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Topics in Asset Pricing and Market Microstructure

by

Anna Amirdjanova

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

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in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Nancy Wallace, Chair

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Spring 2015

Topics in Asset Pricing and Market Microstructure

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Abstract

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Professor Nancy Wallace, Chair

This dissertation addresses various aspects of asset pricing theory in the following three contexts: the case of insider trading (of stocks) with uninformed biased traders, the case of trading of real options (specifically, of the option to sell a real indivisible asset), and the case of house pricing and construction of better house price indices.

Chapter 1 examines the effects of insider trading on uninformed traders with bounded rationality in the context of a continuous-time Kyle-type model with a single perfectly informed risk-neutral agent (insider), a competitive risk-neutral market maker and a set of biased uninformed traders. Two cases of behavioral biases or bounded rationality on the part of the uninformed traders are considered. In the first case the uninformed traders' order flow has a non-zero covariation with a set of public signals (where positive covariation describes aggregate momentum strategies among the uninformed investors in reaction to news, while negative covariation indicates that the uninformed traders are predominantly contrarians). In the second case, the order flow from the uninformed traders has a strictly positive or a strictly negative covariance between its increments and is no longer Markov. The equilibrium strategy of the insider, taking into account such biases, is derived in both cases and the effects of the biases on the equilibrium price of the underlying asset are considered. The question of whether such biases benefit or harm the uninformed traders is answered.

In Chapter 2 a class of mixed stochastic control/optimal stopping problems arising in the problem of finding the best time to sell an indivisible real asset, owned by a risk averse utility maximizing agent, is considered. The agent has power type utility based on the ℓ_α -type aggregator and has access to a frictionless financial market which can be used to partially hedge the risk associated with the real asset if correlations between the financial assets and the real asset value are nonzero. The solution to the problem of finding the optimal time to sell the real asset is characterized in terms of solution to a certain free boundary problem. The latter involves a nonlinear partial differential equation and includes, as special case with $\alpha = 1$, the Hamilton-Jacobi-Bellman equation found in (Evans, Henderson, Hobson, 2008). Comparisons with the case of exponential utility are also given.

Due to lack of data, the U.S. primarily uses repeat-sales indices to measure real-estate returns, despite the serious shortcomings of these indices. Making use of a newly available data set that contains both time-varying characteristics for all properties in the U.S. and transaction details for those properties that traded, in Chapter 3 a new hedonic house-price index is developed that overcomes these shortcomings by allowing house prices and returns to depend on house characteristics and on local and national macroeconomic factors. The index is estimated using Markov Chain Monte Carlo (MCMC) linear filtering techniques and results in significant differences, in both the level and volatility of prices, between the new estimates and those from the Federal Housing Finance Board's weighted-repeat-sales (WRS) price index. This suggests that the new index is significantly superior to repeat-sales indices as a measure of U.S. real-estate returns for economic forecasting, mortgage valuation, and bank stress tests.

To my daughter, Katie

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Chapter 1

Insider Trading: Effects on Uninformed Traders with Bounded Rationality

1.1 Introduction

The discrepancy between the efficient market hypothesis and the empirical evidence, showing real-life deviations of stock prices from their fundamental values, has long been a subject of debate among finance theorists. Among potential explanations for the observed price differences lies the concept of bounded rationality of some investors, a phenomenon which has been well documented in a number of empirical and experimental studies made by behavioral economists.

The idea behind this approach is to view the market as consisting of two types of investors: “arbitrageurs,” who form fully rational expectations about asset returns, and “noise traders” or “liquidity traders” whose trading strategies may be subject to systematic biases. The noise trader approach ((Shleifer and Summers, 1990)) makes an interesting case for why the biases of noise traders may result in substantial mispricing of securities. To a large extent the explanation is based on two key observations. First, in real markets arbitrage is limited and risky. It is limited due to the potential absence of a close substitute portfolio for a given security, so that finitely many arbitrageurs no longer have a perfectly elastic demand for the security at the price of its substitute portfolio. The arbitrage is also risky both due to fundamental risk, since the actual future realization of dividends or news about them can deviate from expectations, and due to the resale price risk, which comes into the picture due to the arbitrageur’s finite horizon. As a result, risk may render arbitrage an ineffective protection against mispricing in general and against eliminating the effects of uninformed trading on prices in particular. Second, the demand shifts of noise traders are often correlated, hence, their trades cannot be expected to cancel out, resulting in aggregate shifts in demand. Many noise traders’ strategies are based on common pseudo-signals, which reflect certain investor

sentiment that is not fully justified by fundamental news. Psychological experiments (Tversky, Kahneman, 1974; Rabin, 1998) show that judgment biases, exhibited by human subjects as a result of information processing, are far from arbitrary and do not represent completely random mistakes. Instead there are often common cognitive biases which are likely to lead to systematic deviations from fully rational behavior and result in correlated demand shifts among the uninformed investors in response to non-fundamental news.

Many such systematic biases have been documented in the empirical literature on financial markets. Overconfidence (see, for example, Alpert and Raiffa, 1982; Barber, Odean, 2001; Malmendier, Tate, 2005), disposition effect (Barberis and Xiong, 2009; Odean, 1998), “trend chasing” based on extrapolation of past time series (Andreassen and Kraus, 1988; Brunnermeier, Nagel, 2004), and overreaction to news, where too little weight is given to long-term indicators about the fundamentals and too much weight is placed on new information (Tversky and Kahneman, 1982; De Bondt, Thaler, 1985; Wang, Burton, Power, 2004; Fang, 2013), are just a sample of judgment biases systematically affecting noise traders. Such biases give rise to a host of popular models (Shiller, 1984) that allow trading on noise and fads rather than on fundamental economic signals. Since the same biases also affect the predictions of market gurus/forecasters who have no inside information but nonetheless often have a substantial influence on other uninformed traders, further amplification of the effects of these biases on the liquidity trading and on correlations among strategies of noise traders ensues.

Although one of the central questions in market microstructure concerns the process by which prices come to reflect private information of the informed investors when trading with the uninformed participants in the market, few analytical studies have examined this question in the context of trading with investors with bounded rationality. One such study analyzed a one-period model where irrational and rational semi-informed traders exchange a risky asset with a competitive market maker and unbiased liquidity traders (Germain, Rousseau, and Vanhems, 2007). In that model the irrational traders can either display an optimistic/pessimistic bias by misperceiving the mean of prior information, or display an overconfidence/underconfidence bias by misperceiving the variance of noise in their private signal (see also Benos, 1998). Under the assumption that the market makers are irrational, the authors show that the moderately underconfident traders outperform rational ones, and that overconfidence generates high trading volume.

Another very recent paper (Kyle, Obizhaeva, Wang, 2013) studied a dynamic continuous-time microstructure model with an infinite time horizon in which risk-averse overconfident oligopolistic agents trade a risky asset in zero net supply against a risk-free asset upon observing both public and private signals. The traders in that model agree to disagree about the precision of their private information, but each trader updates rationally (according to the Bayes’ rule) his or her estimate of the asset value by conditioning on the price history, private information, as well as on the public information. (There are no market makers and no liquidity traders in that model.) In the case when the traders’ beliefs disagree significantly, it is shown that an equilibrium arises in which prices immediately reveal the average of all the traders’ private signals, but the traders continue to trade slowly. It is also shown that trading

too fast may destabilize prices and result in price patterns associated with “flash-crashes.”

Another form of bias, the disposition effect, which is typically defined as the tendency to hold losers and sell winners, in a static setting based on a modification of Glosten and Milgrom’s (1985) model, has been shown (Choi, 2014) to have a two-fold impact on the speed of the price discovery process: the uninformed disposition sales decrease bid-ask spreads and decrease the price impact per sale but at the same time increase the number of uninformed sales. Whether the first or the second effect dominates depends on whether the proportion of informed trading is low or high.

The goal of this paper is to study in a dynamic setting the optimal strategies of the insider and the effects of insider trading on the equilibrium price and distribution of profits/losses incurred by the uninformed market participants in the case when the uninformed traders have certain forms of momentum or contrarian biases.

The issue of identifying the constituency for insider trading laws has been a subject of debate in recent years in view of mounting evidence suggesting that, although insider-trading laws presumably protect small individual liquidity investors, it is the group of semi-informed market professionals (or, more generally, the traders with the best quality of information among the remaining market participants) that profits the most (and disproportionately so) when insiders are banned (see, for example, Tighe and Michener (1994)). Moreover, as shown by Buffa (2009) in a dynamic model of strategic risk averse insider trading, mandatory ex-post disclosure of corporate insider trades reduces the informational efficiency of prices and may make the market less liquid because the insider trades less aggressively in more transparent markets.

Our goal here is to study how certain biases arising in the bounded rationality context, specifically, the momentum and contrarian traits among the liquidity traders, affect the resulting wealth transfers and equilibrium decisions of insiders, and what the short- and long-term effects of such biases are on the informativeness of asset prices. By the same logic as above, if the biases of the liquidity traders allow the market maker to more easily filter out the insider’s information from the total order flow, the market can become more transparent. In this case, the insider could strategically trade less aggressively than in the unbiased case, and the informational efficiency of prices could be lower than in the classical case. Conversely, if the biases of liquidity traders make it harder for the market maker to filter out the insider’s order from the aggregate order flow, the market can become less transparent allowing the insider to trade more aggressively, thus accumulating higher terminal wealth.

I start with a Kyle-Back type order-driven continuous-time microstructure model with anonymous trading of a single risky asset in the presence of a perfectly informed insider and biased liquidity traders, where the asset’s price is set by a risk-neutral competitive market maker who sets the price upon observing the total order flow size and a noisy continuous public signal. The latter public signal is observed by all the market participants but, unlike the market maker who applies filtering methods in order to estimate the value of the asset from the public signal, the liquidity traders submit an aggregate order flow which is a Brownian motion covariating with the noisy public signal. The intuition behind the proposed covariation is that, when the mass of liquidity traders is populated by agents who are more

likely to buy the asset when they observe public news indicating good business performance and who are more likely to sell upon hearing bad news, the covariation is positive. On the other hand, if the liquidity trading is dominated by contrarians, the covariation process can be naturally assumed to be negative. One of the findings in this case is that the price sensitivity to the order flow is a non-monotone function. This is in contrast to the classical cases in Kyle (1985) and Back and Pedersen (1998) where the sensitivity of prices to the total order flow, the Kyle's λ , is constant, or the case developed by Baruch (2002), where the informed trader is risk-averse and the resulting Kyle's λ is monotone decreasing over time. The non-monotonicity of the Kyle's λ also arises in markets with no insiders but with multiple semi-informed traders who privately observe imperfect signals, as shown by Back et al. (2000). However, in the latter case the non-monotonicity is a byproduct of competition among the semi-informed agents, whereas in our case the non-monotone dynamics arises because of non-zero covariation between the liquidity trader's flow and the continuous noisy public signal.

Next I analyze the equilibrium trading strategy of the perfectly informed insider and the price setting dynamics exercised by the risk-neutral market maker in the case when some of the liquidity/noise traders generate an order flow which has a strictly positive or a strictly negative covariance between its time increments. A strictly positive covariance between the time increments of the process implies that (random) changes in the order flow today and tomorrow on average have the same signs, meaning that the underlying liquidity order flow process is persistent, and the respective liquidity traders in aggregate continue to buy more of the asset tomorrow if they are doing so today, or, conversely, continue to sell more of the asset tomorrow if they are doing so today. This represents momentum strategies on behalf of some liquidity traders who, on average, decide to buy more of the asset today based on their observations of an increase in the order flow yesterday. Conversely, if the covariance between the time increments of the liquidity flow is strictly negative, then the corresponding liquidity traders on average act as contrarians, selling the asset today if there was an increase in the order flow yesterday, and vice versa buying the asset today if there was a decrease in the order flow yesterday. Such contrarian behavior may, for example, reflect some mean-reversion beliefs of the corresponding liquidity/noise traders about the demand for the asset and the expected price changes.

Note that a nonzero covariance between the time increments of the liquidity order process implies a non-Markovian nature of the process, so that in aggregate the liquidity/noise traders are no longer memoryless. I show that in this case the linear equilibrium no longer exists, but there exists an equilibrium where the insider's optimal order at any time t is equal to the product of the difference between the realized terminal value of the asset and the price of the asset at time t , multiplied by a known (non-random) intensity function, while the equilibrium price at time t is equal to an integral of a certain kernel λ (representing price sensitivity to the total order flow), but where λ is now a function of both t and the "past" s , and the integral is taken with respect to the total order flow at s , with s ranging from 0 to t .

Moreover, I find that, whereas the contrarian strategies driven by negative covariation

with public signal are beneficial to the noise/liquidity traders (in the sense that the insider can extract less terminal profit from trading in this case than in the case of a liquidity order flow uncorrelated with the public signal), the contrarian behavior, arising when the covariance between the time increments of the liquidity order flow is negative and produces short memory, can be very detrimental to the liquidity traders. By contrast momentum strategies of the liquidity/noise traders hurt the insider's terminal wealth in both scenarios, by allowing the market maker to better filter out the insider's order from the total order flow in comparison with the case of dealing with purely unbiased noise/liquidity traders.

The paper is organized as follows. Section 1.2 describes the model with nonzero covariation between public signals and the liquidity order flow and presents the equilibrium strategy for the insider as well as the equilibrium price set by the market maker. Conditions on the correlation parameters, under which the insider's wealth is maximized, are studied. Section 1.3 describes the insider trading model with no public signals but with the liquidity order flow modeled by a certain Gaussian process with correlated increments, which, depending on the sign of correlations, represents the contrarian and the momentum behavior of the liquidity/noise traders, respectively. The equilibrium insider's strategy and the equilibrium price of the underlying asset are also established, and comparisons with the classical insider trading model are made. Section 1.4 summarizes the conclusions. Proofs are collected in Section 1.5.

1.2 Model with Nonzero Covariation between Public Signal and Liquidity Trading

In this section we study the effects of nonzero covariation between liquidity trading and a noisy multidimensional public signal on the equilibrium price, optimal strategy and expected final profits of the insider in a continuous-time Kyle-type setting. The non-zero covariation between liquidity trading and the public signal allows us to address certain real-life biases among liquidity investors (such as the naive tendency to buy upon hearing good news and sell upon hearing bad news, or, conversely, potential contrarian strategies exercised by liquidity traders).

Suppose a single risky asset is traded in continuous auctions by a monopolistic risk-neutral insider, who is perfectly informed at time 0 about the asset's realized value at time $t = 1$, and uninformed liquidity traders. The period of trading is normalized to $[0, 1)$, where at time 1, the liquidation value of the asset is realized and publicly announced. Let $\tilde{v}$ denote the ex-post liquidation value of the risky asset, where we assume that $\tilde{v} \sim N(p_0, \Sigma^{(0)})$. For simplicity, let us also assume that the risk-free rate is equal to zero. At any auction a competitive risk-neutral market maker, who knows the distribution of $\tilde{v}$ but not the realization, observes not only the total order flow but also a noisy two-dimensional public signal of the liquidation value and then sets the price at which all orders are filled. (At the same time the insider acts strategically by trying to minimize the price impact of his own trading in order to maximize

his final wealth.) Since, in reality, market participants receive (typically noisy) information about the performance of companies and assets from a variety of sources (and many such sources are public), it is natural to incorporate the potential multidimensional nature of the public signal in the model. In what follows we focus on the two-dimensional case since, on the one hand, it provides a nice intuition for an extension to a general multidimensional case and, on the other hand, gives a simpler framework in terms of notation and initial analysis.

The time- t aggregate order made by liquidity traders, who are not strategic and act for exogenous liquidity reasons, is assumed to be given by $du_t = \sigma_t dB^u(t)$ with non-random $\sigma_t > 0$ for all $t \in [0, 1]$. Suppose that the continuous public signal $\{s(t) : t \in [0, 1]\}$ has the dynamics given by

$$ds(t) \equiv \begin{pmatrix} ds_1(t) \\ ds_2(t) \end{pmatrix} = \begin{pmatrix} \tilde{v} \\ \tilde{v} \end{pmatrix} dt + \begin{pmatrix} \eta_1(t) & 0 \\ 0 & \eta_2(t) \end{pmatrix} \begin{pmatrix} dB_1^s(t) \\ dB_2^s(t) \end{pmatrix},$$

where $\{(\eta_1(t), \eta_2(t))' : t \in [0, 1]\}$ is $\mathbb{R}_{++}^2$ -valued non-random continuously differentiable function, and a two-dimensional standard Brownian motion $\{(B_1^s(t), B_2^s(t))'\}$ has a nonzero cross quadratic variation (covariation) with the liquidity flow Brownian motion B^u so that $[B^u, B_j^s]_t = \rho_j(t)dt$ for $j = 1, 2$, where we assume that $\rho(t) \equiv (\rho_1(t), \rho_2(t))'$ is a non-random continuously differentiable $\mathbb{R}^2$ -valued function satisfying $\|\rho(t)\|^2 \equiv \rho_1(t)^2 + \rho_2(t)^2 \leq 1$ for all $t \in [0, 1]$. Let x_t be the insider's order submitted at time t and suppose the insider's wealth is given by $w_t = w_0 + \int_0^t x_{\tau-} dp_{\tau}$, where, without loss of generality, assume that $w_0 = 0$, and where p_t is the asset's price set by the market maker at time t upon observing total order flow $(u_{\tau} + x_{\tau} : 0 \leq \tau \leq t)$ and the history of the public signal, $\{s(\tau) : 0 \leq \tau \leq t\}$, up to time t . Note that, since the insider's trading strategy is assumed to be self-financing, the insider's final wealth can be also written as $w_1 = \int_0^1 (\tilde{v} - p_{\tau-}) dx_{\tau} - [p, x]_1$.

It is natural to define a notion of equilibrium in this model via the following two conditions: (i) Insider's profit maximization:

$$\max_{\{x_{\tau}\}_{\tau \in [t, 1]} \in \mathcal{X}} E[w_1 | \tilde{v}, \{p_{\tau}, s(\tau)\}_{\tau \leq t}], \quad (1.1)$$

where $\mathcal{X}$ is a class of semimartingale processes adapted to the filtration generated by $\{p_{\tau}, s(\tau)\}_{\tau \leq t}$. (ii) Zero-profit condition for the market maker:

$$p_t = E[\tilde{v} | \{u_{\tau} + x_{\tau}, s(\tau)\}_{\tau \in [0, t]}]. \quad (1.2)$$

Similarly to the classical case, we focus on linear equilibria of the form

$$dx_t = \beta_t(\tilde{v} - p_t)dt \quad (1.3)$$

and

$$dp_t = \lambda_t(dx_t + du_t) + \gamma_1(t)(ds_1(t) - p_t dt) + \gamma_2(t)(ds_2(t) - p_t dt) \quad (1.4)$$

for some deterministic functions $\{\beta_t\}, \{\lambda_t\}, \{\gamma_1(t)\}, \{\gamma_2(t)\}$. Note that λ_t is the price sensitivity to the total order flow $y_t := x_t + u_t$, while $\gamma_j(t)$ measures the sensitivity of price to the j th source of the noisy public signal, $j = 1, 2$.

Theorem 1 *Let Σ_t be the solution to the following ordinary differential equation:*

$$\begin{aligned} \sigma_t \left(\frac{\rho_1(t)}{\eta_1(t)} + \frac{\rho_2(t)}{\eta_2(t)} \right) & - \frac{\sigma_t \Sigma_t \sqrt{1 - \|\rho(t)\|^2} \left(\frac{1}{\eta_1(t)^2} + \frac{1}{\eta_2(t)^2} \right)}{\sqrt{-\dot{\Sigma}_t - \Sigma_t^2 \left(\frac{1}{\eta_1(t)^2} + \frac{1}{\eta_2(t)^2} \right)}} \\ & + \frac{d}{dt} \frac{\sigma_t \sqrt{1 - \|\rho(t)\|^2}}{2 \sqrt{-\dot{\Sigma}_t - \Sigma_t^2 \left(\frac{1}{\eta_1(t)^2} + \frac{1}{\eta_2(t)^2} \right)}} = 0, \end{aligned} \quad (1.5)$$

with boundary conditions $\Sigma_0 = \Sigma^{(0)}$ and $\Sigma_1 = 0$. Let λ_t , $(\gamma_1(t), \gamma_2(t))'$ and β_t be the solution of

$$\lambda_t = \frac{\Sigma_t}{\sigma_t(1 - \|\rho(t)\|^2)} \left(\frac{\beta_t}{\sigma_t} - \frac{\rho_1(t)}{\eta_1(t)} - \frac{\rho_2(t)}{\eta_2(t)} \right), \quad (1.6)$$

and for all $j \in \{1, 2\}$,

$$\gamma_j(t) = \frac{\Sigma_t}{\eta_j(t)^2} \left(1 + \frac{\rho_j(t)^2}{1 - \|\rho(t)\|^2} \right) + \frac{\Sigma_t}{(1 - \|\rho(t)\|^2)} \left(\frac{\rho_1(t)\rho_2(t)}{\eta_1(t)\eta_2(t)} - \frac{\beta_t \rho_j(t)}{\sigma_t \eta_j(t)} \right), \quad (1.7)$$

$$\beta_t = (1 - \|\rho(t)\|^2) \frac{\lambda_t \sigma_t^2}{\Sigma_t} + \left(\frac{\rho_1(t)}{\eta_1(t)} + \frac{\rho_2(t)}{\eta_2(t)} \right) \sigma_t \quad (1.8)$$

$$= \sigma_t \sqrt{1 - \|\rho(t)\|^2} \sqrt{-\frac{\dot{\Sigma}_t}{\Sigma_t^2} - \left(\frac{1}{\eta_1(t)^2} + \frac{1}{\eta_2(t)^2} \right)} + \sigma_t \left(\frac{\rho_1(t)}{\eta_1(t)} + \frac{\rho_2(t)}{\eta_2(t)} \right). \quad (1.9)$$

Then the set of functions $(\lambda_t, \gamma_1(t), \gamma_2(t), \beta_t)$ constitutes a linear equilibrium, where $\Sigma_t = \text{Var}(\tilde{v} | \{y_\tau, s(\tau)\}_{0 \leq \tau \leq t})$, i.e., Σ_t is the conditional variance of v given the market maker's information sigma-field at time t .

Remark 1 *The boundary condition $\Sigma_1 = 0$ is natural since it simply states that the insider does not “leave money on the table” just prior to the public announcement of the liquidation value $\tilde{v}$ of the asset. In fact one can show that, if β is the insider's trading intensity, then the conditional variance Σ_t of the liquidation value of the asset, given the market maker's information up to time t , satisfies*

$$\Sigma_t^2 = 2 \left[\int_t^1 \Sigma_s^2 \beta_s ds \right] \lambda_t, \quad (1.10)$$

where the price sensitivity to the total order flow λ is given by (1.6). Hence, $\Sigma_1 = 0$ follows at once from (1.10).

Corollary 1 *From (1.5), (1.6) and (1.8), it follows that the price sensitivity to the total order flow satisfies*

$$\lambda_t = \frac{\sqrt{-\dot{\Sigma}_t - \Sigma_t^2 \left(\frac{1}{\eta_1(t)^2} + \frac{1}{\eta_2(t)^2} \right)}}{\sigma_t \sqrt{1 - \|\rho(t)\|^2}}, \quad (1.11)$$

and λ_t is, in general, neither constant nor monotone even if we keep $\rho_1(t), \rho_2(t), \eta_1(t), \eta_2(t)$ and $\sigma(t)$ constant in time.

In economic terms, λ reflects the market impact of trading by measuring the extent to which buying or selling the asset moves the price against the buyer or seller. Equation (1.11) suggests that the market impact of the insider trading is reduced when the order flow of the uninformed traders is more volatile and when the magnitude of the correlation vector $\|\rho\|$ tends to 0. Hence, one expects the case with the unbiased uninformed trading to be associated with the lowest market impact (even though the conditional variance Σ of the estimated value of the asset depends on the correlation, complicating the formal analysis).

To illustrate this point, I plot in Figure 1.1 the price sensitivity λ in a simple case of constant coefficients $\eta_1 = \eta_2 = 2$, $\rho_2 = 0$ and $\sigma = \Sigma^{(0)} = 1$, but for three different cases of the correlation parameter ρ_1 (with respect to the first public signal): $\rho_1 = 0.9$, $\rho_1 = 0$ and $\rho_1 = -0.9$. In the case when the uninformed have contrarian biases (i.e., $\rho_1 < 0$), we see that the price impact is monotone decreasing over time and starts out the highest at time 0 in comparison with the cases of unbiased and momentum liquidity trading, hence, the strategic insider trades much more cautiously in the beginning of the period in that case. Figure 1.1 also shows that, while the cases with negative and zero correlations result in monotone λ , positive correlation between the public signal and the aggregate order flow of the uninformed traders produces a non-monotone price impact dynamics.

Equation (1.8) shows that the equilibrium strategy of the insider is a weighted sum of the familiar classical Kyle model adverse selection term, $\lambda_t \sigma_t^2 / \Sigma_t$, and two more terms, $\sigma_t / \eta_1(t)$ and $\sigma_t / \eta_2(t)$, where the latter terms reflect the relative volatility of the liquidity flow after controlling for the accuracy of the information contained in the public signals. The latter effects, arising from different (independent) public sources, are additive with weights equal to the correlations between each given public source and the liquidity flow.

