Quantum-field-theory calculation of the two-dimensional Ising model correlation function

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The equivalence between the two-dimensional Ising model and a free fermion field theory is used to rederive various results for the two-spin correlation function in the critical region.

I. INTRODUCTION

The utility of the relation between the two-dimensional Ising model and a relativistic free Fermi field theory[1] for the computation of the spin-spin correlation function, $\langle\sigma_r\sigma_{r'}\rangle$, has been noted recently by several groups.[2,3] In Ref. 2 this technique was used to evaluate the *scaling* part of the correlation function at $T=T_c$ as well as leading corrections in $(T-T_c)|r-r'|$.

The expressions for the correlation functions at $T=T_c$ have been known for some time,[4] as has been the leading term in the $(T-T_c)|\bar{r}-\bar{r}'|$ expansion.[5] The entire perturbation expansion, as well as a closed solution for the scaling region correlation function, was presented in a monumental work of Wu, McCoy, Tracy, and Barouch[6] and earlier references.[7,8]

In this article we obtain a perturbation expansion for the correlation function for $T\to T_c$, $|r-r'|\to\infty$, and $(T-T_c)|r-r'|$ fixed. This result has been obtained previously.[6-8] It may be useful to rederive these results using an approach other than the one of Refs. 6–8. We stay completely within the framework of a relativistic field theory. These techniques may prove to be of use in the study of other lattice field theories, especially in performing mass perturbation where infrared difficulties are severe.

II. MAJORANA FIELD FORMALISM

We wish to review some of the results of Refs. 1 and 2. No details will be presented as this section is intended to establish notation. Our interest is in the two-spin correlation function near the critical point for a square Ising lattice of L/a points on a side; a is the lattice spacing. The critical temperature $1/\beta_c$ is obtained from the transcendental equation

$$\sinh 2\beta_c = 1\,. \tag{2.1}$$

Near this temperature we define

$$m/4=\beta_c-\beta\,, \tag{2.2}$$

and retain only leading terms in m; m positive (negative) corresponds to $T>T_c$ $(T<T_c)$.

In (1) it was shown that this correlation function is given by [Ref. 2, Eq. (29)]

$$\langle\sigma_r\sigma_{r'}\rangle=\left\langle(c_r^\dagger-c_r)\exp\left(i\pi\sum_{s=r+1}^{r'-1}c_s^\dagger c_s\right)(c_{r'}^\dagger+c_{r'})\right\rangle. \tag{2.3}$$

In the above c_r and $c_r^\dagger$ are fermion annihilation and creation operators attached to site r. The sites r and r' are along the same row and the expectation value is in the vacuum state of another set of fermion operators, ξ_q and $\xi_q^\dagger$, defined on the reciprocal lattice $q=2\pi p/L$, $p=0,\pm1,\pm2,\ldots,\pm L/2a$. The ξ's satisfy the usual anticommutation relations

$$\{\xi_q,\xi_{q'}\}=\{\xi_q^\dagger,\xi_{q'}^\dagger\}=0,\quad \{\xi_q,\xi_{q'}^\dagger\}=\delta_{q,q'}\,. \tag{2.4}$$

We still need a relationship between the c's of (2.3) and the ξ's. In order to achieve this we introduce a set of fields defined on a particular row of the lattice:

$$\psi_1(r)=\left(\frac{a}{L}\right)^{1/2}\sum_q\left(\frac{\omega_q+q}{2\omega_q}\right)^{1/2}\epsilon(q)(\xi_qe^{iqr}+\xi_q^\dagger e^{-iqr})\,,$$
$$i\psi_2(r)=-\left(\frac{a}{L}\right)^{1/2}\sum_q\left(\frac{\omega_q-q}{2\omega_q}\right)^{1/2}\epsilon(q)(\xi_qe^{iqr}-\xi_q^\dagger e^{-iqr})\,; \tag{2.5}$$

