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A COMPLEMENTARY DESIGN THEORY FOR DOUBLING

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Chen and Cheng (2006a) discussed the method of doubling for constructing two-level fractional factorial designs. They showed that for $9N/32 \leq n \leq 5N/16$, all minimum aberration designs with N runs and n factors are projections of the maximal design with $5N/16$ factors which is constructed by repeatedly doubling the 2^{5-1} design defined by $I = ABCDE$. This paper develops a general complementary design theory for doubling. For any design obtained by repeated doubling, general identities are established to link the wordlength patterns of each pair of complementary projection designs. A rule is developed for choosing minimum aberration projection designs from the maximal design with $5N/16$ factors. It is further shown that for $17N/64 \leq n \leq 5N/16$, all minimum aberration designs with N runs and n factors are projections of the maximal design with N runs and $5N/16$ factors.

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1 Introduction

Fractional factorial designs are widely used in various experiments and are typically chosen according to the *minimum aberration* criterion. There are many recent researches on the theory and construction of minimum aberration designs; see Wu and Hamada (2000) for a good summary. Chen and Cheng (2006a) discussed the method of doubling for constructing two-level fractional factorial designs, in particular, those of resolution IV. Suppose that X is an $N \times n$ matrix with two distinct entries, 1 and -1 . Then the *double* of X is the $2N \times 2n$ matrix $\begin{pmatrix} X & X \\ X & -X \end{pmatrix}$. Doubling plays an important role in the construction of maximal designs. A regular design of resolution IV or higher is called *maximal* if its resolution reduces to three whenever an extra factor is added. It is obvious that every regular design of resolution IV or higher is a projection of some maximal regular design of resolution IV or higher. Some recent results in the literature of finite projective geometry essentially characterize all maximal regular designs of resolution IV with $n \geq N/4 + 1$. These results imply that for $n \geq N/4 + 1$, a maximal regular design of resolution IV or higher must have

$$n \in \{N/2, 5N/16, 9N/32, 17N/64, 33N/128, \dots\}, \quad (1)$$

where N is a power of 2 and n is the number of factors; see Chen and Cheng (2006a). All these maximal regular designs can be constructed by repeatedly doubling some small designs. For example, the maximal regular design with $N/2$ factors can be constructed by repeatedly doubling the 2^1 design. The maximal regular design with $5N/16$ factors is constructed by repeatedly doubling the 2^{5-1} design defined by $I = ABCDE$. An immediate result is that all regular designs of resolution IV with $5N/16 < n \leq N/2$ must be projections of the maximal regular design with $N/2$

factors and that all regular designs of resolution IV with $9N/32 < n \leq 5N/16$ must be projections of the maximal regular design with either $N/2$ or $5N/16$ factors. Chen and Cheng (2006a) showed that for $9N/32 \leq n \leq 5N/16$, all minimum aberration designs with N runs and n factors are projections of the maximal design with $5N/16$ factors. The motivation of this paper is to develop a complementary design theory that guides the construction of minimum aberration designs from the maximal design with $5N/16$ factors. Previously, Chen and Hedayat (1996) and Tang and Wu (1996) developed a complementary design theory for saturated regular designs of resolution III. For various extensions of this theory including multi-level or mixed-level designs, blocked designs and nonregular designs, see Suen, Chen and Wu (1997), Chen and Cheng (1999), Tang and Deng (1999), Mukerjee and Wu (2001), Xu and Wu (2001), Zhu (2003), and Cheng and Tang (2005). Butler (2003) and Chen and Cheng (2006b) developed a complementary design theory for the maximal regular designs with $N/2$ factors. Note that the maximal regular designs with $N/2$ factors are even designs (i.e., all defining words have even lengths) and are the only even designs that are maximal. We shall call them maximal even designs.

For saturated regular designs of resolution III or maximal even designs, the aforementioned complementary design theories indicate that the wordlength pattern of a design is related to the wordlength pattern of its complement only. The situation is much more complicated for the other maximal designs, wherein the wordlength pattern of a projection design is related to not only the wordlength pattern of its complement but also the structure of the design. A key technique employed here is the Pless power moment identities [Pless (1963)], a fundamental result in coding theory, by which we link moments of Hamming distances between the runs with the wordlength patterns among the factors. For generalizations of the Pless power

moment identities and their applications to factorial designs, see Xu (2003, 2005, 2006).

In Section 2, we review some basic concepts and backgrounds. In Section 3, we develop a general complementary design theory for doubling. For each design constructed by repeated doubling, we express the wordlength pattern of a projection design in terms of a linear combination of the wordlength pattern of its complement and some terms that depend on the frequencies of the columns being doubled. We illustrate how previous complementary design theories for saturated regular designs of resolution III and maximal even designs can be easily derived as special cases of our result. In Section 4, we apply the general theory to the maximal design with $5N/16$ factors and obtain a general rule on constructing its minimum aberration projection designs. As conjectured by Chen and Cheng (2006a), we show that the minimum aberration projection designs have equal or nearly equal frequencies of the columns being doubled. In Section 5, we consider the maximal design with $9N/32$ factors and derive a lower bound for the wordlength patterns of its projections. This lower bound is then used in Section 6 to show that for $17N/64 \leq n \leq 5N/16$, all minimum aberration designs with N runs and n factors are projections of the maximal design with $5N/16$ factors.

