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VARIATIONS OF (α, t) -EULERIAN POLYNOMIALS AND THEIR GAMMA POSITIVITY

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Abstract. In 1977, Carlitz and Scoville introduced the cycle (α, t) -Eulerian polynomials $A_n^{\text{cyc}}(x, y, t \mid \alpha)$ by enumerating permutations with respect to the number of excedances, drops, fixed points, and cycles. In this paper, we propose a nine-variable generalization of the Eulerian polynomials $A_n(u_1, u_2, u_3, u_4, f, g, t \mid \alpha, \beta)$ defined in terms of descent-based statistics of permutations, and we establish a connection formula linking these two families of generalized Eulerian polynomials. By means of this connection formula, we derive explicit exponential generating functions and several γ -positive expressions for various Eulerian-type polynomials. In particular, our results unify and strengthen recent findings of Ji and Ji-Lin. In related work on the transition matrix between the Specht and web bases, Hwang, Jang, and Oh recently introduced the web permutations, which can be described in terms of cycle André permutations. We show that enumerating these permutations with respect to the numbers of drops, fixed points, and cycles yields the normalized γ -vectors of the (α, t) -Eulerian polynomials. Our results thus provide a unified framework encompassing and extending several known results in the literature.

Keywords. (α, t) -Eulerian polynomials, left-to-right maxima, cycle André permutations, cyclic valley-hopping, gamma-positivity

Mathematics Subject Classifications. 05A05, 05A15, 05A19

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1. Introduction

The *Eulerian polynomials* $A_n(x)$ are defined by the exponential generating function [FS70, Pet15]

$$\sum_{n \geq 0} A_n(x) \frac{z^n}{n!} = \frac{1-x}{e^{(x-1)z} - x}. \quad (1.1)$$

These polynomials have a long and rich history. Comprehensive accounts can be found in Foata's survey [Foa10], Stanley's *Enumerative Combinatorics* [Sta12, Sta99], and Petersen's monograph [Pet15]; see also the classic work of Foata and Schützenberger [FS70], which remains a fundamental reference and continues to offer deep insights into the subject.

For a finite set E , let $\mathfrak{S}_E$ denote the set of all permutations of E , and let $\mathfrak{S}_n$ be the symmetric group on $[n] := \{1, 2, \dots, n\}$. For $\sigma \in \mathfrak{S}_n$, an index $1 \leq i \leq n-1$ is called

- a *descent* (**des**) of σ if $\sigma(i) > \sigma(i+1)$;
- an *ascent* (**asc**) of σ if $\sigma(i) < \sigma(i+1)$;
- an *excedance* (**exc**) of σ if $i < \sigma(i)$.

The following combinatorial interpretations of the Eulerian polynomials $A_n(x)$ are well known:

$$A_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{asc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{exc}(\sigma)}. \quad (1.2)$$

Postnikov, Reiner, and Williams [PRW08] introduced the *binomial–Eulerian polynomials*:

$$\tilde{A}_n(x) := \sum_{\sigma \in \mathcal{M}_{n+1}} x^{\text{des}(\sigma)} = 1 + x \sum_{m=1}^n \binom{n}{m} A_m(x), \quad (1.3)$$

where $\mathcal{M}_n$ denotes the set of permutations $\sigma \in \mathfrak{S}_n$ such that the first descent (if any) of σ occurs at the letter n . For example,

$$\mathcal{M}_2 = \{12, 21\}, \quad \mathcal{M}_3 = \{123, 132, 231, 312, 321\}.$$

Any polynomial with real coefficients

$$h(x) = \sum_{i=0}^n h_i x^i$$

satisfying the symmetry condition $h_i = h_{n-i}$ can be written uniquely in the form

$$h(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_k x^k (1+x)^{n-2k}.$$

The coefficients γ_k are called the γ -coefficients of $h(x)$. If all γ_k are nonnegative, then $h(x)$ is said to be γ -positive. In such cases, it is often of particular interest to find combinatorial interpretations of the γ -coefficients. It is known [FS70, PZ21] that the Eulerian polynomials $A_n(x)$ admit the following γ -positive expansion:

$$A_{n+1}(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} 2^j d_{n,j} x^j (1+x)^{n-2j}, \quad (1.4)$$

where $d_{n,j}$ denotes the number of André permutations in $\mathfrak{S}_{n+1}$ with exactly j descents. For recent developments in this direction, we refer the reader to [LZ15, SW20, PZ21], and for a general overview of γ -positivity phenomena to [Ath18, Pet15].

The γ -positivity of the binomial-Eulerian polynomials $\tilde{A}_n(x)$ follows from a general theorem of Postnikov, Reiner, and Williams [PRW08, Section 10.4], which in particular implies that these polynomials are symmetric. Moreover, their coefficients can be expressed beautifully as sums of products of Eulerian numbers and binomial coefficients. Writing

$$A_n(x) = \sum_{k=0}^{n-1} \left\langle n \atop k \right\rangle x^k,$$

Chung, Graham, and Knuth [CGK10] established the following elegant identity (with three different proofs) for integers $a, b \geq 0$:

$$\sum_{k \geq 0} \binom{a+b}{k} \left\langle k \atop a-1 \right\rangle = \sum_{k \geq 0} \binom{a+b}{k} \left\langle k \atop b-1 \right\rangle, \quad (1.5)$$

where $\left\langle 0 \atop 0 \right\rangle = 1$. Shareshian and Wachs [SW20] observed that this identity is precisely the algebraic expression of the palindromicity of the coefficients of $\tilde{A}_n(x)$.

For $\sigma \in \mathfrak{S}_n$, an index $i \in [n]$ is called

- a *drop* (**drop**) of σ if $i > \sigma(i)$;
- a *fixed point* (**fix**) of σ if $i = \sigma(i)$.

We also regard a permutation $\sigma \in \mathfrak{S}_n$ as a word $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$, where $\sigma_i := \sigma(i)$ for $i \in [n]$. A letter σ_i is said to be

- a *left-to-right maximum* (**lmax**) of σ if $\sigma_i > \sigma_j$ for every $j < i$;
- a *right-to-left maximum* (**rmax**) of σ if $\sigma_i > \sigma_j$ for every $j > i$.

In the mid-1970s, Carlitz and Scoville introduced several families of multivariate Eulerian polynomials. Among them are the so-called (α, β) -Eulerian polynomials [CS74], defined by

$$A_n(x, y \mid \alpha, \beta) := \sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\text{asc}(\sigma)} y^{\text{des}(\sigma)} \alpha^{\text{lmax}(\sigma)-1} \beta^{\text{rmax}(\sigma)-1}, \quad (1.6a)$$

and the **cycle** (α, t) –**Eulerian polynomials** [CS77],

$$A_n^{\text{cyc}}(x, y, t \mid \alpha) := \sum_{\sigma \in \mathfrak{S}_n} x^{\text{exc}(\sigma)} y^{\text{drop}(\sigma)} t^{\text{fix}(\sigma)} \alpha^{\text{cyc}(\sigma)}, \quad (1.6b)$$

where $\text{cyc}(\sigma)$ denotes the number of cycles of σ . The first values of these polynomials are as follows:

$$\begin{aligned} A_1^{\text{cyc}}(x, y, t \mid \alpha) &= \alpha t, \\ A_2^{\text{cyc}}(x, y, t \mid \alpha) &= \alpha xy + \alpha^2 t^2, \\ A_3^{\text{cyc}}(x, y, t \mid \alpha) &= \alpha(x + y)xy + 3\alpha^2 xyt + \alpha^3 t^3, \\ A_4^{\text{cyc}}(x, y, t \mid \alpha) &= \alpha xy(x^2 + (3\alpha + 4)xy + y^2) + 4\alpha^2 xy(x + y)t + 6\alpha^3 xyt^2 + \alpha^4 t^4. \end{aligned}$$

Since

$$A_n^{\text{cyc}}(x, y, t \mid \alpha) = y^n A_n^{\text{cyc}}(x/y, 1, t/y \mid \alpha),$$

the polynomials $A_n^{\text{cyc}}(x, y, t \mid \alpha)$ constitute the homogeneous version of $A_n^{\text{cyc}}(x, 1, t \mid \alpha)$, which were studied in detail in [FS70, Chapter 4].

The exponential generating function of the polynomials $A_n^{\text{cyc}}(x, y, t \mid \alpha)$ is well known and takes the form

$$\sum_{n \geq 0} A_n^{\text{cyc}}(x, y, t \mid \alpha) \frac{z^n}{n!} = \left(\frac{(x - y)e^{tz}}{xe^{yz} - ye^{xz}} \right)^\alpha. \quad (1.7a)$$

This identity was implicitly established in Foata–Schützenberger’s monograph [FS70, Chapitre IV, (12)–(13)] via the exponential formula, and later proved analytically by Carlitz and Scoville [CS77, Theorem 1]. A proof using symmetric functions and plethysm was subsequently given by Brenti [Bre00, Proposition 7.3]. As observed in [KZ04], using the exponential formula [FS70, Sta99], Eq. (1.7a) can be rewritten equivalently as either of the following exponential generating functions for derangements.

$$\sum_{n \geq 0} A_n^{\text{cyc}}(x, 1, 0 \mid 1) \frac{z^n}{n!} = \left(1 - \sum_{n \geq 2} (x + x^2 + \cdots + x^{n-1}) \frac{z^n}{n!} \right)^{-1}, \quad (1.7b)$$

$$\sum_{n \geq 0} A_n^{\text{cyc}}(x, 1, 0 \mid -1) \frac{z^n}{n!} = 1 - \sum_{n \geq 2} (x + x^2 + \cdots + x^{n-1}) \frac{z^n}{n!}. \quad (1.7c)$$

Gessel [Ges91] gave an elegant combinatorial proof of (1.7b) in the special case $x = 1$. Later, Kim and Zeng [KZ01] provided a bijective proof of (1.7b). Two further combinatorial proofs of (1.7c) were subsequently presented in [KZ04]. More recently, a multivariable extension of (1.7c) was established in [PZ24].

For a permutation $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$, we say that an index $i \in [n]$ is a

- *cycle peak* (**cpk**) of σ if $\sigma^{-1}(i) < i > \sigma(i)$;
- *cycle valley* (**cval**) of σ if $\sigma^{-1}(i) > i < \sigma(i)$;
- *cycle double ascent* (**cda**) of σ if $\sigma^{-1}(i) < i < \sigma(i)$;
- *cycle double descent* (**cdd**) of σ if $\sigma^{-1}(i) > i > \sigma(i)$.

Note that $\text{cpk}(\sigma) = \text{cval}(\sigma)$. The following refinement of (1.7a) was proved in [Zen93, Th. 1]:

$$1 + \sum_{n \geq 1} \frac{z^n}{n!} \sum_{\sigma \in \mathfrak{S}_n} (u_1 u_2)^{\text{cpk}(\sigma)} u_3^{\text{cda}(\sigma)} u_4^{\text{cdd}(\sigma)} t^{\text{fix}(\sigma)} \alpha^{\text{cyc}(\sigma)} = \left(\frac{(x-y)e^{tz}}{xe^{yz} - ye^{xz}} \right)^\alpha, \quad (1.8)$$

where $xy = u_1 u_2$ and $x + y = u_3 + u_4$.

For $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$ with the convention $\sigma_0 = \sigma_{n+1} = 0$, a letter $\sigma_i \in [n]$ is called a

- *valley* (**val**) of σ if $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$;
- *peak* (**pk**) of σ if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$;
- *double ascent* (**da**) of σ if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$;
- *double descent* (**dd**) of σ if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.

It is clear that the following identities hold

$$\text{val} = \text{pk} - 1, \quad \text{asc} = \text{val} + \text{da}, \quad \text{des} = \text{val} + \text{dd}. \quad (1.9)$$

Recently, Ji [Ji25] introduced the following refinement of the Carlitz–Scoville polynomials $A_n(x, y \mid \alpha, \beta)$, incorporating six permutation statistics:

$$A_n(u_1, u_2, u_3, u_4 \mid \alpha, \beta) := \sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1 u_2)^{\text{val}(\sigma)} u_3^{\text{da}(\sigma)} u_4^{\text{dd}(\sigma)} \alpha^{\text{lmax}(\sigma)-1} \beta^{\text{rmax}(\sigma)-1}. \quad (1.10)$$

Ji's main result [Ji25, Theorem 1.4] gives the following exponential generating function, obtained by means of the context-free grammar calculus.¹ Let

$$F(x, y; t) = \frac{e^{xt} - e^{yt}}{x e^{yt} - y e^{xt}}.$$

¹Ji [Ji25, (3.33)–(3.35)] assumed that α and β are integers in her proof. In that case, an alternative combinatorial interpretation of the same generating function was given earlier by Carlitz and Scoville [CS74, Theorem 5].

Theorem 1.1 (Ji). *We have*

$$\sum_{n \geq 0} A_n(u_1, u_2, u_3, u_4 | \alpha, \beta) \frac{z^n}{n!} = (1 + xF(x, y; z))^{\frac{\alpha+\beta}{2}} (1 + yF(x, y; z))^{\frac{\alpha+\beta}{2}} e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)z}, \quad (1.11)$$

where $x + y = u_3 + u_4$ and $xy = u_1 u_2$.

Ji [Ji25] showed that the generating function (1.11) is related to several families of enumerative polynomials that have appeared in the literature. In a subsequent paper [JL24], using the calculus of context-free grammars together with group actions on permutations, Ji and Lin further established the γ -positivity of two α -analogues of the Eulerian polynomials, as well as an α -analogue of the Chung–Graham–Knuth symmetric Eulerian identity (1.5).

The present paper originates from the desire to provide a more conceptual approach to Ji's generating function (1.11), in the spirit of Foata–Schützenberger's combinatorial proof of (1.1) in [FS70]. To this end, we first observe that Eq. (1.11) can be rewritten in the following form,

$$\sum_{n \geq 0} A_n(u_1, u_2, u_3, u_4 | \alpha, \beta) \frac{z^n}{n!} = e^{(\alpha u_3 + \beta u_4)z} \left(\frac{x - y}{xe^{yz} - ye^{xz}} \right)^{\alpha + \beta} \quad (1.12)$$

$$= \left(\frac{(x - y)e^{u_3 z}}{xe^{yz} - ye^{xz}} \right)^{\alpha} \cdot \left(\frac{(x - y)e^{u_4 z}}{xe^{yz} - ye^{xz}} \right)^{\beta}, \quad (1.13)$$

where the parameters x and y are related by $x + y = u_3 + u_4$ and $xy = u_1 u_2$. Comparing (1.12) with (1.7a), we see that Eq. (1.11) is equivalent to the following connection formula:

$$A_n(u_1, u_2, u_3, u_4 | \alpha, \beta) = A_n^{\text{cyc}} \left(x, y, \frac{\alpha u_3 + \beta u_4}{\alpha + \beta} | \alpha + \beta \right), \quad (1.14)$$

where $xy = u_1 u_2$ and $x + y = u_3 + u_4$.

Our first main result establishes a generalization of Ji's polynomial (1.10) involving three additional variables. This unified framework subsumes the two families of polynomials studied in [Ji25, JL24] and further incorporates several recent results from [Ji25, JL24]. Before stating the result, we introduce some additional definitions. For $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$ with conventions $\sigma_0 = \sigma_{n+1} = 0$, a letter $\sigma_i \in [n]$ is a

- *left-to-right-maximum-peak* (**lmaxpk**) if σ_i is a left-to-right maximum and also a peak;
- *right-to-left-maximum-peak* (**rmaxpk**) if σ_i is a right-to-left maximum and also a peak;
- *left-to-right-maximum-double-ascent* (**lmaxda**) if σ_i is a left-to-right maximum and also a double ascent;
- *right-to-left-maximum-double-descent* (**rmaxdd**) if σ_i is a right-to-left maximum and also a double descent.

Let $\mathbf{u} = (u_1, u_2, u_3, u_4)$ and define the generalized Eulerian polynomial

$$A_n(\mathbf{u}, f, g, t | \alpha, \beta) = \sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1 u_2)^{\text{val}(\sigma)} u_3^{\text{da}(\sigma)} u_4^{\text{dd}(\sigma)} f^{\text{lmaxpk}(\sigma)-1} g^{\text{rmaxpk}(\sigma)-1} \\ \times t^{\text{lmaxda}(\sigma)+\text{rmaxdd}(\sigma)} \alpha^{\text{lmax}(\sigma)-1} \beta^{\text{rmax}(\sigma)-1}. \quad (1.15)$$

Theorem 1.2. *If $xy = u_1 u_2$ and $x + y = u_3 + u_4$, then*

$$A_n(\mathbf{u}, f, g, t | \alpha, \beta) = A_n^{\text{cyc}} \left(x, y, \frac{\alpha u_3 + \beta u_4}{\alpha f + \beta g} t | \alpha f + \beta g \right). \quad (1.16)$$

Combining Theorem 1.2 with (1.7a), we derive immediately the exponential generating function of $A_n(\mathbf{u}, f, g, t | \alpha, \beta)$.

Corollary 1.3. *If $xy = u_1 u_2$ and $x + y = u_3 + u_4$, then*

$$\sum_{n \geq 0} A_n(\mathbf{u}, f, g, t | \alpha, \beta) \frac{z^n}{n!} = e^{(\alpha u_3 + \beta u_4)tz} \left(\frac{x - y}{xe^{yz} - ye^{xz}} \right)^{\alpha f + \beta g}. \quad (1.17)$$

In particular, we have

$$\sum_{n \geq 0} A_n(\mathbf{u}, 1, 1, 1 | \alpha, \beta) \frac{z^n}{n!} = e^{(\alpha u_3 + \beta u_4)z} \left(\frac{x - y}{xe^{yz} - ye^{xz}} \right)^{\alpha + \beta}, \quad (1.18)$$

$$\sum_{n \geq 0} A_n(\mathbf{u}, 0, 1, 1 | \alpha, \beta) \frac{z^n}{n!} = e^{(\alpha u_3 + \beta u_4)z} \left(\frac{x - y}{xe^{yz} - ye^{xz}} \right)^{\beta}. \quad (1.19)$$

Remark 1.4. The polynomials $A_n(\mathbf{u}, 1, 1, 1 | \alpha, \beta)$ and $A_n(\mathbf{u}, 0, 1, 1 | \alpha, \beta)$ correspond, respectively, to Ji's generalized Eulerian polynomial (1.10) and Ji-Lin's binomial-Stirling Eulerian polynomial [JL24, Theorem 1.5].

