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DETERMINATION OF THE OVERALL MODULI IN SECOND ORDER INCOMPRESSIBLE ELASTICITY

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ABSTRACT

The problem of finding the overall second order elastic moduli for a two phase material is considered. The model material consists of an incompressible, isotropic matrix containing spherical inclusions which are also incompressible and isotropic. To evaluate the effective moduli, the second order elastic field of a single inhomogeneity in an infinite matrix under homogeneous displacement boundary conditions is determined. This elastic field is assumed to approximate that of the composite with dilute concentration of the second phase. Subsequently, through the equality of the strain energies of the inhomogeneous material and an equivalent homogeneous material, possessing the overall elastic moduli, explicit expressions for elastic constants are obtained.

1. Introduction

The problem of determining the overall or effective properties of a linearly elastic material composed of several distinct phases has received considerable attention in the last few decades. Either rigorous bounds on the effective properties are established which are based on variational principles [Paul (1960), Hashin-Shtrickman (1963)], or a variety of approximate methods have been proposed, such as self-consistent method and its variants [Budiansky (1965), Hill (1965), Christensen & Lo (1979)], differential formulation [Cleary (1980)], the Mori-Tanaka approach (1983) [Norris (1989)], and the poly-inclusion formalism [Ferrari (1994)]. All these approximate methods are founded on the celebrated Eshelby's solution of the problem of a single inhomogeneity in an infinite matrix [Eshelby (1957)].

In contrast, the problem involving nonlinear elastic materials has hardly been investigated, which is understandable given the inherent mathematical difficulties in the analysis. Nevertheless, using variational principles, estimates for the overall properties of certain classes of composites have been found by Willis (1990), Talbot & Willis (1987), Ponte Castañeda & Willis (1988).

Within the context of second order elasticity, Ogden (1974) determined the overall second order bulk modulus for a composite consisting of dilute concentration of spherical particles. Hashin (1985) studied a similar problem except at finite concentration of inhomogeneities using the composite spheres assemblage model. Most recently Chen & Jiang (1993) considered the problem of finding all of the second order constants for a bi-phase composite. They used a perturbation approach which avoids solving a second order boundary value problem, however, the calculations become so bulky as to prevent them from presenting explicit expressions for the overall moduli.

In this paper we consider a second order elastic composite, consisting of two phases each of which is incompressible and isotropic. The constraint of incompressibility, which approximates most elastomers, reduces the number of elastic constants from five (for an unconstrained isotropic, second order material) to two. Furthermore, the field equations are simplified considerably making it possible for us to obtain explicit expressions for the overall constants of a composite with dilute concentration of inhomogeneities.

In Section 2 the field equations for the first and second order problems are recalled. Then the second order equations are shown to simplify due to the incompressibility condition. This is in part based on the work of Carroll & Rooney (1984). In Section 3, a particular boundary value problem is considered where a body, which is infinite in extent and contains a spherical inhomogeneity, is subjected to homogeneous displacement boundary conditions. The first order solution is recorded and is subsequently used to determine the second order elastic solution. In the first part of Section 4 we show that the incompressibility of the constituent phases implies incompressibility of the effective homogeneous material. In the second part, explicit expressions for the overall constants are obtained through equality of the strain energies of the homogeneous effective medium and the composite. This is in turn based on the result first obtained by Hill (1972) for large deformations. The last Section is devoted to discussing some of the features of the results. In addition, corresponding overall properties for the special case of rigid fillers in an incompressible matrix are obtained.

2. *Formulation and Simplification of the Field Equations*

Let $\mathbf{X}$ and $\mathbf{x}$ denote the position vectors of a material point in a fixed reference configuration and in the current configuration, respectively. The deformation function of the equilibrium problem is defined by $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$. The deformation gradient $\mathbf{F}$ and the displacement gradient $\mathbf{H}$ are given by

$$\mathbf{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}, \qquad \mathbf{H} = \mathbf{F} - \mathbf{I}, \tag{2.1}$$

where $\mathbf{I}$ is the identity tensor. Let $\boldsymbol{\pi}$ represent the first Piola-Kirchhoff stress tensor and $W = W(\mathbf{F})$ represent the strain energy as measured per unit volume of the reference configuration. Then the Lagrangian form of equations of equilibrium, in the absence of any body force, and stress constitutive equations are given as

$$\text{Div}\, \boldsymbol{\pi} = 0, \tag{2.2}$$

$$\boldsymbol{\pi} = \frac{\partial W}{\partial \mathbf{F}}, \tag{2.3}$$

where Div refers to the divergence operator with respect to $\mathbf{X}$. In addition, the constraint of incompressibility requires that

$$\det \mathbf{F} = 1. \tag{2.4}$$

As a result of this constraint the material response consists of a determinate part, depending only on $\underset{\sim}{F}$, and a constraint part involving an unknown scalar field P which is determined from the differential equation and the boundary conditions. Hence (2.2) and (2.3) become

$$\pi_{ij,j} = 0, \tag{2.5}$$

$$\pi_{ij} = -P\, F^{-1}_{ji} + \frac{\partial W}{\partial F_{ij}}. \tag{2.6}$$

Here summation convention is implied by repeated indices. Let $\underset{\sim}{B} = \underset{\sim}{F}\underset{\sim}{F}^{T}$ denote the Cauchy-Green deformation tensor. For isotropic elastic solids the strain energy depends on the principal invariants of $\underset{\sim}{B}$. Therefore using (2.4) we have

$$W = W(I_1, I_2), \tag{2.7}$$

$$I_1 = B_{ii}, \qquad I_2 = B^{*}_{ii} = B^{-1}_{ii}, \qquad I_3 = 1, \tag{2.8}$$

where $\underset{\sim}{B}^{*}$ denotes the adjugate of $\underset{\sim}{B}$. Equations (2.5) and (2.6) hold in general for any nonlinear elastic incompressible material.

We now want to use the method of successive approximations to derive the second order elasticity equations. To that end, let a measure of smallness ε be defined as

$$\varepsilon = \sup_{B} \|\underset{\sim}{H}(\cdot)\|_2, \tag{2.9}$$

where $\|\underset{\sim}{H}\|_2^2 \equiv \mathrm{Tr}\, \underset{\sim}{H}^{T}\underset{\sim}{H}$ and B is the region of space occupied by the body in its undistorted reference configuration. Following Signorini (1936), let us assume that the prescribed surface displacements or tractions on ∂B admit a power series expansion in ε such that

$$x_i - X_i = \varepsilon \bar{u}_i + \varepsilon^2 \bar{v}_i + O(\varepsilon^3) \qquad \text{on } \partial B, \tag{2.10}$$

or

$$\pi_{ij} n_j = \varepsilon \bar{s}_i + \varepsilon^2 \bar{t}_i + O(\varepsilon^3) \qquad \text{on } \partial B, \tag{2.11}$$

where n_i refer to direction cosines for the outward unit normal to ∂B. Under these conditions, Stoppelli (1954) has shown the existence and uniqueness of the solution to (2.5) and (2.6). Furthermore provided ε is sufficiently small, the displacement and stress fields admit an absolutely convergent power series expansion in ε. Thus

$$x_i - X_i = \varepsilon u_i + \varepsilon^2 v_i + O(\varepsilon^3) \qquad \text{in B,} \tag{2.12}$$

$$\pi_{ij} = \varepsilon \sigma_{ij} + \varepsilon^2 \tau_{ij} + O(\varepsilon^3) \qquad \text{in B.} \tag{2.13}$$

When the strain energy is expanded as a power series in I_1 and I_2, Rivlin (1953) has shown that it can be expressed as

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) + O(\varepsilon^4), \tag{2.14}$$

where C_1 and C_2 are material constants. When (2.12) to (2.14) are substituted in (2.4), (2.5) and (2.6), two sets of field equations are obtained which correspond to the first and second order expansions in ε.

The first order equations are

$$u_{i,i} = 0, \tag{2.15}$$

$$\sigma_{ij,j} = 0, \tag{2.16}$$

$$\sigma_{ij} = -p\,\delta_{ij} + 2\,\mu\,e_{ij}, \tag{2.17}$$

subject to the boundary conditions

$$u_i = \bar{u}_i \qquad \text{on } \partial B, \tag{2.18}$$

or

$$\sigma_{ij}\,n_j = \bar{s}_i \qquad \text{on } \partial B, \tag{2.19}$$

where

$$e_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i}) \qquad \text{and} \qquad \mu = 2\,(C_1 + C_2). \tag{2.20}$$

This is the familiar set of equations in linear elasticity for incompressible materials where μ is the shear modulus.