2 Basic concepts and backgrounds

An N -run design with n factors is represented by an $N \times n$ matrix, where each row corresponds to a run and each column corresponds to a factor. A two-level design takes on only two symbols which, for convenience, are denoted by 0 and 1 in the rest of the paper. Regular designs are those which can be constructed by using defining relations and are discussed in many textbooks on experimental

design; see, for example, Raktoe, Hedayat and Federer (1981) and Wu and Hamada (2000). A regular 2^{n-p} design is defined by p defining words and has $N = 2^{n-p}$ runs and n factors. The p defining words together generate $2^p - 1$ defining words. The *resolution* is the length of the shortest defining word. Let A_i be the number of defining words of length i . Then the vector (A_1, \dots, A_n) is called the *wordlength pattern*. The *minimum aberration* criterion introduced by Fries and Hunter (1980) chooses a design by sequentially minimizing A_1, A_2, \dots, A_n . Throughout the paper, we will consider only two-level regular designs.

A two-level design is also called a *binary code* in coding theory. In particular, a two-level regular design is a binary *linear code*, since the row vectors (i.e., runs) form a linear space over the binary field $GF(2) = \{0, 1\}$. For an introduction to coding theory, see MacWilliams and Sloane (1977), Hedayat, Sloane and Stufken (1999, Chap. 4) and van Lint (1999).

For a two-level regular $N \times n$ design D (with symbols 0 and 1), let $w_i(D)$ be *Hamming weight* (i.e., the number of nonzero components) of the i th row of D . For integers k , define the moments

$$M_k(D) = \sum_{i=1}^N w_i(D)^k.$$

For convenience, let $0^0 = 1$ and $\binom{n}{k} = 0$ if $k > n$.

Using the Pless power moment identities [Pless (1963)], we can express M_k as a linear combination of A_1, \dots, A_k as follows; see Xu (2003, 2005).

Lemma 1. *For a two-level regular $N \times n$ design D and positive integers k ,*

$$M_k(D) = N \sum_{i=0}^{\min(n,k)} Q_k(i; n) A_i(D), \quad (2)$$

where $Q_k(i; n) = (-1)^i \sum_{j=0}^k j! S(k, j) 2^{-j} \binom{n-i}{j-i}$, $S(k, j) = (1/j!) \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} i^k$ is a Stirling number of the second kind and $A_0(D) = 1$.

It is easy to verify that $S(k, k) = 1$, $Q_k(k; n) = (-1)^k k! / 2^k$ and $Q_k(i; n) = 0$ when $i > k$.

3 Doubling and complementary designs

For a two-level regular $N_0 \times m_0$ design $X_0 = (x_{ij})$ with $x_{ij} = 0$ or 1 . Define its double as

$$\begin{pmatrix} X_0 & X_0 \\ X_0 & X_0 + 1 \end{pmatrix} \pmod{2}.$$

Let X be the regular $N \times m$ design which is obtained by repeatedly doubling X_0 t times, where $N = 2^t N_0$ and $m = 2^t m_0$.

Suppose that D and \bar{D} are a pair of complementary projection designs of X , i.e., $D \cup \bar{D} = X$, where D has n columns and \bar{D} has u columns, with $n + u = m$. Note that each column of X_0 generates 2^t columns of X . For $i = 1, \dots, m_0$, let f_i be the number of columns of \bar{D} that are generated from the i th column of X_0 . Clearly $\sum_{i=1}^{m_0} f_i = u$.

It is easy to verify the relationship of the Hamming weights of D and \bar{D} as follows:

$$\begin{aligned} w_i(\bar{D}) &= \sum_{j=1}^{m_0} f_j x_{ij} \text{ for } i = 1, \dots, N_0, \\ w_i(D) &= \sum_{j=1}^{m_0} (2^t - f_j) x_{ij} = 2^t w_i(X_0) - w_i(\bar{D}) \text{ for } i = 1, \dots, N_0, \\ w_i(D) + w_i(\bar{D}) &= m/2 \text{ for } i = N_0 + 1, \dots, N. \end{aligned}$$

Then

$$\begin{aligned} M_k(D) &= \sum_{i=1}^N w_i(D)^k = \sum_{i=1}^N [m/2 - w_i(\bar{D})]^k + \Delta_k(D, \bar{D}) \\ &= \sum_{i=1}^N \sum_{j=0}^k \binom{k}{j} (m/2)^{k-j} (-1)^j w_i(\bar{D})^j + \Delta_k(D, \bar{D}) \end{aligned}$$

$$= \sum_{j=0}^k \binom{k}{j} (m/2)^{k-j} (-1)^j M_j(\bar{D}) + \Delta_k(D, \bar{D}),$$

where

$$\Delta_k(D, \bar{D}) = \sum_{i=1}^{N_0} [w_i(D)^k - (m/2 - w_i(\bar{D}))^k] \quad (3)$$

depends on the frequencies f_1, \dots, f_{m_0} and X_0 only. Using Lemma 1, we obtain the following relationship between the wordlength patterns of D and \bar{D} .