Combining Theorem 1.2 and the continued fraction expansion of the ordinary generating function of $A_n^{\text{cyc}}(x, y, t | \alpha)$ in [Zen93, Th. 3] we obtain the following continued fraction formula for the ordinary generating function of $A_n(\mathbf{u}, f, g, t | \alpha, \beta)$.

Corollary 1.5. *We have*

$$\sum_{n=0}^{\infty} A_n(\mathbf{u}, f, g, t | \alpha, \beta) z^n = \frac{1}{1 - b_0 z - \frac{\lambda_1 z^2}{1 - b_1 z - \frac{\lambda_2 z^2}{1 - b_2 z - \dots}}}, \quad (1.20a)$$

where

$$b_k = k(u_3 + u_4) + (\alpha u_3 + \beta u_4)t, \quad (1.20b)$$

$$\lambda_{k+1} = (k + \alpha f + \beta g)(k + 1)u_1 u_2 \quad (k \geq 0). \quad (1.20c)$$

Define the (α, t) -Eulerian polynomials

$$A_n(x, y, t | \alpha) = A_n^{\text{cyc}}\left(x, y, \frac{x+y}{2}t | 2\alpha\right). \quad (1.21)$$

By Theorem 1.2 with $f = g = 1$, $\alpha = \beta$, we obtain the following t -analogue of Ji–Lin’s Theorem 1.6 [JL24].

Corollary 1.6. *If $xy = u_1u_2$ and $x + y = u_3 + u_4$, then*

$$A_n(x, y, t | \alpha) = \sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1u_2)^{\text{val}(\sigma)} u_3^{\text{da}(\sigma)} u_4^{\text{dd}(\sigma)} t^{\text{lmaxda}(\sigma) + \text{rmaxdd}(\sigma)} \alpha^{\text{lmax}(\sigma) + \text{rmax}(\sigma) - 2}. \quad (1.22)$$

In particular, setting $u_1 = u_3 = x$ and $u_2 = u_4 = y$, we have

$$A_n(x, y, t | \alpha) = \sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\text{asc}(\sigma)} y^{\text{des}(\sigma)} t^{\text{lmaxda}(\sigma) + \text{rmaxdd}(\sigma)} \alpha^{\text{lmax}(\sigma) + \text{rmax}(\sigma) - 2}. \quad (1.23)$$

The next theorem gives three different combinatorial interpretations of the normalised γ -positivity of $A_n(x, y, t | \alpha)$.

Theorem 1.7. *For $n \geq 0$, we have*

$$A_n(x, y, t | \alpha) = \sum_{j=0}^{\lfloor n/2 \rfloor} 2^j d_{n,j}(\alpha, t) (xy)^j (x+y)^{n-2j}, \quad (1.24a)$$

where for $0 \leq j \leq \lfloor n/2 \rfloor$,

$$d_{n,j}(\alpha, t) = \sum_{\substack{\sigma \in \mathcal{CA}_n \\ \text{drop}(\sigma) = j}} t^{\text{fix}(\sigma)} \alpha^{\text{cyc}(\sigma)} \quad (1.24b)$$

$$= \sum_{\substack{\sigma \in \mathcal{A}_{n+1}^1 \\ \text{des}(\sigma) = j}} t^{\text{rminda}(\sigma)} \alpha^{\text{rmin}(\sigma) - 1} \quad (1.24c)$$

$$= \sum_{\substack{\sigma \in \mathcal{A}_{n+1}^2 \\ \text{des}(\sigma) = j}} t^{\text{rminda}(\sigma)} \alpha^{\text{rmin}(\sigma) - 1}. \quad (1.24d)$$

The sets $\mathcal{CA}_n$, $\mathcal{A}_n^1$ and $\mathcal{A}_n^2$ in above theorem denote the set of cycle André permutations, André permutations of the first kind and André permutations of the second kind, respectively. The precise definitions of these sets will be given in Section 3.3.

The rest of this paper is organized as follows.

In Section 2, we prove Theorem 1.2 by constructing a combinatorial bijection based on two variants of Foata’s first fundamental transformation (FFT) [Lot83, Chapter 10], and by combining it with the cyclic valley–hopping on permutations introduced by Cooper, Jones, and Zhuang [CJZ20].

In Section 3, we study (α, t) -binomial-Eulerian polynomials $\tilde{A}_n(x, y, t | \alpha)$ and establish a symmetric (α, t) -Eulerian identity. Then we show that the (α, t) -Eulerian polynomials and the (α, t) -binomial-Eulerian polynomials admit remarkable γ -expansions, including t -analogues of Ji and Lin's results [JL24, Theorems 1.5 and 1.6]. Furthermore, we present an (α, t) -analogue of (1.4), together with three combinatorial interpretations of the corresponding (α, t) -analogue of $d_{n,j}$ in terms of cycle André permutations and André permutations of types 1 and 2 (see Theorem 3.19).

In Section 4, we prove Theorem 3.19, establishing the three combinatorial interpretations of the normalized γ -coefficients of the (α, t) -Eulerian polynomials in the models of cycle André permutations and André permutations of types 1 and 2.

In Section 5, using a suitable variant of the FFT, we relate the polynomials $A_n^{\text{cyc}}(x, y, t | \alpha)$ to the enumeration of successions in permutations and derive the exponential generating function for a multivariate statistic involving the number of successions.

Finally, we conclude the paper with a discussion of several open problems.

2. Proof of Theorem 1.2

2.1. Two Foata-type transformations

We recall two variants of the fundamental transformation [Sta12, p. 30], [FS70, Chap. 1]. For a permutation $\sigma \in \mathfrak{S}_n$, the map $\theta_1 : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ (resp. $\theta_2 : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$) is defined as follows:

1. Write the disjoint cycle decomposition of σ , and in each cycle place its largest element at the *beginning* (resp. *end*).
2. Order the resulting cycles from left to right according to the *increasing* (resp. *decreasing*) order of their largest elements. Denote the concatenation of these ordered cycles by $\check{\sigma}$ (resp. $\tilde{\sigma}$).
3. Remove the parentheses of $\check{\sigma}$ (resp. $\tilde{\sigma}$); the resulting word is the one-line notation of a permutation, which we denote by $\theta_1(\sigma)$ (resp. $\theta_2(\sigma)$).

Example 2.1. Let $\sigma = (3\ 1\ 4)(6\ 2\ 5) \in \mathfrak{S}_6$.

- After rotating each cycle so that its largest element is first, we obtain $(4\ 3\ 1)$ and $(6\ 2\ 5)$. Ordering the cycles by increasing largest elements ($4 < 6$) gives $\check{\sigma} = (4\ 3\ 1)(6\ 2\ 5)$, hence

$$\theta_1(\sigma) = 4\ 3\ 1\ 6\ 2\ 5.$$

- After rotating each cycle so that its largest element is last, we obtain $(3\ 1\ 4)$ and $(2\ 5\ 6)$. Ordering the cycles by decreasing largest elements ($6 > 4$) gives $\tilde{\sigma} = (2\ 5\ 6)(3\ 1\ 4)$, hence

$$\theta_2(\sigma) = 2\ 5\ 6\ 3\ 1\ 4.$$

Remark 2.2. Under θ_1 , the set of left-to-right maxima of $\theta_1(\sigma)$ coincides with the set of cycle maxima of σ . Thus the number of cycles of σ equals the number of left-to-right maxima of $\theta_1(\sigma)$. Under θ_2 , the set of right-to-left maxima of $\theta_2(\sigma)$ coincides with the set of cycle maxima of σ . Thus the number of cycles of σ equals the number of right-to-left maxima of $\theta_2(\sigma)$.

2.2. Cyclic valley-hopping

Fix a letter $x \in [n]$. Each permutation $\sigma \in \mathfrak{S}_n$ can be written as the concatenation $\sigma = w_1 w_2 x w_4 w_5$, where w_2 (resp. w_4) is the maximal contiguous subword immediately to the left (resp. right) of x whose letters are all smaller than x . For example, if $\sigma = 934278516$ and $x = 7$, then $w_1 = 9$, $w_2 = 342$, $w_4 = \emptyset$ and $w_5 = 8516$. Define the involution $\xi_x : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ by

$$\xi_x(\sigma) := \begin{cases} w_1 w_4 x w_2 w_5, & \text{if } x \text{ is a double ascent or a double descent of } \sigma; \\ \sigma, & \text{if } x \text{ is a valley or a peak of } \sigma. \end{cases}$$

Here, we use conventions $\sigma(0) = \sigma(n+1) = \infty$. It is known that the involutions ξ_x and ξ_y commute with each other for all $x, y \in [n]$, see [Brä08]. Hence, for any subset $S \subseteq [n]$, we can define an operator $\xi_S := \prod_{x \in S} \xi_x$. As shown by Brändén [Brä08], the involutions ξ_S ($S \subseteq [n]$) define a $\mathbb{Z}_2^n$ -action on $\mathfrak{S}_n$, which are called *modified Foata and Strehl action* or *MFS-action*, see [FS74]. A geometric version of this action was given by Shapiro et al. [SWG83].

Athanasiadis–Savvidou [AS12] defined a cyclic analogue of *valley hopping* for derangements by combining Foata's fundamental transformation and MFS-action, see also Sun–Wang [SW14] and Tirrell–Zhuang [TZ19]. Finally, Cooper, Jones and Zhuang [CJZ20] extended cyclic valley-hopping to permutations by letting the fixed points be unchanged. Define $\psi_x(\sigma) : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ by

$$\psi_x(\sigma) := \begin{cases} \theta_1^{-1} \circ \xi_x \circ \theta_1(\sigma), & \text{if } x \text{ is not a fixed point of } \sigma; \\ \sigma, & \text{if } x \text{ is a fixed point of } \sigma, \end{cases}$$

where we treat the 0-th letter of $\theta_1(\sigma)$ as 0 and the $(n+1)$ -th letter as ∞ . Given a subset $S \subseteq [n]$, define $\psi_S := \prod_{x \in S} \psi_x$. We will call the $\mathbb{Z}_2^n$ -action defined by the involutions $\{\psi_S\}_{S \subseteq [n]}$ *cyclic valley-hopping* or *cyclic MFS-action*.

For example, if $\sigma = 106724548931$ and $S = \{4, 8\}$, then $\check{\sigma} = (5)(642)(9378)(101)$, $\theta_1(\sigma) = 56429378101$, $\xi_S \circ \theta_1(\sigma) = 56249873101$, and $\psi_S(\sigma) = 10496523781$, see Fig. 2.1 for the valley hopping on $\theta_1(\sigma)$.

Define $\text{Cval}(\sigma)$, $\text{Cpk}(\sigma)$, $\text{Cda}(\sigma)$, $\text{Cdd}(\sigma)$ and $\text{Fix}(\sigma)$ to be the set of cyclic valleys, cyclic peaks, cyclic double ascents, cyclic double descents and fixed points of σ .

Proposition 2.3. [CJZ20, Proposition 3 and 4] *For any $S \subseteq [n]$ and $\sigma \in \mathfrak{S}_n$, we have*

1. $\text{Cval}(\psi_S(\sigma)) = \text{Cval}(\sigma)$; $\text{Cpk}(\psi_S(\sigma)) = \text{Cpk}(\sigma)$;
2. $\text{Cda}(\psi_S(\sigma)) = (\text{Cda}(\sigma) \setminus S) \cup (S \cap \text{Cdd}(\sigma))$;
3. $\text{Cdd}(\psi_S(\sigma)) = (\text{Cdd}(\sigma) \setminus S) \cup (S \cap \text{Cda}(\sigma))$;
4. $\text{Fix}(\psi_S(\sigma)) = \text{Fix}(\sigma)$; $\text{cyc}(\psi_S(\sigma)) = \text{cyc}(\sigma)$.

In other words, the cyclic valley-hopping does not affect cyclic valleys, cyclic peaks and fixed points, the number of cycles, but toggles between cyclic double ascents and cyclic double descents. As an application of cyclic valley-hopping, we give a combinatorial proof of the following result, which can be derived by combining formula (1.8) and (1.7a).

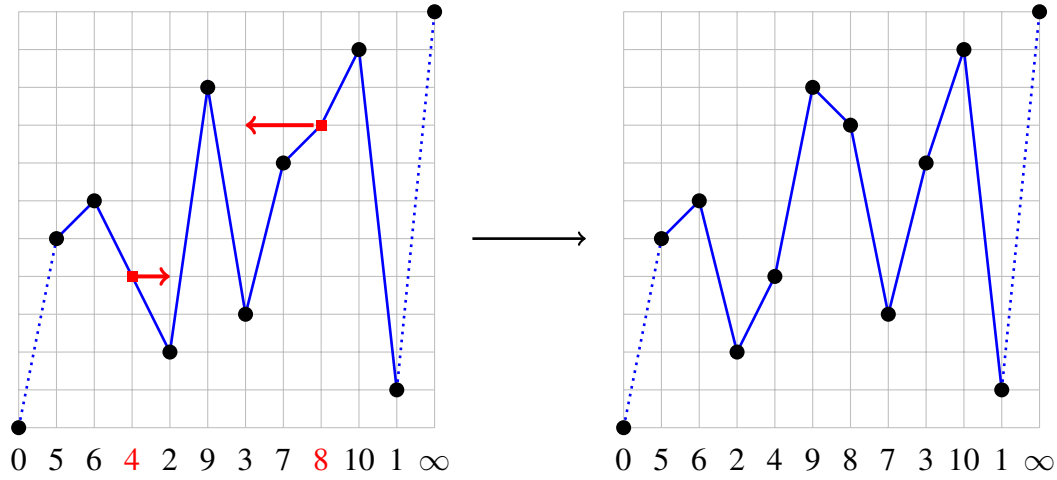


Figure 2.1: Cyclic valley-hopping on $\theta_1(\sigma) = 56429378101$ with $S = \{4, 8\}$ yields $\xi_S \circ \theta_1(\sigma) = 56249873101$.

Theorem 2.4. *If $xy = u_1u_2$ and $x + y = u_3 + u_4$, then*

$$A_n^{\text{cyc}}(x, y, t \mid \alpha) = \sum_{\sigma \in \mathfrak{S}_n} (u_1u_2)^{\text{cpk}(\sigma)} u_3^{\text{cda}(\sigma)} u_4^{\text{cdd}(\sigma)} t^{\text{fix}(\sigma)} \alpha^{\text{cyc}(\sigma)}. \quad (2.1)$$

Proof. Given a subset $S \subseteq [n]$, consider cyclic valley action of $\mathbb{Z}_2^n$ on $\mathfrak{S}_n$ via the functions ψ_S . For a permutation $\sigma \in \mathfrak{S}_n$, denote the orbit of σ under cyclic valley-hopping by $\text{Orb}(\sigma) := \{\psi_S(\sigma) \mid S \subseteq [n]\}$. Let $\bar{\sigma}$ be such a unique element with $\text{cdd}(\bar{\sigma}) = 0$ in $\text{Orb}(\sigma)$. By Proposition 2.3, we have

$$\sum_{\pi \in \text{Orb}(\sigma)} (u_1u_2)^{\text{cpk}(\pi)} u_3^{\text{cda}(\pi)} u_4^{\text{cdd}(\pi)} t^{\text{fix}(\pi)} \alpha^{\text{cyc}(\pi)} = (u_1u_2)^{\text{cpk}(\bar{\sigma})} (u_3 + u_4)^{\text{cda}(\bar{\sigma})} t^{\text{fix}(\bar{\sigma})} \alpha^{\text{cyc}(\bar{\sigma})}. \quad (2.2)$$

By definition, it is clear that for $\sigma \in \mathfrak{S}_n$,

$$\text{exc}(\sigma) = \text{cda}(\sigma) + \text{cpk}(\sigma) \quad \text{and} \quad \text{drop}(\sigma) = \text{cdd}(\sigma) + \text{cval}(\sigma). \quad (2.3)$$

Setting $u_1 = u_3 = x$ and $u_2 = u_4 = y$ in (2.2) yields

$$\sum_{\pi \in \text{Orb}(\sigma)} x^{\text{exc}(\pi)} y^{\text{drop}(\pi)} t^{\text{fix}(\pi)} \alpha^{\text{cyc}(\pi)} = (xy)^{\text{cpk}(\bar{\sigma})} (x + y)^{\text{cda}(\bar{\sigma})} t^{\text{fix}(\bar{\sigma})} \alpha^{\text{cyc}(\bar{\sigma})}. \quad (2.4)$$

Thus, if $u_1u_2 = xy$ and $u_3 + u_4 = x + y$, combining (2.2) and (2.4) we have

$$\sum_{\pi \in \text{Orb}(\sigma)} (u_1u_2)^{\text{cpk}(\pi)} u_3^{\text{cda}(\pi)} u_4^{\text{cdd}(\pi)} t^{\text{fix}(\pi)} \alpha^{\text{cyc}(\pi)} = \sum_{\pi \in \text{Orb}(\sigma)} x^{\text{exc}(\pi)} y^{\text{drop}(\pi)} t^{\text{fix}(\pi)} \alpha^{\text{cyc}(\pi)}. \quad (2.5)$$

Then summing over all the orbits of $\mathfrak{S}_n$ gives Eq. (2.1). $\square$

2.3. Proof of Theorem 1.2

We need the following combinatorial formulation of the binomial theorem.