The second order equations are

$$v_{i,i} = \frac{1}{2}\,u_{i,j}\,u_{j,i}, \tag{2.21}$$

$$\tau_{ij,j} = 0, \tag{2.22}$$

$$\tau_{ij} = -q\,\delta_{ij} + \mu(v_{i,j} + v_{j,i} - u_{j,k}\,u_{k,i}) + p\,u_{j,i} - 8\,C_2\,e_{ik}e_{kj}, \tag{2.23}$$

subject to the boundary conditions

$$v_i = \bar{v}_i \qquad \text{on } \partial B, \tag{2.24}$$

or

$$\tau_{ij}\,n_j = \bar{t}_i \qquad \text{on } \partial B. \tag{2.25}$$

Equation (2.21) represents the incompressibility condition for the second order solution. In the above sets of equations, p and q are the first and second order pressure terms, respectively, in the expansion of P. For traction boundary value problems there is a further condition, known as Signorini's compatibility condition, which ensures existence of the second order solution and is expressed as

$$\int_{\partial B} \varepsilon_{ijk} u_j \sigma_{kl} n_l \, dA = 0. \tag{2.26}$$

This completes formulation of the field equations. We now proceed to simplify the second order equations. Following Chan and Carlson (1970), we introduce the transformation

$$w_i = v_i - \frac{1}{2} u_{i,j} u_j. \tag{2.27}$$

Then (2.21)-(2.23) reduce to

$$w_{i,i} = 0, \tag{2.28}$$

$$\tau_{ij,j} = 0, \tag{2.29}$$

$$\tau_{ij} = -q \, \delta_{ij} + \mu \, (w_{i,j} + w_{j,i}) + \tau^*_{ij}, \tag{2.30}$$

where $\underline{\tau}^*$ is given in terms of the first order solution as

$$\tau^*_{ij} = \mu \, (u_k e_{ij,k} + u_{i,k} e_{kj} + u_{j,k} e_{ki}) - u_{j,k} \sigma_{ki} - 8 \, C_2 \, e_{ik} e_{kj}. \tag{2.31}$$

If (2.30) is substituted in (2.29), there results a body force term given by $\tau^{*}_{ij,j}$. Carroll and Rooney (1984) identified a part of this body force which is conservative and therefore can be absorbed into the unknown second order pressure term q. To see this let

$$\tau^{*}_{ij} = \tau^{+}_{ij} + 4\,\xi\, e_{ik} e_{kj}, \tag{2.31}$$

where

$$\xi = C_1 - C_2 = \frac{1}{2}\mu - 2\,C_2, \tag{2.32}$$

$$\tau^{+}_{ij} = \mu\,(u_k e_{ij,k} - u_{j,k} e_{ki} - u_{k,i} e_{kj}) + p\, u_{j,i}. \tag{2.33}$$

Carroll and Rooney (1984) showed that

$$\tau^{+}_{ij,j} = \frac{1}{2}\,(u_j\, p_{,j} - \mu\, e_{jk} e_{jk})_{,i}. \tag{2.34}$$

Hence

$$\tau^{*}_{ij,j} = -\,q^{+}_{,i} + 4\,\xi\,(e_{ik} e_{kj})_{,j}, \tag{2.35}$$

where

$$q^{+} = -\frac{1}{2}\,(u_j\, p_{,j} - \mu\, e_{jk} e_{jk}). \tag{2.36}$$

Equation (2.35) can be recast in an alternative form which facilitates the computations. Let $\underline{W}$ represent the first order rotation tensor,

$$W_{ij} = \frac{1}{2}\,(u_{i,j} - u_{j,i}). \tag{2.37}$$

Then it can be shown that

$$(e_{ik}e_{kj})_{,j} = e_{ik}e_{kj,j} + \frac{1}{2}(e_{jk}e_{jk})_{,i} + W_{ij,k}\ e_{jk}. \tag{2.38}$$

Thus

$$\tau^*_{ij,j} = -(q^+ + q^*)_{,i} + 4\,\xi\,(e_{ik}e_{kj,j} + W_{ij,k}\ e_{jk}), \tag{2.39}$$

where

$$q^* = -2\,\xi\ e_{ij}e_{ij}. \tag{2.40}$$

We now summarize below the simplified second order equations in light of the preceding results.

$$w_{i,i} = 0, \tag{2.41}$$

$$\tau'_{ij,j} + B_i = 0, \tag{2.42}$$

$$\tau'_{ij} = -q'\ \delta_{ij} + \mu\,(w_{i,j} + w_{j,i}), \tag{2.43}$$

with the second order pressure $q' = q+q^+ +q^*$. The boundary conditions are

$$w_i = \widetilde{w}_i \qquad \text{on } \partial B, \tag{2.44}$$

or

$$\tau'_{ij}\,n_j = \widetilde{t}_i \qquad \text{on } \partial B, \tag{2.45}$$

where

$$B_i = 4\,\xi\,(e_{ik}e_{kj,j} + W_{ij,k}\ e_{jk}), \tag{2.46}$$

$$\widetilde{w}_i = \bar{v}_i - \frac{1}{2}u_{i,j}\,u_j \qquad \text{on } \partial B, \tag{2.47}$$

$$\widetilde{t}_i = \bar{t}_i - (\tau^*_{ij} + q^+\delta_{ij} + q^*\delta_{ij})\,n_j \qquad \text{on } \partial B, \tag{2.48}$$

$$\tau^*_{ij} = \mu\,(u_k\ e_{ij,k} + e_{ik}\ W_{kj} + W_{ik}\ e_{kj}) + p\,(e_{ij} - W_{ij}) + (4\,\xi - 2\,\mu)\ e_{ik}e_{kj}. \tag{2.49}$$

Once the above set of equations are solved for $\underset{\sim}{w}$, the second order displacement and stress fields are given by

$$v_i = w_i + \frac{1}{2} u_{i,j} u_j, \tag{2.50}$$

$$\tau_{ij} = \tau'_{ij} + \tau^*_{ij} + (q^+ + q^*) \delta_{ij}. \tag{2.51}$$

3. Solution of the Problem of Spherical Inhomogeneity In an Infinite Matrix

As discussed in the Introduction, in order to find the overall properties under the assumption of dilute concentration, we will solve a particular boundary value problem in second order elasticity. Consider an infinite body B, consisting of a spherical inhomogeneity of radius R embedded in an otherwise homogeneous matrix. To solve the first order problem we impose homogeneous displacement boundary conditions

$$u_1 = -\frac{E}{2} X_1, \qquad u_2 = -\frac{E}{2} X_2, \qquad u_3 = E X_3 \qquad \text{on } \partial B, \tag{3.1}$$

which are given in terms of the Cartesian coordinates and are symmetric with respect to X_3-axis. Hence, the boundary value problem is axisymmetric as well. The solution to this problem is based on Eshelby's well-known result [Eshelby (1957), Mura (1982)] that the strain field in the inhomogeneity is constant. The first order displacement fields in the two phases are given in terms of spherical coordinates (r,ϑ,η) as

$$u_r = \delta\, r\, (1 - 3 \cos^2\vartheta) \qquad \text{in } \Omega, \tag{3.2}$$

$$u_\vartheta = 3\,\delta\, r \cos\vartheta \sin\vartheta \qquad \text{in } \Omega, \tag{3.3}$$

$$u_r = [-\alpha - \frac{3\gamma}{r^2}(3\cos^2\vartheta - 1) + 3\,\beta \cos^2\vartheta] \,/\, (2\,\mu_m r^2)$$

$$+ E\, r\,(3\cos^2\vartheta - 1)\,/\,2 \qquad \text{in } B\backslash\Omega, \tag{3.4}$$

$$u_\vartheta = -3\cos\vartheta \sin\vartheta \,(\frac{\gamma}{\mu_m r^4} + \frac{rE}{2}) \qquad \text{in } B\backslash\Omega, \tag{3.5}$$

where Ω is the spherical domain of radius R occupied by the inhomogeneity, the subscript m refers to material properties of the matrix, and α, β, γ and δ are constants to be determined by the interface conditions. To obtain the pressure, (2.17) is substituted into (2.16), yielding (in Cartesian coordinates)

$$p_{,i} = 2\,\mu\, e_{ij,j} = \mu\, u_{i,jj}. \tag{3.6}$$

Transforming (3.6) into spherical coordinates and using displacement fields (3.2) to (3.5), the pressure is then obtained by integration. To within a constant, the pressure is

$$p = \beta\,(3\cos^2\vartheta - 1)\,/\,r^3 \qquad \text{in } B\backslash\Omega, \tag{3.7}$$

$$p = 0 \qquad \text{in } \Omega. \tag{3.8}$$

The stress fields in each of the phases, as obtained from (2.17), are

$$\sigma_{rr} = 2\,\mu_f \delta\,(1 - 3\cos^2\vartheta) \qquad \text{in } \Omega, \tag{3.9}$$

$$\sigma_{r\vartheta} = 6\,\mu_f \delta \cos\vartheta \sin\vartheta \qquad \text{in } \Omega, \tag{3.10}$$

$$\sigma_{rr} = (\beta + 2\,\alpha - 9\,\beta\cos^2\vartheta)\,/\,r^3 + (3\cos^2\vartheta - 1)\,(12\,\gamma/\,r^5 + \mu_m E)$$

$$\text{in } B\backslash\Omega, \tag{3.11}$$

$$\sigma_{r\vartheta} = \sin\vartheta\cos\vartheta\,(24\,\gamma/\,r^5 - 3\,\beta/\,r^3 - 3\,\mu_m E) \quad \text{in } B\backslash\Omega, \tag{3.12}$$