Theorem 1. *Let D and \bar{D} be a pair of complementary projection designs of the regular $N \times m$ design X constructed via doubling. For integers $k > 0$,*

$$\sum_{i=0}^k Q_k(i; n) A_i(D) = \sum_{i=0}^k \left[\sum_{j=i}^k \binom{k}{j} \left(\frac{m}{2}\right)^{k-j} (-1)^j Q_j(i; u) \right] A_i(\bar{D}) + \frac{\Delta_k(D, \bar{D})}{N},$$

where $Q_k(i; n)$ is defined in Lemma 1, $\Delta_k(D, \bar{D})$ is defined in (3), $A_0(D) = A_0(\bar{D}) = 1$, $A_i(D) = 0$ for $i > n$ and $A_i(\bar{D}) = 0$ for $i > u$.

Because $Q_k(k; n) = (-1)^k k! / 2^k$, we can express $A_k(D)$ in terms of a linear combination of $A_k(\bar{D}), \dots, A_1(\bar{D})$, plus some terms that depend on the frequencies f_1, \dots, f_{m_0} and X_0 as follows:

$$\begin{aligned} A_k(D) &= (-1)^k A_k(\bar{D}) + c_{k-1} A_{k-1}(\bar{D}) + \dots + c_1 A_1(\bar{D}) + c_0 \\ &+ d_k \Delta_k(D, \bar{D}) + \dots + d_1 \Delta_1(D, \bar{D}), \end{aligned} \quad (4)$$

where c_i and d_i are constants that do not depend on D or \bar{D} . If X has resolution r , then $A_i(D) = A_i(\bar{D}) = 0$ for $1 \leq i < r$ and Theorem 1 implies that $\Delta_i(D, \bar{D})$ is a constant for $1 \leq i < r$. In this paper, we are interested in resolution IV designs. The next result follows from Theorem 1.

Corollary 1. *If X has resolution IV or higher, then $\Delta_1(D, \bar{D}) = 0$, $\Delta_2(D, \bar{D}) = N(2n - m)/4$, $\Delta_3(D, \bar{D}) = 3Nn(2n - m)/8$ and*

$$A_4(D) = A_4(\bar{D}) - (2n - m)(6n^2 + 3m - 2)/24 + 2\Delta_4(D, \bar{D})/(3N). \quad (5)$$

The following two examples illustrate how complementary design theories for saturated regular designs of resolution III and maximal even designs can be derived from Theorem 1.

Example 1. Let X be the $N \times m$ matrix constructed by repeatedly doubling the 2×2 design

$$X_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

t times, where $N = m = 2^{t+1}$. Note that the first column of X is a vector of 0's. Deleting the first column, we obtain a saturated regular design of resolution III, denoted by X' . Following the previous notation, we have $w_1(\bar{D}) = w_1(D) = 0$, $w_2(\bar{D}) = f_2$ and $w_2(D) = 2^t - f_2$. Then

$$\Delta_k(D, \bar{D}) = \sum_{i=1}^2 [w_i(D)^k - (m/2 - w_i(\bar{D}))^k] = (2^t - f_2)^k - (2^t)^k - (2^t - f_2)^k = -2^{tk},$$

which does not depend on the frequencies f_1 and f_2 at all. By (4), sequentially minimizing $A_1(D), A_2(D), \dots$ is equivalent to sequentially minimizing $-A_1(\bar{D}), A_2(\bar{D}), -A_3(\bar{D}), A_4(\bar{D}), \dots$. Note that $A_1(\bar{D})$ is maximized if and only if \bar{D} contains the first column of X , a vector of 0's. Let \bar{D}' be the complement of D with respect to X' , the saturated design of resolution III. It is easy to see that $A_1(\bar{D}') = A_2(\bar{D}') = 0$ and $A_i(\bar{D}) = A_i(\bar{D}') + A_{i-1}(\bar{D}')$ for $1 \leq i \leq u$. Therefore, sequentially minimizing $A_1(D), A_2(D), \dots$ is equivalent to sequentially minimizing $-A_3(\bar{D}'), A_4(\bar{D}'), -A_4(\bar{D}'), A_5(\bar{D}'), \dots$. This result is equivalent to the rules developed by Chen and Hedayat (1996) and Tang and Wu (1996).