Lemma 2.5. *Let E be a finite set, we have*

$$(\alpha + \beta)^{|E|} = \sum_{h: E \rightarrow \{0,1\}} \alpha^{\#\{x \in E : h(x)=0\}} \beta^{\#\{x \in E : h(x)=1\}}. \quad (2.6)$$

Explanation. The exponent $\#\{x \in E : h(x) = 1\}$ counts the number of elements of E sent to 1 (those contributing a factor β), while the remaining elements, sent to 0, contribute a factor α . Each function $h : E \rightarrow \{0, 1\}$ encodes a subset of E , namely the set

$$A_h = \{x \in E : h(x) = 1\}.$$

Summing over all $2^{|E|}$ maps h amounts to summing over all subsets $A_h \subseteq E$, and we obtain the usual binomial expansion

$$(\alpha + \beta)^{|E|} = \sum_{k=0}^{|E|} \binom{|E|}{k} \alpha^{|E|-k} \beta^k. \quad \square$$

For $\pi \in \mathfrak{S}_n$, let $\text{Fix}(\pi)$ denote the set of fixed points of π (that is, the cycles of length 1), and let $\text{Cyc}_{\geq 2}(\pi)$ denote the set of the remaining cycles of π . Then the set of all cycles of π is

$$C(\pi) := \text{Fix}(\pi) \cup \text{Cyc}_{\geq 2}(\pi).$$

A *cycle-decorated permutation* (π, h) is a permutation π in which each cycle is assigned a label in $\{0, 1\}$. We denote by

$$\mathfrak{S}_n^{\text{cd}} := \{(\pi, h) : \pi \in \mathfrak{S}_n, h : C(\pi) \rightarrow \{0, 1\}\}$$

the set of all such pairs. Let $\mathfrak{S}_n^\star$ denote the set of pairs $(\sigma, \tau) \in \mathfrak{S}_A \times \mathfrak{S}_B$, where (A, B) is an ordered set partition of $[n]$. Given a cycle-decorated permutation $(\pi, h) \in \mathfrak{S}_n^{\text{cd}}$, we associate to it a pair (σ, τ) of permutations acting on two disjoint subsets A and B of $[n]$ such that $A \cup B = [n]$, where A (resp. B) is the underlying support of σ (resp. τ). This defines a bijection

$$\Phi : \mathfrak{S}_n^{\text{cd}} \longrightarrow \mathfrak{S}_n^\star, \quad \Phi(\pi, h) = (\sigma, \tau), \quad (2.7)$$

obtained by grouping together the cycles of π according to their decoration: cycles labeled by 0 (resp. 1) form the permutation σ on A (resp. τ on B).

By Theorem 2.4, under the assumption that $xy = u_1u_2$, $x + y = u_3 + u_4$, the right-hand side of equality (1.16) can be written as

$$\begin{aligned} & A_n^{\text{cyc}}(x, y, t(\alpha u_3 + \beta u_4)(\alpha f + \beta g)^{-1} | \alpha f + \beta g) \\ &= \sum_{\pi \in \mathfrak{S}_n} (u_1 u_2)^{\text{cpk}(\pi)} u_3^{\text{cda}(\pi)} u_4^{\text{cdd}(\pi)} (t(\alpha u_3 + \beta u_4))^{\text{fix}(\pi)} (\alpha f + \beta g)^{\text{cyc}(\pi) - \text{fix}(\pi)}. \end{aligned} \quad (2.8)$$

Using (2.6), it is clear that

$$\begin{aligned} & (x_0 + x_1)^{\text{fix}(\pi)} (y_0 + y_1)^{\text{cyc}(\pi) - \text{fix}(\pi)} \\ &= \sum_{h: C(\pi) \rightarrow \{0,1\}} \prod_{i=0}^1 x_i^{\#\{x \in \text{Fix}(\pi) : h(x)=i\}} y_i^{\#\{x \in \text{Cyc}_{\geq 2}(\pi) : h(x)=i\}}. \end{aligned} \quad (2.9)$$

Let E be a finite subset of $\mathbb{Z}_{>0}$. For $\pi \in \mathfrak{S}_E$ define the weight function.

$$w(\pi; \alpha, t, b) = (u_1 u_2)^{\text{cpk}(\pi)} u_3^{\text{cda}(\pi)} u_4^{\text{cdd}(\pi)} t^{\text{fix}(\pi)} b^{\text{cyc}(\pi) - \text{fix}(\pi)} \alpha^{\text{cyc}(\pi)}. \quad (2.10)$$

Since a cycle-decorated permutation (π, h) has the same cycle structure as π and Φ preserves the weight function, applying Eq. (2.9) to (2.8) with $(x_0, x_1) = (t\alpha u_3, t\beta u_4)$ and $(y_0, y_1) = (\alpha f, \beta g)$, we obtain

$$\begin{aligned} A_n^{\text{cyc}} \left(x, y, \frac{t(\alpha u_3 + \beta u_4)}{(\alpha f + \beta g)} \mid \alpha f + \beta g \right) &= \sum_{(\pi, h) \in \mathfrak{S}_n^{\text{cd}}} w(\pi_0; \alpha, u_3 t, f) w(\pi_1; \beta, u_4 t, g) \\ &= \sum_{(\sigma, \tau) \in \mathfrak{S}_n^{\star}} w(\sigma; \alpha, u_3 t, f) w(\tau; \beta, u_4 t, g), \end{aligned} \quad (2.11)$$

where π_0 (resp. π_1) the product of cycles in π with label 0 (resp. 1). The second equation follows from (2.7).

The mapping ρ . Let $\mathfrak{S}_n^{\star}$ denote the set of cycle-decorated permutations (π, h) of $[n]$ with a 2-coloring h of the cycles (equivalently, pairs (σ, τ) of disjoint-support permutations whose product is π , with the cycles of σ colored, say, red and those of τ blue). Define

$$\rho : \mathfrak{S}_n^{\star} \longrightarrow \mathfrak{S}_{n+1}, \quad (\sigma, \tau) \longmapsto \tilde{\pi} := \theta_1(\sigma) (n+1) \theta_2(\tau),$$

where θ_1 and θ_2 are the two transformations introduced in section 2.1.

Since σ and τ are permutations on disjoint subsets A and B of $[n]$ with $A \cup B = [n]$, and θ_1 and θ_2 act locally on each cycle, we may apply θ_1 and θ_2 separately to $\sigma \in \mathfrak{S}_A$ and $\tau \in \mathfrak{S}_B$. Clearly the mapping ρ is a bijection satisfying the following properties:

$$\text{cpk}(\sigma) + \text{cpk}(\tau) = \text{pk}(\tilde{\pi}) - 1, \quad (2.12a)$$

$$\text{cda}(\sigma) + \text{fix}(\sigma) + \text{cda}(\tau) = \text{da}(\tilde{\pi}), \quad (2.12b)$$

$$\text{cdd}(\sigma) + \text{cdd}(\tau) + \text{fix}(\tau) = \text{dd}(\tilde{\pi}), \quad (2.12c)$$

$$\text{fix}(\sigma) + \text{fix}(\tau) = \text{lmaxda}(\tilde{\pi}) + \text{rmaxdd}(\tilde{\pi}), \quad (2.12d)$$

$$\text{cyc}(\sigma) - \text{fix}(\sigma) = \text{lmaxpk}(\tilde{\pi}) - 1, \quad (2.12e)$$

$$\text{cyc}(\tau) - \text{fix}(\tau) = \text{rmaxpk}(\tilde{\pi}) - 1, \quad (2.12f)$$

$$\text{cyc}(\sigma) = \text{lmax}(\tilde{\pi}) - 1, \quad (2.12g)$$

$$\text{cyc}(\tau) = \text{rmax}(\tilde{\pi}) - 1. \quad (2.12h)$$

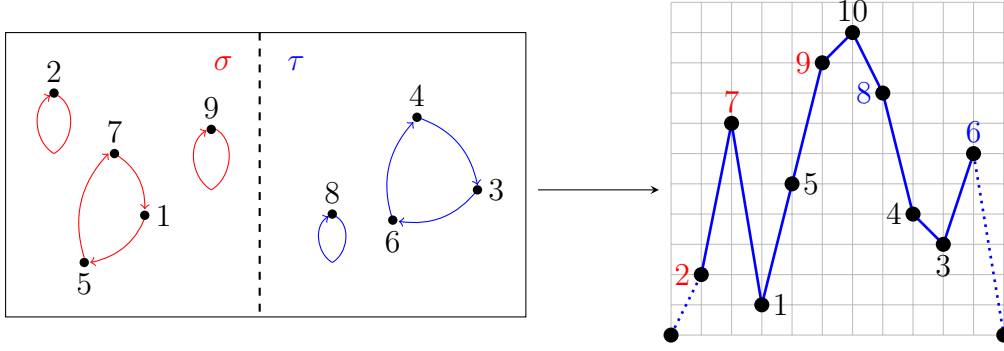


Figure 2.2: The mapping $\rho : (\sigma, \tau) \mapsto \tilde{\pi} = \textcolor{red}{27159} \textcolor{blue}{108436}$.

Since $\text{pk} - 1 = \text{val}$, combining (2.12) with (2.11) we obtain

$$\begin{aligned}
 \sum_{(\sigma, \tau) \in \mathfrak{S}_n^\star} w(\sigma; \alpha, u_3 t, f) w(\tau; \beta, u_4 t, g) \\
 &= \sum_{\tilde{\pi} \in \mathfrak{S}_{n+1}} (u_1 u_2)^{\text{val}(\tilde{\pi})} u_3^{\text{da}(\tilde{\pi})} u_4^{\text{dd}(\tilde{\pi})} f^{\text{lmaxpk}(\tilde{\pi})-1} g^{\text{rmaxpk}(\tilde{\pi})-1} \\
 &\quad \times t^{\text{lmaxda}(\tilde{\pi}) + \text{rmaxdd}(\tilde{\pi})} \alpha^{\text{lmax}(\tilde{\pi})-1} \beta^{\text{rmax}(\tilde{\pi})-1}.
 \end{aligned}$$

This proves (1.16) in view of (1.15). $\square$

We give an example to illustrate the construction of ρ by identifying each element $(\sigma, \tau) \in \mathfrak{S}_n^\star$ with a cycle-decorated permutation (π, h) , where $\pi \in \mathfrak{S}_n$ and h is a 2-coloring of the cycles of π . That is, for each cycle of π , the map h assigns a color in $\{\text{Red}, \text{Blue}\}$.

Example 2.6. Let $(\sigma, \tau) \in \mathfrak{S}_9^\star$ with

$$\sigma = (\textcolor{red}{2})(\textcolor{red}{9})(\textcolor{red}{571}), \quad \tau = (\textcolor{blue}{643})(8).$$

Then $\theta_1(\sigma) = 27159$ and $\theta_2(\tau) = 8436$. Hence

$$\tilde{\pi} = 27159 \textcolor{blue}{108436} \in \mathfrak{S}_{10},$$

as illustrated in Fig. 2.2. As a verification of (2.12), we obtain:

- $\text{cyc}(\sigma) = 3, \quad \text{cyc}(\tau) = 2;$
- $\text{cpk}(\sigma) = |\{7\}| = 1, \quad \text{cpk}(\tau) = |\{6\}| = 1;$
- $\text{cda}(\sigma) = |\{5\}| = 1, \quad \text{cdd}(\tau) = |\{4\}| = 1;$
- $\text{fix}(\sigma) = |\{2, 9\}| = 2, \quad \text{fix}(\tau) = |\{8\}| = 1;$
- $\text{pk}(\tilde{\pi}) = |\{6, 7, 10\}| = 3, \quad \text{val}(\tilde{\pi}) = |\{1, 3\}| = 2;$
- $\text{da}(\tilde{\pi}) = |\{2, 5, 9\}| = 3, \quad \text{dd}(\tilde{\pi}) = |\{4, 8\}| = 2;$

- $\text{lmax}(\tilde{\pi}) = |\{2, 7, 9, 10\}| = 4$, $\text{rmax}(\tilde{\pi}) = |\{6, 8, 10\}| = 3$;
- $\text{lmaxda}(\tilde{\pi}) = |\{2, 9\}| = 2$, $\text{rmaxdd}(\tilde{\pi}) = |\{8\}| = 1$;
- $\text{lmaxpk}(\tilde{\pi}) = |\{7, 10\}| = 2$, $\text{rmaxpk}(\tilde{\pi}) = |\{6, 10\}| = 2$.

3. Applications of the main theorems

In this section, we investigate a special case of $A_n(\mathbf{u}, f, g, t \mid \alpha, \beta)$ namely the (α, t) -binomial-Eulerian polynomials $\tilde{A}_n(x, y, t \mid \alpha)$ and present a symmetric (α, t) -Eulerian identity which is an (α, t) -analogue of Chung–Graham–Knuth’s identity (1.5). In the last two subsections, we further show that both the (α, t) -Eulerian polynomials and the (α, t) -binomial-Eulerian polynomials admit various γ -expansions. Moreover, we provide a combinatorial interpretation of the normalized γ -expansions of $A_n(x, y, t \mid \alpha)$ in terms of cycle André permutations as well as the André permutations of type 1 and type 2.

3.1. (α, t) -binomial-Eulerian polynomials

Define the (α, t) -binomial-Eulerian polynomials

$$\tilde{A}_n(x, y, t \mid \alpha) = A_n^{\text{cyc}}(x, y, (x + y)t \mid \alpha). \quad (3.1)$$

Proposition 3.1. *If $xy = u_1u_2$ and $x + y = u_3 + u_4$, then*

$$\tilde{A}_n(x, y, t \mid \alpha) = \sum_{\sigma \in \mathcal{M}_{n+1}} (u_1u_2)^{\text{val}(\sigma)} u_3^{\text{da}(\sigma)} u_4^{\text{dd}(\sigma)} t^{\text{lmaxda}(\sigma) + \text{rmaxdd}(\sigma)} \alpha^{\text{lmax}(\sigma) + \text{rmax}(\sigma) - 2}. \quad (3.2)$$

In particular, setting $u_1 = u_3 = x$ and $u_2 = u_4 = y$, we have

$$\tilde{A}_n(x, y, t \mid \alpha) = \sum_{\sigma \in \mathcal{M}_{n+1}} x^{\text{asc}(\sigma)} y^{\text{des}(\sigma)} t^{\text{lmaxda}(\sigma) + \text{rmaxdd}(\sigma)} \alpha^{\text{lmax}(\sigma) + \text{rmax}(\sigma) - 2}. \quad (3.3)$$

Proof. By Theorem 1.2 with $f = 0$, $g = 1$, and $\alpha = \beta$, we have

$$A_n^{\text{cyc}}(x, y, (x + y)t \mid \alpha) = A_n(\mathbf{u}, 0, 1, t \mid \alpha, \alpha). \quad (3.4)$$

It follows from (1.15) that $A_n(\mathbf{u}, 0, 1, t \mid \alpha)$ is the enumerative polynomial of permutations σ in $\mathfrak{S}_{n+1}$ such that $\text{lmaxpk}(\sigma) = 1$, this means that $n + 1$ is the unique left-to-right maximum and also a peak of σ , equivalently, $\sigma \in \mathcal{M}_{n+1}$. $\square$

Note that Proposition 3.1 provides a t -analogue of Ji–Lin’s Theorem 1.5 in [JL24]. The first four values of $\tilde{A}_n(x, y, t \mid \alpha)$ are as follows:

$$\begin{aligned} \tilde{A}_1(x, y, t \mid \alpha) &= \alpha t(x + y), \\ \tilde{A}_2(x, y, t \mid \alpha) &= \alpha^2 t^2(x^2 + y^2) + (2\alpha^2 t^2 + \alpha)xy, \\ \tilde{A}_3(x, y, t \mid \alpha) &= \alpha^3 t^3(x^3 + y^3) + (3\alpha^3 t^3 + 3\alpha^2 t + \alpha)(x^2 y + xy^2), \\ \tilde{A}_4(x, y, t \mid \alpha) &= \alpha^4 t^4(x^4 + y^4) + (4\alpha^4 t^4 + 6\alpha^3 t^2 + 4\alpha^2 t + \alpha)(x^3 y + xy^3) \\ &\quad + (6\alpha^4 t^4 + 12\alpha^3 t^2 + 8\alpha^2 t + 3\alpha^2 + 4\alpha)x^2 y^2. \end{aligned}$$

We need the following known properties of the fundamental transformation θ_2 in Section 2.1, see [Sta12, p. 30].

Lemma 3.2. *For $\sigma \in \mathfrak{S}_n$, we have*

- $\text{exc}(\sigma) = \text{asc}(\theta_2(\sigma));$
- $\text{fix}(\sigma) = \text{rmaxdd}(\theta_2(\sigma));$
- $\text{cyc}(\sigma) = \text{rmax}(\theta_2(\sigma)).$

Example 3.3. If $\sigma = 4271365$, then $\tilde{\sigma} = (5\ 3\ 7)(6)(1\ 4)(2)$ and $\sigma' := \theta_2(\sigma) = 5376142$. We verify that

- $\text{exc}(\sigma) = |\{1, 3\}| = 2, \text{fix}(\sigma) = |\{2, 6\}| = 2, \text{cyc}(\sigma) = 4;$
- $\text{asc}(\sigma') = |\{2, 5\}| = 2, \text{rmaxdd}(\sigma') = |\{2, 6\}| = 2, \text{rmax}(\sigma') = |\{2, 4, 6, 7\}| = 4.$

Define two kinds of (α, t) -Eulerian numbers as follows:

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\alpha, t}^{\text{exc}} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = n-k}} \alpha^{\text{cyc}(\sigma)} t^{\text{fix}(\sigma)} \quad (1 \leq k \leq n), \quad (3.5a)$$

and

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\alpha, t}^{\text{asc}} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{asc}(\sigma) = n-k}} \alpha^{\text{rmax}(\sigma)} t^{\text{rmaxdd}(\sigma)} \quad (1 \leq k \leq n). \quad (3.5b)$$

By Lemma 3.2, it is easy to see that the two (α, t) -Eulerian numbers are equal. Let

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\alpha, t} := \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\alpha, t}^{\text{exc}} = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\alpha, t}^{\text{asc}} \quad (3.6)$$

Theorem 3.4. *For integers $a, b \geq 0$, we have*

$$\sum_{k \geq 0} (\alpha t)^{a+b-k} \binom{a+b}{k} \left\langle \begin{matrix} k \\ a \end{matrix} \right\rangle_{\alpha, t} = \sum_{k \geq 0} (\alpha t)^{a+b-k} \binom{a+b}{k} \left\langle \begin{matrix} k \\ b \end{matrix} \right\rangle_{\alpha, t}, \quad (3.7)$$

where $\left\langle \begin{matrix} 0 \\ k \end{matrix} \right\rangle_{\alpha, t} = \left\langle \begin{matrix} k \\ 0 \end{matrix} \right\rangle_{\alpha, t} = \delta_{k,0}$.