where the subscript f refers to the material properties in the inhomogeneity. Finally imposing continuity of u_r, u_ϑ, σ_{rr} and $\sigma_{r\vartheta}$ at the interface, $r=R$, the constants are determined to be

$$\alpha = \beta = 5\,\mu_m(\mu_m - \mu_f)\,R^3\,E\,/\,(3\,\mu_m + 2\,\mu_f), \tag{3.13}$$

$$\gamma = \mu_m(\mu_m - \mu_f)\,R^5\,E\,/\,(3\,\mu_m + 2\,\mu_f), \tag{3.14}$$

$$\delta = -5\,\mu_m E\,/\,(2\,(3\,\mu_m + 2\,\mu_f)). \tag{3.15}$$

With the first order solution known, we now proceed to determine the body force $\underset{\sim}{B}$ (Equation (2.46)) which appears in the second order equations (2.42). Since $\underset{\sim}{e}$ and $\underset{\sim}{W}$ are uniform in the inhomogeneity, it is clear from (2.46) that the body force vanishes there and needs to be determined only in the matrix. For the purpose of computations, it is more convenient to use the cylindrical polar coordinates (ρ,ϑ,z) where

$$r^2 = \rho^2 + z^2\,. \tag{3.16}$$

The body force term is rewritten as

$$B_i = B_i^{(1)} + B_i^{(2)}, \tag{3.17}$$

$$B_i^{(1)} = 4\,\xi_m\,e_{ik}\,e_{kj,j}, \tag{3.18}$$

$$B_i^{(2)} = 4\,\xi_m\,W_{ij,k}\,e_{jk}. \tag{3.19}$$

Using the first order solution in the matrix, as given in (3.4) and (3.5), $\underset{\sim}{e}$ and

$\underset{\sim}{W}$ are evaluated and upon substitution in (3.17) to (3.19) yield

$$B_\rho = 3\,\beta\,\rho\,\xi_m[\,30\,\beta\,r^2z^4 - 180\,\gamma\,z^4 - 5\,\mu_m r^7\,E\,z^2 - 6\,\beta\,r^4z^2$$
$$-8\,\alpha\,r^4z^2 + 75\,\gamma\,r^2z^2 + \mu_m r^9\,E + \alpha\,r^6 - 3\,\gamma\,r^4]/\,\mu_m^2\,r^{14}, \qquad (3.20)$$

$$B_z = \xi_m[(90\,\beta^2\,r^2 - 540\,\beta\,\gamma)z^5 + (-15\,\beta\,\mu_m r^7\,E + (-72\,\beta^2 - 24\,\alpha\,\beta)r^4$$
$$+540\,\beta\,\gamma\,r^2)z^3 + (9\,\beta\,\mu_m r^9\,E + 18\,\alpha\,\beta\,r^6 - 72\,\beta\,\gamma\,r^4)\,z]$$
$$/\,(\mu_m^2\,r^{14}). \qquad (3.21)$$

We now want to determine $\underset{\sim}{w}$ appearing in (2.41) to (2.43). These equations have an alternative representation in terms of Boussinesq potentials, ϕ and ψ, for an axisymmetric problem. We will show that the body force can be further decomposed into a conservative part (which becomes part of the second order pressure) and a component which acts in the z direction only. That is,

$$\underset{\sim}{B} = -\mathrm{Grad}\; q_1 + b(\rho,z)\,\underset{\sim}{e}_z. \qquad (3.22)$$

Then the field equations are

$$\Delta\phi = -z\,b, \qquad \Delta\psi = b, \qquad (3.23)$$
$$2\,\mu\,\underset{\sim}{w} = \mathrm{Grad}(\phi + z\,\psi) - 2\,\psi\,\underset{\sim}{e}, \qquad (3.24)$$
$$\underset{\sim}{\tau}' = -\psi_{,z}\,\underset{\sim}{I} + \mu\,[\mathrm{Grad}\;\underset{\sim}{w} + (\mathrm{Grad}\;\underset{\sim}{w})^T], \qquad (3.25)$$

where Δ denotes the Laplacian. In order to reduce the body force $\underset{\sim}{B}$ to the form of (3.22), B_ρ is integrated with respect to ρ and the result is designated as $-q_1$. Then $\underset{\sim}{B}$ is written as

$$\underset{\sim}{B} = B_\rho \underset{\sim}{e}_\rho + B_z \underset{\sim}{e}_z = -q_{1,\rho} \underset{\sim}{e}_\rho + (B_z + q_{1,z} - q_{1,z}) \underset{\sim}{e}_z$$
$$= -\text{Grad } q_1 + (B_z + q_{1,z}) \underset{\sim}{e}_z = -\text{Grad } q_1 + b \underset{\sim}{e}_z, \tag{3.26}$$

where

$$b \equiv B_z + q_{1,z} = -\xi_m[(36 \beta^2 r^2 - 270 \beta \gamma)z^3 + ((9 \beta^2 - 18 \alpha \beta)r^4$$
$$+36 \beta \gamma r^2)z] / (2 \mu_m^2 r^{12}), \tag{3.27}$$

and the quantity q_1 is given by in spherical coordinates as

$$q_1 = \xi_m[r^2 \cos^2\vartheta (-24 \alpha \mu_m r^7 E - 42 \alpha^2 r^4 + 180 \alpha \gamma r^2)$$
$$+8 \alpha \mu_m r^9 E + r^4(72 \alpha^2 r^2 - 360 \alpha \gamma)\cos^4\vartheta + 4 \alpha^2 r^6$$
$$-9 \alpha \gamma r^4] / (8 \mu_m^2 r^{12}). \tag{3.28}$$

As before, the conservative part of the body force in (3.26) is absorbed in the second order pressure term. Consequently (2.48) and (2.51) are changed to

$$\tilde{t}_i = \bar{t}_i - [\tau^*_{ij} + (q^+ + q^* + q_1)\delta_{ij}] n_i \qquad \text{on } \partial B, \tag{3.29}$$
$$\tau_{ij} = \tau'_{ij} + \tau^*_{ij} + (q^+ + q^* + q_1)\delta_{ij}. \tag{3.30}$$

With the body force given in (3.27), particular solutions for (3.23) are determined to be

$$\phi^p = [3 a_3 \cos^4\vartheta + 6 a_1 \cos^2\vartheta - a_1)r^2 + 12 a_4 \cos^4\vartheta$$
$$+15 a_2 \cos^2\vartheta - a_2] / r^6, \tag{3.31}$$
$$\psi^p = [(3 a_3 \cos^3\vartheta + (-3 a_3 - 2 a_1)\cos \vartheta) r^2 - 4 a_4 \cos^3\vartheta$$
$$+(-3 a_4 - 9 a_2) \cos \vartheta] / r^7, \tag{3.32}$$

where

$$a_1 = 5\ \beta^2\ \xi_m/(8\ \mu_m^2), \qquad a_2 = \beta\ \gamma\ \xi_m/(2\ \mu_m^2),$$

$$a_3 = -3\ \beta^2\ \xi_m/(4\ \mu_m^2), \qquad a_4 = -9\ \beta\ \gamma\ \xi_m/(8\ \mu_m^2). \tag{3.33}$$

The corresponding displacement field, given in (3.24), is then

$$w_r^p = -[(15\ a_3\ \cos^4\vartheta + (6\ a_1 - 9\ a_3)\cos^2\vartheta - 2\ a_1)r^2 + 20\ a_4\cos^4\vartheta$$
$$+ (9\ a_2 - 12\ a_4)\cos^2\vartheta - 3\ a_2] \ / \ (\mu_m r^7), \tag{3.34}$$

$$w_\vartheta^p = -[(9\ a_3\cos^3\vartheta + 6\ a_1\cos\vartheta)\sin\vartheta\ r^2 + (20\ a_4\cos^3\vartheta$$
$$+ 15\ a_2\cos\vartheta)\sin\vartheta] \ / \ (\mu_m r^7), \tag{3.35}$$

where the superscript p stands for the particular solution.

