y_2] + ad^*_{X_1} Y_2 - ad^*_{X_2} Y_1). \quad (3.3.4)$$

Definition 3.3.5. Let π be the standard Poisson bivector on $\mathcal{G}$. For any $\mu \in \mathfrak{g}^*$ we define the (left) dressing vector field as

$$X_\mu := \iota_\mu(\pi).$$

□

Taken together these yield a continuous map $X : \mathfrak{g}^* \widehat{\otimes} \mathbb{C}[\mathcal{G}] \to \mathbb{C}[\mathcal{G}]$ which is a derivation in the right component. Furthermore, one can recover the Poisson bivector π from X. Explicitly, the map

$$m \circ (X_{13} \widehat{\otimes} S_2) \circ (1 \widehat{\otimes} \Delta) : \mathfrak{g}^* \widehat{\otimes} \mathbb{C}[\mathcal{G}] \to \mathbb{C}[\mathcal{G}]$$

factors through $\mathfrak{g}^* \widehat{\otimes} \mathfrak{g}^*$ as in the proof of Proposition 3.2.7, and is dual to the map $\pi : \mathcal{G} \to \bigwedge^2 \mathfrak{g}$. Here Δ is the coproduct on $\mathbb{C}[\mathcal{G}]$, S is the antipode, m is multiplication in $\mathbb{C}[\mathcal{G}]$, and the notation X_{13} means we apply X to the first and third terms of $\mathfrak{g}^* \widehat{\otimes} \mathbb{C}[\mathcal{G}] \widehat{\otimes} \mathbb{C}[\mathcal{G}]$.

Lemma 3.3.6. *Let K be a multiplicative polyvector field. (1) If X is a left-invariant vector field, $\mathcal{L}_X K$ is also left-invariant. Here $\mathcal{L}_X K$ is the Lie derivative of K with respect to X. (2) If $d_e(K) = 0$, then K is identically zero.*

Proof. We take K to be a vector field, the higher rank case being similar.

(1) Left-invariance of X is equivalent to $\Delta \circ X = (1 \widehat{\otimes} X) \circ \Delta$, and multiplicativity of K is equivalent to $\Delta \circ K = (1 \widehat{\otimes} K) \circ \Delta + (K \widehat{\otimes} 1) \circ \Delta$. Then $\mathcal{L}_X K$ is left-invariant by the following equality of maps from $\mathbb{C}[\mathcal{G}]$ to $\mathbb{C}[\mathcal{G}] \widehat{\otimes} \mathbb{C}[\mathcal{G}]$:

$$\begin{aligned}
\Delta \circ \mathcal{L}_X K &= \Delta \circ (X \circ K - K \circ X) \\
&= (1 \widehat{\otimes} X) \circ (K \widehat{\otimes} 1 + 1 \widehat{\otimes} K) \circ \Delta - (K \widehat{\otimes} 1 + 1 \widehat{\otimes} K) \circ (1 \widehat{\otimes} X) \circ \Delta \\
&= (1 \widetilde{\otimes} \mathcal{L}_X K) \circ \Delta
\end{aligned}$$

(2) Since $d_e(K) = 0$, $\mathcal{L}_X K|_e = 0$ for any left-invariant X. But $\mathcal{L}_X K$ is itself left-invariant by (1), hence is identically zero. In particular, since we can integrate the left-invariant vector fields corresponding to the real root spaces, K is invariant under left translation by the corresponding 1-parameter subgroups. Since $\mathcal{G}$ is generated by these subgroups and $H = \exp(\mathfrak{h})$, K is invariant under all left-translations. But K is multiplicative, hence $K|_e = 0$ and K must then be identically zero. □

---

[2] Though one can intrinsically define the Lie algebra structure on $\mathfrak{g}^*$ for an arbitrary Poisson ind-group (Remark 3.2.9), one cannot expect the existence of a corresponding dual group in general, since Lie's third theorem fails in this generality.

Proposition 3.3.7. *The dressing fields X_μ satisfy the twisted multiplicativity condition*

$$X_\mu(gh) = X_\mu(h) + Ad_{h^{-1}}[X_{Ad_{h^{-1}(\mu)}}(g)],$$

and the derivative $d_e X_\mu : \mathfrak{g} \to \mathfrak{g}$ is the coadjoint action ad^*_μ. Moreover, $X : \mathfrak{g}^* \widehat{\otimes} \mathbb{C}[\mathcal{G}] \to \mathbb{C}[\mathcal{G}]$ is the only continuous derivation satisfying these properties.

Proof. Twisted multiplicativity of the dressing fields follows readily from the definition of multiplicativity. Likewise, the fact that $X_\mu = \mathrm{ad}^*_\mu$ follows from unwinding the definition of the bracket on $\mathfrak{g}^*$. We omit the calculations, which resemble those of Proposition 3.2.7 and Lemma 3.3.6.

Suppose $Y : \mathfrak{g}^* \widehat{\otimes} \mathbb{C}[\mathcal{G}] \to \mathbb{C}[\mathcal{G}]$ is a continuous derivation and satisfies the given properties. In the same way that we can recover π from X, we recover a bivector field $\widetilde{Y}$ from Y. The twisted multiplicativity of Y is again equivalent to the multiplicativity of $\widetilde{Y}$, and $d_e \widetilde{Y} = d_e \pi$ since the derivatives of X and Y coincide at the identity. The difference $\pi - \widetilde{Y}$ is then multiplicative bivector field whose derivative at the identity is zero. Then by Lemma 3.3.6 $\pi - \widetilde{Y}$ is identically zero, hence $X = Y$. □

Consider the left action of $\mathcal{G}^\vee$ on $\mathcal{D}/\mathcal{G}^\vee$, and the induced action of $\mathfrak{g}^\vee$ by vector fields. Note that the quotient of $\mathcal{D}/\mathcal{G}^\vee$ exists as an ind-variety; $\mathcal{D}/(\mathcal{B}_- \times \mathcal{B}_+)$ is a product of opposite affine Grassmannians, and $\mathcal{D}/\mathcal{G}^\vee$ is a torus bundle over it (compare with [Kum02, p. 7.2]). The fibers of the projection from $\mathcal{G}$ to $\mathcal{D}/\mathcal{G}^\vee$ are the orbits of right multiplication by $\Gamma := \mathcal{G} \cap \mathcal{G}^\vee$. This intersection is a finite group, specifically the group of square roots of the identity in H. The image of $\mathcal{G}$ in $\mathcal{D}/\mathcal{G}^\vee$ is open by the following proposition and the fact that the quotient map $\mathcal{G} \to \mathcal{G}/\mathcal{B}_\pm$ is open [Kum02, p. 7.4.10].

Proposition 3.3.8. *The image of the multiplication map $\mathcal{G} \times \mathcal{G}^\vee \to \mathcal{D}$, which is the same as the image of $\mathcal{G} \times (\mathcal{B}_- \times \mathcal{B}_+) \to \mathcal{D}$, is the open set $\{(g, g') \mid g^{-1}g' \in \mathcal{G}_0\}$. Here $\mathcal{G}_0$ is the image of $\mathcal{U}_- \times H \times \mathcal{U}_+$ in $\mathcal{G}$ as in Proposition 2.1.6. Similarly, the image of $\mathcal{G}^\vee \times \mathcal{G} \to \mathcal{D}$ is the open set $\{(g, g') \mid g(g')^{-1} \in \mathcal{G}_0\}$.*

Proof. If $(g, g') = (kb_-, kb_+)$ for some $k \in \mathcal{G}$, $(b_-, b_+) \in \mathcal{G}^\vee$, then $g^{-1}g' = b_-^{-1}b_+ \in \mathcal{G}_0$. Conversely, if $g^{-1}g' \in \mathcal{G}_0$ choose $u_\pm \in \mathcal{U}_\pm$ and $h \in H$ such that $g^{-1}g' = u_- h^2 u_+$. Then in $\mathcal{D}$ we have the factorization

$$(g, g') = (gu_-h, gu_-h) \cdot (h^{-1}u_-^{-1}, hu_+),$$

proving the first claim. The second then follows by taking the inverses of the two subsets considered in the first statement. □

In particular the map $\mathcal{G} \to \mathcal{D}/\mathcal{G}^\vee$ induces isomorphisms on the tangent spaces at every point. Thus we can pull back vector fields on $\mathcal{D}/\mathcal{G}^\vee$ to vector fields on $\mathcal{G}$.

Proposition 3.3.9. *Pulling back the vector fields on $\mathcal{D}/\mathcal{G}^\vee$ corresponding to the infinitesimal left action of $\mathfrak{g}^\vee$, we obtain exactly the dressing vector fields on $\mathcal{G}$.*

Proof. We apply the uniqueness statement of Proposition 3.3.7. That these vector fields linearize to the coadjoint action at the identity follows from eq. (3.3.4). Twisted multiplicativity follows from differentiating the following version at the group level.

Consider the open set $\mathcal{D}_0 = \{(g, g') \mid g^{-1}g', g(g')^{-1} \in \mathcal{G}_0\}$. By Proposition 3.3.8, any element of $\mathcal{D}_0$ can be written as $d \cdot g$ for some $(d, g) \in \mathcal{G}^\vee \times \mathcal{G}$. We can also factor it as $g^d \cdot d^g$ for some $(g^d, d^g) \in \mathcal{G} \times \mathcal{G}^\vee$, where g^d and d^g are uniquely defined up to right and left multiplication by Γ, respectively. In particular, the (local) left action of $\mathcal{G}^\vee$ on the image of $\mathcal{G}$ in $\mathcal{D}/\mathcal{G}^\vee$ can be written $\ell_d : g\mathcal{G}^\vee \mapsto g^d\mathcal{G}^\vee$. But now by considering an element of the form ghd, where $g, h \in \mathcal{G}$, we obtain the identity $(g \cdot h)^d = g^d \cdot h^{(d^g)}$. This equality must be taken modulo the action of Γ. However, since Γ is finite it is strictly true in a neighborhood of $e \in \mathcal{G}^\vee$ in the analytic topology, and this is sufficient to obtain the corresponding statement about the infinitesimal action of $\mathfrak{g}^\vee$ as in [LW90]. □

Proof of Theorem 3.3.3. The orbits of the action of $\mathcal{B}_\pm$ on $\mathcal{G}/\mathcal{B}_\pm$ are Schubert cells, which in particular are smooth finite-dimensional subvarieties. It follows straightforwardly that the orbits of the action of $\mathcal{G}^\vee$ on $\mathcal{D}/\mathcal{G}^\vee$ are also smooth finite-dimensional subvarieties, and since $\mathcal{G} \to \mathcal{D}/\mathcal{G}^\vee$ is étale the same is true of the preimages of these orbits in $\mathcal{G}$.

By Proposition 3.3.9, the tangent space to such a preimage at any $g \in \mathcal{G}$ is exactly the span of the dressing vector fields at that point. Note that the span of the $X_\mu|_g$ in $T_g\mathcal{G}$ for $\mu \in \mathfrak{g}^\vee$ is the same as the span of the $X_\mu|_g$ with μ arbitrary, since this subspace is finite-dimensional and $\mathfrak{g}^\vee$ is dense in $\mathfrak{g}^*$. Thus the connected components of the preimages of the $\mathcal{G}^\vee$-orbits in $\mathcal{D}/\mathcal{G}^\vee$ are symplectic leaves of $\mathcal{G}$. But these are exactly the intersections of $\mathcal{G}$ with the double cosets of $\mathcal{G}^\vee$ in $\mathcal{D}$. □

The intersections of $\mathcal{G}$ with the double cosets of $\mathcal{G}^\vee$ are characterized by the following theorem. This was proved in the finite-dimensional case in [KZ02] and [Hof+00], and with Theorem 3.3.3; the proofs given there apply verbatim in the general case.

Theorem 3.3.10. *Given $u, v \in W$, let $H^{u,v} \subset H$ be the subgroup of elements of the form $(\dot{u}^{-1}h^{-1}\dot{u})(\dot{v}^{-1}h\dot{v})$, and let $S^{u,v} = \{g \in G^{u,v} | [\dot{u}^{-1}]_0\dot{v}^{-1}[g\dot{v}^{-1}]_0\dot{v} \in H^{u,v}\}$. Then the intersections of $\mathcal{G}^{u,v}$ with the double cosets of $\mathcal{G}^\vee$ in $\mathcal{D}$ are the subsets $S^{u,v} \cdot h$ for $h \in H$. In particular, the symplectic leaves of a fixed double Bruhat cell are isomorphic with one another.*

Explicit Poisson Brackets on Double Bruhat Cells

Recall from Section 3.3 that the double Bruhat cell $\mathcal{G}^{u,v}$ is a Poisson subvariety of $\mathcal{G}$. By modifying the map $x_\mathbf{i}$ of Definition 4.2.1, we now realize the symplectic leaves of $\mathcal{G}^{u,v}$ (more precisely, their intersections with $\mathcal{G}_i$) as reductions of a Hamiltonian torus action. In particular, we obtain modified factorization coordinates along with explicit formulas for their Poisson brackets. This analysis will be revisited from the point of view of cluster $\mathcal{X}$-coordinates in Section 4.3.

First observe that $SL_2^{(d)}$ has two distinguished symplectic leaves

$$S_+^d = \left\{ \begin{pmatrix} A & B \\ 0 & A^{-1} \end{pmatrix} : A, B \neq 0 \right\}, \quad S_-^d = \left\{ \begin{pmatrix} D^{-1} & 0 \\ C & D \end{pmatrix} : C, D \neq 0 \right\}.$$

The Poisson brackets on S_+^d and S_-^d are given by $\{B, A\} = dAB$ and $\{D, C\} = dCD$, respectively. Now define a symplectic variety

$$S_{\mathbf{i}} := S_{\epsilon(i_1)}^{|d_{i_1}|} \times \cdots \times S_{\epsilon(i_m)}^{|d_{i_m}|},$$

where $\epsilon(i_j)$ is the sign of i_j.

If H_k is the Cartan subgroup of G_{α_k}, we also define two tori

$$H_{\mathbf{i}} := (H/H') \times \prod_{n_{\mathbf{i}}(k)=0} H_k, \quad \widehat{H}_{\mathbf{i}} := \prod_{n_{\mathbf{i}}(k)\neq 0} H_k^{n_{\mathbf{i}}(k)-1}.$$

Here $n_{\mathbf{i}}(k)$ is the total number of times the simple reflection s_k appears in our reduced expressions for u and v, that is,

$$n_{\mathbf{i}}(k) = \#\{j : |i_j| = k, 1 \leq j \leq m\}.$$

As before, $H' = H \cap \mathcal{G}'$ is the subgroup of H generated by the coroots.

Definition 3.3.11. Let $\phi_{\mathbf{i}}$ be the map given by

$$\phi_{\mathbf{i}} : H_{\mathbf{i}} \times S_{\mathbf{i}} \to \mathcal{G}^{u,v}, \quad (a, g_{i_1}, \ldots, g_{i_m}) \mapsto a \cdot \phi_{i_1}(g_{i_1}) \cdots \phi_{i_m}(g_{i_m}).$$

We can define a similar map for the derived subgroup $\mathcal{G}'$ by omitting the H/H' factor in the definition of $H_{\mathbf{i}}$. □

Proposition 3.3.12. *The map $\phi_{\mathbf{i}}$ is Poisson, with $H_{\mathbf{i}}$ being given the trivial Poisson structure. Its image is $\mathcal{G}_{\mathbf{i}}$ and its fibers are the orbits of a simply transitive action of $\widehat{H}_{\mathbf{i}}$.*

Proof. The first assertion follows from Proposition 3.2.14. That the image of $\phi_{\mathbf{i}}$ is $\mathcal{G}_{\mathbf{i}}$ follows from a straightforward comparison of the definitions of $\phi_{\mathbf{i}}$ and $x_{\mathbf{i}}$. We describe the action of $\widehat{H}_{\mathbf{i}}$ by considering each of the $H_k^{n_{\mathbf{i}}(k)-1}$ factors individually. For each k let $j_1 < \cdots < j_{n_{\mathbf{i}}(k)}$ be the indices such that $|i_{j_n}| = k$. Then for any element $t_n^{h_k}$ of the nth H_k factor, where $1 \leq n \leq n(k) - 1$, let

$$\begin{aligned} t_n^{h_k} \cdot (a, g_{i_1}, \ldots, g_{i_m}) = (a, g_{i_1}, \ldots, g_{i_{j_n}} \cdot t_n^{h_k}, \ldots, t_n^{-h_k} \cdot g_{i_\ell} \cdot t_n^{h_k}, \ldots \\ \ldots, t_n^{-h_k} \cdot g_{i_{j_{n+1}}}, \ldots, g_{i_m}). \end{aligned}$$

Here $t_n^{h_k} \cdot g_{i_\ell} \cdot t_n^{-h_k}$ refers to the conjugation action of $\phi_k(H_k)$ on $\phi_{i_\ell}(S_\pm)$. □

In particular, $\phi_{\mathbf{i}}$ induces an isomorphism between the invariant ring $\mathbb{C}[H_{\mathbf{i}} \times S_{\mathbf{i}}]^{\widehat{H}_{\mathbf{i}}}$ and the coordinate ring $\mathbb{C}[\mathcal{G}_{\mathbf{i}}]$. Since we know the Poisson brackets of the coordinate functions on $H_{\mathbf{i}} \times S_{\mathbf{i}}$, we obtain an explicit description of the Poisson structure of $\mathcal{G}_{\mathbf{i}}$.

3.4 Integrable Systems via Affine Double Bruhat Cells

We now turn to our motivating application of the abstract theory of the previous sections, the construction of integrable systems on the reduced Coxeter double Bruhat cells of $\widetilde{LG}$.

Affine Coxeter Double Bruhat Cells

In this section we specialize the discussion of Section 3.3 to the affine case $\mathcal{G}' \cong \widetilde{LG}$, and explicitly calculate the factorization coordinates and their Poisson brackets for a distinguished class of double Bruhat cells. We moreover consider the quotient of $\widetilde{LG}^{u,v}$ by the conjugation action of H, laying the ground for our analysis of the Hamiltonians of the integrable systems constructed in the next section.

Definition 3.4.1. If u and v are Coxeter elements of the affine Weyl group we say that $\widetilde{LG}^{u,v}$ is a *Coxeter double Bruhat cell.* Recall that $w \in W$ is a Coxeter element if in some (hence any) reduced expression for w each simple reflection appears exactly once. □

We may write any reduced word for v as $s_{\sigma(0)} \dots s_{\sigma(r)}$ for some permutation $\sigma \in S_{r+1}$, and likewise any reduced word for u as $s_{\tau(0)} \dots s_{\tau(r)}$ for some permutation τ. Given reduced words for u and v, we will only explicitly write out the factorization coordinates for the unshuffled double reduced word $\mathbf{i} = (s_{\sigma(0)} \dots s_{\sigma(r)} s_{\tau(0)} \dots s_{\tau(r)})$. This will simplify our notation but still let us perform the calculations needed in Section 3.4.

The map $\phi_{\mathbf{i}}$ of Definition 3.3.11 now takes the form

$$\phi_{\mathbf{i}} : (g_{\sigma(0)}, \dots, g_{\sigma(r)}, g'_{\tau(0)}, \dots, g'_{\tau(r)}) \mapsto \phi_{\sigma(0)}(g_{\sigma(0)}) \dots \phi_{\sigma(r)}(g_{\sigma(r)}) \phi_{\tau(0)}(g'_{\tau(0)}) \dots \phi_{\tau(r)}(g'_{\tau(r)}),$$

where

$$(g_{\sigma(0)}, \dots, g_{\sigma(r)}, g'_{\tau(0)}, \dots, g'_{\tau(r)}) \in S_{\mathbf{i}} = S_+^{d_{\sigma(0)}} \times \dots \times S_+^{d_{\sigma(r)}} \times S_-^{d_{\tau(0)}} \times \dots \times S_-^{d_{\tau(r)}}.$$

We will let A_i, B_i and C_i, D_i denote the standard coordinates on $S_+^{d_i}$ and $S_-^{d_i}$, respectively.

Since u and v are Coxeter elements, the torus $\widehat{H}_{\mathbf{i}}$ is equal to $\prod_{k=0}^r H_k$, and its action on $S_{\mathbf{i}}$ is given by

$$t^{h_k} \cdot (g_{\sigma(0)}, \dots, g'_{\tau(r)}) = (g_{\sigma(0)}, \dots, g_k \cdot t^{h_k}, \dots, t^{-h_k} \cdot g_{\sigma(r)} \cdot t^{h_k}, t^{-h_k} \cdot g'_{\tau(0)} \cdot t^{h_k}, \dots$$
$$\dots, t^{-h_k} \cdot g'_k, \dots, g'_{\tau(r)}).$$

To write this in coordinates we introduce the notation $i <_\sigma k$ to mean $\sigma^{-1}(i) < \sigma^{-1}(k)$, or simply that i appears to the left of k in the reduced word for v; likewise we define $i <_\tau k$.

Then we have

$$t^{h_k} : (A_i, B_i) \to \begin{cases} (A_i, B_i) & i <_\sigma k \\ (tA_i, t^{-1}B_i) & i = k \\ (A_i, t^{-C_{ki}}B_i) & i >_\sigma k \end{cases}, \quad (C_i, D_i) \to \begin{cases} (C_i, t^{C_{ki}}D_i) & i <_\tau k \\ (tC_i, tD_i) & i = k \\ (C_i, D_i) & i >_\tau k \end{cases},$$

where C_{ki} is the corresponding entry in the Cartan matrix of $\widetilde{LG}$. If we let

$$T_i = A_i D_i^{-1}, \quad V_i = B_i D_i (\prod_{k<_\sigma i} D_k^{C_{ki}}), \quad W_i = (\prod_{k>_\tau i} A_k^{-C_{ki}}) A_i^{-1} C_i,$$

then

$$\mathbb{C}[\widetilde{LG}_{\mathbf{i}}] \cong \mathbb{C}[S_{\mathbf{i}}]^{\widehat{H}_{\mathbf{i}}} \cong \mathbb{C}[T_0^{\pm 1}, V_0^{\pm 1}, W_0^{\pm 1}, \ldots, T_r^{\pm 1}, V_r^{\pm 1}, W_r^{\pm 1}].$$

In Section 3.4 we will consider the quotient of $\widetilde{LG}^{u,v}$ by the adjoint action of H. This is again a Poisson variety, since H acts by Poisson automorphisms. This is similar to the reduced double Bruhat cells considered in [Zel00; YZ08], though they consider the quotient by left multiplication rather than conjugation. We now derive coordinates on $\widetilde{LG}^{u,v}/H$ along with their Poisson brackets.

If $h^k \in \mathfrak{h}$ satisfies $\alpha_i(h^k) = \delta_{i,k}$, then for $k \neq 0$ we have

$$t^{h^k} : (T_i, V_i, W_i) \to \begin{cases} (T_i, t^{-\theta_k}V_i, t^{\theta_k}W_i) & i = 0 \\ (T_i, tV_i, t^{-1}W_i) & i = k \\ (T_i, V_i, W_i) & i \neq 0, k. \end{cases}$$

Now setting $S_i = V_i W_i$ and $Q = V_0(\prod_{i\neq 0} V_i^{\theta_i})$, a straightforward calculation yields

$$\mathbb{C}[\widetilde{LG}_{\mathbf{i}}/H] \cong \mathbb{C}[T_0^{\pm 1}, S_0^{\pm 1}, \ldots, T_r^{\pm 1}, S_r^{\pm 1}, Q^{\pm 1}]. \tag{3.4.2}$$

The Poisson structure is determined by the pairwise brackets of these generators; the nonzero ones are exactly

$$\begin{gathered} \{S_i, T_k\} = 2d_i S_i T_i \delta_{i,k}, \quad \{Q, T_k\} = d_k \theta_k Q T_k, \\ \{S_i, S_k\} = 2d_k C_{ki}([i >_\sigma k >_\tau i] - [i >_\tau k >_\sigma i]) S_i S_k, \\ \{Q, S_k\} = \left(\sum_{i \neq k} \theta_i d_k C_{ki}([i >_\sigma k >_\tau i] - [i >_\tau k >_\sigma i]) \right) Q S_k. \end{gathered} \tag{3.4.3}$$

Here $[i >_\sigma k >_\tau i]$ is equal to 1 if both $i >_\sigma k$ and $k >_\tau i$, and is equal to 0 otherwise (also recall that $\theta_0 = 1$ by convention).

In particular, though the dimensions of the symplectic leaves of $\widetilde{LG}^{u,v}$ depend on the specific choice of u and v, our computations of the bracket on $\widetilde{LG}_{\mathbf{i}}/H$ imply the following:

Proposition 3.4.4. *The symplectic leaves of $\widetilde{LG}_{\mathbf{i}}/H$ are of dimension $2r+2$, and $Q^2(\prod_k S_k^{-\theta_k})$ is a Casimir.*

Complete Integrability

We first recall the following definition:

Definition 3.4.5. A completely integrable Hamiltonian system on an affine Poisson variety is a collection of Poisson-commuting functions $H_1, \ldots, H_n$ whose associated Hamiltonian vector fields are generically independent, and whose number is half the dimension of a generic symplectic leaf (this is the maximum possible number given the independence requirement). □

Invariant functions on $\widetilde{LG}$ Poisson commute with each other by Proposition 3.2.15, and we will construct such functions as follows. Any regular function on G can be pulled back along the evaluation map $\widetilde{LG} \times \mathbb{C}^* \to G$ to a regular function on $\widetilde{LG} \times \mathbb{C}^*$. Choosing a coordinate z on $\mathbb{C}^*$ identifies the coordinate ring of $\widetilde{LG} \times \mathbb{C}^*$ with the set of regular maps $\widetilde{LG} \to \mathbb{C}[z^{\pm 1}]$. If our original function on G is the character of a representation V, we refer to the resulting map $\widetilde{LG} \to \mathbb{C}[z^{\pm 1}]$ as the *evaluation character* of V. The coefficient of any power of z in an evaluation character is then an invariant scalar function on $\widetilde{LG}$.

Together, all such coefficients of evaluation characters provide an infinite collection of pairwise Poisson-commuting functions on $\widetilde{LG}$. Thus a natural strategy for constructing integrable systems is to restrict these functions to the double Bruhat cells of $\widetilde{LG}$. On a general cell, however, it may be that too few of these functions remain independent to form a maximal set of Poisson-commuting functions. Our main theorem provides a sufficient condition for obtaining an integrable system this way, or more precisely after reducing by the conjugation action of H.

Theorem 3.4.6. *The reduced Coxeter double Bruhat cell $\widetilde{LG}^{u,v}/H$ is the phase space of an integrable system whose Hamiltonians $H_1, \ldots, H_{r+1}$ are coefficients of evaluation characters. We take $H_1, \ldots, H_r$ to be the constant coefficients of the evaluation characters of the r fundamental representations of G, and H_{r+1} to be the z-linear coefficient of the evaluation character of a certain representation V. This is the irreducible representation whose highest weight is in the W-orbit of $\mu := -\sum_{k\neq 0}(\theta_k + \sum_{j>_\sigma k}\theta_j C_{kj})\omega_k$, where the ω_k are the fundamental dominant weights of G and $\theta_0 = 1$.*

Note that in the statement of the theorem we could have taken V to be any sufficiently large representation. The given choice is essentially the minimal possible choice to ensure that H_{r+1} restricts nontrivially to $\widetilde{LG}^{u,v}/H$.

Proof. By Proposition 3.4.4 the symplectic leaves of $\widetilde{LG}^{u,v}/H$ are $(2r+2)$-dimensional, so the stated functions will form an integrable system once we show that their Hamiltonian vector fields remain independent when restricted to $\widetilde{LG}^{u,v}/H$. Since $\widetilde{LG}_{\mathbf{i}}$ is dense in $\widetilde{LG}^{u,v}$ it suffices to consider their restrictions to $\widetilde{LG}_{\mathbf{i}}/H$, where we can use the explicit coordinates given by eq. (3.4.2).

First we show that H_{r+1} is nonzero when restricted to $\widetilde{LG}^{u,v}/H$. We can compute the evaluation character of V by decomposing the action of g with respect to a weight basis. Specifically, let V_λ be the λ-weight space of V, π_λ the projection of V onto V_λ given by the weight space decomposition, and H_λ the regular function defined by $H_\lambda(g) := \mathrm{tr}_{V_\lambda}(\pi_\lambda \circ g)$. Then $H_{r+1} = \sum H_\lambda$, where the sum runs over the nonzero weight spaces of V.

Recall that for any $g \in \widetilde{LG}_{\mathbf{i}}$ we have the factorization

$$g = \phi_{\sigma(0)}(g_{\sigma(0)}) \dots \phi_{\sigma(r)}(g_{\sigma(r)}) \phi_{\tau(0)}(g'_{\tau(0)}) \dots \phi_{\tau(r)}(g'_{\tau(r)}), \tag{3.4.7}$$

where

$$g_i = \begin{pmatrix} A_i & B_i \\ 0 & A_i^{-1} \end{pmatrix}, \quad g'_i = \begin{pmatrix} D_i^{-1} & 0 \\ C_i & D_i \end{pmatrix}.$$

From Lemma 3.4.8 we conclude that the weight spaces in V of weight $\mu + \sum_{k \geq j} \theta_{\sigma(k)} \alpha_{\sigma(k)}$ are nonzero for all j. From this and eq. (3.4.7) we see that for any $v \in V_\mu$, the component of $\phi_{\sigma(j)}(g_j) \dots \dots \phi_{\sigma(r)}(g_r) \cdot v$ of weight $\mu + \sum_{k \geq j} \theta_{\sigma(k)} \alpha_{\sigma(k)}$ is nonzero for all j. Since $s_{\sigma(0)} \dots s_{\sigma(r)}(\mu) = \mu$, it follows that the z-linear term of H_μ contains a monomial whose B_i components are exactly $B_0(\prod_{i \neq 0} B_i^{\theta_i})$. One can compute from the weight spaces involved that this monomial does not depend on the A_i. By inspecting the generators of $\mathbb{C}[\widetilde{LG}_{\mathbf{i}}/H]$ from eq. (3.4.2) we conclude that this monomial must be a scalar multiple of Q. In particular H_μ can be written as a sum of scalar multiple of Qz and other terms not of this form. The reader may check using eq. (3.4.7) that H_λ cannot contain any scalar multiple of Qz unless $\lambda = \mu$. In particular, the z-linear term of the evaluation character is nonzero, since we have ruled out any cancellation of the Qz.

The independence of H_{r+1} and the remaining Hamiltonians follows from the fact that the restriction of H_{r+1} to $\widetilde{LG}_{\mathbf{i}}/H$ is linear in Q, while the other Hamiltonians do not depend on Q. Indeed, suppose M is any monomial in the restriction of an evaluation character to $\widetilde{LG}_{\mathbf{i}}/H$. It is straightforward to see that the power of z accompanying M is the difference of the exponents of B_0 and C_0 in M. Since Q is the only generator of $\mathbb{C}[\widetilde{LG}_{\mathbf{i}}/H]$ whose powers of B_0 and C_0 are distinct, it follows that the z^k-term of an evaluation character has degree k with respect to Q.

Finally, we claim that the Hamiltonians $H_1, \dots, H_r$ are algebraically independent. Decompose each H_i as $J_i + K_i$, where J_i has degree zero with respect to the S_i, and K_i is a sum of monomials of nonzero degree in the S_i. Since H_i is the restriction of a function on $\widetilde{LG}$, $\lim_{B_j, C_j \to 0} H_i$ exists for all j, so these monomials are in fact of positive degree in the S_i.

We claim that the J_i are independent. The projection $\widetilde{H} \to H$ induces an inclusion $\mathbb{C}[H] \subset \mathbb{C}[\widetilde{H}]$, and we identify $\mathbb{C}[\widetilde{H}]$ with $\mathbb{C}[T_0^{\pm 1}, \dots, T_r^{\pm 1}]$ in the obvious way. Then restricting the characters of the i fundamental representations to H and including them in $\mathbb{C}[T_0^{\pm 1}, \dots, T_r^{\pm 1}]$, we obtain exactly the functions J_i; it is a standard result that the restrictions of the fundamental characters to H are independent.

Now suppose there is some polynomial relation among the H_i. That is, for some polynomial p in r variables we have $p(H_1, \dots, H_r) = 0$. For any polynomial p we can consider the decomposition of $p(H_1, \ldots, H_r)$ into a component of degree zero in the S_i and a component which depends nontrivially on the S_i. But the K_i are all of strictly positive degree in the S_i, hence the degree zero part of $p(H_1, \ldots, H_r)$ is exactly $p(J_1, \ldots, J_r)$. Thus $p(H_1, \ldots, H_r) = 0$ implies $p(J_1, \ldots, J_r) = 0$, so p must be identically zero. Finally, one can check using eq. (3.4.3) and Proposition 3.4.4 that for the Hamiltonians $H_1, \ldots, H_{r+1}$, their algebraic independence implies the generic independence of their Hamiltonian vector fields. □

Lemma 3.4.8. *We have $s_{\sigma(j)} \ldots s_{\sigma(r)}(\mu) = \mu + \sum_{k \geq j} \theta_{\sigma(k)} \alpha_{\sigma(k)}$ for all j. Here s_0, α_0 are understood as s_θ, $-\theta$ rather than affine simple roots. In particular, $s_{\sigma(0)} \ldots s_{\sigma(r)}(\mu) = \mu$, since $\theta_0 \alpha_0 = -\sum_{i \neq 0} \theta_i \alpha_i$.*

Proof of Lemma 3.4.8. We induct on j: assuming the statement for $j+1$ we compute that

$$\begin{aligned}
s_{\sigma(j)} \ldots s_{\sigma(r)}(\mu) &= s_{\sigma(j)}(\mu + \sum_{k>j} \theta_{\sigma(k)} \alpha_{\sigma(k)}) \\
&= (\mu + \sum_{k>j} \theta_{\sigma(k)} \alpha_{\sigma(k)}) - \langle \mu + \sum_{k>j} \theta_{\sigma(k)} \alpha_{\sigma(k)} | h_{\sigma(j)} \rangle \alpha_{\sigma(j)} \\
&= \mu + \sum_{k \geq j} \theta_{\sigma(k)} \alpha_{\sigma(k)}
\end{aligned}$$

For $\sigma(j) \neq 0$ the last equality follows from the definition of μ, while for $\sigma(j) = 0$ it follows from calculating that:

$$\begin{aligned}
\langle \mu + \sum_{k >_\sigma 0} \theta_k \alpha_k | h_0 \rangle &= \langle \mu + \sum_{k >_\sigma 0} \theta_k \alpha_k | - \sum_{k \neq 0} d_k \theta_k h_k \rangle \\
&= \sum_{k \neq 0} d_k \theta_k (\theta_k + \sum_{j >_\sigma k} \theta_j C_{kj}) - \sum_{\substack{k \neq 0 \\ j >_\sigma 0}} d_k \theta_k \theta_j C_{kj} \\
&= \sum_{k \neq 0} d_k \theta_k (\theta_k + \sum_{\substack{j \neq 0 \\ j >_\sigma k}} \theta_j C_{kj}) + \sum_{k <_\sigma 0} d_k \theta_k C_{k0} - \sum_{\substack{k \neq 0 \\ j >_\sigma 0}} d_k \theta_k \theta_j C_{kj} \\
&= \frac{1}{2} \sum_{j,k \neq 0} d_k \theta_j \theta_k C_{kj} - \sum_{\substack{j \neq 0 \\ k <_\sigma 0}} d_k \theta_j \theta_k C_{kj} - \sum_{\substack{k \neq 0 \\ j >_\sigma 0}} d_k \theta_k \theta_j C_{kj} \\
&= -1.
\end{aligned}$$

Here we use the fact that $\sum_{j,k \neq 0} d_k \theta_j \theta_k C_{kj} = \langle \theta | h_\theta \rangle = 2$, $C_{k0} = -\sum_{j \neq 0} \theta_j C_{kj}$, and $C_{kk} = 2$. □

Remark 3.4.9. Even for double Bruhat cells on which there are too few independent coefficient functions to obtain an integrable system, it was shown in [Res03] that in the finite-dimensional case one obtains superintegrable systems. This is a stronger statement than simply having a collection of Poisson-commuting functions. In particular, the dynamics are restricted to isotropic analogues of Liouville tori. One expects this to hold in the affine case as well, but we do not pursue this here. □

The Relativistic Periodic Toda System

We now show that the relativistic periodic Toda system of [Rui90] can be realized (up to symplectic reduction) as an affine Coxeter-Toda system of type $A_n^{(1)}$ for a natural choice of Coxeter elements. In canonical coordinates p_k, q_k this system corresponds to the Hamiltonian

$$\sum_{k=0}^{m} e^{hp_k}(1 + h^2 exp(q_{k+1} - q_k)), \tag{3.4.10}$$

where h is a nonzero parameter and we impose the periodic boundary conditions $p_{k+m+1} = p_k$, $q_{k+m+1} = q_k$ [Sur91]. For now we consider the complex form where p_k and q_k take values in $\mathbb{C}$.