If all the correlations between the liquidity flow and public signal sources are zero, the insider's strategy (1.8) is solely determined by the adverse selection term and disregards the input provided by the public signals. Perhaps surprisingly, the price sensitivity to the total order flow is still affected by the presence of the public signals even when the latter are uncorrelated with the liquidity flow. Hence, even if the correlations between various components of the multidimensional public signal and the uninformed order flow are zero, the study of the resulting equilibrium does not directly reduce to the study of its lower dimensional analogue. In particular, even if all the liquidity traders are unbiased—in other words, even if the liquidity traders in aggregate are unaffected by the information contained in *all* the

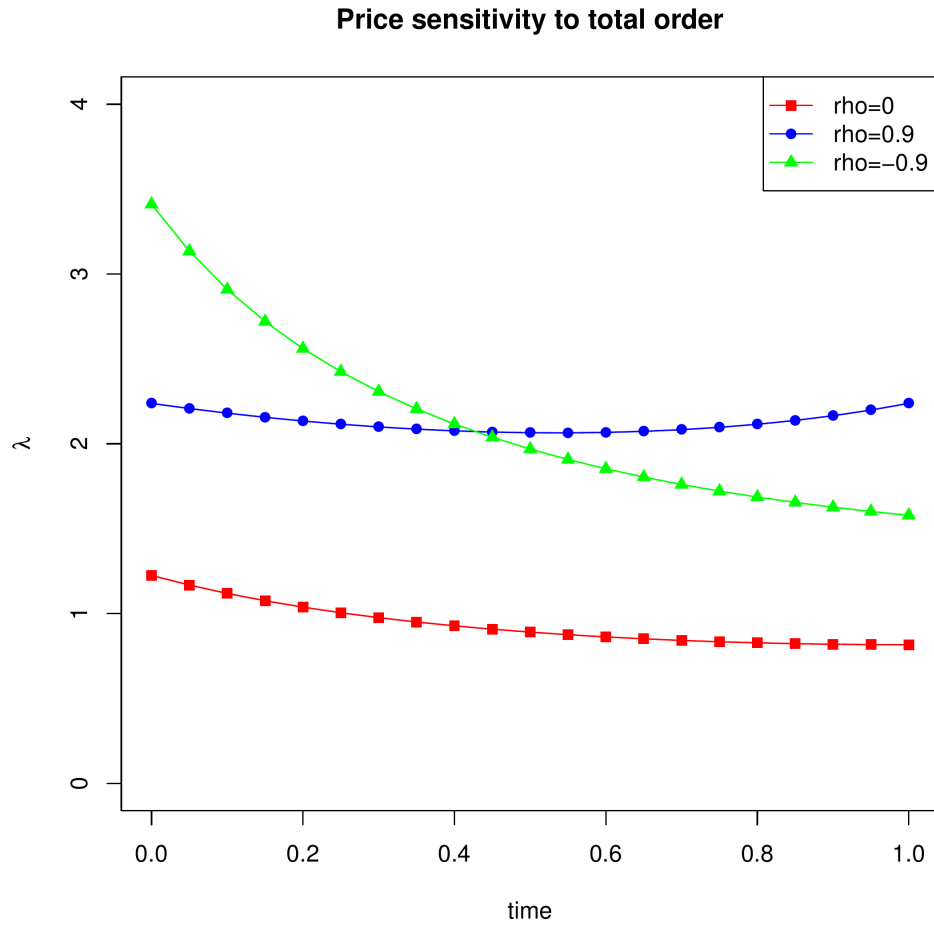


Figure 1.1: Plot of price sensitivity λ to the total order flow as a function of time for different values of correlation ρ_1 .

public signals—the resulting equilibrium still differs from the classical equilibrium in the Kyle model with no public signals.

Note that, if the public signal sources are either all positively correlated or all negatively correlated with the liquidity traders' flow and, in aggregate, the correlations move away from zero, the insider trades more cautiously in order to camouflage himself among the liquidity traders. On the other hand, when the correlations between each public signal source and the liquidity trading take different signs and/or the precision of the public signal drops, the insider can trade less cautiously and with less regard for the presence of the public signal. Figure 1.2 illustrates how the insider's trading intensity changes its shape based on the level of correlation between the liquidity flow and the public signals. It shows that, in the presence of momentum liquidity trading, the insider can afford to trade more aggressively in the beginning of the trading period than in the case of facing contrarian liquidity traders.

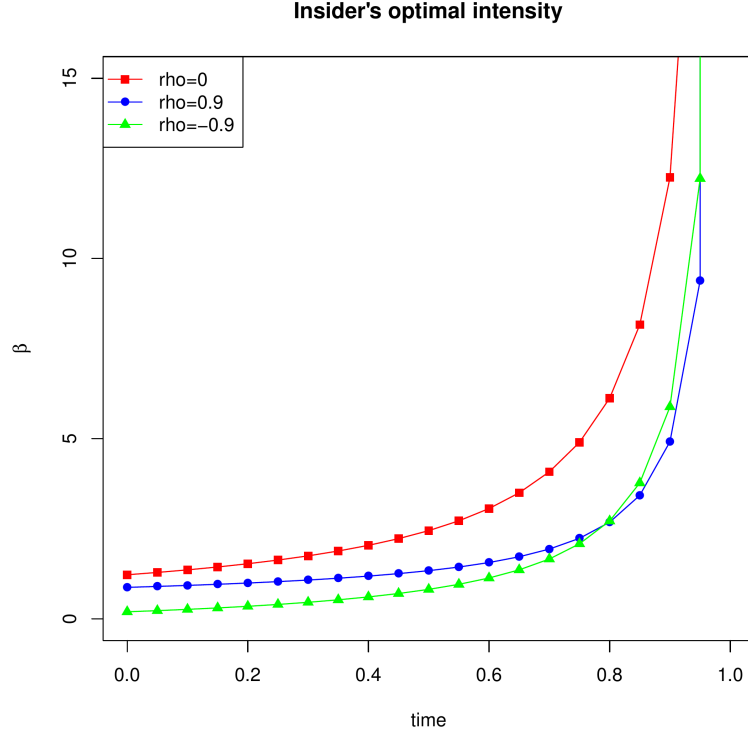


Figure 1.2: Plot of optimal trading intensity β as a function of time for different values of correlation ρ_1 .

However, facing unbiased liquidity traders is the most beneficial option for the insider, overall, as it allows him to trade most profitably early on in the trading period, and the informational efficiency of prices in the unbiased liquidity trading case is the lowest.

Proposition 1 *Insider obtains maximum expected final wealth when the liquidity order flow is uncorrelated with all the public signals.*

The intuition for this result comes from the fact that the existence of a non-zero covariation between the liquidity order flow and the public signal provides additional information to the market maker about the size of the liquidity trading, making it easier for the market maker to filter out the insider's order from the total order flow. As a result, when the covariation parameter moves away from 0, the insider has to trade more conservatively in order to hide his information about the liquidation value of the underlying asset, which in turn results in a lower terminal wealth. In effect, nonzero correlations between the public signals and the order flow of the noise traders provide a coordination mechanism for the paths of the liquidity flow. However, when such correlations come with *different* signs, the coordination mechanism gets distorted, and the insider optimally trades more aggressively. This effect can be seen in Figure 1.3, where the insider's optimal strategy is shown in the cases of correlations

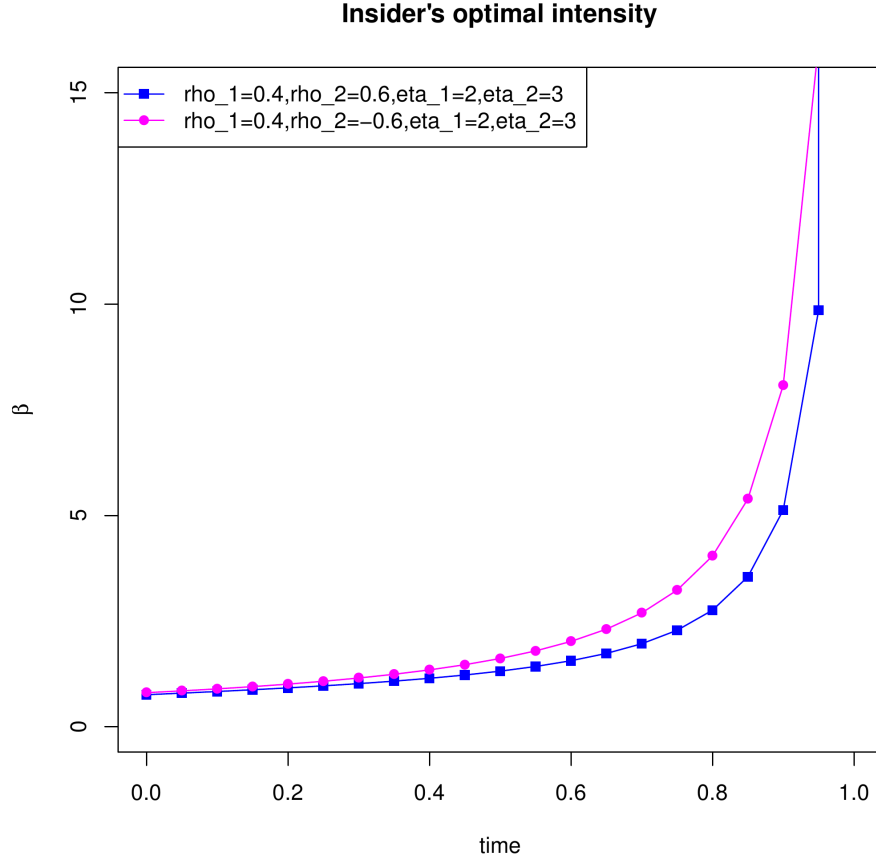


Figure 1.3: Insider's optimal strategy in the case of two-dimensional correlations of the same and opposite signs $((\rho_1, \rho_2) = (0.4, 0.6)$ and $(\rho_1, \rho_2) = (0.4, -0.6)$).

of the same sign and of the opposite signs: $(\rho_1, \rho_2) = (0.4, 0.6)$ and $(\rho_1, \rho_2) = (0.4, -0.6)$, while keeping the rest of the coefficients the same ($\eta_1 = 2$, $\eta_2 = 3$, $\sigma = 1$).

In order to illustrate the effect of correlation between public signals and noise trading on the insider's final wealth, I will consider my earlier example with $\eta_1 = \eta_2 = 2$, $\sigma = 1$ and $\rho_2 = 0$. Figure 1.4 shows the values of the insider's expected final wealth in the three cases: $\rho_1 = 0.9$, $\rho_1 = 0$ and $\rho_1 = -0.9$. Namely, expected final wealth equals 19.65032 for $\rho_1 = 0$, equals 12.85641 for $\rho_1 = 0.9$, and equal 5.125252 for $\rho_1 = -0.9$. This illustrates that the insider can extract maximum profit when liquidity traders are unbiased. On the other hand, the contrarian biases of the liquidity/noise traders are less profitable for the insider than the respective momentum biases, provided that we keep the magnitudes of the correlations the same.

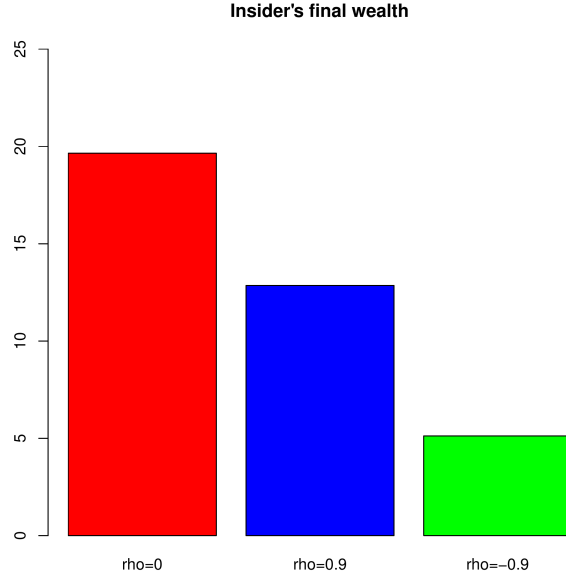


Figure 1.4: Insider's expected final wealth when correlation ρ_1 equals 0.0, 0.9, and -0.9 (while ρ_2 is fixed at zero).

1.3 Liquidity Order Flow with Correlated Increments

Next let us consider the case where the momentum (contrarian) liquidity trading behavior is associated with the positive (respectively, negative) covariance between the current and past disjoint time increments of the liquidity order flow. To simplify the exposition, I assume a classical insider trading setting with no public signals but, in contrast to the classical case, the liquidity order flow is no longer a memoryless process like a Brownian motion, but rather a non-Markov Gaussian process with rough paths. Specifically, a risky asset is traded continuously in the time period $[0, 1)$ in a market with a single perfectly informed insider, a risk-neutral competitively acting market maker, and a mass of uninformed liquidity traders who have limited rationality, where a portion $\alpha \in (0, 1]$ of the uninformed traders has momentum (or, conversely, contrarian) biases while the rest are assumed to be unbiased. The risk-free rate is set equal to 0. At time 1, the liquidation value $\tilde{v}$ of the risky asset becomes public knowledge, while ex-ante the market maker assumes that $\tilde{v} \sim N(p_0, \Sigma^{(0)})$. At time $t \in [0, 1)$ the aggregate order made by the liquidity/noise traders is assumed to have the following form:

$$du_t = \sigma_t(\alpha dB_H^u(t) + (1 - \alpha)dB^u(t)), \quad (1.12)$$

with non-random volatility coefficient $\sigma_t > 0$, $(B^u(t))$ representing a standard Brownian motion driving the orders of the unbiased portion of the liquidity/noise traders, while $(B_H^u(t))$ is a Gaussian process with an additional parameter $H \in (0, 1) \setminus \{1/2\}$, which controls the degree of correlation between the past and present increments of the liquidity order flow generated by the biased uninformed traders. Namely, $(B_H^u(t))_{t \in [0, 1]}$ is a zero mean Gaussian

process with the covariance structure given by

$$\mathbb{E}(B_H^u(t)B_H^u(s)) = \frac{1}{2} \{t^{2H} + s^{2H} - |t - s|^{2H}\}. \quad (1.13)$$

The process $(B_H^u(t))$ is called a fractional Brownian motion with Hurst or persistence parameter $H \in (0, 1)$. This process has been extensively discussed in the financial engineering literature in connection with self-similarity, scaling, long-range dependence properties of various financial time series: stock prices, foreign exchange rates, market indices and commodity prices (Cont, 2005). When $H > 1/2$, $(B_H^u(t))$ represents a persistent process with positively correlated time increments:

$$\mathbb{E} \{(B_H^u(t) - B_H^u(0))(B_H^u(t+h) - B_H^u(t))\} > 0$$

for all $t, h > 0$, whereas the case of $H < 1/2$ produces an anti-persistent process with negatively correlated time increments:

$$\mathbb{E} \{(B_H^u(t) - B_H^u(0))(B_H^u(t+h) - B_H^u(t))\} < 0$$

for all $t, h > 0$. (When $H = 1/2$ this Gaussian process has a covariance function equal to $\min(t, s)$ and orthogonal increments; hence, it reduces to a Brownian motion. In that sense $(B_H^u(t))$ is a natural generalization of the classical Brownian motion to a non-Markov and non-martingale setting.)

Positive correlation between the time increments of B_H^u (when $H > 1/2$) implies that, on average, positive past increases in the order flow of the biased traders are associated with positive increases in that order flow in the current time increment, which, in the context of limited rationality, suggests that (biased) noise traders decide to buy more of the asset today when there is an increase in the liquidity order in the previous time increment, and, conversely, decide to sell more of the asset today when there is a decrease in the liquidity order flow in the previous time increment. This suggests a version of limited rationality where liquidity/noise traders pursue momentum strategies. Conversely, negative correlation between the time increments of B_H^u (with $H \in (0, 1/2)$) suggests that, on average, the biased liquidity traders act as contrarians.

The linear combination of a Brownian motion, representing here the orders of the *unbiased* uninformed traders, and a fractional Brownian motion, describing the noise generated by the *biased* uninformed traders, is known as a mixed fractional Brownian motion and was studied in the mathematical finance literature in connection with the construction of arbitrage opportunities in financial markets where discounted (log) stock prices follow a fractional Brownian motion with drift. Existence of such arbitrage opportunities makes fractional Brownian motions unsuitable candidates for driving stock prices (Rogers, 1997; Bender, Sottinen, Valkeila, 2006). Surprisingly, the *mixed* fractional Brownian motion with $H > 3/4$ does not admit arbitrage (Cheridito, 2001), yet it enjoys many of the desirable properties of the fractional Brownian motion (like long-range dependence/memory, self-similarity, etc.),

making it an interesting potential model for various time-varying processes in the theory of finance and financial engineering.

The goal of this section is to study the equilibrium strategy of the insider and the equilibrium price set by the market maker in the case of $H \in (0, 1/2)$, which is when a portion of the liquidity/noise traders display a form of contrarian behavior (while the rest of the uninformed traders are assumed unbiased/non-strategic), and in the case of $H \in (1/2, 1)$, which is when a portion of the liquidity/noise traders pursue momentum strategies. Moreover, we aim to compare the terminal wealth acquired by the insider under the two regimes, as well as in the classical completely unbiased setting, in order to establish which, if any, momentum or contrarian biases are beneficial to the uninformed liquidity/noise traders, and how the insider needs to adjust his optimal trading strategy when facing such limited rationality behavior on behalf of liquidity traders.

The equilibrium is defined here similarly to the classical case: The insider maximizes his expected profit by maximizing at each point in time the expectation of his terminal wealth w_1 conditional on the (privately known) terminal value of the underlying financial asset and the observed (up to that point in time) history of prices set by the market maker. On the other hand, the price set by the market maker satisfies the zero-profit condition. In other words, at each time t the equilibrium price is equal to the conditional expectation of the underlying asset value $\tilde{v}$ given the total order flow observed by the market maker from time 0 up to time t .

Theorem 2 *A fully linear equilibrium does not exist. But there exists an equilibrium of the form:*

$$dx_t = \beta_t(\tilde{v} - p_t)dt \quad (1.14)$$

and

$$p_t = p_0 + \int_0^t \lambda_{H,\alpha}(t, s)d(x_s + u_s), \quad (1.15)$$

for a certain non-random continuous function (which is the trading intensity on the insider's informational advantage) $\beta_t > 0$ and a certain non-random kernel $\lambda_{H,\alpha}(t, s)$ (which reflects asset price sensitivity to the net order flow), hence, the insider's optimal strategy is still linear in his informational advantage ($\tilde{v} - p_t$) but the equilibrium price is now an integral of a non-random kernel, that depends on the persistence parameter H of the liquidity order flow and reflects the market maker's need to keep track of the entire past history of the total order flow in addition to its present value.

In particular, if all the liquidity traders are biased ($\alpha = 1$) and pursue momentum ($H \in (1/2, 1)$), the equilibrium price sensitivity to the total order flow has the form

$$\lambda_{H,1}(t, s) = g(t, s)f(s), \quad (1.16)$$

where

$$g(t, s) := \frac{\Sigma_t}{\sigma_s} \left[k_H(t, s)a_t - \int_s^t k_H(\tau, s)a'_\tau d\tau \right], \quad (1.17)$$

with nonrandom kernel

$$k_H(t, s) = c_H s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H}, \text{ where } c_H = \frac{1}{2H\Gamma\left(\frac{3}{2}-H\right)\Gamma\left(\frac{1}{2}+H\right)} \quad (1.18)$$

and

$$a_t = \frac{1}{(2-2H)c_H t^{1-2H}} \frac{d}{dt} \int_0^t k_H(t, s) \frac{\beta_s}{\sigma_s} ds, \quad (1.19)$$

and Σ_t is the conditional variance of the asset's terminal value $\tilde{v}$ given the observed history of the total order flow, i.e., $\Sigma_t = \mathbb{E}[\text{Var}(\tilde{v} | \{u_s + x_s, 0 \leq s \leq t\})]$, and where $f(\cdot)$ is a solution of

$$f_\tau - \int_\tau^t g(s, \tau) f_s \beta_s ds = 1_{[0, t]}(\tau), \quad (1.20)$$

for all τ . The equilibrium intensity of trading by the insider, $\beta(\cdot)$, is a solution to the equation:

$$\Sigma_t(\beta) = \frac{2}{\sigma_t} \int_t^1 \beta_r \Sigma_r^2(\beta) \left[a_r(\beta) k_H(r, t) - \int_t^r a'_s(\beta) k_H(s, t) ds \right] dr. \quad (1.21)$$

The above theorem illustrates several important points. First is that a fully linear equilibrium does not arise when the liquidity order flow is no longer memoryless. Instead the equilibrium price requires integration of a price sensitivity kernel λ over time with respect to the total order flow, and the price sensitivity kernel now tracks both the past and present values of the total order flow and decides how much weight to assign to the past values versus the present value of the total order flow in determining the present equilibrium price of the asset. Surprisingly, the equilibrium strategy of the insider can still be linear in his informational advantage $\tilde{v} - p_t$, even though now the insider's optimal trading intensity takes into account the fact that the market maker tracks the entire history of the total order flow when setting the price.

Secondly, if all the noise traders pursue momentum, the above theorem shows in detail how the persistence of the liquidity flow gets incorporated into the equilibrium price and the insider's strategy. This allows a direct comparison of equilibrium regimes for different values of the persistence parameter H . In particular, although the form of the insider's terminal profit makes it difficult to study analytically, numerical evaluations with a constant trading intensity $\beta = \beta(H) > 0$ and a constant volatility $\sigma > 0$ of the liquidity flow lead to the following conclusions.

Proposition 2 (i) Fix the portion of biased liquidity traders $\alpha \in (0, 1]$. Then the insider's profit is a decreasing function of the persistence parameter H . Therefore, when the biased part of the liquidity flow consists of the contrarians (which corresponds to the case of $H \in (0, 1/2)$), the insider's expected profit is larger than the one obtained from trading with the unbiased liquidity traders, while the opposite holds true when the biased portion of the liquidity flow consists of the momentum traders (which is when $H \in (1/2, 1)$). Hence, in the

setting where the liquidity order flow has a non-zero covariance between its increments, the momentum behavior of the liquidity traders is detrimental to the insider, whereas contrarian behavior of the liquidity traders benefits the insider, allowing him to trade more aggressively on the informational advantage.

(ii) If the biased liquidity trading is anti-persistent or contrarian (i.e., $H \in (0, \frac{1}{2})$ is fixed), so that the increments of the biased portion of the liquidity order flow are negatively correlated and its trajectory is rougher than the trajectory of a standard Brownian motion, then as the proportion α of biased liquidity traders increases, so does the terminal wealth of the insider. However, if the liquidity trading is persistent (i.e., $H \in (\frac{1}{2}, 1)$ is fixed), so that momentum biases are at play, then the terminal wealth of the insider drops as the proportion of biased liquidity traders α grows.

Below I illustrate conclusion (i) in the case of momentum liquidity traders ($\alpha = 1$ and $H > 1/2$). For definiteness I fix $\sigma = 1$ and $\Sigma_0 = 1$. If β is the optimal constant trading intensity of the insider, then it maximizes the insider's expected terminal wealth $E[w_1]$, where the latter simplifies to the following function

$$E[w_1] = \beta \int_0^1 (\Sigma_t(\beta) - \Sigma_1(\beta)) dt = \beta \int_0^1 \frac{1}{1 + d_H t^{2-2H} \beta^2} dt - \frac{\beta}{1 + d_H \beta^2}$$

for some constant $d_H := \frac{\Gamma(3/2-H)^3}{2H\Gamma(3-2H)^2\Gamma(1/2+H)}$. Then $E[w_1]$ is continuous in β , and $(E[w_1])|_{\beta=0} = 0$, and $\lim_{\beta \rightarrow \infty} E[w_1] = 0$. Hence $E[w_1]$ attains its maximal value on some $\beta > 0$. In Figure 1.5 we plot the values of $E[w_1]$ as a function of β for four different values of the persistence parameter H of the liquidity flow: $H = 0.6$, $H = 0.7$, $H = 0.8$, and $H = 0.9$. The graphs show that the insider's terminal wealth decreases when persistence of the liquidity order flow H increases, while the optimal intensity of the insider's trading also drops as the persistence bias grows.

Note also that a special case of Theorem 2, namely, with $H > 1/2$ and $\alpha = 1$ (the case of a persistent fBm liquidity flow), was apparently also studied in (Biagini, Hu, Meyer-Brandis and Oksendal, 2012), which is something we learnt after this work was already completed. In this special case, the authors arrive at the same quantitative conclusions as ours, although they lack the behavioral framework and interpretations presented here.

1.4 Conclusions

While Kyle's celebrated model of insider trading is one of the main backbones of the theory of market microstructure, relatively little is known about the effects of behavioral biases or bounded rationality of traders on the conclusions derived from that theory. Yet there is a vast and growing empirical literature documenting the existence of such biases in real world financial markets. This paper is a step towards uniting these two large and important areas of research in finance. Do the biases of liquidity traders matter to the optimal pricing of the underlying assets and to the optimal trading strategies of the informed agents? My results

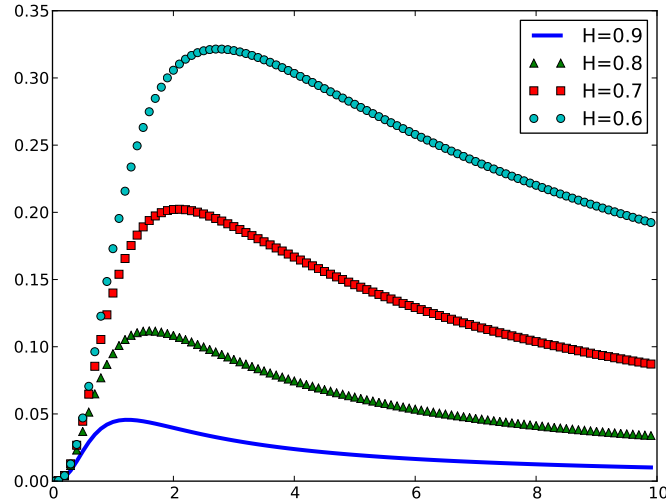


Figure 1.5: Plot of terminal wealth of the insider $E[w_1]$ as a function of trading intensity β for different values of H

show that the answer to this question is a resounding “yes.” However, all biases are not created equal. While some allow the market maker to estimate the true value of the assets much more efficiently, others can be detrimental to filtering out the insider’s information from the noisy stream of market orders made by all the traders. The latter detrimental effect occurs when biases result in an aggregate liquidity flow that is less “smooth” than that associated with the unbiased trading, i.e., the liquidity flow paths are less regular than the paths of Brownian motion. On the other hand, many biases are beneficial to the liquidity traders as they provide a certain level of coordination between noise traders, making the filtering problem facing the market maker easier.

More specifically, whereas the contrarian strategies driven by negative covariation of the liquidity order flow with public signals are useful to the liquidity traders, the contrarian behavior associated with the negatively correlated increments of the liquidity flow allows the insider to extract more profits from the uninformed traders. On the other hand, the momentum biases of the liquidity/noise traders are beneficial to the uninformed traders in both scenarios by allowing the market maker to better filter out the private information contained in the insider’s order and, hence, forcing the insider to trade more conservatively in order to hide his informational advantage.