The displacement boundary conditions for the second order problem, given in (2.47), are chosen so as to make $\underline{w}$ vanish at infinity. Therefore, it is necessary that

$$\bar{v}_i = \frac{1}{2}\ u_{i,j}\ u_j \qquad \text{on } \partial B. \tag{3.36}$$

Using (3.1), in Cartesian coordinates, this gives

$$\bar{v}_1 = \frac{E^2}{4}\ X_1, \quad \bar{v}_2 = \frac{E^2}{4}\ X_2, \quad \bar{v}_3 = E^2\ X_3 \qquad \text{on } \partial B. \tag{3.37}$$

The homogeneous solutions to (3.23) are two harmonic potentials ϕ and ψ which admit the following product representation [Sternberg, Eubanks and Sadowsky (1952)]

$$\phi_n = r^{-n-1}\, p_n(\cos\vartheta), \qquad \psi_n = r^{-n-1}\, p_n(\cos\vartheta),$$
$$n = 0, \pm 1, \pm 2, \ldots \tag{3.38}$$

where p_n is the Legendre polynomial of degree n. The appropriate potentials are those which have the same angular dependence as the particular solution in (3.34) and (3.35). Once these harmonic potentials are identified, use of (3.25) leads to the following solutions in the matrix and the inhomogeneity.

$$2\,\mu_m\, w_r^h = -m_1\, p_0/\, r^2 - (3\, m_2/\, r^4 + 6\, m_4/\, r^2) p_2 - (5\, m_3/\, r^6 + 20\, m_5/\, r^4) p_4 \quad \text{in } B\backslash\Omega, \tag{3.39}$$

$$2\,\mu_m\, w_\vartheta^h = \sin\vartheta\, [-m_2\, p_2'/\, r^4 - (m_3/\, r^6 + 2\, m_5/\, r^4) p_4'] \quad \text{in } B\backslash\Omega, \tag{3.40}$$

$$2\,\mu_f\, w_r = 2(f_1 r - 3\, f_4 r^3) p_2 + 4\, (f_2 r^3 - 5\, f_5 r^5) p_4 \quad \text{in } \Omega, \tag{3.41}$$

$$2\,\mu_f\, w_\vartheta = \sin\vartheta\, [(-f_1 r + 5\, f_4 r^3) p_2' + (-f_2 r^3 + 7\, f_5 r^5) p_4'] \quad \text{in } \Omega, \tag{3.42}$$

where $p_n' = \dfrac{dp_n}{d\cos\vartheta}$. The coefficients $m_1,\ldots,m_5$ and $f_1,\ldots,f_5$ are a set of, as yet, undetermined constants associated with the matrix and inclusion response, respectively.

The second order pressure q' is determined using (2.42) and (2.43). We also note that

$$\underset{\sim}{w} = \underset{\sim}{w}^h + \underset{\sim}{w}^p \qquad \text{in } B\backslash\Omega. \tag{3.43}$$

The displacement fields obtained above are used to find

$$q' = [\cos^2\vartheta\ (18\ m_4 r^2 - 210\ m_5) - 6\ m_4 r^2 + 245\ m_5 \cos^4\vartheta + 21\ m_5] / (2\ r^5) + [(24\ a_3 \cos^4\vartheta + (-27\ a_3 - 12\ a_1)\cos^2\vartheta + 3\ a_3 + 2\ a_1) r^2 - 40\ a_4 \cos^4\vartheta + (-12\ a_4 - 72\ a_2)\cos^2\vartheta + 3\ a_4 + 9\ a_2] / r^8 \quad \text{in } B\backslash\Omega, \tag{3.44}$$

$$q' = -\frac{55}{8} f_5 r^4 (35\cos^4\vartheta - 30\cos^2\vartheta + 3) - \frac{21}{2} f_4 r^2 (3\cos^2\vartheta - 1) \quad \text{in } \Omega \tag{3.45}$$

The stress field $\underline{\tau}'$ can now be evaluated using (2.43). We record below only the relevant components of it which are used in determining the constants.

$$\tau'_{rr} = [(840\ m_4 p_2 + 70\ m_1 p_0) r^5 + (4760\ m_5 p_4 + 420\ m_2 p_2) r^3 + (1392\ a_3 p_4 + (750\ a_3 + 1120\ a_1) p_2 + (-42\ a_3 - 70\ a_1) p_0) r^2 + 1050\ m_3 p_4 r + 1920\ a_4 p_4 + (600\ a_4 + 1260\ a_2) p_2 + (-315\ a_4 - 525\ a_2) p_0] / (35\ r^8) \quad \text{in } B\backslash\Omega, \tag{3.46}$$

$$\tau'_{\vartheta r} = [105\ m_4 \mu_m r^5 p'_2 + (525\ m_5 \mu_m p'_4 + 104\ m_2 \mu_m p'_2) r^3 + (114\ a_3 p'_4 + (180\ a_3 + 280\ a_1) p'_2) r^2 + 210\ m_3 \mu_m p'_4\ r + 240\ a_4 p'_4 + (460\ a_4 + 805\ a_2) p'_2] \sin\vartheta / (35\ \mu_m r^8) \quad \text{in } B\backslash\Omega, \tag{3.47}$$

$$\tau'_{rr} = 110\ f_5 r^4 p_4 + (12\ f_2 p_4 + 24\ f_4 p_2) r^2 + 2\ f_1 p_2 + 3\ f_3 p_0 \quad \text{in } \Omega, \tag{3.48}$$

$$\tau'_{\vartheta r} = [(24\ f_5 r^4 - 3\ f_2 r^2) p'_4 + (8\ f_4 r^2 - f_1) p'_2] \sin\vartheta \quad \text{in } \Omega. \tag{3.49}$$

The second order displacement $\underline{v}$ and stress field $\underline{\tau}$ are given by (2.50) and (3.30). There are various terms in these expressions which arise exclusively out of the first order solution and are computed below.

The expressions for $\underline{\tau}^*$ and q^+, as given by (2.31) and (2.36), are evaluated using the first order displacement fields. The relevant components of $\underline{\tau}^*$ and q^+

are found to be

$$\tau^*_{rr} = -[((3\ \mu_m^2 - 6\ \xi_m \mu_m^2) r^{10} \cos^2\vartheta + (\mu_m^3 - 2\ \xi_m \mu_m^2) r^{10})\ E^2$$
$$+ (((108\ \alpha\ \xi_m \mu_m - 117\ \alpha\ \mu_m^2) r^7 + (810\ \gamma\ \mu_m^2 - 720\ \gamma\ \xi_m \mu_m) r^5 \cos^4\vartheta$$
$$+ ((93\ \alpha\ \mu_m^2 - 84\ \alpha\ \xi_m \mu_m) r^7 + (576\ \gamma\ \xi_m \mu_m - 648\ \gamma\ \mu_m^2) r^5) \cos^2\vartheta$$
$$+ (8\ \alpha\ \xi_m \mu_m - 8\ \alpha\ \mu_m^2) r^7 + (54\ \gamma\ \mu_m^2 - 48\ \gamma\ \xi_m \mu_m) r^5)\ E$$
$$+ ((18\ \alpha^2 \mu_m - 54\ \alpha^2 \xi_m) r^4 + (576\ \alpha\ \gamma\ \xi_m - 99\ \alpha\ \gamma\ \mu_m) r^2 + 270\ \gamma^2 \mu_m$$
$$- 1440\ \gamma^2 \xi_m) \cos^4\vartheta + ((30\ \alpha^2 \xi_m - 9\ \alpha^2 \mu_m) r^4 + (36\ \alpha\ \gamma\ \mu_m$$
$$- 288\ \alpha\ \gamma\ \xi_m) r^2 - 108\ \gamma^2 \mu_m + 576\ \gamma^2 \xi_m) \cos^2\vartheta + (3\ \alpha^2 \mu_m - 8\ \alpha^2 \xi_m) r^4$$
$$+ (96\ \alpha\ \gamma\ \xi_m - 21\ \alpha\ \gamma\ \mu_m) r^2 + 54\ \gamma^2 \mu_m - 288\ \gamma^2 \xi_m]/(2\ \mu_m^2 r^{10}) \qquad \text{in } B\backslash\Omega$$

(3.50)

$$\tau^*_{\vartheta r} = 3 \cos\vartheta \sin\vartheta\ [24\ \beta\ \mu_m r^7 \cos^2\vartheta\ E - 225\ \gamma\ \mu_m r^5 \cos^2\vartheta\ E$$
$$- 5\ \beta\ \mu_m r^7 E - 3\ \alpha\ \mu_m r^7 E + 85\ \gamma\ \mu_m r^5 E + 6\ \beta^2 r^4 \cos^2\vartheta$$
$$- 102\ \beta\ \gamma\ r^2 \cos^2\vartheta + 150\ \gamma^2 \cos^2\vartheta - 2\ \alpha\ \beta\ r^4 + 34\ \alpha\ \gamma\ r^2 - 30\ \gamma^2]$$
$$/(4\ \mu_m r^{10}) - 3(4\ \xi_m - 2\ \mu_m) \cos\vartheta \sin\vartheta\ (\mu_m r^5 E + \beta\ r^2 - 8\ \gamma)$$
$$[\mu_m r^5 E - 3\ \beta\ r^2 \cos^2\vartheta + 15\ \gamma \cos^2\vartheta + \alpha\ r^2 - 3\ \gamma]\ /(4\ \mu_m^2\ r^{10})$$
$$- [3\ \beta \cos\vartheta\ (3 \cos^2\vartheta - 1) \sin\vartheta\ (\mu_m r^5 E + 2\ \beta\ r^2 - 8\ \gamma)]$$
$$/\ (2\ \mu_m r^8) \qquad \text{in } B\backslash\Omega, \qquad (3.51)$$

$$\tau^*_{rr} = \delta^2\ (4\ \xi_f - 2\ \mu_f)(3 \cos^2\vartheta + 1) \qquad \text{in } \Omega, \qquad (3.52)$$
$$\tau^*_{\vartheta r} = -3\ \delta^2\ (4\ \xi_f - 2\ \mu_f) \cos\vartheta \sin\vartheta \qquad \text{in } \Omega, \qquad (3.53)$$