Proof. We prove (3.7) with $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\alpha, t} = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\alpha, t}^{\text{exc}}$. As $\text{exc} + \text{drop} + \text{fix} = n$ over $\mathfrak{S}_n$, it follows from (1.6b) that

$$A_n^{\text{cyc}}(x, y, ty \mid \alpha) = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{exc}(\sigma)} y^{\text{drop}(\sigma) + \text{fix}(\sigma)} t^{\text{fix}(\sigma)} \alpha^{\text{cyc}(\sigma)} = \sum_{k=1}^n \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\alpha, t}^{\text{exc}} x^{n-k} y^k. \quad (3.8)$$

$n \backslash k$	1	2	3	4
1	αt			
2	α	$\alpha^2 t^2$		
3	α	$\alpha + 3\alpha^2 t$	$\alpha^3 t^3$	
4	α	$4\alpha + 3\alpha^2 + 4\alpha^2 t$	$\alpha + 4\alpha^2 t + 6\alpha^3 t^2$	$\alpha^4 t^4$

Table 3.1: The first values of $\langle \frac{n}{k} \rangle_{\alpha, t}$ for $1 \leq k \leq 4$.

By expanding $A_n^{\text{cyc}}(x, y, t(x+y) | \alpha)$ according to a partition of the fixed points of permutations in $\mathfrak{S}_n$ into two types, called *blue* and *red*, we deduce from (3.1) and (3.8) the following binomial identity:

$$\begin{aligned}
 \tilde{A}_n(x, y, t | \alpha) &= A_n^{\text{cyc}}(x, y, t(x+y) | \alpha) \\
 &= \sum_{m=0}^n \binom{n}{m} (\alpha t x)^{n-m} A_m^{\text{cyc}}(x, y, t y | \alpha) \\
 &= \sum_{m=0}^n \binom{n}{m} (\alpha t)^{n-m} \sum_{l=1}^m \langle \frac{m}{l} \rangle_{\alpha, t}^{\text{exc}} x^{n-l} y^l,
 \end{aligned} \tag{3.9}$$

where $\langle \frac{0}{k} \rangle_{\alpha, t}^{\text{exc}} = \langle \frac{k}{0} \rangle_{\alpha, t}^{\text{exc}} = \delta_{k,0}$.

The polynomial $A_n^{\text{cyc}}(x, y, t(x+y) | \alpha)$ is symmetric in x and y , because the inversion map $\vartheta : \sigma \mapsto \sigma^{-1}$ on $\mathfrak{S}_n$ interchanges the numbers of excedances and drops, while leaving the number of fixed points invariant, that is,

$$(\text{exc}, \text{drop}, \text{fix})(\sigma) = (\text{drop}, \text{exc}, \text{fix})(\sigma^{-1}). \tag{3.10}$$

Extracting the coefficients of $x^a y^b$ and $x^b y^a$ with $n = a + b$ in Eq. (3.9) we obtain Eq. (3.7) for $\langle \frac{n}{k} \rangle_{\alpha, t}^{\text{exc}}$. $\square$

Remark 3.5. The equidistribution (3.10) also follows from the γ -formula in Proposition 3.8. Identity (3.7) is an (α, t) -analogue of (1.5). When $t = 1$, by complement operation $\sigma \mapsto \sigma^c$, identity (3.7) with $\langle \frac{n}{k} \rangle_{\alpha, 1}^{\text{asc}}$ yields Ji and Lin's Theorem 4.1 in [JL24].

Remark 3.6. If $\sigma \in \mathfrak{S}_n$ with $\sigma_1 = n$, then $\text{lmaxda}(\sigma) = 0$ and $\text{lmax}(\sigma) = 1$. From (3.3), we have

$$\begin{aligned}
 \hat{A}_n(x, y, t | \alpha) &:= \sum_{\substack{\sigma \in \mathcal{M}_{n+1} \\ \sigma_1 = n+1}} x^{\text{asc}(\sigma)} y^{\text{des}(\sigma)} t^{\text{rmaxdd}(\sigma)} \alpha^{\text{rmax}(\sigma)-1} \\
 &= \sum_{\sigma \in \mathfrak{S}_n} x^{\text{asc}(\sigma)} y^{\text{des}(\sigma)+1} t^{\text{rmaxdd}(\sigma)} \alpha^{\text{rmax}(\sigma)} \\
 &= \sum_{k=1}^n \langle \frac{n}{k} \rangle_{\alpha, t}^{\text{asc}} x^{n-k+1} y^k.
 \end{aligned} \tag{3.11}$$

A similar proof of (3.7) with $\langle \frac{n}{k} \rangle_{\alpha, t}^{\text{asc}}$ can be given using (3.11).

3.2. γ -positivity of (α, t) -Eulerian polynomials

In this section, we investigate the γ -coefficients of the two families of (α, t) -Eulerian polynomials, namely

$$\tilde{A}_n(x, y, t | \alpha) \quad \text{and} \quad A_n(x, y, t | \alpha).$$

We shall show that their γ -coefficients are polynomials in $\mathbb{N}[\alpha, t]$ by providing several combinatorial interpretations. Moreover, we derive explicit expressions for their exponential generating functions.

Lemma 3.7. *For any variable f , we have*

$$\begin{aligned} A_n(x, y, 0, x + y, f, 1, t | \alpha, \alpha) &= \sum_{\substack{\sigma \in \mathfrak{S}_{n+1} \\ \text{da}(\sigma)=0}} (xy)^{\text{asc}(\sigma)} (x + y)^{n-2\text{asc}(\sigma)} f^{\text{lmaxpk}(\sigma)-1} t^{\text{rmaxdd}(\sigma)} \alpha^{\text{lmax}(\sigma)+\text{rmax}(\sigma)-2} \end{aligned} \quad (3.12a)$$

$$= \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{cda}(\sigma)=0}} (xy)^{\text{exc}(\sigma)} (x + y)^{n-2\text{exc}(\sigma)} t^{\text{fix}(\sigma)} \alpha^{\text{cyc}(\sigma)} (f + 1)^{\text{cyc}(\sigma)-\text{fix}(\sigma)}. \quad (3.12b)$$

$$\begin{aligned} A_n(x, y, x + y, 0, f, 1, t | \alpha, \alpha) &= \sum_{\substack{\sigma \in \mathfrak{S}_{n+1} \\ \text{dd}(\sigma)=0}} (xy)^{\text{des}(\sigma)} (x + y)^{n-2\text{des}(\sigma)} f^{\text{lmaxpk}(\sigma)-1} t^{\text{lmaxda}(\sigma)} \alpha^{\text{lmax}(\sigma)+\text{rmax}(\sigma)-2} \end{aligned} \quad (3.13a)$$

$$= \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{cdd}(\sigma)=0}} (xy)^{\text{drop}(\sigma)} (x + y)^{n-2\text{drop}(\sigma)} t^{\text{fix}(\sigma)} \alpha^{\text{cyc}(\sigma)} (f + 1)^{\text{cyc}(\sigma)-\text{fix}(\sigma)}. \quad (3.13b)$$

$$\begin{aligned} A_n\left(1, xy, \frac{x+y}{2}, \frac{x+y}{2}, f, 1, t | \alpha, \alpha\right) &= \sum_{\sigma \in \mathfrak{S}_{n+1}} (xy)^{\text{val}(\sigma)} \left(\frac{x+y}{2}\right)^{n-2\text{val}(\sigma)} f^{\text{lmaxpk}(\sigma)-1} t^{\text{lmaxda}(\sigma)+\text{rmaxdd}(\sigma)} \alpha^{\text{lmax}(\sigma)+\text{rmax}(\sigma)-2} \end{aligned} \quad (3.14a)$$

$$= \sum_{\sigma \in \mathfrak{S}_n} (xy)^{\text{cpk}(\sigma)} \left(\frac{x+y}{2}\right)^{n-2\text{cpk}(\sigma)} (2t)^{\text{fix}(\sigma)} \alpha^{\text{cyc}(\sigma)} (f + 1)^{\text{cyc}(\sigma)-\text{fix}(\sigma)}. \quad (3.14b)$$

Proof. First of all, by definition (see Section 1), it is easy to verify the following identities:

$$\text{val} = \text{pk} - 1, \text{asc} = \text{val} + \text{da}, \text{des} = \text{val} + \text{dd}; \quad (3.15a)$$

$$\text{cval} = \text{cpk}, \text{exc} = \text{cval} + \text{cda}, \text{drop} = \text{cpk} + \text{cdd}; \quad (3.15b)$$

$$\text{fix} + \text{cdd} + \text{cda} = n - \text{cpk} - \text{cval} = n - 2\text{cpk}. \quad (3.15c)$$

When $\alpha = \beta$ and $g = 1$, Theorem 1.2 (with $xy = u_1 u_2$ and $x + y = u_3 + u_4$) becomes

$$A_n(u_1, u_2, u_3, u_4, f, 1, t | \alpha, \alpha) = A_n^{\text{cyc}}\left(x, y, \frac{x+y}{f+1} t | \alpha(f+1)\right). \quad (3.16)$$

- (i) Let $\mathbf{u} = (x, y, 0, x + y)$ in Eq. (3.16), then the left-hand side becomes $A_n(x, y, 0, x + y, f, 1, t \mid \alpha, \alpha)$, which is equal to (3.12a) by definition (1.15) and (3.15), while the right-hand side of Eq. (3.16) becomes (3.12b) by Theorem 2.4.
- (ii) Let $\mathbf{u} = (x, y, x + y, 0)$ in Eq. (3.16), similarly, we prove Eq. (3.13a) and (3.13b).
- (iii) Let $\mathbf{u} = (1, xy, \frac{x+y}{2}, \frac{x+y}{2})$ in Eq. (3.16), then the left-hand side becomes

$$A_n \left(1, xy, \frac{x+y}{2}, \frac{x+y}{2}, f, 1, t \mid \alpha, \alpha \right),$$

which is equal to (3.14a) by definition (1.15) and (3.15), while the right-hand side of (3.16) becomes (3.14b) by Theorem 2.4. $\square$

For convenience of description of the γ -coefficients, we shall define three types of subsets of $\mathfrak{S}_n$ with respect to cyclic statistics and linear statistics of permutations, respectively.

The cycle type subsets of $\mathfrak{S}_n$:

$$\mathfrak{S}_{n, \text{exc}=j}^{\text{cda}=0} := \{\sigma \in \mathfrak{S}_n : \text{cda}(\sigma) = 0 \text{ and } \text{exc}(\sigma) = j\}; \quad (3.17a)$$

$$\mathfrak{S}_{n, \text{drop}=j}^{\text{cdd}=0} := \{\sigma \in \mathfrak{S}_n : \text{cdd}(\sigma) = 0 \text{ and } \text{drop}(\sigma) = j\}; \quad (3.17b)$$

$$\mathfrak{S}_n^{\text{cpk}=j} := \{\sigma \in \mathfrak{S}_n : \text{cpk}(\sigma) = j\}. \quad (3.17c)$$

The linear counterpart subsets of $\mathfrak{S}_n$:

$$\mathfrak{S}_{n, \text{asc}=j}^{\text{da}=0} := \{\sigma \in \mathfrak{S}_n : \text{da}(\sigma) = 0 \text{ and } \text{asc}(\sigma) = j\}; \quad (3.18a)$$

$$\mathfrak{S}_{n, \text{des}=j}^{\text{dd}=0} := \{\sigma \in \mathfrak{S}_n : \text{dd}(\sigma) = 0 \text{ and } \text{des}(\sigma) = j\}; \quad (3.18b)$$

$$\mathfrak{S}_n^{\text{pk}=j} := \{\sigma \in \mathfrak{S}_n : \text{pk}(\sigma) = j\}; \quad (3.18c)$$

and the subsets of $\mathcal{M}_n$:

$$\mathcal{M}_{n, \text{asc}=j}^{\text{da}=0} := \{\sigma \in \mathcal{M}_n : \text{da}(\sigma) = 0 \text{ and } \text{asc}(\sigma) = j\}; \quad (3.19a)$$

$$\mathcal{M}_{n, \text{des}=j}^{\text{dd}=0} := \{\sigma \in \mathcal{M}_n : \text{dd}(\sigma) = 0 \text{ and } \text{des}(\sigma) = j\}; \quad (3.19b)$$

$$\mathcal{M}_n^{\text{pk}=j} := \{\sigma \in \mathcal{M}_n : \text{pk}(\sigma) = j\}. \quad (3.19c)$$

In view of (3.1), we see that $\tilde{A}_n(x, y, t \mid \alpha)$ is symmetric in x and y , and the first few expansions in elementary symmetric polynomials are as follows:

$$\tilde{A}_1(x, y, t \mid \alpha) = \alpha t(x + y),$$

$$\tilde{A}_2(x, y, t \mid \alpha) = \alpha^2 t^2 (x + y)^2 + \alpha xy,$$

$$\tilde{A}_3(x, y, t \mid \alpha) = \alpha^3 t^3 (x + y)^3 + (\alpha + 3\alpha^2) t xy(x + y),$$

$$\tilde{A}_4(x, y, t \mid \alpha) = \alpha^4 t^4 (x + y)^4 + (6\alpha^3 + 4\alpha^2 + \alpha) t^2 xy(x + y)^2 + (3\alpha^2 + 2\alpha)(xy)^2.$$

From Lemma 3.7, we derive the following combinatorial interpretations of the coefficients in the expansion in elementary symmetric polynomials of $\tilde{A}_n(x, y, t \mid \alpha)$ in terms of linear and cyclic statistics.

Proposition 3.8. *For $n \geq 1$, we have*

$$\tilde{A}_n(x, y, t | \alpha) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \tilde{\gamma}_{n,j}(\alpha, t) (xy)^j (x+y)^{n-2j}, \quad (3.20)$$

where for $0 \leq j \leq \lfloor n/2 \rfloor$,

$$\tilde{\gamma}_{n,j}(\alpha, t) = \sum_{\sigma \in \mathcal{M}_{n+1, \text{asc}=j}^{\text{da}=0}} \alpha^{\text{rmax}(\sigma)-1} t^{\text{rmaxdd}(\sigma)} \quad (3.21a)$$

$$= \sum_{\sigma \in \mathfrak{S}_{n, \text{exc}=j}^{\text{cda}=0}} \alpha^{\text{cyc}(\sigma)} t^{\text{fix}(\sigma)} \quad (3.21b)$$

$$= \sum_{\sigma \in \mathcal{M}_{n+1, \text{des}=j}^{\text{dd}=0}} \alpha^{\text{lmax}(\sigma)+\text{rmax}(\sigma)-2} t^{\text{lmaxda}(\sigma)} \quad (3.21c)$$

$$= \sum_{\sigma \in \mathfrak{S}_{n, \text{drop}=j}^{\text{cdd}=0}} \alpha^{\text{cyc}(\sigma)} t^{\text{fix}(\sigma)} \quad (3.21d)$$

$$= 2^{2j-n} \sum_{\sigma \in \mathcal{M}_{n+1}^{\text{pk}=j}} \alpha^{\text{lmax}(\sigma)+\text{rmax}(\sigma)-2} t^{\text{lmaxda}(\sigma)+\text{rmaxdd}(\sigma)} \quad (3.21e)$$

$$= 2^{2j-n} \sum_{\sigma \in \mathfrak{S}_n^{\text{cpk}=j}} \alpha^{\text{cyc}(\sigma)} (2t)^{\text{fix}(\sigma)}. \quad (3.21f)$$

Proof. Recall (see (3.1)) that

$$\tilde{A}_n(x, y, t | \alpha) = A_n^{\text{cyc}}(x, y, (x+y)t | \alpha). \quad (3.22)$$

From the above formula and Eq. (3.16) with $f = 1$, $u_1 u_2 = xy$ and $u_3 + u_4 = x + y$, we obtain

$$\tilde{A}_n(x, y, t | \alpha) = A_n(u_1, u_2, u_3, u_4, 0, 1, t | \alpha, \alpha). \quad (3.23)$$

Combining (3.23) with $u_3 = 0$ (resp. $u_4 = 0$) and (3.12) (resp. (3.13)) with $f = 0$, we obtain the twin interpretations (3.21a) and (3.21b) (resp. (3.21c) and (3.21d)). Note that we obtain (3.21a) by using the fact that if $\sigma \in \mathcal{M}_n$ and $\text{da}(\sigma) = 0$, then $\text{lmax}(\sigma) = 1$.

Similarly, let $u_1 = 1$, $u_2 = xy$ and $u_3 = u_4 = \frac{x+y}{2}$ in (3.23), we obtain

$$\tilde{A}_n(x, y, t | \alpha) = A_n\left(1, xy, \frac{x+y}{2}, \frac{x+y}{2}, 0, 1, t | \alpha, \alpha\right). \quad (3.24)$$

Combining (3.24) and (3.14), we obtain the twin interpretations (3.21e) and (3.21f). $\square$

Remark 3.9. Previously, Shin–Zeng [SZ12, Theorem 11] proved the $t = 0$ case of (3.21b), i.e.,

$$A_n^{\text{cyc}}(x, y, 0 | \alpha) = \sum_{\sigma \in \mathfrak{S}_{n, \text{fix}=0}^{\text{cda}=0}} \alpha^{\text{cyc}(\sigma)} (xy)^{\text{exc}(\sigma)} (x+y)^{n-2\text{exc}(\sigma)}, \quad (3.25)$$

where $\mathfrak{S}_{n, \text{fix}=0}^{\text{cda}=0} := \{\sigma \in \mathfrak{S}_n : \text{cda}(\sigma) = 0 \text{ and } \text{fix}(\sigma) = 0\}$.

In fact, it is easy to derive the general case (3.21b) from (3.25) as in the following. Counting the permutations in $\mathfrak{S}_n$ by the number of fixed points and plugging (3.25), we obtain

$$\begin{aligned} A_n^{\text{cyc}}(x, y, t(x+y) \mid \alpha) &= \sum_{k=0}^n \binom{n}{k} (\alpha t(x+y))^k A_{n-k}^{\text{cyc}}(x, y, 0 \mid \alpha) \\ &= \sum_{k=0}^n \binom{n}{k} (\alpha t)^k \sum_{\sigma \in \mathfrak{S}_{n-k, \text{fix}=0}^{\text{cda}=0}} \alpha^{\text{cyc}(\sigma)} (xy)^{\text{exc}(\sigma)} (x+y)^{n-2\text{exc}(\sigma)} \\ &= \sum_{j=0}^{\lfloor n/2 \rfloor} \left(\sum_{\sigma \in \mathfrak{S}_{n, \text{exc}=j}^{\text{cda}=0}} \alpha^{\text{cyc}(\sigma)} t^{\text{fix}(\sigma)} \right) (xy)^j (x+y)^{n-2j}. \end{aligned} \quad (3.26)$$

An equivalent form of (3.26) appeared in [MMYY24, Theorem 3.4] and [PS24, Theorem 7]. One can also derive this result from [CJZ20, Theorem 6]. When $t = 1$, by complement operation $\sigma \mapsto \sigma^c$, the three combinatorial interpretations (3.21a), (3.21c) and (3.21e) yield the three combinatorial interpretations of the γ -coefficients in [JL24, Theorem 2.1].