It is also interesting to note that, in contrast to the classical case, both of my models with momentum/contrarian biases produce a price sensitivity λ which is non-constant and non-monotone and, in the case when the liquidity flow has a memory, λ becomes a deterministic kernel of two variables tracking past as well as present prices. Surprisingly, even in the case when the liquidity flow has memory, the equilibrium strategy of the insider can still

be linear in his informational advantage (i.e., linear in $(\tilde{v} - p_t)$). On the other hand, when biases reflect non-zero covariation with public signals, the insider's intensity of trading is a sum of a familiar term measuring adverse selection and the terms measuring the volatility of the liquidity flow relative to the volatilities of the public signals. As volatilities of public signals increase and/or correlations between noise trading and the public signals weaken, the insider's trading intensity is largely determined by the adverse selection term. But as the volatilities of public signals decrease and/or correlations between noise trading move away from zero, the additional terms may "overshadow" the adverse selection term. However, when public signals have correlations of different signs with the liquidity flow, the insider's trading intensity may still reduce to just the adverse selection term (which happens when $\rho_1(t)/\eta_1(t) = -\rho_2(t)/\eta_2(t)$) but the intensity will be smaller than in the classical case.

1.5 Proofs

Proof of Theorem 1

The first step is to show that, given β_t , the linear pricing rule (1.4) with λ_t and $(\gamma_1(t), \gamma_2(t))$ defined by (1.6) and (1.7), respectively, satisfies the zero-profit condition for the market maker. The second step is to prove that β is an optimal strategy for the insider provided that the market maker updates the price using λ_t and $(\gamma_1(t), \gamma_2(t))$ defined by (1.6) and (1.7).

Note that in the first step the market maker solves a stochastic linear filtering problem. Recall the following general multivariate Kalman filtering result: Assume $X(t_0) \sim N(m_0, \Pi_0)$ and $X(t_0)$ is independent of a standard Brownian motion $(B(t))$ and

$$\begin{cases} dX(t) = A(t)X(t)dt + F(t)dB(t) \\ dY(t) = C(t)X(t)dt + D(t)dB(t). \end{cases}$$

Then the optimal filter (equivalently, the mean-square estimate) of unknown $(X(t))$ given the observation process up to time t , namely, the process $\hat{X}(t) := E[X(t)|\{Y_s\}_{s \in [0, t]}]$, satisfies the following equation

$$d\hat{X}(t) = A(t)\hat{X}(t)dt + L(t)(dY(t) - C(t)\hat{X}(t)dt), \quad (1.22)$$

where $L(t) := (\Sigma(t)C^T(t) + F(t)D^T(t))(D(t)D^T(t))^{-1}$ and $\hat{X}(t_0) = m_0$ and $\Sigma(t)$ is the conditional variance of the filter, i.e.,

$$\Sigma(t) = E \left[(X(t) - \hat{X}(t))(X(t) - \hat{X}(t))' \right],$$

which is the solution to the matrix-valued Riccati equation of the form

$$\frac{d\Sigma(t)}{dt} = A(t)\Sigma + \Sigma A^T(t) + F(t)F^T(t) \quad (1.23)$$

$$- (\Sigma C^T(t) + F(t)D^T(t)) [D(t)D^T(t)]^{-1} (C(t)\Sigma + D(t)F^T(t)) \quad (1.24)$$

with $\Sigma(t_0) = \Pi_0$. In the above, B is a standard multidimensional Brownian motion (thus, coordinates of B are mutually independent). Since in our model $d[B^u, B_j^s] = \rho_j(t)dt$, it is useful to note that B^u has the decomposition $B^u(t) = \rho_1(t)B_1^s(t) + \rho_2(t)B_2^s(t) + \sqrt{1 - \rho_1(t)^2 - \rho_2(t)^2}Z(t)$ for some standard one-dimensional Brownian motion ($Z(t)$) independent of $(B_1^s(t), B_2^s(t))$. Then the market maker's observation process is of the form:

$$\begin{aligned} d \begin{bmatrix} s_1(t) \\ s_2(t) \\ a_t \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \\ \beta_t \end{bmatrix} \tilde{v} dt \\ &+ \begin{bmatrix} \eta_1(t) & 0 & 0 \\ 0 & \eta_2(t) & 0 \\ \sigma_t \rho_1(t) & \sigma_t \rho_2(t) & \sigma_t \sqrt{1 - \rho_1(t)^2 - \rho_2(t)^2} \end{bmatrix} d \begin{bmatrix} B_1^s(t) \\ B_2^s(t) \\ Z(t) \end{bmatrix}, \end{aligned}$$

where $da_t := dy_t + \beta_t p_t dt$. Thus, applying above linear filtering result with $A(t) \equiv 0$, $F(t) \equiv 0$, $C(t) = \begin{bmatrix} 1 \\ 1 \\ \beta_t \end{bmatrix}$ and

$$D(t) = \begin{bmatrix} \eta_1(t) & 0 & 0 \\ 0 & \eta_2(t) & 0 \\ \sigma_t \rho_1(t) & \sigma_t \rho_2(t) & \sigma_t \sqrt{1 - \rho_1(t)^2 - \rho_2(t)^2} \end{bmatrix},$$

we obtain the following analogue of (1.22) for $p_t = E[\tilde{v} | \{(y_\tau, s(\tau)) : 0 \leq \tau \leq t\}]$:

$$dp_t = L(t) \left\{ d \begin{bmatrix} s_1(t) \\ s_2(t) \\ a_t \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ \beta_t \end{bmatrix} p_t dt \right\} = L(t) \begin{bmatrix} ds_1(t) - p_t dt \\ ds_2(t) - p_t dt \\ dy_t \end{bmatrix}, \quad (1.25)$$

where

$$\begin{aligned} L(t) &= \Sigma_t(1, 1, \beta_t)(D(t)D^T(t))^{-1} \\ &= \frac{\Sigma_t(1, 1, \beta_t)}{(1 - \|\rho(t)\|^2)\eta_1(t)^2\eta_2(t)^2} \\ &\times \begin{bmatrix} (\eta_2(t))^2(1 - (\rho_2(t))^2) & \eta_1(t)\eta_2(t)\rho_1(t)\rho_2(t) & -\frac{\eta_1(t)}{\sigma_t}\rho_1(t)(\eta_2(t))^2 \\ \eta_1(t)\eta_2(t)\rho_1(t)\rho_2(t) & (\eta_1(t))^2(1 - (\rho_1(t))^2) & -\frac{\eta_2(t)}{\sigma_t}\rho_2(t)(\eta_1(t))^2 \\ -\frac{\eta_1(t)}{\sigma_t}\rho_1(t)(\eta_2(t))^2 & -\frac{\eta_2(t)}{\sigma_t}\rho_2(t)(\eta_1(t))^2 & \frac{(\eta_1(t))^2(\eta_2(t))^2}{\sigma_t^2} \end{bmatrix} \\ &= \frac{\Sigma_t}{(1 - \|\rho(t)\|^2)[\eta_1(t)\eta_2(t)]^2} \\ &\times \begin{bmatrix} \eta_2(t)^2(1 - \rho_2(t)^2) + \eta_1(t)\eta_2(t)\rho_1(t)\rho_2(t) - \beta_t \frac{\eta_1(t)}{\sigma_t}(\eta_2(t))^2\rho_1(t) \\ \eta_1(t)\eta_2(t)\rho_1(t)\rho_2(t) + (\eta_1(t))^2(1 - \rho_1(t)^2) - \beta_t \frac{\eta_2(t)}{\sigma_t}(\eta_1(t))^2\rho_2(t) \\ -\frac{\eta_1(t)}{\sigma_t}(\eta_2(t))^2\rho_1(t) - \frac{\eta_2(t)}{\sigma_t}(\eta_1(t))^2\rho_2(t) + \beta_t \frac{[\eta_1(t)\eta_2(t)]^2}{\sigma_t^2} \end{bmatrix}^T \end{aligned}$$

implying that (1.4) holds with $(\gamma_1(t), \gamma_2(t), \lambda_t) = (L_1(t), L_2(t), L_3(t))$ so that (1.6) and (1.7) are satisfied. Moreover, Σ_t is the variance of the optimal filter and satisfies the Riccati equation of the form:

$$\dot{\Sigma}_t = -\Sigma_t^2(1, 1, \beta_t)(D(t)D^T(t))^{-1} \begin{pmatrix} 1 \\ 1 \\ \beta_t \end{pmatrix}^T, \quad \text{with } \Sigma_0 = \Sigma^{(0)},$$

which, after some algebra and in view of (1.6), reduces to

$$\dot{\Sigma}_t = -\Sigma_t^2 \left(\frac{1}{(\eta_1(t))^2} + \frac{1}{(\eta_2(t))^2} \right) - \sigma_t^2(1 - \|\rho(t)\|^2)\lambda_t^2, \quad (1.26)$$

thus,

$$\lambda_t = \frac{\sqrt{-\dot{\Sigma}_t - \Sigma_t^2 \left(\frac{1}{(\eta_1(t))^2} + \frac{1}{(\eta_2(t))^2} \right)}}{\sigma_t \sqrt{(1 - \|\rho(t)\|^2)}}. \quad (1.27)$$

From (1.6) and (1.27), it follows that (1.8) and (1.9) hold. This completes step 1 of the proof.

To prove step 2, note that (1.5), (1.6) and (1.7) imply that

$$\frac{1}{\lambda_t}(\gamma_1(t) + \gamma_2(t)) = \frac{d}{dt} \left(\frac{1}{2\lambda_t} \right), \quad (1.28)$$

or, equivalently,

$$\frac{\dot{\lambda}_t}{2\lambda_t} = -(\gamma_1(t) + \gamma_2(t)),$$

implying that $\lambda_t = \text{const} \cdot e^{-2(\gamma_1(t) + \gamma_2(t))}$ and, since $\lambda_0 > 0$, it follows that $\lambda_t > 0$ for all t . Next define function $J(t, p)$ by

$$J(t, p) := \frac{(\tilde{v} - p)^2}{2\lambda_t} + \int_t^1 \frac{\lambda_u^2 \sigma_u^2 + [\gamma_1(u)\eta_1(u)]^2 + [\gamma_2(u)\eta_2(u)]^2 + 2\lambda_u \sigma_u \sum_{j=1}^2 \gamma_j(u)\eta_j(u)\rho_j(u)}{2\lambda_u} du.$$

Then

$$\begin{aligned} \frac{\partial}{\partial t} J(t, p) = & - \frac{(v - p)^2}{2\lambda_t^2} \dot{\lambda}_t \\ & - \frac{\lambda_t^2 \sigma_t^2 + [\gamma_1(t)\eta_1(t)]^2 + [\gamma_2(t)\eta_2(t)]^2 + 2\lambda_t \sigma_t \sum_{j=1}^2 \gamma_j(t)\eta_j(t)\rho_j(t)}{2\lambda_t}, \end{aligned}$$

$$\text{and } \frac{\partial}{\partial p} J(t, p) = -\frac{(\tilde{v} - p)}{\lambda_t}, \text{ and } \frac{\partial^2}{\partial p^2} J(t, p) = \frac{1}{\lambda_t}.$$

Using Itô's formula and the self-financing condition for wealth, we obtain that

$$\begin{aligned} (w_1 + J(1, p_1)) - (w_0 + J(0, p_0)) &= \int_0^1 (\tilde{v} - p_{t-}) dx_t - [p, x]_1^c \\ &+ \int_0^1 \left\{ \frac{\partial}{\partial t} J(t, p_{t-}) dt + \frac{\partial}{\partial t} J(t, p_{t-}) dp_t \right\} \\ &+ \frac{1}{2} \int_0^1 \frac{\partial^2}{\partial p^2} J(t, p_{t-}) d[p, p]_t^c + \sum_{0 \leq t \leq 1} \left\{ \Delta J(t, p_t) - \frac{\partial}{\partial p} J(t, p_{t-}) \Delta p_t - \underbrace{\Delta p_t}_{=\lambda_t \Delta x_t} \Delta x_t \right\}, \end{aligned}$$

where

$$\begin{aligned} [p, x]_1^c &= \int_0^1 d[p, x]_t^c = \int_0^1 \lambda_t d[y, x]_t^c + \gamma_1(t) \eta_1(t) d[B_1^s, x]_t^c + \gamma_2(t) \eta_2(t) d[B_2^s, x]_t^c \\ &= \int_0^1 \lambda_t (d[x, x]_t^c + \sigma_t d[B^u, x]_t^c) + \gamma_1(t) \eta_1(t) d[B_1^s, x]_t^c + \gamma_2(t) \eta_2(t) d[B_2^s, x]_t^c, \\ d[p, p]_t^c &= \lambda_t^2 d[x, x]_t^c + \{ \lambda_t^2 \sigma_t^2 + (\gamma_1(t) \eta_1(t))^2 + (\gamma_2(t) \eta_2(t))^2 \\ &+ 2\lambda_t \sigma_t [\gamma_1(t) \eta_1(t) \rho_1(t) + \gamma_2(t) \eta_2(t) \rho_2(t)] \} dt \\ &+ 2\lambda_t \gamma_1(t) \eta_1(t) d[B_1^s, x]_t^c + 2\lambda_t \gamma_2(t) \eta_2(t) d[B_2^s, x]_t^c + 2\lambda_t^2 \sigma_t d[B^u, x]_t^c. \end{aligned}$$

Then

$$\begin{aligned} w_1 - J(0, p_0) &= -J(1, p_1) - \frac{1}{2} \int_0^1 \lambda_t d[x, x]_t^c \\ &+ \sum_{0 \leq t \leq 1} \left\{ \Delta J(t, p_t) - \frac{\partial}{\partial p} J(t, p_{t-}) \Delta p_t - \lambda_t (\Delta x_t)^2 \right\} \\ &- \int_0^1 (\tilde{v} - p_{t-}) \left\{ \sigma_t dB^u(t) + \frac{\gamma_1(t) \eta_1(t)}{\lambda_t} dB_1^s(t) + \frac{\gamma_2(t) \eta_2(t)}{\lambda_t} dB_2^s(t) \right\}. \end{aligned}$$

Note that, conditional on $\{\tilde{v}, (p_t)_{t \leq \tau}\}$, $M^u(\tau) := \int_0^\tau (\tilde{v} - p_{t-}) \sigma_t dB^u(t)$ is a zero-mean martingale with respect to the sigma-field generated by $(B^u(t) : 0 \leq t \leq \tau)$ (and the latter is independent of $(B_1^s(t), B_2^s(t) : 0 \leq t \leq 1)$). On the other hand, $M^s(\tau) := \int_0^\tau (\tilde{v} - p_{t-}) \left\{ \frac{\gamma_1(t) \eta_1(t)}{\lambda_t} dB_1^s(t) + \frac{\gamma_2(t) \eta_2(t)}{\lambda_t} dB_2^s(t) \right\}$ is a zero-mean martingale with respect to the sigma-field generated by $\{\tilde{v}, \{p_t, s(t)\}_{0 \leq t \leq \tau}\}$. Moreover,

$$\begin{aligned} \Delta J(t, p_t) - \left(\frac{\partial}{\partial p} J(t, p_{t-}) \right) \Delta p_t &= \frac{1}{2\lambda_t} [(\tilde{v} - p_t)^2 - (\tilde{v} - p_{t-})^2] - \left(-\frac{(\tilde{v} - p_{t-})}{\lambda_t} \Delta p_t \right) \\ &= -\frac{\Delta p_t}{2\lambda_t} (2\tilde{v} - p_t - p_{t-}) + \frac{(\tilde{v} - p_{t-})}{\lambda_t} \Delta p_t \\ &= -\frac{\Delta p_t}{2\lambda_t} (2(\tilde{v} - p_{t-}) - \Delta p_t) + \frac{(\tilde{v} - p_{t-})}{\lambda_t} \Delta p_t \\ &= \frac{(\Delta p_t)^2}{2\lambda_t} = \frac{\lambda_t (\Delta x_t)^2}{2}, \end{aligned}$$

implying that

$$\begin{aligned}
 w_1 - J(0, p_0) &= -J(1, p_1) - M^u(1) - M^s(1) \\
 &\quad - \frac{1}{2} \int_0^1 \lambda_t d[x, x]_t^c - \sum_{0 \leq t \leq 1} \frac{\lambda_t (\Delta x_t)^2}{2} \\
 &= -J(1, p_1) - \frac{1}{2} \int_0^1 \lambda_t d[x, x]_t - M^u(1) - M^s(1). \tag{1.29}
 \end{aligned}$$

Since $\lambda_t > 0$ for all $t \geq 0$, it follows that $J(1, p_1) \geq 0$ and the equality to zero holds if and only if $p_1 = v$ almost surely. Similarly, since $\lambda_t > 0$ for all $t \geq 0$, it follows that $\int_0^1 \lambda_t d[x, x]_t \geq 0$ and the equality to zero holds if and only if $[x, x]_t = 0$ for all $t \geq 0$, i.e., if and only if $(x_t)_{t \in [0,1]}$ is a continuous process of bounded variation. Therefore,

$$w_1 \leq J(0, p_0) - M^u(1) - M^s(1) \quad (\text{with probability } 1) \tag{1.30}$$

and the expected value of w_1 is maximized on $(x(t); t \in [0, 1])$ such that x is a continuous process of bounded variation and such that the corresponding price process $p_t \rightarrow \tilde{v}$ almost surely, as $t \uparrow 1$. Finally, note that the strategy $x_t = \int_0^t \beta_\tau (\tilde{v} - p_\tau) d\tau$ is continuous with bounded variation and implies that p_t converges to $\tilde{v}$ as $t \uparrow 1$ almost surely since $\Sigma_1 = 0$. Therefore, $x_t = \int_0^t \beta_\tau (\tilde{v} - p_\tau) d\tau$ is an optimal strategy for a given triple $(\lambda_t, \gamma_1(t), \gamma_2(t))$. $\square$

Proof of Theorem 2

Since the liquidity order flow is given by

$$du_t = \sigma_t(\alpha dB_H^u(t) + (1 - \alpha)dB^u(t)),$$

and the insider's strategy is assumed here to be of the form $dx_t = (\tilde{v} - p_t)\beta_t dt$, where the insider's trading intensity $\beta_t > 0$ is a continuous deterministic function which maximizes the insider's terminal wealth, then the total order flow $y_t = u_t + x_t$ satisfies

$$dy_t = (\tilde{v} - E[\tilde{v} | \{y_s, 0 \leq s \leq t\}]) \beta_t dt + \sigma_t(\alpha dB_H^u(t) + (1 - \alpha)dB^u(t)). \tag{1.31}$$

Let $V(t) := \alpha B_H^u(t) + (1 - \alpha)B^u(t)$ and $\mathcal{F}_t^y := \sigma(y_s, 0 \leq s \leq t)$, then

$$dy_t = (\tilde{v} - E[\tilde{v} | \mathcal{F}_t^y]) \beta_t dt + \sigma_t dV(t), \tag{1.32}$$

where $(V(t))$ is a mean zero Gaussian process with covariance function

$$\begin{aligned}
 E[V(t)V(s)] &= \alpha^2 E[B_H^u(t)B_H^u(s)] + (1 - \alpha)^2 \min(t, s) \\
 &= \alpha^2 \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}) + (1 - \alpha)^2 \min(t, s).
 \end{aligned}$$

Also

$$V(t) = \alpha \int_0^t k_H(t, s) d\hat{B}(s) + (1 - \alpha)B^u(t)$$

for some standard Brownian motion $(\hat{B}(t))_{t \in [0,1]}$ independent of $(B^u(t))_{t \in [0,1]}$, and where $k_H(\cdot, \cdot)$ is a suitable deterministic kernel whose precise form depends on whether $H > 1/2$ or $H < 1/2$. For example, for $H > 1/2$,

$$k_H(t, s) = \frac{1}{2H\Gamma(\frac{3}{2} - H)\Gamma(\frac{1}{2} + H)} s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H}. \quad (1.33)$$

I will show later that there exists a solution to equation (1.31) which can be represented as an innovation process arising in the context of a certain filtering problem. But for the moment let us focus on the optimization problem faced by the insider. Note that, by integration by parts,

$$\begin{aligned} w_t &= w_0 + \int_0^t x_s dp_s = w_0 + x_t p_t - \int_0^t p_s dx_s \\ &= w_0 + p_t \int_0^t (\tilde{v} - p_s) \beta_s ds - \int_0^t p_s (\tilde{v} - p_s) \beta_s ds \\ &= w_0 + \int_0^t ((\tilde{v} - p_s) - (\tilde{v} - p_t)) (\tilde{v} - p_s) \beta_s ds \\ &= w_0 + \int_0^t (\tilde{v} - p_s)^2 \beta_s ds - \int_0^t (\tilde{v} - p_t) (\tilde{v} - p_s) \beta_s ds, \end{aligned}$$

so that

$$E[w_1] = w_0 + \int_0^1 E[(\tilde{v} - p_t)^2] \beta_t dt - \int_0^1 E[(\tilde{v} - p_1)(\tilde{v} - p_t)] \beta_t dt.$$

By properties of conditional expectations and since $p_t = E[\tilde{v} | \mathcal{F}_t^y]$ is a square-integrable martingale,

$$\begin{aligned} E[(\tilde{v} - p_1)(\tilde{v} - p_t)] &= E[\tilde{v}^2] - E[p_t^2] - E[p_1^2] + E[p_1 p_t] \\ &= E[\tilde{v}^2] - E[p_1^2] \\ &= E[\tilde{v}^2] - \left(E[\tilde{v}^2] - E\left[(\tilde{v} - E[\tilde{v} | \mathcal{F}_1^y])^2 \right] \right) \\ &= E\left[(\tilde{v} - E[\tilde{v} | \mathcal{F}_1^y])^2 \right]. \end{aligned}$$

Hence, the insider maximizes (over continuous $\beta_t > 0$)

$$E[w_1] = w_0 + \int_0^1 (\Sigma_t(\beta) - \Sigma_1(\beta)) \beta_t dt,$$

where

$$\Sigma_t(\beta) := E\left[(\tilde{v} - E[\tilde{v} | \mathcal{F}_t^y])^2 \right],$$

is the conditional variance of $\tilde{v}$ and the dependence on β is due to the dependence of the total order flow on the insider's trading intensity. Note that it suffices to find a β which maximizes

$$J(\beta) := \int_0^1 \Sigma_t(\beta) \beta_t dt \quad (1.34)$$

and such that $\Sigma_1(\beta) = 0$. Let ϵ be an arbitrary small number and $(\xi_t)_{t \in [0,1]}$ be an arbitrary smooth function. First, I will apply a perturbation argument, i.e., find a β satisfying

$$\left. \frac{d}{d\epsilon} J(\beta + \epsilon \xi) \right|_{\epsilon=0} = 0, \quad (1.35)$$

and then verify that Σ_1 vanishes on such a β . Note that the latter will also imply that $p_1 = \tilde{v}$. In order to identify Σ_t as a function of β more explicitly and to show that the solution to equation (1.31) exists, consider the following auxiliary filtering problem: the “signal” $\hat{x}_t = \tilde{v}$ for all $t \in [0, 1]$, unobserved directly, needs to be estimated on the basis of the observed process $d\hat{y}_t = \hat{x}_t \beta_t dt + \sigma_t dV_t$ for $t \in [0, 1]$, with $\hat{y}_0 = 0$. Then $d\tilde{y}_t := d\hat{y}_t - E[\hat{x}_t \beta_t | \mathcal{F}_t^{\hat{y}}] dt$ is the innovations process for this problem, and it is easy to see that

$$d\tilde{y}_t = (\tilde{v} - E[\tilde{v} | \mathcal{F}_t^{\hat{y}}]) \beta_t dt + \sigma_t dV_t.$$

Comparison of the latter equation with (1.32) reveals that the innovation process $\tilde{y}$ is a solution to (1.32) provided that $\mathcal{F}_t^{\tilde{y}} = \mathcal{F}_t^{\hat{y}}$.

Let us show that $\mathcal{F}_t^{\tilde{y}} = \mathcal{F}_t^{\hat{y}}$. Without loss of generality assume that $E[\tilde{v}] = 0$ and $\text{Var}[\tilde{v}] = 1$. The inclusion $\mathcal{F}_t^{\tilde{y}} \subset \mathcal{F}_t^{\hat{y}}$ is immediate by the definition of the innovation process. The converse inclusion can be established by construction of the fundamental martingale associated to the process V_t (namely, by finding a Gaussian martingale which generates the same filtration as V). For example, in the case of $\alpha = 1$ and $H > 1/2$, the fundamental martingale associated to V_t has the form

$$N_t^* = \int_0^t k_H(t, s) dB_s^H,$$

where k_H is the kernel from (1.33). In what follows, for definiteness I focus the rest of the discussion of the proof on the latter case of $\alpha = 1$ and $H > 1/2$, even though similar arguments but with a different set of kernels can be made for other choices of $\alpha \in (0, 1]$ and $H \in (0, 1)$. In particular, the necessary details on the construction of fundamental martingales associated with a mixed fractional Brownian motion in the general case of $\alpha \in (0, 1]$ and $H \in (0, 1)$ can be found in (Cai, Chigansky, Kleptsyna, 2013).