Example 2. Let X be the $N \times m$ matrix constructed by repeatedly doubling the 2^1 design

$$X_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

t times, where $N = 2^{t+1}$ and $m = 2^t$. According to Chen and Cheng (2006a), X is the maximal even design. Following the previous notation, we have $w_1(\overline{D}) = w_1(D) = 0$, $w_2(\overline{D}) = u$ and $w_2(D) = 2^t - u$. Then

$$\Delta_k(D, \overline{D}) = \sum_{i=1}^2 [w_i(D)^k - (m/2 - w_i(\overline{D}))^k] = (2^t - u)^k - (2^{t-1})^k - (2^{t-1} - u)^k,$$

which is a constant. Since X is an even design, $A_i(D) = A_i(\overline{D}) = 0$ for all odd i . Then by (4), sequentially minimizing $A_4(D), A_6(D), \dots$ is equivalent to sequentially minimizing $A_4(\overline{D}), A_6(\overline{D}), \dots$. Combining this with the fact that all regular designs of resolution IV with $5N/16 < n \leq N/2$ must be projections of the maximal even design X , i.e., they must be even designs, we reach the conclusion that for $5N/16 < n \leq N/2$, D has minimum aberration among all regular designs with N runs and n factors if and only if \overline{D} has minimum aberration among all even designs. This is indeed Theorem 3 of Butler (2003). Chen and Cheng (2006b) derived explicit relationships between the wordlength patterns of complementary projection designs of a maximal even design.

4 Complementary projection designs of the maximal regular design of resolution IV with $5N/16$ factors

Let X_0 be the unique 16×5 design of resolution V defined by $I = ABCDE$. Repeatedly doubling X_0 t times, we obtain the maximal regular design of resolution IV with $5N/16$ factors, denoted by X , where $N = 16 \cdot 2^t$. Following the previous notation, the Hamming weights of \overline{D} (i.e., $w_i(\overline{D})$) are 0, $f_i + f_j$ ($i < j$) and $u - f_i$ ($i = 1, \dots, 5$). The Hamming weights of D (i.e., $w_i(D)$) are 0, $2^{t+1} - f_i - f_j$ ($i < j$)

and $2^{t+2} - u + f_i$ ($i = 1, \dots, 5$). Then

$$\begin{aligned}\Delta_k(D, \overline{D}) &= \sum_{i=1}^5 \left[(2^{t+2} - u + f_i)^k - (m/2 - u + f_i)^k \right] \\ &+ \sum_{1 \leq i < j \leq 5} \left[(2^{t+1} - f_i - f_j)^k - (m/2 - f_i - f_j)^k \right] - (m/2)^k,\end{aligned}$$

where $m = 5 \cdot 2^t$. According to Corollary 1, $\Delta_1(D, \overline{D}) = 0$, $\Delta_2(D, \overline{D}) = 2^{t+2}(2n - 5 \cdot 2^t)$ and $\Delta_3(D, \overline{D}) = 3 \cdot 2^{t+1}n(2n - 5 \cdot 2^t)$, which can be verified easily. For $k \geq 4$, $\Delta_k(D, \overline{D})$ depends on the frequencies f_1, \dots, f_5 . The next lemma states that $\Delta_4(D, \overline{D})$ achieves the minimum value if and only if all f_i are equal or nearly equal.

Lemma 2. *When $u \leq 15 \cdot 2^{t-3}$, $\Delta_4(D, \overline{D})$ achieves the minimum value if and only if pairwise differences of the frequencies f_1, \dots, f_5 do not exceed one, i.e., $|f_i - f_j| \leq 1$ for all $i < j$.*

Proof. We show that $\Delta_4(D, \overline{D})$ does not have minimum if $|f_i - f_j| > 1$ for some $i < j$. Without loss of generality, assume that $f_1 - f_2 > 1$ is the largest among all possible differences $f_i - f_j$. Then $f_1 \geq u/5$ and $3 \cdot 2^t + 2f_1 + 2f_2 - 2u > 3 \cdot 2^t - 8u/5 \geq 0$. Consider another pair of complementary projection designs D' and \overline{D}' , whose frequencies are $f_1 - 1, f_2 + 1, f_3, f_4, f_5$. It can be verified that

$$\Delta_4(D, \overline{D}) - \Delta_4(D', \overline{D}') = 3 \cdot 2^{t+2}(f_1 - f_2 - 1)(3 \cdot 2^t + 2f_1 + 2f_2 - 2u) > 0.$$

That is, $\Delta_4(D, \overline{D}) > \Delta_4(D', \overline{D}')$. This completes the proof. \square

Note that $\Delta_k(D, \overline{D})$ are uniquely determined for all $k \geq 4$ when $\Delta_4(D, \overline{D})$ achieves the minimum value. Theorem 1 and Lemma 2 together lead to the following result.

Theorem 2. *Let D and \overline{D} be a pair of complementary projection designs of the maximal regular design X of resolution IV with $16 \cdot 2^t$ runs and $5 \cdot 2^t$ factors. When*

$25 \cdot 2^{t-3} \leq n \leq 5 \cdot 2^t$, D has minimum aberration among all possible n -factor projections of X if the following two conditions hold

(i) $A_4(\overline{D}), -A_5(\overline{D}), A_6(\overline{D}), -A_7(\overline{D}), \dots$ are sequentially minimized.