Consider the double Bruhat cell of $\widetilde{LSL_n}$ with u and v both equal to the element $s_0 s_1 \cdots s_n$, where the simple roots of SL_n are numbered in the usual way. We note that from the computations in Section 3.4 it follows that the symplectic leaves of this cell are already $(2r+2)$-dimensional, so the corresponding Coxeter-Toda system is integrable before reduction by H.

If $H_1 \in \mathbb{C}[(\widetilde{LSL_n})_{\mathbf{i}}]$ is the Hamiltonian obtained from the constant term of the character of the defining representation of SL_n, a simple calculation yields that

$$H_1 = \sum_{i=0}^{n} T_i T_{i-1}^{-1}(1 + S_i), \tag{3.4.11}$$

where T_{-1} and S_{-1} are read as T_n and S_n.

To connect this with the relativistic Toda system, we introduce auxiliary variables $c_0, \dots, c_n, d_0, \dots, d_n$, on which we define a Poisson structure by setting

$$\{c_k, d_k\} = 2c_k d_k, \quad \{c_k, d_{k+1}\} = -2c_k d_{k+1}, \quad \{c_k, c_{k+1}\} = -2c_k c_{k+1},$$

with all other brackets among the generators equal to zero (here d_{n+1} and c_{n+1} are understood as d_0 and c_0). The algebra $\mathbb{C}[c_0^{\pm 1}, d_0^{\pm 1}, \dots, c_n^{\pm 1}, d_n^{\pm 1}]$ is then the coordinate ring of a $(2n+2)$-dimensional Poisson torus with $2n$-dimensional symplectic leaves.

Now observe that this Poisson variety can be obtained as a reduction of both $(\widetilde{LSL_n})_{\mathbf{i}}$ and the phase space of the relativistic Toda system (for $m = n$ and $h = 2$). That is, we have surjective Poisson maps given by

$$c_i \mapsto S_i T_i T_{i-1}^{-1}, \quad d_i \mapsto T_i T_{i-1}^{-1} \quad \text{and} \quad c_i \mapsto 4e^{2p_i - q_i + q_{i+1}}, \quad d_i \mapsto e^{2p_i}.$$

Moreover, the following proposition is clear from eqs. (3.4.10) and (3.4.11):

Proposition 3.4.12. *The Hamiltonian*

$$H_1 = \sum_{i=0}^{n} c_i + d_i$$

pulls back to the Hamiltonians of the relativistic Toda and Coxeter-Toda systems under the maps given above, hence defines a Hamiltonian system which is a common reduction of these two integrable systems.

Finally, we recall that the relativistic Toda system is usually defined on the real phase space with canonical coordinates p_k, q_k. Because of the exponentials in the Hamiltonian, the corresponding real slice of the Coxeter-Toda phase space is the subset of $(\widetilde{LSL_n})_{\mathbf{i}}$ on which the factorization coordinates take *positive* real values. This totally positive part of the double Bruhat cell has many interesting combinatorial properties and was the principal motivation for [FZ99]. Thus in the present context we find that total positivity arises naturally when we compare our construction with the usual real form of the relativistic Toda system.

Chapter 4

Cluster Duality and Kac-Moody Groups

4.1 Introduction

The goals of this chapter are to exhibit the Chamber Ansatz of [FZ99] as an example of duality between cluster varieties, and to extend the construction of cluster structures on double Bruhat cells of algebraic groups to the setting of symmetrizable Kac-Moody groups.

The dsicovery of cluster algebras by Fomin and Zelevinsky was precipitated in part by their analysis of the identities satisfied by generalized minors encountered in the study of double Bruhat cells [FZ99]. These minors were used to write explicit formulas for the inverses of certain birational parametrizations of these cells, generalizing the Chamber Ansatz previously introduced in the context of unipotent cells [BFZ96; BZ97]. After the axiomatization of cluster algebras in [FZ02], these generalized minors were reinterpreted as cluster variables in an upper cluster algebra structure on the coordinate ring of the double Bruhat cell [BFZ05].

Soon after [FZ02] it was discovered that the combinatorial data encoding a cluster algebra encodes a second, dual type of algebraic structure, variously called coefficients or Y-variables [FZ07], τ-coordinates [GSV03], and $\mathcal{X}$-coordinates [FG09]. The two structures may be regarded as a dual pair of varieties covered by toric charts and connected by a regular map, which in a precise sense is a geometrization of the natural map from a lattice with a skew-symmetric form to its dual. Concrete instances of this map include the projection from decorated Teichmüller space to Teichmüller space [FG07] and the transformation of T-system solutions to corresponding Y-system solutions [KNS11]. In [FG06a] a class of $\mathcal{X}$-coordinates were constructed on the double Bruhat cells of the adjoint form of a semisimple algebraic group. These are given by another family of birational parametrizations of the cell, related to those studied in [FZ99] but defined in terms of coweight subgroups rather than one-parameter unipotent subgroups. However, the relationship between these $\mathcal{X}$-coordinates and the cluster variables of [BFZ05] was not studied explicitly.

Our first main result is to demonstrate that the generalized Chamber Ansatz of [FZ99], when expressed in terms of the coweight parametrization of a double Bruhat cell, is in fact an instance of the map between dual cluster varieties. In particular, this change of variables turns the initially opaque formulas of [FZ99] into ones whose form is completely intuitive from the perspective of the general theory. Moreover, we prove this in the setting of an arbitrary symmetrizable Kac-Moody group, generalizing along the way many previous results of [FZ99; BFZ05; FG06a] on the double Bruhat cells of semisimple algebraic groups. In particular, we show that the coordinate rings of all such double Bruhat cells are upper cluster algebras, verifying a conjecture of [BFZ05].

Theorem. (4.3.2) *The double Bruhat cells $G^{u,v}, G^{uv}_{\mathrm{Ad}}$ of a symmetrizable Kac-Moody group and its adjoint form have the structure of a dual pair of cluster varieties. This identifies the twist map of [FZ99] and its infinite-dimensional generalization with the natural map between dual cluster varieties, up to the addition of nondegenerate terms intertwining frozen variables. The Poisson structure on G^{uv}_{Ad} inherited from the standard r-matrix Poisson structure of Section 3.2 coincides with the canonical cluster Poisson structure.*

Whereas cluster variables are motivated by the theory of canonical bases, $\mathcal{X}$-coordinates are more natural from the perspective of Poisson geometry. In particular, an exchange matrix endows the corresponding $\mathcal{X}$-coordinates with a canonical Poisson bracket, which in the case of double Bruhat cells coincides with that induced by the standard Poisson structure on the group. The characters of the group restrict to Poisson-commuting functions on the double Bruhat cell, and in some cases form a completely integrable system [Hof+00; Res03]. Many interesting examples come from non-unipotent cells in affine Kac-Moody groups (as in Section 3.4), and this is one of our main motivations for studying double Bruhat cells in this generality. Moreover, this context calls specific attention to role of the coweight parametrization, in that the resulting $\mathcal{X}$-coordinates provide the link between these systems and those constructed from the dimer partition function of a bipartite torus graph [FM13; GK11].

4.2 Coordinates on Double Bruhat Cells

When G is a semisimple algebraic group, each double Bruhat cell $G^{u,v}$ is endowed with several natural families of coordinate systems. To any double reduced word for (u, v) is associated a parametrization of $G^{u,v}$ by one-parameter simple root subgroups, the definition of which is motivated by the theory of total positivity [FZ99]. In [FG06a], a modified version of this parametrization was introduced on the adjoint form of G using coweight subgroups; the resulting coordinates are convenient for working with the standard Poisson bracket, and transform as cluster $\mathcal{X}$-coordinates as the double reduced word is varied.

Explicitly describing the inverse maps to these parametrizations amounts to solving certain factorization problems in the group. In the case of one-parameter simple root subgroups the solution was found in terms of twisted generalized minors in [FZ99]. In Section 4.2 we extend this result to the setting of symmetrizable Kac-Moody groups, after generalizing the various coordinates as necessary in Section 4.2. In Section 4.2 we use this to solve the corresponding factorization problem for the coweight parametrization. In the process we will directly recover the entries of the exchange matrix defined in [BFZ05].

Double Reduced Words and Parametrizations

Let G be a symmetrizable Kac-Moody group and $G^{u,v}$ a fixed double Bruhat cell. A *double reduced word* $\mathbf{i} = (i_1, \ldots, i_m)$ for (u, v) is a shuffle of a reduced word for u written in the alphabet $\{-1, \ldots, -r\}$ and a reduced word for v written in the alphabet $\{1, \ldots, r\}$.

Definition 4.2.1. Let $\mathbf{i}$ be a double reduced word for (u, v), and set $m = \ell(u) + \ell(v)$. Let $T_{\mathbf{i}}$ denote the complex torus $(\mathbb{C}^*)^{m+\widetilde{r}}$ with coordinates $t_1, \ldots, t_{m+\widetilde{r}}$. Then we have a map $x_{\mathbf{i}} : T_{\mathbf{i}} \to G$ given by

$$x_{\mathbf{i}} : (t_i, \ldots, t_{m+\widetilde{r}}) \mapsto x_{i_1}(t_1) \cdots x_{i_m}(t_m) t_{m+1}^{\alpha_1^\vee} \cdots t_{m+\widetilde{r}}^{\alpha_{\widetilde{r}}^\vee}.$$

Here $x_i(t)$ and $x_{-i}(t)$ denote the one-parameter subgroups corresponding to α_i and $-\alpha_i$, respectively. When G is an algebraic group this was defined in [FZ99], where the following result was also proved. $\square$

Proposition 4.2.2. *The map $x_{\mathbf{i}}$ is an open immersion from $T_{\mathbf{i}}$ to $G^{u,v}$.*

Proof. First we show that the image of $x_{\mathbf{i}}$ is contained in $G^{u,v}$. For each $1 \leq i \leq r$, we have $x_i(t) \in \mathcal{B}_+$ and $x_{-i}(t) \in \mathcal{B}_+ s_i \mathcal{B}_+$. Thus if $k_1 < \cdots < k_{\ell(u)} \subset \{1, \ldots, m\}$ are the indices of the negative entries in $\mathbf{i}$,

$$x_{\mathbf{i}}(t_1, \ldots, t_{m+\widetilde{r}}) \in \mathcal{B}_+ \cdots \mathcal{B}_+ s_{i_{k_1}} \mathcal{B}_+ \cdots \mathcal{B}_+ s_{i_{k_{\ell(u)}}} \mathcal{B}_+ \cdots \mathcal{B}_+.$$

Recall that for $w, w' \in W$,

$$\mathcal{B}_+ w \mathcal{B}_+ \cdot \mathcal{B}_+ w' \mathcal{B}_+ = \mathcal{B}_+ ww' \mathcal{B}_+$$

whenever $\ell(ww') = \ell(w) + \ell(w')$ [Kum02, p. 5.1.3]. Thus in particular $x_{\mathbf{i}}(t_1, \ldots, t_{m+\widetilde{r}}) \in \mathcal{B}_+ u \mathcal{B}_+$, and by the same argument $x_{\mathbf{i}}(t_1, \ldots, t_{m+\widetilde{r}}) \in \mathcal{B}_- v \mathcal{B}_-$.

Suppose that

$$x_{\mathbf{i}}(t_1, \ldots, t_{m+\widetilde{r}}) = x_{\mathbf{i}}(t'_1, \ldots, t'_{m+\widetilde{r}})$$

but $(t_1, \ldots, t_{m+\widetilde{r}}) \neq (t'_1, \ldots, t'_{m+\widetilde{r}})$, and let k be the smallest index such that $t_k \neq t'_k$. If $k > m$ this is a contradiction, since an element of H factors uniquely as a product of coroot subgroups.

On the other hand, if $k \leq m$, then $\mathbf{i}' := (i_k, \ldots, i_m)$ is a double reduced word for some (u', v'), and $x_{\mathbf{i}'}(t_k, \ldots, t_{m+\widetilde{r}}) = x_{\mathbf{i}'}(t'_k, \ldots, t'_{m+\widetilde{r}})$. Multiplying both sides on the left by $x_{i_k}(-t'_k)$, we obtain

$$x_{\mathbf{i}'}(t_k - t'_k, \ldots, t_{m+\widetilde{r}}) = x_{\mathbf{i}''}(t'_{k+1}, \ldots, t'_{m+\widetilde{r}}),$$

where $\mathbf{i}'' := (i_{k+1}, \dots, i_m)$. But by the first part of the proposition the left and right sides lie in different double Bruhat cells, hence by contradiction $x_{\mathbf{i}}$ must be injective. But an injective regular map between smooth complex varieties of the same dimension is an open immersion, and the proposition follows. □

A closely related family of parametrizations was introduced in [FG06a] for semisimple algebraic groups. Whereas so far we have taken G to be simply-connected, to describe these $\mathcal{X}$-coordinates we must consider its adjoint version. When the Cartan matrix is not of full rank and the center of G is positive-dimensional, we will abuse terminology and use G_{Ad} to denote a variant of the adjoint group.

Recall from Section 2.1 that the fundamental weight basis of P induces a dual basis of the cocharacter lattice $\mathrm{Hom}(\mathbb{C}^*, H)$. We denote it by $\alpha_1^\vee, \dots, \alpha_{\widetilde{r}}^\vee$ since the first r are exactly the coroots of G. In parallel with this we define elements $\alpha_{r+1}, \dots, \alpha_{\widetilde{r}}$ of P by

$$\alpha_i = D \sum_{j=1}^{r} d_j^{-1} C_{ij} \omega_j,$$

where D is the least common integer multiple of $d_1, \dots, d_r$. Then $\oplus_{1 \le i \le \widetilde{r}} \mathbb{Z}\alpha_i$ is a full rank sublattice of P, and its kernel $\{h \in H | h^{\alpha_i} = 1, 1 \le i \le \widetilde{r}\}$ is a discrete subgroup of the center of G. We let G_{Ad} denote the quotient of G by this discrete subgroup. Of course, if C has full rank this is exactly the adjoint form of G.

If H_{Ad} is the image of H in G_{Ad}, the character lattice of H_{Ad} is canonically isomorphic with $\oplus_{1 \le i \le \widetilde{r}} \mathbb{Z}\alpha_i$. In particular, the cocharacter lattice of H_{Ad} inherits a dual basis $\omega_1^\vee, \dots, \omega_{\widetilde{r}}^\vee$ of *fundamental coweights* such that $\langle \alpha_i | \omega_j^\vee \rangle = \delta_{i,j}$ for $1 \le i, j \le \widetilde{r}$. We will denote elements of the corresponding one-parameter subgroups of H_{Ad} by $t^{\omega_i^\vee}$, where $t \in \mathbb{C}^*$; in other words, $t^{\omega_i^\vee}$ is defined so that

$$(t^{\omega_i^\vee})^{\alpha_j} = t^{\delta_{ij}}.$$

We can now define $C_{ij} := \langle \alpha_j | \alpha_i^\vee \rangle$ for all $1 \le i, j \le \widetilde{r}$. The definitions of α_i for $i > r$ are chosen exactly to obtain the following proposition, which the reader may easily verify.

Proposition 4.2.3. *The $\widetilde{r} \times \widetilde{r}$ integer matrix with entries C_{ij} is nondegenerate and symmetrizable (with $d_i = D$ for $i > r$). Moreover, the coweights and coroots are related by*

$$\alpha_i^\vee = \sum_{j=1}^{\widetilde{r}} C_{ij} \omega_j^\vee.$$

Example 4.2.4. Let G be the untwisted affine Kac-Moody group corresponding to a simply-connected simple algebraic group $\mathring{G}$. That is, G is the semidirect product of $\mathbb{C}^*$ and the universal central extension of the group of regular maps from $\mathbb{C}^*$ to $\mathring{G}$. Then the center $Z(\mathring{G})$ of $\mathring{G}$ sits inside G as constant maps, and we may choose the fundamental coweights so that $G_{\mathrm{Ad}} = G/Z(\mathring{G})$. □

Definition 4.2.5. Let $\mathbf{i} = (i_1, \ldots, i_m)$ be a double reduced word for (u, v), and let I denote the index set $I = \{-\widetilde{r}, \ldots, -1\} \cup \{1, \ldots, m\}$. Let $\mathcal{X}_\mathbf{i}$ denote the torus $(\mathbb{C}^*)^I$ with coordinates $\{X_i\}_{i \in I}$. We will write $E_i := x_i(1)$ for $i \in \{\pm 1, \ldots, \pm r\}$. Then we have a map $x_\mathbf{i} : \mathcal{X}_\mathbf{i} \to G^{u,v}_{\mathrm{Ad}}$ given by

$$x_\mathbf{i} : (X_{-\widetilde{r}}, \ldots, X_m) \mapsto X_{-\widetilde{r}}^{\omega^\vee_{\widetilde{r}}} \cdots X_{-1}^{\omega^\vee_1} E_{i_1} X_1^{\omega^\vee_{|i_1|}} \cdots E_{i_j} X_j^{\omega^\vee_{|i_j|}} \cdots E_{i_m} X_m^{\omega^\vee_{|i_m|}}.$$

□

Though we have also used $x_\mathbf{i}$ to denote the map of Definition 4.2.1, it will always be clear from the context which we mean. The following proposition may be deduced straightforwardly from Theorem 5.2.8.

Proposition 4.2.6. *The map $x_\mathbf{i} : \mathcal{X}_\mathbf{i} \to G^{u,v}_{\mathrm{Ad}}$ is an open immersion. Moreover, the restriction of the quotient map $\pi_G : G^{u,v} \to G^{u,v}_{\mathrm{Ad}}$ to $T_\mathbf{i}$ is a finite covering of $\mathcal{X}_\mathbf{i}$.*

In particular, the t_i and X_i may be regarded as implicitly defined rational coordinates on $G^{u,v}$ and $G^{u,v}_{\mathrm{Ad}}$. In [FZ99], the former coordinates were explicitly described in the semisimple case in terms of a certain family of generalized minors whose definition we now recall.

Given an index $1 \leq k \leq m$ and a double reduced word $\mathbf{i}$, we define two Weyl group elements

$$u_{<k} := s_{i_1}^{\frac{1}{2}(1-\epsilon_1)} \cdots s_{i_{(k-1)}}^{\frac{1}{2}(1-\epsilon_{(k-1)})}, \quad v_{>k} := s_{i_n}^{\frac{1}{2}(\epsilon_n+1)} \cdots s_{i_{k+1}}^{\frac{1}{2}(\epsilon_{(k+1)}+1)},$$

where ϵ_k is equal to 1 if $i_k > 0$ and -1 if $i_k < 0$. In short, $u_{<k}$ is the part of the reduced word for u whose indices in $\mathbf{i}$ are less than k, and $v_{>k}$ is the inverse of the part of the reduced word for v whose indices in $\mathbf{i}$ are greater than k. For purposes of the following definition, we will also set $v_{>k} = v^{-1}$ if $k < 0$.

Definition 4.2.7. If $\mathbf{i} = (i_1, \ldots, i_m)$ is a double reduced word for (u, v), let I denote the index set $\{-\widetilde{r}, \ldots, -1\} \cup \{1, \ldots, m\}$ and let $i_k = k$ for $k < 0$. Then to each $k \in I$ we associate a generalized minor

$$A_{k,\mathbf{i}} := \Delta^{\omega_{|i_k|}}_{u_{\leq k}, v_{>k}}.$$

When the choice of double reduced word is clear we will abbreviate this to A_k. □

Remark 4.2.8. One may define the postive part $G^{u,v}_{>0}$ of $G^{u,v}$ as the image of $\mathbb{R}^{m+\widetilde{r}}_{>0} \subset T_\mathbf{i}$ in $G^{u,v}$; when G is a semisimple algebraic group this is an important object in the theory of total positivity, the study of which motivated the work [FZ99]. Though total positivity will not play a direct role in the present article, we note in passing that the above definition of $G^{u,v}_{>0}$ agrees with the analogous definition in terms of the coweight parametrization. That is, if $g \in G^{u,v}_{>0}$ it follows straightforwardly that $\pi_G(g) \in G^{u,v}_{\mathrm{Ad}}$ is in the image of $\mathbb{R}^{m+\widetilde{r}}_{>0} \subset \mathcal{X}_\mathbf{i}$. □

The Twist Isomorphism

To precisely describe the relationships among the various coordinates introduced in Section 4.2, we will require a certain isomorphism of inverse double Bruhat cells, called the twist map in [FZ99]. In this section we recall its key properties, which extend readily to the setting of Kac-Moody groups.

Definition 4.2.9. We write $x \mapsto x^\theta$ for the automorphism of G which acts as follows on the Cartan subgroup and Chevalley generators:

$$a^\theta = a^{-1} \quad (a \in H), \quad x_i(t)^\theta = x_{-i}(t) \quad (1 \leq i \leq r).$$

□

Definition 4.2.10. For any $u, v \in W$, the *twist map* $\zeta^{u,v} : G^{u,v} \to G^{u^{-1},v^{-1}}$ is defined by

$$\zeta^{u,v} : x \mapsto \left([\overline{u}^{-1}x]_-^{-1}\overline{u}^{-1}x\overline{v^{-1}}[x\overline{v^{-1}}]_+^{-1}\right)^\theta. \tag{4.2.11}$$

□

Proposition 4.2.12. *The twist map $\zeta^{u,v}$ is an isomorphism of $G^{u,v}$ and $G^{u^{-1},v^{-1}}$ whose inverse is $\zeta^{u^{-1},v^{-1}}$.*

Proof. That $\zeta^{u,v}$ is well-defined on $G^{u,v}$ follows from Corollary 2.1.11. To see that $x' = \zeta^{u,v}(x) \in B_-\dot{v}^{-1}B_-$, we simplify eq. (4.2.11) as

$$x' = \left([\overline{u}^{-1}x]_0[\overline{u}^{-1}x]_+y_-^{-1}\right)^\theta \overline{v}^{-1} \in G_0\overline{v}^{-1},$$

where $y_- = \pi_-(x)$ as in Corollary 2.1.11. In particular,

$$[x'\overline{v}]_+ = (y_-^{-1})^\theta \in N_-(v)^\theta = N_+(v^{-1}), \tag{4.2.13}$$

hence $x' \in B_-\dot{v}^{-1}B_-$. Similarly one can see that

$$[\overline{\overline{u}}x']_- = (y_+^{-1})^\theta \in N_-(u^{-1}), \tag{4.2.14}$$

hence $x' \in B_+\dot{u}^{-1}B_+$. But now the fact that $\zeta^{u,v}$ and $\zeta^{u^{-1},v^{-1}}$ are inverse to each other follows from plugging our expressions for $[x'\overline{v}]_+$ and $[\overline{\overline{u}}x']_-$ into the definition of $\zeta^{u^{-1},v^{-1}}$ and simplifying. □

Proposition 4.2.15. *The twist map $\zeta^{u,v}$ restricts to an isomorphism of the open sets $G_0^{u,v}$ and $G_0^{u^{-1},v^{-1}}$. Moreover, if $x \in G_0^{u,v}$, $x' = \zeta^{u,v}(x)$, we have*

$$[x']_0 = [\overline{u}^{-1}x]_0^{-1}[x]_0[x\overline{v^{-1}}]_0^{-1}. \tag{4.2.16}$$

Proof. We can rewrite eq. (4.2.11) as

$$x' = \left([\overline{u}^{-1}x]_0[\overline{u}^{-1}x]_+x^{-1}[x\overline{v^{-1}}]_-[x\overline{v^{-1}}]_0\right)^\theta,$$

and the proposition follows from taking the Cartan part of each side. □

If $w = s_{i_1}\cdots s_{i_{\ell(w)}}$ is a reduced word for $w \in W$, we define Weyl group elements

$$w_{<k} := s_{i_1}\cdots s_{i_{k-1}}, \quad w_{>k} := s_{i_{\ell(w)}}\cdots s_{i_k},$$

and similarly $w_{\leq k}$, $w_{\geq k}$.

Proposition 4.2.17. *If $x \in G_0^{u,v}$, $x' = \zeta^{u,v}(x)$, and $1 \leq j \leq \widetilde{r}$,*

$$\Delta^{\omega_j}_{v_{>k},e}(y_-) = \frac{\Delta^{\omega_j}_{e,v_{\leq k}}(x')}{\Delta^{\omega_j}_{e,v}(x')}, \quad \Delta^{\omega_j}_{e,u_{<k}}(y_+) = \frac{\Delta^{\omega_j}_{u_{\geq k},e}(x')}{\Delta^{\omega_j}_{u^{-1},e}(x')}.$$

Proof. First we claim that if $y_\pm = \pi_\pm(x)$ and $y'_\pm = \pi_\pm(x')$, then

$$y'_+ = \overline{\overline{u}}^{-1}(y_+^{-1})^\theta\overline{\overline{u}}, \quad y'_- = \overline{v}(y_-^{-1})^\theta\overline{v}^{-1}.$$

This follows straightforwardly from eq. (4.2.13) and eq. (4.2.14).

We can use these identities to write

$$\Delta^{\omega_j}_{v_{>k},e}(y_-) = \Delta^{\omega_j}(\overline{\overline{v_{\leq k}}}^{-1}\overline{\overline{v}}y_-) = \Delta^{\omega_j}(\overline{\overline{v_{\leq k}}}^{-1}(y_-'^{-1})^\theta\overline{\overline{v}}).$$

One can check that $\Delta^{\omega_j}((g^{-1})^\theta) = \Delta^{\omega_j}(g)$ for all $g \in G$, hence

$$\Delta^{\omega_j}(\overline{\overline{v_{\leq k}}}^{-1}(y_-'^{-1})^\theta\overline{\overline{v}}) = \Delta^{\omega_j}(\overline{v}^{-1}y'_-\overline{v_{\leq k}}).$$

By Corollary 2.1.11, $x' = b_-\overline{v}^{-1}y'_-$ for some $b_- \in B_-$. Then

$$\Delta^{\omega_j}(\overline{v}^{-1}y'_-\overline{v_{\leq k}}) = \Delta^{\omega_j}(b_-^{-1}x'\overline{v_{\leq k}}) = [b_-]_0^{-\omega_j}\Delta^{\omega_j}(x'\overline{v_{\leq k}}).$$

Now since $\overline{v}^{-1}y'_-\overline{v} \in N_+$,

$$\Delta^{\omega_j}_{e,v}(x') = \Delta^{\omega_j}(b_-\overline{v}^{-1}y'_-\overline{v}) = [b_-]_0^{\omega_j}.$$

But then

$$[b_-]_0^{-\omega_j}\Delta^{\omega_j}(x'\overline{v_{\leq k}}) = \frac{\Delta^{\omega_j}_{e,v_{\leq k}}(x')}{\Delta^{\omega_j}_{e,v}(x')},$$

proving the first part of the proposition. The remaining statement then follows by essentially the same argument. □

Factorization in Unipotent Groups

In Theorem 4.2.24 we derive expressions for the t_i as Laurent monomials in the twists of the A_i, generalizing the main result of [FZ99] to the Kac-Moody setting. The strategy of the proof is the same as in the finite-dimensional case. We build up to the main theorem by solving a series of more elementary factorization problems, starting with the factorization of the unipotent subgroup $N_-(w)$ as a product of one-parameter subgroups. This in turn lets us solve the factorization problem for the unipotent cell $N_+^w := N_+ \cap B_-\dot{w}B_-$. From here we can extract the solution for a general double Bruhat cell by reducing to the case of an "unmixed" double reduced word.

For $w \in W$, recall the unipotent group $N_-(w) = N_- \cap \dot{w}^{-1}N_+\dot{w}$ and fix a reduced word $w = s_{i_1} \cdots s_{i_n}$. For short we will write

$$w_k := w_{\geq k} = s_{i_n} \cdots s_{i_k}.$$

Now define one-parameter subgroups

$$y_k(p_k) = \overline{w_{k+1}} x_{-i_k}(p_k) \overline{w_{k+1}}^{-1},$$

where we take $w_{n+1} = e$.

Lemma 4.2.18. *For any $p_k \in \mathbb{C}$ we have*

$$\overline{w_m}^{-1} y_k(p_k) \overline{w_m} \in \begin{cases} N_- & m > k \\ N_+ & m \leq k. \end{cases}$$

Proof. Follows straightforwardly from the standard fact that if $\ell(ws_i) > \ell(w)$ for some $w \in W$, then $w(\alpha_i)$ is again a positive root. $\square$

Proposition 4.2.19. *The map $y_{\mathbf{i}} : \mathbb{C} \to N_-(w)$ given by*

$$(p_1, \ldots, p_n) \mapsto y = y_1(p_1) \cdots y_n(p_n)$$

is an isomorphism. Its inverse is given explicitly by

$$p_k = \Delta^{\omega_{i_k}}_{w_k, w_{k+1}}(y).$$

Proof. That $y_{\mathbf{i}}$ is an isomorphism is well-known [GLS11, p. 5.2]. Let $y_k = y_k(p_k)$ be as in Lemma 4.2.18, and

$$y_{<k} = y_1 \cdots y_{k-1}, \quad y_{>k} = y_{k+1} \cdots y_n.$$

In particular,

$$y = y_{<k} \cdot y_k \cdot y_{>k}.$$

It follows from Lemma 4.2.18 that

$$\overline{w_k}^{-1} y_{<k} \overline{w_k} \in N_-, \quad \overline{w_{k+1}}^{-1} y_{>k} \overline{w_{k+1}} \in N_+.$$

But we then have

$$\begin{aligned}\Delta^{\omega_{i_k}}_{w_k,w_{k+1}}(y) &= \Delta^{\omega_{i_k}}((\overline{w_k}^{-1}y_{<k}\overline{w_k})\overline{w_k}^{-1}y_k\overline{w_{k+1}}(\overline{w_{k+1}}^{-1}y_{>k}\overline{w_{k+1}}))\\ &= \Delta^{\omega_{i_k}}(\overline{w_k}^{-1}y_k\overline{w_{k+1}})\\ &= \Delta^{\omega_{i_k}}(\overline{s_{i_k}}^{-1}x_{-i_k}(p_k))\\ &= p_k.\end{aligned}$$

The first two lines follow from the definitions of the generalized minors, while the last is a simple computation in SL_2 representation theory (similar to eq. (2.1.24)). □

Factorization in Unipotent Cells

We can now solve the factorization problem for the unipotent cell $N^w_+ := N_+ \cap B_-\dot{w}B_-$. Given a reduced word $w = s_{i_1}\cdots s_{i_n}$, N^w_+ has a birational parametrization

$$(\mathbb{C}^*)^n \to N^w_+, \quad (t_1,\dots,t_n) \mapsto x_{i_1}(t_1)\cdots x_{i_n}(t_n).$$

The inverse map is described in Proposition 4.2.23, which relies on the following two lemmas.

Lemma 4.2.20. *Let $1 \le i \le r$. Then any $x \in N_-$ can be written as $\overline{s_i}x'\overline{s_i}^{-1}x_{-i}(t)$ for some $x' \in N_-$ and $t \in \mathbb{C}$. Morevover, t is given by*

$$t = \Delta^{\omega_i}_{s_i,e}(x).$$

Proof. That g admits such an expression is an immediate consequence of Proposition 2.1.9. To verify that t is given by the stated formula, we check that

$$\begin{aligned}\Delta^{\omega_i}_{s_i,e}(x) &= \Delta^{\omega_i}(x'\overline{s_i}x_{-i}(t))\\ &= \Delta^{\omega_i}(\overline{s_i}x_{-i}(t))\\ &= t.\end{aligned}$$

The last line is another simple SL_2 computation. □

Lemma 4.2.21. *Let $x = x_{i_1}(t_1)\cdots x_{i_n}(t_n) \in N^w_+$ and $x' = x_{i_2}(t_2)\cdots x_{i_n}(t_n) \in N^{w'}_+$. Here $w' = s_{i_1}w$, and $\mathbf{i}' = (i_2,\dots,i_n)$ is a reduced word for w'. Let $p_2,\dots,p_n$ be complex numbers such that $y' = \pi_-(x') = y_{\mathbf{i}'}(p_2,\dots,p_n)$. Then*

$$y = \pi_-(x) = y_{\mathbf{i}}(p_1,\dots,p_n),$$

where

$$p_1 := \Delta^{\omega_{i_1}}_{s_{i_1},e}(x_{-i_1}([\overline{\overline{w'}}y']_0^{-\alpha_{i_1}}t_1^{-1})[\overline{\overline{w'}}y']_-^{-1}).$$

Moreover, t_1 can be recovered as

$$t_1 = [\overline{\overline{w'}}y']_0^{\omega_{i_1}-\alpha_{i_1}}[\overline{\overline{w}}y]_0^{-\omega_{i_1}}.$$

Proof. We denote $y_{\mathbf{i}}(p_1, \ldots, p_n)$ by $\widetilde{y}$ during the proof. To show $y = \widetilde{y}$ it suffices to show that $\overline{\overline{w}}\widetilde{y} \in G_0$ and $[\overline{\overline{w}}\widetilde{y}]_+ = x$, or equivalently that $\overline{\overline{w}}\widetilde{y}x^{-1} \in B_-$. Now one can calculate that

$$\overline{\overline{w}}\widetilde{y}x^{-1} = \overline{s_{i_1}}^{-1}x_{-i_1}(p_1)[\overline{\overline{w'}}y']_-[\overline{\overline{w'}}y']_0x_1(-t_1). \tag{4.2.22}$$

Applying Lemma 4.2.20 to $x_{-i_1}([\overline{\overline{w'}}y']_0^{-\alpha_{i_1}}t_1^{-1})[\overline{\overline{w'}}y']_-^{-1}$, we know that

$$x_{-i_1}([\overline{\overline{w'}}y']_0^{-\alpha_{i_1}}t_1^{-1})[\overline{\overline{w'}}y']_-^{-1} = \overline{s_{i_1}}y''\overline{s_{i_1}}^{-1}x_{-i_1}(p_1)$$

for some $y'' \in N_-$. Combining this with eq. (4.2.22) lets us write

$$\begin{aligned}\overline{\overline{w}}\widetilde{y}x^{-1} &= (y'')^{-1}\overline{s_{i_1}}^{-1}x_{-i_1}([\overline{\overline{w'}}y']_0^{-\alpha_{i_1}}t_1^{-1})[\overline{\overline{w'}}y']_0x_{i_1}(-t_1)\\ &= (y'')^{-1}\overline{s_{i_1}}^{-1}[\overline{\overline{w'}}y']_0x_{-i_1}(t_1^{-1})x_{i_1}(-t_1)\\ &= (y'')^{-1}\overline{s_{i_1}}^{-1}[\overline{\overline{w'}}y']_0\overline{s_{i_1}}t_1^{-\alpha_{i_1}^\vee}x_{-i_1}(-t_1^{-1}) \in B_-.\end{aligned}$$

The last line can be checked directly in $\varphi_{i_1}(SL_2)$.