$$q^+ = [3\ \mu_m^2\ r^{10} E^2 + (315\ \gamma\ \mu_m r^5 \cos^4\vartheta + (-270\ \gamma\ \mu_m r^5 \cos^2\vartheta$$
$$+ 27\ \gamma\ \mu_m r^5) E + (45\ \beta^2 r^4 - 225\ \beta\ \gamma\ r^2 + 405\ \gamma^2) \cos^4\vartheta$$
$$+ (-27\ \alpha\ \beta\ r^4 + (108\ \alpha - 18\ \beta) \gamma\ r^2 - 90\ \gamma^2) \cos^2\vartheta + 6\ \alpha^2 r^4$$
$$+ (-9\ \beta - 36\ \alpha) \gamma\ r^2 + 117\ \gamma^2]\ /\ (4\ \mu_m r^{10}) \qquad \text{in } B\backslash\Omega, \qquad (3.54)$$

$$q^{+} = 3\,\delta^2\,\mu_f \qquad \text{in } \Omega. \tag{3.55}$$

To compute the second order displacements, as given in (2.50), let

$$y_i = u_{i,j}\,u_j. \tag{3.56}$$

Then

$$\begin{aligned} y_r = {} & [(3\,\mu_m^2\,r^{10}\cos^2\vartheta + \mu_m^2 r^{10})E^2 + ((135\,\gamma\,\mu_m r^5 - 27\,\alpha\,\mu_m r^7)\cos^4\vartheta \\ & + (24\,\alpha\,\mu_m r^7 - 108\,\gamma\,\mu_m r^5)\cos^2\vartheta - \alpha\,\mu_m r^7 + 9\,\gamma\,\mu_m r^5)E \\ & + (-18\,\alpha^2 r^4\ 126\,\alpha\,\gamma\,r^2 - 180\,\gamma^2)\cos^4\vartheta + (12\,\alpha^2 r^4 - 72\,\alpha\,\gamma\,r^2 \\ & + 72\,\gamma^2)\cos^2\vartheta - 2\,\alpha^2 r^4 + 18\,\alpha\,\gamma\,r^2 - 36\,\gamma^2]/(4\,\mu_m^2 r^9) \qquad \text{in } B\backslash\Omega, \end{aligned} \tag{3.57}$$

$$\begin{aligned} y_\vartheta = {} & -(3\,\mu_m^2 r^{10} E^2 + ((18\,\alpha\,\mu_m r^7 - 180\,\gamma\,\mu_m r^5)\cos^3\vartheta + (72\,\gamma\,\mu_m r^5 \\ & - 6\,\alpha\,\mu_m r^7)E + ((90\,\gamma^2 - 54\,\alpha\,\gamma\,r^2)\cos^3\vartheta + (18\,\alpha\,\gamma\,r^2 - 18\,\gamma^2)) \\ & \sin\vartheta\,\cos\vartheta\ /\ (4\,\mu_m^2\,r^9) \qquad \text{in } B\backslash\Omega, \end{aligned} \tag{3.58}$$

$$y_r = 3\,\delta^2 r\,\cos^2\vartheta + \delta^2 r \qquad \text{in } \Omega, \tag{3.59}$$

$$y_\vartheta = -3\,\delta^2 r\,\cos\vartheta\,\sin\vartheta \qquad \text{in } \Omega. \tag{3.60}$$

We now require that v_r, v_ϑ, τ_{rr} and $\tau_{r\vartheta}$ be continuous at the interface $\partial\Omega$ $(r=R)$. This leads to the following conditions

$$[\![w_i]\!] = -\frac{1}{2}\,[\![u_{i,j}\,u_j]\!] \qquad \text{on } \partial\Omega, \tag{3.61}$$

$$[\![\tau'_{ij} n_j]\!] = -[\![\tau^*_{ij} n_j + (q^{+} + q^{*} + q_1) n_i]\!] \qquad \text{on } \partial\Omega, \tag{3.62}$$

where $[\![\cdot]\!]$ represents the jump discontinuity. By matching various angular dependencies, these equations provide ten linearly independent algebraic equations in the ten unknowns $f_1,\ldots,f_5$ and $m_1,\ldots,m_5$. They are solved using the symbolic manipulator VAXIMA and the resulting expressions, being somewhat lengthy, are given in the Appendix.

This completes the second order solution, for the particular boundary value problem considered. We proceed to homogenization in the subsequent section.

4. Homogenization

The problem of determination of the effective properties for nonlinear elastic materials was addressed by Hill (1972). He established an overall constitutive law for the nonlinear material by relating the volume average of the strain energy to that of the deformation gradient. Cowin (1977) has also addressed the same topic.

Let the volume average of any field quantity $\underset{\sim}{G}$ be defined as

$$\bar{\underset{\sim}{G}} = \frac{1}{V}\int_B \underset{\sim}{G}(\underset{\sim}{X})\, d\underset{\sim}{X}, \tag{4.1}$$

where V is the volume of B. Under homogeneous displacement boundary conditions

$$\underset{\sim}{x} = \bar{\underset{\sim}{F}}(t)\, \underset{\sim}{X} \qquad \text{on } \partial B, \tag{4.2}$$

the average strain energy function, $U \equiv \overline{W(\underset{\sim}{F})}$, is given by [Hill (1972)]

$$U = U(\bar{F}). \tag{4.3}$$

This result provides the basis for the homogenization procedure that follows.

Before proceeding further, however, we would show that if both the matrix and the inhomogeneity are incompressible then the effective homogeneous medium is also incompressible. To prove this we write

$$\det F \, \delta_{AB} = F_{iA} \, F^*_{iB}, \tag{4.4}$$

where F^* denotes the adjugate of F. Then

$$(\det F \, \delta_{AB})_{,B} = (F_{iA} F^*_{iB})_{,B} = F^*_{iB} F_{iA,B} + F^*_{iB,B} F_{iA}. \tag{4.5}$$

But

$$(\det F \, \delta_{AB})_{,B} = (\det F)_{,A} = \frac{\partial \det F}{\partial F_{iD}} F_{iD,A} = F^*_{iD} F_{iD,A}$$

$$= F^*_{iD} F_{iA,D}, \tag{4.6}$$

where we used the fact that $F_{iA,D} = F_{iD,A}$. Equating (4.5) and (4.6) gives

$$F_{iA} F^*_{iB,B} = 0 \qquad \text{or} \qquad F^T(\text{Div } F^*) = 0. \tag{4.7}$$

Since F^T is nonsingular, the above implies that

$$\text{Div } F^* = 0 \qquad \text{or} \qquad F^*_{iA,A} = (J \, X_{A,i})_{,A} = 0. \tag{4.8}$$

Using the assumption that $J = 1$, the above gives

$$X_{A,iA} = 0. \tag{4.9}$$

Then incompressibility of both phases and the boundary condition (4.2) along with the above imply

$$\begin{aligned} 0 &= \int_B \dot{F}_{iA} F^{-1}_{Ai} \, d\underline{X} = \int_B \dot{x}_{i,A} X_{A,i} \, d\underline{X} = \int_B (\dot{x}_i X_{A,i})_{,A} \, d\underline{X} \\ &= \int_{\partial B} \dot{x}_i X_{A,i} N_A \, d\underline{A} = \int_{\partial B} \dot{\bar{F}}_{iB} X_B X_{A,i} N_A \, d\underline{A} \\ &= \dot{\bar{F}}_{iB} \int_B \delta_{BA} X_{A,i} \, d\underline{X} = V \dot{\bar{F}}_{iB} \overline{F^{-1}_{Bi}} = V \dot{\bar{F}}_{iB} \bar{F}^{-1}_{Bi}, \end{aligned} \tag{4.10}$$

where the last equality is obtained by using the identity $\underline{I} = \overline{\underline{F}\,\underline{F}^{-1}}$ and invoking the boundary condition (4.2). Equation (4.10) establishes the desired result.

We now need to record the expression for strain energy function, W, in second order elasticity. This has been reported in various sources such as Toupin and Bernstein (1961) and Johnson (1985). The expression given in the former can be written in terms of the Lagrangian strain tensor as

$$W = \frac{1}{2} L^1_{ijkl} E_{ij} E_{kl} + \frac{1}{6} L^2_{ijklmn} E_{ij} E_{kl} E_{mn}, \tag{4.11}$$

where

$$L^1_{ijkl} = \lambda \, \delta_{ij} \, \delta_{kl} + 2 \mu \, I_{ijkl}, \tag{4.12}$$

$$\begin{aligned} L^2_{ijklmn} = \nu_1 \delta_{ij} \delta_{kl} \delta_{mn} &+ 2 \nu_2 (\delta_{ij} \, I_{klmn} + \delta_{kl} \, I_{mnij} + \delta_{mn} \, I_{ijkl}) \\ &+ 8 \nu_3 \, I_{ijklmn}), \end{aligned} \tag{4.13}$$

$$I_{ijklmn} = \frac{1}{4}(\delta_{ik} I_{jlmn} + \delta_{jl} I_{ikmn} + \delta_{il} I_{jkmn} + \delta_{jk} I_{ilmn}), \tag{4.14}$$

$$I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{4.15}$$

To find the corresponding expression for the incompressible case (4.11) is expanded using the above relations. In addition, the incompressibility condition implies

$$E_{ii} = H_{ii} + \frac{1}{2} H_{ij} H_{ij} = O(\varepsilon^2). \tag{4.16}$$

Then (4.11) reduces to

$$W = \mu I_{ijkl} E_{ij} E_{kl} + \frac{4}{3} \nu_3 I_{ijklmn} E_{ij} E_{kl} E_{mn} + O(\varepsilon^4), \tag{4.17}$$

which upon using (4.14) and (4.15) simplifies to

$$W = \mu E_{ij} E_{ji} + \frac{4}{3} \nu_3 E_{im} E_{in} E_{mn} = \mu \operatorname{Tr} \underline{E}^2 + \frac{4}{3} \nu_3 \operatorname{Tr} \underline{E}^3. \tag{4.18}$$

This expression is also reported in Ogden (1984).