(ii) $|f_i - f_j| \leq 1$ for all $i < j$.

5 Complementary projection designs of the maximal regular design of resolution IV with $9N/32$ factors

There is a unique maximal regular design of resolution IV with $9N/32$ factors [Chen and Cheng (2006a)]. For easy presentation, we construct one as follows. Let B be the 2^{7-4} design defined by $4 = 13, 5 = 123, 6 = 12, 7 = 23$. The defining contrast subgroup of B is

$$\begin{aligned} I &= 134 = 245 = 356 = 467 = 571 = 612 = 723 = 1234567 \\ &= 1235 = 2346 = 3457 = 4561 = 5672 = 6713 = 7124. \end{aligned}$$

It is easy to verify that B is cyclic and is equivalent to the 8-run Plackett-Burman design. Let

$$X_0 = \begin{pmatrix} B & 0_8 & 0_8 \\ B & 1_8 & 1_8 \\ B+1 & 0_8 & 1_8 \\ B+1 & 1_8 & 0_8 \end{pmatrix} \pmod{2},$$

where 0_8 and 1_8 are vectors of eight 0's and 1's. It can be verified that X_0 is a maximal 32×9 design of resolution IV and its defining contrast subgroup is

$$\begin{aligned} I &= 1235 = 2346 = 3457 = 4561 = 5672 = 6713 = 7124 \\ &= 13489 = 24589 = 35689 = 46789 = 57189 = 61289 = 72389 = 123456789. \end{aligned}$$

Note that there are seven words of length four not containing factors 8 and 9 and that factors 1–7 are cyclic (i.e., replacing factors 1–7 with 2–7, 1 yields the same design). Repeatedly doubling X_0 yields the maximal regular design of resolution IV with $9N/32$ factors.

The explicit expression of $\Delta_k(D, \overline{D})$ is quite complicated. Nevertheless, it is cyclic in f_1, \dots, f_7 and symmetric in f_8 and f_9 . In particular, with some tedious algebra, we have

$$\begin{aligned} \Delta_4(D, \overline{D}) &= 9534T^4 - 8T^3(535F_1 + 511G_1) \\ &+ 12T^2(51F_1^2 + 3F_2 + 94F_1G_1 + 47G_1^2 + 7G_2) \\ &- 8T(4F_1^3 + 9F_1^2G_1 + 3F_2G_1 + 12G_1G_2 + 9F_1G_1^2 + 3F_1G_2 - 8G_3) \\ &+ 48T(f_1f_3f_4 + f_2f_4f_5 + f_3f_5f_6 + f_4f_6f_7 + f_5f_7f_1 + f_6f_1f_2 + f_7f_2f_3) \end{aligned}$$

where $T = 2^t$, $F_1 = \sum_{i=1}^7 f_i$, $F_2 = \sum_{i=1}^7 f_i^2$, $G_1 = f_8 + f_9$, $G_2 = f_8^2 + f_9^2$, and $G_3 = f_8^3 + f_9^3$.

The next lemma gives conditions for $\Delta_4(D, \overline{D})$ to achieve the minimum value.

Lemma 3. *Let D and \overline{D} be a pair of complementary projection designs of the maximal regular design with $32 \cdot 2^t$ runs and $9 \cdot 2^t$ factors defined in this section.*

(a) *When $u \leq 3 \cdot 2^{t-1}$, necessary conditions for $\Delta_4(D, \overline{D})$ to achieve the minimum value are (i) $f_8 = f_9 = 0$ and (ii) $|f_i - f_j| \leq 1$ for all $1 \leq i < j \leq 7$.*

(b) *When $u \leq 3 \cdot 2^{t-1}$,*

$$\Delta_4(D, \overline{D}) \geq 2^{t+1}(760n^3 - 5400n^22^t + 17380n2^{2t} - 39477 \cdot 2^{3t})/49. \quad (6)$$

Proof. (a) Let $T = 2^t$ and $F(f_1, \dots, f_9) = \Delta_4(D, \overline{D})$. Without loss of generality, assume that f_1 is the smallest among all f_i , $1 \leq i \leq 7$. It can be verified that

$$F(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9) - F(f_1 + f_8, f_2, f_3, f_4, f_5, f_6, f_7, 0, f_9)$$

$$\begin{aligned}
&= 192T^3 f_8 - 24T^2(7f_1 + 4f_2 + 4f_3 + 4f_4 + 4f_5 + 4f_6 + 4f_7)f_8 \\
&+ 48T f_8(f_1 f_2 + f_1 f_3 + f_2 f_3 + f_1 f_4 + f_2 f_4 + f_1 f_5 + f_2 f_5 + f_3 f_5 + f_4 f_5 + f_1 f_6 \\
&\quad + f_3 f_6 + f_4 f_6 + f_5 f_6 + f_1 f_7 + f_2 f_7 + f_3 f_7 + f_4 f_7 + f_6 f_7 + f_1 f_9) \\
&> 192T^3 f_8 - 24T^2(5u)f_8 \geq 0
\end{aligned}$$