If we take the H-components of each side, we see further that

$$[\overline{\overline{w}}y]_0 = \overline{s_{i_1}}^{-1}[\overline{\overline{w'}}y']_0\overline{s_{i_1}}t_1^{-\alpha_{i_1}^\vee}.$$

The last assertion then follows by applying the character ω_{i_1} to each side. □

Proposition 4.2.23. *Let $t_1, \ldots, t_n$ be nonzero complex numbers and let $x = x_{i_1}(t_1)\cdots x_{i_n}(t_n) \in N_+^w$. Then*

$$t_k = \frac{1}{\Delta^{\omega_{i_k}}_{w_k,e}(y)\Delta^{\omega_{i_k}}_{w_{k+1},e}(y)}\prod_{\substack{1\le j\le\widetilde{r}\\ j\neq i_k}}(\Delta^{\omega_j}_{w_{k+1},e}(y))^{-C_{j,i_k}},$$

where $y = \pi_-(x) \in N_-(w)$ and $w_k = s_{i_n}\cdots s_{i_k}$.

Proof. Let

$$x_{\ge k} := x_{i_k}(t_k)\cdots x_{i_n}(t_n), \quad y_{\ge k} = \overline{w_k}[x_{\ge k}\overline{w_k}]_+\overline{w_k}^{-1}, \quad z_{\ge k} = \overline{w_k}^{-1}y_{\ge k}.$$

Then applying Lemma 4.2.21 to $x_{\ge k}$ we obtain

$$t_k = [z_{\ge(k+1)}]_0^{\omega_{i_k}-\alpha_{i_k}}[z_{\ge k}]_0^{-\omega_{i_k}}.$$

We claim then that $[z_{\ge k}]_0 = [\overline{w_k}^{-1}y]_0$. This follows from

$$\begin{aligned}\overline{w_k}^{-1}y &= (\overline{w_k}^{-1}y_{<k}\overline{w_k})\overline{w_k}^{-1}y_{\ge k}\\ &= (\overline{w_k}^{-1}y_{<k}\overline{w_k})z_{\ge k},\end{aligned}$$

and the observation that

$$(\overline{w_k}^{-1} y_{<k} \overline{w_k}) \in N_-$$

which follows from Lemma 4.2.18. But then

$$\begin{aligned}
t_k &= [\overline{w_{k+1}}^{-1} y]_0^{(\omega_{i_k} - \alpha_{i_k})} [\overline{w_k}^{-1} y]_0^{-\omega_{i_k}} \\
&= [\overline{w_{k+1}}^{-1} y]_0^{(\omega_{i_k} - \sum_{1 \le j \le \widetilde{r}} C_{j,i_k} \omega_j)} [\overline{w_k}^{-1} y]_0^{-\omega_{i_k}} \\
&= [\overline{w_{k+1}}^{-1} y]_0^{(-\omega_{i_k} - \sum_{j \neq i_k} C_{j,i_k} \omega_j)} [\overline{w_k}^{-1} y]_0^{-\omega_{i_k}} \\
&= \frac{1}{\Delta^{\omega_{i_k}}_{w_k,e}(y) \Delta^{\omega_{i_k}}_{w_{k+1},e}(y)} \prod_{\substack{1 \le j \le \widetilde{r} \\ j \neq i_k}} (\Delta^{\omega_j}_{w_{k+1},e}(y))^{-C_{j,i_k}},
\end{aligned}$$

completing the proof. $\square$

Factorization in Double Bruhat Cells

We now turn to the factorization problem in an arbitrary double Bruhat cell $G^{u,v}$.

Let $\mathbf{i} = (i_1, \ldots, i_m)$ be a double reduced word for (u, v). For $1 \le j \le m$ and $k \in I = \{-\widetilde{r}, \ldots, -1\} \cup \{1, \ldots, m\}$, we define[1]

$$\Psi_{j,k} := -\epsilon_j \epsilon_k \big([j = k] + [j = k^+]\big) + \frac{C_{|i_k|,|i_j|}}{2} \Bigg(\epsilon_j (\epsilon_{k^+} - \epsilon_k)[k^+ < j] - (1 + \epsilon_j \epsilon_k)[k < j < k^+] \Bigg);$$

let us explain the notation. For an index $k \in I$, we let

$$k^+ := \min\{\ell \in I : \ell > k, |i_\ell| = |i_k|\},$$

setting $k^+ = m + 1$ if there are no such ℓ (recall that we set $i_k = k$ for $k < 0$). Also recall that ϵ_k is equal to 1 if $i_k > 0$ and -1 if $i_k < 0$, with $\epsilon_{m+1} = 1$ for purposes of the above formula. Note that $\Psi_{j,k}$ can only take the values 0, ± 1, and $\pm C_{|i_k|,|i_j|}$.

For $k \in I$, recall the generalized minors

$$A_k := A_{k,\mathbf{i}} = \Delta^{\omega_{|i_k|}}_{u_{\le k}, v_{>k}}$$

from Definition 4.2.7. We let $x \mapsto x^\iota$ denote the involutive antiautomorphism of G determined by

$$a^\iota = a^{-1} \text{ for } a \in H, \quad x_i(t)^\iota = x_i(t) \text{ for } 1 \le i \le r.$$

It is clear that ι restricts to an isomorphism of $G^{u,v}$ and $G^{u^{-1},v^{-1}}$, hence in particular $\zeta^{u^{-1},v^{-1}} \circ \iota$ is an automorphism of $G^{u,v}$.

[1] Recall that if $P(x_1, \ldots)$ is a boolean function of some variables $\{x_1, \ldots\}$, $[P(x_1, \ldots)]$ denotes the integer-valued function of the x_i whose value is 1 when P is true and 0 when P is false.

Theorem 4.2.24. *Let G be a symmetrizable Kac-Moody group, $u, v \in W$, and $\mathbf{i} = (i_1, \ldots, i_m)$ a double reduced word for (u, v). Then if $x = x_{\mathbf{i}}(t_1, \ldots, t_{m+\widetilde{r}})$ and $x' = (\zeta^{u^{-1},v^{-1}} \circ \iota)(x)$, we have*

$$t_j = \prod_{k \in I} A_k(x')^{\Psi_{j,k}} \tag{4.2.25}$$

for $1 \leq j \leq m$, and

$$t_{m+j} = \prod_{\substack{k \in I \\ |i_k| = j}} A_k(x')^{\frac{1}{2}(\epsilon_{k^+} - \epsilon_k)}. \tag{4.2.26}$$

for $1 \leq j \leq r$.[2]

Proof. The double reduced word $\mathbf{i} = (i_1, \ldots, i_m)$ for (u, v) induces an opposite double reduced word $\mathbf{i}^{\mathrm{op}} = (j_1, \ldots, j_m)$ for (u^{-1}, v^{-1}), by setting $j_k = i_{m+1-k}$. Let $k^{\mathrm{op}} := m + 1 - k$ and $t'_k := t_{k^{\mathrm{op}}}$, so that

$$x^\iota = t_{m+\widetilde{r}}^{-\alpha_{\widetilde{r}}^\vee} \cdots t_{m+1}^{-\alpha_1^\vee} x_{j_1}(t'_1) \cdots x_{j_m}(t'_m).$$

We first consider the case where $\mathbf{i}$ is "unmixed"; that is, $k < \ell$ whenever $\epsilon_k > 0$ and $\epsilon_\ell < 0$. Then $x^\iota \in G_0^{u,v}$ and $[x^\iota]_0 = t_{m+\widetilde{r}}^{-\alpha_{\widetilde{r}}^\vee} \cdots t_{m+1}^{-\alpha_1^\vee}$. By Propositions 4.2.12 and 4.2.15 we have

$$t_{m+j} = [x^\iota]_0^{-\omega_j} = [\overline{u}^{-1} x']_0^{\omega_j} [x']_0^{-\omega_j} [x' \overline{v^{-1}}]_0^{\omega_j}.$$

One can then check that this agrees with eq. (4.2.26) in this case.

Next observe that since $\mathbf{i}$ is unmixed, $y_- := \pi_-(x^\iota)$ is equal to $\pi_-([x^\iota]_+)$, and

$$[x^\iota]_+ = x_{j_{\ell(v)^{\mathrm{op}}}}(t'_{\ell(v)^{\mathrm{op}}}) \cdots x_{j_m}(t'_m) \in N_+^{v^{-1}}.$$

For $1 \leq k \leq \ell(v)$, we can use Proposition 4.2.23 to obtain

$$t_k = t'_{k^{\mathrm{op}}} = \frac{1}{\Delta^{\omega_{i_k}}_{(v^{-1})_{>(k+1)^{\mathrm{op}}},e}(y_-) \Delta^{\omega_{i_k}}_{(v^{-1})_{>k^{\mathrm{op}}},e}(y_-)} \left(\prod_{\substack{1 \leq j \leq \widetilde{r} \\ j \neq i_k}} (\Delta^{\omega_j}_{(v^{-1})_{>k^{\mathrm{op}}},e}(y_-))^{-C_{j,|i_k|}} \right).$$

Applying Proposition 4.2.17 to each term and using the observation that $(v^{-1})_{\leq k^{\mathrm{op}}} = v_{\geq k}$, we can rewrite this as

$$t_k = \frac{1}{\Delta^{\omega_{i_k}}_{e,v_{\geq(k+1)}}(x') \Delta^{\omega_{i_k}}_{e,v_{\geq k}}(x')} \left(\prod_{\substack{1 \leq j \leq \widetilde{r} \\ j \neq i_k}} \Delta^{\omega_j}_{e,v_{\geq k}}(x')^{-C_{j,|i_k|}} \right) \left(\prod_{1 \leq j \leq \widetilde{r}} \Delta^{\omega_j}_{e,v^{-1}}(x')^{C_{j,|i_k|}} \right)$$

[2] Though equivalent to [FZ99, Theorem 1.9] in finite type, the formulation here differs slightly to better match the conventions of [BFZ05]. The statement in [FZ99] does not involve ι, and correspondingly the t_i are expressed in terms of cluster variables on the inverse double Bruhat cell $G^{u^{-1},v^{-1}}$. Also, our definition of $\Psi_{j,k}$ differs from the corresponding definition in [FZ99] in order to facilitate the proof of Proposition 4.2.28.

Using the fact that $\mathbf{i}$ is unmixed, one checks that this is equivalent to

$$t_k = A_k(x')^{-1}A_{k^-}(x')^{-1}\Bigg(\prod_{\substack{\ell\in I\\ \ell<k<\ell^+}} A_\ell(x')^{-C_{|i_\ell|,|i_k|}}\Bigg)\Bigg(\prod_{1\le j\le\widetilde{r}} A_{-j}(x')^{C_{j,|i_k|}}\Bigg).$$

Here $k^-\in I$ is defined by $(k^-)^+=k$. Again, the reader may check that this expression agrees with eq. (4.2.25) in this case.

For $\ell(v)<k\le m$, we note that $\pi_+(x^\iota)=\pi_+([x^\iota]_-)$ and if $a=t_{m+1}^{\alpha_1^\vee}\cdots t_{m+\widetilde{r}}^{\alpha_{\widetilde{r}}^\vee}$,

$$[x^\iota]_- = x_{j_1}(a^{\alpha_{|j_1|}}t'_1)\cdots x_{j_{\ell(u)}}(a^{\alpha_{|j_{\ell(u)}|}}t'_{\ell(u)}).$$

From here eq. (4.2.25) follows by a similar argument as above, again invoking Propositions 4.2.17 and 4.2.23. One arrives at

$$\begin{aligned} t_k = A_k(x')^{-1}A_{k^-}(x')^{-1}&\Bigg(\prod_{\substack{\ell\in I\\ \ell<k<\ell^+}} A_\ell(x')^{-C_{|i_\ell|,|i_k|}}\Bigg)\Bigg(\prod_{\ell:\ell^+>m} A_\ell(x')^{C_{|i_\ell|,|i_k|}}\Bigg)\times\\ &\times\Bigg(\prod_{\ell\in I} A_\ell(x')^{-\frac{1}{2}C_{|i_\ell|,|i_k|}(\epsilon_{\ell^+}-\epsilon_\ell}\Bigg),\end{aligned}$$

which agrees with eq. (4.2.25) given that $\mathbf{i}$ is unmixed.

Now suppose two double reduced words $\mathbf{i}$ and $\mathbf{i}'$ differ only by the exchange of two consecutive positive and negative indices. That is, for some $1\le k<m$ and $1\le i,j\le r$ we have

$$i_k = i'_{k+1} = j,\quad i_{k+1}=i'_k=-i.$$

We claim that if the theorem holds for $\mathbf{i}$ it also holds for $\mathbf{i}'$. Specifically, suppose that

$$x = x_{\mathbf{i}}(t_1,\ldots,t_{m+\widetilde{r}}) = x_{\mathbf{i}'}(t'_1,\ldots,t'_{m+\widetilde{r}}),$$

and that the t_ℓ satisfy eqs. (4.2.25) and (4.2.26). Then we claim the t'_ℓ also satisfy eqs. (4.2.25) and (4.2.26) with respect to the $A_{\ell,\mathbf{i}'}$.

This is trivial unless $i=j$. In that case, a straightforward computation in $\varphi_i(SL_2)$ yields that

$$\begin{gathered} t'_{m+i}=t_{m+i}(1+t_kt_{k+1}),\quad t'_{m+\ell}=t_{m+\ell} \text{ for } \ell\ne i,\\ t'_\ell=t_\ell \text{ for } \ell<k,\quad t'_\ell=t_\ell(1+t_kt_{k+1})^{\epsilon_\ell C_{|i,i_\ell|}},\text{ for } k+1<\ell\le m,\\ t'_k=t_{k+1}(1+t_kt_{k+1})^{-1},\quad t'_{k+1}=t_k(1+t_kt_{k+1}).\end{gathered}$$

Using the expression for $(1+t_kt_{k+1})$ provided by Lemma 4.2.27 and simplifying the result, one can then check directly that eqs. (4.2.25) and (4.2.26) hold for the t'_ℓ. But then since the image of $x_{\mathbf{i}}$ intersects the image of $x_{\mathbf{i}'}$ along a dense subset, we conclude that eqs. (4.2.25) and (4.2.26) hold for all points in the image of $x_{\mathbf{i}'}$. $\square$

Lemma 4.2.27. *Suppose Theorem 4.2.24 holds for a double reduced word* $\mathbf{i}$ *with* $i_k = -i_{k+1} = i$ *for some* $1 \leq i \leq r$. *Let* $\mathbf{i}'$ *be the double reduced word obtained by exchanging* i_k *and* i_{k+1}. *Then for* $x = x_{\mathbf{i}}(t_1, \ldots, t_{m+\widetilde{r}})$ *and* $x' = \zeta^{u^{-1},v^{-1}} \circ \iota$ *we have*

$$1 + t_k t_{k+1} = \frac{A_{k,\mathbf{i}}(x')A_{k,\mathbf{i}'}(x')}{A_{k^-,\mathbf{i}}(x')A_{k+1,\mathbf{i}}(x')}.$$

Proof. Letting $u' = u_{<k}, v' = v_{>(k+1)}$, we first calculate that

$$A_{k,\mathbf{i}} = \Delta^{\omega_i}_{u',v'}, \quad A_{k,\mathbf{i}'} = \Delta^{\omega_i}_{u's_i,v's_i},$$
$$A_{k+1,\mathbf{i}} = \Delta^{\omega_i}_{u's_i,v'}, \quad A_{k^-,\mathbf{i}} = \Delta^{\omega_i}_{u',v's_i}.$$

Using eq. (4.2.25) and the fact that $\epsilon_k = -\epsilon_{k+1} = 1$, we also have

$$\begin{aligned} 1 + t_k t_{k+1} &= 1 + A_{k+1,\mathbf{i}}(x')^{-1} A_{k^-,\mathbf{i}}(x')^{-1} \left(\prod_{\ell < k < \ell^+} A_{\ell,\mathbf{i}}(x')^{-C_{|i_\ell|,i}} \right) \\ &= \frac{\Delta^{\omega_i}_{u's_i,v'}(x')\Delta^{\omega_i}_{u',v's_i}(x') + \prod_{\substack{1 \leq j \leq \widetilde{r} \\ j \neq i}} \Delta^{\omega_j}_{u',v'}(x')^{-C_{ji}}}{\Delta^{\omega_i}_{u's_i,v'}(x')\Delta^{\omega_i}_{u',v's_i}(x')}. \end{aligned}$$

But then by Proposition 2.1.21 this yields

$$1 + t_k t_{k+1} = \frac{\Delta^{\omega_i}_{u',v'}(x')\Delta^{\omega_i}_{u's_i,v's_i}(x')}{\Delta^{\omega_i}_{u's_i,v'}(x')\Delta^{\omega_i}_{u',v's_i}(x')},$$

and the lemma follows. ☐

$\mathcal{X}$-coordinates and Generalized Minors

Recall that the coweight parametrization $x_{\mathbf{i}} : \mathcal{X}_{\mathbf{i}} \to G^{u,v}_{\mathrm{Ad}}$ of Definition 4.2.5 yields a set $\{X_i\}_{i \in I}$ of rational coordinates on $G^{u,v}_{\mathrm{Ad}}$. Since the image of $T_{\mathbf{i}}$ in $G^{u,v}$ is a finite cover of $\mathcal{X}_i$ in $G^{u,v}$, the pullbacks of the X_i to $G^{u,v}$ are Laurent monomials in the t_i, and, by Theorem 4.2.24, in the twisted generalized minors. In this section we derive explicit formulas for this, rewriting the generalized Chamber Ansatz of [FZ99] in terms of the X_i. We will see that the resulting formula recovers the exchange matrix defined in [BFZ05].

Proposition 4.2.28. *Fix a double reduced word* $\mathbf{i}$ *for* (u, v), *let* $\{X_i\}_{i \in I}$ *be the corresponding rational coordinates on* $G^{u,v}_{\mathrm{Ad}}$, *and let* $\{A_i\}_{i \in I}$ *be the corresponding generalized minors on* $G^{u,v}$. *Then if* $p_G : G \to G_{\mathrm{Ad}}$ *is the composition of the automorphism* $\iota \circ \zeta^{u,v}$ *of* $G^{u,v}$ *with the quotient map* $G \to G_{\mathrm{Ad}}$, *we have*

$$p_G^*(X_j) = \prod_{k \in I} A_k^{\widetilde{B}_{j,k}}.$$

Here $\widetilde{B} = B + M$, where B and M are the $I \times I$ matrices given by[3]

$$B_{jk} = \frac{C_{|i_k|,|i_j|}}{2}\Big(\epsilon_j[j = k^+] - \epsilon_k[j^+ = k] + \epsilon_j[k < j < k^+][j > 0] - \epsilon_{j^+}[k < j^+ < k^+][j^+ \leq m]$$
$$- \epsilon_k[j < k < j^+][k > 0] + \epsilon_{k^+}[j < k^+ < j^+][k^+ \leq m]\Big)$$

and

$$M_{jk} = \frac{1}{2}C_{|i_k|,|i_j|}\Big([j^+, k^+ > m] + [j, k < 0]\Big).$$

Proof. Recall from Proposition 4.2.6 that the image of $T_{\mathbf{i}}$ in $G^{u,v}$ is a finite cover of $\mathcal{X}_{\mathbf{i}}$ in $G^{u,v}_{\mathrm{Ad}}$ under the quotient map. Thus it follows from Theorem 4.2.24 that there exists some integer matrix N such that

$$p_G^*(X_j) = \prod_{k \in I} A_k^{N_{jk}}.$$

To compute N, define new variables $t'_1, \ldots, t'_{m+\widetilde{r}}$ by

$$t'_k = \prod_{\substack{j<k \\ |i_j|=|i_k|}} X_j^{\epsilon_k}.$$

Here if $k > m$ we set $|i_k| = k - m$ and $\epsilon_k = +1$. The t'_k are uniquely determined by the requirement that

$$X_{-\widetilde{r}}^{\omega^\vee_{\widetilde{r}}} \cdots X_{-1}^{\omega^\vee_1} E_{i_1} X_1^{\omega^\vee_{|i_1|}} \cdots E_{i_m} X_m^{\omega^\vee_{|i_m|}} = x_{i_1}(t'_1) \cdots x_{i_m}(t'_m) \prod_{k=1}^{\widetilde{r}} (t'_{m+k})^{\omega^\vee_k}.$$

Moreover, inverting this change of variables one finds that

$$X_j = \prod_{1 \leq k \leq m+\widetilde{r}} (t'_k)^{D_{jk}}, \tag{4.2.29}$$

where D is the integer matrix with rows labelled by I, columns labelled by $1, \ldots, m + \widetilde{r}$, and

$$D_{jk} = ([j^+ = k] - [j = k])\epsilon_k.$$

We now compare the t'_k with the coordinates t_k on $G^{u,v}$ induced from

$$x_{\mathbf{i}} : (t_1, \ldots, t_{m+\widetilde{r}}) \mapsto x_{i_1}(t_1) \cdots x_{i_m}(t_m) \prod_{k=1}^{\widetilde{r}} (t_{m+k})^{\alpha^\vee_k}.$$

---

[3] We keep the notation introduced at the beginning of Section 4.2.

If $\pi_G : G^{u,v} \to G^{u,v}_{\mathrm{Ad}}$ is the quotient map, then we can check that

$$\pi_G^* t_j' = \prod_{k=1}^{m+\widetilde{r}} t_k^{E_{jk}}, \tag{4.2.30}$$

where E is the $(m+\widetilde{r}) \times (m+\widetilde{r})$ matrix given by

$$E_{jk} = \delta_{jk}[j \le m] + C_{|i_k|,|i_j|}[j,k > m].$$

By Theorem 4.2.24 we have

$$(\iota \circ \zeta^{u,v})^* t_j = \prod_{k \in I} A_k^{F_{j,k}}, \tag{4.2.31}$$

where $F_{j,k}$ is the integer matrix with rows labelled by $1, \dots, m+\widetilde{r}$, columns labelled by I, and

$$F_{jk} = [j \le m]\Psi_{j,k} + \frac{1}{2}[j > m][|i_j| = |i_k|](\epsilon_{k^+} - \epsilon_k).$$

Here $\Psi_{j,k}$ is as in Section 4.2, and if $k_+ > m$ for some $k \in I$, we set $\epsilon_{k^+} = +1$.

We can now compute N by multiplying the matrices D, E, and F, and simplifying the resulting conditional expression. Before doing any serious simplification, a straightforward initial calculation yields

$$N_{jk} = [j^+ \le m]\epsilon_{j^+}\Psi_{j^+,k} - [j > 0]\epsilon_j \Psi_{j,k} + \frac{C_{|i_k|,|i_j|}}{2}[j^+ > m](\epsilon_{k^+} - \epsilon_k). \tag{4.2.32}$$

Unwinding the definition of Ψ we see that

$$\epsilon_j \Psi_{j,k} = \frac{C_{|i_k|,|i_j|}}{2}\Big(-\epsilon_k[j = k] - \epsilon_k[j = k^+] - (\epsilon_j + \epsilon_k)[k < j < k^+] + (\epsilon_{k^+} - \epsilon_k)[k^+ < j]\Big).$$

Plugging this and the corresponding expression for $\epsilon_{j^+}\Psi_{j^+,k}$ into eq. (4.2.32), we obtain

$$\begin{aligned} N_{jk} = \frac{C_{|i_k|,|i_j|}}{2}\Big(&\epsilon_k[j = k]\big([j > 0] - [j^+ \le m]\big) - \epsilon_k[j^+ = k] + \epsilon_k[j = k^+] \\ &+ (\epsilon_j + \epsilon_k)[k < j < k^+][j > 0] - (\epsilon_{j^+} + \epsilon_k)[k < j^+ < k^+][j^+ \le m] \\ &+ (\epsilon_{k^+} - \epsilon_k)\big([k^+ < j^+][j^+ \le m] - [k^+ < j][j > 0] + [j^+ > m]\big)\Big). \end{aligned} \tag{4.2.33}$$

The reader may verify that for any $j, k \in I$,

$$\begin{aligned} &[k^+ < j^+][j^+ \le m] - [k^+ < j][j > 0] + [j^+ > m] \\ &\qquad = [j < k^+ < j^+][k^+ \le m] + [j = k^+] + [j^+, k^+ > m]. \end{aligned}$$

This identity lets us rewrite eq. (4.2.33) as

$$\begin{aligned}N_{jk} = \frac{C_{|i_k|,|i_j|}}{2}\Big(& [j=k]\big([j<0]+[j^+>m]\big)+\epsilon_j[j=k^+]-\epsilon_k[j^+=k] \\ & +(1-\epsilon_k)[j^+,k^+>m][j\neq k]+\epsilon_j[k<j<k^+][j>0] \\ & -\epsilon_{j^+}[k<j^+<k^+][j^+\leq]+\epsilon_{k^+}[j<k^+<j^+][k^+\leq m] \\ & +\epsilon_k\big([k<j<k^+][j>0]-[k<j^+<k^+][j^+\leq m] \\ & -[j<k^+<j^+][k^+\leq m]\big)\Big).\end{aligned} \tag{4.2.34}$$

By another boolean computation the reader may check that

$$\begin{aligned}&[k<j<k^+][j>0]-[k<j^+<k^+][j^+\leq m]-[j<k^+<j^+][k^+\leq m] \\ &\qquad = -[j<k<j^+][k>0]+[j\neq k]\big([j^+,k^+>m]-[j,k<0]\big)\end{aligned}$$

for any $j,k\in I$. But now we can use this to rewrite eq. (4.2.34) as

$$\begin{aligned}N_{jk} &= \frac{C_{|i_k|,|i_j|}}{2}\Big([j^+,k^+>m]+[j,k<0]+\epsilon_j[j=k^+]-\epsilon_k[j^+=k]+\epsilon_j[k<j<k^+][j>0] \\ &\qquad -\epsilon_{j^+}[k<j^+<k^+][j^+\leq]-\epsilon_k[j<k<j^+][k>0]+\epsilon_{k^+}[j<k^+<j^+][k^+\leq m]\Big) \\ &= \widetilde{B}_{j,k},\end{aligned}$$

completing the proof. □

4.3 Double Bruhat Cells as Dual Cluster Varieties

Corresponding to a double reduced word for (u,v) we associated in Section 4.2 a collection of generalized minors. In [FZ99] it was discovered that as the double reduced word is varied, these collections vary by certain subtraction-free relations, which served as prototypes for the cluster algebra exchange relations introduced in [FZ02]. In [BFZ05] it was shown that the generalized minors are organized into an upper cluster algebra structure on the coordinate ring of a double Bruhat cell in a semisimple algebraic group; in this section we extend this result to the double Bruhat cells of any symmetrizable Kac-Moody group.

In fact, the cluster algebra associated with a double Bruhat cell is encoded by an exchange matrix we have already seen, when we computed the inverse of the coweight parametrization in Section 4.2. This is an instance of a general phenomenon, that one can define $\mathcal{X}$-coordinates from cluster variables via the monomial transformation defined by the exchange matrix. In the present situation, however, this is reversed: we start with independently defined cluster variables and $\mathcal{X}$-coordinates, and derive this monomial transformation directly from the Chamber Ansatz. We summarize our main results in Theorem 4.3.2, which relates the simply-connected and adjoint forms of the double Bruhat cell and the twist map as a pair of dual cluster varieties and the natural map between them [FG09].

Seeds Associated with Double Reduced Words

Before reinterpreting the results of Section 4.2 in terms of cluster algebras, let us explain how to associate a seed $\Sigma_{\mathbf{i}}$ with any double reduced word $\mathbf{i}$ for (u, v). This allows us to state the main result, Theorem 4.3.2, which incorporates the generalized minors and twist map into a modified cluster ensemble in the sense of Proposition 2.2.12.

Definition 4.3.1. Let $\mathbf{i}$ be a double reduced word for (u, v), and let $m = \ell(u) + \ell(v)$. We define a seed $\Sigma_{\mathbf{i}}$ as follows. The index set is $I = \{-\widetilde{r}, \dots, -1\} \cup \{1, \dots, m\}$, and an index $k \in I$ is frozen if either $k < 0$ or $k^+ > m$. To each index $k > 0$ is associated a weight $1 \le |i_k| \le \widetilde{r}$, which we extend to $k < 0$ by setting $|i_k| = |k|$. The exchange matrix $B := B_{\mathbf{i}}$ is defined by

$$\begin{aligned} b_{jk} = \frac{C_{|i_k|,|i_j|}}{2} \Big(& \epsilon_j [j = k^+] - \epsilon_k [j^+ = k] \\ & + \epsilon_j [k < j < k^+][j > 0] - \epsilon_{j^+} [k < j^+ < k^+][j^+ \le m] \\ & - \epsilon_k [j < k < j^+][k > 0] + \epsilon_{k^+} [j < k^+ < j^+][k^+ \le m] \Big). \end{aligned}$$

We let $d_k = d_{|i_k|}$, where the right-hand side refers to the symmetrizing factors of the Cartan matrix. One easily checks that the skew-symmetrizability of B follows from the symmetrizability of the Cartan matrix. □

Note that the exchange matrix defined in [BFZ05] is equal to the transpose of the matrix formed by the unfrozen rows of B. Our main results are summarized in the following theorem.

Theorem 4.3.2. *Let G be a symmetrizable Kac-Moody group, $u, v \in W$ elements of its Weyl group, and $\mathbf{i}$ a double reduced word for (u, v). Consider the seed $\Sigma_{\mathbf{i}}$ defined in Definition 4.3.1 and let $\mathcal{A}_{|\Sigma_{\mathbf{i}}|}$, $\mathcal{X}_{|\Sigma_{\mathbf{i}}|}$ be the associated complex $\mathcal{A}$- and $\mathcal{X}$-spaces. Let M be the $I \times I$ matrix with entries*

$$M_{jk} = \frac{1}{2} C_{|i_k|,|i_j|} \Big([j^+, k^+ > m] + [j, k < 0] \Big),$$

and let $p_G : G^{u,v} \to G^{u,v}_{\mathrm{Ad}}$ be the composition of the automorphism $\iota \circ \zeta^{u,v}$ of $G^{u,v}$ from Theorem 4.2.24 and the quotient map from G to G_{Ad}.

1. *There is a regular map $a_{|\Sigma_{\mathbf{i}}|} : \mathcal{A}_{|\Sigma_{\mathbf{i}}|} \to G^{u,v}$ which identifies the generalized minors of Definition 4.2.7 with the corresponding cluster variables on $\mathcal{A}_{\Sigma_{\mathbf{i}}}$. It induces an isomorphism of $\mathbb{C}[G^{u,v}]$ and the upper cluster algebra $\mathbb{C}[\mathcal{A}_{|\Sigma_{\mathbf{i}}|}]$.*

2. *There is a regular map $x_{|\Sigma_{\mathbf{i}}|} : \mathcal{X}_{|\Sigma_{\mathbf{i}}|} \to G^{u,v}_{\mathrm{Ad}}$ which extends the map $\mathcal{X}_{\Sigma_{\mathbf{i}}} \to G^{u,v}_{\mathrm{Ad}}$ of Definition 4.2.5. It is Poisson with respect to the standard Poisson-Lie structure on G_{Ad} and the Poisson structure on $\mathcal{X}_{|\Sigma_{\mathbf{i}}|}$ defined by the exchange matrix B.*

3. *The matrix* $\widetilde{B} = B + M$ *has integer entries, hence there is an associated regular map* $p_M : \mathcal{A}_{|\Sigma_{\mathbf{i}}|} \to \mathcal{X}_{|\Sigma_{\mathbf{i}}|}$*. These maps together form a commutative diagram:*

$$\begin{tikzcd} \mathcal{A}_{|\Sigma_{\mathbf{i}}|} \arrow[r, "a_{|\Sigma_{\mathbf{i}}|}"] \arrow[d, "p_M"] & G^{u,v} \arrow[d, "p_G"] \\ \mathcal{X}_{|\Sigma_{\mathbf{i}}|} \arrow[r, "x_{|\Sigma_{\mathbf{i}}|}"] & G^{u,v}_{\mathrm{Ad}}. \end{tikzcd}$$

The proof will occupy the rest of the chapter. We treat each statement separately, as Theorems 4.3.11, 4.3.16 and 4.3.17.

Remark 4.3.3. In general the map p_0 between dual cluster varieties has positive-dimensional fibers, and its image is a symplectic leaf of the $\mathcal{X}$-space. However, it is clear from Proposition 4.2.28 that p_M is a finite covering map. Thus it is natural to summarize Theorem 4.3.2 as saying that the double Bruhat cells $G^{u,v}$, $G^{u,v}_{\mathrm{Ad}}$ are dual cluster varieties and the map p_G is a nondegenerate version of the natural map, differing only in how the frozen $\mathcal{A}$- and $\mathcal{X}$-variables are related.

This statement should be understood with the caveat that the maps $a_{|\Sigma_{\mathbf{i}}|}$, $x_{|\Sigma_{\mathbf{i}}|}$ are typically not biregular; rather, the complement of their images will have codimension at least 2. In addition, the scheme $\mathcal{X}_{|\Sigma|}$ is not separated in general. Thus while the restriction of $x_{|\Sigma_{\mathbf{i}}|}$ to any individual torus $\mathcal{X}_\Sigma$ is injective, this is not obviously the case for the entire map $x_{|\Sigma_{\mathbf{i}}|}$. □

Example 4.3.4. The exact form of the modified exchange matrix $\widetilde{B}$ is clarified by considering the degenerate example where u and v are the identity. The relevant double Bruhat cells are then the Cartan subgroups H and H_{Ad}, and the cluster variables and $\mathcal{X}$-coordinates are their respective coroot and coweight coordinates. The change of variables between these is the Cartan matrix, and this is exactly what the definition of $\widetilde{B}$ reduces to in this case (note that the twist map is trivial when u and v are).

The theorem then says that in general to get the twisted change of variables matrix, we add to the exchange matrix a copy of the Cartan matrix split in half between the "left" and "right" frozen variables. As a typical example, let u and v be Coxeter elements of the affine group of type $A_1^{(1)}$. For the natural choice of fundamental weights the extended Cartan matrix is

$$C = \begin{pmatrix} 2 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

□

Example 4.3.5. If we take $\mathbf{i} = (-1, -2, 1, 2)$, then from the definitions one checks that

$$B = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ \frac{1}{2} & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 2 & -1 \\ \frac{1}{2} & 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

$$\widetilde{B} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 2 & -1 \\ 1 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix}.$$

Note in particular that while B is degenerate, reflecting the fact that the symplectic leaves of $G_{\mathrm{Ad}}^{u,v}$ have positive codimension, $|\det \widetilde{B}| = 2$, reflecting the fact that p_G is a double cover. Furthermore, $\widetilde{B}$ has integral entries, while B may in general have half-integral entries where both the row and column correspond to frozen variables. □

Remark 4.3.6. When G is not of finite type, it is sometimes convenient to distinguish between two different versions of its adjoint form. What we have so far called G_{Ad} we will sometimes refer to as the maximal adjoint form $G_{\mathrm{Ad}}^{\max}$ (so $\{\omega_i\}_{i=1}^{\widetilde{r}}$ is a basis of its Cartan subgroup's cocharacter lattice), while by the minimal adjoint form $G_{\mathrm{Ad}}^{\min}$ we will mean the quotient of G by $Z(G)$ (so $\{\omega_i\}_{i=1}^{r}$ is a basis of its Cartan subgroup's cocharacter lattice). For example, if C is of untwisted affine type, G' is a central extension of the group $L\mathring{G}$ of regular maps from $\mathbb{C}^*$ to a simple Lie group $\mathring{G}$, and G is the semidirect product $G' \rtimes \mathbb{C}^*$. $G_{\mathrm{Ad}}^{\max}$ is then quotient of G by $Z(\mathring{G})$, embedded as constant maps, while $G_{\mathrm{Ad}}^{\min}$ is the semidirect product$(L\mathring{G}/Z(\mathring{G})) \rtimes \mathbb{C}^*$.