We would like to relate the second order constant ξ, which we have been using in the previous section, to the corresponding one in (4.18), namely ν_3. Recall that W is given in terms of C_1 and C_2 by (2.14). Once the Lagrangian strain $\underline{E}$ is expressed in terms of $\underline{C}$ then the principal invariants of $\underline{C}$ may be used to obtain

$$\operatorname{Tr} \underline{E}^2 = \frac{1}{2}(H_{ij}H_{ij} + H_{ij}H_{ji} + 2 H_{ij}H_{ik}H_{jk}) + O(\varepsilon^4), \tag{4.19}$$

$$\operatorname{Tr} \underline{E}^3 = \frac{1}{8}(-6 H_{ii} + 3 H_{ij}H_{ji} + 6 H_{ij}H_{ik}H_{jk}) + O(\varepsilon^4). \tag{4.20}$$

These are substituted in (4.18) and the result is compared to the expression for W given in (2.14) to conclude

$$\mu = 2\,(C_1 + C_2), \tag{4.21}$$

$$\nu_3 = -2\,(C_1 + 2\,C_2). \tag{4.22}$$

Recalling (2.32) the above yields

$$\xi = \nu_3 + \frac{3}{2}\,\mu. \tag{4.23}$$

It is the pair of effective constants μ and ξ which will be determined in what follows.

Next we express the strain energy W in terms of the first and second order displacement fields. To that end, let

$$e^{(1)}_{ij} \equiv e_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i}), \tag{4.24}$$

$$e^{(2)}_{ij} \equiv \frac{1}{2}\,(v_{i,j} + v_{j,i}). \tag{4.25}$$

These are then used in (4.19) and (4.20) to obtain

$$\mathrm{Tr}\,\underset{\sim}{E}^2 = \varepsilon\, e^{(1)}_{ik} e^{(1)}_{ik} + \varepsilon^2\,(2\,u_{i,k} e^{(2)}_{ik} + e^{(1)}_{ik} u_{j,i} u_{j,k}), \tag{4.26}$$

$$\mathrm{Tr}\,\underset{\sim}{E}^3 = \varepsilon^2\,\mathrm{Tr}\,\underset{\sim}{e}^3. \tag{4.27}$$

Then the strain energy function in (4.18) is written as

$$W = \varepsilon W_1 + \varepsilon^2 W_2, \tag{4.28}$$

where

$$W_1 = \mu e_{ik} e_{ik} = \mu \operatorname{Tr} \underset{=}{e}^2, \tag{4.29}$$

$$\begin{aligned} W_2 &= \mu(2 u_{i,k} e_{ik}^{(2)} + u_{j,i} u_{j,k} e_{ik}) + \frac{4}{3} \nu_3 \operatorname{Tr} \underset{=}{e}^3 \\ &= \mu (2 u_{i,k} e_{ik}^{(2)} + u_{j,i} u_{j,k} e_{ik}) + \frac{4}{3} (\xi - \frac{3}{2} \mu) \operatorname{Tr} \underset{=}{e}^3. \end{aligned} \tag{4.30}$$

We assume the strain energy function in the effective homogeneous medium has the same form as in the composite, with the overall constants μ and ξ. To determine these constants we consider a spherical body of radius R_0 made of the matrix material which contains a spherical inhomogeneity of radius R. This gives the volumetric concentration $c \equiv \dfrac{R^3}{R_0^3}$. A similar body, made entirely of the homogeneous effective material, is also considered. Under identical boundary conditions the total strain energy of the two materials are equated to yield the expressions for the overall constants. The dilute concentration limit is obtained by expanding these expressions in powers of c and retaining only the terms of up to order one in c.

In particular, considering the first order case, the displacements in the matrix given in (3.4) and (3.5) are evaluated at $r=R_0$ and then imposed, as displcement boundary conditions, on the boundary of the homogeneous body. The solution to this problem is found to be

$$u_r^h = (2 g_1 r - 6 g_3 r^3)(3 \cos^2\vartheta - 1) / (4 \mu), \tag{4.31}$$

$$u_\vartheta^h = 3 (5 g_3 r^3 - g_1 r) \cos \vartheta \sin \vartheta / (2 \mu), \tag{4.32}$$

where

$$g_1 = (2\,\mu\,\mu_m E\, R_0^{\,5} + 5\,\alpha\,\mu\, R_0^{\,2} - 21\,\gamma\,\mu) \,/\, (2\,\mu_m R_0^{\,5}), \tag{4.33}$$

$$g_3 = (\alpha\,\mu\, R_0^{\,2} - 5\,\gamma\,\mu) \,/\, (2\,\mu\, R_0^{\,7}), \tag{4.34}$$

and the superscript h refers to the homogeneous material. Substituting the above in (4.29) an expression for W_1 is obtained which upon integration over the spherical region $0 \le r \le R_0$, yields

$$U_1^h = \pi\,\mu\,(10\,\mu_m^2\, E^2 R_0^{\,10} + 8\,\alpha\,\mu_m E\, R_0^{\,7} + 13\,\alpha^2 R_0^{\,4} - 114\,\alpha\,\gamma\, R_0^{\,2} + 285\,\gamma^2) \,/\, (5\,\mu_m^2 R_0^{\,7}). \tag{4.35}$$

The corresponding expression for the inhomogeneous material is obtained by using the appropriate first order displacement field, (3.2) to (3.5), and performing the integration in the inhomogeneity and the matrix. The result is found to be

$$U_1 = 2\,\pi(5\,\mu_m^2\, E^2 R_0^{\,10} - 6\,\beta^2 R_0^{\,4} + 48\,\beta\,\gamma\, R_0^{\,2} - 120\,\gamma^2) \,/\, (5\,\mu_m R_0^{\,7}) - 2\,\pi(5\,\mu_m^2\, R^{10} E^2 - 6\,\beta^2 R^4 + 48\,\beta\,\gamma\, R^2 - 120\,\gamma^2) \,/\, (5\,\mu_m R^7) + 8\,\pi\,\delta^2 \mu_f R^3. \tag{4.36}$$

The effective shear modulus μ is then determined by equating (4.35) and (4.36). The resulting expression is precisely the result of Eshelby (1957). When μ is expanded in powers of c and terms up to the first order are retained, the familiar expression for the overall shear modulus [Christensen (1979)] results,

$$\mu = \mu_m - [5\,\mu_m(\mu_m - \mu_f)\,c] \,/\, (3\,\mu_m + 2\,\mu_f). \tag{4.37}$$

To find the second order constant, the second order solution in the matrix, as given by (3.34), (3.35), (3.39) and (3.40), is evaluated at $r=R_0$ and then imposed on the boundary of the homogeneous body. The displacement field is then found to be

$$w_r^h = -[(175\ h_4\ \mu^2 r^5 - 35\ h_2\ \mu^2 r^3)\cos^4\vartheta + (-150\ h_4\ \mu^2 r^5 + (18\ h_3 + 30\ h_2)\mu^2 r^3 - 6\ h_1 \mu^2 r)\cos^2\vartheta + 15\ h_4\ \mu^2 r^5 + (-6\ h_3 - 3\ h_2)\ \mu^2 r^3 + 2\ h_1\ \mu^2 r]\ /\ (4\ \mu^3), \tag{4.38}$$

$$w_\vartheta^h = [(245\ h_4\ \mu^2 r^5 - 35\ h_2\ \mu^2 r^3)\cos^3\vartheta + ((21\ e_3^2\ \xi - 105\ h_4\ \mu^2) r^5 + (30\ h_3 + 15\ h_2)\mu^2 r^3 - 6\ h_1 \mu^2 r)\cos\vartheta]\ \sin\vartheta\ /\ (4\ \mu^3), \tag{4.39}$$

where

$$h_1 = -[210\ \mu^3 m_4 R_0^{\ 5} + 147\ \mu^3 m_2 R_0^{\ 3} + (288\ a_3 + 448\ a_1)\mu^3 R_0^{\ 2} + (480\ a_4 + 840\ a_2)\mu^3]\ /\ (28\ \mu^2 \mu_m R_0^{\ 8}), \tag{4.40}$$

$$h_2 = -(1269\ \mu\ m_5 R_0^{\ 3} + 480\ a_3 \mu\ R_0^{\ 2} + 385\ \mu\ m_3 R_0 + 768\ a_4 \mu)/(56\ \mu_m R_0^{\ 10}), \tag{4.41}$$

$$h_3 = -[42\ \mu^3 m_4 R_0^{\ 5} + 35\ \mu^3 m_2 R_0^{\ 3} + (72\ a_3 + 112\ a_1)\mu^3 R_0^{\ 2} + (128\ a_4 + 224\ a_2)\mu^3]\ /\ (28\ \mu^2 \mu_m R_0^{\ 10}), \tag{4.42}$$

$$h_4 = -(980\ \mu\ m_5 R_0^{\ 3} + 384\ a_3 \mu\ R_0^{\ 2} + 315\ \mu\ m_3 R_0 + 640\ a_4 \mu)\ /\ (280\ \mu_m R_0^{\ 12}). \tag{4.43}$$

The complete second order solution in the homogeneous medium is obtained by using (2.50). Finally, the relevant part of the strain energy, W_2, is determined by integrating (4.3) over the homogeneous body.