From Lemma 3.7, we obtain the following combinatorial interpretations of the coefficients in the expansion of $A_n(x, y, t \mid \alpha) = A_n^{\text{cyc}}(x, y, \frac{x+y}{2}t \mid 2\alpha)$ in elementary symmetric polynomials, expressed in terms of linear and cyclic statistics.

Proposition 3.10. *For $n \geq 0$, we have*

$$A_n(x, y, t \mid \alpha) = \sum_{j=0}^{\lfloor n/2 \rfloor} \gamma_{n,j}(\alpha, t) (xy)^j (x+y)^{n-2j}, \quad (3.27)$$

where for $0 \leq j \leq \lfloor n/2 \rfloor$,

$$\gamma_{n,j}(\alpha, t) = \sum_{\sigma \in \mathfrak{S}_{n+1, \text{asc}=j}^{\text{da}=0}} \alpha^{\text{lmax}(\sigma) + \text{rmax}(\sigma) - 2t^{\text{rmaxdd}(\sigma)}} \quad (3.28a)$$

$$= \sum_{\sigma \in \mathfrak{S}_{n, \text{exc}=j}^{\text{cda}=0}} 2^{\text{cyc}(\sigma) - \text{fix}(\sigma)} \alpha^{\text{cyc}(\sigma)} t^{\text{fix}(\sigma)} \quad (3.28b)$$

$$= \sum_{\sigma \in \mathfrak{S}_{n+1, \text{des}=j}^{\text{dd}=0}} \alpha^{\text{lmax}(\sigma) + \text{rmax}(\sigma) - 2t^{\text{lmaxda}(\sigma)}} \quad (3.28c)$$

$$= \sum_{\sigma \in \mathfrak{S}_{n, \text{drop}=j}^{\text{cdd}=0}} 2^{\text{cyc}(\sigma) - \text{fix}(\sigma)} \alpha^{\text{cyc}(\sigma)} t^{\text{fix}(\sigma)} \quad (3.28d)$$

$$= 2^{2j-n} \sum_{\sigma \in \mathfrak{S}_{n+1}^{\text{pk}=j+1}} \alpha^{\text{lmax}(\sigma) + \text{rmax}(\sigma) - 2t^{\text{lmaxda}(\sigma) + \text{rmaxdd}(\sigma)}} \quad (3.28e)$$

$$= 2^{2j-n} \sum_{\sigma \in \mathfrak{S}_n^{\text{cpk}=j}} (2\alpha)^{\text{cyc}(\sigma)} t^{\text{fix}(\sigma)}. \quad (3.28f)$$

$n \setminus j$	0	1	2
0	1		
1	αt		
2	$\alpha^2 t^2$	2α	
3	$\alpha^3 t^3$	$2\alpha + 6\alpha^2 t$	
4	$\alpha^4 t^4$	$2\alpha + 8\alpha^2 t + 12\alpha^3 t^2$	$4\alpha + 12\alpha^2$
5	$\alpha^5 t^5$	$2\alpha + 10\alpha^2 t + 20\alpha^3 t^2 + 20\alpha^4 t^3$	$16\alpha + 40\alpha^2 + 20\alpha^2 t + 60\alpha^3 t$

Table 3.2: The first values of $\gamma_{n,j}(\alpha, t)$ for $0 \leq 2j < n \leq 5$.

Proof. Recall (see (1.21)) that

$$A_n(x, y, t \mid \alpha) = A_n^{\text{cyc}} \left(x, y, \frac{x+y}{2} t \mid 2\alpha \right).$$

From the above formula and Eq. (3.16) with $f = 1$, $u_1 u_2 = xy$ and $u_3 + u_4 = x + y$, we derive

$$A_n(x, y, t \mid \alpha) = A_n(u_1, u_2, u_3, u_4, 1, 1, t \mid \alpha, \alpha). \quad (3.29)$$

Combining (3.29) with $u_3 = 0$ (resp. $u_4 = 0$) and (3.12) (resp. (3.13)) with $f = 1$, we obtain the twin interpretations (3.28a) and (3.28b) (resp. (3.28c) and (3.28d)). Specializing equation (3.29) with $u_1 = 1$, $u_2 = xy$ and $u_3 = u_4 = \frac{x+y}{2}$, we obtain

$$A_n(x, y, t \mid \alpha) = A_n \left(1, xy, \frac{x+y}{2}, \frac{x+y}{2}, 1, 1, t \mid \alpha, \alpha \right). \quad (3.30)$$

Combining (3.30) and (3.14), we obtain the twin interpretations (3.28e) and (3.28f). $\square$

The first few values of $\gamma_{n,j}(\alpha, t)$ are given in Table 3.2.

Theorem 3.11. For $n \geq 0$, we have

$$A_n \left(x, y, t \mid \frac{\alpha + \beta}{2} \right) = \sum_{j=0}^{\lfloor n/2 \rfloor} \gamma_{n,j} \left(\frac{\alpha + \beta}{2}, t \right) (xy)^j (x+y)^{n-2j}, \quad (3.31a)$$

where for $0 \leq j \leq \lfloor n/2 \rfloor$,

$$\gamma_{n,j} \left(\frac{\alpha + \beta}{2}, t \right) = 2^{2j-n} \sum_{\sigma \in \mathfrak{S}_{n+1}^{\text{pk}=j+1}} \alpha^{\text{lmax}(\sigma)-1} \beta^{\text{rmax}(\sigma)-1} t^{\text{lmaxda}(\sigma) + \text{rmaxdd}(\sigma)}. \quad (3.31b)$$

Proof. By (1.21), we have

$$A_n \left(x, y, t \mid \frac{\alpha + \beta}{2} \right) = A_n^{\text{cyc}} \left(x, y, \frac{x+y}{2} t \mid \alpha + \beta \right). \quad (3.32)$$

Let $\mathbf{u} = (1, xy, \frac{x+y}{2}, \frac{x+y}{2})$, it follows from Theorem 1.2 and definition (1.15) that

$$\begin{aligned} & A_n^{\text{cyc}}\left(x, y, \frac{x+y}{2}t \mid \alpha + \beta\right) \\ &= A_n\left(1, xy, \frac{x+y}{2}, \frac{x+y}{2}, 1, 1, t \mid \alpha, \beta\right) \\ &= \sum_{\sigma \in \mathfrak{S}_{n+1}} (xy)^{\text{val}(\sigma)} (x+y)^{n-2\text{val}(\sigma)} t^{\text{lmaxda}(\sigma)+\text{rmaxdd}(\sigma)} \alpha^{\text{lmax}(\sigma)-1} \beta^{\text{rmax}(\sigma)-1} 2^{2\text{val}(\sigma)-n}. \end{aligned} \quad (3.33)$$

As $\text{val} = \text{pk} - 1$, combining (3.32) and (3.33) yields (3.31b). $\square$

Remark 3.12. Theorem 3.11 provides a t -analogue of Ji's Theorem 1.8 [Ji25]. Setting $\alpha = \beta$ in equation (3.31b) we obtain (3.28e). Combining (3.28e) with $\alpha \rightarrow \frac{\alpha+\beta}{2}$ and (3.31b) we obtain

$$\begin{aligned} & \sum_{\sigma \in \mathfrak{S}_{n+1}^{\text{pk}=j+1}} (2\alpha)^{\text{lmax}(\sigma)-1} (2\beta)^{\text{rmax}(\sigma)-1} t^{\text{lmaxda}(\sigma)+\text{rmaxdd}(\sigma)} \\ &= \sum_{\sigma \in \mathfrak{S}_{n+1}^{\text{pk}=j+1}} (\alpha + \beta)^{\text{lmax}(\sigma)+\text{rmax}(\sigma)-2} t^{\text{lmaxda}(\sigma)+\text{rmaxdd}(\sigma)}. \end{aligned} \quad (3.34)$$

This identity is far from trivial. When $t = 1$, Dong, Lin and Pan [DLP24, Theorem 1.5] provided a combinatorial proof of Eq. (3.34). Finding a purely combinatorial explanation of Eq. (3.34) would be of independent interest.

Define the generating polynomials of the two kinds of γ -vectors in (3.20) and (3.27)

$$G_n^{\text{cyc}}(x, t, \alpha) := \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \tilde{\gamma}_{n,j}(\alpha, t) x^j = \sum_{\sigma \in \mathfrak{S}_n^{\text{cda}=0}} x^{\text{exc}(\sigma)} t^{\text{fix}(\sigma)} \alpha^{\text{cyc}(\sigma)}, \quad (3.35a)$$

$$G_n(x, t, \alpha) := \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,j}(\alpha, t) x^j = \sum_{\sigma \in \mathfrak{S}_{n+1}^{\text{da}=0}} x^{\text{asc}(\sigma)} \alpha^{\text{lmax}(\sigma)+\text{rmax}(\sigma)-2} t^{\text{rmaxdd}(\sigma)}, \quad (3.35b)$$

where

$$\mathfrak{S}_n^{\text{cda}=0} := \{\sigma \in \mathfrak{S}_n : \text{cda}(\sigma) = 0\} \quad \text{and} \quad \mathfrak{S}_n^{\text{da}=0} := \{\sigma \in \mathfrak{S}_n : \text{da}(\sigma) = 0\}. \quad (3.35c)$$

Theorem 3.13. Let $u = \sqrt{1 - 4x}$. We have

$$1 + \sum_{n \geq 1} G_n^{\text{cyc}}(x, t, \alpha) \frac{z^n}{n!} = \left(\frac{u e^{(t-\frac{1}{2})z}}{u \cosh(uz/2) - \sinh(uz/2)} \right)^\alpha, \quad (3.36a)$$

$$1 + \sum_{n \geq 1} G_n(x, t, \alpha) \frac{z^n}{n!} = \left(\frac{u e^{\frac{1}{2}(t-1)z}}{u \cosh(uz/2) - \sinh(uz/2)} \right)^{2\alpha}. \quad (3.36b)$$

Proof. With $y = 1$ and $s = \frac{x}{(1+x)^2}$, the two γ -expansions (3.20) and (3.27) can be written, respectively, as follows,

$$A_n^{\text{cyc}}(x, 1, t(x+1) | \alpha) = (1+x)^n G_n^{\text{cyc}}(s, t, \alpha), \quad (3.37a)$$

$$A_n(x, 1, t | \alpha) = (1+x)^n G_n(s, t, \alpha). \quad (3.37b)$$

Plugging (3.37a) in (1.7a) yields

$$1 + \sum_{n \geq 1} G_n^{\text{cyc}}(s, t, \alpha) \frac{z^n}{n!} = \left(\frac{(x-1)e^{tz}}{xe^{z/(1+x)} - e^{xz/(1+x)}} \right)^\alpha, \quad (3.37c)$$

After a little computation, the right-hand sides of (3.36a) and (3.37c) agree under substitution $u = \frac{1-x}{1+x} = \sqrt{1-4s}$.

From (1.17) with $u_1 = u_3 = x$, $u_2 = u_4 = f = g = 1$, we derive

$$1 + \sum_{n \geq 1} A_n(x, 1, t | \alpha) \frac{z^n}{n!} = e^{\alpha(x+1)tz} \left(\frac{x-1}{xe^z - e^{xz}} \right)^{2\alpha}. \quad (3.37d)$$

Formula (3.36b) can be proved similarly by plugging (3.37b) in (3.37d). $\square$

Remark 3.14. Formula (3.36a) appeared in [MMYY24, Eq. (13)] with a different proof, and (3.36b) is a t -analogue of Carlitz and Scoville's formula [CS74, Eq. (4.10)].

3.3. γ -vector of (α, t) -Eulerian polynomials and cycle André permutations

For $n \geq 0$ and $0 \leq j \leq \lfloor n/2 \rfloor$, let

$$d_{n,j}(\alpha, t) = \gamma_{n,j}(\alpha, t)/2^j. \quad (3.38)$$

Then, Eq. (3.27) reads

$$A_n(x, y, t | \alpha) = \sum_{j=0}^{\lfloor n/2 \rfloor} 2^j d_{n,j}(\alpha, t) (xy)^j (x+y)^{n-2j}. \quad (3.39)$$

From Table 3.2 we derive the first values of $d_{n,j}(\alpha, t)$ in Table 3.3. A glance at these values foretells that the polynomials $d_{n,j}(\alpha, t)$ have nonnegative integral coefficients. Indeed, from Corollary 1.5, we derive the continued fraction

$$\sum_{n=0}^{\infty} \sum_{j=0}^{\lfloor n/2 \rfloor} d_{n,j}(\alpha, t) x^j z^n = \frac{1}{1 - b_0 z - \frac{\lambda_1 z^2}{1 - b_1 z - \frac{\lambda_2 z^2}{1 - b_2 z - \dots}}}, \quad (3.40a)$$

$n \setminus j$	0	1	2
0	1		
1	αt		
2	$\alpha^2 t^2$	α	
3	$\alpha^3 t^3$	$\alpha + 3\alpha^2 t$	
4	$\alpha^4 t^4$	$\alpha + 4\alpha^2 t + 6\alpha^3 t^2$	$\alpha + 3\alpha^2$
5	$\alpha^5 t^5$	$\alpha + 5\alpha^2 t + 10\alpha^3 t^2 + 10\alpha^4 t^3$	$4\alpha + 10\alpha^2 + 5\alpha^2 t + 15\alpha^3 t$

Table 3.3: The values of $d_{n,j}(\alpha, t)$ for $0 \leq 2j \leq n \leq 5$.

where

$$b_k = k + \alpha t, \quad (3.40b)$$

$$\lambda_{k+1} = \binom{k+1}{2} x + \alpha(k+1)x \quad (k \geq 0). \quad (3.40c)$$

It follows that the polynomials $d_{n,j}(\alpha, t)$ are elements in $\mathbb{N}[\alpha, t]$. The aim of this section is to provide three combinatorial interpretations in terms of cycle André and André permutations.

Note that when $x = \alpha = t = 1$ Eq.(3.39) is equivalent to the γ -expansion of Eulerian polynomials (1.4) and $d_{n,j}(1, 1)$ is the number of André permutations of $[n+1]$ with j descents.

There are many equivalent ways to define an André permutation. Following Foata–Strehl [FS74] and Heteyi–Reiner [HR98], we define André permutations in terms of x -factorization.

Definition 3.15. Let $\sigma = \sigma_1 \sigma_2 \dots \sigma_n \in \mathfrak{S}_n$ and let $x = \sigma_i$ for $i \in [n]$. The x -factorization of σ is given by $\sigma = u \lambda(x) x \rho(x) v$, where

- (1) $\lambda(x) = \sigma_j \dots \sigma_{i-1}$ with $\sigma_l > x$ for $j \leq l \leq i-1$, $u = \sigma_1 \dots \sigma_{j-1}$, and $\sigma_{j-1} < x$.
- (2) $\rho(x) = \sigma_{i+1} \dots \sigma_k$ with $\sigma_l > x$ for $i+1 \leq l \leq k$, $v = \sigma_{k+1} \dots \sigma_n$, and $\sigma_{k+1} < x$.

Here, any of words $u, \lambda(x), \rho(x), v$ may be the empty.

Example 3.16. If $\sigma = 8\ 10\ 14\ 5\ 11\ 3\ 6\ 12\ 2\ 7\ 9 \in \mathfrak{S}_{12}$, then, the 3-factorization of σ is

$$\underbrace{8\ 10\ 1}_u \underbrace{4\ 5\ 11}_{\lambda(3)} 3 \underbrace{6\ 12}_{\rho(3)} \underbrace{2\ 7\ 9}_v.$$

Definition 3.17 ([HR98]). Let $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$. Say that σ is an **André permutation of the first kind** (resp. **second kind**) if σ has no double descents, i.e., $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$, and each x -factorisation $u \lambda(x) x \rho(x) v$ of σ has property

- $\lambda(x) = \emptyset$ if $\rho(x) = \emptyset$,
- $\max(\lambda(x)) < \max(\rho(x))$ (resp. $\min(\rho(x)) < \min(\lambda(x))$) if $\rho(x) \neq \emptyset$ and $\lambda(x) \neq \emptyset$.

$n = 1:$	1;	$n = 2:$	12;
$n = 3:$	123, 213;		
$n = 4:$	1234, 1324, 2314, 2134, 3124;		
$n = 5:$	12345, 12435, 13425, 23415, 13245, 14235, 34125, 24135,		
	23145, 21345, 41235, 31245, 21435, 32415, 41325, 31425.		

Table 3.4: André permutations of the first kind of length ≤ 5 .

$n = 1:$	1;	$n = 2:$	12;
$n = 3:$	123, 312;		
$n = 4:$	1234, 1423, 3412, 4123, 3124;		
$n = 5:$	12345, 12534, 14523, 34512, 15234, 14235, 34125, 45123,		
	35124, 51234, 41235, 31245, 51423, 53412, 41523, 31524.		

Table 3.5: André permutations of the second kind of length ≤ 5 .

Let $\mathcal{A}_n^1$ (resp. $\mathcal{A}_n^2$) be the set of André permutations of the first (resp. second) kind in $\mathfrak{S}_n$. For $n \in [5]$, these permutations are given in Tables 3.4 and 3.5. It is known [FS70] that the cardinality of $\mathcal{A}_n^1$ (resp. $\mathcal{A}_n^2$) is the *Euler number* E_n , which is the n th coefficient in the Taylor expansion of $\sec(z) + \tan(z)$, namely,

$$\sum_{n \geq 0} E_n \frac{z^n}{n!} = \sec(z) + \tan(z). \quad (3.41)$$

The sequence E_n 's starts with 1, 1, 2, 5, 16, 61, 271, $\dots$, see [Slo, A138265].

Let $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$ with $\sigma_0 = \sigma_{n+1} = 0$. A right-to-left minimum (**rmin**) of σ is an element σ_i such that $\sigma_j > \sigma_i$ if $n \geq j > i \geq 1$. A letter $\sigma_i \in [n]$ is a right-to-left-minimum-double-ascent (**rminda**) if it is a double ascent and also a right-to-left minimum. Hwang et al. [HJO23] introduced cycle André permutations to characterise the so-called *Web permutations*.