If $\mathbf{i}$ is a double reduced word for u, v, we have minimal and maximal seeds $\Sigma_{\mathbf{i}}^{\min}$, $\Sigma_{\mathbf{i}}^{\max}$ with respective index sets

$$I_{\min} := \{-r, \ldots, -1\} \cup \{1, \ldots, m\}, \quad I_{\max} := \{-\widetilde{r}, \ldots, -(r+1)\} \cup I_{\min},$$

and exchange matrices as in Definition 4.3.1. Definition 4.2.5 now yields charts $\mathcal{X}_{\Sigma_{\mathbf{i}}^{\min}} \hookrightarrow (G_{\mathrm{Ad}}^{\min})^{u,v}$ and $\mathcal{X}_{\Sigma_{\mathbf{i}}^{\max}} \hookrightarrow (G_{\mathrm{Ad}}^{\max})^{u,v}$, while Definition 4.2.7 yields charts $\mathcal{A}_{\Sigma_{\mathbf{i}}^{\min}} \hookrightarrow (G')^{u,v}$ and $\mathcal{A}_{\Sigma_{\mathbf{i}}^{\max}} \hookrightarrow G^{u,v}$ (where G' is the derived subgroup of G). Theorem 4.3.2 can be extended to assert commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{A}_{\Sigma_{\mathbf{i}}^{\min}} \arrow[r, hook] \arrow[d, hook] & \mathcal{A}_{\Sigma_{\mathbf{i}}^{\max}} \arrow[r, two heads, "p^{\mathrm{mod}}_{\Sigma_{\mathbf{i}}^{\max}}"] \arrow[d, hook] & \mathcal{X}_{\Sigma_{\mathbf{i}}^{\max}} \arrow[r, two heads] \arrow[d, hook] & \mathcal{X}_{\Sigma_{\mathbf{i}}^{\min}} \arrow[d, hook] \\
(G')^{u,v} \arrow[r, hook] & G^{u,v} \arrow[r, two heads, "p_G"] & (G^{\max}_{\mathrm{Ad}})^{u,v} \arrow[r, two heads] & (G^{\min}_{\mathrm{Ad}})^{u,v}.
\end{tikzcd}$$

Here the top left and top right maps are induced by the inclusion of lattices $\mathbb{Z}I_{\min} \hookrightarrow \mathbb{Z}I_{\max}$ following Remark 2.2.5. $\square$

Cluster Transformations of $\mathcal{X}$-coordinates

Recall that in Definition 4.2.5 we constructed an explicit regular map $x_{\Sigma_{\mathbf{i}}} : \mathcal{X}_{\Sigma_{\mathbf{i}}} \to G^{u,v}_{\mathrm{Ad}}$ (from now on we identify the tori $\mathcal{X}_{\mathbf{i}}$ and $\mathcal{X}_{\Sigma_{\mathbf{i}}}$ in the obvious way). If Σ' is obtained from $\Sigma_{\mathbf{i}}$ by a single mutation, we now show that this extends to a regular map $\mathcal{X}_{\Sigma'} \to G^{u,v}_{\mathrm{Ad}}$, compatible with the cluster transformation between $\mathcal{X}_{\Sigma_{\mathbf{i}}}$ and $\mathcal{X}_{\Sigma'}$. This generalizes a closely related statement in [Zel00, p. 4.4].

Proposition 4.3.7. *Let $\Sigma_{\mathbf{i}}$ be the seed associated with a double reduced word $\mathbf{i}$, and $\mathcal{X}_k := \mathcal{X}_{\mu_k(\Sigma_{\mathbf{i}})}$ for some index $k \in I_u$. There is a unique regular map $x_k : \mathcal{X}_k \to G^{u,v}_{\mathrm{Ad}}$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{X}_{\Sigma_{\mathbf{i}}} \arrow[rr, dashed, "\mu_k"] \arrow[dr, "x_{\Sigma_{\mathbf{i}}}"'] & & \mathcal{X}_k \arrow[dl, "x_k"] \\
& G^{u,v}_{\mathrm{Ad}} &
\end{tikzcd}$$

Proof. First note that since μ_k and $x_{\Sigma_{\mathbf{i}}}$ are birational, there is a unique rational map x_k making the diagram commute; the claim is that this is in fact regular.

We will let $Y_i := X_i'$ denote the $\mathcal{X}$-coordinates on $\mathcal{X}_k$. The cluster transformation eq. (2.2.8) lets us express the X_i as rational functions of the Y_i, and with this in mind we write the rational map x_k as

$$(Y_{-\widetilde{r}}, \ldots, Y_m) \mapsto X_{-\widetilde{r}}^{\omega^\vee_{\widetilde{r}}} \cdots X_{-1}^{\omega^\vee_1} E_{i_1} X_1^{\omega^\vee_{|i_1|}} \cdots X_m^{\omega^\vee_{|i_m|}} \tag{4.3.8}$$

Note that if $i > k^+$ or $i^+ < k$, we have $Y_i = X_i$ by eq. (2.2.8) and Definition 4.3.1. In particular, the corresponding terms in eq. (4.3.8) do not affect whether or not the overall expression defines a regular map. Thus it suffices to consider the case where $k = 1$ and $k^+ = m$, to which we will now restrict our attention (given this, we will write i in place of $|i_1| = |i_m|$).

Define rational maps $g_j : \mathcal{X}_1 \to G$ by

$$g_j = \begin{cases} \left(\prod_{j \in I} X_j^{\omega^\vee_{|i_j|}}\right) x_{i_1}(X_1^{-\epsilon_1} X_m^{-\epsilon_1}) x_{i_m}(X_m^{-\epsilon_m}) & j = 1 \\ x_{i_m}(-X_m^{-\epsilon_m}) x_{i_j} \left(\prod_{\substack{j \leq \ell < m \\ |i_j| = |i_\ell|}} X_\ell^{-\epsilon_j}\right) x_{i_m}(X_m^{-\epsilon_m}) & 1 < j \leq m, \end{cases}$$

again interpreting the X_i as rational functions of the Y_i on the right-hand side. Then

$$X_{-\widetilde{r}}^{\omega^\vee_{\widetilde{r}}} \cdots X_{-1}^{\omega^\vee_1} E_{i_1} X_1^{\omega^\vee_{|i_1|}} \cdots X_m^{\omega^\vee_{|i_m|}} = g_1 \cdots g_m,$$

so it suffices to prove that each g_j is regular (and that their product lands in $G^{u,v}_{\mathrm{Ad}}$). The details of the argument depend on the signs of i_1 and i_m, so we consider the distinct cases separately.

Case 1, $i_1 = i_m = i$: First consider g_1. By Definition 4.3.1 we have $b_{-i,1} = -1$ and $b_{m,1} = 1$, hence

$$X_{-i} = Y_{-i} Y_1 (1 + Y_1)^{-1}, \quad X_m = Y_m (1 + Y_1).$$

Thus

$$\left(\prod_{\substack{j \in I \\ |i_j| = i}} X_j^{\omega^\vee_i}\right) = \left(Y_{-i} Y_1 (1 + Y_1)^{-1}\right)^{\omega^\vee_i} Y_1^{-\omega^\vee_i} \left(Y_m (1 + Y_1)\right)^{\omega^\vee_i}$$

$$= (Y_{-i} Y_m)^{\omega^\vee_i},$$

which is a regular function of the Y_j.

In fact, for any $1 \leq j \leq \widetilde{r}$ such that $i \neq j$, there are as many indices $k \in I$ with $|i_k| = j$ and $b_{k,1} > 0$ as there are with $|i_k| = j$ and $b_{k,1} < 0$. One has $b_{k,1} > 0$ exactly either when $1 < k < k^+ < m$ and $\epsilon_k = -\epsilon_{k^+} = -1$, or when $k = -j$, $1 < k^+ < m$, and $\epsilon_{k^+} = 1$. Similarly $b_{k,1} < 0$ exactly either when $1 < k < k^+ < m$ and $\epsilon_k = -\epsilon_{k^+} = 1$, or when $1 < k < m < k^+$ and $\epsilon_k = 1$. One can check that the latter situations are in bijection with the former.

If $|i_k| = j$ for some index $k \in I$, we have

$$X_k = \begin{cases} Y_k (1 + Y_1)^{-C_{ij}} & b_{k,1} > 0 \\ Y_k Y_1^{-C_{ij}} (1 + Y_1)^{C_{ij}} & b_{k,1} < 0 \\ Y_k & b_{k,1} = 0. \end{cases}$$

But then by the above remark the positive and negative powers of $(1 + Y_1)$ in

$$\prod_{\substack{k \in I \\ |i_k| = j}} X_k^{\omega^\vee_j}$$

cancel each another out, leaving a total expression which depends regularly on the Y_k. Since this holds for all $1 \leq j \leq \widetilde{r}$, it follows that $\prod_{j\in I} X_j^{\omega^\vee_{|i_j|}}$ is a regular function of the Y_k. Furthermore, we have

$$\begin{aligned} x_{i_1}(X_1^{-\epsilon_1} X_m^{-\epsilon_1}) x_{i_m}(-X_m^{-\epsilon_m}) &= x_i\big(Y_1 Y_m^{-1}(1+Y_1)^{-1}\big) x_i\big(Y_m^{-1}(1+Y_1)^{-1}\big) \\ &= x_i(Y_m^{-1}), \end{aligned}$$

and it follows that g_1 is regular.

Now consider g_j for $j > 1$. If $\epsilon_j = -1$, then by following a similar analysis as above one sees that $\prod_{\substack{j\leq \ell<m \\ |i_j|=|i_\ell|}} X_\ell^{-\epsilon_j}$ is actually a regular function of the Y_k, since all $(1+Y_1)$ terms cancel out. Since in this case the E_i terms commute with E_{i_j}, it follows that g_j is regular.

If $\epsilon_j = 1$, then $\prod_{\substack{j\leq \ell<m \\ |i_j|=|i_\ell|}} X_\ell^{-\epsilon_j}$ is equal to $(1+Y_1)^{-C_{i,|i_j|}}$ times some Laurent monomial q in the Y_k. But then

$$x_i\big(-Y_m^{-1}(1+Y_1)^{-1}\big) x_{i_j}\big(q(1+Y_1)^{-C_{i,|i_j|}}\big) x_i\big(Y_m^{-1}(1+Y_1)^{-1}\big)$$

is regular by Lemma 4.3.9.

Case 2, $i_1 = i, i_m = -i$: Again, first consider g_1. Now $b_{-i,1}$ and $b_{m,1}$ are both equal to -1, so

$$X_{-i} = Y_{-i} Y_1 (1+Y_1)^{-1} \text{ and } X_m = Y_m Y_1 (1+Y_1)^{-1}.$$

Thus

$$\begin{aligned} \prod_{\substack{j\in I \\ |i_j|=i}} X_j^{\omega^\vee_{|i_j|}} &= \Big(Y_{-i} Y_1 (1+Y_1)^{-1}\Big)^{\omega_i^\vee} Y_1^{-\omega_i^\vee} \Big(Y_m Y_1 (1+Y_1)^{-1}\Big)^{\omega_i^\vee} \\ &= \Big(Y_{-i} Y_m (1+Y_1)^{-2}\Big)^{\omega_i^\vee}. \end{aligned}$$

This time for any $1 \leq j \leq \widetilde{r}$ with $j \neq i$, there is exactly one more index $k \in I$ with $|i_k| = j$ and $b_{k,1} > 0$ than there is with $|i_k| = j$ and $b_{k,1} < 0$. One has $b_{k,1} > 0$ exactly when either $1 < k < m$ and $\epsilon_k = -1$, or $k = -j$ with either $k^+ > m$ or $1 < k^+ < m$ and $\epsilon_{k^+} = 1$. On the other hand $b_{k,1} < 0$ if and only if $1 < k < k^+ < m$ and $\epsilon_k = -\epsilon_{k^+} = 1$. Thus

$$\prod_{\substack{k\in I \\ |i_k|=j}} X_k^{\omega_j^\vee}$$

is the product of $(1+Y_1)^{-C_{ij}\omega_j^\vee}$ and a term which is regular in the Y_k.

It follows that $\prod_{j\in I} X_j^{\omega^\vee_{|i_j|}}$ is the product of a regular term and

$$\prod_{1\leq j\leq \widetilde{r}} (1+Y_1)^{-C_{ij}\omega_j^\vee} = (1+Y_1)^{-\alpha_i^\vee}.$$

Finally g_1 itself is then the product of a regular term and

$$\begin{aligned}(1+Y_1)^{-\alpha_i^\vee} x_{i_1}(X_1^{-\epsilon_1}X_m^{-\epsilon_1})x_{i_m}(X_m^{-\epsilon_m}) &= (1+Y_1)^{-\alpha_i^\vee} x_i\big(Y_m^{-1}(1+Y_1)\big)x_{-i}\big(Y_mY_1(1+Y_1)^{-1}\big)\\ &= \varphi_i\begin{pmatrix} 1 & Y_m^{-1} \\ Y_1Y_m & 1+Y_1\end{pmatrix},\end{aligned}$$

hence is regular.

Now consider g_j for $j>1$. This time if $\epsilon_j = 1$, $\prod_{\substack{j\leq \ell<m\\ |i_j|=|i_\ell|}} X_\ell^{-\epsilon_j}$ is a Laurent monomial in the Y_k, the $(1+Y_1)$ terms cancelling. If $\epsilon_j=-1$, the relevant expression becomes

$$x_{-i}\big(-Y_mY_1(1+Y_1)^{-1}\big)x_{i_j}\big(q(1+Y_1)^{-C_{i,|i_j|}}\big)x_{-i}\big(Y_mY_1(1+Y_1)^{-1}\big)$$

for some Laurent monomial q in the Y_k. Again, this is regular by Lemma 4.3.9.

The remaining cases of $i_1=i_m=-i$ and $i_1=-i_m=-i$ do not differ substantively from the above two; the details are left to the reader.

It is clear that the image of $\mathcal{X}_1$ in G_{Ad} lands in the closure of $G^{u,v}_{\mathrm{Ad}}$. Consider the extension of the regular map $p_G : G^{u,v} \to G^{u,v}_{\mathrm{Ad}}$ to a rational map between their closures. By Propositions 2.2.12 and 4.2.28 we can write the rational functions $p_G^*(Y_i)$ on $\overline{G^{u,v}}$ as Laurent monomials in A_1' and the A_i with $i\neq 1$, where A_1' is the rational function on $G^{u,v}$ obtained by eq. (2.2.7). Since p_G is a finite covering map, by Proposition 4.2.28 the determinant D of the matrix $\widetilde{B}$ is a nonzero integer. In particular, we can write each $(A_i)^D$ with $i\neq 1$ as a Laurent monomial in the $p_G^*(Y_i)$. But the generalized minors $\Delta^{\omega_i}_{u,e}$ and $\Delta^{\omega_i}_{e,v^{-1}}$ are frozen cluster variables, hence their Dth powers can be expressed as Laurent monomials in the $p_G^*(Y_i)$. Thus these powers, hence the minors themselves, are nonvanishing on $p_G^{-1}(\mathcal{X}_1)$. Since p_G is the composition of a biregular automorphism of $G^{u,v}$ and the quotient map $\pi_G : G^{u,v}\to G^{u,v}_{\mathrm{Ad}}$, it follows that these minors do not vanish on $\pi_G^{-1}(\mathcal{X}_1)$. The fact that the image of $\mathcal{X}_1$ lies in $G^{u,v}$ then follows by Lemma 4.3.10. □

The following result was proved in finite type in [Zel00, Lemma 4.4]. However, the proof in loc. cited does not extend to the general case, as it involves exponentiating Lie algebra elements which in general have components in imaginary root spaces.

Lemma 4.3.9. *For distinct $1\leq i,j\leq r$ the map $\mathbb{C}^*\times\mathbb{C}\to N_\pm$ given by*

$$(p,q)\mapsto x_{\pm i}(p^{-1})x_{\pm j}(p^{-C_{ij}}q)x_{\pm i}(-p^{-1})$$

extends to a regular map $\mathbb{C}^2\to N_\pm$.

Proof. We prove the statement for N_+; the N_- version then follows after applying the involution θ. Recall from [Kum02, p. 7.4] that the map

$$N_+\to\bigoplus_{1\leq i\leq\widetilde{r}} L(\omega_i)^\vee,\quad n\mapsto n\cdot(v_1,\dots,v_{\widetilde{r}})$$

is a closed embedding of ind-varieties, where v_i is the lowest-weight vector of $L(\omega_i)^\vee$. Thus it suffices to show that

$$(p,q) \mapsto x_i(p^{-1})x_j(p^{-C_{ij}}q)x_i(-p^{-1}) \cdot v_k$$

extends regularly to $p = 0$ for all $1 \le k \le \widetilde{r}$. This is immediate unless k is equal to i or j.

If $k = j$, then

$$x_i(p^{-1})x_j(p^{-C_{ij}}q)x_i(-p^{-1}) \cdot v_j = x_i(p^{-1}) \cdot (v_j + p^{-C_{ij}}qe_jv_j),$$

where e_j is jth the positive Chevalley generator. Since e_jv_j is a lowest-weight vector for the $\varphi_i(SL_2)$-subrepresentation it generates and $\langle -\omega_j + \alpha_j | \alpha_i^\vee \rangle = C_{ij}$, we have

$$\begin{aligned} x_i(p^{-1}) \cdot (v_j + p^{-C_{ij}}qe_jv_j) &= \sum_{n=0}^{\infty} p^{-n}\frac{e_i^n}{n!}(v_j + p^{-C_{ij}}qe_jv_j) \\ &= v_j + \sum_{n=0}^{-C_{ij}} p^{-C_{ij}-n}\frac{qe_i^ne_j}{n!}v_j. \end{aligned}$$

Since this last expression depends only on nonnegative powers of p, the claim follows.

If $k = i$, a similar calculation yields

$$\begin{aligned} x_i(p^{-1})x_j(p^{-C_{ij}}q)x_i(-p^{-1}) \cdot v_i &= x_i(p^{-1})x_j(p^{-C_{ij}}q) \cdot (v_i - p^{-1}e_iv_i) \\ &= x_i(p^{-1}) \cdot \left(v_i - \sum_{n=0}^{-C_{ij}} p^{-1-nC_{ij}}\frac{q^ne_j^ne_i}{n!}v_i \right). \end{aligned}$$

If $n > 0$, $e_j^ne_iv_i$ is a lowest-weight vector for the $\varphi_i(SL_2)$-subrepresentation it generates. Otherwise, $-\omega_i + n\alpha_j$ would have a nonzero weight space in $L(\omega_i)^\vee$, which would generate a nontrivial $\varphi_j(SL_2)$-representation containing v_i, a contradiction.

Since $\langle -\omega_i + \alpha_i + n\alpha_j | \alpha_i^\vee \rangle = 1 + nC_{ij}$,

$$x_i(p^{-1}) \cdot p^{-1-nC_{ij}}\frac{q^ne_j^ne_i}{n!}v_i = \sum_{m=0}^{-1-nC_{ij}} p^{-1-nC_{ij}-m}\frac{q^ne_i^me_j^ne_i}{m!n!}v_i.$$

But since $-1 - nC_{ij} - m \ge 0$ for all $m \le -1 - nC_{ij}$, the right hand side depends only on nonnegative powers of p. But $x_i(p^{-1})x_j(p^{-C_{ij}}q)x_i(-p^{-1}) \cdot v_i$ is a sum of such terms with $n > 0$ and

$$x_i(p^{-1}) \cdot (v_i - p^{-1}e_iv_i) = v_i,$$

hence extends to a regular map at $p = 0$. □

Lemma 4.3.10. *The closure of $G^{u,v}$ in G is*

$$\overline{G^{u,v}} = \bigsqcup_{\substack{u' \le u \\ v' \le v}} G^{u',v'},$$

where we use the Bruhat order on W. If $x \in \overline{G^{u,v}}$, then $x \in G^{u,v}$ if and only if $\Delta^{\omega_i}_{u,e}(x) \neq 0$ and $\Delta^{\omega_i}_{e,v^{-1}}(x) \neq 0$ for all $1 \leq i \leq \widetilde{r}$.[4]

Proof. The decomposition of $\overline{G^{u,v}}$ follows easily from the corresponding statement about Schubert varieties [Kum02, p. 7.1]. It is also clear from their definitions that the stated generalized minors do not vanish on $G^{u,v}$. Thus we must show that if $x \in \overline{G^{u,v}} \setminus G^{u,v}$, one of the stated minors vanishes on it.

Suppose that $u' \leq u$ in the Bruhat order. By definition, there exist positive real roots $\beta_1, \dots, \beta_k$ such that $u = u'r_1 \cdots r_k$, where $r_j \in W$ is the reflection

$$r_j : \lambda \mapsto \lambda - \langle \lambda | \beta_j^\vee \rangle \beta_j.$$

Here $\beta_j^\vee$ is the positive coroot associated with β_j. Moreover, these satisfy $\ell(u'r_1) < \ell(u'r_1r_2) < \cdots < \ell(u)$, which in particular implies that $u'r_1 \cdots r_{j-1}(\beta_j) > 0$ for all j [Kum02, p. 1.3.13].

If $u' \leq u$, we claim that for each ω_i,

$$u'(\omega_i) - u(\omega_i) \in \bigoplus_{1 \leq j \leq r} \mathbb{N}\alpha_j.$$

For any $1 < j \leq r$ we have

$$u'r_1 \cdots r_{j-1}(\omega_i) - u'r_1 \cdots r_j(\omega_i) = \langle \omega_i | \beta_j^\vee \rangle u'r_1 \cdots r_{j-1}(\beta_j).$$

But then

$$\begin{aligned} u'(\omega_i) - u(\omega_i) &= \sum_{1<j\leq r} \big(u'r_1 \cdots r_{j-1}(\omega_i) - u'r_1 \cdots r_j(\omega_i) \big) \\ &= \sum_{1<j\leq r} \langle \omega_i | \beta_j^\vee \rangle u'r_1 \cdots r_{j-1}(\beta_j), \end{aligned}$$

which is indeed a sum of positive roots with nonnegative coefficients. Furthermore, if u' is strictly less than u in the Bruhat order, $u'(\omega_i) - u(\omega_i)$ must be nonzero for some $1 \leq i \leq r$. But then for any $x \in B_+ u' B_+$, we have $\Delta^{\omega_i}_{u,e}(x) = 0$. A straightforward adaptation of this argument implies that for any $x \in B_- v' B_-$ with $v' < v$, $\Delta^{\omega_i}_{e,v^{-1}}(x) = 0$ for some $1 \leq i \leq r$, and the lemma follows. □

Cluster Transformations of Generalized Minors

Recall that to a double reduced word $\mathbf{i}$ we associated in Definition 4.2.7 a collection $\{A_i\}_{i \in I}$ of generalized minors. In this section we identify these with the cluster variables corresponding to the seed $\Sigma_{\mathbf{i}}$ and study their cluster transformations.

[4] In finite type a stronger version of this is stated in [BFZ05, Proposition 2.8], following from the proof of [FZ00, Proposition 3.3].

Theorem 4.3.11. *There is a regular map $a_{|\Sigma_{\mathbf{i}}|}: \mathcal{A}_{|\Sigma_{\mathbf{i}}|} \to G^{u,v}$ which identifies the generalized minors of Definition 4.2.7 with the corresponding cluster variables on $\mathcal{A}_{\Sigma_{\mathbf{i}}}$. This map induces an isomorphism of $\mathbb{C}[G^{u,v}]$ and the upper cluster algebra $\mathbb{C}[\mathcal{A}_{|\Sigma_{\mathbf{i}}|}]$.*

When G is a semisimple algebraic group, this is the content of [BFZ05, p. 2.10]. As in loc. cited, the proof we give is modelled on that of a closely related result in [Zel00], which treats the case of reduced double Bruhat cells. Most of the work is delegated to a series of lemmas that take up the bulk of the section; first we show how these lemmas assemble into the proof of Theorem 4.3.11.

Proof of Theorem 4.3.11. By Lemma 4.3.12, Proposition 2.2.11 applies to $\Sigma_{\mathbf{i}}$, hence

$$\mathbb{C}[\mathcal{A}_{|\Sigma_{\mathbf{i}}|}] = \mathbb{C}[\mathcal{A}_{\Sigma_{\mathbf{i}}}] \cap \bigcap_{k \in I_u} \mathbb{C}[\mathcal{A}_k].$$

On the other hand, by Lemma 4.3.15, the maps $a_{\Sigma_{\mathbf{i}}}: \mathcal{A}_{\Sigma_{\mathbf{i}}} \to G^{u,v}$, $a_k: \mathcal{A}_k \to G^{u,v}$ induce an isomorphism

$$\mathbb{C}[G^{u,v}] \cong \mathbb{C}[\mathcal{A}_{\Sigma_{\mathbf{i}}}] \cap \bigcap_{k \in I_u} \mathbb{C}[\mathcal{A}_k].$$

Then since $G^{u,v}$ is an affine variety (Proposition 2.1.12), we have $G^{u,v} \cong \operatorname{Spec} \mathbb{C}[\mathcal{A}_{|\Sigma_{\mathbf{i}}|}]$. But then $a_{|\Sigma_{\mathbf{i}}|}$ is just the canonical map $\mathcal{A}_{|\Sigma_{\mathbf{i}}|} \to \operatorname{Spec} \mathbb{C}[\mathcal{A}_{|\Sigma_{\mathbf{i}}|}]$. □

Lemma 4.3.12. *The submatrix of B formed by its unfrozen rows has full rank.*

Proof. First let

$$I_+ = \{k \in I : k^- \in I_u\}.$$

We claim the submatrix of B whose rows are those indexed by I_u and whose columns are indexed by I_+ is lower triangular with nonzero diagonal entries. The diagonal entries are of the form b_{k,k^+}, hence equal to ± 1 by Definition 4.3.1. On the other hand if an entry $b_{k,\ell}$ of this submatrix lies above the diagonal then $\ell > k^+$. Again, from the definition of B we must have $b_{k,\ell} = 0$. Thus this square submatrix has full rank, and it follows that the matrix formed by the unfrozen rows has full rank. □

Lemma 4.3.13. *For each unfrozen index $k \in I$, let A'_k be the rational function on $G^{u,v}$ obtained from the exchange relation*

$$A'_k = A_k^{-1}\Bigg(\prod_{b_{kj}>0} A_j^{b_{kj}} + \prod_{b_{kj}<0} A_j^{-b_{kj}}\Bigg).$$

Then A'_k is in fact regular.

Proof. It suffices to consider the case $k = 1$, $k^+ = m$, where we will in fact show that A'_1 is the restriction to $G^{u,v}$ of a strongly regular function on G. In the general case, consider the double reduced word $\mathbf{i}' = (i_k, \dots, i_{k^+})$. Then one has

$$A'_{k,\mathbf{i}}(x) = A'_{1,\mathbf{i}'}(\overline{u_{<k}}^{-1} x \overline{v_{>k^+}}),$$

hence $A'_{k,\mathbf{i}}$ is the restriction of a strongly regular function if $A'_{1,\mathbf{i}'}$ is.

We obtain the following formulas for A'_1 depending on the signs of i_1 and i_m. We will let $E_\pm = \{1 < j < m | \epsilon_j = \pm 1\}$, $J_\pm = \{|i_j| | 1 \leq j < m, j_- < 0\}$, and $i := |i_1| = |i_m|$.

Case 1, $i_1 = i_m = i$

$$A'_1 \Delta^{\omega_i}_{e,s_i} = \Delta^{\omega_i}_{e,v^{-1}} \prod_{\substack{k \in E_+ \\ k^+ \notin E_+}} (\Delta^{\omega_{|i_k|}}_{u_{\leq k}, v_{>k}})^{-C_{|i_k|,i}} + \Delta^{\omega_i}_{e,e} \prod_{\substack{k \in E_+ \\ k^- \notin E_+}} (\Delta^{\omega_{|i_k|}}_{u_{<k}, v_{\geq k}})^{-C_{|i_k|,i}}$$

Case 2, $i_1 = i_m = -i$

$$A'_1 \Delta^{\omega_i}_{s_i,e} = \Delta^{\omega_i}_{u,e} \prod_{\substack{k \in E_- \\ k^- \notin E_-}} (\Delta^{\omega_{|i_k|}}_{u_{<k}, v_{>k}})^{-C_{|i_k|,i}} + \Delta^{\omega_i}_{e,e} \prod_{\substack{k \in E_- \\ k^+ \notin E_-}} (\Delta^{\omega_{|i_k|}}_{u_{\leq k}, v_{>k}})^{-C_{|i_k|,i}}$$

Case 3, $i_1 = i$**,** $i_m = -i$

$$\begin{aligned} A'_1 \Delta^{\omega_i}_{e,e} = \Delta^{\omega_i}_{e,v^{-1}} \Delta^{\omega_i}_{u,e} \prod_{\substack{k \in E_+ \\ k^+ \in E_-}} (\Delta^{\omega_{|i_k|}}_{u_{\leq k}, v_{>k}})^{-C_{|i_k|,i}} & \\ + \left(\prod_{\substack{k \in E_- \\ k^- \notin E_-}} (\Delta^{\omega_{|i_k|}}_{u_{\leq k}, v_{>k}})^{-C_{|i_k|,i}} \right) & \left(\prod_{j \in [1,\widetilde{r}] \setminus J_-} (\Delta^{\omega_j}_{e,v^{-1}})^{-C_{ij}} \right) \end{aligned}$$

Case 4, $i_1 = -i$**,** $i_m = i$

$$\begin{aligned} A'_1 \Delta^{\omega_i}_{s_i,s_i} = \Delta^{\omega_i}_{e,s_i} \Delta^{\omega_i}_{s_i,e} \prod_{\substack{k \in E_- \\ k^+ \in E_+}} (\Delta^{\omega_{|i_k|}}_{u_{\leq k}, v_{>k}})^{-C_{|i_k|,i}} & \\ + \left(\prod_{\substack{k \in E_+ \\ k^+ \notin E_+}} (\Delta^{\omega_{|i_k|}}_{u_{\leq k}, v_{>k}})^{-C_{|i_k|,i}} \right) & \left(\prod_{j \in [1,\widetilde{r}] \setminus J_+} (\Delta^{\omega_j}_{e,v^{-1}})^{-C_{ij}} \right) \end{aligned}$$

We now impose the further assumption that $j < k$ for all $j \in E_+$, $k \in E_-$, before returning to the general case. Letting $S_\pm = \{|i_k| : k \in E_\pm\} \subset [1, \widetilde{r}]$, we can then simplify the above formulas as:

Case 1, $i_1 = i_m = i$

$$A'_1 \Delta^{\omega_i}_{e,s_i} = \Delta^{\omega_i}_{e,v^{-1}} \prod_{\ell \in S_+} (\Delta^{\omega_\ell}_{e,e})^{-C_{\ell i}} + \Delta^{\omega_i}_{e,e} \prod_{\ell \in S_+} (\Delta^{\omega_\ell}_{e,v^{-1}})^{-C_{\ell i}}$$

Case 2, $i_1 = i_m = -i$

$$A'_1 \Delta^{\omega_i}_{s_i,e} = \Delta^{\omega_i}_{u,e} \prod_{\ell \in S_-} (\Delta^{\omega_\ell}_{e,e})^{-C_{\ell i}} + \Delta^{\omega_i}_{e,e} \prod_{\ell \in S_-} (\Delta^{\omega_\ell}_{u,e})^{-C_{\ell i}}$$

Case 3, $i_1 = i$**,** $i_m = -i$

$$A'_1 \Delta^{\omega_i}_{e,e} = \Delta^{\omega_i}_{e,v^{-1}} \Delta^{\omega_i}_{u,e} \prod_{\ell \in S_+ \cap S_-} (\Delta^{\omega_\ell}_{e,e})^{-C_{\ell i}} + \left(\prod_{\ell \in S_-} (\Delta^{\omega_\ell}_{u,e})^{-C_{\ell i}} \right) \left(\prod_{\ell \in ([1,\widetilde{r}] \setminus S_-) \cup S_+} (\Delta^{\omega_\ell}_{e,v^{-1}})^{-C_{\ell i}} \right)$$

Case 4, $i_1 = -i$**,** $i_m = i$

$$A'_1 \Delta^{\omega_i}_{s_i,s_i} = \Delta^{\omega_i}_{e,s_i} \Delta^{\omega_i}_{s_i,e} + \left(\prod_{\ell \in [1,\widetilde{r}] \setminus \{i\}} (\Delta^{\omega_\ell}_{e,e})^{-C_{\ell i}} \right)$$

In each case, one can apply Proposition 2.1.21 to deduce that A'_1 is indeed regular. For example, in case 1, multiplying both sides of the above equation by

$$\prod_{j \in [1,\widetilde{r}] \setminus (\{i\} \cup S_+)} (\Delta^{\omega_j}_{e,e})^{-C_{ji}} = \prod_{j \in [1,\widetilde{r}] \setminus (\{i\} \cup S_+)} (\Delta^{\omega_j}_{e,v^{-1}})^{-C_{ji}}$$

we obtain

$$\begin{aligned}
A'_1 \Delta^{\omega_i}_{e,s_i} & \left(\prod_{j \in [1,\widetilde{r}] \setminus (\{i\} \cup S_+)} (\Delta^{\omega_j}_{e,e})^{-C_{ji}} \right) \\
&= \Delta^{\omega_i}_{e,v^{-1}} \prod_{\ell \in [1,\widetilde{r}] \setminus \{i\}} (\Delta^{\omega_\ell}_{e,e})^{-C_{\ell i}} + \Delta^{\omega_i}_{e,e} \prod_{\ell \in [1,\widetilde{r}] \setminus \{i\}} (\Delta^{\omega_\ell}_{e,v^{-1}})^{-C_{\ell i}} \\
&= \Delta^{\omega_i}_{e,v^{-1}} (\Delta^{\omega_i}_{e,e} \Delta^{\omega_i}_{s_i,s_i} - \Delta^{\omega_i}_{e,s_i} \Delta^{\omega_i}_{s_i,e}) + \Delta^{\omega_i}_{e,e} (\Delta^{\omega_i}_{e,s_i} \Delta^{\omega_i}_{s_i,v^{-1}} - \Delta^{\omega_i}_{s_i,s_i} \Delta^{\omega_i}_{e,v^{-1}}) \\
&= \Delta^{\omega_i}_{e,s_i} (\Delta^{\omega_i}_{e,e} \Delta^{\omega_i}_{s_i,v^{-1}} - \Delta^{\omega_i}_{s_i,e} \Delta^{\omega_i}_{e,v^{-1}}).
\end{aligned}$$

By Proposition 2.1.20, $\Delta^{\omega_i}_{e,s_i}$ is a prime element of $\mathbb{C}[G]$ distinct from the $\Delta^{\omega_j}_{e,e}$ for $j \neq i$, hence $\prod_{j \in [1,\widetilde{r}] \setminus (\{i\} \cup S_+)} (\Delta^{\omega_j}_{e,e})^{-C_{ji}}$ must divide $(\Delta^{\omega_i}_{e,e} \Delta^{\omega_i}_{s_i,v^{-1}} - \Delta^{\omega_i}_{s_i,e} \Delta^{\omega_i}_{e,v^{-1}})$ in $\mathbb{C}[G]$. But then

$$A'_1 = (\Delta^{\omega_i}_{e,e} \Delta^{\omega_i}_{s_i,v^{-1}} - \Delta^{\omega_i}_{s_i,e} \Delta^{\omega_i}_{e,v^{-1}}) / \left(\prod_{j \in [1,\widetilde{r}] \setminus (\{i\} \cup S_+)} (\Delta^{\omega_j}_{e,e})^{-C_{ji}} \right)$$

is indeed an element of $\mathbb{C}[G]$. We omit the remaining cases, which may be dealt with using the same strategy.

Now suppose $\mathbf{i}$ and $\mathbf{i}'$ are two double reduced word differing only in that $i_k = i'_{k+1} = j$ and $i_{k+1} = i'_k = -j'$ for some $1 \leq k < m$ and $1 \leq j, j' \leq r$. We claim that if $A'_{1,\mathbf{i}}$ is regular, so is $A'_{1,\mathbf{i}'}$. This is straightforward unless $j = j'$ and $C_{ji} \neq 0$, so we restrict our attention to this case. The argument in each of the above cases is essentially the same, so we will only consider Case 1 in detail.