Let $y_t^* := \int_0^t k_H(t, s) \sigma_s^{-1} d\hat{y}_s$. Then

$$y_t^* = \int_0^t k_H(t, s) \tilde{v} \frac{\beta_s}{\sigma_s} ds + \int_0^t k_H(t, s) dB_s^H = \int_0^t k_H(t, s) \tilde{v} \frac{\beta_s}{\sigma_s} ds + N_t^*,$$

hence, (y_t^*) is a semimartingale generating the same filtration as $(\hat{y}_t)$, and its quadratic variation process satisfies

$$d\langle y^*, y^* \rangle_s = d\langle N^*, N^* \rangle_s = (2 - 2H)c_H s^{1-2H} ds.$$

Define

$$a_s = a_s(\beta) := \frac{d}{(2 - 2H)c_H s^{1-2H} ds} \int_0^s k_H(s, z) \frac{\beta_z}{\sigma_z} dz$$

for all $s \in [0, 1]$. Then, by (Kleptsyna, Le Breton, Roubaud, 2000), conditional variance $\hat{\Sigma}_t := E \left[\left(\tilde{v} - E[\tilde{v} | \mathcal{F}_t^{\hat{y}}] \right)^2 \right]$ satisfies

$$\hat{\Sigma}_t = \left(\hat{\Sigma}_0^{-1} + \int_0^t (2 - 2H)c_H s^{1-2H} a_s^2 ds \right)^{-1}, \quad t \in [0, 1], \quad (1.36)$$

for some positive constant $\hat{\Sigma}_0$ and, if $\hat{p}_t := E[\tilde{v} | \mathcal{F}_t^{\hat{y}}]$, then

$$\hat{p}_t = \hat{\Sigma}_t \hat{\Sigma}_0^{-1} \left(\hat{p}_0 + \hat{\Sigma}_0 \int_0^t a_s dy_s^* \right), \quad t \in [0, 1].$$

By Lemma 5.3 in (Kleptsyna, Le Breton, Roubaud, 2000), it follows that

$$\hat{p}_t = \hat{\Sigma}_t \hat{\Sigma}_0^{-1} \hat{p}_0 + \int_0^t \hat{g}(t, s) d\hat{y}_s,$$

where $\hat{p}_0 = E[\tilde{v} | \mathcal{F}_0^{\hat{y}}] = E[\tilde{v}] = 0$ and

$$\hat{g}(t, s) := \frac{\hat{\Sigma}_t}{\sigma_s} \left[k_H(t, s) a_t - \int_s^t k_H(z, s) a'_z dz \right].$$

Then for any (deterministic) smooth function f_t ,

$$\begin{aligned} \int_0^t f_s d\tilde{y}_s &= \int_0^t f_s (d\hat{y}_s - \hat{p}_s \beta_s ds) \\ &= \int_0^t f_s d\hat{y}_s - \int_0^t f_s \beta_s \left(\int_0^s \hat{g}(s, z) d\hat{y}_z \right) ds \\ &= \int_0^t \left(f_z - \int_z^t \hat{g}(s, z) f_s \beta_s ds \right) d\hat{y}_z. \end{aligned}$$

Since we want to find a representation of $\hat{y}$ in terms of $\tilde{y}$, it is necessary to find a solution $\{f_z\}$ to the equation

$$f_z - \int_z^t \hat{g}(s, z) f_s \beta_s ds = 1_{[0, t]}(z).$$

But classical results on Volterra integral equations ensure that such a solution exists provided that

$$\int_0^t \int_z \hat{g}^2(s, z) \beta_s^2 dz ds < \infty \quad (1.37)$$

for all $t \in (0, 1)$. But, under assumption that $\beta(\cdot)/\sigma(\cdot)$ is a twice continuously differentiable function on $[0, t]$ for all $t < 1$, one can show that condition (1.37) is indeed satisfied. Hence, there exists a deterministic f such that $\hat{y}_t = \int_0^t f_s d\tilde{y}_s$, implying that $\mathcal{F}_t^{\hat{y}} \subset \mathcal{F}_t^{\tilde{y}}$. Thus, $\mathcal{F}_t^{\hat{y}} = \mathcal{F}_t^{\tilde{y}}$, which, in turn, implies that $\tilde{y}$ is a solution to the total order dynamics equation given by (1.32). Note that, if the solution to (1.32) is unique, then we at once obtain that $y = \tilde{y}$ (and, consequently, $\hat{\Sigma} = \Sigma$, $\hat{g} = g$, $\hat{\Sigma} = \Sigma$, and $\hat{p} = p$). On the other hand, since

$$d\hat{y}_t = dy_t + p_t \beta_t dt,$$

then at each time t the market maker knows $\mathcal{F}_t^{\hat{y}}$, implying that $d\tilde{y}_t = dy_t$ for all t . Hence, from now on we do not distinguish between y and $\tilde{y}$.

Recall that we need to find a continuous positive trading intensity $\beta > 0$, satisfying (1.35) (and verify that Σ_1 vanishes on such a β). First, note that

$$\frac{d}{d\epsilon} a_t(\beta + \epsilon\xi) = \frac{1}{(2 - 2H)c_H s^{1-2H}} \frac{d}{ds} \int_0^s k_H(s, z) \frac{\xi_z}{\sigma_z} dz,$$

hence,

$$\left(\frac{d}{d\epsilon} \int_0^t (a_s(\beta + \epsilon\xi))^2 (2 - 2H)c_H s^{1-2H} ds \right) \Big|_{\epsilon=0} = 2 \int_0^t a_s(\beta) \left(\frac{d}{ds} \int_0^s k_H(s, z) \frac{\xi_z}{\sigma_z} dz \right) ds.$$

Next, from the above and (1.36), it follows that

$$\begin{aligned} \left(\frac{d}{d\epsilon} \Sigma_t(\beta + \epsilon\xi) \right) \Big|_{\epsilon=0} &= -2\Sigma_t^2 \int_0^t a_s \frac{d}{ds} \left(\int_0^s k_H(s, z) \frac{\xi_z}{\sigma_z} dz \right) ds \\ &= -2\Sigma_t^2 a_t \int_0^t k_H(t, z) \frac{\xi_z}{\sigma_z} dz + 2\Sigma_t^2 \int_0^t a'_s \left(\int_0^s k_H(s, z) \frac{\xi_z}{\sigma_z} dz \right) ds \\ &= 2\Sigma_t^2 \int_0^t \frac{\xi_z}{\sigma_z} \left\{ \int_z^t a'_s k_H(s, z) ds - a_t k_H(t, z) \right\} dz \\ &= 2\Sigma_t^2 \int_0^t \frac{\xi_z}{\sigma_z} \ell(t, z) dz, \end{aligned}$$

where $\ell(t, z) := \int_z^t a'_s k_H(s, z) ds - a_t k_H(t, z)$. Therefore,

$$\begin{aligned} \left(\frac{d}{d\epsilon} J(\beta + \epsilon\xi) \right) \Big|_{\epsilon=0} &= \int_0^1 \frac{d}{d\epsilon} [\Sigma_t(\beta + \epsilon\xi)(\beta_t + \epsilon\xi_t)] dt \Big|_{\epsilon=0} \\ &= \int_0^1 \beta_t \left(2\Sigma_t^2(\beta) \int_0^t \frac{\xi_z}{\sigma_z} \ell(t, z) dz \right) dt + \int_0^1 \Sigma_t(\beta) \xi_t dt \\ &= 2 \int_0^1 \frac{\xi_z}{\sigma_z} \left(\int_z^1 \beta_t \Sigma_t^2(\beta) \ell(t, z) dt \right) dz + \int_0^1 \Sigma_z(\beta) \xi_z dz. \end{aligned}$$

Since (ξ_z) is arbitrary, it follows that, if β maximizes $J(\beta)$, then $\left(\frac{d}{d\epsilon}J(\beta + \epsilon\xi)\right)\big|_{\epsilon=0} = 0$, which happens if and only if

$$\Sigma_z(\beta) = -\frac{2}{\sigma_z} \int_z^1 \beta_t \Sigma_t^2(\beta) \ell(t, z) dt$$

for all $z \in [0, 1]$, and, in particular, $\Sigma_1(\beta) = 0$. The desired result then follows in view of $\Sigma_1(\beta) \geq 0$ and $\beta \geq 0$. $\square$

Chapter 2

Optimal Selling Time for Real Indivisible Assets: When Should You Sell Your Mansion?

2.1 Introduction

One of the key features that makes real estate the largest yet one of the most poorly understood classes of investments in the world is the shortage (often, the lack) of transactions data which arises, even in a world with perfect information sharing, from the fact that real properties are infrequently traded assets. This feature is naturally shared by many popular investment vehicles (from rare coins and art collections to non-traded REITs) but also becomes prominent with almost any asset class once transactions costs are high or whenever other strong disincentives to frequent trading are present. The resulting pricing of such assets is often very challenging due to incompleteness of the market and inability to hedge idiosyncratic risk.

This paper is concerned with a practical problem of finding an optimal time to irreversibly sell a real indivisible asset in the setting where a risk-averse utility maximizing agent, the owner of the real asset, has access to a frictionless financial market whose fluctuations may (or may not) be correlated with the real asset's price. The presence of the financial market allows the agent to eliminate systematic risk, while the right to sell the real asset reduces the agent's exposure to idiosyncratic risk. The latter part of risk cannot be fully eliminated until the real asset is finally sold, but if the real asset has a higher return than the financial assets, the agent may do better by holding onto the real asset for longer. The agent continuously (in time) assesses both his financial investment opportunities and the real asset's price and makes instantaneous decision on whether to sell the real asset. It is assumed that there is a market on which the real asset can be irreversibly sold (say, a luxury real estate market, or an art and antiques market). However, the price of the real asset is potentially affected by either the asset's rarity or its historical and cultural features. As a result the value of

the real asset cannot be assumed to be spanned by securities traded in the financial market, giving rise to an incomplete market.

In this setting it is instructive to think of the agent's wealth as the price of his financial portfolio plus the option or right to sell the real asset and that option carries its own value that needs to be determined. This research is therefore generally motivated by problems in real options (see, for example (Dixit, Pindyck, 1994) and (Vollert, 2003) for a survey of results in that field). In the real options literature managerial decisions about when to invest or sell various services and facilities are treated as options on the underlying real asset. However, much of the existing literature on the subject assumes market completeness with only a few notable exceptions. The latter include (Smith, Nau, 1995), which considers a problem of pricing the option to invest in a real asset in the presence of both systematic and private risks in the discrete-time binomial framework, also (Miao, Wang, 2007), where a real option with the payoff in the form of a stream of cash-flows is considered in a setting with an agent who maximizes his expected exponential utility of consumption over time. There are also two papers, namely (Henderson, 2007) and (Evans, Henderson, Hobson, 2008), which are very closely related to the research presented here and we will discuss them in detail.

In (Henderson, 2007) the indivisible asset sale problem for an agent with exponential utility is considered and the author obtains explicit formulas for the value function and the utility indifference price of the real asset. The conclusion of the paper is that the combination of incompleteness and risk aversion reduces the value of the real asset and accelerates sale. In the nondegenerate case, where the agent neither optimally sells the real asset at time zero nor keeps holding it forever, the optimal strategy is to sell once the value of the real asset hits a certain constant critical threshold, and that threshold scales nicely with the coefficient of risk aversion (namely, multiplication of the risk aversion parameter by a certain factor reduces that critical threshold by the same factor). Note also that in this case the optimal strategy is independent of the agent's financial wealth, which is due to the fact that financial wealth factors out of the problem reducing the dimension of the Hamilton-Jacobi-Bellman equation (HJB).

In (Evans, Henderson, Hobson, 2008) the problem is revisited but under the arguably more realistic assumption that the agent has CRRA preferences. However, analytically the problem becomes much more intricate, and the HJB equation associated with the mixed stochastic control/optimal stopping problem cannot be solved directly to obtain a closed-form expression for the value function. The authors start by characterizing the value function as the unique solution to a certain free boundary problem but then show that the free boundary (which determines the optimal time to sell) is a solution to a transcendental equation, whose form is derived explicitly. It turns out that the optimal strategy in this case is to sell the real asset once the *ratio* of the real asset's value to the agent's financial wealth exceeds a certain constant threshold, however, the economic interpretation of that threshold is far from obvious.

We should note here that, from an analytic point of view, optimal selling time problems are effectively first-passage time problems for certain diffusion processes which model relevant statistics about prices in given economies of interest. As is known from stochastic analysis,

except for only a few special cases (namely, a standard Brownian motion with constant drift and a geometric Brownian motion (GBM)) there are no explicit expressions for distribution functions (or their Laplace transforms) of first-passage times to a given constant barrier (not to mention moving barriers) for general diffusions. However, there are some implicit characterizations of these distributions as solutions to certain integral equations. This is consistent with the results found in the above two papers in the sense that the combined use of geometric Brownian motions in modelling asset prices and exponential utility in modelling preferences produces a model with explicit solutions to the optimal selling time problem but represents a rare exception to the general rule that no such explicit expressions for solutions typically exist. On the other hand, the situation described in the case of CRRA utility results in a statistic which is a ratio of two GBMs and the resulting optimal stopping threshold can be characterized only implicitly as the unique solution to a certain integral equation.

This paper is primarily motivated by two considerations. One is to extend the results of (Evans, Henderson, Hobson, 2008) to include more general and realistic preferences and to discuss, however briefly, the effect of incorporating transaction costs, arising from the real asset sale, in the model. The second goal, on the other hand, is far more ambitious and will not be addressed here now but which we keep in mind during the analysis, is to establish how curvature and other properties of the agent's utility function and diffusion parameters (or infinitesimal generator) of the underlying price processes of the financial and real assets affect the optimal selling strategy. The latter problem generally involves finding the optimal choice of the statistic in the decision (stopping) rule plus characterization of the critical threshold. Of the two components, the general form of the statistic appears more amenable to investigation and is perhaps more important from the economic perspective. On the other hand, the corresponding representations of critical thresholds as solutions of integral equations are generally likely to exist and, although not explicit, such representations are nonetheless very valuable from practical standpoint since they allow to approximate the desired critical values numerically and consequently aid in visualizing the effects of various economic parameters on the optimal time of sale.

The paper is organized as follows. Section 2.2 formally states the model and describes the associated optimal stopping problem. Then the solution to the optimal selling time problem is described in terms of solution to a nonlinear PDE with free boundary and comparisons with the results of (Evans, Henderson, Hobson, 2008) and (Henderson, 2007) are made. The conclusions of our model, which assumes agent's preferences built upon CRRA-type function but with the ℓ_α -type ($0 < \alpha \leq 1$) aggregator, are that in the nondegenerate case the agent should sell the real asset once the ratio of the real asset's value to the agent's financial wealth becomes large and hits a certain critical threshold but the HJB equation is highly nonlinear and for $\alpha \neq 1$ has a more complex structure than the one found in (Evans, Henderson, Hobson, 2008), which changes the optimal time of sale of the real asset. Section 2.3 is devoted to the analysis of "house flipping" phenomenon.

We conclude the paper with a brief discussion on the choice of model used for pricing of the real asset, which up to that point in the paper was assumed to follow the GBM with some known parameters. This is obviously an unrealistic model in practice but was

a convenient and natural starting point in the analysis of optimal stopping/selling time problem. However, a more attractive approach to pricing of real assets traded infrequently is obtained by choosing a similar asset or index, but which is traded frequently, and using it for hedging/pricing purposes. Naturally, the higher is the correlation between the real asset in question and its frequently traded counterpart, the better is the hedge and the more accurate is the price. Unfortunately, the actual correlation between the infrequently-traded real asset and its actively traded counterpart is typically unknown and cannot be statistically tested, although, intuitively, if the two assets share a significant number of key characteristics, their prices at any given point in time are expected to be similar. In the context of luxury real estate, for example, this points to the idea of combining hedonic modeling with estimated time-evolution of real estate indices (whose trajectories tend to aggregate information about changes in supply and demand in the housing market over time, as well as changes in technology, constructions costs and other related dynamic features) in order to create meaningful dynamic models of property-specific real estate prices for properties which are rarely, if ever, sold.

2.2 Model Description and Optimal Stopping Problem

Consider an agent endowed with a single indivisible unit of a real asset whose value is given by random process $(Y_t)_{t \geq 0}$ modelled by geometric Brownian motion (GBM) with volatility $\sigma > 0$ and drift $\nu \in \mathbb{R}$:

$$dY_t = Y_t(\sigma dB_t + \nu dt), \quad Y_0 = y. \quad (2.1)$$

In addition the agent can invest his wealth in N risky financial assets with price processes $(P_t^1, \dots, P_t^N)_{t \geq 0}$, governed by

$$dP_t^i = P_t^i \left(\sum_{j=1}^N \Sigma^{ij} dW_t^j + \mu^i dt \right), \quad (2.2)$$

where $\Sigma = (\Sigma^{ij})_{1 \leq i, j \leq N}$ is an invertible volatility matrix, $W = (W_t^1, \dots, W_t^N)_{t \geq 0}$ is a N -dim standard Brownian motion, and a riskless bond paying interest rate r with price dynamics

$$dI_t = I_t r dt. \quad (2.3)$$

The agent may choose to sell the real asset at any point in time (with the decision horizon being assumed unbounded), but the sale is irreversible (in particular, once sold, the real asset cannot be bought back) and results in a single lump-sum payment. Consequently, the agent faces incomplete market, however, the agent's right to sell limits the idiosyncratic risk associated with the real asset. In addition, the Wiener processes B and W are generally assumed to be correlated, thus the agent may have access to partial hedging via (complete) financial market. Specifically, let us assume that $dB_t = \sum_{j=1}^N \rho_j dW_t^j + (1 - \rho^T \rho) d\bar{W}_t$, where

$\bar{W}$ is a standard Brownian motion independent of W and $\rho = (\rho_1, \dots, \rho_N)$ is the vector of correlation coefficients between B and the components of W . Clearly, if at time t the agent holds a portfolio $(\theta_t^1, \dots, \theta_t^N)$ of risky financial assets (where θ_t^i denotes the proportion of financial wealth invested in the i th risky financial asset), then the agent's self-financing wealth process $(X_t)_{t \geq 0}$, by Itô's formula, satisfies the stochastic differential equation (SDE) of the form:

$$dX_t = X_t \left[\sum_{i,j=1}^N \theta_t^i \Sigma^{ij} (dW_t^j + \lambda_j dt) + r dt \right], \quad (2.4)$$

with $X_0 = x$ for some $x \in \mathbb{R}_+$, where $\lambda = (\lambda_1, \dots, \lambda_N) \equiv \Sigma^{-1}(\mu - r1)$ with the interpretation that λ_j represents the market price of risk derived from motion of W^j . Here the financial market is assumed to be frictionless, and we exclude the possibility of borrowing against the value of the real asset in order to invest in the financial market until the real asset is actually sold.

At the time τ of sale of the real asset the risk-averse utility-maximizing infinitely-lived agent receives $(1 - \delta)Y_\tau$, thus the model assumes proportional transaction costs (i.e. $0 \leq \delta < 1$) from sale of the real asset. However, at first we are focusing on the case of $\delta = 0$, with the goal of subsequently relaxing that assumption. Thus, under the assumed absence of transaction costs, the agent wants to find an optimal time to sell the real asset in order to maximize his expected utility:

$$V_* \equiv V_*(x, y) = \sup_{\tau \in \mathcal{T}} \sup_{\theta \in \mathcal{A}_\tau} \mathbb{E}^{x,y} U(\tau, X_\tau, Y_\tau), \quad (2.5)$$

where U is of the form:

$$U(t, x, y) = \frac{\alpha e^{-\beta t}}{1 - R} \left([x^\alpha + y^\alpha]^{1/\alpha} \right)^{1-R}, \quad (2.6)$$

and $\mathcal{A}_\tau$ denotes the set of admissible trading strategies (defined up to sale time τ of the real asset), and $\mathcal{T}$ is the set of all stopping times τ such that $\tau \leq \inf\{t > 0 : X_t = 0\}$ (with the convention that infimum of an empty set is equal to $+\infty$), and the agent's initial wealth x is assumed to be strictly positive. Thus, in the case of non-zero transaction costs, the agent can pay them from the proceeds that he receives from the sale of the real asset.

In the agent's utility function (2.6) parameter $\alpha \in (0, 1]$ governs the degree of intratemporal substitutability between the real asset's value and the financial wealth. Here $\beta > 0$ is the usual time impatience parameter and will later be chosen so that the objective function is horizon-unbiased in the sense of (Evans, Henderson, Hobson, 2008). (See also (Henderson, Hobson, 2007) for a detailed discussion of the importance of using horizon-unbiased utility functions from the economics' point of view in order to avoid artificial bias in optimal stopping problems. Intuitively, one should choose β in such a way that, in the absence of the real asset, $\arg \max_{\tau \in \mathcal{T}} V_* = \mathcal{T}$. An alternative way of stating the latter is to say that there is no preferred horizon in the sense that the solution of $\sup_{\theta} \mathbb{E}[U(\tau, X_\tau, Y_\tau)]$ does not depend on τ . From technical point of view, β is chosen so that the objective function is a supermartingale

under every admissible trading strategy $\theta \in \mathcal{A}_r$ and a (local) martingale under the optimal strategy. We will see that $\beta = (\frac{\lambda^T \lambda}{2(R+\alpha-1)} + r)(1-R)$ is precisely the choice that results in the desired horizon-unbiased property in our model.) Finally, parameter $R \in (0, 1)$ controls the utility's curvature (thus, reflects the agent's risk aversion) and is chosen so that $\frac{1-R}{\alpha} \in (0, 1)$.

Based on microeconomic evidence (e.g., see (Flavin, Nakagawa, 2008) for empirical tests using household-level data with preferences (on consumption and housing) which are integrated versions of (2.6)), utilities of the form (2.6) frequently outperform habit formation models (with habits defined as in (Campbell, Cochrane, 1999)), as well as classical CARA and CRRA, in terms of applicability to real-life data. Note also that the choice of $\alpha = 1$ reduces the model to the classical CRRA type, with the aggregator being a simple sum of the financial wealth and the real asset value, but still yields a very rich framework and nontrivial solution. The latter case of $\alpha = 1$ in the absence of transaction costs was solved in (Evans, Henderson, Hobson, 2008) and is closely related to this work.

Note that the general problem introduced here with the N risky financial assets can be reduced to that of a single risky traded asset with Sharpe ratio λ and driven by a single Brownian motion with correlation ρ to Brownian motion B driving the dynamics of the real asset's value Y . Thus, without loss of generality (but with a slight abuse of notation), the system (2.2) is replaced by a univariate price dynamics of the form:

$$dP_t = P_t(\Sigma(dW_t + \lambda dt) + r dt),$$

where $\Sigma > 0$ and λ are now scalar, and W is a one-dimensional Brownian motion such that $\text{corr}(W_t, B_t) = \rho$ for all $t \geq 0$. Consequently, (2.4) reduces to the following financial wealth process dynamics:

$$dX_t = X_t(\theta_t \Sigma(dW_t + \lambda dt) + r dt).$$

It is also convenient to introduce the Sharpe ratio $\xi = (\nu - r)/\sigma$ of the real asset and rewrite (2.26) as follows:

$$dY_t = Y_t(\sigma(dB_t + \xi dt) + r dt).$$

Optimal Selling Time As Solution of Free-Boundary Problem

Fix $R \in (0, 1)$ and $\alpha \in (0, 1)$ such that $(1-R)/\alpha \in (0, 1)$. Suppose $|\rho| < 1$. Define parameters

$$\gamma = \frac{2}{1-\rho^2} \left(\frac{\xi - \lambda\rho}{\alpha\sigma} \right)$$

and

$$\eta = 1 + \frac{1}{1-\rho^2} \left[\frac{\lambda}{(\alpha-1+R)\sigma} - \rho \right]^2.$$

Consider the following free boundary problem: Solve on $[0, z^*]$, where z^* is the (unknown) free boundary, the following equation

$$\begin{aligned} 0 = & (z^2 H''(z))^2 + k_1 z^3 H''(z) H'(z) + k_2 z^2 H''(z) H(z) + k_3 (z H'(z))^2 \\ & + k_4 z H'(z) H(z) + k_5 (H(z))^2, \end{aligned} \quad (2.7)$$

where $k_1 = \gamma + \frac{2(\alpha-1+R)}{\alpha^3}$, $k_2 = -\frac{(1-R)}{\alpha^2} \left\{ \eta(\alpha + R - 1) + \frac{1-\alpha}{1-\rho^2} \right\}$ and

$$k_3 = -\frac{1}{\alpha^2} \left\{ (\alpha + R - 1)^2(\eta - 1) - (\alpha + 2R - 1)\gamma\alpha + \frac{(1-\alpha)^2}{1-\rho^2} + 2(\alpha + R - 1)\frac{1-\alpha}{1-\rho^2} \left[1 - \frac{\lambda\rho}{\sigma(\alpha+R-1)} \right] \right\},$$

$$k_4 = -\frac{1-R}{\alpha^3} \left[\frac{(1-\alpha)\lambda^2}{(1-\rho^2)(\alpha+R-1)\sigma^2} (1 + 2\sigma\rho(\alpha + R - 1)) + R\left(\gamma\alpha - \frac{1-\alpha}{1-\rho^2}\right) \right], \quad k_5 = \frac{(1-\alpha)\lambda^2(1-R)^2}{\alpha^4(\alpha+R-1)\sigma^2(1-\rho^2)},$$

subject to smooth fit and value matching

$$\text{at } z = z^* : \quad H = \frac{\alpha(1+z^*)^{\frac{1-R}{\alpha}}}{1-R}, \quad H' = (1+z^*)^{\frac{(1-R)}{\alpha}-1}, \quad (2.8)$$

and subject to the following non-negativity and boundedness condition

$$0 \leq z^{-\frac{(1-R)}{\alpha}} \left[H(z) - \frac{\alpha(1+z)^{\frac{1-R}{\alpha}}}{1-R} \right] \leq C \quad \text{on } [0, z^*]. \quad (2.9)$$

Let us also extend the definition of H to the entire $\mathbb{R}_+$ by setting $H(z) = \frac{\alpha(1+z)^{\frac{1-R}{\alpha}}}{1-R}$ on $z > z^*$.

Assumption 1 Assume that for parameter values such that $0 < \gamma < \frac{\alpha-1+R}{\alpha^2}\eta + \frac{1-\alpha}{\alpha(1-\rho^2)}$ there exists a concave increasing solution to the free boundary problem (2.7) subject to (2.8) and (2.9), and that for this solution the free boundary z^* satisfies $z^* > \gamma/(R - \gamma)$.

Then we can prove the main theorem of the paper, which represents the solution to the problem of finding the optimal time to sell the real asset, i.e. (2.5), in terms of the solution to a free boundary problem described above.

Theorem 3 Suppose $0 < R < 1$, $0 < \alpha < 1$ and $0 < \frac{1-R}{\alpha} < 1$. Then solution to (2.5) has three regimes:

- (i) If $\gamma \leq 0$ then the optimal selling time $\tau^* \equiv 0$ and $V_*(x, y) = \frac{\alpha}{1-R}(x^\alpha + y^\alpha)^{(1-R)/\alpha}$;
- (ii) If $0 < \gamma < \frac{\alpha-1+R}{\alpha^2}\eta + \frac{1-\alpha}{\alpha(1-\rho^2)}$ and assume that Assumption 1 is satisfied. Then $V_*(x, y) = x^{1-R}H(y^\alpha/x^\alpha)$, where H is the solution to the free boundary problem (2.7).
- (iii) If $\gamma > \frac{\alpha-1+R}{\alpha^2}\eta + \frac{1-\alpha}{\alpha(1-\rho^2)}$ then there is no finite optimal time to sell and $V_*(x, y)$ is infinite.