$\epsilon(q)=q/|q|$ and ω_q is related to m [Eq. (2.2)],

$$\omega_q=(q^2+m^2)^{1/2}\,. \tag{2.6}$$

Finally,

$$c_r=\frac{e^{-i\pi/4}}{\sqrt{2}}[\psi_1(r)+i\psi_2(r)]\,. \tag{2.7}$$

In the thermodynamic limit $(L\to\infty)$ the summation over q may be replaced by an integration,

$$\psi_1(r)=\frac{1}{\sqrt{2\pi}}\int_{-\pi/a}^{\pi/a}dq\left(\frac{\omega_q+q}{2\omega_q}\right)^{1/2}\epsilon(q)(\tilde{\xi}_q e^{iqr}+\tilde{\xi}_q^\dagger e^{-iqr}),$$

$$\tag{2.8}$$

$$i\psi_2(r)=\frac{-1}{\sqrt{2\pi}}\int_{-\pi/a}^{\pi/a}dq\left(\frac{\omega_q-q}{2\omega q}\right)^{1/2}\epsilon(q)(\tilde{\xi}_q e^{iqr}-\tilde{\xi}_q^\dagger e^{-iqr}),$$

where the $\tilde{\xi}$'s satisfy the anticommutation relations similar to those of Eq. (2.4) with the Kronecker δ replaced by the Dirac δ function. As we shall be interested in large separations we shall let a tend to zero whenever no ultraviolet divergence occurs.

The natural connection between the fields defined on different rows is through the transfer matrix. The relevant part of this matrix W is

$$W=\exp(-H),\qquad H=\sum_q \omega_q(\xi_q^\dagger\xi_q-\tfrac{1}{2}). \tag{2.9}$$

Introducing the Euclidean time development of the fields by

$$\psi_\alpha(r,t)=e^{Ht}\psi_\alpha(r)e^{-Ht}, \tag{2.10}$$

we find that in the limit $a\to 0$ the ψ's satisfy

$$-i\frac{\partial\psi_1}{\partial t}+\frac{\partial\psi_1}{\partial r}=m\psi_2,\qquad i\frac{\partial\psi_2}{\partial t}+\frac{\partial\psi_2}{\partial r}=m\psi_1. \tag{2.11}$$

ψ_1 and ψ_2 may be viewed as the two components of a Euclidean Majorana field of mass m.

III. CORRELATION FUNCTION—GENERAL FORMULATION

Returning to Eq. (2.3) we observe that

$$(c_r^\dagger-c_r)(c_{r'}^\dagger+c_{r'})=i\exp\left[-\frac{i\pi}{2}(c_r^\dagger-c_r)(c_{r'}^\dagger+c_{r'})\right], \tag{3.1}$$

and that the correlation function is

$$\langle\sigma_r\sigma_{r'}\rangle=i\left\langle\exp\left[-\frac{i\pi}{2}(c_r^\dagger-c_r)(c_r^\dagger+c_{r'})+i\pi\sum_{s=r+1}^{r'-1}c_s^\dagger c_s\right]\right\rangle. \tag{3.2}$$

From (2.5) and (2.7) we note that the c's are proportional to $\sqrt{a}$ and in the limit $r'-r$ large the first term in the exponent of (3.2) may be neglected, while the second one becomes an integral.

As

$$c_s^\dagger c_s=\tfrac{1}{2}(\psi_1-i\psi_2)(\psi_1+i\psi_2)=\tfrac{1}{2}(1+i[\psi_1,\psi_2]), \tag{3.3}$$

we obtain

$$\langle\sigma_r\sigma_{r'}\rangle=\left|\left\langle\exp\left\{-\frac{\pi}{2}\int_r^{r'}dx[\psi_1(x),\psi_2(x)]\right\}\right\rangle\right|. \tag{3.4}$$

When dealing with square roots of various quantities, we keep in mind that the correlation function is positive,[9] and ensure this by taking the absolute value in (3.4).