Let P_1 and P_2 (P_1' and P_2') be the two monomials appearing in the right-hand side of the exchange relation defining $A'_{1,\mathbf{i}}$ ($A'_{1,\mathbf{i}'}$). We must show that $\Delta^{\omega_i}_{e,s_i}$ divides $P_1' + P_2'$ in $\mathbb{C}[G^{u,v}]$ given that it divides $P_1 + P_2$.

If $u' = u_{\leq k}$, $v' = v_{>k}$, one can check that

$$P_1' + P_2' = \frac{\left(P_1(\Delta^{\omega_j}_{u',v's_j}\Delta^{\omega_j}_{u's_j,v'})^{-C_{ji}} + P_2(\Delta^{\omega_j}_{u',v'}\Delta^{\omega_j}_{u's_j,v's_j})^{-C_{ji}}\right)}{((\Delta^{\omega_j}_{u',v's_j})^{[k^-\notin E_+]}(\Delta^{\omega_j}_{u's_j,v'})^{[k^++\in E_+]}\Delta^{\omega_j}_{u',v'})^{-C_{ji}}}.$$

Here, e.g., $[k^- \in E_+]$ is the function which is 1 if $k^- \in E_+$, and 0 otherwise. By Proposition 2.1.20, $\Delta^{\omega_i}_{e,s_i}$ and the denominator of the right-hand side are relatively prime, so it suffices to show that $\Delta^{\omega_i}_{e,s_i}$ divides the numerator. This in turn is equivalent to showing that $\Delta^{\omega_i}_{e,s_i}$ divides

$$(\Delta^{\omega_j}_{u',v's_j}\Delta^{\omega_j}_{u's_j,v'})^{-C_{ji}} - (\Delta^{\omega_j}_{u',v'}\Delta^{\omega_j}_{u's_j,v's_j})^{-C_{ji}},$$

or simply that it divides

$$\Delta^{\omega_j}_{u',v's_j}\Delta^{\omega_j}_{u's_j,v'} - \Delta^{\omega_j}_{u',v'}\Delta^{\omega_j}_{u's_j,v's_j}.$$

But since $\Delta^{\omega_i}_{e,s_i} = \Delta^{\omega_i}_{u',v'}$, this follows from Proposition 2.1.21. $\square$

Lemma 4.3.14. *There is an open immersion $a_{\Sigma_\mathbf{i}} : \mathcal{A}_{\Sigma_\mathbf{i}} \to G^{u,v}$ such that the generalized minors A_i from Definition 4.2.7 pull back to the corresponding cluster variables on $\mathcal{A}_{\Sigma_\mathbf{i}}$. If $k \in I_u$ is any unfrozen index and $\mathcal{A}_k := \mathcal{A}_{\mu_k(\Sigma_\mathbf{i})}$, then there is also an open immersion $a_k : \mathcal{A}_k \to G^{u,v}$ forming a commutative diagram*

$$\begin{tikzcd}
\mathcal{A}_{\Sigma_\mathbf{i}} \arrow[rr, dashed, "\mu_k"] \arrow[dr, "a_{\Sigma_\mathbf{i}}"'] & & \mathcal{A}_k \arrow[dl, "a_k"] \\
& G^{u,v}. &
\end{tikzcd}$$

In particular, the regular functions $\{A_i | i \in I, i \neq k\} \cup \{A_k'\} \subset \mathbb{C}[G^{u,v}]$ pull back to the corresponding cluster variables on $\mathcal{A}_k$.

Proof. The existence of the stated map $a_{\Sigma_\mathbf{i}}$ follows readily from Theorems 5.2.8 and 4.2.24. Moreover, $a_{\Sigma_\mathbf{i}}$ is birational, hence there is a unique rational map a_k making the given diagram commute; we claim it is in fact regular.

There is a commutative square

$$\begin{tikzcd}
\mathcal{A}_k \arrow[r, dashed, "a_k"] \arrow[d, two heads, "p_M'"] & G^{u,v} \arrow[d, two heads, "p_G"] \\
\mathcal{X}_k \arrow[r, "x_k"] & G^{u,v}_{\mathrm{Ad}},
\end{tikzcd}$$

where x_k is the regular map defined in Proposition 4.3.7. Since a_k is birational and the remaining maps are regular and dominant, the diagram embeds $\mathbb{C}[\mathcal{X}_k]$ and $\mathbb{C}[G^{u,v}_{\mathrm{Ad}}]$ as subalgebras of the function field $\mathbb{C}(\mathcal{A}_k)$. Moreover, we have $\mathbb{C}[G^{u,v}_{\mathrm{Ad}}] \subset \mathbb{C}[\mathcal{X}_k]$ inside $\mathbb{C}(\mathcal{A}_k)$.

Since p'_M is finite and $\mathcal{A}_k$ is normal, $\mathbb{C}[\mathcal{A}_k]$ is the integral closure of $\mathbb{C}[\mathcal{X}_k]$ in $\mathbb{C}(\mathcal{A}_k)$. For the same reason, $\mathbb{C}[G^{u,v}]$ is the integral closure of $\mathbb{C}[G^{u,v}_{\mathrm{Ad}}]$ in $\mathbb{C}(\mathcal{A}_k)$. But then the containment $\mathbb{C}[G^{u,v}_{\mathrm{Ad}}] \subset \mathbb{C}[\mathcal{X}_k]$ inside $\mathbb{C}(\mathcal{A}_k)$ implies a containment $\mathbb{C}[G^{u,v}] \subset \mathbb{C}[\mathcal{A}_k]$ of their integral closures, and it follows that a_k is regular.

It is clear from the construction that a_k pulls back the regular functions $\{A_i | i \in I, i \neq k\} \cup \{A'_k\}$ on $G^{u,v}$ to the corresponding cluster variables on $\mathcal{A}_k$. It follows in particular that a_k is injective. But an injective birational morphism of smooth varieties is an open immersion, and the proposition follows. □

Lemma 4.3.15. *Let $U \subset G^{u,v}$ be the open subset*

$$U := \mathcal{A}_{\Sigma_{\mathbf{i}}} \cup \bigcup_{k \in I_u} \mathcal{A}_k,$$

where we identify $\mathcal{A}_{\Sigma_{\mathbf{i}}}$, $\mathcal{A}_k := \mathcal{A}_{\mu_k(\Sigma_{\mathbf{i}})}$ with their images in $G^{u,v}$ following Lemma 4.3.14. Then the complement of U in $G^{u,v}$ has complex codimension greater than 1.

Proof. We first claim that the unfrozen generalized minors A_k are distinct irreducible elements of $\mathbb{C}[G^{u,v}]$, while the frozen ones are units. If k is frozen, either $k < 0$ or $k^+ = m+1$. In the former case, $A_k = \Delta^{\omega_{|i_k|}}_{e,v^{-1}}$, while in the latter $A_k = \Delta^{\omega_{|i_k|}}_{u,e}$. But in either case the fact that A_k is nonvanishing on $G^{u,v}$ follows easily from the definition of the generalized minors.

Observe then that a Laurent monomial $M = \prod_{k \in I} A_k^{n_k}$ in the initial cluster variables is regular on $G^{u,v}$ if and only if $n_k \geq 0$ for all unfrozen k. This follows from the definition of A'_k, since M is regular on $\mathcal{A}_k$ and hence expressible as a Laurent polynomial in A'_k and the A_i with $i \neq k$. Suppose then that for some unfrozen index k we can write A_k as a product of two regular functions $P, Q \in \mathbb{C}[G^{u,v}]$. Clearly P and Q are themselves Laurent monomials in the A_i. But since $PQ = A_k$, one of them must only involve frozen variables, hence is a unit in $\mathbb{C}[G^{u,v}]$. The fact that they are distinct is clear since their restrictions to $\mathcal{A}_{\Sigma_{\mathbf{i}}}$ are distinct.

We now claim that each A'_k is the product of some irreducible element $A''_k \in \mathbb{C}[G^{u,v}]$ and a Laurent monomial in the A_i with $i \neq k$. For suppose P is an irreducible factor of A'_k. Then P must be expressible as a Laurent monomial in A'_k and the A_i with $i \neq k$, since it divides A'_k. On the other hand, since P is regular on $\mathcal{A}_{\Sigma_{\mathbf{i}}}$, it follows from the definition of A'_k that A'_k appears with a nonnegative exponent in this monomial expression. But then in the prime factorization of A'_k there is exactly one irreducible factor such that this exponent is 1, and the statement follows. Again, it is clear that this irreducible element A''_k is distinct from the A_i since their restrictions to $\mathcal{A}_k$ are distinct.

Finally, we observe that the complement $G^{u,v} \setminus U$ is the locus where either A_j and A_k vanish for two distinct $j, k \in I$, or A''_k and A_k vanish for some $k \in I_u$. Let $x \in G^{u,v}$ be any element in the complement of U. Since $x \notin \mathcal{A}_{\Sigma_{\mathbf{i}}}$, $A_k(x)$ must equal zero for some $k \in I_u$. But $x \notin \mathcal{A}_k$, so either $A''_k(x) = 0$ or $A_j(x) = 0$ for some $j \neq k$. Thus $G^{u,v} \setminus U$ is the union of finitely many subvarieties cut out by two distinct irreducible equations, and the lemma follows. □

Theorem 4.3.16. *There is a regular map $x_{|\Sigma_{\mathbf{i}}|} : \mathcal{X}_{|\Sigma_{\mathbf{i}}|} \to G^{u,v}_{\mathrm{Ad}}$ extending the map $\mathcal{X}_{\Sigma_{\mathbf{i}}} \to G^{u,v}_{\mathrm{Ad}}$ of Definition 4.2.5. We have a commutative diagram*

$$\begin{tikzcd} \mathcal{A}_{|\Sigma_{\mathbf{i}}|} \arrow[r, "a_{|\Sigma_{\mathbf{i}}|}"] \arrow[d, "p_M"] & G^{u,v} \arrow[d, "p_G"] \\ \mathcal{X}_{|\Sigma_{\mathbf{i}}|} \arrow[r, "x_{|\Sigma_{\mathbf{i}}|}"] & G^{u,v}_{\mathrm{Ad}}, \end{tikzcd}$$

where p_M and p_G are as defined in Theorem 4.3.2

Proof. It follows from Proposition 4.2.28 that p_M is well-defined and that there is a rational map $x_{|\Sigma_{\mathbf{i}}|}$ making the diagram commute. Let Σ' be any seed mutation equivalent to $\Sigma_{\mathbf{i}}$ and let x' be the restriction of this rational map to $\mathcal{X}_{\Sigma'}$; it will follow that $x_{|\Sigma_{\mathbf{i}}|}$ is regular if we show that each such x' is regular.

We have a commutative diagram

$$\begin{tikzcd} \mathcal{A}_{\Sigma'} \arrow[r, "a'"] \arrow[d, two heads, "p'_M"] & G^{u,v} \arrow[d, two heads, "p_G"] \\ \mathcal{X}_{\Sigma'} \arrow[r, dashed, "x'"] & G^{u,v}_{\mathrm{Ad}}, \end{tikzcd}$$

where a' is the restriction of $a_{|\Sigma_{\mathbf{i}}|}$ to $\mathcal{A}_{\Sigma'}$. If we pull back $\mathbb{C}[G^{u,v}_{\mathrm{Ad}}]$ along $x' \circ p'_M$ to the function field $\mathbb{C}(\mathcal{A}_{\Sigma'})$, we see that its image is contained in $\mathbb{C}(\mathcal{X}_{\Sigma'})$. On the other hand, if we perform the same pullback along $p_G \circ a'$, we see that the image of $\mathbb{C}[G^{u,v}_{\mathrm{Ad}}]$ is contained in $\mathbb{C}[\mathcal{A}_{\Sigma'}]$. Since p'_M is surjective, any rational function on $\mathcal{X}_{\Sigma'}$ which pulls back to a regular function on $\mathcal{A}_{\Sigma'}$ must have been regular on $\mathcal{X}_{\Sigma'}$. Thus the intersection of $\mathbb{C}(\mathcal{X}_{\Sigma'})$ and $\mathbb{C}[\mathcal{A}_{\Sigma'}]$ in $\mathbb{C}(\mathcal{A}_{\Sigma'})$ is exactly $\mathbb{C}[\mathcal{X}_{\Sigma'}]$. Thus x' pulls back $\mathbb{C}[G^{u,v}_{\mathrm{Ad}}]$ to $\mathbb{C}[\mathcal{X}_{\Sigma'}]$, hence is regular. □

Poisson Brackets of $\mathcal{X}$-coordinates

We now complete the proof of Theorem 4.3.2, demonstrating that the map $x_{|\Sigma_{\mathbf{i}}|} : \mathcal{X}_{|\Sigma_{\mathbf{i}}|} \to G^{u,v}_{\mathrm{Ad}}$ is Poisson. First we recall some rudiments of Poisson-Lie theory [CP94].

Any symmetrizable Kac-Moody group G is a Poisson ind-algebraic group in a canonical way (see Section 3.2). That is, its coordinate ring is equipped with a continuous Poisson bracket such that the multiplication map $G \times G \to G$ is Poisson. The double Bruhat cells of G are Poisson subvarieties, and on any given double Bruhat cell H acts transitively on the set of symplectic leaves by left multiplication. This standard Poisson structure is characterized by the fact that the maps

$$\varphi_i : SL_2^{d_i} \to G$$

are Poisson. Here $SL_2^{d_i}$ refers to the following Poisson-Lie structure on SL_2: if we write

$$SL_2 = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : AD - BC = 1 \right\},$$

then the brackets of the coordinate functions on $SL_2^{d_i}$ are given by

$$\{B, A\} = \frac{d_i}{2} AB, \quad \{B, D\} = -\frac{d_i}{2} BD, \quad \{B, C\} = 0,$$
$$\{C, A\} = \frac{d_i}{2} AC, \quad \{C, D\} = -\frac{d_i}{2} CD, \quad \{D, A\} = d_i BC.$$

The Cartan subgroup of G is a Poisson-Lie subgroup endowed with the trivial Poisson structure. Then since the kernel of $G \to G_{\mathrm{Ad}}$ is a discrete subgroup of H, G_{Ad} in turn inherits the standard Poisson structure from G.

Theorem 4.3.17. *The regular map $x_{|\Sigma_{\mathbf{i}}|} : \mathcal{X}_{|\Sigma_{\mathbf{i}}|} \to G_{\mathrm{Ad}}^{u,v}$ defined in Theorem 4.3.16 is Poisson.*[5]

Proof. Since $\mathcal{X}_{\Sigma_{\mathbf{i}}}$ is dense in $\mathcal{X}_{|\Sigma_{\mathbf{i}}|}$, it suffices to check that the original map $\mathcal{X}_{\Sigma_{\mathbf{i}}} \to G_{\mathrm{Ad}}^{u,v}$ is Poisson. Thus if $\{,\}_G$ denotes the restriction of the standard Poisson bracket on $G_{\mathrm{Ad}}^{u,v}$, we must check that

$$\{X_j, X_k\}_G = b_{jk} d_k X_j X_k$$

for all $j, k \in I$. We recall that the upper and lower Borel subgroups of SL_2^d are Poisson subgroups. For $1 \le k \le m$ let B_{i_k} denote the positive Borel subgroup of $SL_2^{d_{|i_k|}}$ if $\epsilon_k = 1$, and its negative Borel subgroup if $\epsilon_k = -1$. There is then a Poisson map

$$m_{\mathbf{i}} : H \times B_{i_1} \times \cdots \times B_{i_m} \to G_{\mathrm{Ad}}^{u,v}$$

given by the maps $\varphi_{|i_k|}$ and multiplication in G_{Ad}, and whose image coincides with $\mathcal{X}_{\Sigma_{\mathbf{i}}}$. We define coordinates P_k, Q_k on each B_{i_k} by

$$B_{i_k} = \left\{ \begin{pmatrix} P_k & Q_k \\ 0 & P_k^{-1} \end{pmatrix} : (P_k, Q_k) \in \mathbb{C}^* \times \mathbb{C} \right\}$$

for $\epsilon_k = +1$ and

$$B_{i_k} = \left\{ \begin{pmatrix} P_k & 0 \\ Q_k & P_k^{-1} \end{pmatrix} : (P_k, Q_k) \in \mathbb{C}^* \times \mathbb{C} \right\}$$

for $\epsilon_k = -1$. In either case the Poisson bracket on $H \times B_{i_1} \times \cdots \times B_{i_m}$ is given by

$$\{P_j, Q_k\} = \frac{d_{|i_k|}}{2} P_k Q_k \delta_{jk}.$$

[5] In finite type this is the result of [FG06a, Proposition 3.11].

Since $m_{\mathbf{i}}$ is dominant and Poisson, the brackets among the X_i are determined by the brackets of their pullbacks along $m_{\mathbf{i}}$. Moreover, since the coordinate functions on H are Casimirs, it suffices to consider the restrictions of these pullbacks to $B_{i_1} \times \cdots \times B_{i_m}$.

Note that

$$\begin{aligned}\varphi_{|i_k|}(B_{i_k}) &= P_k^{\alpha^\vee_{|i_k|}}(P_k^{-1}Q_k^{\epsilon_k})^{\omega^\vee_{|i_k|}}E_{i_k}(P_kQ_k^{-\epsilon_k})^{\omega^\vee_{|i_k|}} \\ &= \left(\prod_{\substack{j\neq|i_k| \\ 1\leq j\leq \widetilde{r}}} P_k^{C_{|i_k|,|i_j|}\omega_j^\vee}\right)(P_kQ_k^{\epsilon_k}))^{\omega^\vee_{|i_k|}}E_{i_k}(P_kQ_k^{-\epsilon_k})^{\omega^\vee_{|i_k|}}.\end{aligned}$$

Then writing out $m_{\mathbf{i}}$ explicitly and comparing with Definition 4.2.5 one obtains

$$m_{\mathbf{i}}^* X_j = (P_jQ_j^{-\epsilon_j})^{[j>0]}(P_{j^+}Q_{j^+}^{\epsilon_{j^+}})^{[j^+\leq m]}\left(\prod_{\substack{j<k<j^+ \\ k>0}} P_k^{C_{|i_k|,|i_j|}}\right).$$

But now one can check directly that

$$\begin{aligned}\frac{\{X_j,X_k\}_G}{X_jX_k} &= \epsilon_j d_k[j=k^+] - \epsilon_k d_k[j^+=k] + \epsilon_j d_j\frac{C_{kj}}{2}[k<j<k^+][j>0] \\ &\quad - \epsilon_{j^+}d_j\frac{C_{kj}}{2}[k<j^+<k^+][j^+\leq m] - \epsilon_k d_k\frac{C_{kj}}{2}[j<k<j^+][k>0] \\ &\quad + \epsilon_{k^+}d_k\frac{C_{kj}}{2}[j<k^+<j^+][k^+\leq m] \\ &= b_{jk}d_k.\end{aligned}$$

□

Chapter 5

Q-Systems, Factorization Dynamics, and the Twist Automorphism

5.1 Introduction

The goals of this chapter are to realize the cluster structures associated with Q-systems as amalgamations of those on double Bruhat cells, use this to identify Q-system dynamics with those of a factorization mapping (hence deduce their integrability), relate these to the Fomin-Zelevinsky twist automorphism, and provide cluster realizations of twisted Q-systems.

Q-systems are nonlinear recurrence relations associated with affine Dynkin diagrams, arising in the Bethe ansatz and the representation theory of Yangians and quantum loop algebras [KR87; Nak03; Her06; Her10]. There is by now a large literature related to them and their relatives (see [KNS11, Section 13] for a survey), and in particular it was discovered in [Ked08; DK09] that they may be realized as sequences of cluster transformations in certain cluster algebras. In this chapter we provide concrete realizations of these cluster algebras in terms of double Bruhat cells and their amalgamations. The relevant sequences of cluster transformations are then identified with factorization mappings on quotients of double Bruhat cells, leading to their discrete integrability. Moreover, these sequences provide an alternate description of the Fomin-Zelevinsky twist automorphism in terms of cluster transformations, yielding explicit formulas relating twisted and untwisted cluster variables.

Theorem. (5.2.4, 5.3.6) *The conjugation quotient $G^{c,c}/H$ has a natural cluster structure obtained from that of $G^{c,c}$ by amalgamation. Its exchange matrix is of the form*

$$B_C := \begin{pmatrix} 0 & C^t \\ -C^t & 0 \end{pmatrix},$$

where C is the Cartan matrix of G. Up to normalization, there is a Q-system which can be realized by exchange relations in the corresponding cluster algebra; its type is the affinization of that of G when this is simply-laced, otherwise it is of a twisted type related to that of G by folding.

When G is of type A_n this reformulates a result of [GSV11], and our use of amalgamation to construct cluster structures on adjoint quotients derives from the construction of [FM13]. When G is not simply-laced, this provides a novel cluster algebraic realization of the Q-systems of twisted type, though the cluster structures associated in [DK09] to Q-systems of nonsimply-laced untwisted type do not fit into our framework. We note that in the context of double Bruhat cells, what arises more naturally are the Y-system analogues of Q-systems, which differ by a standard change of variables. In different language, we work directly with $\mathcal{X}$-coordinates rather than cluster variables; this is essential in using amalgamation to form the quotient cluster structures we need.

Given the above result, the sequence of mutations underlying the Q-system gives rise to a corresponding sequence of cluster transformations on $G^{c,c}/H$.

Theorem. (5.2.8, 5.3.7) *Under the identification of their associated cluster structures, the dynamics of the Q-system correspond to those of a certain factorization mapping on the quotient $G^{c,c}/H$. In particular, these Q-systems are discrete integrable in the Liouville sense.*

Factorization mappings play an important role in discrete integrable systems, analogous to that of Lax forms in continuous-time integrable systems [DLT89; MV91; Ves91]. Given a rule for factoring a group element g as a product $g = hk$, one defines a corresponding factorization mapping by $g \mapsto kh$, typically restricted to some subvariety of G. The factorization relevant for our purposes is defined via the decomposition of an element into opposite Borel subgroups, which is unambiguously defined up to conjugation by H. In addition to making contact with Q-systems, the requirement that c be a Coxeter element guarantees that the invariant functions on G descend to form an integrable system on $G^{c,c}/H$, which has a natural symplectic structure [Hof+00]. The factorization mapping manifestly preserves these invariant functions, hence as observed in [Hof+00] is discrete integrable in the Liouville sense. The discrete integrability of the corresponding Q-system then follows as a corollary of our setup; in type A_n this integrability is well-known from a number of different perspectives [GSV11; DK10]. In fact, $G^{c,c}/H$ is also equipped with an integrable system (a generalization of the relativistic periodic Toda lattice) when G is an affine Kac-Moody group [Wil13b], and inherits a quotient cluster structure as well.

Theorem. (5.2.8) *If G is an affine Kac-Moody group, the factorization mapping on $G^{c,c}/H$ is again equivalent to an integrable mutation sequence in a quotient cluster structure.*

In type $A_n^{(1)}$ a generalization of this is treated in [FM13], and is related to the Hirota bilinear difference equation (or octahedron recurrence). In other simply-laced affine types it is related to the analogues of Q-systems for quantum toroidal algebras [Her07].

Since amalgamation commutes with mutation in a suitable sense, our setup also gives rise to a distinguished sequence of cluster transformations on $G^{c,c}$ itself. This turns out to be closely related to the Fomin-Zelevinsky twist automorphism, which relates the cluster variables and factorization parametrization associated with a double reduced word.

Theorem. (5.4.1) *The twist automorphism of $G^{c,c}$ maps the toric chart associated with any seed to the chart obtained from the mutation sequence associated with the factorization mapping on $G^{(c,c)}/H$. This holds when G is any symmetrizable Kac-Moody group, and yields explicit formulas expressing twisted cluster variables as Laurent monomials in the untwisted cluster variables of a different cluster.*

Versions of the twist map exist on many varieties of Lie-theoretic origin with natural cluster structures. This result parallels similar ones for unipotent cells [GLS12] and Grassmannians [MS13], which show that certain twisted cluster variables differ by a change of coefficients from the untwisted cluster variables obtained from a distinguished sequence of mutations.

Our interest in understanding properties of the exchange matrices B_C also comes from their appearance (in the simply-laced case) as BPS quivers of pure $\mathcal{N} = 2$ gauge theories [Ali+11; CD12]. In this setting the BPS spectrum of an $\mathcal{N} = 2$ theory is encoded as a rational torus automorphism, the monodromy operator or spectrum generator, which in the presence of certain finiteness properties is a mutation-periodic sequence of cluster transformations (often called a maximal green sequence in the cluster algebra literature). For pure $\mathcal{N} = 2$ gauge theories, this mutation sequence is in fact an iteration of the Q-system sequence [Ali+11], hence in particular is itself discrete integrable.

5.2 Factorization Dynamics as Cluster Transformations

In this section we discuss factorization mappings from the perspective of cluster transformations. To any Cartan matrix C we associate a seed Σ_C with a canonical mutation-periodic sequence. We realize this seed as an amalgamation of a Coxeter double Bruhat cell, which can be identified with its quotient under conjugation by the Cartan subgroup. We show that the mutation-periodic sequence corresponds to a factorization mapping on this quotient. In finite type this mapping is known to be discrete integrable [Hof+00], and we show it is also integrable in affine type. We will freely use the notation and concepts introduced in Section 4.2 and Remark 4.3.6.

Definition 5.2.1. For any symmetrizable r-by-r Cartan matrix C, let Σ_C be the seed with $I_C = (I_C)_u = \{1, \ldots, 2r\}$, exchange matrix

$$B_C := \begin{pmatrix} 0 & C^t \\ -C^t & 0 \end{pmatrix},$$

and d_i derived from the symmetrizers of C in the obvious way. We let $\widehat{\mu}$ be the mutation sequence $\mu_1 \circ \cdots \circ \mu_r$ of Σ_C, and σ the permutation of I interchanging i and $i + r$. □

Proposition 5.2.2. *The mutation sequence $\widehat{\mu}$ is a σ-period of Σ_C, that is*

$$\widehat{\mu}(B_C)_{ij} = (B_C)_{\sigma(i)\sigma(j)}.$$

Proof. Since $(B_C)_{\sigma(i)\sigma(j)} = -(B_C)_{ij}$, we must check that $\widehat{\mu}(B_C) = -B_C$. This is immediate for the top-left and off-diagonal r-by-r blocks of $\widehat{\mu}(B_C)$. We then calculate that

$$\begin{aligned}
\widehat{\mu}(B_C)_{i+r,j+r} =& \frac{1}{2}\sum_{1\leq k\leq r}(C_{k,i}|C_{j,k}| - |C_{k,i}|C_{j,k}) \\
=& \frac{1}{2}\sum_{k=i,j}(C_{k,i}|C_{j,k}| - |C_{k,i}|C_{j,k}) \\
=& 0.
\end{aligned}$$

□

Fix a Coxeter element $c = s_1 \cdots s_r$ in the Weyl group associated with C, and a double reduced word $\mathbf{i} = (-1, \ldots, -r, 1, \ldots, r)$ for $u = v = c$. In fact the essential content of this section and the next hold when u and v are possibly distinct Coxeter elements, see Remark 5.2.5. When C is not of finite type, G_{Ad} will refer to the minimal form of the adjoint group associated with C, and $\Sigma_{\mathbf{i}}$ to the corresponding minimal seed. Note that $I_{\mathbf{i}} = \{-1, \ldots, -r\} \cup I_C$.

Lemma 5.2.3. *Let C_U^t, C_L^t be the upper- and lower-triangular $r \times r$ matrices with 1's on the diagonal such that $C_U^t + C_L^t = C^t$. That is,*

$$(C_U^t)_{ij} = \delta_{ij} + [i < j]C_{ji}, \quad (C_L^t)_{ij} = \delta_{ij} + [i > j]C_{ji}.$$

Then the exchange matrix of $\Sigma_{\mathbf{i}}$ has the form

$$B_{\Sigma_{\mathbf{i}}} = \begin{pmatrix} C_U^t - \frac{1}{2}C^t & C_L^t & 0 \\ -C_U^t & 0 & -C_L^t \\ 0 & C_U^t & C_L^t - \frac{1}{2}C^t \end{pmatrix},$$

where we have ordered the indices as $-1, \ldots, -r, 1, \ldots, 2r$.

Proof. Can be checked directly from Definition 4.3.1. □

For any $u, v \in W$, we denote by $G_{\mathrm{Ad}}^{u,v}/H_{\mathrm{Ad}}$ the quotient of $G_{\mathrm{Ad}}^{u,v}$ under conjugation by H_{Ad}, with the following caveat. If $\mathbf{j}$ is any double reduced word for u, v, then since H_{Ad} is generated by coweight subgroups and $X_k^{\omega^\vee_{|i_k|}}$ commutes with E_j for $|j| \neq |i_k|$, it follows from the definition of $x_{\mathbf{j}}$ that the conjugation action of H_{Ad} preserves the image of $\mathcal{X}_{\Sigma_{\mathbf{j}}}$, and that a good geometric quotient $\mathcal{X}_{\Sigma_{\mathbf{j}}}/H_{\mathrm{Ad}}$ exists. In fact, from eq. (2.2.8) it is clear that for any seed Σ' mutation-equivalent to $\Sigma_{\mathbf{j}}$, the corresponding chart $\mathcal{X}_{\Sigma'} \subset G_{\mathrm{Ad}}^{u,v}$ has a good quotient by H_{Ad}. These charts cover an open subset of $G_{\mathrm{Ad}}^{u,v}$ whose complement is of codimension at least 2, hence this open subset also has a good quotient by H_{Ad}. The question of whether or not the whole cell $G_{\mathrm{Ad}}^{u,v}$ admits a good quotient will not be relevant for our purposes, so we will simply write $G_{\mathrm{Ad}}^{u,v}/H_{\mathrm{Ad}}$ with the understanding that we may need to restrict to an open subset.

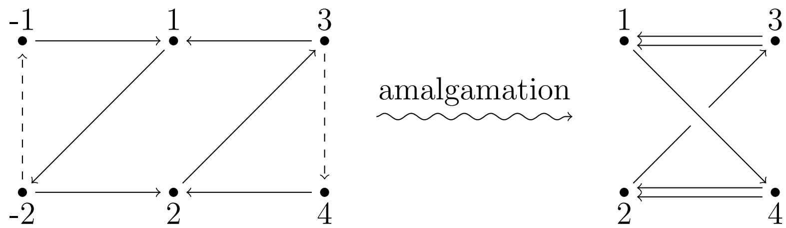


Figure 5.1: The quivers of $\Sigma_{\mathbf{i}}$ and Σ_C when C is of type A_2. The dashed arrows correspond to entries of $B_{\mathbf{i}}$ equal to $\pm\frac{1}{2}$; since they connect frozen vertices they do not affect the structure of cluster transformations, but record the Poisson brackets among frozen variables. The amalgamation itself "glues together" some of the frozen variables: -1 to 3 and -2 to 4.

Theorem 5.2.4. *The seed Σ_C is the amalgamation of $\Sigma_{\mathbf{i}}$ along the map $\pi : I_{\mathbf{i}} \twoheadrightarrow I_C$ given by*

$$\pi(k) = \begin{cases} k & k > 0 \\ |k| + r & k < 0. \end{cases}$$

The map $x_{\mathbf{i}} : \mathcal{X}_{\Sigma_{\mathbf{i}}} \hookrightarrow G_{\mathrm{Ad}}^{c,c}$ descends to an open immersion $\mathcal{X}_{\Sigma_C} \hookrightarrow G_{\mathrm{Ad}}^{c,c}/H_{\mathrm{Ad}}$ intertwining the quotient and amalgamation maps:

$$\begin{tikzcd} \mathcal{X}_{\Sigma_{\mathbf{i}}} \arrow[r, "x_{\mathbf{i}}"] \arrow[d, two heads, "\pi"] & G_{\mathrm{Ad}}^{c,c} \arrow[d, two heads, "\pi"] \\ \mathcal{X}_{\Sigma_C} \arrow[r, "x_{\mathbf{i}}"] & G_{\mathrm{Ad}}^{c,c}/H_{\mathrm{Ad}}. \end{tikzcd}$$

Proof. Using Lemma 5.2.3, one can immediately verify that the hypothesis of Definition 2.2.14 are satisfied by Σ_C, $\Sigma_{\mathbf{i}}$, and π. The conjugation-invariant subalgebra $\mathbb{C}[\mathcal{X}_{\Sigma_{\mathbf{i}}}]^{H_{\mathrm{Ad}}}$ is manifestly generated by the X_i, $X_{-i}X_{i+r}$, and their inverses for $1 \leq i \leq r$. But this is equal to $\pi^*\mathbb{C}[\mathcal{X}_{\Sigma_C}]$, hence we obtain the map $\mathcal{X}_{\Sigma_C} \hookrightarrow G_{\mathrm{Ad}}^{c,c}/H_{\mathrm{Ad}}$. □

Remark 5.2.5. If $\mathbf{j}$ is any double reduced word for $u, v \in W$, the conjugation action of H_{Ad} on $G_{\mathrm{Ad}}^{u,v}$ will always have a comparably simple expression in the associated $\mathcal{X}$-coordinates. However, it is not always the case that quotient map $\mathcal{X}_{\Sigma_{\mathbf{j}}} \twoheadrightarrow \mathcal{X}_{\Sigma_{\mathbf{j}}}/H_{\mathrm{Ad}}$ is an amalgamation map. For example, if $u = c$ but $v = e$, the hypotheses of Definition 2.2.14 will not be satisfied by the quotient map. However, if u and v are (possibly distinct) Coxeter elements, there will be a unique amalgamation $\widetilde{\Sigma}$ of $\Sigma_{\mathbf{j}}$ and isomorphism $\mathcal{X}_{\widetilde{\Sigma}} \xrightarrow{\sim} \mathcal{X}_{\Sigma_{\mathbf{j}}}/H_{\mathrm{Ad}}$ intertwining the quotient and amalgamation maps from $\mathcal{X}_{\Sigma_{\mathbf{j}}}$. In fact, when u and v are Coxeter elements conjugate to c, the reader can check that the resulting seed $\widetilde{\Sigma}$ is mutation-equivalent to Σ_C. For GL_n, this was previously observed (from a different point of view) in [GSV11]. □

Recall that an integrable system on a (smooth) symplectic variety is a Poisson-commutative subalgebra of its coordinate ring whose differentials generically span Lagrangian subspaces of its cotangent spaces, inducing a Lagrangian foliation of an open subset. By an integrable system on a Poisson variety we will mean an algebra of functions which restricts to an integrable system on a generic symplectic leaf.

Proposition 5.2.6. ([Hof+00],[Wil13a]) *If C is of finite or affine type, the restrictions of the conjugation-invariant functions on G_{Ad} form an integrable system on $G_{\mathrm{Ad}}^{c,c}/H_{\mathrm{Ad}}$.*

Proof. We only comment that the affine case treated in [Wil13a] and Section 3.4 is slightly different from the present one, though the proof there extends straightforwardly. In loc. cited it was shown that the invariants restrict to form an integrable system on $(G')^{c,c}/H$, where G' is the central extension of the algebraic loop group $L\mathring{G}$. This is actually more delicate, as its symplectic leaves are of dimension $2r+2$, rather than $2r$ (where r is the rank of $\mathring{G}$). For the present case the needed Hamiltonians are derived from the invariant ring $\mathbb{C}[\mathring{G}]^{\mathring{G}}$: we pull back this subalgebra along the evaluation map $L\mathring{G}\times\mathbb{C}^*\to\mathring{G}$ and take the component invariant under the $\mathbb{C}^*$ action (in particular they extend to functions on the semidirect product $L\mathring{G}\rtimes\mathbb{C}^*$). The Hamiltonians for groups of twisted affine type may be produced similarly by embedding them into algebraic loop groups as subgroups invariant under a diagram automorphism. □

We recall the following basic result about cluster structures of double Bruhat cells; we omit its extension to the Kac-Moody case, which is straightforward.