Proof

First note that $U(t, X_t, Y_t) = \frac{e^{-\beta t}}{1-R}(\hat{X}_t + \hat{Y}_t)^{1-\hat{R}}$, where $\hat{R} = 1 - (1 - R)/\alpha$ (where, by assumption on R and α , it follows that $\hat{R} \in (0, 1)$), and processes $\hat{X}, \hat{Y}$ are defined by $\hat{X}_t := X_t^\alpha$ and $\hat{Y}_t := Y_t^\alpha$ for all $t \geq 0$. By Itô's formula, the dynamics of $\hat{X}$ is of the form:

$$d\hat{X}_t = \hat{X}_t \left(\alpha\theta_t \Sigma(dW_t + \lambda dt) + \alpha r dt + \frac{1}{2}\alpha(\alpha - 1)\theta_t^2 \Sigma^2 dt \right), \quad (2.10)$$

and, similarly, the evolution of $\hat{Y}$ is given by

$$d\hat{Y}_t = \hat{Y}_t \left(\alpha \sigma (dB_t + \xi dt) + \alpha r dt + \frac{1}{2} \alpha (\alpha - 1) \sigma^2 dt \right). \quad (2.11)$$

By the multivariate Itô formula, the semimartingale $Q_t := \frac{e^{-\beta t}}{1-\hat{R}} (\hat{X}_t + \hat{Y}_t)^{1-\hat{R}}$ has the decomposition $dQ_t = dM_t - A_t dt$, where the (local) martingale part is given by

$$dM_t = \frac{Q_t(1-\hat{R})}{\hat{X}_t + \hat{Y}_t} \left(\hat{X}_t \alpha \theta_t \Sigma dW_t + \hat{Y}_t \alpha \sigma dB_t \right),$$

whereas process A_t in the bounded variation part of the decomposition is of the form:

$$\begin{aligned} A_t &= Q_t \left\{ \beta - \frac{1-\hat{R}}{\hat{X}_t + \hat{Y}_t} \left(\hat{X}_t \alpha \theta_t \Sigma \lambda + \hat{X}_t \alpha r + \hat{X}_t \frac{1}{2} \alpha (\alpha - 1) \theta_t^2 \Sigma^2 + \hat{Y}_t \alpha \sigma \xi + \hat{Y}_t \alpha r + \hat{Y}_t \frac{1}{2} \alpha (\alpha - 1) \sigma^2 \right) \right. \\ &\quad \left. - \frac{1}{2} \frac{(1-\hat{R})(-\hat{R})}{(\hat{X}_t + \hat{Y}_t)^2} \underbrace{\left(\hat{X}_t^2 \alpha^2 \theta_t^2 \Sigma^2 + \hat{Y}_t^2 \alpha^2 \sigma^2 + 2 \hat{X}_t \hat{Y}_t \alpha^2 \theta_t \Sigma \sigma \rho \right)}_{=(\hat{X}_t \alpha \theta_t \Sigma + \hat{Y}_t \alpha \sigma \rho)^2 + \hat{Y}_t^2 \alpha^2 \sigma^2 (1-\rho^2)} \right\} \\ &= \frac{Q_t \hat{R} (1-\hat{R})}{2} \left\{ \left[\frac{\hat{X}_t \alpha \theta_t \Sigma + \hat{Y}_t \alpha \sigma \rho}{\hat{X}_t + \hat{Y}_t} \right]^2 - \frac{2\lambda}{\hat{R}} \left[\frac{\hat{X}_t \alpha \theta_t \Sigma + \hat{Y}_t \alpha \sigma \rho}{\hat{X}_t + \hat{Y}_t} \right] + \frac{\hat{Y}_t \alpha \sigma}{\hat{X}_t + \hat{Y}_t} \left(\frac{2\lambda \rho}{\hat{R}} - \frac{2\xi}{\hat{R}} \right) \right. \\ &\quad \left. + \frac{2\beta}{\hat{R}(1-\hat{R})} + \frac{\alpha(1-\alpha)}{\hat{R}} \left(\frac{\hat{X}_t \theta_t^2 \Sigma^2 + \hat{Y}_t \sigma^2}{\hat{X}_t + \hat{Y}_t} \right) - \frac{2\alpha r}{\hat{R}} + \frac{\hat{Y}_t^2 \alpha^2 \sigma^2 (1-\rho^2)}{(\hat{X}_t + \hat{Y}_t)^2} \right\} \\ &= \frac{Q_t \hat{R} (1-\hat{R})}{2} \left\{ \left[\frac{\hat{X}_t \alpha \theta_t \Sigma + \hat{Y}_t \alpha \sigma \rho}{\hat{X}_t + \hat{Y}_t} - \frac{\lambda}{\hat{R}} \right]^2 + \frac{\hat{Y}_t \alpha^2 \sigma^2 (1-\rho^2)}{(\hat{X}_t + \hat{Y}_t)^2} \underbrace{\left[\hat{Y}_t - \frac{2(\hat{X}_t + \hat{Y}_t)}{\alpha \hat{R}} \cdot \frac{(\xi - \rho \lambda)}{\sigma(1-\rho^2)} \right]}_{=\hat{Y}_t - \frac{\gamma(\hat{X}_t + \hat{Y}_t)}{\hat{R}}} \right. \\ &\quad \left. + \frac{\alpha(1-\alpha)}{\hat{R}} \left(\frac{\hat{X}_t \theta_t^2 \Sigma^2 + \hat{Y}_t \sigma^2}{\hat{X}_t + \hat{Y}_t} \right) + \underbrace{\left[\frac{2\beta}{\hat{R}(1-\hat{R})} - \frac{\lambda^2}{\hat{R}^2} - \frac{2\alpha r}{\hat{R}} \right]}_{=0} \right\}, \end{aligned}$$

where the last term is zero by the horizon-unbiased choice of $\beta = (\frac{\lambda^2}{2\hat{R}} + r\alpha)(1-\hat{R})$, since $\beta = (1-R) \left(\frac{\lambda^2}{2\alpha(1-\frac{1-R}{\alpha})} + r \right) = (1-R) \left(r + \frac{\lambda^2}{2(\alpha-(1-R))} \right)$. Therefore, since $\alpha \in (0, 1]$ and $\hat{R} \in (0, 1)$ and processes $(\hat{X}_t), (\hat{Y}_t), (Q_t)$ all have strictly positive trajectories, then if $\gamma \leq 0$ it follows that $A_t > 0$ for all $t \geq 0$. Since M is bounded from below then $A_t > 0$ implies that Q_t is a supermartingale for any admissible strategy (θ_t) . Thus, for $\gamma \leq 0$ the optimal stopping time is $\tau^* = 0$ and the desired conclusion (i) holds.

Next, suppose $0 < \gamma < \frac{\alpha-1+R}{\alpha^2}\eta + \frac{1-\alpha}{\alpha(1-\rho^2)}$. Define $S_t = e^{-\beta t} \hat{X}_t^{1-\hat{R}} H(\hat{Z}_t)$, where $\hat{Z}_t = \hat{Y}_t / \hat{X}_t$. Then, in the region $\hat{Z}_t > z^* > \gamma / (R - \gamma)$ we have that $e^{-\beta t} \hat{X}_t^{1-\hat{R}} H(\hat{Z}_t) \equiv e^{-\beta t} (\hat{X}_t + \hat{Y}_t)^{1-\hat{R}} / (1 - \hat{R})$ and

$$\hat{Y}_t - \frac{\gamma(\hat{X}_t + \hat{Y}_t)}{\hat{R}} = \hat{X}_t \left[\left(1 - \frac{\gamma}{\hat{R}}\right) \hat{Z}_t - \frac{\gamma}{\hat{R}} \right] = \frac{\hat{X}_t}{\hat{R}} \left[(\hat{R} - \gamma) \hat{Z}_t - \gamma \right] = \frac{\hat{X}_t}{\hat{R}} \left[\frac{(\hat{R} - \gamma)}{\gamma} \hat{Z}_t - 1 \right] > 0,$$

thus, by the same argument as before, it follows that S_t is a supermartingale for any admissible strategy θ .

Next, consider the region $\hat{Z}_t < z^*$. By Itô's formula,

$$\begin{aligned} dS_t = S_t \left\{ \right. & - \beta dt + \frac{1 - \hat{R}}{\hat{X}_t} d\hat{X}_t + \frac{H'(\hat{Z}_t)}{H(\hat{Z}_t)} d\hat{Z}_t \\ & + \frac{(1 - \hat{R})(-\hat{R})}{2\hat{X}_t^2} d[\hat{X}, \hat{X}]_t + \frac{H''(\hat{Z}_t)}{2H(\hat{Z}_t)} d[\hat{Z}, \hat{Z}]_t + \frac{(1 - \hat{R})H'(\hat{Z}_t)}{\hat{X}_t H(\hat{Z}_t)} d[\hat{X}, \hat{Z}]_t \left. \right\}, \end{aligned}$$

where $[\hat{X}, \hat{Z}]_t$ denotes the cross quadratic variation process of $\hat{X}$ and $\hat{Z}$, and $[\hat{X}, \hat{X}]_t$ and $[\hat{Z}, \hat{Z}]_t$ are the quadratic variation processes of $\hat{X}$ and $\hat{Z}$, respectively. Note that

$$\begin{aligned} d\hat{Z}_t = & \hat{Z}_t \left\{ \alpha \sigma dB_t - \alpha \theta_t \Sigma dW_t \right. \\ & \left. + \left(\alpha \sigma \xi - \alpha \theta_t \Sigma \lambda + \frac{1}{2} \alpha (\alpha - 1) (\sigma^2 - \theta_t^2 \Sigma^2) + \alpha^2 (\theta_t^2 \Sigma^2 - \sigma \rho \theta_t \Sigma) \right) dt \right\} \end{aligned}$$

therefore, $d[\hat{Z}, \hat{Z}]_t = \hat{Z}_t^2 \alpha^2 (\sigma^2 + \theta_t^2 \Sigma^2 - 2\sigma \theta_t \Sigma \rho) dt = \hat{Z}_t^2 \alpha^2 \{ \sigma^2 (1 - \rho^2) + (\sigma \rho - \theta_t \Sigma)^2 \} dt$ and $d[\hat{X}, \hat{Z}]_t = \hat{Z}_t \hat{X}_t \alpha^2 (\sigma \theta_t \Sigma \rho - \theta_t^2 \Sigma^2) dt$. Then we can write the (local) semimartingale decomposition $dS_t = dM_t^H - C_t dt$, where

$$dM_t^H = S_t \left\{ (1 - \hat{R}) \alpha \theta_t \Sigma dW_t + \frac{H'(\hat{Z}_t)}{H(\hat{Z}_t)} \hat{Z}_t (\alpha \sigma dB_t - \alpha \theta_t \Sigma dW_t) \right\}$$

and

$$\begin{aligned} C_t = S_t \left\{ \right. & \beta - (1 - \hat{R}) \left(\alpha \theta_t \Sigma \lambda + r \alpha + \frac{1}{2} \alpha (\alpha - 1) \theta_t^2 \Sigma^2 \right) - \frac{H'(\hat{Z}_t)}{H(\hat{Z}_t)} \hat{Z}_t (\alpha \sigma \xi - \alpha \theta_t \Sigma \lambda) \\ & - \frac{H'(\hat{Z}_t)}{H(\hat{Z}_t)} \hat{Z}_t \left(\frac{1}{2} \alpha (\alpha - 1) (\sigma^2 - \theta_t^2 \Sigma^2) + \alpha^2 (\theta_t^2 \Sigma^2 - \sigma \rho \theta_t \Sigma) \right) + \frac{(1 - \hat{R}) \hat{R}}{2} \alpha^2 \theta_t^2 \Sigma^2 \\ & \left. - \frac{H''(\hat{Z}_t)}{2H(\hat{Z}_t)} \hat{Z}_t^2 \alpha^2 \{ \sigma^2 (1 - \rho^2) + (\sigma \rho - \theta_t \Sigma)^2 \} - \frac{(1 - \hat{R}) H'(\hat{Z}_t)}{H(\hat{Z}_t)} \hat{Z}_t \alpha^2 (\sigma \theta_t \Sigma \rho - \theta_t^2 \Sigma^2) \right\}, \end{aligned}$$

where, since H is assumed to be a concave increasing function satisfying non-negativity and boundedness condition, it is easy to check that C_t is a non-negative process which is zero for the optimal choice of θ by (2.7). By the same argument as the one presented earlier, it follows that S_t is a supermartingale. Finally, the value matching/smooth fit condition ensures that no extra terms appear at $\hat{Z}_t = z^*$ during application of Itô formula. It follows by (2.9) that

$$\mathbb{E}\left[e^{-\beta\tau}\frac{(X_\tau^\alpha + Y_\tau^\alpha)^{1-\hat{R}}}{1-\hat{R}}\right] \leq \mathbb{E}[S_\tau] \leq S_0 = \hat{X}_0^{1-\hat{R}}H(\hat{Y}_0/\hat{X}_0) = x^{1-R}H(y^\alpha/x^\alpha),$$

hence, $V_* \leq x^{1-R}H(y^\alpha/x^\alpha)$.

Similarly to the argument presented in (Evans, Henderson, Hobson, 2008), we can show that there is a strategy for which the value function gets arbitrarily close to the stated bound. Consider stopping time of the form $\tau^* = \inf\{t > 0 : \hat{Z}_t \geq z^*\}$. If for $t < \tau^*$ the financial portfolio strategy is given by

$$\theta = \frac{\lambda(1-\hat{R})H - (\lambda + \hat{R}\alpha\sigma\rho)Z_tH' - \alpha\sigma\rho Z_t^2H''}{\Sigma\left\{-\alpha Z_t^2H'' - (2\alpha\hat{R} + 1 - \alpha)Z_tH' + (1 - \alpha + \alpha\hat{R})(1 - \hat{R})H\right\}} \quad (2.12)$$

then, by (2.7), $C_{t \wedge \tau^*} \equiv 0$. Then let $\tau_K = \tau^* \wedge K \wedge \inf\{t > 0 : \hat{Y}_t = K\}$. For above choice of θ , $S_0 = \mathbb{E}_{S_{\tau_K}} = \mathbb{E}[e^{-\beta\tau_K}\hat{X}_{\tau_K}^{1-\hat{R}}H(\hat{Z}_{\tau_K})]$ and $V_* \geq \mathbb{E}[e^{-\beta\tau_K}\hat{X}_{\tau_K}^{1-\hat{R}}(1 + \hat{Z}_{\tau_K})^{1-\hat{R}}/(1 - \hat{R})]$, thus,

$$\begin{aligned} x^{1-R}H(y^\alpha/x^\alpha) - V_* &\leq \mathbb{E}\left\{e^{-\beta\tau_K}\hat{Y}_{\tau_K}^{1-\hat{R}}\hat{Z}_{\tau_K}^{-(1-\hat{R})}\left[H(\hat{Z}_{\tau_K}) - \frac{(1 + \hat{Z}_{\tau_K})^{1-\hat{R}}}{(1 - \hat{R})}\right]; \tau_K < \tau\right\} \\ &\leq C\mathbb{E}[e^{-\beta\tau_K}\hat{Y}_{\tau_K}^{1-\hat{R}}; \tau_K < \tau] \quad (\text{by boundedness condition}) \\ &= C\mathbb{E}[e^{-\beta\tau_K}Y_{\tau_K}^{1-R}; \tau_K < \tau] \\ &= Cy^{1-R}\mathbb{E}_Q[e^{-\Lambda\tau_K}; \tau_K < \tau], \end{aligned}$$

where Q is defined via a change of measure by $dQ/dP = e^{(\Lambda-\beta)t}Y_t^{1-R}/y^{1-R}$. Note that

$$\mathbb{E}[Y_t^{1-R}/y^{1-R}] = e^{\left[(1-R)(\sigma\xi+r) + \frac{1}{2}(1-R)(-R)\sigma^2\right]t},$$

thus,

$$\Lambda - \beta = -(1-R)\left[\sigma\xi + r - \frac{R\sigma^2}{2}\right], \quad \text{i.e.} \quad \Lambda = \beta + (1-R)\left[-\sigma\xi - r + \frac{R\sigma^2}{2}\right].$$

After some algebra, it follows that

$$\Lambda = \frac{\alpha(1-R)\sigma^2}{2}(1-\rho^2)\left[\frac{(\alpha-1+R)}{\alpha^2}\eta - \gamma + \frac{(1-\alpha)}{\alpha(1-\rho^2)}\right],$$

and the assumption that $\gamma < \frac{\alpha-1+R}{\alpha^2}\eta + \frac{1-\alpha}{\alpha(1-\rho^2)}$ ensures that $\Lambda > 0$. The result follows by letting $K \rightarrow \infty$.

Finally suppose $\gamma > \frac{\alpha-1+R}{\alpha^2}\eta + \frac{1-\alpha}{\alpha(1-\rho^2)}$. Consider stopping time $\tau = T$. Then

$$\begin{aligned} e^{-\beta\tau} \frac{\alpha(X_\tau^\alpha + Y_\tau^\alpha)^{(1-R)/\alpha}}{1-R} &\geq e^{-\beta T} \frac{\alpha(Y_T^\alpha)^{(1-R)/\alpha}}{1-R} \\ &= \alpha e^{-\beta T} \frac{Y_T^{1-R}}{1-R} \end{aligned} \quad (2.13)$$

$$\begin{aligned} &= \alpha \frac{e^{-\beta T} y^{1-R}}{1-R} e^{(1-R)\sigma B_T - \frac{(1-R)^2\sigma^2 T}{2} + (1-R)\left[\sigma\xi + r - \frac{R\sigma^2}{2}\right]T} \\ &= \alpha \frac{y^{1-R}}{1-R} e^{(1-R)\sigma B_T - \frac{(1-R)^2\sigma^2 T}{2}} e^{-\left\{\beta + (1-R)\left[-\sigma\xi - r + \frac{R\sigma^2}{2}\right]\right\}T} \\ &= \alpha \frac{y^{1-R}}{1-R} \underbrace{e^{(1-R)\sigma B_T - \frac{(1-R)^2\sigma^2 T}{2}}}_{\text{exponential martingale}} e^{-\Lambda T}. \end{aligned} \quad (2.14)$$

Since $\gamma > \frac{\alpha-1+R}{\alpha^2}\eta + \frac{1-\alpha}{\alpha(1-\rho^2)}$ then $\Lambda < 0$, thus the expected value of the right-hand side of (2.14) converges to infinity as $T \rightarrow \infty$. $\square$

The economic interpretation of this theorem is that for $\gamma \leq 0$ the real asset depreciates relative to bond, under the minimal martingale measure, and it is optimal to sell the real asset immediately. If $\gamma > \frac{\alpha-1+R}{\alpha^2}\eta + \frac{1-\alpha}{\alpha(1-\rho^2)}$ the real asset's value grows much faster than the financial wealth, the value function is infinite and it is never optimal to sell the real asset at any finite stopping time. Finally, if $0 < \gamma < \frac{\alpha-1+R}{\alpha^2}\eta + \frac{1-\alpha}{\alpha(1-\rho^2)}$, the case is non-degenerate and it is optimal to sell the real asset once it becomes too large part relative to the agent's wealth in the financial portfolio.

Optimal Selling Time as Solution of Integral Equation

Theorem 4 Fix $\gamma > 0$ and $\eta > 1$. Consider the free boundary-value problem (2.7)–(2.9). Let

$$\begin{aligned} a(v; R; \alpha) : &= -v(k_1 - 2) - \frac{k_2}{1-R} \\ &\quad - \sqrt{\left(v(k_1 - 2) + \frac{k_2}{1-R}\right)^2 - 4\left\{v^2(1 - k_1 + k_3) + \frac{v}{1-R}(k_4 - k_2) + \frac{k_5}{(1-R)^2}\right\}} \end{aligned} \quad (2.15)$$

and

$$\psi(z) := \frac{\ln(\alpha)}{1-R} + \frac{1}{\alpha} \ln(1+z) - \int_0^{\frac{1}{\alpha} - \frac{1}{(1+z)}} \frac{2v}{a(v; R; \alpha) - 2(1-R)v^2} dv.$$

Assume that $\psi(z) = 0$ has a unique positive solution. Then the free boundary-value problem (2.7)–(2.9) has a classical solution and the desired free boundary z^* is the solution to $\psi(z) = 0$.

Proof

Since the ODE (2.7) is scale-invariant, it is useful to make a change of variables $z = e^u$. Also define $f(z) := H(z) - \frac{\alpha}{1-R}$. Then equation (2.7) takes the form

$$\begin{aligned} 0 = & \left(\frac{d^2 f}{du^2} \right)^2 - (2 - k_1) \left(\frac{d^2 f}{du^2} \right) \left(\frac{df}{du} \right) + \left(\frac{d^2 f}{du^2} \right) \frac{k_2}{1-R} (\alpha + (1-R)f) \\ & + \left(\frac{df}{du} \right)^2 (1 - k_1 + k_3) + \left(\frac{df}{du} \right) \left[-\frac{k_2}{1-R} (\alpha + (1-R)f) + \frac{k_4}{1-R} (\alpha + (1-R)f) \right] \\ & + \frac{k_5}{1-R} (\alpha + (1-R)f)^2, \end{aligned} \quad (2.16)$$

subject to the boundary conditions

$$\text{at } u = u^*: \quad f = \frac{\alpha(1 + e^{u^*})^{\frac{1-R}{\alpha}} - \alpha}{1-R}, \quad \frac{df}{du} = e^{u^*} (1 + e^{u^*})^{\frac{(1-R)}{\alpha} - 1},$$

where $z^* = e^{u^*}$, and $f \rightarrow 0$ as $u \rightarrow -\infty$ because of (2.9). Upon introducing the new dependent variable $g = df/du$, we obtain that $d^2 f/du^2 = g dg/df$ and equation (2.16) reduces to

$$g^2 \left(\frac{dg}{df} \right)^2 + \left[(k_1 - 2)g^2 + \frac{k_2}{1-R} (\alpha + (1-R)f)g \right] \left(\frac{dg}{df} \right) + c = 0, \quad (2.17)$$

where $c := (1 - k_1 + k_3)g^2 + \frac{(k_4 - k_2)}{1-R} (\alpha + (1-R)f)g + \frac{k_5}{(1-R)^2} [\alpha + (1-R)f]^2$. Consider the following two branches of solutions to (2.17):

$$2g \frac{dg}{df} = - \left[(k_1 - 2)g + \frac{k_2}{1-R} (\alpha + (1-R)f) \right] \pm \sqrt{\left[(k_1 - 2)g + \frac{k_2(\alpha + (1-R)f)}{1-R} \right]^2 - 4c}. \quad (2.18)$$

The corresponding boundary condition is as follows:

$$\text{at } f = f^*, \quad g = \frac{(1-R)f^*}{\alpha} + 1 - \left(\frac{(1-R)f^*}{\alpha} + 1 \right)^{1 - \frac{\alpha}{1-R}}, \quad (2.19)$$

where $f^* := \frac{\alpha(1+e^{u^*})^{\frac{1-R}{\alpha}} - \alpha}{1-R} = \frac{\alpha(1+z^*)^{\frac{1-R}{\alpha}} - \alpha}{1-R}$.

Next let us change dependent variable to $v = g/(\alpha + (1-R)f)$ and independent variable to $w = \alpha + (1-R)f$. Then equation (2.18) becomes

$$2(1-R)v(v + w \frac{dv}{dw}) = a_{\pm}(v; R; \alpha) \quad (2.20)$$

with $a_{-}(v; R; \alpha)$ defined in (2.15) and $a_{+}(v; R; \alpha)$ defined similarly but with a “+” sign in front of the square root term. Thus,

$$w \frac{dv}{dw} = \frac{a_{\pm}(v; R; \alpha) - 2(1-R)v^2}{2(1-R)v},$$

hence,

$$\frac{d(\ln(w))}{dv} = \frac{1}{w} \frac{dw}{dv} = \frac{2(1-R)v}{a_{\pm}(v; R; \alpha) - 2(1-R)v^2}.$$

Integrating this equation subject to required boundary conditions gives

$$\int_0^{\frac{1}{\alpha} - \left(\frac{(1-R)f^* + \alpha}{\alpha}\right)^{-\frac{\alpha}{1-R}}} \frac{2v(1-R)}{a_{\pm}(v; R; \alpha) - 2(1-R)v^2} dv = \ln(\alpha + (1-R)f^*).$$

□

Conjecture 1 *In the non-degenerate case, the optimal threshold to sell the real asset, z^* , is monotone decreasing in both the risk aversion coefficient R and the correlation parameter ρ . Thus, the more risk averse is the agent the earlier he sells the real asset, provided all other parameters are fixed. On the other hand, the smaller is the correlation between the real asset behavior and the financial markets, the more significant is the idiosyncratic component of risk contained in the real asset, which accelerates the sale of the real asset by the risk-averse investor.*

Comparison with CARA Case

It is instructive to compare our results to those obtained in the exponential (CARA) case in (Henderson, 2007). Consider the objective function

$$U(t, x, y) = -\frac{1}{\zeta} \exp\left(-\zeta(x+y)e^{-rt} + \frac{\lambda^2 t}{2}\right) \quad (2.21)$$

and consider the problem (2.5) but with (2.21) in place of (2.6). Define $\chi = 2(\xi - \lambda\rho)/\sigma$. Then, similarly to our results above, there are three possible regimes:

- If $\chi \leq 0$, then the problem is degenerate and the optimal strategy is to sell the real asset at time zero.
- If $\chi \geq 1$, then there is no optimal strategy in the sense that one should delay the sale indefinitely.
- If $0 < \chi < 1$, then the optimal sale time $\tau^* = \inf\{t > 0 : e^{-rt}Y_t \geq y^*\}$ where y^* is the unique solution in $(0, \infty)$ to the equation:

$$y^* = \frac{1}{\zeta(1-\rho^2)} \ln\left(1 + \frac{\zeta(1-\rho^2)y^*}{1-\chi}\right). \quad (2.22)$$

thus, in contrast to our case, in the exponential case, financial wealth factors out of the problem making solution explicit and simpler to interpret. For example, a direct examination of (2.22) shows that for the CARA case multiplication of the risk aversion parameter by a factor of c (i.e. $\zeta \mapsto c\zeta$) results in a decrease of the critical threshold y^* by the same factor c (i.e. $y^* \mapsto y^*/c$). As is noted in (Evans, Henderson, Hobson, 2008), even when $\alpha = 1$, the effect of a change in risk aversion for power utility is already much more complicated.