A similar procedure in the inhomogeneous medium yields the strain energy which, upon being equated to that of the homogeneous medium, results in an expression for ξ in which μ is eliminated using (4.37). Once this is expanded in powers of c and terms of order up to one are retained we find the relatively simple expression

$$\xi = \xi_m + 5\,(126\,\mu_m^4 - 63\,\mu_f\,\mu_m^3 - 200\,\xi_m\,\mu_m^3 + 350\,\xi_f\,\mu_m^3 - 27\,\mu_f^2\,\mu_m^2 - 240\,\xi_m\,\mu_f\,\mu_m^2 - 36\,\mu_f^3\,\mu_m + 30\,\xi_m\,\mu_f^2\,\mu_m + 60\,\xi_m\,\mu_f^3)\,c \,/\, [14\,(3\,\mu_m + 2\,\mu_f)^3]. \tag{4.44}$$

This is the expression for the overall second order constant in terms of the elastic properties of the constituents and the volume fraction under the dilute concentration assumption.

5. Discussion

The expression obtained above for the overall second order constant can be reduced to the case of rigid spherical fillers in an incompressible matrix by taking its limit as μ_f goes to infinity. We find that

$$\xi_r = \xi_m + \frac{15}{28}\,(5\,\xi_m - 3\,\mu_m)\,c. \tag{5.1}$$

This provides a good approximation for rubber composites containing rigid inclusions.

As mentioned in the Introduction, Ogden (1974) determined the overall second order bulk modulus. He found that, when the embedded phase consists of the rigid particles, the overall bulk modulus is less than that of the matrix. This is in contrast to the behavior of the first order bulk modulus which increases under similar conditions.

We now examine whether similar behavior holds for the second order modulus ξ. The following typical values of the two elastic constants are taken from Hashin (1985).

$$C_1 = 0.5 \text{ MPa}, \qquad C_2 = 0.05 \text{ MPa}. \tag{5.2}$$

Then using (2.20), (2.32) and (4.22) we find

$$\mu_m = 1.1 \text{ MPa}, \qquad \nu_{3,m} = -1.2 \text{ MPa}, \qquad \xi_m = 0.45 \text{ MPa}. \tag{5.3}$$

The second order constant ν_3 is generally found to be negative. Using (5.1) it is then found that

$$\xi = \xi_m - 0.56\ c, \tag{5.4}$$

which indicates that $\xi < \xi_m$, similar to what is observed for the second order bulk modulus.

From the derivation presented here, it is clear that finding explicit expressions for the overall second order constants for an unconstrained material involves prohibitively lengthy computations. The advantages of using the constraint of incompressibility are that, on the one hand, the number of elastic constants is reduced from five to two and, on the other hand, the presence of the undetermined pressure term allows one to absorb the conservative part of the body force thereby simplifying the task of obtaining the second order solution.

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Appendix

The constants determined from equations (3.61) and (3.62) are given as follows.

$$m_1 = 0,$$

$$\begin{aligned} m_2 = & -(846\ \xi_m\mu_m^6 + 705\ \xi_m\mu_f\ \mu_m^5 + 72\ j\ \xi_m\mu_m^5 - 10\ j^2\mu_m^5 \\ & - 1880\ \xi_m\mu_f^2\ \mu_m^4 + 132\ j\ \xi_m\mu_f\ \mu_m^4 - 235\ j^2\mu_f\ \mu_m^4 - 674\ j^2\ \xi_m\mu_m^4 \\ & + 1400\ j^2\xi_f\ \mu_m^4 - 1175\ \xi_m\mu_f^3\ \mu_m^3 - 28\ j\ \xi_m\mu_f^2\ \mu_m^3 - 200\ j^2\mu_f^2\ \mu_m^3 \\ & - 1183\ j^2\xi_m\mu_f\ \mu_m^3 + 2800\ j^2\ \xi_f\mu_f\ \mu_m^3 + 544\ j^2\ \xi_m\mu_m^3 \\ & + 940\ \xi_m\mu_f^4\ \mu_m^2 - 128\ j\ \xi_m\mu_f^3\ \mu_m^2 + 445\ j^2\mu_f^3\ \mu_m^2 - 512\ j^2\xi_m\mu_f^2\mu_m^2 \\ & + 68\ j^2\xi_m\mu_f\ \mu_m^2 + 564\ \xi_m\mu_f^5\ \mu_m - 48\ j\ \xi_m\mu_f^4\ \mu_m \\ & - 2631\ j^2\xi_m\mu_f^3\ \mu_m - 1768\ j^2\xi_m\mu_f^2\ \mu_m + 800\ j^2\xi_m\mu_f^4 \\ & + 1156\ j^2\xi_m\mu_f^3)\ R^5E^2\ /\ (14\ h\ j^4), \end{aligned}$$

$$\begin{aligned} m_3 = & \ 6\ (\mu_m - \mu_f)\ (2160\ \xi_m\mu_m^5 + 3096\ \xi_m\mu_f\ \mu_m^4 - 4320\ j\ \xi_m\mu_m^4 \\ & + 5520\ j^2\ \mu_m^4 - 1128\ \xi_m\mu_f^2\ \mu_m^3 - 10512\ j\ \xi_m\mu_f\ \mu_m^3 \\ & + 21885\ j^2\ \mu_f\ \mu_m^3 - 23400\ j^2\ \xi_m\mu_m^3 - 3072\ \xi_m\mu_f^3\ \mu_m^2 \\ & - 8256\ j\ \xi_m\mu_f^2\ \mu_m^2 + 15120\ j^2\ \mu_f^2\ \mu_m^2 - 120660\ j^2\ \xi_m\mu_f\ \mu_m^2 \\ & + 21080\ j^2\ \xi_m\mu_m^2 - 1056\ \xi_m\mu_f^4\ \mu_m - 2112\ j\ \xi_m\mu_f^3\ \mu_m \\ & + 183090\ j^2\xi_m\mu_f^2\ \mu_m + 80445\ j^2\ \xi_m\mu_f\ \mu_m - 33990\ j^2\ \xi_m\mu_f^3 \\ & -101525\ j^2\ \xi_m\mu_f^2)R^7E^2\ /\ [7\ j^4(488\ \mu_m^2 + 1611\ \mu_f\mu_m + 1030\ \mu_f^2)], \end{aligned}$$

$$
\begin{aligned}
m_4 = {} & (6768\ \xi_m \mu_m^6 + 3525\ \xi_m \mu_f\ \mu_m^5 + 576\ j\ \xi_m \mu_m^5 + 720\ j^2\ \mu_m^5 \\
& - 13630\ \xi_m \mu_f^2\ \mu_m^4 + 876\ j\ \xi_m \mu_f\ \mu_m^4 + 45\ j^2\ \mu_f \mu_m^4 - 5392\ j^2\ \xi_m \mu_m^4 \\
& + 5600\ j^2\ \xi_f \mu_m^4 - 6815\ \xi_m \mu_f^3\ \mu_m^3 - 284\ j\ \xi_m \mu_f^2\ \mu_m^3 - 2250\ j^2\ \mu_f^2 \mu_m^3 \\
& - 9539\ j^2\ \xi_m \mu_f\ \mu_m^3 + 14000\ j^2\ \xi_f \mu_f\ \mu_m^3 + 1632\ j^2\ \xi_m \mu_m^3 \\
& + 6580\ \xi_m \mu_f^4\ \mu_m^2 - 864\ j\ \xi_m \mu_f^3\ \mu_m^2 + 1485\ j^2\ \mu_f^3\ \mu_m^2 \\
& + 9514\ j^2\ \xi_m \mu_f^2\ \mu_m^2 + 2244\ j^2\ \xi_m \mu_m^2\ \mu_f + 3572\ \xi_m\ \mu_f^5\ \mu_m \\
& - 304\ j\ \xi_m \mu_m \mu_f^4 - 14343\ j^2 \xi_m \mu_m \mu_f^3 - 9384\ j^2\ \xi_m \mu_m \mu_f^2 + 160\ j^2\ \xi_m \mu_f^4 \\
& + 5508\ j^2\ \xi_m \mu_f^3)\ R^3 E^2\ /\ (84\ h\ j^4),
\end{aligned}
$$