Proposition 5.2.7. ([FG06a]) *Suppose that $\mathbf{i}=(i_1,\dots,i_m)$, $\mathbf{i}'=(i'_1,\dots,i'_m)$ differ by swapping two adjacent indices differing only by a sign. That is, for some $1\le k<m$, $i_k=-i_{k+1}$, and*

$$i'_\ell=\begin{cases}-i_\ell & \ell=k,k+1\\ i_\ell & \text{otherwise.}\end{cases}$$

Then the corresponding sets of $\mathcal{X}$-coordinates on $G_{\mathrm{Ad}}^{u,v}$ differ by the cluster transformation at k:

$$\begin{tikzcd}
\mathcal{X}_{\Sigma_{\mathbf{i}}} \arrow[rr, dashed, "\mu_k"] \arrow[dr, "x_{\mathbf{i}}"'] & & \mathcal{X}_{\Sigma_{\mathbf{i}'}} \arrow[dl, "x_{\mathbf{i}'}"] \\
& G_{\mathrm{Ad}}^{u,v} &
\end{tikzcd}$$

Theorem 5.2.8. *The cluster automorphism $\widehat{\mu}_\sigma$ of $\mathcal{X}_{\Sigma_C}$ coincides with the restriction of the following rational automorphism of $G_{\mathrm{Ad}}^{c,c}/H_{\mathrm{Ad}}$. Given $g\in G_{\mathrm{Ad}}^{c,c}/H_{\mathrm{Ad}}$, there will generically be unique elements $h_1,h_2\in H_{\mathrm{Ad}}$ such that, up to conjugation by H_{Ad},*

$$g=\left(\Big(\overset{\curvearrowright}{\prod_{1\le i\le r}}E_i\Big)h_1\right)\left(\Big(\overset{\curvearrowright}{\prod_{1\le i\le r}}F_i\Big)h_2\right).$$

The rational automorphism of $G^{c,c}_{\mathrm{Ad}}/H_{\mathrm{Ad}}$ is then the factorization mapping

$$g = \left((\overset{\curvearrowright}{\prod_{1\le i\le r}} E_i) h_1 \right) \left((\overset{\curvearrowright}{\prod_{1\le i\le r}} F_i) h_2 \right) \mapsto \left((\overset{\curvearrowright}{\prod_{1\le i\le r}} F_i) h_2 \right) \left((\overset{\curvearrowright}{\prod_{1\le i\le r}} E_i) h_1 \right),$$

taken up to conjugation by H_{Ad}. Here the product notation indicates we order the terms from left to right by increasing i. In particular, $\widehat{\mu}_\sigma$ preserves the restrictions of any conjugation-invariant functions on G_{Ad}, and in finite or affine type is discrete integrable in the Liouville sense.

Proof. By Proposition 5.2.7, the $\mathcal{X}$-coordinates on $\mathcal{X}_{\Sigma_{\mathbf{i}}}$ and $\mathcal{X}_{\Sigma'_{\mathbf{i}}}$ (where $\Sigma'_{\mathbf{i}} = \widehat{\mu}(\Sigma_{\mathbf{i}})$) are related by

$$\left(\overset{\curvearrowright}{\prod_{1\le i\le r}} X_{-i}^{\omega_i^\vee} \right) \left(\overset{\curvearrowright}{\prod_{1\le i\le r}} F_i X_i^{\omega_i^\vee} \right) \left(\overset{\curvearrowright}{\prod_{1\le i\le r}} E_i X_{i+r}^{\omega_i^\vee} \right) = \left(\overset{\curvearrowright}{\prod_{1\le i\le r}} (X'_{-i})^{\omega_i^\vee} \right) \left(\overset{\curvearrowright}{\prod_{1\le i\le r}} E_i (X'_i)^{\omega_i^\vee} \right) \left(\overset{\curvearrowright}{\prod_{1\le i\le r}} F_i (X'_{i+r})^{\omega_i^\vee} \right).$$

It is straightforward to see that each of the seeds $\mu_k \circ \cdots \circ \mu_r(\Sigma_{\mathbf{i}})$ satisfy the hypotheses of Definition 2.2.14 with respect to $\pi : I_{\mathbf{i}} \twoheadrightarrow I_C$, hence we can apply Proposition 2.2.16 to obtain

$$\begin{tikzcd} \mathcal{X}_{\Sigma_{\mathbf{i}}} \arrow[r, dashed, "\widehat{\mu}"] \arrow[d, two heads, "\pi"] & \mathcal{X}_{\Sigma'_{\mathbf{i}}} \arrow[d, two heads, "\pi"] \\ \mathcal{X}_{\Sigma_C} \arrow[r, dashed, "\widehat{\mu}"] & \mathcal{X}_{\Sigma'_C}. \end{tikzcd}$$

In particular, the $\mathcal{X}$-coordinates on $\mathcal{X}_{\Sigma_C}$ and $\mathcal{X}_{\Sigma'_C}$ are related by

$$\left(\overset{\curvearrowright}{\prod_{1\le i\le r}} F_i X_i^{\omega_i^\vee} \right) \left(\overset{\curvearrowright}{\prod_{1\le i\le r}} E_i X_{i+r}^{\omega_i^\vee} \right) = \left(\overset{\curvearrowright}{\prod_{1\le i\le r}} E_i (X'_i)^{\omega_i^\vee} \right) \left(\overset{\curvearrowright}{\prod_{1\le i\le r}} F_i (X'_{i+r})^{\omega_i^\vee} \right),$$

up to conjugation by H_{Ad}.

The isomorphism $\mathcal{X}_{\Sigma'_C} \xrightarrow{\sim} \mathcal{X}_{\Sigma_C}$ given by σ then induces a rational automorphism of $G^{c,c}_{\mathrm{Ad}}/H_{\mathrm{Ad}}$ through

$$\left(\overset{\curvearrowright}{\prod_{1\le i\le r}} E_i (X'_i)^{\omega_i^\vee} \right) \left(\overset{\curvearrowright}{\prod_{1\le i\le r}} F_i (X'_{i+r})^{\omega_i^\vee} \right) \mapsto \left(\overset{\curvearrowright}{\prod_{1\le i\le r}} F_i (X'_{i+r})^{\omega_i^\vee} \right) \left(\overset{\curvearrowright}{\prod_{1\le i\le r}} E_i (X'_i)^{\omega_i^\vee} \right).$$

But this is just the map described in the theorem, with $h_1 = \prod (X'_i)^{\omega_i^\vee}$ and $h_2 = \prod (X'_{i+r})^{\omega_i^\vee}$. That $\widehat{\mu}_\sigma$ preserves invariant functions is clear, hence we obtain discrete integrability in finite and affine types by Proposition 5.2.6. Note that in affine type even though the symplectic leaves of $\mathcal{X}_{\Sigma_C}$ are of positive codimension, $\widehat{\mu}_\sigma$ preserves the distinguished symplectic leaf hence restricts to an integrable symplectomorphism of it. □

5.3 Q-Systems and Discrete Integrability

Q-systems are nonlinear recurrence relations associated with affine Dynkin diagrams $X_N^{(\kappa)}$. We review their normalized versions and cluster-algebraic realizations following [Ked08; DK09], which we extend to include twisted types. In twisted and simply-laced untwisted types these systems are encoded by the seeds Σ_C studied in the previous section. The Q-system itself is realized by a sequence of cluster transformations coinciding with that of the corresponding factorization mapping, though realized by cluster variables rather than $\mathcal{X}$-coordinates. Since the relevant exchange matrix is nondegenerate, the two sets of variables differ by a finite map, leading to the discrete integrability of these Q-systems.

Recall that affine Dynkin diagrams are classified by pairs of a finite-type diagram X_N and an automorphism of order κ. This induces an automorphism of the simple Lie algebra of type X_N, whose invariant subalgebra is also simple and whose type we denote by Y_M. Clearly for untwisted types ($\kappa = 1$) we have $X_N = Y_M$, while for twisted types the correspondence is given below. It is summarized by the fact that the Langlands dual of $X_N^{(\kappa)}$ is the affinization of the Langlands dual of Y_M.

$X_N^{(\kappa)}$	$A_{2r-1}^{(2)}$	$D_{r+1}^{(2)}$	$E_6^{(2)}$	D_4^3
Y_M	C_r	B_r	F_4	G_2

Definition 5.3.1. The Q-system of type $X_N^{(\kappa)}$ is the following recurrence relation in the commuting variables $\{Q_n^{(a)}\}$, where $n \in \mathbb{Z}$ is a discrete "time" variable and a is an index labeled by the roots of Y_M. If $X_N^{(\kappa)}$ is of untwisted simply-laced type and C the Cartan matrix of type X_N, the corresponding Q-system is

$$(Q_n^{(a)})^2 = Q_{n-1}^{(a)} Q_{n+1}^{(a)} + \prod_{b \neq a} (Q_n^{(a)})^{-C_{ba}}.$$

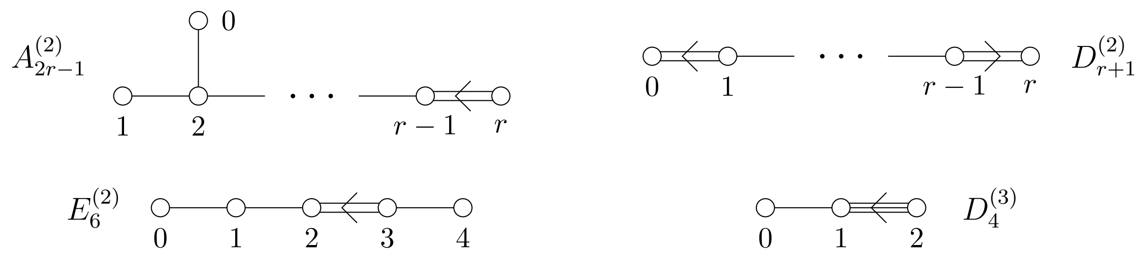


Figure 5.2: Affine Dynkin diagrams of twisted type and enumerations of their vertices. The diagram Y_M is the subdiagram whose nodes have nonzero labels.

For $X_N^{(\kappa)}$ of twisted type, the corresponding Q-systems are as follows [Hat+02; Her10]:

$$A_{2r-1}^{(2)} \begin{cases} (Q_n^{(a)})^2 = Q_{n-1}^{(a)} Q_{n+1}^{(a)} + Q_n^{(a-1)} Q_n^{(a+1)} & 1 \leq a < r \\ (Q_n^{(r)})^2 = Q_{n-1}^{(r)} Q_{n+1}^{(r)} + (Q_n^{(r)})^2 \end{cases}$$

$$D_{r+1}^{(2)} \begin{cases} (Q_n^{(a)})^2 = Q_{n-1}^{(a)} Q_{n+1}^{(a)} + Q_n^{(a-1)} Q_n^{(a+1)} & 1 \leq a < r-1 \\ (Q_n^{(r-1)})^2 = Q_{n-1}^{(r-1)} Q_{n+1}^{(r-1)} + Q_n^{(r-2)} (Q_n^{(r)})^2 \\ (Q_n^{(r)})^2 = Q_{n-1}^{(r)} Q_{n+1}^{(r)} + Q_n^{(r-1)} \end{cases}$$

$$E_6^{(2)} \begin{cases} (Q_n^{(1)})^2 = Q_{n-1}^{(1)} Q_{n+1}^{(1)} + Q_n^{(2)} \\ (Q_n^{(2)})^2 = Q_{n-1}^{(2)} Q_{n+1}^{(2)} + Q_n^{(1)} Q_n^{(3)} \\ (Q_n^{(3)})^2 = Q_{n-1}^{(3)} Q_{n+1}^{(3)} + (Q_n^{(2)})^2 Q_n^{(4)} \\ (Q_n^{(4)})^2 = Q_{n-1}^{(4)} Q_{n+1}^{(4)} + Q_n^{(3)} \end{cases}$$

$$D_4^3 \begin{cases} (Q_n^{(1)})^2 = Q_{n-1}^{(1)} Q_{n+1}^{(1)} + Q_n^{(2)} \\ (Q_n^{(2)})^2 = Q_{n-1}^{(2)} Q_{n+1}^{(2)} + (Q_n^{(1)})^3 \end{cases}$$

Here we set $Q_n^{(0)} = 1$ and enumerate the roots of Y_M as in fig. 5.2. □

We omit the definition of the Q-systems of nonsimply-laced untwisted type, as they lie outside the scope of our main result. Also absent from the above discussion is the twisted type $A_{2n}^{(2)}$; its relationship with the corresponding finite type is more subtle, and it does not admit an interpretation in terms of cluster transformations.[1] Thus when referring to a generic twisted type $X_N^{(\kappa)}$ we will tacitly assume it is not of type $A_{2n}^{(2)}$.

The correspondence between $X_N^{(\kappa)}$ and Y_M allows us to write the above Q-systems uniformly as follows:

Proposition 5.3.2. *Let $X_N^{(\kappa)}$ be of twisted type or simply-laced untwisted type, and C the Cartan matrix of the associated finite type Y_M. Then the Q-system of type $X_N^{(\kappa)}$ may be*

[1] It contains the relation $(Q_n^{(r)})^2 = Q_{n-1}^{(r)} Q_{n+1}^{(r)} + Q_n^{(r-1)} Q_n^{(r)}$, whose terms cannot be rearranged into an exchange relation since $Q_n^{(r)}$ appears on both sides.

written as

$$(Q_n^{(a)})^2 = Q_{n-1}^{(a)} Q_{n+1}^{(a)} + \prod_{b \neq a} (Q_n^{(a)})^{-C_{ba}}.$$

Proof. Follows by inspection of the above list and the definition of Y_M. $\square$

To realize Q-systems in terms of cluster transformations, it is convenient to replace them with certain normalized, but equivalent, Q-systems. These normalized variables differ from those of the usual Q-system via rescaling by certain roots of unity.

Proposition 5.3.3. ([Ked08; DK09]) *The normalized Q-system*

$$\widetilde{Q}_{n-1}^{(a)} \widetilde{Q}_{n+1}^{(a)} = (\widetilde{Q}_n^{(a)})^2 + \prod_{b \neq a} (\widetilde{Q}_n^{(b)})^{-C_{ba}} \tag{5.3.4}$$

is equivalent to the ordinary Q-system under the rescaling $\widetilde{Q}_n^{(a)} = \epsilon_a Q_n^{(a)}$, where the $\epsilon_a \in \mathbb{C}$ are defined by $\prod_{1 \leq a \leq r} \epsilon_a^{C_{ab}} = -1$ for all $1 \leq b \leq r$.

Proof. Note that the existence of such ϵ_a follows from the nondegeneracy of C. The derivation of eq. (5.3.4) is then straightforward. $\square$

Remark 5.3.5. The normalized Q-systems also have a direct interpretation in terms of T-systems. These are relations among q-characters of Kirillov-Reshetikhin modules, in variables $\{T_n^{(a)}(u)\}$ where n and a are as before and $u \in \mathbb{C}$ is a spectral parameter. In the simply-laced case, the relations are

$$T_n^{(a)}(u+1) T_n^{(a)}(u-1) = T_{n-1}^{(a)}(u) T_{n+1}^{(a)}(u) + \prod_{b \neq a} (T_n^{(b)}(u))^{-C_{ba}}.$$

By forgetting the spectral parameter u, we obtain the usual Q-system, but by forgetting instead the parameter n we obtain the normalized Q-system. A similar statement holds for the twisted case, with some subtlety in that we must only consider u modulo a certain additive constant. $\square$

Given a finite-type Cartan matrix C, we let $A_k^{(1)}, \ldots, A_k^{(2r)}$ denote the cluster variables associated with the seed $\widehat{\mu}_\sigma^k(\Sigma_C)$ for $k \in \mathbb{Z}$. Recall from Definition 5.2.1 that the exchange matrix of Σ_C is

$$B_C := \begin{pmatrix} 0 & C^t \\ -C^t & 0 \end{pmatrix},$$

the mutation sequence $\widehat{\mu}$ is $\mu_1 \circ \cdots \circ \mu_r$, and σ interchanges i and $i+r$. As elements of the (upper) cluster algebra $\mathbb{C}[\mathcal{A}_{|\Sigma_C|}]$ the relations among the $A_k^{(i)}$ are in fact equivalent to normalized Q-systems under the identification $A_k^{(i)} \mapsto \widetilde{Q}_k^{(i)}$. Note that $A_k^{(i+r)} = A_{k+1}^{(i)}$ for $1 \leq i \leq r$, so we lose no information by restricting our attention to $A_k^{(1)}, \ldots, A_k^{(r)}$.

Theorem 5.3.6. *Let C be a finite-type Cartan matrix, and $A_k^{(1)}, \dots, A_k^{(r)}$ cluster variables associated with $\widehat{\mu}_\sigma^k(\Sigma_C)$.*

1. ([Ked08; DK09]) *If C is of simply-laced type X_N, the relations among the cluster variables $A_k^{(i)}$ coincide with those of the normalized Q-system of type $X_N^{(1)}$.*

2. *If C is of nonsimply-laced type Y_M, the relations among the cluster variables $A_k^{(i)}$ coincide with those of the normalized Q-system of the associated twisted type $X_N^{(\kappa)}$.*

Proof. Given the definition of the normalized Q-systems in eq. (5.3.4), this is a straightforward check involving the definition of the exchange matrix B_C and the cluster automorphism $\widehat{\mu}_\sigma$. □

Theorem 5.3.7. *For $X_N^{(\kappa)}$ of twisted type or simply-laced untwisted type, the corresponding Q-system is discrete integrable in the Liouville sense.*

Proof. The statement should be understood in light of Theorem 5.3.6, which says that incrementing the discrete time variable n of the (normalized) Q-system is equivalent to expanding the rational symplectomorphism $\widehat{\mu}_\sigma$ of $\mathcal{A}_{\Sigma_C}$ in terms of cluster variables. Since the matrix B_C is nondegenerate, the canonical map $p_{\Sigma_C} : \mathcal{A}_{\Sigma_C} \to \mathcal{X}_{\Sigma_C}$ is a finite cover. In particular, $\mathcal{A}_{\Sigma_C}$ inherits from $\mathcal{X}_{\Sigma_C}$ a symplectic structure and the integrable system of Proposition 5.2.6. Since $p_{\Sigma_C} : \mathcal{A}_{\Sigma_C} \to \mathcal{X}_{\Sigma_C}$ intertwines the associated automorphisms $\widehat{\mu}_\sigma$ of $\mathcal{A}_{\Sigma_C}$ and $\mathcal{X}_{\Sigma_C}$, and the latter preserves the integrable system on $\mathcal{X}_{\Sigma_C}$ by Theorem 5.2.8, the former is also discrete integrable. Since the normalized and unnormalized Q-systems differ by an invertible rescaling, the integrability of the normalized Q-system implies that of the unnormalized version. □

5.4 The Twist Automorphism

Since amalgamation commutes with mutation, the mutation sequence of Σ_C studied in the previous sections lifts to a mutation sequence on the double Bruhat cell $G^{c,c}$ itself. We now show that this sequence is intimately connected with the twist automorphism of $G^{c,c}$. Specifically, any two clusters related by the corresponding sequence of cluster transformations are also mapped to each other by the twist automorphism. Equivalently, the twist pulls back cluster variables to cluster monomials of the seed obtained by this mutation sequence. While these pullbacks are generally not cluster variables, the unfrozen cluster variables are taken to monomials with only a single unfrozen factor, so in this sense the twist acts by a change of coefficients. From the perspective of Poisson geometry this is quite natural; it is known that the twist automorphism is Poisson [GSV03], hence both twisted and untwisted cluster variables have quadratic brackets with respect to the standard Poisson-Lie structure.

Theorem 5.4.1. *Let G be a symmetrizable Kac-Moody group, τ the twist automorphism of $G^{c,c}$, and $\mathcal{A}_\Sigma \subset G^{c,c}$ the toric chart associated with a seed Σ. Then τ restricts to an* *isomorphism of $\mathcal{A}_\Sigma$ onto $\mathcal{A}_{\widehat{\mu}(\Sigma)}$, where $\widehat{\mu} = \mu_1 \circ \cdots \mu_r$ is the mutation sequence consisting of a single mutation at each unfrozen index. In particular, if $\{A_i\}$ and $\{A_i'\}$ are the cluster variables associated with Σ and $\widehat{\mu}(\Sigma)$, respectively, then the $\{A_i'\}$ and the twisted cluster variables $\{\tau^*(A_i)\}$ are Laurent monomials in each another. If Σ is the seed associated with the double reduced word $\mathbf{i} = (-1, \ldots, -r, 1, \ldots, r)$, this transformation is explicitly given by*

$$A_i' = \prod_{j \in I} (\tau^* A_j)^{M_{ij}},$$

where M is the $I \times I$ matrix with entries

$$M_{j,k} = \begin{cases} \langle \omega_{|i_j|} | \alpha^\vee_{|i_k|} \rangle \quad (= \delta_{jk}) & 1 \le j, k \le r \\ \langle c\omega_{|i_j|} | \alpha^\vee_{|i_k|} \rangle & j > r \text{ and } k < 0 \\ \langle c^{-1}\omega_{|i_j|} | \alpha^\vee_{|i_k|} \rangle & j < 0, \text{ and } k > r \text{ or } k < -r \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From Lemma 5.4.6 and Theorem 4.3.2 it follows immediately that

$$A_i' = \prod_{j \in I} (\tau^* A_j)^{(NB_\Sigma^{\mathrm{mod}})_{ij}},$$

where N is the matrix of Lemma 5.4.6 and B_Σ^{mod} is the modified exchange matrix associated with Σ as in Theorem 4.3.2. Most of the difficulty in verifying that the product of N and B_Σ^{mod} is the given matrix M is encapsulated in Lemma 5.4.7. For example, for $1 \le i, k \le r$, we may use it to compute

$$\begin{aligned} (NB_\Sigma^{\mathrm{mod}})_{i+r,-k} &= \langle (c\omega_i) - \omega_i | \omega_k^\vee + \sum_{j<k} C_{kj} \omega_j^\vee \rangle \\ &= \langle (c\omega_i) - \omega_i | \alpha_k^\vee - (\omega_k^\vee + \sum_{j>k} C_{kj} \omega_j^\vee) \rangle \\ &= \langle (c\omega_i) - \omega_i | \alpha_k^\vee \rangle + \delta_{ik} \\ &= \langle c\omega_i | \alpha_k^\vee \rangle. \end{aligned}$$

Given that $M = NB_\Sigma^{\mathrm{mod}}$, the theorem follows by verifying that M satisfies the hypotheses of Lemma 5.4.3 with respect to the exchange matrices B_Σ and $B_{\widehat{\mu}(\Sigma)}$. Note that $B_{\widehat{\mu}(\Sigma)} = -B_\Sigma$, as $\widehat{\mu}(\Sigma)$ is associated with the double reduced word $(1, \ldots, r, -1, \ldots, -r)$. This computation then parallels that of M itself, again with Lemma 5.4.7 being the core of the calculation. □

Remark 5.4.2. If C is of finite type, the decomposition of M into r-by-r blocks is

$$M = \begin{pmatrix} 0 & 0 & c^{-1} \\ 0 & \mathrm{Id} & 0 \\ c & 0 & 0 \end{pmatrix}.$$

Here we express c as a matrix via its action on the fundamental weight basis, and order the indices by $(-1, \ldots, -r, 1, \ldots, 2r)$. □

Lemma 5.4.3. *Let Σ, $\widetilde{\Sigma}$ be two seeds with the same index set I and unfrozen subset I_u. For an invertible $I \times I$ matrix M, let $\varphi_M : \mathcal{A}_{\widetilde{\Sigma}} \xrightarrow{\sim} \mathcal{A}_\Sigma$ be the isomorphism defined by*

$$\varphi_M^*(A_i) = \prod_{j \in I} \widetilde{A}_j^{M_{ij}}. \tag{5.4.4}$$

Suppose that M satisfies the following conditions:

1. $\widetilde{B}_{ij} = (BM)_{ij}$ *when i is unfrozen.*
2. $M_{ij} = \delta_{ij}$ *when j is unfrozen.*

In particular $B_{ij} = \widetilde{B}_{ij}$ when i and j are both unfrozen, hence Σ and $\widetilde{\Sigma}$ are of the same cluster type. Then we have:

1. *The map φ_M extends to an isomorphism between $\mathcal{A}_{\mu_k(\widetilde{\Sigma})}$ and $\mathcal{A}_{\mu_k(\Sigma)}$ for any unfrozen index k. Specifically, if M' is the $I \times I$ matrix defined by*

$$M'_{ij} = \begin{cases} M_{ij} & i \neq k \\ 2\delta_{kj} - M_{kj} + \sum_{\ell \in I}([B_{k\ell}M_{\ell j}]_- - [B_{k\ell}]_- M_{\ell j}) & i = k, \end{cases} \tag{5.4.5}$$

then the corresponding isomorphism $\varphi_{M'} : \mathcal{A}_{\mu_k(\widetilde{\Sigma})} \xrightarrow{\sim} \mathcal{A}_{\mu_k(\Sigma)}$ satisfies

$$\begin{tikzcd} \mathcal{A}_{\widetilde{\Sigma}} \arrow[r, "\varphi_M"] \arrow[d, dashed, "\mu_k"] & \mathcal{A}_\Sigma \arrow[d, dashed, "\mu_k"] \\ \mathcal{A}_{\widetilde{\Sigma}'} \arrow[r, "\varphi_{M'}"] & \mathcal{A}_{\Sigma'}. \end{tikzcd}$$

2. *If $B_{ij} = 0$ when i and j are both unfrozen (so Σ, $\widetilde{\Sigma}$ are of cluster type A_1^n), then φ_M extends to an isomorphism of $\mathcal{A}$-spaces and upper cluster algebras.*

Proof. To prove the first claim one must check that for any cluster variable A'_i on $\mathcal{A}_{\Sigma'}$, we have $\varphi_{M'}^* A'_i = (\mu_k \circ \varphi_M \circ \mu_k)^* A'_i$. The condition that $M_{ij} = \delta_{ij}$ when j is unfrozen ensures this holds for $i \neq k$. The condition that $\widetilde{B}_{kj} = (BM)_{kj}$ ensures $(\mu_k \circ \varphi_M \circ \mu_k)^* A'_k$ is a Laurent monomial in the cluster variables on $\mathcal{A}_{\widetilde{\Sigma}'}$, and the given formula for M' follows from explicitly calculating this composition using eqs. (2.2.7) and (5.4.4).

The second claim follows inductively once we establish that M' satisfies the same hypotheses as M, but with respect to the seeds Σ', $\widetilde{\Sigma}'$. That $M'_{ij} = \delta_{ij}$ when j is unfrozen can be checked generally without any assumptions on the cluster type of Σ. On the other hand, a direct computation reveals that B_{ij} vanishing when i and j are unfrozen is a sufficient condition to ensure $\widetilde{B}'_{ij} = (B'M')_{ij}$ when i is unfrozen. □

When C is not of finite type, we take G_{Ad} to be the maximal form of the adjoint group in the following statement.

Lemma 5.4.6. *Let $\mathcal{X}_{\mathbf{i}} \subset G^{c,c}_{\mathrm{Ad}}$ be the toric chart associated with the double reduced word $\mathbf{i} = (-1, \ldots, -r, 1, \ldots, r)$, and $\mathcal{A}_{\mathbf{i}'} \subset G^{c,c}$ the chart associated with $\mathbf{i}' = (1, \ldots, r, -1, \ldots, -r)$. Then the quotient map $\pi : G^{c,c} \twoheadrightarrow G^{c,c}_{\mathrm{Ad}}$ restricts to a finite cover of $\mathcal{A}_{\mathbf{i}'}$ onto $\mathcal{X}_{\mathbf{i}}$. Equivalently, the (pullbacks to $G^{c,c}$ of the) $\mathcal{X}$-coordinates associated with $\mathbf{i}$ are Laurent monomials in the untwisted cluster variables associated with $\mathbf{i}'$. In fact,*

$$A_i = \prod_{j\in I} (\pi^* X_j)^{N_{ij}},$$

where

$$N_{jk} = \begin{cases} \langle c\omega_{|i_j|} | \omega^\vee_{|i_k|}\rangle & j > r,\ k < 0 \\ \langle c^{-1}\omega_{|i_j|} | \omega^\vee_{|i_k|}\rangle & j < 0,\ k > r \\ \langle \omega_{|i_j|} | \omega^\vee_{|i_k|}\rangle & \textit{otherwise.} \end{cases}$$

Proof. By Definition 4.2.7 the cluster variables associated with $\mathbf{i}'$ are generalized minors of the form $\Delta^{\omega_i}_{e,c^{-1}}$, $\Delta^{\omega_i}_{e,e}$, and $\Delta^{\omega_i}_{c,e}$. Calculating the matrix N consists of evaluating such minors on an element of the form

$$g = \left(\prod^{\curvearrowright}_{1\le i\le \widetilde{r}} X_{-i}^{\omega_i^\vee}\right)\left(\prod^{\curvearrowright}_{1\le i\le r} F_i X_i^{\omega_i^\vee}\right)\left(\prod^{\curvearrowright}_{1\le i\le r} E_i X_{i+r}^{\omega_i^\vee}\right).$$

This involves fractional powers of the X_i, since the coweight subgroups themselves do not act on the fundamental representations, but only covering groups of them.

By definition $\Delta^{\omega_i}_{e,c^{-1}}(g) = \langle v_i | g \overline{s_r} \cdots \overline{s_1} v_i\rangle$, where v_i is a highest weight vector of the fundamental representation of highest weight ω_i. The key point is that while the action of E_i or F_i on a vector of weight ω is in general a sum of components with weights of the form $\omega + n\alpha_i$, many of these can be discarded in the computation of a given generalized minor. For example, one can check inductively that for $1 \le k \le r$,

$$\begin{aligned} &\Delta^{\omega_i}_{e,c^{-1}}(g) \\ &\quad = \langle v_i| \left(\prod^{\curvearrowright}_{1\le i\le \widetilde{r}} X_{-i}^{\omega_i^\vee}\right)\left(\prod^{\curvearrowright}_{1\le i\le r} F_i X_i^{\omega_i^\vee}\right)\left(\prod^{\curvearrowright}_{1\le i\le k} E_i X_{i+r}^{\omega_i^\vee}\right) \overline{s_k} \cdots \overline{s_1} v_i\rangle \left(\prod_{j=k+1}^{r} X_{j+r}^{\langle s_j\cdots s_1\omega_i|\omega_j^\vee\rangle}\right), \end{aligned}$$

and from this that

$$\begin{aligned} \Delta^{\omega_i}_{e,c^{-1}}(g) &= \langle v_i| \left(\prod^{\curvearrowright}_{1\le i\le \widetilde{r}} X_{-i}^{\omega_i^\vee}\right)\left(\prod^{\curvearrowright}_{1\le i\le r} F_i X_i^{\omega_i^\vee}\right) v_i\rangle \left(\prod_{j=1}^{r} X_{j+r}^{\langle s_j\cdots s_1\omega_i|\omega_j^\vee\rangle}\right) \\ &= \left(\prod_{j=1}^{\widetilde{r}} X_{-j}^{\langle \omega_i|\omega_j^\vee\rangle}\right)\left(\prod_{j=1}^{r} X_{j}^{\langle \omega_i|\omega_j^\vee\rangle}\right)\left(\prod_{j=1}^{r} X_{j+r}^{\langle s_j\cdots s_1\omega_i|\omega_j^\vee\rangle}\right) \end{aligned}$$

Since

$$\langle c^{-1}\omega_i|\omega_j^\vee\rangle = \langle s_j\cdots s_1\omega_i|\omega_j^\vee\rangle = \langle s_r\cdots s_1\omega_i|\omega_j^\vee\rangle,$$

we obtain the stated values of N_{jk} when $j < 0$. Note that up to a scalar factor this expression depends on choosing $\overline{s_i}$ as the representative of s_i in G. The remaining entries of N can be computed following the same logic. $\square$

Lemma 5.4.7. *For $1 \leq i, k \leq r$, the Coxeter element $c = s_1 \cdots s_r$ satisfies*

$$\langle (c\omega_i) - \omega_i | \omega_k^\vee + \sum_{j>k} C_{kj}\omega_j^\vee \rangle = -\delta_{ik},$$
$$\langle (c^{-1}\omega_i) - \omega_i | \omega_k^\vee + \sum_{j<k} C_{kj}\omega_j^\vee \rangle = -\delta_{ik}.$$

Proof. The two statements are equivalent by reversing the labeling of the simple roots, so it suffices to prove the first. The claim is immediate if $k \geq i$. For $k < i$, note that

$$\langle (c\omega_i) - \omega_i | \omega_k^\vee + \sum_{j>k} C_{kj}\omega_j^\vee \rangle = \langle (s_k \cdots s_i\omega_i) - \omega_i | \omega_k^\vee + \sum_{j>k} C_{kj}\omega_j^\vee \rangle.$$

A simple induction yields

$$s_k \cdots s_i \omega_i = \omega_i + \sum_{j=k}^{i} \left(\sum_{a_1 = j < \cdots < a_\ell = i} (-1)^\ell \prod_{m=1}^{\ell-1} C_{a_m, a_{m+1}} \right) \alpha_j,$$

where the sum is taken over increasing sequences of any length from j to i, and the product is taken to equal 1 when $\ell = 1$. From this we compute that $\langle (s_k \cdots s_i\omega_i) - \omega_i | \omega_k^\vee + \sum_{j>k} C_{kj}\omega_j^\vee \rangle$ is equal to

$$\left(\sum_{a_1 = k < \cdots < a_\ell = i} (-1)^\ell \prod_{m=1}^{\ell-1} C_{a_m, a_{m+1}} \right) + \sum_{j=k+1}^{i} \left(\sum_{a_1 = j < \cdots < a_\ell = i} (-1)^\ell \prod_{m=1}^{\ell-1} C_{a_m, a_{m+1}} \right) C_{kj},$$

which vanishes since the two sums cancel. $\square$

Example 5.4.8. The simplest example is SL_2, where c is the nonidentity element of W and $\mathbf{i} = (-1, 1)$, $\mathbf{i}' = (1, -1)$ are the only double reduced words for (c, c). Their respective cluster variables are just matrix entries:

$$(A_{-1}, A_1, A_2) = (\Delta_{12}, \Delta_{22}, \Delta_{21}), \quad (A'_{-1}, A'_1, A'_2) = (\Delta_{12}, \Delta_{11}, \Delta_{21}).$$

The parametrization associated with $\mathbf{i}$ is

$$x_{\mathbf{i}} \colon (X_{-1}, X_1, X_2) \mapsto (X_{-1}X_1X_2)^{-\frac{1}{2}} \begin{pmatrix} X_{-1}X_1X_2 & X_{-1}X_1 \\ X_1X_2 & 1 + X_1 \end{pmatrix}.$$

From this we can directly evaluate the matrix N of Lemma 5.4.6, and along with the matrix B_Σ^{mod} we have

$$N = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \quad B_\Sigma^{\mathrm{mod}} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

From this we compute the matrix M of Theorem 5.4.1, and the matrix M' of eq. (5.4.5):

$$M = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad M' = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

Theorem 5.4.1 then says that the twisted cluster variables are determined from these by

$$A'_i = \prod_{j \in I} (\tau^* A_j)^{M_{ij}}, \quad A_i = \prod_{j \in I} (\tau^* A'_j)^{M'_{ij}}. \tag{5.4.9}$$

On the other hand, by expanding **??** we compute the following explicit formula for the twist:

$$\tau \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} db^{-1}c^{-1} & b^{-1} \\ c^{-1} & d \end{pmatrix}.$$

From this we can compute the twisted cluster variables directly:

$$(\tau^* A_{-1}, \tau^* A_1, \tau^* A_2) = (\Delta_{21}^{-1}, \Delta_{11}, \Delta_{12}^{-1}), \quad (\tau^* A'_{-1}, \tau^* A'_1, \tau^* A'_2) = (\Delta_{21}^{-1}, \Delta_{12}^{-1}\Delta_{22}\Delta_{21}^{-1}, \Delta_{12}^{-1}).$$

Of course, this agrees with eq. (5.4.9), noting that M and M' are each their own inverses. $\square$

Chapter 6

Integrable Systems, Canonical Bases, and $\mathcal{N} = 2$ Field Theory

6.1 Introduction

The goals of this chapter are to identify the Hamiltonians of the open quadratic Toda system as generating functions of Euler characteristics of quiver Grassmannians, hence heuristically as generalized canonical basis elements, and explain how such an expression is predicted by the appearance of the relevant cluster structures in supersymmetric gauge theory.