when $u \leq 3 \cdot 2^{t-1}$. Therefore, a necessary condition for $\Delta_4(D, \bar{D})$ to achieve the minimum value is $f_8 = 0$. By the symmetry of f_8 and f_9 , another necessary condition is $f_9 = 0$. Next, we show that $F(f_1, \dots, f_9)$ can not be the minimum if $|f_i - f_j| > 2$ for some $1 \leq i < j \leq 7$. Without loss of generality, assume that $f_2 - f_1$ is the largest among all possible $f_i - f_j$ for $1 \leq i, j \leq 7$; for the other cases, the proof is the same due to the special structure of X_0 . It is sufficient to show that if $f_2 - f_1 \geq 2x > 0$, then

$$F(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9) > F(f_1 + x, f_2 - x, f_3, f_4, f_5, f_6, f_7, f_8, f_9). \quad (7)$$

It is straightforward to verify that

$$\begin{aligned}
&F(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9) - F(f_1 + x, f_2 - x, f_3, f_4, f_5, f_6, f_7, f_8, f_9) \\
&= 24Tx[(f_2 - f_1 - x)(3T - 2(f_6 + f_8 + f_9)) - 2(f_3 - f_5)(f_4 - f_7)]. \quad (8)
\end{aligned}$$

By the assumption that $f_2 - f_1$ is the largest among all $f_i - f_j$ for $1 \leq i, j \leq 7$, it is obvious that

$$-2(f_3 - f_5)(f_4 - f_7) \geq -(f_2 - f_1)(f_3 + f_4 + f_5 + f_7).$$

If $f_2 - f_1 \geq 2x > 0$, then $f_2 - f_1 - x \geq (f_2 - f_1)/2$ and

$$\begin{aligned}
&(f_2 - f_1 - x)(3T - 2(f_6 + f_8 + f_9)) - 2(f_3 - f_5)(f_4 - f_7) \\
&\geq ((f_2 - f_1)/2)(3T - 2(f_6 + f_8 + f_9)) - (f_2 - f_1)(f_3 + f_4 + f_5 + f_7)
\end{aligned}$$

$$\begin{aligned}
&= ((f_2 - f_1)/2)(3T - 2(f_6 + f_8 + f_9 + f_3 + f_4 + f_5 + f_7)) \\
&> (f_2 - f_1)(3T - 2u)/2 \geq 0,
\end{aligned} \tag{9}$$

provided that $u \leq 3T/2 = 3 \cdot 2^{t-1}$. Then (7) follows from (8) and (9). This completes the proof.

(b) Following the proof of (a), when $f_i = u/7 = (9 \cdot 2^t - n)/7$ for $i = 1, \dots, 7$ and $f_8 = f_9 = 0$, $F(f_1, \dots, f_9)$ achieves the minimum value, which is the lower bound in (6). The lower bound is achievable only if u is a multiple of 7. \square

A result similar to Theorem 2 can be developed. However, such a result is not interesting because, as we will show in the next section, projection designs of the maximal regular design with $9N/32$ factors often do not have minimum aberration. The following lower bound of A_4 is needed in the development.

Lemma 4. *Let D be an n -factor projection design of the maximal regular design of resolution IV with $32 \cdot 2^t$ runs and $9 \cdot 2^t$ factors. For $n \geq 15 \cdot 2^{t-1}$, $A_4(D) \geq L(n)$, where*

$$L(n) = \frac{196n + 172n^3 - (882 + 2646n + 2754n^2)2^t + (11907 + 17380n)2^{2t} - 39477 \cdot 2^{3t}}{1176}.$$

Proof. It follows from Corollary 1, Lemma 3(b) and the fact $A_4(\overline{D}) \geq 0$. \square

6 Some results on minimum aberration designs

Chen and Cheng (2006a) showed that for $9N/32 \leq n \leq 5N/16$, minimum aberration designs are projection designs of the maximal regular design of resolution IV with $5N/16$ factors. Here we strengthen their result and show that this is also true for $17N/64 \leq n < 9N/32$.

We first establish a general upper bound of wordlength patterns for projection designs.

Lemma 5. *Let X be a regular design of resolution r with m factors having $A_r(X)$ words of length r . For $r \leq n \leq m$, there exists a projection design D of resolution r with n factors having at most*

$$A_r(D) \leq A_r(X) \binom{n}{r} / \binom{m}{r}$$

words of length r .