Definition 3.18 ([HJO23]). Let $A := \{a_1, \dots, a_k\}$ be a set of k positive integers. Let $C = (a_1 \cdots a_k)$ be a cycle (cyclic permutation) of A with $a_1 = \min\{a_1, \dots, a_k\}$. Then, cycle C is called an **André cycle** if the word $a_2 \cdots a_k$ is an André permutation of the first kind. We say that a permutation is a **cycle André permutation** if it is a product of disjoint André cycles.

Let $\mathcal{CA}_n$ be the set of cycle André permutations of $[n]$. For example, $(1\ 5\ 6)(2\ 7\ 4\ 8)(3)$ is a cycle André permutation, but $(1\ 2\ 5\ 3)(4)$ is not because the $\rho(3) = \emptyset$ and $\lambda(3) = 5$ in 3-factorisation of 253.

Theorem 3.19. For $n \geq 0$ and $0 \leq j \leq \lfloor n/2 \rfloor$, the normalized γ -coefficient $d_{n,j}(\alpha, t)$ in (3.39) has the following combinatorial interpretations

$$d_{n,j}(\alpha, t) = \sum_{\substack{\sigma \in \mathcal{CA}_n \\ \text{drop}(\sigma)=j}} t^{\text{fix}(\sigma)} \alpha^{\text{cyc}(\sigma)} \quad (3.42a)$$

$$= \sum_{\substack{\sigma \in \mathcal{A}_{n+1}^1 \\ \text{des}(\sigma)=j}} t^{\text{rminda}(\sigma)} \alpha^{\text{rmin}(\sigma)-1} \quad (3.42b)$$

$$= \sum_{\substack{\sigma \in \mathcal{A}_{n+1}^2 \\ \text{des}(\sigma)=j}} t^{\text{rminda}(\sigma)} \alpha^{\text{rmin}(\sigma)-1}. \quad (3.42c)$$

Theorem 1.7 follows from Theorem 3.19 together with (3.39). We shall prove Theorem 3.19 in the next section. Our proof proceeds as follows. First, we establish (3.42a) by showing that the exponential generating functions of both sides coincide. Next, we deduce (3.42b) from (3.42a) by constructing a bijection $\Phi : \mathcal{CA}_n \rightarrow \mathcal{A}_{n+1}^1$. Finally, we derive (3.42c) from (3.42b) by means of a bijection $\Psi : \mathcal{A}_{n+1}^1 \rightarrow \mathcal{A}_{n+1}^2$ built upon André trees.

4. Cycle André and André permutations: Proof of Theorem 3.19

4.1. Counting André permutations of type 1 by des and lmax

When $x = t = 1$, let $\gamma_{n,j}(\alpha) := \gamma_{n,j}(\alpha, 1)$ and

$$A_{n,k}(\alpha) := \sum_{\substack{\sigma \in \mathfrak{S}_{n+1} \\ \text{des}(\sigma)=k}} \alpha^{\text{lmax}(\sigma)+\text{rmax}(\sigma)-2}, \quad (4.1a)$$

then, Eq. (3.27) can be rephrased as

$$\sum_{k=0}^n A_{n,k}(\alpha) y^k = \sum_{j=0}^{\lfloor n/2 \rfloor} \gamma_{n,j}(\alpha) y^j (1+y)^{n-2j}. \quad (4.1b)$$

By considering the position of the letter 1 in permutations of $\mathfrak{S}_{n+2}$, one readily verifies that the coefficients $A_{n,k}(\alpha)$ obey the following recurrence relation:

$$A_{n+1,k}(\alpha) = (k + \alpha) A_{n,k}(\alpha) + (n + 1 - k + \alpha) A_{n,k-1}(\alpha), \quad (4.1c)$$

where $A_{0,0}(\alpha) = 1$ and $A_{0,-1}(\alpha) = 0$, for $0 \leq k \leq n$.

Lemma 4.1. For $n \geq 0$ and $0 \leq j \leq \lfloor n/2 \rfloor$, the γ -coefficients $\gamma_{n,j}(\alpha)$ satisfy the recurrence relation

$$\gamma_{n+1,j}(\alpha) = (\alpha + j) \gamma_{n,j}(\alpha) + 2(n + 2 - 2j) \gamma_{n,j-1}(\alpha), \quad (4.2)$$

with $\gamma_{0,0}(\alpha) = 1$ and $\gamma_{0,-1}(\alpha) = 0$. Equivalently, setting $d_{n+1,j}(\alpha) = \gamma_{n+1,j}(\alpha)/2^j$, then

$$d_{n+1,j}(\alpha) = (\alpha + j) d_{n,j}(\alpha) + (n + 2 - 2j) d_{n,j-1}(\alpha), \quad (4.3)$$

with $d_{0,0}(\alpha) = 1$ and $d_{0,-1}(\alpha) = 0$.

Proof. We verify recurrence (4.2) by induction on n . Since $\gamma_{2,0}(\alpha) = \alpha^2$, $\gamma_{1,0}(\alpha) = \alpha$ and $\gamma_{1,-1}(\alpha) = 0$, this is obvious when $n = 1$. If $n \geq 2$ and $0 \leq k \leq n$, Eq. (4.1b) is equivalent to

$$A_{n,k}(\alpha) = \sum_{0 \leq j \leq k} \binom{n-2j}{k-j} \gamma_{n,j}(\alpha). \quad (4.4)$$

Substituting (4.4) in (4.1c) and using (4.2), we derive

$$\begin{aligned} & \sum_{0 \leq j \leq k} \binom{n+1-2j}{k-j} \left((\alpha+j) \gamma_{n,j}(\alpha) + 2(n+2-2j) \gamma_{n,j-1}(\alpha) \right) \\ &= (k+\alpha) \sum_{0 \leq j \leq k} \binom{n-2j}{k-j} \gamma_{n,j}(\alpha) + (n+1-k+\alpha) \sum_{0 \leq j \leq k-1} \binom{n-2j}{k-1-j} \gamma_{n,j}(\alpha). \end{aligned} \quad (4.5)$$

Extracting the coefficients of $\gamma_{n,j}(\alpha)$ we obtain:

$$\begin{aligned} & (\alpha+j) \binom{n+1-2j}{k-j} + 2(n-2j) \binom{n-1-2j}{k-j-1} \\ &= (k+\alpha) \binom{n-2j}{k-j} + (n+1-k+\alpha) \binom{n-2j}{k-1-j}, \end{aligned} \quad (4.6)$$

which can be verified by direct algebraic manipulation. $\square$

Lemma 4.2. For $n \geq 0$, we have

$$\sum_{j=0}^{\lfloor n/2 \rfloor} d_{n,j}(\alpha) x^j = \sum_{\sigma \in \mathcal{A}_{n+1}^1} x^{\text{des}(\sigma)} \alpha^{\text{lmax}(\sigma)-1}. \quad (4.7)$$

Proof. Let

$$\sum_{\sigma \in \mathcal{A}_{n+1}^1} x^{\text{des}(\sigma)} \alpha^{\text{lmax}(\sigma)-1} = \sum_{j=0}^{\lfloor n/2 \rfloor} d_{n,j}^*(\alpha) x^j. \quad (4.8)$$

We shall prove that $d_{n,j}^*(\alpha)$ satisfies the same recurrence relation (4.3), i.e.,

$$d_{n+1,j}^*(\alpha) = (\alpha+j) d_{n,j}^*(\alpha) + (n+2-2j) d_{n,j-1}^*(\alpha). \quad (4.9)$$

The left-hand side of equation (4.9) counts permutations $\sigma \in \mathcal{A}_{n+2}^1$ with j descents considering the weights of $\alpha^{\text{lmax}-1}$. Deleting the letter 1 in $\sigma \in \mathcal{A}_{n+2}^1$ yields also an André permutation of $\{2, 3, \dots, n+2\}$. Thus, we can obtain such a permutation uniquely in one of two ways. For the first way, choose a permutation $\sigma = \sigma_1 \cdots \sigma_{n+1} \in \mathcal{A}_E^1$ with j descents where $E = \{2, 3, \dots, n+2\}$, and insert 1 after σ_i if i is descent, or insert 1 at the beginning. Note that letter 1 will contribute a left-to-right maximum at the beginning, while in other positions not. Thus, there are $j+1$ ways to insert 1, so we obtain by this factor $(\alpha+j) d_{n,j}^*(\alpha)$. For the second way, choose $\sigma = \sigma_1 \cdots \sigma_{n+1} \in \mathcal{A}_E^1$ with $j-1$ descents where $E = \{2, 3, \dots, n+2\}$, and insert 1 after σ_i if i is an ascent and $i-1$ is not a descent. Since an André permutation has no double descents, thus there are $n-2(j-1)$ ways to insert 1, so we obtain the second factor $(n+2-2j) d_{n,j-1}^*(\alpha)$. $\square$

Combining Eq. (4.2), (4.3) and Lemma 4.2, we derive the following exponential generating function from (3.36b).

Theorem 4.3. *We have*

$$\sum_{n \geq 0} \sum_{\sigma \in \mathcal{A}_{n+1}^1} x^{\text{des}(\sigma)} \alpha^{\text{lmax}(\sigma)-1} \frac{z^n}{n!} = \left(\frac{\sqrt{2x-1} \sec(\frac{z}{2}\sqrt{2x-1})}{\sqrt{2x-1} - \tan(\frac{z}{2}\sqrt{2x-1})} \right)^{2\alpha}. \quad (4.10)$$

Remark 4.4. When $\alpha = 1$, (4.10) was first proved by Foata and Schützenberger [FS73], other proofs are given by Foata and Han [FH01], and Chow and Shiu [CS11, Theorem 2.1].

Remark 4.5. It would be interesting to extend the above approach by incorporating a suitable parameter t .

4.2. Counting André cycles by drop

The following is an alternative definition of André permutations of the first kind.

Lemma 4.6 ([FH16]). *Say that the empty word e and each one-letter word are both André permutations of the first kind. Next, if $w = w_1 w_2 \cdots w_n$ ($n \geq 2$) is a permutation of a set of positive integers $Y = \{y_1 < y_2 < \cdots < y_n\}$, write $w = v \min(w) v'$. Then, w is an André permutation of the first kind, if both v and v' are André permutation of the first kind, with $\max(v) < \max(v')$.*

Define the exponential generating function

$$E(x; z) := \sum_{n \geq 0} \left(\sum_{\sigma \in \mathcal{A}_{n+1}^1} x^{\text{des}(\sigma)} \right) \frac{z^n}{n!}. \quad (4.11a)$$

Then, Eq. (4.10) with $\alpha = 1$ yields

$$E(x; z) = \left(\frac{\sqrt{2x-1} \sec(\frac{z}{2}\sqrt{2x-1})}{\sqrt{2x-1} - \tan(\frac{z}{2}\sqrt{2x-1})} \right)^2. \quad (4.11b)$$

We give a combinatorial proof of the following lemma, which can also be derived from (4.11b).

Lemma 4.7. *We have*

$$E(x; z) = \exp \left(z + x \int_0^z \left(\int_0^v E(x; u) du \right) dv \right). \quad (4.11c)$$

Proof. Let

$$E_n(x) := \sum_{\sigma \in \mathcal{A}_n^1} x^{\text{des}(\sigma)}. \quad (4.12a)$$

By Lemma 4.6, any permutation $w \in \mathcal{A}_{n+2}^1$ can be written as $w = v1v'$ where v and v' are André permutations of the first kind. Let $j+1$ be the position of 1. Then, classifying the permutations in $\mathcal{A}_{n+2}^1$ by the index $j \in \{0, 1, \dots, n\}$, we have

$$E_{n+2}(x) = E_{n+1}(x) + x \sum_{j=1}^n \binom{n}{j} E_j(x) E_{n+1-j}(x), \quad (4.12b)$$

in other words,

$$\sum_{n \geq 0} E_{n+2}(x) \frac{z^n}{n!} = \left(\sum_{n \geq 0} E_{n+1}(x) \frac{z^n}{n!} \right) \left(1 + x \sum_{n \geq 1} E_n(x) \frac{z^n}{n!} \right). \quad (4.12c)$$

As $E(x; z) = \sum_{n \geq 0} E_{n+1}(x) \frac{z^n}{n!}$, the above equation can be written as

$$\frac{\partial E(x; z)}{\partial z} = E(x; z) \left(1 + x \int_0^z E(x; u) du \right), \quad (4.12d)$$

which is the equation obtained by differentiating the two sides of (4.11c) with respect to z . As the two sides of (4.11c) agree at $z = 0$, we are done. $\square$

Let $\mathcal{AC}_n$ be the set of André cycles of $[n]$, see Definition 3.18.

Lemma 4.8. *We have*

$$\sum_{n \geq 2} \sum_{\sigma \in \mathcal{AC}_n} x^{\text{drop}(\sigma)} \frac{z^n}{n!} = x \int_0^z \left(\int_0^t E(x; u) du \right) dt. \quad (4.13)$$

Proof. For $\sigma = (1 a_2 \dots a_{n-1} n) \in \mathcal{AC}_n$ with $n \geq 2$, let $\zeta(\sigma) = (a_2 - 1)(a_3 - 1) \dots (n - 1) \in \mathcal{A}_{n-1}^1$. It is clear that the mapping $\zeta : \mathcal{AC}_n \rightarrow \mathcal{A}_{n-1}^1$ is a bijection satisfying

$$\text{drop}(\sigma) = \text{des}(\zeta(\sigma)) + 1. \quad (4.14)$$

Therefore,

$$\begin{aligned} \sum_{n \geq 2} \sum_{\sigma \in \mathcal{AC}_n} x^{\text{drop}(\sigma)} \frac{z^n}{n!} &= \sum_{n \geq 2} \sum_{\sigma \in \mathcal{A}_{n-1}^1} x^{\text{des}(\sigma) + 1} \frac{z^n}{n!} \\ &= x \sum_{n \geq 0} E_{n+1}(x) \frac{z^{n+2}}{(n+2)!}. \end{aligned} \quad (4.15)$$

By (4.11a), the last series is equal to the right-hand side of (4.13). $\square$

4.3. Proof of Theorem 3.19 (3.42a)

By Lemma 4.8 we have

$$\begin{aligned} \sum_{n \geq 1} \sum_{\sigma \in \mathcal{AC}_n} x^{\text{drop}(\sigma)} t^{\text{fix}(\sigma)} \frac{z^n}{n!} &= tz + \sum_{n \geq 2} \sum_{\sigma \in \mathcal{AC}_n} x^{\text{drop}(\sigma)} \frac{z^n}{n!} \\ &= (t-1)z + \left(z + x \int_0^z \left(\int_0^t E(x; u) du \right) dt \right). \end{aligned} \quad (4.16)$$

Recall that a cycle André permutation is a product of disjoint André cycles, and the set of cycle André permutations of $[n]$ is denoted by $\mathcal{CA}_n$. Combining (4.16), the exponential formula and Lemma 4.7 we obtain

$$\sum_{n \geq 0} \sum_{\sigma \in \mathcal{CA}_n} x^{\text{drop}(\sigma)} t^{\text{fix}(\sigma)} \alpha^{\text{cyc}(\sigma)} \frac{z^n}{n!} = e^{\alpha(t-1)z} (E(x; z))^\alpha. \quad (4.17)$$

Substituting (4.11b) in (4.17) yields

$$\sum_{n \geq 0} \sum_{\sigma \in \mathcal{CA}_n} x^{\text{drop}(\sigma)} t^{\text{fix}(\sigma)} \alpha^{\text{cyc}(\sigma)} \frac{z^n}{n!} = e^{(t-1)\alpha z} \left(\frac{\sqrt{2x-1} \sec(\frac{z}{2}\sqrt{2x-1})}{\sqrt{2x-1} - \tan(\frac{z}{2}\sqrt{2x-1})} \right)^{2\alpha}. \quad (4.18)$$

On the other hand, from Proposition 3.10 and Theorem 3.13 we derive the generating function

$$\sum_{n \geq 0} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} d_{n,j}(\alpha, t) x^j \frac{z^n}{n!} = e^{(t-1)\alpha z} \left(\frac{\sqrt{2x-1} \sec(\frac{z}{2}\sqrt{2x-1})}{\sqrt{2x-1} - \tan(\frac{z}{2}\sqrt{2x-1})} \right)^{2\alpha}. \quad (4.19)$$

Eq. (3.42a) then follows by comparing the above two generating functions. $\square$

4.4. Proof of Theorem 3.19 (3.42b)

We define a map $\phi : \mathcal{CA}_n \rightarrow \mathcal{A}_{n+1}^1$ as follows: let $\sigma = C_1 C_2 \dots C_l \in \mathcal{CA}_n$, where $C_1, \dots, C_l$ are André cycles ordered from left to right in increasing order of their smallest letters, which are at the end of each cycle; erasing the parentheses in σ and appending letter $n+1$ to the end results in $\phi(\sigma)$.

It is not difficult to verify that ϕ is a bijection (See, for example, the proof of Theorem 4.1 in Hwang–Jang–Oh [HJO23]). Moreover, similar to Lemma 3.2, we have

$$(\text{drop}, \text{fix}, \text{cyc}) \sigma = (\text{des}, \text{rminda}, \text{rmin} - 1) \phi(\sigma). \quad (4.20)$$

Thus, combining (4.20) with (3.42a) we prove (3.42b). $\square$

Example 4.9. If $\sigma = (5\ 6\ 1)(7\ 4\ 8\ 2)(3) \in \mathcal{CA}_8$, then $\sigma' := \phi(\sigma) = 561748239 \in \mathcal{A}_9^1$. Moreover,

- $\text{drop}(\sigma) = |\{6, 7, 8\}| = 3$, $\text{fix}(\sigma) = |\{3\}| = 1$ and $\text{cyc}(\sigma) = 3$;
- $\text{des}(\sigma') = |\{2, 4, 6\}| = 3$, $\text{rminda}(\sigma') = |\{3\}| = 1$ and $\text{rmin}(\sigma') = |\{1, 2, 3, 9\}| = 4$.