Rewriting (3.4) as

$$\langle\sigma_r\sigma_{r'}\rangle=\left|\left\langle\exp\left\{-\int dt\,dx\,A(x,t)\tfrac{1}{2}[\psi_1,\psi_2]\right\}\right\rangle\right| \tag{3.5}$$

with

$$A(x,t)=\pi\delta(t)\theta(x-r)\theta(r'-x) \tag{3.6}$$

suggests the interaction representation for the coupling of the Majorana field to an external potential $A(r,t)$. Connecting (3.5) to an anticommuting path integral, it becomes apparent that

$$\langle\sigma_r\sigma_{r'}\rangle=\{\det[S^{-1}(m,A)S(m,0)]\}^{1/2}. \tag{3.7}$$

$S(m,0)$ is the propagator for the free massive field while $S(m,A)$ is the same propagator in the presence of the external potential A.[10] Normally, when dealing with Dirac fields it is the determinant and not its square root that would appear in (3.7); however, we are dealing with a Majorana field with half the degrees of freedom of a Dirac field. As discussed above, we will take the positive square root. We shall study some of these propagators in the next section.

IV. PROPAGATORS

The form of Eq. (2.11) suggests that we introduce the complex variable z and its complex conjugate $\bar{z}$,

$$z=x+it,\qquad \bar{z}=x-it. \tag{4.1}$$

Equation (2.11) takes on the form

$$2\partial\psi_1=m\psi_2,\qquad 2\bar{\partial}\psi_2=m\psi_1, \tag{4.2}$$

where $\partial=\partial/\partial z$ and $\bar{\partial}=\partial/\partial\bar{z}$. The interaction with the external potential implied in (3.5) adds $A(x,t)$ to the mass terms and the field equations become

$$2\partial\psi_1=(m-A)\psi_2,\qquad 2\bar{\partial}\psi_2=(m-A)\psi_1. \tag{4.3}$$

From these we deduce that the propagator $S(m,A;z,z')$ satisfies

$$\begin{bmatrix} 2\partial & A-m \\ A-m & 2\bar{\partial} \end{bmatrix} S=\delta^2(z-z') \tag{4.4}$$

with $\delta^2(z-z')=\delta(x-x')\delta(t-t')$. We shall now concentrate on the propagators for the case $A=0$ and for the massless case in the presence of A.

A. $A(x,t)=0$

We are interested in the propagator for a free Majorana field, $S(m,0;z,z')$, satisfying (4.4) with $A=0$. Writing

$$S(m,0;z,z')=\int \frac{d^2k}{(2\pi)^2}\, e^{ik\cdot(z-z')}\tilde{S}_m(k)\,, \tag{4.5}$$

we find that

$$\tilde{S}_m(k)=\frac{1}{k^2+m^2}\begin{pmatrix} k^0-ik^1 & -m \\ -m & -k^0-ik^1 \end{pmatrix} \tag{4.6}$$

and

$$S(m,0;z,z')=\frac{m}{2\pi}\begin{bmatrix} \dfrac{z-z'}{|z-z'|}K_1(m|z-z'|) & -K_0(m|z-z'|) \\ -K_0(m|z-z'|) & \dfrac{\bar{z}-\bar{z}'}{|z-z'|}K_1(m|z-z'|) \end{bmatrix}. \tag{4.7}$$

K_0 and K_1 are the modified Bessel functions. For small values of their arguments these functions have the expansions

$$K_0(x)=-(\ln\tfrac{1}{2}x+\gamma)(1+\tfrac{1}{4}x^2+\cdots)+\tfrac{1}{4}x^2+\cdots\,, \tag{4.8}$$

$$K_1(x)=\frac{1}{x}(1-\tfrac{1}{4}x^2+\cdots)+(\ln\tfrac{1}{2}x+\gamma)(\tfrac{1}{2}x+\cdots)\,;$$

γ is Euler's constant equal to 0.577. . . .

In the $m=0$ limit the propagator becomes

$$S(0,0;z-z')=\frac{1}{2\pi}\begin{pmatrix} 1/(\bar{z}-\bar{z}') & 0 \\ 0 & 1/(z-z') \end{pmatrix}. \tag{4.9}$$

Comparing the above with (4.4) we find that

$$\partial\,\frac{1}{\pi}\,\frac{1}{\bar{z}-\bar{z}'}=\delta^2(z-z')\,,$$
$$\bar{\partial}\,\frac{1}{\pi}\,\frac{1}{z-z'}=\delta^2(z-z')\,, \tag{4.10}$$

which are just a restatement of Cauchy's theorem.