Proof. We show that such a design can be constructed by deleting one factor at a time. First, there must be a factor that appears in at least $r \cdot A_r(X)/m$ words of length r . Deleting this factor yields a design with $m - 1$ factors having at most $A_r(X) - r \cdot A_r(X)/m = A_r(X)(m - r)/m$ words of length r . Repeating this procedure until we get a design D with n factors. Then

$$A_r(D) \leq A_r(X) \frac{m - r}{m} \frac{m - 1 - r}{m - 1} \cdots \frac{n + 1 - r}{n + 1} = A_r(X) \frac{\binom{n}{r}}{\binom{m}{r}}.$$

□

For the maximal regular design X of resolution IV with $5N/16$ factors, Chen and Cheng (2006a) showed that $A_4(X) = (65 \cdot 2^{3t-2} - 75 \cdot 2^{2t-2} + 5 \cdot 2^{t-1})/6$. Applying Lemma 5, we obtain the following upper bound of A_4 for minimum aberration designs.

Corollary 2. *For $N = 16 \cdot 2^t$, $t \geq 0$, and $n \leq 5N/16$, there exists a regular $N \times n$ design D of resolution IV having*

$$A_4(D) \leq ((65 \cdot 2^{3t-2} - 75 \cdot 2^{2t-2} + 5 \cdot 2^{t-1})/6) \binom{n}{4} / \binom{5 \cdot 2^t}{4}.$$

Corollary 2 is used next to show that for $N/4 + 1 \leq n \leq 5N/16$, projection designs of the maximal regular designs of resolution IV with $N/2$ or $9N/32$ factors do not have minimum aberration.

Lemma 6. *For $N = 16 \cdot 2^t$, $t \geq 0$, and $N/4 + 1 \leq n \leq 5N/16$, projection designs of the maximal regular design of resolution IV with $N/2$ factors do not have minimum aberration.*

Proof. Let D be an n -factor projection design of the maximal regular design of resolution IV with $N/2 = 2^{t+3}$ factors. It follows from (2.2) and (4.3) of Chen and Cheng (2006a) that

$$A_4(D) \geq \left[\binom{n}{2}^2 / (2^{t+3} - 1) - \binom{n}{2} \right] / 6.$$

It is sufficient to show that the upper bound given in Corollary 2 is less than the above lower bound, that is,

$$\left[(65 \cdot 2^{3t-2} - 75 \cdot 2^{2t-2} + 5 \cdot 2^{t-1}) / 6 \right] \binom{n}{4} / \binom{5 \cdot 2^t}{4} < \left[\binom{n}{2}^2 / (2^{t+3} - 1) - \binom{n}{2} \right] / 6.$$

Equivalently,

$$\left[(65 \cdot 2^{3t-2} - 75 \cdot 2^{2t-2} + 5 \cdot 2^{t-1}) \right] (n-2)(n-3) - \left[\frac{6n(n-1)}{2^{t+3}-1} - 12 \right] \binom{5 \cdot 2^t}{4} < 0. \quad (10)$$

Given t , the left side is a polynomial in n of degree 2. It can easily be verified that (10) holds for $t \geq 0$ and $4 \cdot 2^t + 1 \leq n \leq 5 \cdot 2^t$. \square

The condition given in Lemma 6 is conservative for $t \geq 1$. Indeed (10) holds for $n = 8-10$ and $t = 1$, $n = 13-20$ and $t = 2$, $n = 20-40$ and $t = 3$, and $n = 31-80$ and $t = 4$.

Lemma 7. For $N = 32 \cdot 2^t$, $t \geq 0$ and $N/4+1 \leq n \leq 9N/32$, projection designs of the maximal regular design of resolution IV with $9N/32$ factors do not have minimum aberration.

Proof. It is sufficient to show that the upper bound (with $N = 32 \cdot 2^t$) given in Corollary 2 is less than the lower bound given in Lemma 4, that is,

$$U(n) = \left[(65 \cdot 2^{3t+1} - 75 \cdot 2^{2t} + 5 \cdot 2^t) / 6 \right] \binom{n}{4} / \binom{5 \cdot 2^{t+1}}{4} < L(n), \quad (11)$$

where $L(n)$ is defined in Lemma 4. Let $T = 2^t$. Given $t \geq 0$, $F(n) = L(n) - U(n)$ is a polynomial of degree 4. It is sufficient to show that $F(n)$ increases strictly when $8T \leq n \leq 9T$ and that $F(8T + 1) > 0$. Let $F^{(i)}(n)$ be the i th derivative of $F(n)$. It is straightforward to verify that $F^{(3)}(9T) > 0$ and $F^{(4)}(n) < 0$ when $8T \leq n \leq 9T$; therefore, $F^{(3)}(n) > 0$ for $8T \leq n \leq 9T$. It is easy to verify that $F^{(2)}(8T) > 0$, which implies $F^{(2)}(n) > 0$ for $8T \leq n \leq 9T$. It is also easy to verify that $F^{(1)}(8T) > 0$, leading to $F^{(1)}(n) > 0$ for $8T \leq n \leq 9T$. Therefore, $F(n)$ increases strictly when $8T \leq n \leq 9T$. Finally, it can be verified that $F(8T + 1) > 0$. This completes the proof. \square

The condition given in Lemma 7 is conservative for $t \geq 1$. Indeed (11) holds for $n = 16-18$ and $t = 1$, $n = 30-36$ and $t = 2$, and $n = 58-72$ and $t = 3$.