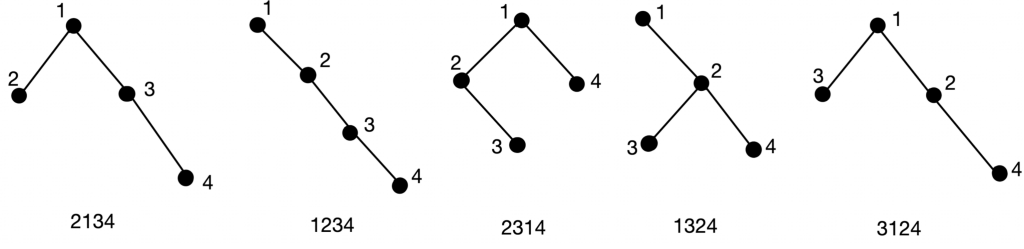


Figure 4.1: The André trees in $\mathcal{T}_4^1$ and associated André permutations in $\mathcal{A}_4^1$.

4.5. Proof of Theorem 3.19 (3.42c)

We construct a bijection $\Phi : \mathcal{A}_n^1 \rightarrow \mathcal{A}_n^2$, which preserves the triple statistic $(\text{des}, \text{rminda}, \text{rmin})$.

4.5.1 Binary and André trees

A binary tree T on a finite ordered set S is (recursively) defined as follows:

- if $S = \emptyset$, then $T = \emptyset$,
- otherwise, T is a triple (T_1, r, T_2) with $r \in S$ and T_1 and T_2 are binary trees on S_1 and S_2 , respectively such that $(\{r\}, S_1, S_2)$ is a partition of S (possibly with empty S_1 or S_2).

For the binary tree $T = (T_1, r, T_2)$ on S , the element r is called *root* of the tree, S the set of *nodes* or *vertices* of T , T_1 (resp. T_2) is the left (resp. right) subtree of T . A binary tree is called an *increasing binary tree*, if the vertices along any path from the root are increasing. A vertex of T without successor is called *leaf*, otherwise it is an *interior vertex*. Let $\text{leaf}(T)$ be the number of leaves of T .

Definition 4.10 ([Dis13]). An increasing binary tree T on a finite ordered set is an **André tree of the first (resp. second) kind**, if

- no vertex of T has only left successor;
- and if x is an interior vertex of T with left successor y and right successor z , then

$$\max(T(y)) < \max(T(z)) \quad (\text{resp. } y > z),$$

where $T(a)$ denotes the subtree rooted at a and $\max(T(a))$ is the largest vertex of $T(a)$.

The set of André trees of the first kind (resp. second kind) on $[n]$ is denoted $\mathcal{T}_n^1$ (resp. $\mathcal{T}_n^2$).

There is a well-known mapping sending permutations to increasing binary trees [Sta12, Section 1.5], which we recall below. Let $\sigma = \sigma_1 \cdots \sigma_n$ be a permutation of $[n]$. We define inductively a binary tree T_σ as follows. Let $x = \sigma_i$ be the smallest letter of σ , and factorize

$$\sigma = u x v, \quad \text{where } u = \sigma_1 \cdots \sigma_{i-1} \quad \text{and} \quad v = \sigma_{i+1} \cdots \sigma_n.$$

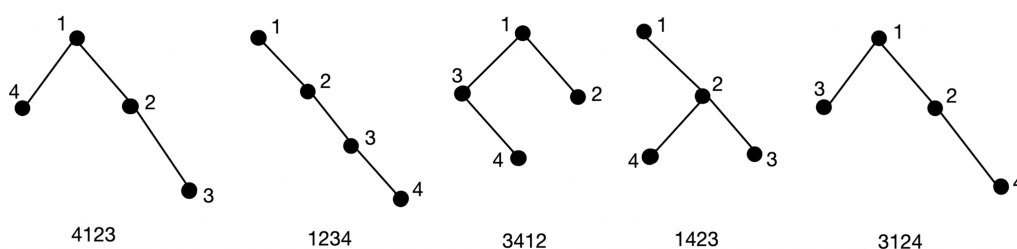


Figure 4.2: The André trees in $\mathcal{T}_4^2$ and associated André permutations in $\mathcal{A}_4^2$.

Then x is taken as the root of T_σ , the left subtree T_u is constructed recursively from u , and the right subtree T_v from v . If σ is empty, we define T_σ to be the empty tree.

The mapping $\omega : \sigma \mapsto T_\sigma$ is thus a bijection from $\mathfrak{S}_n$ onto the set of increasing binary trees on $[n]$. To recover σ from T_σ , one performs a *symmetric order traversal* (or *inorder traversal*), that is, a recursive visit of the vertices of T_σ defined as follows: first visit in symmetric order the left subtree of the root (if it exists); then visit the root itself; and finally visit in symmetric order the right subtree of the root (if it exists).

By restricting this mapping on André permutations, we obtain a bijection from André permutations to André trees:

$$\omega : \mathcal{A}_n^i \rightarrow \mathcal{T}_n^i, \quad i = 1, 2. \quad (4.21)$$

Example 4.11. The André trees in $\mathcal{T}_4^1$ and $\mathcal{T}_4^2$ and their associated permutations are given in Figs. 4.1 and 4.2, respectively.

Given an increasing binary tree T , the *right face* of T is the path from the root to a leaf by performing only right-steps. Denote by $\text{rface}(T)$ the number of vertices in the right face of T . Let $\text{rface}'(T)$ be the number of interior vertices having only a right successor in the right face of T . For example, in Fig. 4.3, we have $\text{rface}'(T_\sigma) = |\{4, 8\}| = 2$ and $\text{rface}(T_\sigma) = |\{1, 3, 4, 5, 8, 14\}| = 6$.

It is straightforward to verify the following properties, some of which can be found in [Sta12, Section 1.5].

Lemma 4.12. Let $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$ and $T_\sigma := \omega(\sigma)$, with $\sigma_0 = \sigma_{n+1} = 0$. Then,

- letter i is a double ascent of σ , iff vertex i of T_σ has only a right successor;
- letter i is a double descent of σ , iff vertex i of T_σ has only a left successor;
- letter i is a valley of σ , iff vertex i of T_σ has left and right successors;
- letter i is a peak of σ , iff vertex i of T_σ is a leaf;
- letter i is a right-to-left minimum of σ , iff vertex i is in the right face of T_σ .

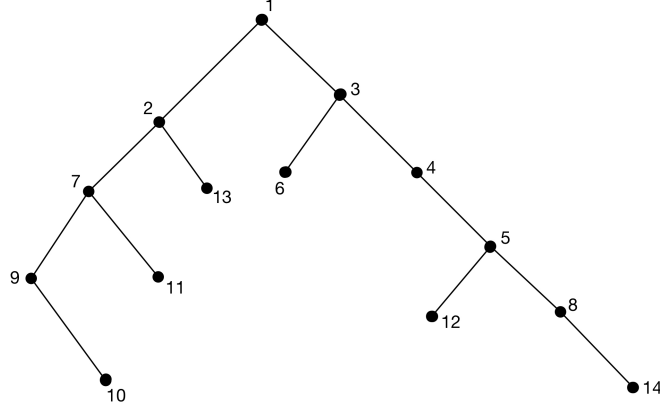


Figure 4.3: The binary tree T_σ with $\sigma = 9\ 10\ 7\ 11\ 2\ 13\ 1\ 6\ 3\ 4\ 12\ 5\ 8\ 14 \in \mathcal{A}_{14}^1$.

Proposition 4.13. For $i = 1, 2$ and $\sigma \in \mathcal{A}_n^i$ with $T_\sigma = \omega(\sigma)$, the following relation holds:

$$(\text{des}, \text{rminda}, \text{rmin}) \sigma = (\text{leaf} - 1, \text{rface}', \text{rface}) T_\sigma. \quad (4.22)$$

Proof. In Definition 3.17, we define André permutations by the so-called x -factorizations of permutations. We only prove the $i = 1$ case. If $\sigma \in \mathcal{A}_n^1$, then $\text{dd}(\sigma) = 0$, so (see (1.9))

$$\text{des}(\sigma) = \text{val}(\sigma) + \text{dd}(\sigma) = \text{pk}(\sigma) - 1 = \text{leaf}(T_\sigma) - 1;$$

besides, it follows from Lemma 4.12 that $\text{rmin}(\sigma) = \text{rface}(T_\sigma)$ and $\text{rminda}(\sigma) = \text{rface}'(T_\sigma)$. $\square$

Example 4.14. If $\sigma = 9\ 10\ 7\ 11\ 2\ 13\ 1\ 6\ 3\ 4\ 12\ 5\ 8\ 14 \in \mathcal{A}_{14}^1$, then the binary tree T_σ is given in Fig. 4.3. Moreover,

- $\text{des}(\sigma) = |\{2, 4, 6, 8, 11\}| = 5$,
 $\text{rminda}(\sigma) = |\{4, 8\}| = 2$, and
 $\text{rmin}(\sigma) = |\{1, 3, 4, 5, 8, 14\}| = 6$;
- $\text{leaf}(T_\sigma) = |\{6, 10, 11, 12, 13, 14\}| = 6$,
 $\text{rface}'(T_\sigma) = |\{4, 8\}| = 2$, and
 $\text{rface}(T_\sigma) = |\{1, 3, 4, 5, 8, 14\}| = 6$.

4.5.2 Construction of bijection Φ .

Let $\mathcal{T}_n$ be the set of all increasing binary trees on $[n]$. First, for $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$, let $T_\sigma = \omega(\sigma) \in \mathcal{T}_n$ and define

$$S_{T_\sigma} = \{i : \text{the vertex } \sigma_i \text{ has two successors in } T_\sigma\}. \quad (4.23)$$

Let $i \in S_{T_\sigma}$, and l_{σ_i} and r_{σ_i} denote the left and right successors of σ_i , respectively. Define the operator ϕ_i on T_σ as follows:

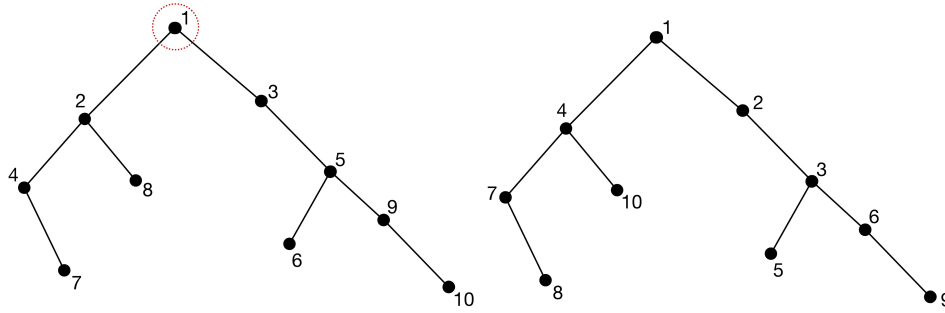


Figure 4.4: Two increasing binary trees T_σ with $\sigma = 47281365910$ (Left) and $\phi_5(T_\sigma) = T_\tau$ with $\tau = 78410125369$.

- if $l_{\sigma_i} > r_{\sigma_i}$, then $\phi_i T_\sigma = T_\sigma$;
- if $l_{\sigma_i} < r_{\sigma_i}$, then exchange l_{σ_i} with $\max(T(r_{\sigma_i}))$ to obtain a new tree T' , and then rearrange the vertices of $T'(l_{\sigma_i})$ and $T'(r_{\sigma_i})$ so that each subtree preserves its internal relative order.

Example 4.15. If $\sigma = 47281365910 \in \mathfrak{S}_{10}$, then $T_\sigma = \omega(\sigma)$ is shown at the left of Fig. 4.4. As $\sigma_5 = 1$ and $l_1 = 2 < r_1 = 3$, we have $\phi_5(T_\sigma) = T_\tau$ with $\tau = 78410125369$.

Lemma 4.16. Let $T_\sigma \in \mathcal{T}_n$ be an increasing tree with $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$ and let $S_{T_\sigma} = \{i : \text{the vertex } \sigma_i \text{ has two successors in } T_\sigma\}$. Then, for any $i, j \in S_{T_\sigma}$, the operators ϕ_i and ϕ_j commute, namely,

$$\phi_i(\phi_j(T_\sigma)) = \phi_j(\phi_i(T_\sigma)). \quad (4.24)$$

Proof. If ϕ_i (or ϕ_j) is an identity, then Eq. (4.24) is trivial. Otherwise, without loss of generality, we assume that $\sigma_i > \sigma_j$. If σ_i is not a descendant of σ_j , then ϕ_i and ϕ_j operate on two disjoint vertex subsets of T_σ and the commutativity is obvious. We suppose that σ_i is a descendant of l_{σ_j} , and

- the vertex set of $T(l_{\sigma_j})$ (resp. $T(r_{\sigma_j})$) is $A_{l_{\sigma_j}} := \{a_1 < a_2 < \dots < a_m\}$ (resp. $B_{r_{\sigma_j}} := \{b_1 < b_2 < \dots < b_l\}$) with $a_1 = l_{\sigma_j}$ (resp. $b_1 = r_{\sigma_j}$);
- the vertex set of $T(l_{\sigma_i})$ (resp. $T(r_{\sigma_i})$) is $A_{l_{\sigma_i}} := \{a_{t_1} < a_{t_2} < \dots < a_{t_p}\}$ (resp. $A_{r_{\sigma_i}} := \{a_{h_1} < a_{h_2} < \dots < a_{h_q}\}$) with $a_{t_1} = l_{\sigma_i}$ (resp. $a_{h_1} = r_{\sigma_i}$). Note that these two sets are subsets of $A_{l_{\sigma_j}}$.

1. We firstly consider $\phi_i(\phi_j(T_\sigma))$. By definition, the vertices of the $T(l_{\sigma_j})$ and $T(r_{\sigma_j})$ are changed according to the following substitutions,

$$\begin{array}{cccccccccccc} a_1 & a_2 & a_3 & \cdots & a_{m-1} & a_m & & b_1 & b_2 & b_3 & \cdots & b_{l-1} & b_l \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \text{and} & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\ a_2 & a_3 & a_4 & \cdots & a_m & b_l & & a_1 & b_1 & b_2 & \cdots & b_{l-2} & b_{l-1} \end{array} \quad (4.25)$$

In particular, as $A_{l_{\sigma_i}}$ and $A_{r_{\sigma_i}}$ are subsets of $A_{l_{\sigma_j}}$, under ϕ_j the vertices of $T(l_{\sigma_i})$ and $T(r_{\sigma_i})$ are changed as follows,

$$\begin{array}{ccccccccc} a_{t_1} & a_{t_2} & \cdots & a_{t_{p-1}} & a_{t_p} & & a_{h_1} & a_{h_2} & \cdots & a_{h_{q-1}} & a_{h_q} \\ \downarrow & \downarrow & \cdots & \downarrow & \downarrow & & \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\ a_{t_1+1} & a_{t_2+1} & \cdots & a_{t_{p-1}+1} & a_{t_p+1} & \text{and} & a_{h_1+1} & a_{h_2+1} & \cdots & a_{h_{p-1}+1} & a_{h_q+1} \end{array}, \quad (4.26)$$

where if $a_{h_q} = a_m$, then $a_{h_q+1} = b_l$. Next, proceeding by ϕ_i , the vertices of $\phi_j(T(l_{\sigma_i}))$ and $\phi_j(T(r_{\sigma_i}))$ are changed according to

$$\begin{array}{ccccccccc} a_{t_1+1} & a_{t_2+1} & \cdots & a_{t_{p-1}+1} & a_{t_p+1} & & a_{h_1+1} & a_{h_2+1} & \cdots & a_{h_{q-1}+1} & a_{h_q+1} \\ \downarrow & \downarrow & \cdots & \downarrow & \downarrow & & \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\ a_{t_2+1} & a_{t_3+1} & \cdots & a_{t_p+1} & a_{h_q+1} & \text{and} & a_{t_1+1} & a_{h_1+1} & \cdots & a_{h_{q-2}+1} & a_{h_{q-1}+1} \end{array}. \quad (4.27)$$

2. Now, we consider $\phi_j(\phi_i(T_\sigma))$. Under ϕ_i , the vertices of $T(l_{\sigma_i})$ and $T(r_{\sigma_i})$ are changed according to the following substitutions,

$$\begin{array}{ccccccccc} a_{t_1} & a_{t_2} & \cdots & a_{t_{p-1}} & a_{t_p} & & a_{h_1} & a_{h_2} & \cdots & a_{h_{q-1}} & a_{h_q} \\ \downarrow & \downarrow & \cdots & \downarrow & \downarrow & & \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\ a_{t_2} & a_{t_3} & \cdots & a_{t_p} & a_{h_q} & \text{and} & a_{t_1} & a_{h_1} & \cdots & a_{h_{q-2}} & a_{h_{q-1}} \end{array}. \quad (4.28)$$

Note that the remaining vertices are not changed under ϕ_i . Next, we proceed by ϕ_j , the vertices of $\phi_i(T(l_{\sigma_j}))$ and $\phi_i(T(r_{\sigma_j}))$ are changed by (4.25), then the vertices of $\phi_i(T(l_{\sigma_i}))$ and $\phi_i(T(r_{\sigma_i}))$ are changed as follows

$$\begin{array}{ccccccccc} a_{t_2} & a_{t_3} & \cdots & a_{t_p} & a_{h_q} & & a_{t_1} & a_{h_1} & \cdots & a_{h_{q-2}} & a_{h_{q-1}} \\ \downarrow & \downarrow & \cdots & \downarrow & \downarrow & & \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\ a_{t_2+1} & a_{t_3+1} & \cdots & a_{t_p+1} & a_{h_q+1} & \text{and} & a_{t_1+1} & a_{h_1+1} & \cdots & a_{h_{q-2}+1} & a_{h_{q-1}+1} \end{array}. \quad (4.29)$$

Comparing the bottom rows (4.27) and (4.29), we prove (4.24). $\square$

By lemma 4.16, we can define the operator $\Psi := \prod_{i \in S_T} \phi_i$ for any $T \in \mathcal{T}_n^1$.