$$
\begin{aligned}
m_5 = {} & -9\ (\mu_m - \mu_f)(1152\ \xi_m \mu_m^5 + 1608\ \xi_m \mu_m^4\ \mu_f - 2304\ j\ \xi_m \mu_m^4 \\
& + 1480\ j^2\ \mu_m^4 - 616\ \xi_m \mu_m^3 \mu_f^2 - 5520\ j\ \xi_m \mu_m^3 \mu_f + 6875\ j^2\ \mu_m^3 \mu_f \\
& - 12480\ j^2 \xi_m \mu_m^3 - 1600\ \xi_m \mu_m^2\ \mu_f^3 - 4288\ j\ \xi_m \mu_m^2\ \mu_f^2 + 4980\ j^2\ \mu_m^2\ \mu_f^2 \\
& - 74620\ j^2 \xi_m \mu_m^2 \mu_f + 6200\ j^2 \xi_m \mu_m^2 - 544\ \xi_m \mu_m \mu_f^4 - 1088\ j\ \xi_m \mu_m \mu_f^3 \\
& + 86650\ j^2 \xi_m \mu_m \mu_f^2 + 36425\ j^2 \xi_m \mu_m \mu_f + 3090\ j^2 \xi_m \mu_f^3 \\
& - 42625\ j^2 \xi_m \mu_f^2)\ R^5 E^2 / [14\ j^4 (488\ \mu_m^2 + 1611\ \mu_f \mu_m + 1030\ \mu_f^2)],
\end{aligned}
$$

$$
\begin{aligned}
f_1 = {} & -\mu_f (8037\ \xi_m \mu_m^6 + 1410\ \xi_m \mu_m^5\ \mu_f + 684\ j\ \xi_m\ \mu_m^5 + 705\ j^2\ \mu_m^5 \\
& - 14335\ \xi_m \mu_m^4\ \mu_f^2 + 804\ j\ \xi_m \mu_m^4\ \mu_f + 930\ j^2\ \mu_m^4\ \mu_f + 1597\ j^2\ \xi_m \mu_m^4 \\
& - 700\ j^2\ \xi_f\ \mu_m^4 - 4700\ \xi_m \mu_m^3\ \mu_f^3 - 416\ j\ \xi_m \mu_m^3\ \mu_f^2 - 1875\ j^2\ \mu_m^3\ \mu_f^2 \\
& - 8426\ j^2 \xi_m \mu_m^3 \mu_f + 5600\ j^2 \xi_f \mu_m^3 \mu_f - 1632\ j^2 \xi_m \mu_m^3 + 6580\ \xi_m \mu_m^2 \mu_f^4 \\
& - 816\ j\ \xi_m \mu_m^2\ \mu_f^3 + 240\ j^2\ \mu_m^2\ \mu_f^3 + 7161\ j^2 \xi_m \mu_m^2\ \mu_f^2 + 4896\ j^2 \xi_m \mu_m^2\ \mu_f \\
& + 3008\ \xi_m \mu_m \mu_f^5 - 256\ j\ \xi_m \mu_m\ \mu_f^4 - 5232\ j^2\ \xi_m \mu_m \mu_f^3 - 4896\ j^2\ \xi_m \mu_m\ \mu_f^2 \\
& + 1632\ j^2 \xi_m\ \mu_f^3)\ E^2\ /\ (28\ h\ j^4\ \mu_m),
\end{aligned}
$$

$$
\begin{aligned}
f_2 = {} & 15\,\mu_f\,(\mu_m - \mu_f)\,(1836\,\xi_m\,\mu_m^5 + 2340\,\xi_m\,\mu_f\,\mu_m^4 - 3672\,j\,\xi_m\,\mu_m^4 \\
& - 1530\,j^2\,\mu_m^4 - 1056\,\xi_m\,\mu_f^2\,\mu_m^3 - 8352\,j\,\xi_m\,\mu_f\,\mu_m^3 - 1695\,j^2\,\mu_f\,\mu_m^3 \\
& + 34278\,j^2\xi_m\,\mu_m^3 - 2352\,\xi_m\,\mu_f^3\,\mu_m^2 - 6240\,j\xi_m\,\mu_f^2\,\mu_m^2 - 240\,j^2\,\mu_f^2\,\mu_m^2 \\
& - 47217\,j^2\,\xi_m\mu_f\mu_m^2 - 16120\,j^2\,\xi_m\mu_m^2 - 768\,\xi_m\mu_f^4\mu_m - 1536\,j\,\xi_m\mu_f^3\mu_m \\
& + 16899\,j^2\,\xi_m\,\mu_f^2\,\mu_m + 24645\,j^2\,\xi_m\,\mu_f\,\mu_m - 8525\,j^2\,\xi_m\,\mu_f^2\,)\,E^2 \\
& /\,(\,7\,j^4\,\mu_m\,(488\,\mu_m^2 + 1611\,\mu_f\,\mu_m + 1030\,\mu_f^2\,)\,R^2),
\end{aligned}
$$

$$
\begin{aligned}
f_3 = {} & -\,(1251\,\xi_m\,\mu_m^4 - 834\,\xi_m\,\mu_f\,\mu_m^3 + 30\,j^2\,\mu_m^3 - 1529\,\xi_m\,\mu_f^2\,\mu_m^2 \\
& - 90\,j^2\mu_f\mu_m^2 + 21\,j^2\xi_m\mu_m^2 + 400\,j^2\xi_f\mu_m^2 + 556\,\xi_m\mu_f^3\mu_m + 60\,j^2\mu_f^2\,\mu_m \\
& + 358\,j^2\,\xi_m\,\mu_f\,\mu_m + \xi_m\,\mu_f^4 - 179\,j^2\,\xi_m\,\mu_f^2\,)\,E^2\,/\,(24\,j^4),
\end{aligned}
$$

$$
\begin{aligned}
f_4 = {} & -\,\mu_f\,(\,1269\,\xi_m\,\mu_m^6 + 108\,j\,\xi_m\,\mu_m^5 - 15\,j^2\,\mu_m^5 - 2115\,\xi_m\,\mu_f^2\,\mu_m^4 \\
& + 108\,j\xi_m\mu_f\mu_m^4 + 60\,j^2\mu_f\mu_m^4 + 909\,j^2\xi_m\mu_m^4 - 700\,j^2\xi_f\mu_m^4 \\
& - 470\,\xi_m\mu_f^3\mu_m^3 - 72\,j\xi_m\mu_f^2\mu_m^3 - 75\,j^2\mu_f^2\mu_m^3 - 1092\,j^2\xi_m\mu_f\mu_m^3 \\
& - 544\,j^2\xi_m\mu_m^3 + 940\,\xi_m\mu_f^4\mu_m^2 - 112\,j\xi_m\mu_f^3\mu_m^2 + 30\,j^2\mu_f^3\mu_m^2 \\
& + 757\,j^2\xi_m\mu_f^2\mu_m^2 + 952\,j^2\xi_m\mu_f\mu_m^2 + 376\,j\xi_m\mu_f^5\mu_m - 32\,j\xi_m\mu_f^4\mu_m \\
& + 126\,j^2\xi_m\mu_f^3\mu_m - 272\,j^2\xi_m\mu_f^2\mu_m - 136\,j^2\xi_m\mu_f)\,E^2/(28\,h\,j^4\mu_m R^2),
\end{aligned}
$$

$$
\begin{aligned}
f_5 = {} & 9\mu_f(\mu_m - \mu_f)\,(396\,\xi_m\mu_m^5 + 492\,\xi_m\mu_f\mu_m^4 - 792\,j\xi_m\mu_m^4 - 330\,j^2\mu_m^4 \\
& - 232\,\xi_m\mu_f^2\mu_m^3 - 1776\,j\xi_m\mu_f\mu_m^3 - 355\,j^2\mu_f\mu_m + 10350\,j^2\xi_m\mu_m^3 \\
& - 496\,\xi_m\mu_f^3\mu_m^2 - 1312\,j\xi_m\mu_f^2\mu_m^2 - 50\,j^2\mu_f^2\mu_m^2 - 6400\,j^2\xi_m\mu_f\mu_m^2 \\
& - 4960\,j^2\xi_m\mu_m^2 - 160\,\xi_m\mu_f^4\mu_m - 320\,j\xi_m\mu_f^3\mu_m - 3110\,j^2\xi_m\mu_f^2\mu_m \\
& + 3410\,j^2\xi_m\mu_f\mu_m + 1550\,j^2\xi_m\mu_f^2\,)\,E^2 \\
& /(\,7\,j^4\mu_m\,(\,488\,\mu_m^2 + 1611\,\mu_f\mu_m + 1030\,\mu_f^2)\,R^4),
\end{aligned}
$$

where

$$j \equiv 3\,\mu_m + 2\,\mu_f,$$

$$h \equiv 8\,\mu_m^2 + 19\,\mu_f\mu_m + 8\,\mu_f^2.$$