Given a quiver Q, there is a close relationship between its representation theory and the associated cluster algebra. In particular, there is a natural bijection between the set of non-initial cluster variables and the set of rigid indecomposable representations (with suitable relations imposed in the presence of oriented cycles). The expansion of a cluster variable in terms of the initial cluster is completely determined by the structure of the associated representation, being expressible as a generating function of Euler characteristics of its quiver Grassmannians called the cluster character.

A primary motivation for the axiomatization of cluster algebras is to codify and abstract part of the combinatorial structure of various examples of canonical bases. However, while the cluster variables of a cluster algebra are to be regarded as prototypes of canonical basis elements, in general they do not span it as a vector space and so do not encapsulate the complete structure of a canonical basis. Nonetheless, in some cases where an interesting a priori definition of a complete canonical basis of a cluster algebra is known, such as the dual semicanonical basis of a unipotent cell, the basis elements which are not cluster variables are still cluster characters (necessarily of nonrigid modules). Thus cluster characters provide a flexible heuristic notion of a generalized canonical basis element, encompassing but extending nontrivially the notion of a cluster variable. The main theorem of this chapter asserts that the Hamiltonians of the quadratic open Toda systems studied in [GSV11; Hof+00] and chapter 5 are in fact cluster characters, hence should be regarded as generalized canonical basis elements.

Theorem. (6.5.1) *The Hamiltonians of the quadratic A_n open Toda systems are cluster characters of nonrigid modules of the associated Jacobian algebra.*

Recall that a potential W on a quiver Q is a formal sum of oriented cycles, and the Jacobian algebra of a quiver with potential is the quotient of the path algebra $\mathbb{C}Q$ by the cyclic derivatives of W. The proof of the above theorem relates the internal structure of the relevant Jacobian algebra to a combinatorial model for computing the Hamiltonians of the quadratic Toda system. This model realizes the Hamiltonians as weighted sums of paths in an associated planar network, a point of view emphasized by [GSV11].

Though not needed directly in its proof, we argue in the last section that the most compelling conceptual point of view on this result is that of nonabelian Hodge theory. In particular, we argue that the double Bruhat cell $SL_{n+1}^{c,c}/H$ should be interpreted as a moduli space of flat connections with irregular singularities, while the network used to compute the Hamiltonians is the 1-skeleton of the spectral curve of the associated Hitchin system. As functions on a space of flat connections, the Hamiltonians themselves become traces of holonomies around closed loops. Such functions are the most basic geometric examples of canonical basis elements, yielding an intuitive explanation for why these Hamiltonians should be expressible as cluster characters. Crucial to this point of view is the appearance of the relevant cluster structure in 4d $\mathcal{N}=2$ field theory. It is only by noticing that the relevant quiver coincides with the BPS quiver of pure $\mathcal{N}=2$ Yang-Mills theory that we are able to connect our double Bruhat cell to an irregular moduli space; the mathematics literature does not contain a sufficiently general treatment of cluster structures in the presence of irregular singularities to encompass this example.

6.2 Jacobian Algebras and Cluster Characters

In this section we recall the Jacobian algebra of a quiver with potential, the proper generalization of the path algebra of an acyclic quiver to the case of quivers with oriented cycles [DWZ08]. We also recall the cluster character of a module, a generating function of the Euler characteristics of its quiver Grassmannians [Pal08].

Given a quiver Q, a representation of Q is the assignment of a vector space M_v to every vertex v of Q, and a linear map $M_{s(a)} \to M_{t(a)}$ to every arrow a with source $s(a)$ and target $t(a)$. The path algebra $\mathbb{C}Q$ is the space of linear combinations of (possibly length zero) paths in Q, with multiplication given by composition. That is, the product pq of two paths is zero if $t(q) \neq s(p)$ and is their composition otherwise. There is an equivalence between left $\mathbb{C}Q$-modules and representations of Q.

The completed path algebra $\widehat{\mathbb{C}Q}$ is the completion of $\mathbb{C}Q$ with respect to the ideal generated by the arrows. A potential W is an element of $Pot(\mathbb{C}Q)$, the closure in $\widehat{\mathbb{C}Q}$ of the ideal generated by all nontrivial cyclic paths in $\mathbb{C}Q$. Given an arrow a of Q, the cyclic derivative $\partial_a : Pot(\mathbb{C}Q) \to \widehat{\mathbb{C}Q}$ is the unique continuous linear map such that

$$\partial_a(c) = \sum_{c=paq} qp,$$

for any cycle c, where the sum is taken over all decompositions of c with p, q being possibly lazy paths. We call a pair (Q, W) a quiver with potential, and always assume W contains no 2-cycles. The Jacobian algebra $J(Q, W)$ is the quotient of $\widehat{\mathbb{C}Q}$ by the closure of the ideal generated by all cyclic derivatives of W (we often write J when Q and W are understood). We say the quiver with potential (Q, W) is Jacobi finite if J is finite-dimensional, and always assume this is the case.

We write J-mod for the category of finite-dimensional left J-modules; equivalently this is the category of finite-dimensional representations of Q satisfying the relations imposed by the cyclic derivatives of W. Given a labeling of the vertices of Q by $\{1, \ldots, n\}$, we write S_i for the simple J-module supported at the ith vertex of Q and P_i for its projective cover.

In this section, to be more in line with the standard conventions on cluster characters, we notate cluster variables by lower-case letters x_i and $\mathcal{X}$-coordinates by lower case letters y_i. That is, if Q_{ij} is the number of edges from i to j minus the number from j to i, and we define a seed by $B_{ij} = Q_{ji}$ (note the transposition of the indices), we now denote the cluster variables A_i by x_i, and the $\mathcal{X}$-coordinates X_i by y_i. We will also abuse our notation slightly and conflate y_i with its pullback $\prod_{j=1}^n x_j^{Q_{ji}}$ to $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ when this meaning is clear.

Definition 6.2.1. Let M be a left J-module and

$$P_M^1 \to P_M^0 \to M \to 0$$

the first two terms of a minimal projective resolution. The index ind_M is the class $[P_0^M] - [P_1^M]$ in $K_0(\mathrm{proj}\, J)$, the Grothendieck group of the category of projective left J-modules. If $\mathrm{ind}_M = \sum_{i=1}^n a_i[P_i]$, we write $x^{\mathrm{ind}_M} = \prod_{i=1}^n x_i^{a_i}$ □

The Grothendieck group $K_0(J\text{-mod})$ has a basis given by the classes of the simple modules S_i, and using this we identify $K_0(J\text{-mod})$ with $\mathbb{Z}^n$ and the class of a module with its dimension vector. Given a dimension vector $e \in \mathbb{Z}^n$ and a J-module M, the quiver Grassmannian $\mathrm{Gr}_e\, M$ is the variety of e-dimensional subrepresentations of M. It is a projective variety naturally embedded in the usual vector space Grassmannian of M.

Definition 6.2.2. The cluster character $CC(M)$ of a J-module M is the Laurent polynomial

$$CC(M) = x^{-\mathrm{ind}_M} \sum_{e \in K_0(J\text{-mod})} \chi(\mathrm{Gr}_e\, M) y^{[M]-e} \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Here χ is the topological Euler characteristic, and for a class $e = \sum_{i=1}^n b_i[S_i]$ we write $y^e = \prod_{i=1}^n y_i^{b_i}$. Note that if N is an e-dimensional submodule of M, $[M] - e$ is the class of M/N. □

This definition is simpler but more limited in scope than that of [Pal08]. A richer picture is provided by the cluster category $\mathcal{C}$, a triangulated 2-Calabi-Yau category which unlike J-mod is in a suitable sense independent of the choice of a particular initial seed. The choice of initial seed given by Q determines a so-called cluster-tilting object T of $\mathcal{C}$, and we have an equivalence J-mod $\cong \mathcal{C}/\langle \Sigma T \rangle$, where Σ is the suspension functor of $\mathcal{C}$ and $\langle \Sigma T \rangle$ the ideal of all morphisms factoring through the additive subcategory generated by ΣT. As we will only be concerned with cluster characters relative to a particular initial cluster, the category J-mod is rich enough for our purposes. Note that we also work with left rather than right modules and dualize the conventions of [Pal08] as needed.

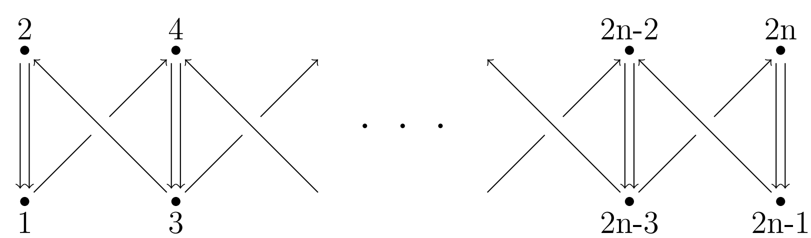
The notion of a cluster character originates in [CC06] for Dynkin quivers, and is treated in increasing generality in [CK06; Pal08; Pla11]. The definition is motivated by the following fundamental property:

Theorem 6.2.3. *For a suitable potential, the cluster character defines a bijection between rigid indecomposables $J(Q, W)$-modules and non-initial cluster variables of the cluster algebra associated with Q, extending to a bijection between rigid modules and the cluster monomials of non-initial clusters.*

For Dynkin quivers, the cluster monomials form a basis of their cluster algebra. However, in general cluster monomials do not span their cluster algebra as a vector space, and the issue of extending them to a complete basis is a fundamental one. One approach is to describe a class of modules containing the rigid ones such that their cluster characters extend the set of cluster monomials to a basis. In particular, the dual semicanonical basis of the coordinate ring of a unipotent cell of a Kac-Moody group is of this form [GLS12].

6.3 The Jacobian Algebra of Q_n

We now study in detail the Jacobian algebra of the quiver Q_n associated with the cluster structure on $SL^{c,c}_{n+1}/H$ described in Section 5.2. We change our indexing slightly so that the vertices of Q_n are indexed as follows:



The signed adjacency matrix of Q_n is (up to reindexing) the skew-symmetric matrix B_{A_n} introduced in Section 5.2. We label the edges of Q_n as follows: for $i \in \{1, \ldots, n\}$ the two vertical arrows from $2i$ to $2i-1$ are labeled a_i and b_i, for $i \in \{2, \ldots, n\}$ the leftward diagonal arrows from $2i-1$ to $2i-2$ are labeled ℓ_i, and for $i \in \{1, \ldots, n-1\}$ the rightward diagonal arrows from $2i-1$ to $2i+2$ are labeled r_i.

We will consider the potential

$$W = \sum_{i=1}^{n-1} a_i \ell_{i+1} b_{i+1} r_i - b_i \ell_{i+1} a_{i+1} r_i,$$

so for each edge in the A_n Dynkin diagram there is a pair of cycles in W. The cyclic derivatives of W are as follows:

$$\begin{aligned}
\partial_{a_i} W &= \ell_{i+1} b_{i+1} r_i - r_{i-1} b_{i-1} \ell_i \\
\partial_{b_i} W &= r_{i-1} a_{i-1} \ell_i - \ell_{i+1} a_{i+1} r_i \\
\partial_{\ell_i} W &= b_i r_{i-1} a_{i-1} - a_i r_{i-1} b_{i-1} \\
\partial_{r_i} W &= a_i \ell_{i+1} b_{i+1} - b_i \ell_{i+1} a_{i+1}.
\end{aligned}$$

Here any terms involving nonexistent edges such as r_n or a_0 are understood to be zero.

We can understand the structure of the resulting Jacobian algebra J explicitly as follows. Since the above relations are all either a difference of two paths or a single path, J inherits from $\mathbb{C}Q$ a basis indexed by certain equivalence classes of paths. Generally, suppose an ideal I of a path algebra $\mathbb{C}Q$ is generated by a set of relations of this form, that is

$$I = \langle p_1 - p_1', \ldots, p_m - p_m', q_1, \ldots, q_\ell \rangle$$

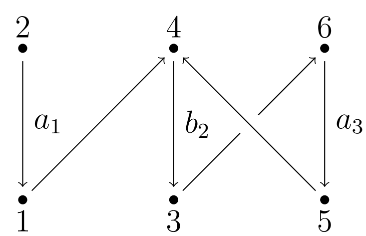
for some paths p_i, q_i such that each pair p_i, p_i' has the same source and target. Then $\mathbb{C}Q/I$ has a basis formed by the nonzero images of paths in $\mathbb{C}Q$. An element of this basis is indexed by the set of paths mapping to it, which is an equivalence class of the relation

$$\alpha \sim \beta \iff \alpha = a p_i b,\ \beta = a p_i' b \text{ for some paths } a, b \text{ and some index } i.$$

The equivalence classes corresponding to basis elements of $\mathbb{C}Q/I$ are those not intersecting I, that is those with no representatives containing some q_j as a subpath.

Let us describe these equivalence classes explicitly for the above potential on Q_n. First, note that any path in Q_n is a sequence of edges that are alternately vertical (an a_i or b_i) and diagonal (an ℓ_i or r_i). Ignoring the indices, this is a perfect shuffle of a word in the alphabet $\{a, b\}$ and a word in the alphabet $\{\ell, r\}$. Since the starting vertex of a path determines both which indices appear and whether the shuffled word starts with a vertical or diagonal edge, the data of a path is exactly the data of its starting vertex and a pair of words in the alphabets $\{a, b\}$ and $\{\ell, r\}$. For example, the following path in Q_3 with starting vertex 2 is associated to the words aba and $\ell r r$:

$\ell_3 a_3 r_2 b_2 r_1 a_1 =$



Conversely, a pair of words and a starting vertex corresponds to an honest path in Q_n if the lengths of the two words satisfy an obvious compatibility condition, and if the choice of starting vertex does not force any of the edges to have an invalid index (such as a_{n+1}).

Viewing the data of a path in Q_n this way makes it easy to understand the relations imposed by W. They assert that two paths are equivalent if their associated words differ by a pair of permutations. The induced basis of J is labeled by the resulting equivalence classes, which are determined by the data of a starting vertex and the total number of times each letter appears in the words of any of its representative paths. Suppose a path has starting vertex either $2i$ or $2i-1$ for some $i \in \{1, \ldots, n\}$, and that x and y are the number of times it traverses an ℓ or r edge, respectively. Then its equivalence class is associated with a basis element of J if and only if either $x < i$ and $y < n+1-i$. Informally, if you change the path so that it takes all its right steps before its left steps (or vice-versa), it shouldn't fall off the edge of Q_n.

The projective module P_i is the subspace of J spanned by paths with starting vertex i. A path starting at i and ending at vertex j is an element of the subspace $(P_i)_j$ supported at j.

Definition 6.3.1. For each $i \in \{1, \ldots, n\}$, define a $\mathbb{P}^1$-family of modules M_i^λ as follows. Given projective coordinates $\lambda = (\lambda_1 : \lambda_2)$ we embed $P_{2i-1} \hookrightarrow P_{2i}$ by sending the generator of P_{2i-1} (the length zero path at vertex $2i-1$) to the element $\lambda_1 a_i + \lambda_2 b_i \in (P_{2i})_{2i-1}$. The module M_i^λ is then the cokernel of this map. □

From now on we will denote by $\nu_n : \{1, \ldots, n\} \to \{1, \ldots, n\}$ the Nakayama involution $\nu_n(i) = n+1-i$.

Proposition 6.3.2. *The module M_i^λ has a basis $\mathcal{B}_i = \{b_{(x,y,v)}\}$ indexed by*

$$\{(x,y,v)|x,y \in \mathbb{N}, v \in \{2(i+y-x), 2(i+y-x)-1\}, x < i, y < \nu_n(i)\}.$$

We let $b_{(x,y,v)}$ be any nonzero element which is the image in M_i^λ of a path with starting vertex $2i$, ending vertex v, and x and y the number of times it traverses an ℓ or r edge, respectively (different paths of this form have images in M_i^λ differing by a scalar, and we choose one arbitrarily). This basis has the property that the image of any element under the linear map associated with an arrow of Q_n is a scalar multiple of another basis element.

Proof. The argument is essentially the same as that for why the quotient $\mathbb{C}Q/I$ inherited a basis from $\mathbb{C}Q$ when I was generated by relations of the form $p_i - p_i'$. Any two paths associated with the data (x,y,v) as described differ only in the order of a and b edges they traverse. The relations imposed by W asserted that two such paths give rise to the same element of P_i if they traverse an a edge the same number of times (hence a b edge the same number of times, since they correspond to the same (x,y,v)). For $\lambda_1, \lambda_2 \neq 0$, taking the quotient by P_{2i-1} imposes the relation that in M_i^λ two such paths differ by $(-\frac{\lambda_1}{\lambda_2})^k$ when one traverses an a edge k more times than the other. For $\lambda_1 = 0$ (resp. $\lambda_2 = 0$), there is a unique such path with nonzero image in M_i^λ, the one which only traverse a edges (resp. b edges).

The fact that the elements of $\mathcal{B}_i$ are compatible with the arrow maps of M_i^λ in the stated way is an immediate property of elements which are images of paths. □

The compatibility of $\mathcal{B}_i$ with the arrow maps of M_i^λ lets us completely and explicitly understand the submodule structure of M_i^λ. To this end we associate the following graph with the module M_i^λ.

Definition 6.3.3. Let G_i denote the directed graph with vertices the elements of $\mathcal{B}_i$, and an arrow from $b_{(x,y,v)}$ to $b_{(x',y',v')}$ if there exists an arrow e of Q_n such that $e \cdot b_{(x,y,v)}$ is a nonzero scalar multiple of $b_{(x',y',v')}$. We say a subgraph of G_i is admissible if it has the following property: if $b_{(x,y,v)}$ is a vertex of Γ and there is an edge from $b_{(x,y,v)}$ to $b_{(x',y',v')}$ in G_i, then $b_{(x',y',v')}$ is a vertex of Γ and this edge is an edge of Γ. □

Proposition 6.3.4. *Submodules of M_i^λ are in bijection with admissible subgraphs Γ of G_i. The submodule N_Γ corresponding to an admissible subgraph Γ is the subspace spanned by the basis elements at its vertices.*

Proof. It is immediate that the stated correspondence defines a bijection between admissible subgraphs and the set of submodules which are spanned as vector spaces by a subset of $\mathcal{B}_i$; what we must show is that every submodule of M_i^λ has this property. To do this we show that for any submodule N and any vertex v of Q_n, the subspace N_v is preserved by a nilpotent endomorphism E_v of $(M_i^\lambda)_v$ which forces it to be spanned by a subset of $\mathcal{B}_i$.

If $v = 2j$ for some $j \in \{1,\dots,n\}$, we let $E_v = \ell_{j+1}a_{j+1}r_j a_j$ if $\lambda_2 \neq 0$ and $E_v = \ell_{j+1}b_{j+1}r_j b_j$ otherwise. Here we identify arrows of Q_n with their corresponding endomorphisms of M_i^λ; the separate definition when $\lambda_2 = 0$ is needed since the a arrows act by zero in that case. Similarly, if $v = 2j-1$ we let $E_v = a_j\ell_{j+1}a_{j+1}r_j$ if $\lambda_2 \neq 0$ and $E_v = b_j\ell_{j+1}b_{j+1}r_j$ otherwise.

The action of the E_v on the basis $\mathcal{B}_i$ is especially simple, namely $E_v b_{(x,y,v)}$ is a nonzero scalar multiple of $b_{(x+1,y+1,v)}$ unless $x = i$ or $y = \nu_n(i)$, in which case $E_v b_{(x,y,v)} = 0$. In particular, up to normalization and ordering of the $b_{(x,y,v)}$, E_v is equivalent to the standard shift matrix. Of course, if N is a submodule of M_i^λ, then N_v must be invariant under the action of E_v, and it follows from the form of E_v that N_v is spanned by a subset of $\mathcal{B}_i$. □

It is useful to visualize the graph G_i as follows. Defining a map $\mathcal{B}_i \hookrightarrow \mathbb{Z}^2$ by $b_{(x,y,v)} \mapsto (y-x, -(y+x)+(v-(i+y-x)))$ and drawing $\mathbb{Z}^2$ as a grid in the plane in the usual way, we obtain a planar realization of G_i where all arrows are directed downward. The admissible subgraphs Γ are then just subgraphs that are "downward closed".

6.4 Hamiltonians and Nonintersecting Paths

In this section we discuss the quadratic A_n Toda Hamiltonians and their explicit expression in terms of cluster coordinates. In particular, we explain a combinatorial model that allows us to write these Hamiltonians as weighted sums of nonintersecting paths in a planar network.

Throughout this chapter we fix the double reduced word

$$\mathbf{i} = (1,-1,2,-2,\dots,r,-r)$$

for (c, c), with c the standard Coxeter element. This yields an indexing matching that used in Section 6.3 for the vertices of the quiver Q_n; that is, we have $(B_{\mathbf{i}})_{ij} = (Q_n)_{ij}$ where $B_{\mathbf{i}}$ is the exchange matrix of the seed associated with $\mathbf{i}$ and $(Q_n)_{ij}$ is the signed number of arrows from i to j in the quiver Q_n. Associated to $\mathbf{i}$ is a map $\mathcal{X}_{\mathbf{i}} \to PSL_{n+1}^{c,c}/H$ defined by

$$(y_1, \ldots, y_{2n}) \mapsto E_1 y_1^{\omega_1^\vee} F_1 y_2^{\omega_1^\vee} \cdots E_n y_{2n-1}^{\omega_n^\vee} F_{2n} y_{2n}^{\omega_{2n-1}^\vee}.$$

Again, we use somewhat different notation in this chapter than previous ones: we write y_i instead of X_i, so $\mathcal{X}_{\mathbf{i}} = \operatorname{Spec} \mathbb{C}[y_1^{\pm 1}, \ldots, y_{2n}^{\pm 1}]$. Note that the maps defined by the double reduced words $(1, -1, 2, -2, \ldots, r, -r)$ and $(1, \ldots, r, -1, \ldots, -r)$ are essentially the same, differing only in the indexing of their coordinates.

The Hamiltonian H_i is the pullback to $\mathcal{X}_{\mathbf{i}}$ of the character of the fundamental representation $\bigwedge^i \mathbb{C}^{n+1}$. Since $\mathcal{X}_{\mathbf{i}}$ maps to the adjoint form of the group, the H_i will necessarily involve fractional powers of the y_i. However, the natural choice of positive real part of $\mathcal{X}_{\mathbf{i}}$ determines a canonical choice of root $y_i^{\frac{1}{n+1}}$. More precisely, we define formal coordinates $y_i^{\frac{1}{n+1}}$ on a torus $\widehat{\mathcal{X}}_{\mathbf{i}} = \operatorname{Spec} \mathbb{C}[(y_1^{\frac{1}{n+1}})^{\pm 1}, \ldots, (y_{2n}^{\frac{1}{n+1}})^{\pm 1}]$. This has a covering map $\widehat{\mathcal{X}}_{\mathbf{i}} \twoheadrightarrow \mathcal{X}_{\mathbf{i}}$ defined implicitly via the map $\widehat{\mathcal{X}}_{\mathbf{i}} \to SL_{n+1}^{c,c}/H$ given by

$$(y_1^{\frac{1}{n+1}}, \ldots, y_{2n}^{\frac{1}{n+1}}) \mapsto E_1 (y_1^{\frac{1}{n+1}})^{(n+1)\omega_1^\vee} \cdots F_{2n} (y_{2n}^{\frac{1}{n+1}})^{(n+1)\omega_{2n-1}^\vee}.$$

We now explain a combinatorial description of the map $\mathcal{X}_{\mathbf{i}} \to PSL_{n+1}^{c,c}/H$ (or more precisely, of $\widehat{\mathcal{X}}_{\mathbf{i}} \twoheadrightarrow SL_{n+1}^{c,c}/H$) in terms of a directed network $\mathcal{N}_{\mathbf{i}}$. The network $\mathcal{N}_i$ is a directed graph embedded in a disk with $n+1$ "input" vertices and $n+1$"output" vertices on the boundary of the disk. The sets of inputs and outputs will each be labeled by $\{1, \ldots, n+1\}$. We draw the inputs along the right boundary of the disk with their indices increasing as one moves downward along the boundary, then draw the outputs along the left side so that inputs and outputs of the same index have the same vertical height. We draw a horizontal directed edge from each input to the output of the same index.

For each index i_k in $\mathbf{i}$ we draw an internal edge from the $|i_k|$th horizontal edge to the $(|i_k| + 1)$th horizontal edge if $i_k > 0$, and from the $(|i_k| + 1)$th horizontal edge to the $|i_k|$th horizontal edge if $i_k < 0$. We draw these so that the source of the jth internal edge is on the left of the target kth internal edge for $j < k$. We draw these internal edges with a slant so they are always directed to the left; with this convention we may omit drawing the directions on edges, since they are always directed to the left. Each internal edge thus corresponds to an index in $\{1, \ldots, 2n\}$, and we will label the region to the right of an internal edge by the corresponding variable y_i.

The network provides the following combinatorial description of the map $\widehat{\mathcal{X}}_{\mathbf{i}} \twoheadrightarrow SL_{n+1}^{c,c}/H$. More precisely, this map factors through $SL_{n+1}^{c,c}$, and we describe the image of $\widehat{\mathcal{X}}_{\mathbf{i}}$ in SL_{n+1} directly as a family of $(n+1) \times (n+1)$ matrices. The (i, j) entry of a matrix will be a weighted sum over all directed paths from the ith input to the jth output. The weight of the bottom horizontal path (the unique path from the $(n+1)$th input to the $(n+1)$th output) is

$$y_1^{\frac{-1}{n+1}} y_2^{\frac{-1}{n+1}} \cdots y_{2n-1}^{\frac{-n}{n+1}} y_{2n}^{\frac{-n}{n+1}}.$$

The weights of other paths are then determined by the following rule. Two paths p,p' can be viewed as elements of $H_1(\mathcal{N}_{\mathbf{i}}, \partial\mathcal{N}_{\mathbf{i}})$, and if the difference $p-p'$ is a cycle oriented counterclockwise around the region labeled y_i, then the weight of p is y_i times the weight of p'. In other words, if p differs from p' only in that it goes above the region y_i rather than below it, than its weight is y_i times that of p'.

That this network prescription is indeed consistent with the definition of the map $\widehat{\mathcal{X}}_{\mathbf{i}} \twoheadrightarrow SL_{n+1}^{c,c}/H$ follows from two observations. First, the internal edges describe the actions of the E_i and F_i in the standard basis of $\mathbb{C}^{n+1}$. Second, each coweight subgroup can be written as

$$y^{\omega_k^\vee} = y^{\frac{-k}{n+1}} \begin{pmatrix} y & 0 & & \cdots & & 0 \\ 0 & \ddots & & & & \\ & & y & & & \\ \vdots & & & 1 & & \vdots \\ & & & & \ddots & 0 \\ 0 & & & \cdots & 0 & 1 \end{pmatrix},$$

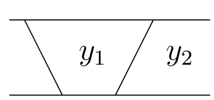
where the diagonal matrix on the right hand side has its first k entries equal to y and its last $n+1-k$ equal to 1.

We also have a combinatorial description of how elements of SL_{n+1} in the image of $\widehat{\mathcal{X}}_{\mathbf{i}}$ act on the other fundamental represention $\bigwedge^i \mathbb{C}^{n+1}$. The standard basis of $\bigwedge^i \mathbb{C}^{n+1}$ is indexed by i-element subsets of $\{1,\ldots,n+1\}$, and the family $\widehat{\mathcal{X}}_{\mathbf{i}} \subset SL_{n+1}^{c,c}$ acts on V_{ω_i} by matrices whose entries are weighted sums of i-tuples of directed nonintersecting paths. The weight of an i-tuple of paths is the product of the weights of each path.

We will say a directed path in $\mathcal{N}_{\mathbf{i}}$ is cyclic if its input and output have the same index. Such paths give rise to cycles on $\overline{\mathcal{N}}_{\mathbf{i}}$, where $\overline{\mathcal{N}}_{\mathbf{i}}$ is the closed graph obtained by gluing the ith input to the ith output. The following observation is immediate:

Proposition 6.4.1. *The Hamiltonian H_i, defined as the pullback to $\widehat{\mathcal{X}}_{\mathbf{i}}$ of the character of $\bigwedge^i \mathbb{C}^{n+1}$, is the weighted sum of all i-tuples of nonintersecting cyclic paths in $\mathcal{N}_{\mathbf{i}}$.*

Example 6.4.2. Let us illustrate the above discussion for SL_2. On the left below we have the relevant network and on the right the family of matrices it parametrizes. As all the edges are directed leftward, we omit specifically notating the directions of the edges of the network. It is convenient to pull out an overall scalar factor equal to the weight of the lowest horizontal path, since with this normalization the weights of all paths become polynomials in the y_i.

$y_1^{-\frac{1}{2}} y_2^{-\frac{1}{2}} \Bigg($  $\Bigg) = y_1^{-\frac{1}{2}} y_2^{-\frac{1}{2}} \begin{pmatrix} y_2 + y_1 y_2 & 1 \\ y_2 & 1 \end{pmatrix}.$

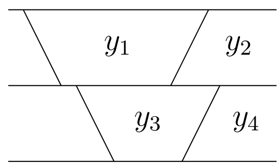
Computing the Hamiltonian H_1 requires taking the trace of the matrix on the right, which is the weighted sum of the three distinct cyclic paths on the left. We find that

$$\begin{aligned} H_1 &= y_1^{-\frac{1}{2}} y_2^{-\frac{1}{2}} (1 + y_2 + y_1 y_2) \\ &= x_1 x_2^{-1} (1 + y_2 + y_1 y_2), \end{aligned}$$

where $y_1 = x_2^2$, $y_2 = x_1^{-2}$. □

Example 6.4.3. Below are the network and corresponding family of matrices for SL_3:

$$y_1^{-\frac{1}{3}} y_2^{-\frac{1}{3}} y_3^{-\frac{2}{3}} y_4^{-\frac{2}{3}} \left(\quad \right)$$



$$= y_1^{-\frac{1}{3}} y_2^{-\frac{1}{3}} y_3^{-\frac{2}{3}} y_4^{-\frac{2}{3}} \begin{pmatrix} y_2 y_3 y_4 + y_1 y_2 y_3 y_4 & y_4 + y_3 y_4 & 1 \\ y_2 y_3 y_4 & y_4 + y_3 y_4 & 1 \\ 0 & y_4 & 1 \end{pmatrix}.$$

There are two Hamiltonians H_1 and H_2 corresponding to the fundamental and anti-fundamental representations, respectively. The former is a weighted sum of the five cyclic paths, while the latter is a weighted sum of the five nonintersecting pairs of cyclic paths:

$$\begin{aligned} H_1 &= y_1^{-\frac{1}{3}} y_2^{-\frac{1}{3}} y_3^{-\frac{2}{3}} y_4^{-\frac{2}{3}} (1 + y_4 + y_3 y_4 + y_2 y_3 y_4 + y_1 y_2 y_3 y_4) \\ &= x_3 x_4^{-1} (1 + y_4 + y_3 y_4 + y_2 y_3 y_4 + y_1 y_2 y_3 y_4) \\ H_2 &= y_1^{-\frac{2}{3}} y_2^{-\frac{2}{3}} y_3^{-\frac{1}{3}} y_4^{-\frac{1}{3}} (1 + y_2 + y_1 y_2 + y_1 y_2 y_4 + y_1 y_2 y_3 y_4) \\ &= x_1 x_2^{-1} (1 + y_2 + y_1 y_2 + y_1 y_2 y_4 + y_1 y_2 y_3 y_4). \end{aligned}$$

Here we have $y_1 = x_2^2 x_4^{-1}$, $y_2 = x_1^{-2} x_3$, $y_3 = x_2^{-1} x_4^2$, and $y_4 = x_1 x_3^{-2}$. □

6.5 Hamiltonians and Cluster Characters

In this section we prove our main result, realizing the Hamiltonians of the quadratic A_n Toda system as cluster characters of the quiver Q_n. Recall that by $\nu_n : \{1, \ldots, n\} \to \{1, \ldots, n\}$ we denote the Nakayama involution $\nu_n(i) = n + 1 - i$.

Theorem 6.5.1. *For each $i \in \{1, \ldots, n\}$ we have $H_i = CC(M^\lambda_{\nu_n(i)})$.*

Proof. There are two components to the proof. First, we prove that the index of $M^\lambda_{\nu_n(i)}$ agrees with the corresponding quantity appearing in H_i. Second, we construct a bijection between nonintersecting i-tuples of cyclic paths in $\mathcal{N}_{\mathbf{i}}$ and admissible subgraphs of $G_{\nu_n(i)}$, and show that this takes weights of paths to dimension vectors of quotient modules.

Let $x^{\mathrm{ind}_{H_i}}$ be the Laurent monomial in $x_1, \ldots, x_{2n}$ defined by the property that $H_i = x^{\mathrm{ind}_{H_i}} p(y_1, \ldots, y_{2n})$, where $p(y_1, \ldots, y_{2n}) \in \mathbb{C}[y_1, \ldots, y_{2n}]$ has constant term 1; we must show that $x^{\mathrm{ind}_{H_i}} = x^{\mathrm{ind}_{M^\lambda_{\nu_n(i)}}}$. From the network representation of H_i it is clear that $x^{\mathrm{ind}_{H_i}}$ is the weight of the lowest i-tuple of cyclic paths. Equivalently, it is the contribution to the trace of the action of

$$y_1^{\omega_1^\vee} y_2^{\omega_1^\vee} \cdots y_{2n-1}^{\omega_n^\vee} y_{2n}^{\omega_n^\vee}$$

on the lowest weight space of $\bigwedge^i \mathbb{C}^{n+1}$ (which has weight $-\omega_{\nu_n(i)}$).