Theorem 3. For $N = 32 \cdot 2^t$, $t \geq 0$, and $17N/64 \leq n \leq 5N/16$, a minimum aberration design with N runs and n factors must be a projection of the maximal regular design of resolution IV with $5N/16$ factors.

Proof. As explained in the introduction, for $17N/64 \leq n \leq 5N/16$, every regular design must be a projection of maximal regular designs of resolution IV with $N/2$, $5N/16$, $9N/32$ or $17N/64$ factors. Lemmas 6 and 7 state that projection designs of

maximal regular designs of resolution IV with $N/2$ or $9N/32$ factors do not have minimum aberration. We only need show that maximal designs of resolution IV with $17N/64$ factors do not have minimum aberration. According to Chen and Cheng (2006a), there are five maximal designs of resolution IV with $17N/64$ factors that are constructed by repeatedly doubling five maximal regular 64×17 designs of resolution IV. According to Block and Mee (2003), all five maximal regular 64×17 designs do not have minimum aberration. Then it follows from Corollary 2.4 of Chen and Cheng (2006a) that all five maximal regular designs of resolution IV with $17N/64$ factors do not have minimum aberration. This completes the proof. \square

Theorems 2 and 3 together yield the following result on minimum aberration designs.

Theorem 4. *Let D and \bar{D} be a pair of complementary projection designs of the maximal regular design of resolution IV with $5N/16$ factors. For $N = 32 \cdot 2^t$, $t \geq 0$, and $17N/64 \leq n \leq 5N/16$, D has minimum aberration among all regular designs with N runs and n factors if \bar{D} satisfies the following two conditions:*

(i) $A_4(\bar{D}), -A_5(\bar{D}), A_6(\bar{D}), -A_7(\bar{D}), \dots$ are sequentially minimized among all $(5N/16 - n)$ -factor projection designs of the maximal regular design with $5N/16$ factors.

(ii) $|f_i - f_j| \leq 1$ for all $i < j$.

Finally, we present the choice of complementary designs \bar{D} that satisfy the two conditions in Theorem 4 for $u = 1-11$. Let X be the maximal regular design of resolution IV with $N = 16 \cdot 2^t$ runs and $5 \cdot 2^t$ factors constructed by repeatedly doubling X_0 as in Section 4. For $i = 1, \dots, 5$, the i th column of X_0 generates columns $5k + i$, $0 \leq k \leq 2^t - 1$, of X . For $u = 1, 2, 3, 4$, the choice of \bar{D} is straightforward.

For instance, we can choose \overline{D} as the first u columns of X , which satisfies both conditions. For $u = 5$, $\overline{D} = \{1, 2, 3, 4, 5\}$ (i.e., the first 5 columns of X) satisfies both conditions because $A_4(\overline{D}) = 0$ is minimized and $A_5(\overline{D}) = 1$ is maximized. For $u = 6-11$ and $N \geq 128$, let $S = \{1, 2, 3, 4, 5, 6, 12, 18, 24, 30, 31\}$ and \overline{D} be the projection design of X whose column indexes are the first u elements of S . Clearly \overline{D} satisfies the second condition. It can be verified (via complete computer search) that $A_4(\overline{D}) = 0$ is minimized and $A_5(\overline{D})$ is maximized among all possible regular designs. Furthermore, such designs are unique up to isomorphism and therefore satisfy both conditions except for $u = 9$. By Theorem 4, the complement of \overline{D} with respect to X has minimum aberration among all regular designs provided $n = 5N/16 - u \geq 17N/64$. For $u = 9$ and $N \geq 128$, there are two nonisomorphic designs with $A_4(\overline{D}) = 0$ and maximum $A_5(\overline{D}) = 2$ as follows:

$$\overline{D}_1 = \{1, 2, 3, 4, 5, 6, 12, 18, 24\}, A_4 = 0, A_5 = 2, A_6 = 1, A_7 = 0, A_8 = 0, A_9 = 0;$$

$$\overline{D}_2 = \{1, 2, 3, 4, 5, 6, 12, 23, 39\}, A_4 = 0, A_5 = 2, A_6 = 0, A_7 = 0, A_8 = 1, A_9 = 0.$$

The second design has a smaller A_6 and should be chosen as \overline{D} .

As a numerical illustration, consider the maximal regular design of resolution IV with 256 runs and 80 factors. For $u = 1-11$, by Theorem 4, deleting \overline{D} with u columns as described above from this maximal design yields minimum aberration designs among all regular designs with $n = 69-79$ factors. Block (2003) previously considered the construction of minimum aberration designs from this maximal regular design via a naive projection (i.e., deleting one column at a time). Our theoretic result confirms that the designs given by Block (2003) with $n = 69, 70, 72-79$ factors have minimum aberration among all possible regular designs. However, for $n = 71$, the design given by Block (2003) has the same wordlength pattern as the

complement of \overline{D}_1 and does not have minimum aberration.

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