Nonzero Transactions Cost

Finally, coming back to the question of non-zero proportional transactions costs, in the case when the agent is allowed to pay them from proceeds from sale, problem (2.5) is effectively replaced with

$$V_* \equiv V_*(x, y) = \sup_{\tau \in \mathcal{T}} \sup_{\theta \in \mathcal{A}_\tau} \mathbb{E}^{x, y} U(\tau, X_\tau, (1 - \delta)Y_\tau), \quad (2.23)$$

where $d(1 - \delta)Y_t = (1 - \delta)dY_t = (1 - \delta)Y_t(\sigma dB_t + \nu dt)$, so $(1 - \delta)Y_t$ is a GBM with the same parameters as Y but with a starting value of $(1 - \delta)y$ in place of y at time zero. Thus, the results of Theorem 3, obtained in the zero transactions costs case, extend immediately to the case with non-zero proportional transactions costs (with $\delta \in (0, 1)$) but with the adjustment that y in Theorem 3 is now replaced with $(1 - \delta)y$. Similarly, the critical threshold z^* for the ratio of the statistic Y_t^α / X_t^α in the optimal strategy is accordingly rescaled by factor $(1 - \delta)^{-\alpha}$. This means that in the presence of non-zero transaction costs (in the nondegenerate case) the real asset will be sold later than in the case of zero transactions costs.

Utility Indifference Pricing of the Right to Sell

Given the agent's problem (2.5) described in the previous section, it is natural to ask how much the agent's right to sell the real asset is worth to him. As is common, we can put a price on that right to sell by solving a certainty equivalence problem and use the principle of utility indifference. Namely, the certainty equivalence price $p = p(X_t, Y_t, t)$ at time t is defined as the solution to the following equation:

$$U(t, X_t + p, 0) = \sup_{\tau} \sup_{\theta \in \mathcal{A}_\tau} \mathbb{E}[U(\tau, X_\tau, Y_\tau) | X_t, Y_t]. \quad (2.24)$$

Then, by the arguments found in the proof of Theorem 3, we arrive at the following result:

Theorem 5 *For $0 < \gamma < \frac{\alpha-1+R}{\alpha^2}\eta + \frac{1-\alpha}{\alpha(1-\rho^2)}$, the utility indifference price of the right to irreversibly sell the indivisible real asset is given by*

$$p = X_t \left\{ \left[\frac{(1-R)}{\alpha} H(Y_t^\alpha / X_t^\alpha) \right]^{\frac{1}{1-R}} - 1 \right\}. \quad (2.25)$$

2.3 House Flipping

Next let us consider the problem of a “house flipper,” who is the same risk-averse utility-maximizing agent as before but now endowed with a finite set of D real indivisible assets, whose values have the same average rate of appreciation and share the same source of idiosyncratic risk but have different volatilities $0 < \sigma_1 < \dots < \sigma_D < \infty$. For each $i \in \{1, \dots, D\}$, the value of the i th real asset is given by random process $(Y_t^i)_{t \geq 0}$ modelled by geometric Brownian motion (GBM) with volatility $\sigma_i > 0$ and drift $\nu \in \mathbb{R}$:

$$dY_t^i = Y_t^i(\sigma_i dB_t + \nu dt), \quad Y_0 = y^i. \quad (2.26)$$

In addition, as before, the agent can invest his wealth in N risky financial assets with price processes $(P_t^1, \dots, P_t^N)_{t \geq 0}$, governed by (2.2), and riskless bond with price dynamics (2.3). The agent wants to find an optimal strategy to sell the real assets in order to maximize his expected utility.

Conjecture 2 *Let τ_i be the optimal time to sell indivisible real asset i , $i \in \{1, \dots, D\}$. Then, in the non-degenerate parameter case, $\tau_D < \tau_{D-1} < \dots < \tau_1$ with probability one, i.e. it is optimal for the agent to sell first the real asset with the highest volatility once its value hits a certain sufficiently high threshold, then sell the real asset with the next highest volatility once it is sufficiently valuable, and so on until the least volatile real asset is sold at time $\tau_1 = \max\{\tau_i\}_{i=1}^D$ when the value of that real asset is sufficiently high relative to the value of the agent's financial portfolio. Moreover, the optimal times to sell the real assets can be described as solutions to a set of free boundary value problems analogous to the one discussed in Section 2.2.*

Chapter 3

The Myth of the Constant-Quality Home: A New, Unbiased House-Price Index

3.1 Introduction

Housing represents about half of total U.S. household net worth and almost two thirds of the median household's total wealth (Bertaut, McCluer, 2002; Iacoviello, 2011). It is well documented that house prices significantly affect consumption levels, so good estimates of house prices and their dynamics over time are crucial to understanding and forecasting both housing returns and economic conditions.¹ House prices are also a critical ingredient in mortgage valuation² and in current bank stress tests.³ However, despite their importance, it is difficult to measure house prices at a national or regional level. Houses are heterogeneous, so the value of one house is at best a noisy signal of the value of another. Moreover, unlike exchange-traded securities, we can only observe the value of a house when it sells, an event that in general happens very infrequently.⁴ As a result, constructing a real-estate price index requires us to make assumptions about the relation between returns on different properties, and somehow “splice” them together. These difficulties are compounded by the historically limited data available on housing characteristics in the U.S.

¹See, for example, (Case, Quigley, Shiller, 2005; Case, Quigley, Shiller, 2011; Reinhart, Rogoff, 2009; Campbell, Cocco, 2007; Chen, Michaux, Roussanov, 2013; Iacoviello, 2011; Corder, 2010; Corder, Roberts, 2008; Benito, Wood, 2005; Benito, Thompson, Waldron, Wood, 2006; BOE, 2010).

²See, for example, (Kau, Keenan, Muller, Epperson, 1995; Downing, Stanton, Wallace, 2005; Schwartz, Torous, 1992; Schwartz, Torous, 1993).

³See, for example, (FRB, 2013; Wall, 2013). Recent discussion in the popular press include Sam Fleming, “Bank stress tests focus on housing and interest rate shocks,” *Financial Times*, March 27, 2014, and “A harsh light: A handful of banks are caught short by the Fed’s annual stress test,” *Economist*, March 29, 2014.

⁴Using data from the 2011 American Housing Survey (AHS), (Emrath, 2013) estimates that the typical single-family home-buyer will remain in their home for between 13 and 16 years.

Unlike almost every other country,⁵ the U.S. has historically used *repeat-sales indices*⁶ as a measure of house prices despite serious shortcomings, which include:

- A large fraction of transactions is ignored, since houses are only included if they sell more than once during the sample period.
- Home characteristics and the relative prices of each characteristic are assumed to remain constant over time.⁷
- Prices are assumed to move in exactly the same way for houses of different types (e.g., two- versus five-bedroom houses).

An important reason for the dominance of repeat-sales indices in the U.S., despite these drawbacks, has been the historical unavailability of good home-characteristic data. The growing adoption of electronic recording of housing transaction and excise tax data at the municipality level has improved the situation, but even so, until now it has been impossible to obtain accurate home characteristics.⁸ This changed with the arrival of a new database from DataQuick in October 2013, which for the first time provided sufficient data to fully explore alternatives to repeat-sales indices.⁹ In particular, the database contains hedonic characteristics for *all* houses in a given region in each year (not just those that sell in each period, and not just a single snapshot).

Making use of this new data set, we develop a new hedonic house-price index that overcomes the shortcomings of repeat-sales indices by using all transactions (not just repeat sales) and by allowing house prices and returns to depend on home characteristics and on local and national macroeconomic factors. Our methodology also allows us to estimate both time-series and cross-sectional volatility in the prices of an individual home, something that the averaging process inherent in the construction of a repeat-sales index makes impossible. We estimate the index using linear filtering techniques, and find significant differences, in both the level and volatility of prices, between our estimates and those from the Federal

⁵(Gourieroux, LaFerrere, 2006) report that France, Hong Kong, Norway, Great Britain, Sweden, and Switzerland use hedonic indices (in which prices are estimated as functions of the characteristics of houses), while Germany, Austria, Belgium, Canada, Spain, and Holland use indices constructed from the mean or median value of observed transaction prices.

⁶The three primary indices in the U.S.—the S&P/Case-Shiller index; the Federal Housing Finance Agency (FHFA) index; and the HPI of CoreLogic—are all based on the Case-Shiller repeat-sales index methodology (Case, Shiller, 1987; Case, Shiller, 1989; Calhoun, 1996), which is based in turn on the methodology of (Bailey, Muth, Nourse, 1963).

⁷Both of these assumptions have been consistently rejected in empirical studies including, among others, (Meese, Wallace, 1997; Gatzlaff, Haurin, 1997; Gatzlaff, Haurin, 1998; Clapham, Englund, Quigley, Redfearn, 2006; Downing, Wallace, 2007).

⁸Several data vendors have provided home characteristics, but until now these have been just a single snapshot, making it impossible to track changes in these characteristics over time.

⁹The dataset was constructed by DataQuick from their historical backups after we explained to them the problems with existing data provided by both them and other vendors in the U.S.

Housing Finance Board's benchmark repeat-sales index. This shows that our index is significantly superior to repeat-sales indices as a measure of U.S. real-estate returns for economic forecasting, mortgage valuation, and bank stress tests. It is also a useful automatic valuation tool.

This chapter is organized as follows. Section 3.2 contains a brief discussion of existing house-price indices. Section 3.3 describes the new data set. Section 3.4 outlines construction of the new house price index. Section 3.5 presents estimation results, while Section 3.6 presents the conclusions.

3.2 Existing Indices

Repeat-sales indices

- Methodology: See (Case, Shiller, 1987; Case, Shiller, 1989; Calhoun, 1996; Bailey, Muth, Nourse, 1963).
- Detailed description of OFHEO index methodology: (Calhoun, 1996).
- Detailed description of Case-Shiller index methodology: (Jones, 2014).
- Comparison of OFHEO with Case-Shiller: (OFHEO, 2008; Aubuchon, Wheelock, 2008; Noeth, Sengupta, 2011)

Unlike every other country, the U.S. relies heavily on **repeat-sales** indices. The three primary indices are all of this type: the S&P/Case-Shiller index; the Federal Housing Finance Agency (FHFA) index; and the HPI of CoreLogic. All of these use the Case-Shiller repeat-sales index methodology (Case, Shiller, 1987; Case, Shiller, 1989; Calhoun, 1996), which is in turn based on a methodology developed by (Bailey, Muth, Nourse, 1963). While the exact methodologies differ slightly, this is mainly in how observations are weighted in the analysis. In constructing the FHFA index, for example (Calhoun, 1996), write the price of house i at time t as

$$p_{it} = \log(P_{it}) = \beta_t + H_{it} + N_{it},$$

where P_{it} is the house price, H_{it} is a Gaussian random walk, and N_{it} is a white-noise term. The total (log) return for a house selling in periods s and t is

$$\begin{aligned} y_i &= p_{it} - p_{is} \\ &= (\beta_t + H_{it} + N_{it}) - (\beta_s + H_{is} + N_{is}). \end{aligned}$$

We can write this as

$$y_i = \sum_{\tau=0}^T (\beta_\tau + H_{i\tau} + N_{i\tau}) D_{i\tau} = \sum_{\tau=0}^T \beta_\tau D_{i\tau} + \epsilon_i, \quad (3.1)$$

where

$$D_{i\tau} = \begin{cases} +1 & \text{if house } i \text{ sold for a second time at time } \tau, \\ -1 & \text{if house } i \text{ sold for the first time at time } \tau, \\ 0 & \text{otherwise.} \end{cases} \quad (3.2)$$

$$\epsilon_i = (H_{it} - H_{is}) + (N_{it} - N_{is}), \quad \text{so} \quad (3.3)$$

$$\text{var}(\epsilon_i) = A(t - s) + B(t - s)^2 + C. \quad (3.4)$$

The index levels, β_τ , can be estimated by regressing y_i on D_i using OLS, but since the ϵ_i are heteroskedastic, estimation is usually done in two stages, using GLS, to improve efficiency. Specifically, the first stage is to estimate β_τ using OLS from

$$y_i = \sum_{\tau=0}^T \beta_\tau D_{i\tau} + \epsilon_i.$$

Now define the estimated squared residual,

$$\begin{aligned} d_i^2 &= [y_i - \hat{y}_i]^2 \\ &= [y_i - \hat{\beta}_t + \hat{\beta}_s]^2. \end{aligned}$$

From above, we know that

$$E(d_i^2) = A(t - s) + B(t - s)^2 + 2C.$$

The parameters A , B and C can be estimated by regressing d_i^2 on $(t - s)$, $(t - s)^2$ and a constant.¹¹ In the final stage, we weight each observation i by $1/\sqrt{\hat{d}_i^2}$, and estimate β_i again using OLS regression, this time with the weighted observations.

Despite their dominance in the U.S., no other country uses repeat-sales house-price indices, due to their significant shortcomings.

Sample size In estimating Equation (3.1) to calculate a repeat-sales index, a given house only enters the sample if it has transacted at least twice. Any house that has sold only once during the sample period will thus be excluded completely, as will all new houses (because all new units are single sales), even though in some geographic areas and periods new-home sales can be 20% or more of all housing transactions. In our sample from 2003–2012, repeat sales were, on average, only 23% of the overall arms-length transaction volume.¹² The small

¹⁰Under the assumptions made so far, $B = 0$. This was loosened by (Abraham, Schauman, 1991) to allow volatility to increase less slowly than linearly as the time interval between sales increases.

¹¹The estimate for parameter C often turns out to be negative.

¹²The repeat sales volume was 22.61% in Alameda County, 22.33% in Contra Costa County, 25.30% in Los Angeles County, 21.83% in San Diego County, 26.53% in San Francisco County 23.42%, and 23.67% in Santa Clara County.

sample sizes mean that minor changes to the way the index is constructed can have a large impacts on results. For example, Figures 3.1 and 3.2 show the huge differences that existed between the FHFA and Case-Shiller indices during the financial crisis. The main difference

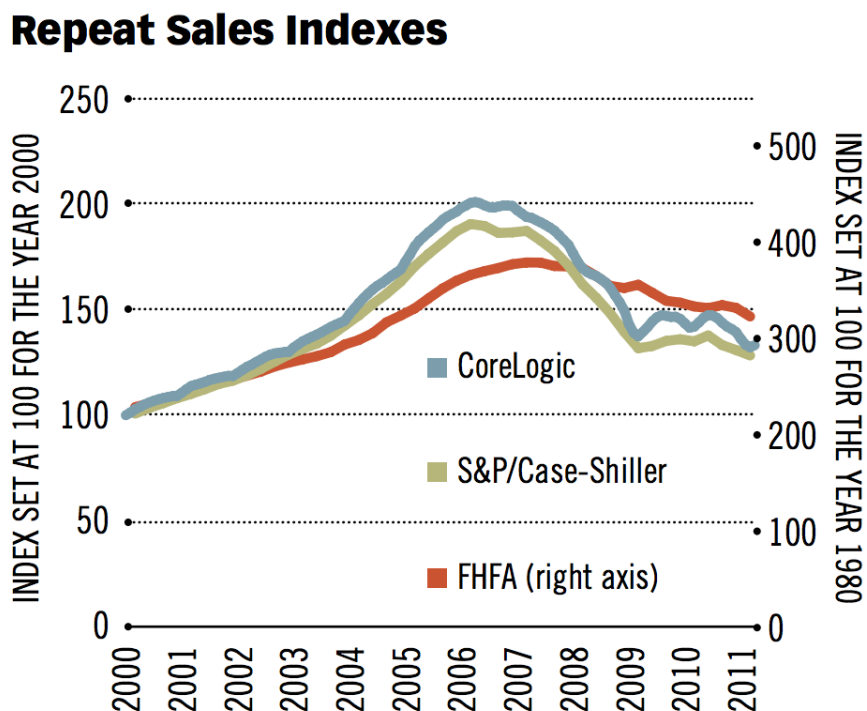


Figure 3.1: Figure 2 from (Noeth, Sengupta, 2011)

between the two indices (OFHEO, 2008; Aubuchon, Wheelock, 2008; Noeth, Sengupta, 2011) is that FHFA collects data from conforming mortgages only, so the other indices have more higher-priced homes. In addition, Case-Shiller and CoreLogic value-weight transactions.

Volatility and Dynamics For mortgage valuation, stress testing, etc., in addition to knowing what house prices are today, we also need the distribution of **future** house prices. However, when we estimate a repeat-sales index, each period's index is treated as a **constant**. There is no consideration of inter-period dynamics. It is common to assume house prices follow (say) a geometric Brownian motion, and to estimate volatility by looking at period-by-period changes in these indices. However, there are many problems with this approach, which will lead to this approach significantly under-estimating house-price volatility. In particular, the construction of the index automatically induces smoothing; in addition, the index is attempting to measure returns on a somewhat diversified portfolio of real estate, which will have a lower volatility than the individual houses in the portfolio (even if the index is perfectly measured). This is noted in (Schneeweis, Crowder, Kazemi, 2010) and

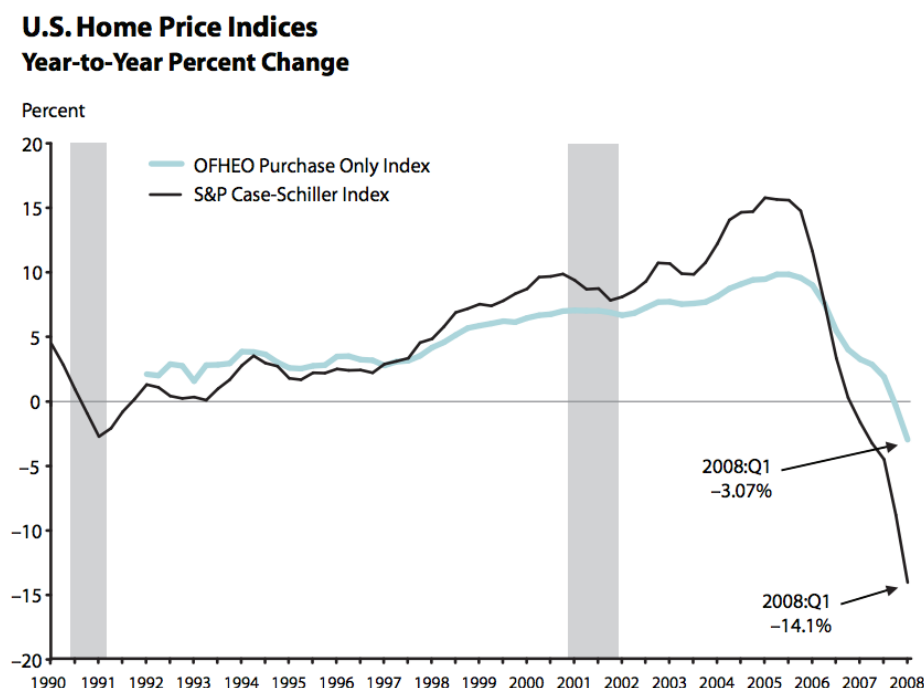


Figure 3.2: Figure 1 from (Aubuchon, Wheelock, 2008)

(Stanton, Wallace, CMBX, 2014). Another implication of these points is that we cannot use the properties of the index to test for (e.g.) serial correlation in house prices.

Non-Constant Quality and Quantity of Housing The houses that enter the sample will in general be of various types (e.g., two- versus three-bedroom, one- versus two-story). Unless we are willing to argue that the average rate of appreciation on all houses of all types is the same, a repeat-sales index thus represents some sort of randomly blended appreciation rate on houses of different types. The empirical evidence to date finds that different housing attributes have different appreciation rates (Meese, Wallace, 1997; Gatzlaff, Haurin, 1997; Gatzlaff, Haurin, 1998), suggesting that this may be a serious problem. Moreover, even the attributes of a given house do not remain fixed over time, because of remodeling and new construction. In the pre-crisis period, aggregate residential investment accounted for nearly six percent of quarterly GDP in 2004 and the first two quarters of 2005, or approximately 35 percent of quarterly gross private domestic investment.¹³

¹³See Bureau of Economic Analysis, Table 1.1.5, Gross Domestic Product.

3.3 Data

In this study we use two different data sets obtained by DataQuick from county assessors' (or equivalent) offices across all available counties in the U.S. The first is the DataQuick historical transaction data, which has been widely used in published academic and practitioner studies focusing on the estimation of house prices and price indices. Despite the widespread use of the DataQuick historical transaction data, there are significant problems with its use to estimate hedonic or repeat-sales price indices. In particular, the home characteristic data in this database are static observations at a single date, so there is no way to track changes to the characteristics of a property (e.g., the addition of a new bedroom or second floor). Moreover, there is no record of the exact date that corresponds to the housing characteristics reported for each property. As a result, such data cannot be used to estimate hedonic price indices, because the reported transaction prices in the DataQuick transaction data cannot be linked to the characteristics of the property at the time of sale. Even for estimating repeat-sales indices, there will be sizable biases introduced whenever there are changes in property characteristics.¹⁴ To offset these limitations of the DataQuick historical transaction data, we augment it with a second, newly available DataQuick data set, released in October 2013.¹⁵ This important data set contains an annual electronic snapshot of the property characteristics and the parcel-specific records that are recorded, usually on paper, by the county assessors' offices. For the first time, it is now possible to construct a panel of the characteristics of the stock of *all* housing in the United States from 2003–2012. These data include a parcel-specific time series for all parcels in all U.S. counties, and include all recording activities associated with foreclosure auctions, quitclaims, short sales and mortgage lien recording, along with any changes in the physical characteristics of the property (due to remodeling). For the purposes of this study, we focus on existing single-family residential houses for arms-length transactions including foreclosure auction sales.¹⁶ We exclude transactions that are quitclaims and short sales because such transaction are not arms-length and are instead “valuations” that are a function of the outstanding liens on the property, or write downs on those liens, and require approval of the mortgage lender.

A first important advantage of merging the DataQuick assessor data panel with the DataQuick transaction data is that we now have an accurate representation of the stock of housing within geographic designations. Given the population of the housing stock, we also know which houses in the stock transact, the exact date of those transactions, and the corresponding characteristics of the house within the year of its sale. These data thus allow us to avoid problems associated with sample-selection bias arising from the estimation of

¹⁴Although DataQuick includes an indicator for whether there was a change in square footage on the property, this indicator is also static and undated, so it cannot be used in WRS estimates with any accuracy.

¹⁵The release of this new dataset followed several conversations between us and them on the drawbacks of overwriting the characteristic data each year.

¹⁶For the current version of the paper, we use data from six large urban counties in California (Alameda, Contra Costa, Los Angeles, San Diego, San Francisco, and Santa Clara). We exclude newly built houses because a large fraction of these new properties have significant lags in the first recorded sale.

price indices using only the set of houses that sell within a given period.

A second important advantage of these merged data is that we can control for the evolution of the characteristics of the stock over time. Because California counties require building permits to remodel houses, the DataQuick assessor data panel allows us to track the timing of remodeling activity and the nature of the remodel on specific parcels. For each house in the stock of single family residential houses in each county, we track the changes in the number of bathrooms, bedrooms, number of rooms, square footage of the structure, and the lot size from 2003–2012. Starting in 2003, we record house-specific changes in any of these characteristics and, not surprisingly, most of these changes occur simultaneously for given properties. This characteristic of our data set allows us to control for the evolution of the supply of housing services independently from the evolution of prices, something that neither WRS nor hedonic price indices estimated from the static DataQuick price history data can do. For this reason, many of these studies wrongly attribute all of the evolution of the price index to changes in prices since the previously existing data sets do not allow controls for changes in housing characteristics.¹⁷

We focus on two Southern California coastal counties, Los Angeles and San Diego Counties, and four Northern California coastal counties, San Francisco, Santa Clara, Contra Costa and Alameda Counties. These counties are of particular interest because of their size and their use as a laboratory of study in numerous other studies (Korteweg, Sorensen, 2013; Landvoight, Piazzesi, Schneider, 2013; Piazzesi, Schneider, Stroebel, 2013; Ghysels, Plazzi Torous, Valkanov, 2012). Our augmented DataQuick data allows us to extend earlier studies by controlling for the dynamics of both the quantity of house services and the price of single family residential houses in markets where there is a lot of remodeling activity.

Table 3.1 provides summary statistics for the balanced time series panels of single family residential houses and their characteristics as found in the DataQuick assessor panel data. As shown in the table, Los Angeles County has the largest stock of single family residential dwellings and San Francisco has the smallest stock of such houses. The more suburban markets of Contra Costa, San Diego, and Santa Clara Counties are shown to have houses with larger average square footage, larger average lot sizes, and more bedrooms than the other counties. These counties also have the newest housing stock where the average year built is 1973 in Contra Costa County, 1966 in San Diego, and 1967 in Santa Clara County. As shown in Table 3.1, the characteristics of the single family residential housing stock in San Francisco Counties is quite different from the other counties. The San Francisco Assessor office does not keep data on the number of bedrooms for houses, however, the square footage, the lot sizes, and the average number of bathrooms is smaller. San Francisco has the oldest single family residential housing stock with an average construction year of 1932. The single family residential houses in Los Angeles and Alameda are similar, both in terms of the vintage of construction and the average structure and lot size square footage, average number of bedrooms and number of bathrooms.

¹⁷This significant limitation is also a problem with studies that apply the American Corelogic transaction data and with studies using data from Trulia.

Table 3.1: Summary statistics for existing stock and arms-length transactions of single-family residential houses located in Alameda, Contra Costa, Los Angeles, San Diego, San Francisco, and Santa Clara Counties from 2003–2012.