Since $y_{2i-1} = \prod_j x_{2j}^{C_{ij}}$ and $y_{2i} = \prod_j x_{2j-1}^{-C_{ij}}$, where C is the A_n Cartan matrix, we have

$$\begin{aligned} y_1^{\omega_1^\vee} y_2^{\omega_1^\vee} \cdots y_{2n-1}^{\omega_n^\vee} y_{2n}^{\omega_n^\vee} &= \prod_i (y_{2i-1}y_{2i})^{\omega_i^\vee} \\ &= \prod_{i,j} (x_{2j-1}^{-1}x_{2j})^{C_{ij}\omega_i^\vee} \\ &= \prod_j (x_{2j-1}^{-1}x_{2j})^{\alpha_j^\vee}. \end{aligned}$$

But on the lowest weight space this acts by the scalar

$$\prod_j (x_{2j-1}^{-1}x_{2j})^{\langle \alpha_j^\vee | -\omega_{\nu_n(i)}\rangle} = x_{2\nu_n(i)-1} x_{2\nu_n(i)}^{-1},$$

which is equal to $x^{\mathrm{ind}_{M^\lambda_{\nu_n(i)}}}$ since $M^\lambda_{\nu_n(i)}$ is defined by a projective resolution of the form

$$0 \to P_{2\nu_n(i)-1} \to P_{2\nu_n(i)} \to M^\lambda_{\nu_n(i)} \to 0.$$

Now we turn to the bijection between i-tuples of nonintersecting paths in $\mathcal{N}_{\mathbf{i}}$ and admissible subgraphs of $G_{\nu_n(i)}$. Recall that the vertices of $G_{\nu_n(i)}$ are the elements $\{b_{(x,y,v)}\}$ of a basis of $M^\lambda_{\nu_n(i)}$ indexed by tuples

$$\{(x,y,v) | x, y \in \mathbb{N}, v \in \{2(\nu_n(i)+y-x), 2(\nu_n(i)+y-x)-1\}, x < \nu_n(i), y < i\}.$$

For each fixed value $y \in \{0, \ldots, i-1\}$, $G_{\nu_n(i)}$ has $2\nu_n(i)$ vertices of the form $b_{(x,y,v)}$, for which the possible values of v are $\{2y+1, 2y+2, \ldots, 2y+2\nu_n(i)-1, 2y+2\nu_n(i)\}$ (note that the value of x is determined by those of y and v). Recall that $b_{(x,y,v)}$ is the image of an element of $\mathbb{C}Q_n$ corresponding to a path that has ending vertex v, and x and y the number of times it traverses an ℓ or r edge, respectively. For fixed y, it follows that there is an arrow from $b_{(x,y,v)}$ to $b_{(x',y,v')}$ in $G_{\nu_n(i)}$ if and only if $v' = v-1$ (since such an arrow corresponds to either a vertical or leftward arrow of Q_n). In particular, given an admissible graph Γ, for each $y \in \{0, \ldots, i-1\}$ there is at most one "maximal" value of v for which $b_{(x,y,v)}$ is a vertex of Γ; that is, such that $b_{(x,y,v)}$ is a vertex of Γ but $b_{(x',y,v+1)}$ is not (including the case $v = 2(y+\nu_n(i))$ when there is no such vertex of $G_{\nu_n(i)}$). Let us call maximal value $v(\Gamma, y)$; if there are no vertices of Γ with the given value of y we set $v(\Gamma, y) = 2y$, so by a slight abuse of notation we may have $v(\Gamma, 0) = 0$ even though 0 does not label an actual vertex of Q_n.

The set $\{v(\Gamma, y)\}_{y=0}^{i-1}$ completely determines the graph Γ, though an arbitrary collection of vertices of Q_n need not correspond to an actual admissible graph.

We use the data $\{v(\Gamma, y)\}_{y=0}^{i-1}$ to assign an i-tuple of nonintersecting cyclic paths in $\mathcal{N}_{\mathbf{i}}$ to the graph Γ. First note the following bijection between $\{0, 1, \ldots, 2n\}$ and the set of cyclic paths in $\mathcal{N}_{\mathbf{i}}$. We associate a cyclic path with the largest value of v such that the face labeled by y_v lies above it. To the top cyclic path, which lies above all such faces, we associate the index 0. To an admissible graph Γ we now assign the i-tuple of cyclic paths associated with the set $\{v(\Gamma, y)\}_{y=0}^{i-1}$. We must show that these paths do not intersect, and that any nonintersecting i-tuple of cyclic paths arises this way.

We have already described all arrows between vertices $b_{(x,y,v)}$, $b_{(x',y',v')}$ of $G_{\nu_n(i)}$ for which $y = y'$. From the path description of this basis, it also follows that if $G_{\nu_n(i)}$ contains an arrow from $b_{(x,y,v)}$ to $b_{(x',y',v')}$ and $y' \neq y$, we must have $v = 2(y-x)-1$ and $b_{(x',y',v')} = b_{(x,y+1,v+3)}$. From this we arrive at a necessary and sufficient condition for a set of vertices to be of the form $\{v(\Gamma, y)\}_{y=0}^{i-1}$ for an admissible graph Γ: for each $y < i-1$ we should have $v(\Gamma, y+1) \geq v(\Gamma, y)+3$ if $v(\Gamma, y)$ is odd and $v(\Gamma, y+1) \geq v(\Gamma, y)+2$ if $v(\Gamma, y)$ is even. Under our bijection between cyclic paths in $\mathcal{N}_{\mathbf{i}}$ and elements of $\{0, 1, \ldots, 2n\}$, it follows easily that this corresponds exactly to the condition that an i-tuple of cyclic paths be nonintersecting.

All that remains to be shown is that if N_Γ is the submodule associated with an admissible graph Γ, the dimension vector of $M^\lambda_{\nu_n(i)}/N_\Gamma$ agrees with the weight of the i-tuple of paths associated with $\{v(\Gamma, y)\}_{y=0}^{i-1}$. More precisely, we must verify that the following two monomials coincide. First is $y^{[M^\lambda_{\nu_n(i)}/N_\Gamma]}$, where we write $y^{[L]} = \prod_i y_i^{a_i}$ for a class $[L] = \sum_i a_i[S_i] \in K_0(J\text{-mod})$. Second is the ratio of the weight of the i-tuple associated to Γ and that of the lowest i-tuple, that is the i-tuple associated with $\{2\nu_n(i), \ldots, 2n-2, 2n\}$. This normalization arises because while H_i is a weighted sum of i-tuples of cyclic paths, to compare H_i with $CC(M^\lambda_{nu_n(i)})$ we must pull out a factor of $x^{\mathrm{ind}_{H_i}}$, which is the weight of the lowest i-tuple. Explicitly, for each $1 \leq j \leq i$ let $m_j(y_1, \ldots, y_{2n})$ be the product of all y_v whose associated face lies between the jth path from the top of our given i-tuple and the jth path from the top of the lowest i-tuple; the ratio of the weights of the two i-tuples is the product of the m_j.

Now it is clear that

$$y^{[M^\lambda_{\nu_n(i)}/N_\Gamma]} = \prod_{b_{(x,y,v)} \in G_{\nu_n(i)} \setminus \Gamma} y_v,$$

the product being taken over all vertices of $G_{\nu_n(i)}$ which are not vertices of Γ. But it is easy to check that

$$\prod_{b_{(x,j-1,v)} \in G_{\nu_n(i)} \setminus \Gamma} y_v = m_j(y_1, \ldots, y_{2n}),$$

concluding the proof. $\square$

Example 6.5.2. Below we have the graph G_3 associated to the 18-dimensional representation M_3^λ of the quiver Q_5. There are 61 submodules corresponding to 61 admissible subgraphs. For example, the zero submodule contributes a term $y_1y_2y_3^2y_4^2y_5^3y_6^3y_7^2y_8^2y_9y_{10}$ to $CC(M_3^\lambda)$. There are three "chains" along which y is constant between the bottom-left and top-right of the graph. Given an admissible subgraph Γ, the highest vertex it contains in each of the three chains indicates the position of one of a triple of nonintersecting cyclic paths in the network $\mathcal{N}_{\mathbf{i}}$.

$$\begin{tikzcd}
 & & (0,0,6) \arrow[d] & & \\
 & & (0,0,5) \arrow[dl] \arrow[dr] & & \\
 & (1,0,4) \arrow[d] & & (0,1,8) \arrow[d] & \\
 & (1,0,3) \arrow[dl] \arrow[dr] & & (0,1,7) \arrow[dl] \arrow[dr] & \\
(2,0,2) \arrow[d] & & (1,1,6) \arrow[d] & & (0,2,10) \arrow[d] \\
(2,0,1) \arrow[dr] & & (1,1,5) \arrow[dl] \arrow[dr] & & (0,2,9) \arrow[dl] \\
 & (2,1,4) \arrow[d] & & (1,2,8) \arrow[d] & \\
 & (2,1,3) \arrow[dr] & & (1,2,7) \arrow[dl] & \\
 & & (2,2,6) \arrow[d] & & \\
 & & (2,2,5) & &
\end{tikzcd}$$

$\square$

6.6 Irregular Flat Connections and $\mathcal{N} = 2$ Field Theory

In this section we discuss the results of this chapter from the point of view of nonabelian Hodge theory. We interpret the double Bruhat cell $SL_{n+1}^{c,c}/H$ as a moduli space of flat connections with irregular singularities, and the network $\overline{\mathcal{N}}_{\mathbf{i}}$ used to compute the Hamiltonians as the 1-skeleton of the spectral curve of the associated Hitchin system. The Hamiltonians themselves become traces of holonomies around closed loops, providing a geometric reason for their interpretation as canonical basis elements (hence their realization as cluster characters). We also explain how this viewpoint is intimately tied to that of 4d $\mathcal{N} = 2$ field theory, wherein this particular irregular Hitchin system plays a fundamental role, the Seiberg-Witten system of $\mathcal{N} = 2$ Yang-Mills theory.

Recall that the nonabelian Hodge correspondence identifies the moduli space $\mathcal{M}_{GL_n}(\mathcal{C})$ of flat rank-n vector bundles on a Riemann surface $\mathcal{C}$ with a corresponding moduli space $\mathcal{M}_{\mathrm{Higgs}}(\mathcal{C})$ of Higgs bundles, certain $\mathfrak{gl}_n$-valued 1-forms on $\mathcal{C}$. The latter is the phase space of the Hitchin system, a Lagrangian fibration $\mathcal{M}_{\mathrm{Higgs}}(\mathcal{C}) \twoheadrightarrow \mathcal{B}$ where $\mathcal{B}$ is a space of polydifferentials on $\mathcal{C}$. The fiber over a point $u \in \mathcal{B}$ is the Jacobian of a spectral curve Σ_u, which is naturally embedded in $T^*\mathcal{C}$ as a branched cover of $\mathcal{C}$. Both $\mathcal{M}_{GL_n}(\mathcal{C})$ and $\mathcal{M}_{\mathrm{Higgs}}(\mathcal{C})$

have holomorphic Poisson structures but are not complex-analytically equivalent; rather, they can be regarded as two different complex structures on a single hyperkahler space.

One of the insights of [GMN13] is that for a generic $u \in \mathcal{B}$ there is a class of open holomorphic Poisson embeddings $\mathcal{M}_{GL_1}(\Sigma_u) \hookrightarrow \mathcal{M}_{GL_n}(\mathcal{C})$ depending on a phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Varying $u \in \mathcal{B}$ and θ one (conjecturally, in general) obtains the toric charts comprising a cluster atlas on $\mathcal{M}_{GL_n}(\mathcal{C})$. This recovers and extends many constructions of [FG06b] from a complementary point of view. In particular, although the cluster structure lives naturally on the space $\mathcal{M}_{GL_n}(\mathcal{C})$ of flat connections, the cluster charts themselves originate on the other side of the nonabelian Hodge correspondence, being most naturally defined in terms of the spectral curves Σ_u.

An embedding $\mathcal{M}_{GL_1}(\Sigma_u) \hookrightarrow \mathcal{M}_{GL_n}(\mathcal{C})$ is more or less equivalent to a rule for expressing the GL_n-holonomy around a closed loop in $\mathcal{C}$ in terms of GL_1-holonomies around closed loops in Σ_u. This rule may be described in terms of a combinatorial object called a spectral network. This consists of a special a family of paths, or walls, drawn on $\mathcal{C}$ and labeled locally by ordered pairs ij of sheets of the spectral curve. To a closed loop γ in $\mathcal{C}$ is associated family of loops in Σ_u, determined by the pattern of how γ crosses the walls of the network, and the matrix entries of the GL_n-holonomy around γ are sums of GL_1-holonomies around these loops in Σ_u.

The essential detail for us is that the holonomy around γ is produced along with an explicit factorization as a product of diagonal matrices and elementary matrices E_{ij} for each ij-wall crossed by γ, multiplied in the order in which they are crossed. In this way the formal features of the map $\mathcal{M}_{GL_1}(\Sigma_u) \hookrightarrow \mathcal{M}_{GL_n}(\mathcal{C})$ coincide with those of the network description of the cluster coordinates on $SL_{n+1}^{c,c}/H$, where elements of $SL_{n+1}^{c,c}/H$ were described via a factorization into diagonal and elementary matrices. Moreover, the matrix entries of both an element of $SL_{n+1}^{c,c}/H$ and a holonomy around a loop in $\mathcal{C}$ are expressed as weighted sums of 1-cycles, either of the closed network $\overline{\mathcal{N}}_{\mathbf{i}}$ or the spectral curve Σ_u, respectively.

In fact, the cluster structure we have studied on $SL_{n+1}^{c,c}/H$ can be seen as a particular instance of one arising from a moduli space of flat connections, once irregular singularities are allowed. These moduli spaces (and their cluster structures) play an important role in 4d $\mathcal{N} = 2$ quantum field theory, the following aspects of which are relevant to our discussion. Associated to an $\mathcal{N} = 2$ theory is an algebraic integrable system, its Seiberg-Witten system, which we write as a Lagrangian fibration $\mathcal{M} \twoheadrightarrow \mathcal{B}$ and whose spectral curves are also called the Seiberg-Witten curves of the theory. Physically, $\mathcal{B}$ is a space of vacua, the Coulomb branch of the theory. To theories satisfying certain finiteness conditions there is associated a quiver, its BPS quiver Q. More precisely, one has a quiver for each generic $u \in \mathcal{B}$ and phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, but all are mutation equivalent. The vertices of Q are in bijection with a distinguished homology basis of the Seiberg-Witten curve Σ_u, its edges encoding their intersection numbers.

When the Seiberg-Witten system is a Hitchin system with singularities the framework of [GMN13] described above produces a cluster chart on $\mathcal{M}$ with coordinates labeled by vertices of Q. Many fundamental $\mathcal{N} = 2$ theories are of this type, but generally require the consideration of irregular singularities. Since the BPS quiver Q can often be determined by

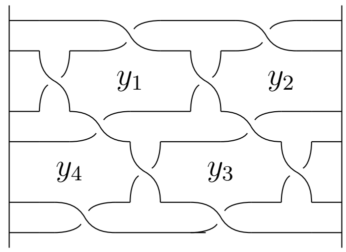


Figure 6.1: A Seiberg-Witten curve of $\mathcal{N} = 2$ SU(3) Yang-Mills, projected onto an unwrapped cylinder. The Coulomb branch is the space $\{u = u_1(\frac{dz}{z})^2 + (\frac{1}{z} + u_0 + z)(\frac{dz}{z})^3\}$ of cubic differentials on $\mathbb{CP}^1$, parametrized by $(u_0, u_1) \in \mathbb{C}^2$ [GMN13]. The spectral curve $\Sigma_u \subset T^*\mathbb{CP}^1$ is the solution set of $\lambda^3 + \lambda u_1(\frac{dz}{z})^2 + (\frac{1}{z} + u_0 + z)(\frac{dz}{z})^3 = 0$, where λ is a coordinate on the cotangent fibers of $\mathbb{CP}^1$. These are genus two curves realized as three-sheeted branched covers of $\mathbb{CP}^1$, with two punctures over 0 and ∞ (Σ_u has cyclic monodromy around these points). In the picture, the punctures are blown up to boundary components. The homology cycles labeled by the y_i have intersection numbers given by the BPS quiver Q_3. The planar realization identifies 1-skeleton of Σ_u with the corresponding closed network $\overline{\mathcal{N}}_{\mathbf{i}}$ of Section 6.4 (it only defined up to the action of the Torelli group).

physical considerations unrelated to the associated Seiberg-Witten geometry, this essentially leads to specific predictions about cluster structures on irregular moduli spaces more general than those considered in the mathematics literature.

The quiver Q_n relevant to $SL_{n+1}^{c,c}/H$ in fact arises as a basic example of a BPS quiver, that of pure $\mathcal{N} = 2$ $SU(n+1)$ Yang-Mills theory. We can use this to identify the cluster charts on $SL_{n+1}^{c,c}/H$ with those on the relevant moduli space, which is a space of flat connections on $\mathbb{CP}^1$ with irregular singularities at two points. Such a flat connection is essentially just the data of the holonomy around the unique nontrivial closed cycle in $\mathcal{C}$ (neglecting Stokes data, which in principle is encoded in the fact that the holonomy produced is well-defined up to conjugation by H rather than merely by G). The network $\overline{\mathcal{N}}_{\mathbf{i}}$ is thus identified with the 1-skeleton of a spectral curve of the associated Hitchin system. The Hamiltonians H_i are then the traces of the unique nontrivial holonomy in the fundamental representations, producing a geometric reason for their interpretation as canonical basis elements.

Bibliography

[Ali+11] M. Alim et al. “N = 2 Quantum Field Theories and Their BPS Quivers”. In: *Preprint (arXiv:1112.3984v1)* (2011), pp. 1–93. arXiv:arXiv:1112.3984v1.

[BFZ05] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky. “Cluster algebras III: Upper bounds and double Bruhat cells”. In: *Duke Mathematical Journal* 126.1 (2005), pp. 1–52. arXiv:0305434v3 [arXiv:math].

[BFZ96] A. Berenstein, S. Fomin, and A. Zelevinsky. “Parametrizations of Canonical Bases and Totally Positive Matrices”. In: *Adv. Math.* 122.1 (1996), pp. 49–149. URL: http://math.uoregon.edu/$\sim$arkadiy/bfz.pdf.

[BZ97] A. Berenstein and A. Zelevinsky. “Total Positivity in Schubert Varieties”. In: *Comment. Math. Helv.* 72.1 (May 1997), pp. 128–166. ISSN: 0010-2571. DOI: 10.1007/PL00000363. URL: http://www.ems-ph.org/doi/10.1007/PL00000363.

[CC06] P. Caldero and F. Chapoton. “Cluster algebras as Hall algebras of quiver representations”. In: *Comment. Math. Helv.* 81 (2006), pp. 595–616. arXiv:0410187v2 [arXiv:math].

[CD12] S. Cecotti and M. Del Zotto. “4d N = 2 gauge theories and quivers: the non-simply laced case”. In: *J. High Energy Phys.* 10 (2012), pp. 190–224. arXiv:arXiv:1207.7205v1.

[CK06] P. Caldero and B. Keller. “From triangulated categories to cluster algebras. II”. In: *Ann. Sci. École Norm. Sup. (4)* 39.6 (2006), pp. 983–1009. arXiv:0510251v2 [arXiv:math].

[CNV10] S. Cecotti, A. Neitzke, and C. Vafa. “R-twisting and 4d/2d correspondences”. In: *Preprint (arXiv:1006.3435v2)* (2010), pp. 1–160. arXiv:arXiv:1006.3435v2. URL: http://arxiv.org/abs/1006.3435.

[CP94] V. Chari and A. Pressley. *A Guide to Quantum Groups*. Cambridge: Cambridge University Press, 1994. URL: http://books.google.com/books?hl=en&lr=&id=bn5GNkLfnsAC&oi=fnd&pg=PA1&dq=Guide+to+Quantum+Groups&ots=VFBwjveqoe&sig=VPp7qSEBN-urbVm2kXVR0Po6UYc.

[DK09] P. Di Francesco and R. Kedem. "Q-systems as cluster algebras II: Cartan matrix of finite type and the polynomial property". In: *Lett. Math. Phys.* 89.3 (2009), pp. 183–216. arXiv:arXiv:0803.0362v2.

[DK10] P. Di Francesco and R. Kedem. "Q-systems, heaps, paths and cluster positivity". In: *Communications in Mathematical Physics* (2010). arXiv:arXiv:0811.3027v2. URL: http://link.springer.com/article/10.1007/s00220-009-0947-5.

[DLT89] P. Deift, L.C. Li, and C. Tomei. "Matrix factorizations and integrable systems". In: *Comm.Pure Appl. Math* 42.4 (1989), pp. 443–521. URL: http://onlinelibrary.wiley.com/doi/10.1002/cpa.3160420405/abstract.

[Dri88] V. Drinfel'd. "Quantum groups". In: *J. Soviet Math.* 41.2 (Apr. 1988), pp. 898–915. ISSN: 0090-4104. DOI: 10.1007/BF01247086. URL: http://www.springerlink.com/content/r56104684r703807/.

[DWZ08] H. Derksen, J. Weyman, and A. Zelevinsky. "Quivers with potentials and their representations. I. Mutations". In: *Selecta Math. (N.S.)* 14.1 (2008), pp. 59–119. arXiv:arXiv:0704.0649v4.

[FG06a] V.V. Fock and A.B. Goncharov. "Cluster X-varieties, amalgamation, and Poisson-Lie groups". In: *Algebraic geometry and number theory*. Vol. 253. Progr. Math. Boston, MA: Birkhauser, 2006, pp. 27–68. arXiv:0508408v2 [arXiv:math]. URL: http://www.springerlink.com/index/k40570kw3j577708.pdf.

[FG06b] V.V. Fock and A.B. Goncharov. "Moduli spaces of local systems and higher Teichmüller theory". In: *Publ. Math. Inst. Hautes Etudes Sci.* 103 (2006), pp. 1–211. arXiv:0311149v4 [arXiv:math]. URL: http://link.springer.com/article/10.1007/s10240-006-0039-4.

[FG07] V.V. Fock and A.B. Goncharov. "Dual Teichmuller and Lamination Spaces". In: *Handbook of Teichmuller Theory*. Vol. I. IRMA Lect. Math. Theor. Phys. 1. Zurich: Eur. Math. Soc., 2007, pp. 647–684. arXiv:0510312v1 [arXiv:math]. URL: http://arxiv.org/abs/math/0510312.

[FG09] V.V. Fock and A.B. Goncharov. "Cluster ensembles, quantization and the dilogarithm". In: *Ann. Sci. Ec. Norm. Super.* 42.6 (2009), pp. 865–930. arXiv:0311245v7 [arXiv:math]. URL: http://arxiv.org/abs/math/0311245.

[FGT08] S. Fishel, I. Grojnowski, and C. Teleman. "The Strong Macdonald Conjecture and Hodge Theory on the Loop Grassmannian". In: *Ann. of Math.* 168.1 (2008), pp. 175–220. URL: http://arxiv.org/abs/math/0411355.

[FM13] V.V. Fock and A. Marshakov. "Loop groups, clusters, dimers and integrable systems". In: *Preprint (arXiv:1401.1606)* (2013), pp. 1–58.

[Fom10] S. Fomin. "Total positivity and cluster algebras". In: *Proceedings of the International Congress of Mathematicians. Volume II*. New Delhi: Hindustan Book Agency, 2010, pp. 125–145. arXiv:arXiv:1005.1086v2. URL: http://arxiv.org/abs/1005.1086.

[FZ00] S. Fomin and A. Zelevinsky. "Recognizing Schubert Cells". In: *J. Algebraic Combin.* 12.1 (2000), pp. 37–57. arXiv:9807079v2 [arXiv:math]. URL: http://www.springerlink.com/index/P76688345582VL06.pdf.

[FZ02] S. Fomin and A. Zelevinsky. "Cluster algebras I: foundations". In: *J. Amer. Math. Soc.* 15.2 (2002), pp. 497–529. arXiv:0104151 [arXiv:math]. URL: http://arxiv.org/abs/math/0104151.

[FZ07] S. Fomin and A. Zelevinsky. "Cluster algebras IV: coefficients". In: *Compos. Math.* 143.01 (2007), pp. 112–164. arXiv:0602259 [arXiv:math]. URL: http://arxiv.org/abs/math/0602259.

[FZ99] Sergey Fomin and Andrei Zelevinsky. "Double Bruhat cells and total positivity". In: *Journal of the American Mathematical Society* 12.2 (1999), pp. 335–380. arXiv:9802056v1 [arXiv:math]. URL: http://www.ams.org/journals/jams/1999-12-02/S0894-0347-99-00295-7/S0894-0347-99-00295-7.pdf.

[GHK13] M. Gross, P. Hacking, and S. Keel. "Birational geometry of cluster algebras". In: *Preprint (arXiv:1309.2573)* (2013), pp. 1–50. arXiv:arXiv:1309.2573v1. URL: http://arxiv.org/abs/1309.2573.

[GK11] A. B. Goncharov and R. Kenyon. "Dimers and Cluster Integrable Systems". In: *Preprint arXiv:1107.5588* (2011). arXiv:arXiv:1107.5588v1. URL: http://arxiv.org/abs/1107.5588.

[GLS11] C. Geiss, B. Leclerc, and J. Schröer. "Kac-Moody Groups and Cluster Algebras". In: *Adv. Math.* 228.1 (2011), pp. 329–433. arXiv:arXiv:1001.3545v3. URL: http://arxiv.org/abs/1001.3545.

[GLS12] C. Geiss, B. Leclerc, and J. Schröer. "Generic bases for cluster algebras and the Chamber Ansatz". In: *J. Amer. Math. Soc.* 25.1 (2012), pp. 21–76. arXiv:arXiv:1004.2781v3. URL: http://www.ams.org/jams/2012-25-01/S0894-0347-2011-00715-7/S0894-0347-2011-00715-7.pdf.

[GMN13] D. Gaiotto, G. Moore, and A. Neitzke. "Spectral networks". In: *Ann. Henri Poincaré* 14.7 (2013), pp. 1643–1731. arXiv:arXiv:1204.4824v1. URL: http://link.springer.com/article/10.1007/s00023-013-0239-7.

[GSV03] M. Gekhtman, M. Shapiro, and A. Vainshtein. "Cluster Algebras and Poisson Geometry". In: *Mosc. Math. J.* 3.3 (2003), pp. 899–934. arXiv:0208033v2 [arXiv:math]. URL: http://onlinelibrary.wiley.com/doi/10.1002/cbdv.200490137/abstracthttp://arxiv.org/abs/math/0208033.

[GSV11] M. Gekhtman, M. Shapiro, and A. Vainshtein. "Generalized Bäcklund-Darboux transformations for Coxeter-Toda flows from a cluster algebra perspective". In: *Acta Math.* 206.2 (2011), pp. 245–310. arXiv:arXiv:0906.1364v4. URL: http://www.springerlink.com/index/E701800L35151798.pdfhttp://link.springer.com/article/10.1007/s11511-011-0063-1.

[Hat+02] G. Hatayama et al. "Paths, crystals and fermionic formulae". In: *MathPhys Odyssey*. Vol. 23. Prog. Math. Phys. Boston, MA: Birkhauser Boston, 2002, pp. 205–272. arXiv:0102113v1 [arXiv:math]. URL: http://link.springer.com/chapter/10.1007/978-1-4612-0087-1_9.

[Her06] D. Hernandez. "The Kirillov-Reshetikhin conjecture and solutions of T-systems". In: *J. Reine Angew. Math.* 596 (2006), pp. 63–87. arXiv:0501202v3 [arXiv:math]. URL: http://www.degruyter.com/view/j/crll.2006.2006.issue-596/crelle.2006.052/crelle.2006.052.xml.

[Her07] D. Hernandez. "Drinfeld coproduct, quantum fusion tensor category and applications". In: *Proc. Lond. Math. Soc. (3)* 95.3 (2007), pp. 567–608. arXiv:0504269v3 [arXiv:math]. URL: http://plms.oxfordjournals.org/content/95/3/567.short.

[Her10] D. Hernandez. "Kirillov-Reshetikhin conjecture: the general case". In: *Int. Math. Res. Not.* 1 (2010), pp. 149–193. arXiv:arXiv:0704.2838v3. URL: http://imrn.oxfordjournals.org/content/2010/1/149.short.

[Hof+00] Tim Hoffmann et al. "Factorization Dynamics and CoxeterToda Lattices". In: *Communications in Mathematical Physics* 212.2 (2000), pp. 297–321. arXiv:9906013v2 [arXiv:solv-int]. URL: http://www.springerlink.com/index/3NR2V0VP934LWFAE.pdf.

[Kac94] V. Kac. *Infinite-Dimensional Lie Algebras*. 3rd. Cambridge: Cambridge University Press, 1994, p. 400. ISBN: 0521466938. URL: http://books.google.com/books?id=kuEjSb9teJwC&pgis=1.

[Ked08] R. Kedem. "Q -systems as cluster algebras". In: *J. Phys. A* 41.19 (2008), 14 pp. arXiv:arXiv:0712.2695v3.

[KNS11] A. Kuniba, T. Nakanishi, and J. Suzuki. "T-systems and Y-systems in integrable systems". In: *J. Phys. A* 44.10 (2011), 146 pp. arXiv:arXiv:1010.1344v3. URL: http://iopscience.iop.org/1751-8121/44/10/103001.

[KP83a] V. Kac and D. Peterson. "Infinite Flag Varieties and Conjugacy Theorems". In: *Proc. Nat. Acad. Sci. U.S.A.* 80.6 (Mar. 1983), pp. 1778–1782. ISSN: 0027-8424. URL: http://www.pubmedcentral.nih.gov/articlerender.fcgi?artid=393689&tool=pmcentrez&rendertype=abstracthttp://www.pnas.org/content/80/6/1778.short.

[KP83b] V. Kac and D. Peterson. "Regular Functions on Certain Infinite-Dimensional Groups". In: *Arithmetic and Geometry*. Vol. II. Progr. Math. Boston: Birkhauser, 1983, pp. 141–166.

[KR87] A.N. Kirillov and N. Reshetikhin. "Representations of Yangians and multiplicities of occurrence of the irreducible components of the tensor product of representations of simple Lie algebras". In: *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 160 (1987), pp. 211–221. URL: http://www.ams.org/journals/ert/2003-007-12/S1088-4165-03-00164-X/S1088-4165-03-00164-X.pdfhttp://www.springerlink.com/index/3X849473W32M56T5.pdf.

[KS96] Y. Kosmann-Schwarzbach. "Lie Bialgebras, Poisson Lie Groups and Dressing Transformations". In: *Integrability of Nonlinear Systems.* Lecture Notes in Phys. Berlin: Springer, 1996, pp. 104–170. URL: http://www.springerlink.com/index/3h880058821k22v7.pdf.

[Kum02] S. Kumar. *Kac-Moody Groups, Their Flag Varieties, and Representation Theory.* Vol. 204. Progr. Math. Boston, MA: Birkhauser, 2002, 606 pp. ISBN: 0817642277. URL: http://books.google.com/books?hl=en&lr=&id=DigiIbYhFhMC&oi=fnd&pg=PR11&dq=Kac-Moody+groups,+their+flag+varieties,+and+representation+theory&ots=56w9hM6voG&sig=iOPOro-ID_XFxYCfIfJJzRL70oQhttp://books.google.com/books?hl=en&lr=&id=DigiIbYhFhMC&oi=fnd&pg=PR11&dq=Kac-Moody+groups,+their+flag+varieties,+and+representation+theory&ots=56w9mQ5ysA&sig=cohN5iecOSsPfo864tZ7eaQ5MLg.

[KZ02] M. Kogan and A. Zelevinsky. "On Symplectic Leaves and Integrable Systems in Standard Complex Semisimple Poisson-Lie Groups". In: *Int. Math. Res. Not.* 2002.32 (2002), pp. 1685–1702. arXiv:0203069v1 [arXiv:math]. URL: http://imrn.oxfordjournals.org/content/2002/32/1685.short.

[Lec10] B. Leclerc. "Cluster algebras and representation theory". In: *Proceedings of the International Congress of Mathematicians. Volume IV.* New Delhi: Hindustan Book Agency, 2010, pp. 2471–2488. arXiv:arXiv:1009.4552v1. URL: http://arxiv.org/abs/1009.4552.

[LW90] J. Lu and A. Weinstein. "Poisson Lie Groups, Dressing Transformations, and Bruhat Decompositions". In: *J. Differential Geom.* 31.2 (1990), pp. 501–526. ISSN: 0022040X.

[MS13] R. Marsh and J. Scott. "Twists of Plücker coordinates as dimer partition functions". In: *Preprint (arXiv:1309.6630v1)* (2013), pp. 1–39. arXiv:arXiv:1309.6630v1. URL: http://www1.maths.leeds.ac.uk/$\sim$marsh/research_articles/pp43.pdf.

[MV91] J. Moser and A. Veselov. "Discrete versions of some classical integrable systems and factorization of matrix polynomials". In: *Comm. Math. Phys.* 139.2 (1991), pp. 217–243. URL: http://link.springer.com/article/10.1007/BF02352494.

[Nak03] H. Nakajima. “t-analogs of q-characters of Kirillov-Reshetikhin modules of quantum affine algebras”. In: *Represent. Theory* 7 (2003), pp. 259–274. arXiv:0204185v2 [arXiv:math]. URL: http://www.ams.org/journals/ert/2003-007-12/S1088-4165-03-00164-X/S1088-4165-03-00164-X.pdf.

[Pal08] Y. Palu. “Cluster characters for 2-Calabi-Yau triangulated categories”. In: *Ann. Inst. Fourier* 58.6 (2008), pp. 2221–2248. arXiv:0703540v3 [arXiv:math].

[Pla11] P. Plamondon. “Cluster characters for cluster categories with infinite-dimensional morphism spaces”. In: *Adv. Math.* 227.1 (2011), pp. 1–39. arXiv:arXiv:1002.4956v2.

[Res03] N Reshetikhin. “Integrability of characteristic Hamiltonian systems on simple Lie groups with standard Poisson Lie structure”. In: *Communications in mathematical physics* 242.1 (2003), pp. 1–29. arXiv:0103147v1 [arXiv:math]. URL: http://www.springerlink.com/index/FB6MVVH0VBJ8FXDK.pdf.

[RSTS94] A. Reyman and M. Semenov-Tian-Shansky. “Group-theoretical Methods in the Theory of Finite-Dimensional Integrable Systems”. In: *Dynamical Systems VII: Integrable Systems, Nonholonomic Dynamical Systems*. Vol. 1. Encyclopedia of Mathematical Sciences. Springer, 1994, p. 341. ISBN: 3540181768. URL: http://books.google.com/books?hl=en&lr=&id=9MD-jTkkGdAC&pgis=1.

[Rui90] S. Ruijsenaars. “Relativistic Toda systems”. In: *Comm. in Math. Phys.* 133.2 (Oct. 1990), pp. 217–247. ISSN: 0010-3616. DOI: 10.1007/BF02097366. URL: http://www.springerlink.com/index/10.1007/BF02097366.

[Sha81] I. Shafarevich. “On Some Infinite-Dimensional Groups II”. In: *Izv. Akad. Nauk. SSSR Ser. Mat.* 45.1 (1981), pp. 214–226.

[Sur91] Y. Suris. “Algebraic Structure of Discrete-Time and Relativistic Toda Lattices”. In: *Phys. Lett. A* 156.9 (July 1991), pp. 467–474. ISSN: 03759601. DOI: 10.1016/0375-9601(91)90181-7. URL: http://linkinghub.elsevier.com/retrieve/pii/0375960191901817http://www.sciencedirect.com/science/article/pii/0375960191901817.

[Van01] P. Vanhaecke. *Integrable Systems in the Realm of Algebraic Geometry*. Lecture Notes in Mathematics. Berlin: Springer-Verlag, 2001, p. 256. URL: http://books.google.com/books?hl=en&lr=&id=gOseWkOhgWwC&oi=fnd&pg=PR5&dq=Integrable+Systems+in+the+realm+of+Algebraic+Geometry&ots=VSY3RPq0zz&sig=YBe2fy94-2wZEwiRlF83lanFO3A.

[Ves91] A. Veselov. “Integrable maps”. In: *Russ. Math. Surv.* 46.1 (1991), pp. 1–51. URL: http://www.turpion.org/php/reference.phtml?journal_id=rm&paper_id=2856&volume=46&issue=5&type=xrf.

[Wil13a] H. Williams. “Cluster ensembles and Kac-Moody groups”. In: *Adv. Math.* 247 (2013), pp. 1–40. arXiv:arXiv:1210.2533. URL: http://www.sciencedirect.com/science/article/pii/S0001870813002557.

[Wil13b] H. Williams. “Double Bruhat cells in Kac-Moody groups and integrable systems”. In: *Lett. Math. Phys.* 103.4 (2013), pp. 389–419. arXiv:arXiv:1204.0601v1. URL: http://link.springer.com/article/10.1007/s11005-012-0604-3.

[YZ08] S. Yang and A. Zelevinsky. “Cluster algebras of finite type via coxeter elements and principal minors”. In: *Transform. Groups* 13.3-4 (2008), pp. 855–895. arXiv:arXiv:0804.3303v2. URL: http://www.springerlink.com/index/312114112X978772.pdf.

[Zel00] A. Zelevinsky. “Connected components of real double Bruhat cells”. In: *Internat. Math. Res. Notices* 2000.21 (2000), pp. 1131–1154. arXiv:0003231v2 [arXiv:math]. URL: http://imrn.oxfordjournals.org/content/2000/21/1131.short.