Lemma 4.17. *The operator Ψ is a bijection from $\mathcal{T}_n^1$ to $\mathcal{T}_n^2$. Moreover, if $T_\sigma = \omega(\sigma)$ with $\sigma \in \mathcal{A}_n^1$, then*

$$(\text{leaf}, \text{rfac}'', \text{rfac}) T_\sigma = (\text{leaf}, \text{rfac}'', \text{rfac}) \Psi(T_\sigma). \quad (4.30)$$

Proof. By Definition 4.10 and the construction of Ψ , for each interior vertex of $\Psi(T_\sigma)$ with two successors, its left successor is greater than its right successor. Hence $\Psi(T_\sigma) \in \mathcal{T}_n^2$. For $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$, let $T_\sigma = \omega(\sigma) \in \mathcal{T}_n$ and define

$$S_{T_\sigma} = \{i : \text{the vertex } \sigma_i \text{ has two successors in } T_\sigma\}. \quad (4.31)$$

If $i \in S_{T_\sigma}$, let l_{σ_i} and r_{σ_i} be the left and right successors of σ_i , respectively. We define the operator ϕ_i^{-1} on the vertices of T_σ as follows:

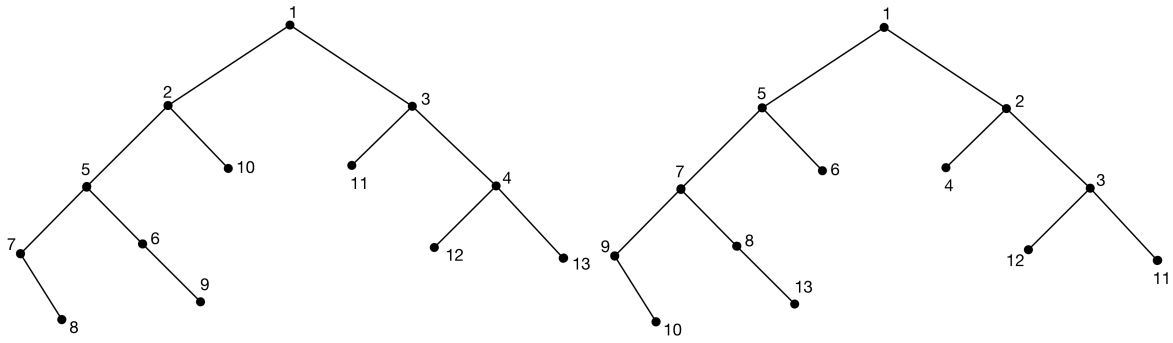


Figure 4.5: (Left) the André tree $T_\sigma \in \mathcal{T}_{13}^1$ with associated $\sigma = 78569210111312413 \in \mathcal{A}_{13}^1$; (Right) the transformed tree $T_\tau \in \mathcal{T}_{13}^2$ under $\Psi = \prod_{i \in S_{T_\sigma}} \phi_i$, where $\tau = 91078135614211312$ and $S_{T_\sigma} = \{3, 6, 8, 10, 12\}$.

- if $\max(T(l_{\sigma_i})) < \max(T(r_{\sigma_i}))$, then $\phi_i^{-1}T_\sigma = T_\sigma$;
- if $\max(T(l_{\sigma_i})) > \max(T(r_{\sigma_i}))$, then exchange r_{σ_i} with $\max(T(l_{\sigma_i}))$ to obtain a new tree T' , and then rearrange the vertices of $T'(l_{\sigma_i})$ and $T'(r_{\sigma_i})$, so that they keep their same relative orders.

By a similar argument to that in the proof of Lemma 4.16, we can show that for any $i, j \in S_{T_\sigma}$ the operators ϕ_i^{-1} and ϕ_j^{-1} commute. Let $T_{\hat{\sigma}} := \Psi(T_\sigma) \in \mathcal{T}_n^2$, with

$$S_{T_{\hat{\sigma}}} = \{i : \hat{\sigma}_i \text{ is a vertex of } T_{\hat{\sigma}} \text{ with 2 successors}\}.$$

Since the operator Ψ merely permutes the labels of vertices of T_σ , we have $S_T = S_{T_{\hat{\sigma}}}$. Define $\Psi^{-1} := \prod_{i \in S_T} \phi_i^{-1}$, then

$$\Psi^{-1}(\Psi(T_\sigma)) = \prod_{i \in S_T} (\phi_i^{-1} \circ \phi_i)(T_\sigma) = \text{id}(T_\sigma) = T_\sigma.$$

Thus, Ψ is an injection from $\mathcal{T}_n^1$ to $\mathcal{T}_n^2$. As $|\mathcal{T}_n^1| = |\mathcal{T}_n^2| = E_n$, see [Dis13, FS73], hence Ψ is a bijection.

Since Ψ does not alter the structure the tree and preserves the relative order within each subtree, Eq. (4.30) follows immediately. $\square$

Proof of Theorem 3.19 (3.42c). It is clear that $\Phi := \omega^{-1} \circ \Psi \circ \omega$ is a bijection from $\mathcal{A}_n^1$ to $\mathcal{A}_n^2$. Combining (4.30) with Proposition 4.13, we prove (3.42c). $\square$

Example 4.18. If $\sigma = 78569210111312413 \in \mathcal{A}_{13}^1$, then $T_\sigma \in \mathcal{T}_{13}^1$, see Fig. 4.5. As $S_T = \{3, 6, 8, 10, 12\}$, let $\Psi = \prod_{i \in S_{T_\sigma}} \phi_i$ we obtain $\Psi(T_\sigma) = T_\tau \in \mathcal{T}_{13}^2$ and finally

$$\tau = \omega^{-1} \circ \Psi \circ \omega(\sigma) = 91078135614211312 \in \mathcal{A}_{13}^2.$$

Remark 4.19. Foata and Han gave another bijection from $\mathcal{A}_n^1$ to $\mathcal{A}_n^2$ in [FH16, Section 3], but their bijection does not keep the triple statistic $(\text{des}, \text{rminda}, \text{rmin})$.

5. Link to successions

Let $\sigma \in \mathfrak{S}_n$, an index $i \in [n - 1]$ is called a

- *succession* (**suc**) of σ if $\sigma(i + 1) = \sigma(i) + 1$;
- *big ascent* (**basc**) of σ if $\sigma(i + 1) \geq \sigma(i) + 2$.

Let $\text{Suc}(\sigma)$ and $\text{Basc}(\sigma)$ be the sets of successions and big ascents of σ , respectively. Furthermore, we define the sets

- $\text{Basc}_v(\sigma) := \{\sigma(i + 1) : \sigma(i) + 1 < \sigma(i + 1), i \in [n - 1]\}$,
- $\text{Des}_v(\sigma) := \{\sigma(i + 1) : \sigma(i) > \sigma(i + 1), i \in [n - 1]\}$,
- $\text{Suc}_v(\sigma) := \{\sigma(i + 1) : \sigma(i) + 1 = \sigma(i + 1), i \in [n - 1]\}$.

For $\sigma \in \mathfrak{S}_n$ we define

$$\begin{aligned}\widehat{\text{Fix}}(\sigma) &:= \{i : \sigma(i) = i, i \in \{2, \dots, n\}\}, \\ \widehat{\text{Exc}}_v(\sigma) &:= \{\sigma(i) : \sigma(i) > i, i \in \{2, \dots, n\}\}, \\ \text{Drop}_v(\sigma) &:= \{\sigma(i) : \sigma(i) < i, i \in [n]\},\end{aligned}$$

and denote the corresponding cardinalities by $\widehat{\text{fix}}(\sigma)$, $\widehat{\text{exc}}(\sigma)$ and $\text{drop}(\sigma)$, respectively.

We recall a bijection φ in [CHZ97, Section 5]. For $\sigma = \sigma_1 \sigma_2 \dots \sigma_n \in \mathfrak{S}_n$ we set $\sigma' = \sigma_2 \dots \sigma_n \sigma_1$ and define $\varphi(\sigma) = \theta_2(\sigma')$, where θ_2 is the second variant of the fundamental transformation, see Section 2.1.

Example 5.1. If $\sigma = 142836759 \in \mathfrak{S}_9$, then $\sigma' = 428367591$ and

$$\widetilde{\sigma'} = (14389)(567)(2), \quad \varphi(\sigma) = 143895672.$$

We have $\widehat{\text{Exc}}_v(\sigma) = \{4, 8\}$, $\text{Drop}_v(\sigma) = \{2, 3, 5\}$, $\widehat{\text{Fix}}(\sigma) = \{6, 7, 9\}$, $\text{Basc}_v(\varphi(\sigma)) = \{4, 8\}$, $\text{Des}_v(\varphi(\sigma)) = \{2, 3, 5\}$, and $\text{Suc}_v(\varphi(\sigma)) = \{6, 7, 9\}$.

A pair of consecutive integers $(i, i + 1)$ is called a *cyclic succession* of $\sigma \in \mathfrak{S}_n$ if $\sigma(i) = i + 1$ with $i \in [n - 1]$.

Theorem 5.2. The mapping φ is a bijection on $\mathfrak{S}_n$ such that for all $\sigma \in \mathfrak{S}_n$, $\varphi(\sigma)(1) = \sigma(1)$ and

$$(\widehat{\text{Exc}}_v, \text{Drop}_v, \widehat{\text{Fix}}) \sigma = (\text{Basc}_v, \text{Des}_v, \text{Suc}_v) \varphi(\sigma). \quad (5.1)$$

In particular, we have $(\widehat{\text{exc}}, \text{drop}, \widehat{\text{fix}}) \sigma = (\text{basc}, \text{des}, \text{suc}) \varphi(\sigma)$.

Proof. Clearly the mappings $\sigma \rightarrow \widetilde{\sigma'}$ and $\sigma \rightarrow \varphi(\sigma)$ have the following properties:

- $i \in \widehat{\text{Fix}}(\sigma)$ if and only if the pair $(i - 1, i)$ is a cyclic succession of $\widetilde{\sigma'}$, which is equivalent to $i \in \text{Suc}_v(\varphi(\sigma))$.

- If $\sigma(i) \in \text{Drop}_v(\sigma)$, then $i - 1 \geq \sigma(i)$. So the pair $(i - 1, \sigma(i))$ is a cyclic descent or fixed point of σ' . This is equivalent to say that $\sigma(i) \in \text{Des}_v(\varphi(\sigma))$.
- If $\sigma(i) \in \widehat{\text{Exc}}_v(\sigma)$, then $(i-1)+1 < \sigma(i)$. This is equivalent to say that the pair $(i-1, \sigma(i))$ is a big ascent of $\varphi(\sigma)$.

Summarising we have (5.1). □

Remark 5.3. The bijection φ provides a combinatorial proof of Theorem 9 in [MQYY25].

Define the polynomials

$$F_n(x, y, t | \alpha) = \sum_{\sigma \in \mathfrak{S}_n} x^{\widehat{\text{exc}}(\sigma)} y^{\text{drop}(\sigma)} t^{\widehat{\text{fix}}(\sigma)} \alpha^{\text{cyc}(\sigma)}.$$

Theorem 5.4. For integer $n \geq 0$, we have

$$xF_{n+1}(x, y, t | \alpha) = A_{n+1}^{\text{cyc}}(x, y, t | \alpha) + \alpha(x - t)A_n^{\text{cyc}}(x, y, t | \alpha), \quad (5.2a)$$

where $A_0(x, y, t | \alpha) = 1$, and

$$\sum_{n \geq 0} F_{n+1}(x, y, t | \alpha) \frac{z^n}{n!} = \alpha e^{(y-t)z} \left(\frac{(x-y)e^{tz}}{xe^{yz} - ye^{xz}} \right)^{1+\alpha}. \quad (5.2b)$$

Proof. Note that

$$\begin{aligned} \sum_{\substack{\sigma \in \mathfrak{S}_{n+1} \\ \sigma(1)=1}} x^{\widehat{\text{exc}}(\sigma)} y^{\text{drop}(\sigma)} t^{\widehat{\text{fix}}(\sigma)} \alpha^{\text{cyc}(\sigma)} &= \alpha t A_n^{\text{cyc}}(x, y, t | \alpha), \\ \sum_{\substack{\sigma \in \mathfrak{S}_{n+1} \\ \sigma(1)=1}} x^{\widehat{\text{exc}}(\sigma)} y^{\text{drop}(\sigma)} t^{\widehat{\text{fix}}(\sigma)} \alpha^{\text{cyc}(\sigma)} &= \alpha A_n^{\text{cyc}}(x, y, t | \alpha). \end{aligned}$$

According to the position of 1 in a permutation, which is 1 or greater than 1, we have

$$\begin{aligned} A_{n+1}^{\text{cyc}}(x, y, t | \alpha) &= \alpha t A_n^{\text{cyc}}(x, y, t | \alpha) + x \sum_{\substack{\sigma \in \mathfrak{S}_{n+1} \\ \sigma(1) > 1}} x^{\widehat{\text{exc}}(\sigma)} y^{\text{drop}(\sigma)} t^{\widehat{\text{fix}}(\sigma)} \alpha^{\text{cyc}(\sigma)}, \\ F_{n+1}(x, y, t | \alpha) &= \alpha A_n^{\text{cyc}}(x, y, t | \alpha) + \sum_{\substack{\sigma \in \mathfrak{S}_{n+1} \\ \sigma(1) > 1}} x^{\widehat{\text{exc}}(\sigma)} y^{\text{drop}(\sigma)} t^{\widehat{\text{fix}}(\sigma)} \alpha^{\text{cyc}(\sigma)}. \end{aligned}$$

Combining the above two equations yields recurrence (5.2a), which is equivalent to

$$x \sum_{n \geq 1} F_n(x, y, t | \alpha) \frac{z^n}{n!} = \sum_{n \geq 1} A_n^{\text{cyc}}(x, y, t | \alpha) \frac{z^n}{n!} + \alpha(x - t) \sum_{n \geq 1} A_{n-1}^{\text{cyc}}(x, y, t | \alpha) \frac{z^n}{n!}. \quad (5.3)$$

Invoking the generating function (1.7a) and applying $\frac{\partial}{\partial z}$ to (5.3) we obtain

$$x \sum_{n \geq 1} F_n(x, y, t | \alpha) \frac{z^{n-1}}{(n-1)!} = \left(\frac{\partial}{\partial z} + \alpha(x - t) \right) \left(\frac{(x-y)e^{tz}}{xe^{yz} - ye^{xz}} \right)^\alpha,$$

which can be shown (or by Maple) to be equivalent to (5.2b). □

Remark 5.5. By Theorem 5.2, when $y = \alpha = 1$, and substitute t by xt , Eq. (5.2b) reduces to Roselle's generating function for the statistic (asc, suc) [Ros68], namely,

$$\sum_{n \geq 0} \sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\text{asc}(\sigma)} t^{\text{suc}(\sigma)} \frac{z^n}{n!} = \frac{(1-x)^2 e^{(xt+1)z}}{(e^{xz} - xe^z)^2}. \quad (5.4)$$

6. Concluding remarks

A permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ of an ordered set is said to be *(increasing) alternating* (or *up-down*) if $\sigma_1 < \sigma_2 > \sigma_3 < \cdots$. We denote by Alt_n the set of all up-down (alternating) permutations in $\mathfrak{S}_n$. A cycle (cyclic permutation) $(\sigma_1 \sigma_2 \cdots)$ of an ordered set is said to be *up-down* if $\sigma_1 < \sigma_2 > \sigma_3 < \cdots$ and σ_1 is the smallest element. A permutation π is said to be *cycle-up-down* if it is a product of disjoint up-down cycles. Let Δ_n be the set of cycle-up-down permutations of $[n]$. Deutsch and Elizalde [DE11] proved the following bivariate generating function

$$\sum_{n \geq 0} \sum_{\pi \in \Delta_n} t^{\text{fix}(\pi)} \alpha^{\text{cyc}(\pi)} \frac{z^n}{n!} = \frac{e^{\alpha(t-1)z}}{(1 - \sin z)^\alpha}. \quad (6.1)$$

When $x = 1$, Eq. (4.18) reduces to

$$\sum_{n \geq 0} \sum_{\sigma \in \mathcal{CA}_n} t^{\text{fix}(\sigma)} \alpha^{\text{cyc}(\sigma)} \frac{z^n}{n!} = \frac{e^{\alpha(t-1)z}}{(1 - \sin z)^\alpha}. \quad (6.2)$$

In [FH16, Section 2], Foata and Han constructed a bijection $\eta : \mathcal{A}_{n+1}^1 \rightarrow \text{Alt}_{n+1}$ preserving the first letter. Applying the mapping η to each cycle of $\sigma \in \mathcal{CA}_n$ immediately yields a bijection $\tilde{\eta} : \mathcal{CA}_n \rightarrow \Delta_n$ such that $(\text{fix}, \text{cyc}) \sigma = (\text{fix}, \text{cyc}) \tilde{\eta}(\sigma)$.

Hwang, Jang and Oh [HJO23, Corollary 4.4] proved the $t = 1$ case of (6.2). By Theorem 3.19, we also have

$$\sum_{n \geq 0} \sum_{\sigma \in \mathcal{A}_{n+1}^i} t^{\text{rminda}(\sigma)} \alpha^{\text{rmin}(\sigma)-1} \frac{z^n}{n!} = \frac{e^{\alpha(t-1)z}}{(1 - \sin z)^\alpha}, \quad (6.3)$$

where $i = 1, 2$.

Disanto [Dis13, Proposition 1] proved the $t = 1$ case of (6.3) and asked for a bijection from $\mathcal{A}_{n+1}^1$ to Δ_n enlightening the correspondence between rmin and cyc . In fact, combining the bijections $\phi^{-1} : \mathcal{A}_{n+1}^1 \rightarrow \mathcal{CA}_n$ in Section 4.4 and $\tilde{\eta} : \mathcal{CA}_n \rightarrow \Delta_n$, we obtain a bijection $\Xi := \tilde{\eta} \circ \phi^{-1} : \mathcal{A}_{n+1}^1 \rightarrow \Delta_n$ such that $(\text{rminda}, \text{rmin}) \sigma = (\text{fix}, \text{cyc}) \Xi(\sigma)$ for $\sigma \in \mathcal{A}_{n+1}^1$.

Problem 6.1. Find a suitable statistic $\widehat{\text{drop}}$ over Δ_n and bijection $\Lambda : \mathcal{CA}_n \rightarrow \Delta_n$ such that $\text{drop}(\sigma) = \widehat{\text{drop}}(\Lambda(\sigma))$ for $\sigma \in \mathcal{CA}_n$.

In view of the work [PZ21], where a (p, q) -analogue of (1.4) is proved, the following question is natural.

Problem 6.2. Find a combinatorial explanation of the identity $\gamma_{n,j}(\alpha, t) = 2^j d_{n,j}(\alpha, t)$ for $0 \leq j \leq \lfloor n/2 \rfloor$.

Remark 6.3. After the initial preprint of this paper was posted, Dong, Lin and Pan [DLP25] have resolved the above problems.

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