	Mean	Standard Deviation	Mean	Standard Deviation
	Alameda		Contra Costa	
Structure Square Footage	1,743.05	900.55	1,828.79	805.37
Number of Bathrooms	1.99	0.89	2.19	0.79
Number of Bedrooms	3.17	0.92	3.40	0.85
Lot Size Square Footage	6,987.93	13,076.47	8,807.51	11,201.89
Year Built	1958.68	28.78	1973.60	23.00
Price (\$)	550,299.28	361,461.07	470,422.4	395,990.83
Arms-length Transactions	123,554		153,701	
	Los Angeles		San Diego	
Structure Square Footage	1,623.30	800.84	1,783.38	949.65
Number of Bathrooms	1.97	0.91	2.12	0.84
Number of Bedrooms	3.06	0.88	3.25	0.86
Lot Size Square Footage	8,404.70	9,487.27	22,927.14	53,937.76
Year Built	1950.49	15.66	1966.89	17.66
Price (\$)	595,518.31	724,380.42	548,628.56	538,721.78
Arms-length Transactions	286,407		103,749	
	San Francisco		Santa Clara	
Structure Square Footage	1,570.41	821.86	1,810.72	788.61
Number of Bathrooms	1.54	0.83	2.20	0.78
Number of Bedrooms	N.A.	N.A.	3.48	0.87
Lot Size Square Footage	2,875.63	1,186.66	9,909.73	22,662.21
Year Built	1931.91	21.18	1967.30	21.96
Price (\$)	896,693.78	968,338.08	752,923.12	664,236.88
Arms-length Transactions	27,614		113,776	

As previously discussed, an important advantage of our data is that we can match the characteristics of houses to their transaction price, so we can estimate a well-defined hedonic price index. A second important advantage is that our data allow us to monitor the permitted remodeling that has occurred for each house in each of our six counties.¹⁸ We define remodeling as the change in the number of bedrooms, number of bathrooms, square footage, and/or the lot size of the property. Our panel is from 2003–2012, so the 2003 characteristics of the house are the benchmark by which we measure whether annual changes have occurred for each house characteristic. We measure whether, by how much, and in what ways each house in the county has changed from 2004–2013.

Table 3.2 presents the total number of houses that were remodeled from 2004–2012. As shown in the table, the rate of remodeling ranges from a low of about 6.74% of houses in the newer housing stock counties, such as Contra Costa County, over the period to a high of about 10.16% of houses in Alameda County. Here again, we see that the more suburban counties of Santa Clara and San Diego have on average the lowest rates of remodeling, whereas the more urban counties such as Alameda, San Francisco, and Los Angeles appear to have significantly higher remodeling rates.

Table 3.2: Cumulative rates of remodeling for the existing stock of single family residential houses located in Alameda, Contra Costa, Los Angeles, San Diego, San Francisco, and Santa Clara Counties from 2004–2012.

	Number	Percent
Alameda County, N = 274,665		
Remodeling 2004–2012	27,903	10.16
Contra Costa, N = 254,998		
Remodeling 2004–2012	17,183	6.74
Los Angeles County, N = 924,612		
Remodeling 2004–2012	85,908	9.29
San Diego County, N = 309,250		
Remodeling 2004–2012	26,970	8.72
San Francisco County, N = 93,656		
Remodeling 2004–2012	9,034	9.65
Santa Clara County, N = 281,153		
Remodeling 2004–2012	23,159	8.24

Overall, the summary statistics reinforce the need for appropriate methods to control for heterogeneity in the dynamics of both prices and characteristics of single family residential housing. Accounting for shifts in preferences for the functional composition of individual

¹⁸Our data set does not allow us to observe non-permitted remodeling or permitted remodeling that does not change the square footage or measurable features of the houses. For this reason, the data provide a conservative representation of the extent of remodeling in the housing stock.

housing units, clearly requires an econometric estimation strategies for house price indices that accounts for the composition of attributes of the houses that trade. Only in this way can observed prices changes be accurately decomposed into the component associated with true price shocks and the component associated with the evolution of, or shocks to, the characteristics of the houses that trade.

3.4 A New, Dynamic House Price Estimator

Write log-price, $y_{i,t}$, as

$$y_{i,t} = Ax_{i,t} + \overbrace{B\xi_t + \alpha_t}^{\text{house price index}} + \mu_i + \epsilon_{i,t}, \quad (3.5)$$

$$= X_{i,t}\beta + \alpha_t + \mu_i + \epsilon_{i,t}, \quad (3.6)$$

$$\alpha_t = \rho\alpha_{t-1} + \eta_t, \quad (3.7)$$

where

$$\epsilon_{i,t} \sim \text{i.i.d. } N(0, \sigma_\epsilon^2), \quad (3.8)$$

$$\mu_i \sim \text{i.i.d. } N(0, \sigma_\mu^2), \quad (3.9)$$

$$\eta_{i,t} \sim \text{i.i.d. } N(0, \sigma_\eta^2). \quad (3.10)$$

- $x_{i,t}$ is a set of home hedonics.
- ξ_t is a set of macro-fundamentals.
- $X_{i,t} = (x_{i,t}, \xi_t)$, the combined set of regressors, and $\beta = (A^\top, B^\top)^\top$.
- μ_i is a house-specific random effect (due to differences in unobservable hedonics).
- α_t is the unexplained (and unobserved) portion of the index.

Estimation

This random-effects model can be written as a standard linear, Gaussian state-space model if we augment the definition of the state, α_t , to include β and μ :

$$\alpha_t = \begin{pmatrix} \alpha_t \\ \beta_{1,t} \\ \beta_{2,t} \\ \vdots \\ \beta_{k,t} \\ \mu_{1,t} \\ \mu_{2,t} \\ \vdots \\ \mu_{I,t} \end{pmatrix} \equiv \begin{pmatrix} \alpha_t \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_I \end{pmatrix} = \begin{pmatrix} \rho\alpha_{t-1} + \eta_t \\ \beta_{1,t-1} \\ \beta_{2,t-1} \\ \vdots \\ \beta_{k,t-1} \\ \mu_{1,t-1} \\ \mu_{2,t-1} \\ \vdots \\ \mu_{I,t-1} \end{pmatrix}.$$

In principle we could estimate the model by, for example, maximizing the likelihood function using a Kalman Filter (Kalman, Bucy, 1961). However, we do not do it this way due to the high dimensionality of the problem, the presence of lots of missing data, and a desire to be able to handle more general settings later (e.g., non-Gaussian distributions). Instead, we estimate the model in a Bayesian framework using Markov Chain Monte Carlo (MCMC) methods. We present a brief overview of these methods here. For a very readable textbook treatment, see (Robert, Casella, 2004).¹⁹

Bayesian statistics

Unlike in classical statistics, where parameters are regarded as constants, in Bayesian statistics we explicitly treat the parameters being estimated, θ , as random variables. Before seeing the data, y , we assume θ comes from the prior distribution $p(\theta)$. Estimation involves calculating the posterior distribution $p(\theta | y)$ using Bayes Theorem,

$$p(\theta | y) = \frac{p(y | \theta)p(\theta)}{p(y)} = \frac{p(y | \theta)p(\theta)}{\int_{\theta} p(y | \theta)p(\theta) d\theta}. \quad (3.11)$$

Given this distribution, we can (either analytically or by sampling) calculate the posterior mean (or median) and variance of θ , calculate the probability that $\theta_1 > 2$, etc.

There is a long (and often acrimonious) debate between classical and Bayesian statisticians, which we do not intend to contribute to. For our purposes, we note that Bayesian and classical methods are often very close. For example, from Equation 3.11, if we assume a flat prior, $p(\theta) = \text{constant}$, the mode of the posterior distribution in a Bayesian setting is identical to the classical maximum likelihood estimate of θ . Two particularly attractive features of Bayesian estimators are

¹⁹Other references on MCMC in general include (Casella, George, 1992; Chib, Greenberg, 1995; Chib, Greenberg, 1996; Tanner, Wong, 1987). (Johannes, Polson, 2009) present a good overview of these methods in a finance context.

1. Since we have the *exact* posterior distribution, we don't need to rely on asymptotic results.
2. Unlike most classical estimators, which are obtained by numerically optimizing (often badly behaved) nonlinear objective functions, no optimization is required in Bayesian estimation.

On the other hand, except in simple cases, we usually cannot calculate the exact posterior in closed form, especially if the model involves non-standard distributions, latent variables, etc. This is where Markov Chain Monte Carlo (MCMC) methods come in, allowing us to sample from complex posteriors even when we can't write the distribution down in closed form, and making it much easier to estimate models with latent variables such as stochastic volatility (Jacquier, Polson, Rossi, 1994).

Markov Chain Monte Carlo (MCMC)

MCMC methods are based on the Clifford-Hammersley theorem, which shows that under weak conditions, the joint distribution of two sets of variables, $p(\theta_1, \theta_2)$, can be recovered from the two conditional distributions, $p(\theta_1 | \theta_2)$ and $p(\theta_2 | \theta_1)$. Often, the joint distribution is very high dimensional and impossible to sample from directly (or even compute in closed form), while the conditional distributions are much simpler. The standard MCMC approach is to use the Gibbs sampler (Casella, George, 1992) to generate a Markov chain as follows:

- Pick initial values θ_1^0 and θ_2^0 .
- For $i = 1, \dots, N$:
 1. Draw $\theta_1^i \sim p(\theta_1 | \theta_2^{i-1})$.
 2. Draw $\theta_2^i \sim p(\theta_2 | \theta_1^i)$.

Under weak conditions, the steady-state distribution of the Markov chain $\{(\theta_1^i, \theta_2^i) : i = 1, 2, \dots\}$ is $p(\theta_1, \theta_2 | y)$. Estimation involves constructing a long Markov chain, throwing away a large number of initial values to avoid dependence on initial conditions, and using the remaining values to estimate this steady-state distribution.

Estimating the index

If we knew the values of the random intercepts, $\boldsymbol{\mu}$, we could write the model in state-space form as

$$\begin{aligned} y_t - \boldsymbol{\mu} &= \mathbf{Z}_t \boldsymbol{\alpha}_t + \epsilon_t; & \epsilon_t &\sim N(0, \sigma_\epsilon^2 \mathbf{I}); \\ \boldsymbol{\alpha}_t &= \mathbf{T} \boldsymbol{\alpha}_{t-1} + \mathbf{G} \eta_t; & \eta_t &\sim N(0, \sigma_\eta^2); \end{aligned}$$

where

$$\boldsymbol{\alpha}_t = \begin{bmatrix} \beta \\ \alpha_t \end{bmatrix}, \quad \mathbf{Z}_t = \begin{bmatrix} X_t & 1 \end{bmatrix}, \quad \mathbf{T}_t = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \rho \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}.$$

This immediately suggests splitting the parameters so that $\boldsymbol{\mu}$ is in a separate block from the others, as much work has been done on sampling and Bayesian estimation for state space models, including (Fruhwirth-Schnatter, 1994; Carter, Kohn, 1994; Durbin, Koopman, 2002; Strickland, Turner, Denham, Mengersen, 2009). We follow the approach outlined in (Strickland, Turner, Denham, Mengersen, 2009), and implemented using the Python package PySSM (Strickland, Burdett, Mengersen, Denham, 2014). We split the parameters and latent variables into four blocks:

1. $\boldsymbol{\alpha} \equiv \{\beta_k : k = 1, \dots, K\} \cup \{\alpha_t : t = 1, \dots, T\}$: The augmented state, including regression coefficients.
2. $\boldsymbol{\theta} = \{\sigma_\epsilon, \sigma_\eta, \rho\}$: Variances plus AR parameter.
3. $\boldsymbol{\mu} = \{\mu_i : i = 1, \dots, N\}$: House-specific intercepts (random effects).
4. σ_μ : Prior variance of random effects.

Extending the two-block Gibbs sampler described above, we construct a Markov chain by drawing one set of values for each block in turn, taking the values of the other blocks as given, subject to one point worth noting. The sampled values of $\boldsymbol{\alpha}$ and $\boldsymbol{\theta}$ in the Gibbs sampler will tend to be highly correlated, which can cause the algorithm to converge very slowly. In such cases it is preferable, where possible, to sample from the joint distribution of the two blocks, which we do by writing the joint distribution as the product

$$p(\boldsymbol{\alpha}, \boldsymbol{\theta} \mid y, \boldsymbol{\mu}, \sigma_\mu) = p(\boldsymbol{\alpha} \mid y, \boldsymbol{\theta}, \boldsymbol{\mu}, \sigma_\mu) p(\boldsymbol{\theta} \mid y, \boldsymbol{\mu}, \sigma_\mu).$$

We first sample from the marginal distribution $p(\boldsymbol{\theta} \mid y, \boldsymbol{\mu}, \sigma_\mu)$ and then from $p(\boldsymbol{\alpha} \mid y, \boldsymbol{\theta}, \boldsymbol{\mu}, \sigma_\mu)$. The marginal posterior of $\boldsymbol{\theta}$ can be written in terms of the likelihood,

$$p(\boldsymbol{\theta} \mid y, \boldsymbol{\mu}, \sigma_\mu) \propto p(y \mid \boldsymbol{\theta}, \boldsymbol{\mu}, \sigma_\mu) p(\boldsymbol{\theta}),$$

where

$$p(y \mid \boldsymbol{\theta}, \boldsymbol{\mu}, \sigma_\mu) = \int p(y \mid \boldsymbol{\alpha}, \boldsymbol{\theta}, \boldsymbol{\mu}, \sigma_\mu) p(\boldsymbol{\alpha} \mid \boldsymbol{\theta}, \boldsymbol{\mu}, \sigma_\mu) d\boldsymbol{\alpha}.$$

(Koopman, Durbin, 2000) show how to calculate the likelihood in closed form. The algorithm overall then looks like this:

1. Sample $\boldsymbol{\alpha}^j$ from $p(\boldsymbol{\alpha} \mid y, X, \boldsymbol{\theta}^{j-1}, \boldsymbol{\mu}^{j-1}, \sigma_\mu^{j-1})$.
There is a sizeable literature presenting various “simulation smoothing” algorithms for sampling the state vector in a linear state-space model from its full posterior distribution (Fruhwirth-Schnatter, 1994; Carter, Kohn, 1994; Durbin, Koopman, 2002; Strickland, Turner, Denham, Mengersen, 2009). We use the algorithm described in Section 3.1.2 (Strickland, Turner, Denham, Mengersen, 2009).

2. Sample $\boldsymbol{\theta}^j$ from $p(\boldsymbol{\theta} \mid y, X, \boldsymbol{\mu}^{j-1}, \sigma_\mu^{j-1})$.

Assuming a flat prior for $\boldsymbol{\theta}$, we have

$$p(\boldsymbol{\theta} \mid y, X, \boldsymbol{\mu}^{j-1}, \sigma_\mu^{j-1}) \propto p(y \mid X, \boldsymbol{\theta}, \boldsymbol{\mu}^{j-1}, \sigma_\mu^{j-1}),$$

the likelihood of the observed data. To sample from this distribution, we use a random-walk Metropolis-Hastings algorithm (RWMH) (Robert, Casella, 2004).

3. Sample $\boldsymbol{\mu}^j$ from $p(\boldsymbol{\mu} \mid y, X, \boldsymbol{\alpha}^j, \boldsymbol{\theta}^j, \sigma_\mu^{j-1})$.

Each observed house price yields a noisy signal for μ_i :

$$m_{i,t} = y_{i,t} - X_t \beta_b - \alpha_{b,t} \sim N(\mu_i, \sigma_\epsilon^2).$$

Assuming a normal prior, $\mu_i \sim N(0, \sigma_\mu)$, then using standard results the posterior is also normal,

$$(\mu_i \mid y, X, \boldsymbol{\alpha}^j, \boldsymbol{\theta}^j, \sigma_\mu^{j-1}) \sim N \left(\bar{m}_i \times \frac{\frac{N_i}{\sigma_\epsilon^2}}{\frac{N_i}{\sigma_\epsilon^2} + \frac{1}{\sigma_\mu^2}}, \left(\frac{1}{\sigma_\mu^2} + \frac{N_i}{\sigma_\epsilon^2} \right)^{-1} \right), \quad \text{where}$$

$$\bar{m}_i = \frac{\sum_{j=1}^{N_i} m_{i,t_j}}{N_i},$$

N_i = number of sales of property i .

4. Sample σ_μ^j from $p(\sigma_\mu \mid y, X, \boldsymbol{\alpha}^j, \boldsymbol{\theta}^j, \boldsymbol{\mu}^j)$.

Using standard results, if the prior for σ_μ^2 is inverse gamma with parameters α and β , the posterior is also inverse gamma, with parameters

$$\alpha + \frac{N}{2} \quad \text{and} \quad \beta + \frac{\sum_{i=1}^N \mu_i^2}{2}.$$

Hedonic Characteristics

Each house is defined by its county and a set of house-specific characteristics at time t , $x_i(t) = (s_i(t), \ell_i(t), N_i(t), n_i(t), a_i(t))'$, where

$s_i(t)$ = square footage,

$\ell_i(t)$ = lot size,

$N_i(t)$ = number of bedrooms,

$n_i(t)$ = number of bathrooms,

$a_i(t)$ = age.

Each observed house i at time t belongs to basket b_N , defined by the number of bedrooms $N_i(t) = N \in \{1, 2, 3, 4^+\}$.

Macroeconomic variables

The macroeconomic variables used in the estimation are

- California population.
- The ten-year Treasury rate.
- The regional unemployment rate.²⁰

3.5 Results

We estimate the index for each of our six counties: Alameda, Contra Costa, Los Angeles, San Diego, Santa Clara, and San Francisco. For each county, we estimate separate indices for each number of bedrooms (1, 2, 3 and 4 or more).²¹ For each county and house type, we estimate two indices, one including and one excluding the macroeconomic variables. Tables 3.3 and 3.4 summarize the results.

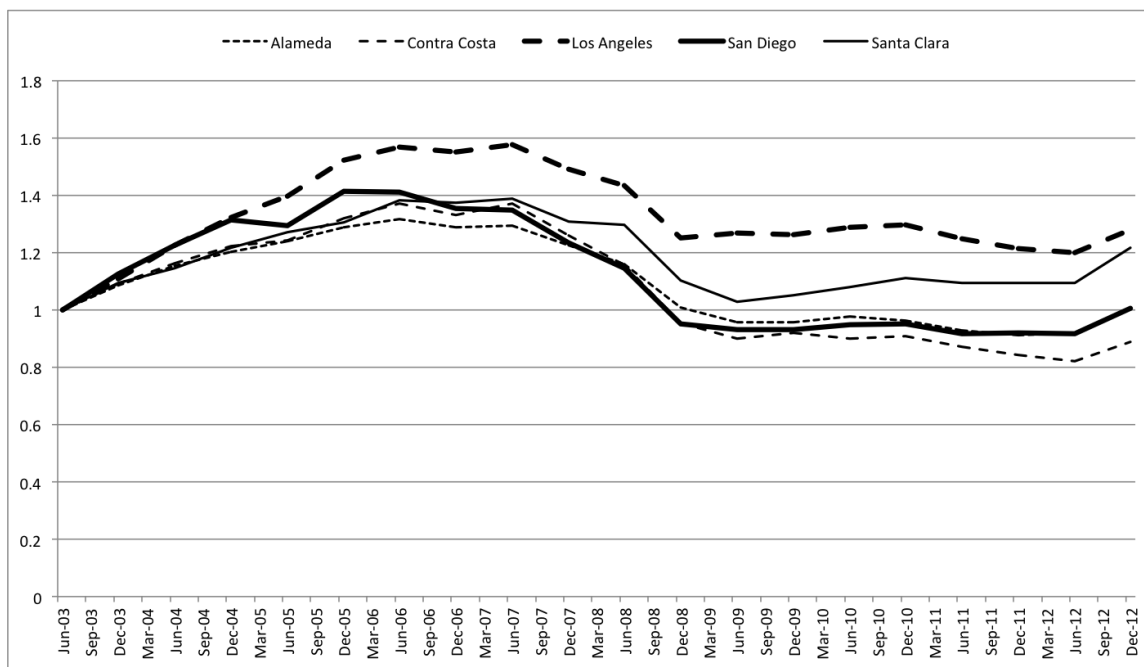


Figure 3.3: Dynamic model for two bedroom houses in Alameda, Contra Costa, Los Angeles, San Diego, and Santa Clara Counties using the specification with macro fundamentals

²⁰We have data for LA, SD and SF counties. The other counties are assigned the unemployment rate of whichever of these three counties is closest.

²¹San Francisco does not provide the number of bedrooms for each house, so for this county we estimate indices for house with 1 and 2+ bathrooms.

	ρ	σ_μ	σ_η	$A_{lotsize}$	A_{sqft}
Alameda					
Two Bedroom	0.973977*** (0.000000)	0.175162*** (0.000000)	0.147233*** (0.000000)	0.000018 (0.000013)	0.000533*** (0.000070)
Three Bedroom	0.998717*** (0.000001)	0.179783*** (0.000356)	0.154124*** (0.000043)	0.000004 (0.000012)	0.000533*** (0.000051)
Contra Costa					
Two Bedroom	0.998316*** (0.000001)	0.161232*** (0.014706)	0.161221*** (0.020739)	0.000018 (0.000013)	0.000229** (0.000077)
Three Bedroom	0.998333*** (0.000000)	0.191722*** (0.000000)	0.124500*** (0.000000)	0.000011*** (0.000002)	0.000229*** (0.000037)
Los Angeles					
Two Bedroom	0.998799*** (0.000251)	0.123394*** (0.012907)	0.143950*** (0.008744)	0.000012 (0.000013)	0.000249*** (0.000072)
Three Bedroom	0.995732*** (0.000478)	0.123394*** (0.003363)	0.170166*** (0.044814)	0.000011 (0.000013)	0.000235 (0.000235)
San Diego					
Two Bedroom	0.998577*** (0.000151)	0.153260*** (0.003655)	0.152162*** (0.006966)	0.000016** (0.000006)	0.000244*** (0.000071)
Three Bedroom	0.998554*** (0.000215)	0.131608*** (0.005796)	0.185035*** (0.033310)	0.000003 (0.000005)	0.000244*** (0.000044)
Santa Clara					
Two Bedroom	0.998909*** (0.000000)	0.166418*** (0.000000)	0.146325*** (0.000000)	0.000012 (0.000019)	0.000232** (0.000114)
Three Bedroom	0.997846*** (0.000000)	0.147231*** (0.000000)	0.213377*** (0.000000)	0.000002 (0.000008)	0.000246*** (0.000037)
San Francisco					
One Bath	0.998947*** (0.000000)	0.182058*** (0.000000)	0.150279*** (0.000000)	0.000006 (0.000023)	0.000216*** 0.000048
Two Bath	0.998722*** (0.000000)	0.190944*** (0.000000)	0.121983*** (0.000000)	0.000036 (0.000028)	0.000155*** (0.000057)

Table 3.3: Dynamic model for houses located in Alameda, Contra Costa, Los Angeles, San Diego, San Francisco, Santa Clara and San Francisco Counties using the specification without macro fundamentals.

	ρ	σ_μ	σ_η	$A_{lotsize}$	A_{sqft}	$B_{CA-Pop.}$	$B_{10Yr-Rate}$	B_{Unemp}
Alameda								
Two Bedroom	0.9368*** (0.0485)	0.1830*** (0.0121)	0.1013*** (0.0290)	0.00001 (0.00001)	0.0002*** (0.0001)	0.2881*** (0.0607)	0.0767 (0.0544)	-0.0406 (0.0323)
Three Bedroom	0.9620*** (0.0676)	0.1727*** (0.0138)	0.0849*** (0.0280)	0.00023*** (0.00001)	0.0002*** (0.0000)	0.1969** (0.0708)	0.1132** (0.0491)	-0.0297 (0.0310)
Contra Costa								
Two Bedroom	0.9200*** (0.1067)	0.1559*** (0.0172)	0.1345*** (0.0395)	0.00001 (0.00001)	0.0002*** (0.0001)	0.2749*** (0.0686)	0.1041 (0.0796)	-0.0487 (0.0446)
Three Bedroom	0.9531*** (0.0765)	0.1877*** (0.0154)	0.1107*** (0.0288)	0.00001*** (0.00000)	0.0002*** (0.0000)	0.2263*** (0.1008)	0.0900 (0.0588)	-0.0433 (0.0302)
Los Angeles								
Two Bedroom	0.9017*** (0.1368)	0.1198*** (0.0197)	0.1059** (0.0382)	0.00001 (0.00001)	0.0003*** (0.0001)	0.3287*** (0.0705)	0.1816** (0.0622)	-0.0560 (0.0364)
Three Bedroom	0.9511*** (0.0851)	0.1094*** (0.1008)	0.0862*** (0.0246)	0.00001 (0.00001)	0.0002*** (0.0000)	0.2637*** (0.0664)	0.0650 (0.0468)	-0.0429 (0.0285)
San Diego								
Two Bedroom	0.8905*** (0.1387)	0.1511* (0.0904)	0.1134** (0.0478)	0.00000 (0.00000)	0.0002* (0.0001)	0.3497*** (0.0671)	0.1004 (0.0729)	-0.0629 (0.0434)
Three Bedroom	0.9502*** (0.0844)	0.1457 (0.0904)	0.1050*** (0.0338)	0.00000 (0.00000)	0.0002 (0.0007)	0.2449** (0.0934)	0.0648 (0.1927)	-0.0471 (0.1692)
Santa Clara								
Two Bedroom	0.8906*** (0.1368)	0.1619*** (0.0266)	0.1228** (0.0542)	0.00001 (0.00002)	0.0002*** (0.0000)	0.2044*** (0.0619)	0.0855 (0.0910)	-0.0510 (0.0457)
Three Bedroom	0.9699*** (0.0576)	0.1455*** (0.0142)	0.0780*** (0.0189)	0.00000 (0.00001)	0.0002*** (0.0000)	0.2044*** (0.0619)	0.0376 (0.0432)	-0.0316 (0.0231)
San Francisco								
One Bath	0.9243*** (0.1110)	0.1742*** (0.0265)	0.0857** (0.0407)	0.00001 (0.00002)	0.0002*** (0.0000)	0.2099*** (0.0570)	0.0743 (0.0649)	-0.0315 (0.0360)
Two Bath	0.9120*** (0.1060)	0.1955*** (0.0314)	0.0983** (0.0492)	0.00004 (0.00003)	0.0001** (0.0001)	0.2498*** (0.0525)	0.0635 (0.0704)	-0.0332 (0.0412)

Table 3.4: Dynamic model for houses located in Alameda, Contra Costa, Los Angeles, San Diego, San Francisco, Santa Clara and San Francisco Counties using the specification with macro fundamentals.

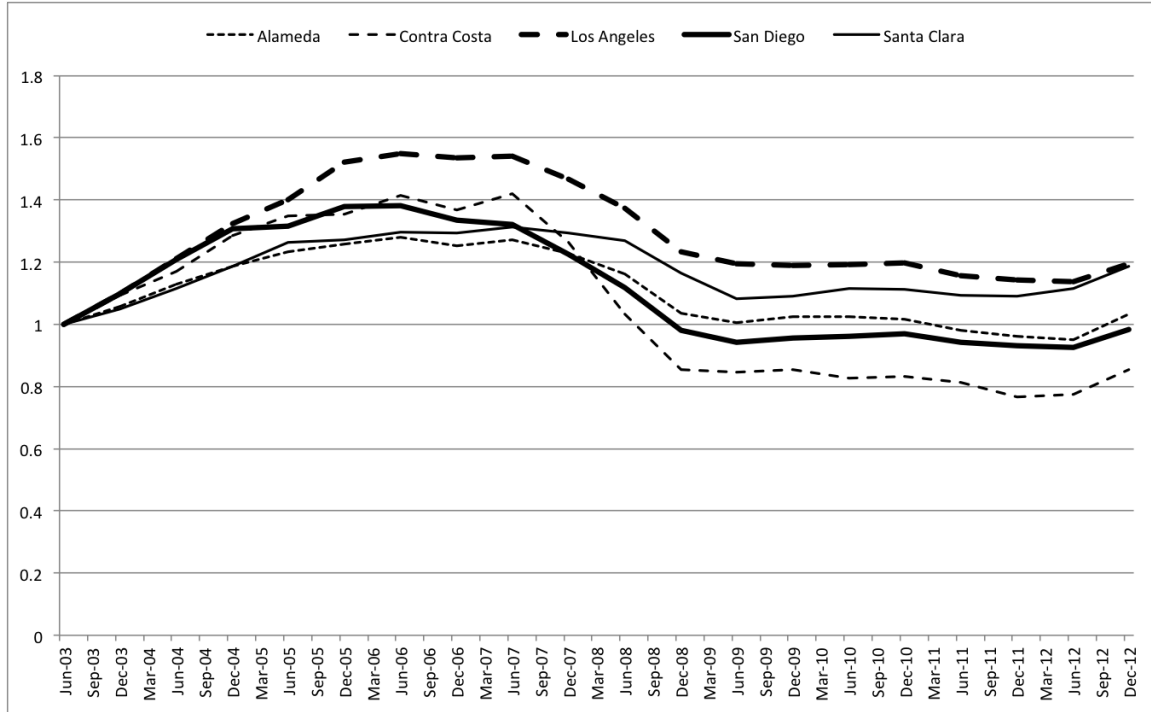


Figure 3.4: Dynamic model for three bedroom houses in Alameda, Contra Costa, Los Angeles, San Diego, and Santa Clara Counties using the specification with macro fundamentals

3.6 Conclusions

In this work we apply a new data set, developed by DataQuick using historical county assessor's office records, that contains time-varying property characteristics over a ten-year period for all single-family residential housing units in the California counties of Alameda, Contra Costa, Los Angeles, San Diego, and San Francisco. We merge these data with DataQuick's historical transaction data, which, by construction, has never had a time-consistent link between home-characteristics and recorded transaction prices. Because this new merged data set represents the entire stock of houses in each of these counties, we are able to avoid the traditional sample-selection problems associated with the estimation of empirical price indices using only transaction data. Instead, we develop a model of house price dynamics that incorporates parsimonious controls for the intensity of sales and remodels of homes in an estimator where the drift and volatility of the price index are time varying. Our model allows channels for the hedonic characteristics of the house, for a parsimonious representation of the macro-fundamental determinants of house price formation, the idiosyncratic components of house price volatility, and the volatility of the house prices around the index for baskets of houses defined by their number of bedrooms.²² We then apply classical linear-filtering techniques for these baskets of houses to obtain the unobservable transition equation for the

²²The baskets are defined by the number of bathrooms in the case of San Francisco.

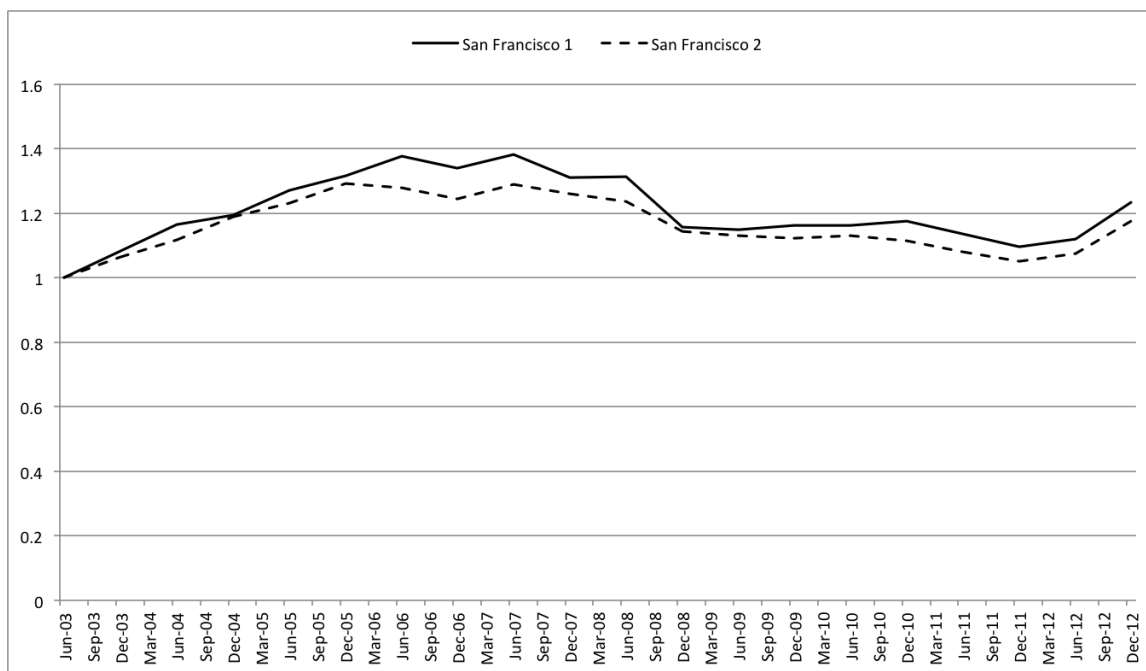


Figure 3.5: Dynamic model for one and two bathroom houses in San Francisco using the specification with macro fundamentals

price indices for each of these baskets. Because our estimator includes a decomposition of house price volatility into a time series component (the volatility of the basket) and cross-sectional component (the idiosyncratic cross-sectional component of volatility for houses in the basket), our estimator reveals significant differences in both the level and volatility of our price indices compared with the Federal Housing Finance Board's estimate of annual house price volatility of their WRS price index for California. This shows that our index is superior to repeat-sales indices as a measure of U.S. real-estate returns for economic forecasting, mortgage valuation, and bank stress tests.

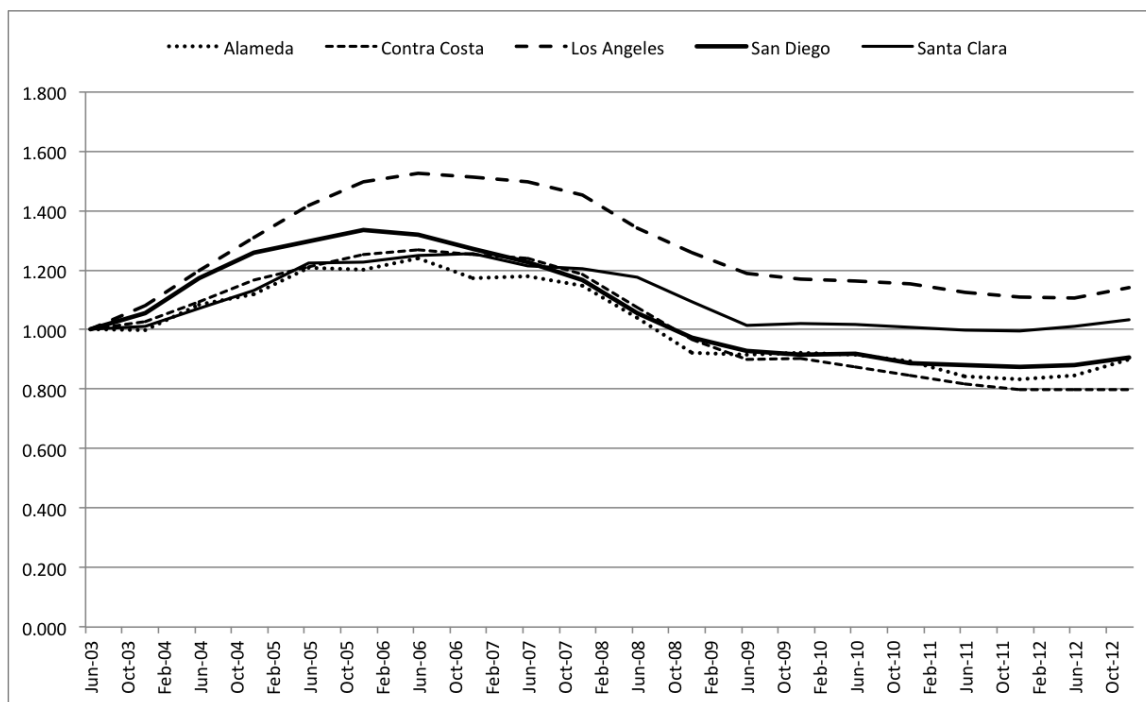


Figure 3.6: Dynamic model for two bedroom houses in Alameda, Contra Costa, Los Angeles, San Diego, and Santa Clara Counties using the specification without macro fundamentals

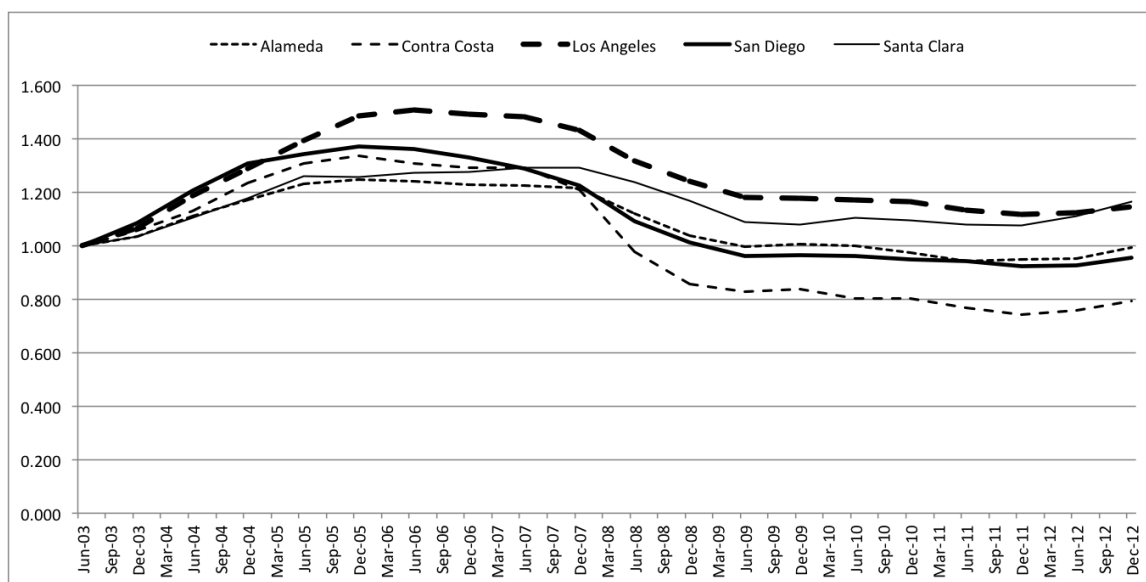


Figure 3.7: Dynamic model for three bedroom houses in Alameda, Contra Costa, Los Angeles, San Diego, and Santa Clara Counties using the specification without macro fundamentals

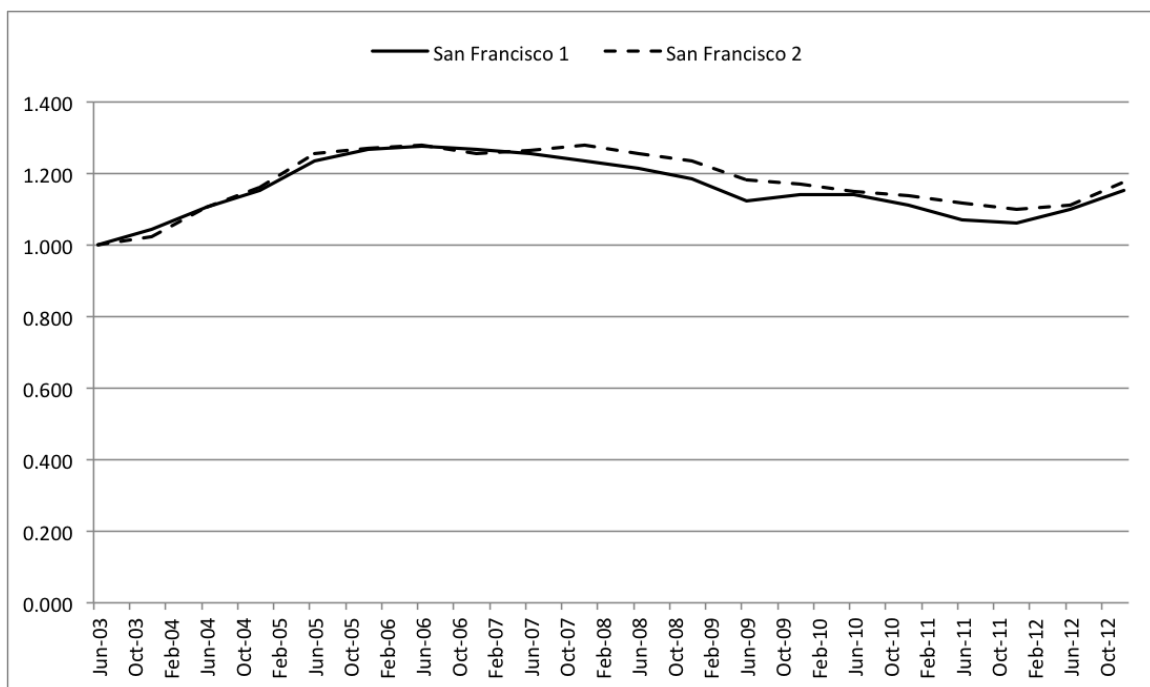


Figure 3.8: Dynamic model for one and two bathroom houses in San Francisco using the specification without macro fundamentals

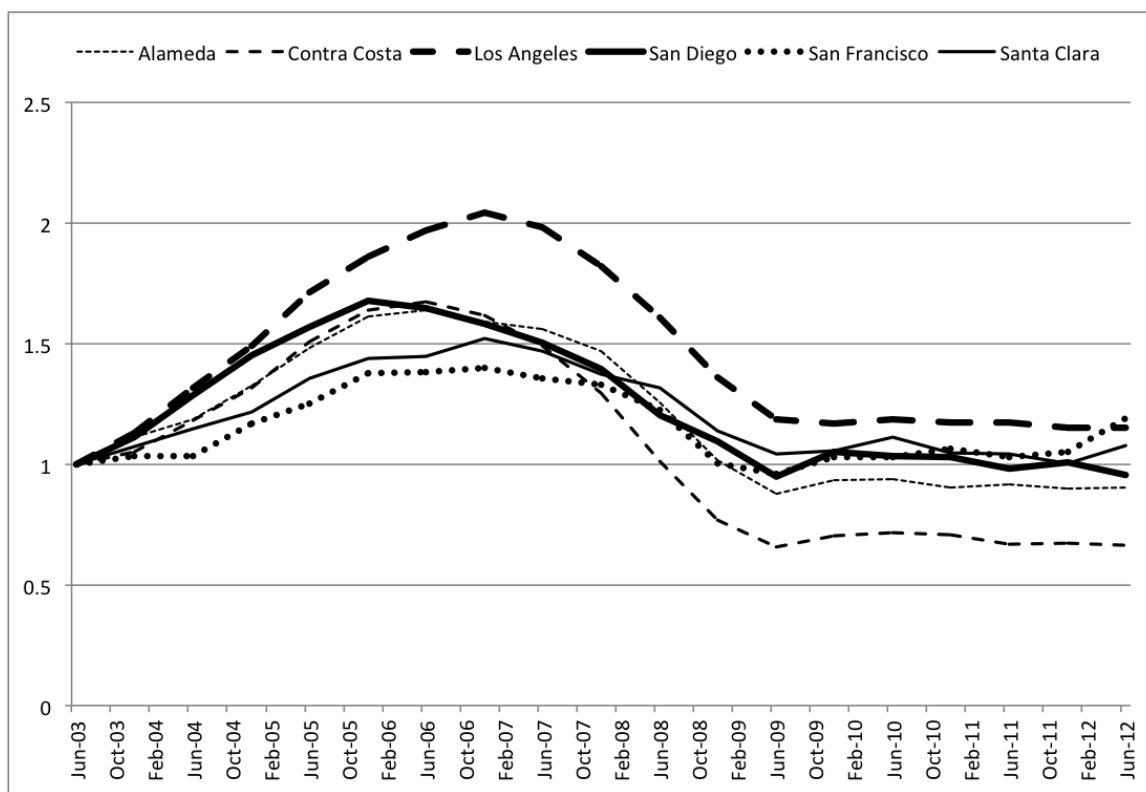


Figure 3.9: Weighted Repeat Sales model Alameda, Contra Costa, San Francisco, Los Angeles, San Diego, and Santa Clara Counties

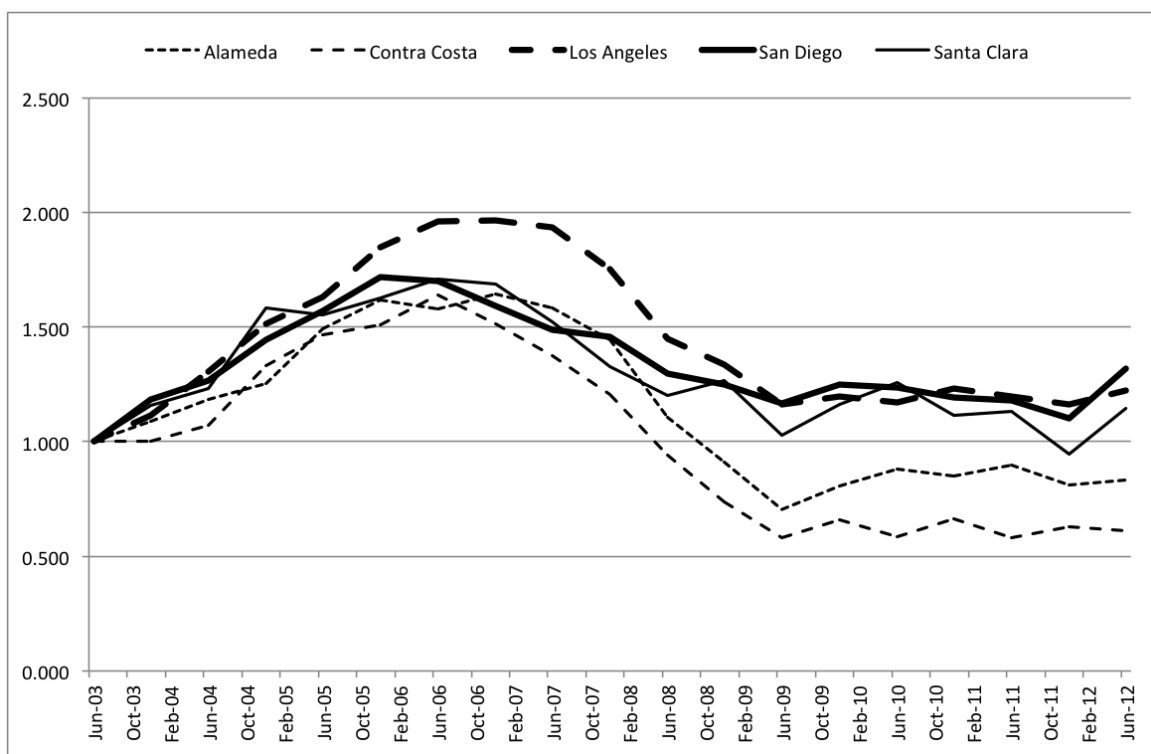


Figure 3.10: Weighted Repeat Sales model for two bedroom houses in Alameda, Contra Costa, Los Angeles, San Diego, and Santa Clara Counties

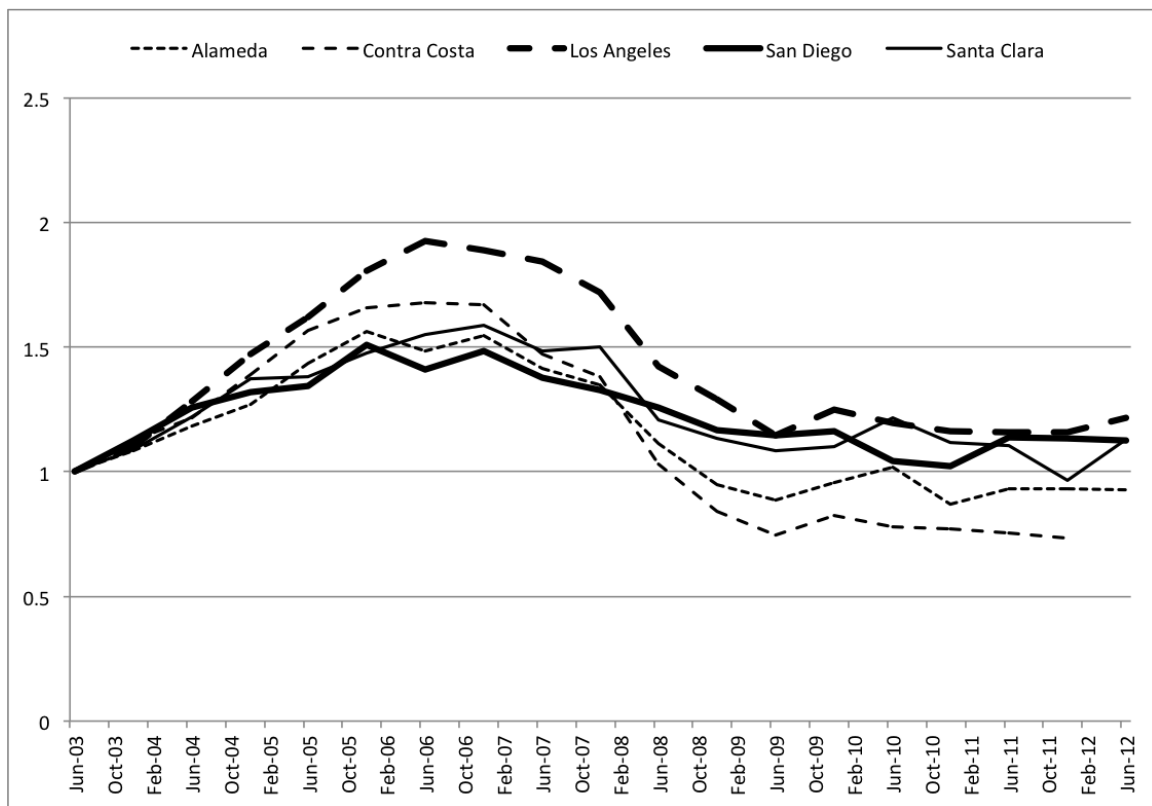


Figure 3.11: Weighted Repeat Sales model for three bedroom houses in Alameda, Contra Costa, Los Angeles, San Diego, and Santa Clara Counties

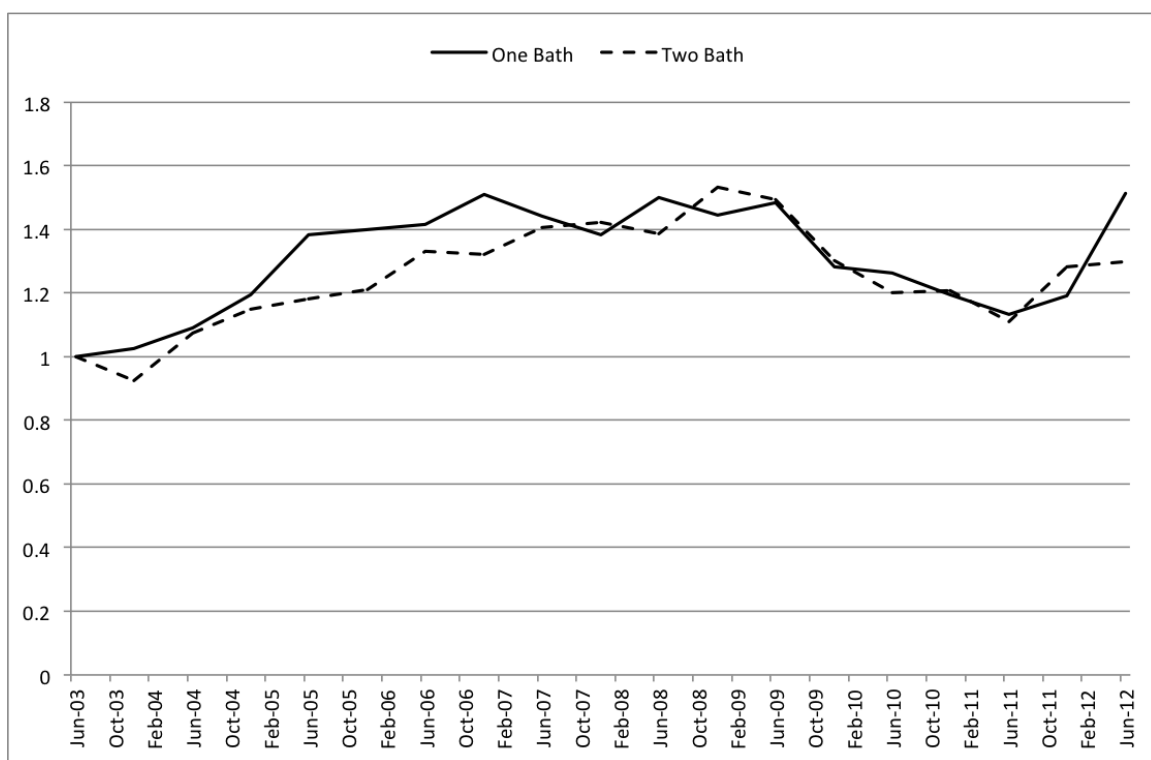


Figure 3.12: Weighted Repeat Sales for one and two baths houses San Francisco no correction for remodeling

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