B. $m=0,\ A(x,t)\neq 0$

The equation for the propagator in this situation is

$$\begin{bmatrix} 2\partial & A \\ A & 2\bar{\partial} \end{bmatrix} S(0,A;z,z')=\delta^2(z-z')\,. \tag{4.11}$$

Let us introduce the complex-conjugation operator K, and as A is real and concentrated on the real axis, i.e., $KA=AK=A$, we note that the matrix

$$G=\frac{1}{\sqrt{2}}\begin{bmatrix} K & K \\ 1 & -1 \end{bmatrix} \tag{4.12}$$

diagonalizes the differential equation (4.11):

$$\begin{bmatrix} 2\partial & A \\ A & 2\bar{\partial} \end{bmatrix}=G\begin{bmatrix} 2\bar{\partial}+A & 0 \\ 0 & 2\bar{\partial}-A \end{bmatrix}G^{-1}. \tag{4.13}$$

The propagator may be expressed in terms of the solutions to

$$(2\bar{\partial}\pm A)S'_\pm=\delta^2(z-z')\,. \tag{4.14}$$

This suggests that

$$S'_\pm(z,z')=f_\pm(z)\,\frac{1}{2\pi}\,\frac{1}{z-z'}\,f_\pm^{-1}(z') \tag{4.15}$$

with $f_\pm(z)$ satisfying

$$(2\bar{\partial}\pm A)f_\pm=0 \tag{4.16}$$

or

$$2\bar{\partial}\ln f_\pm=\mp A\,. \tag{4.17}$$

As (4.10) provides us with Green's functions for the differential operators in the complex plane we obtain

$$\ln f_\pm(z)=\mp\frac{1}{2\pi}\int dx'dt'\,\frac{A(x',t')}{z-z'}\,. \tag{4.18}$$

With $A(x,t)$ as given in (3.6) we find that

$$\ln f_\pm(z) = \mp \tfrac{1}{2} \ln \frac{z - r'}{z - r} \tag{4.19}$$

or

$$f_+(z) = f^{-1}(z) = f_-^{-1}(z)\,, \qquad f(z) = \left(\frac{z - r'}{z - r}\right)^{1/2}. \tag{4.20}$$

Summarizing, the propagator is

$$S(0,A;z,z') = \frac{G}{2\pi}\begin{bmatrix} \dfrac{f^{-1}(z)f(z')}{z - z'} & 0 \\ 0 & \dfrac{f(z)f^{-1}(z')}{z - z'} \end{bmatrix} G^{-1}. \tag{4.21}$$

It will prove useful in the next section to generalize the above results to the case where A is multiplied by a real constant λ. It follows immediately that

$$S(0,\lambda A;z,z') = \frac{G}{2\pi}\begin{bmatrix} \dfrac{f^{-\lambda}(z)f^{\lambda}(z')}{z - z'} & 0 \\ 0 & \dfrac{f^{\lambda}(z)f^{-\lambda}(z')}{z - z'} \end{bmatrix} G^{-1}. \tag{4.22}$$

V. CORRELATION FUNCTION—CRITICAL TEMPERATURE, $m = 0$

Though this situation was discussed in Ref. 2, we repeat it using the Majorana formalism. As we shall be discussing singular products, we must at times remember that in reality we are dealing with a cutoff field theory, the cutoff provided by the lattice spacing a. For example, we shall encounter the massless propagator at zero separation; returning to the discrete case, it is easy to see that the correct replacement in this limit is

$$\lim_{z\to z'} S(0,0;z,z') \sim -i/a\,. \tag{5.1}$$

In order to make the above more precise we would have to treat both x and t as discrete. We do not pursue this further and thus abandon the calculation of the overall magnitude of the correlation function, and satisfy ourselves with its functional dependence. This is analogous to any renormalization calculation in a local field theory, where the magnitude of the field itself is arbitrary and fixed only by placing some conditions on the propagators.

From (3.7) we infer that for $m = 0$

$$\ln\langle\sigma_r, \sigma_{r'}\rangle = \tfrac{1}{2}\,\mathrm{Tr}\,\ln[S^{-1}(0,\lambda A)S(0,0)]\,, \tag{5.2}$$

where for the moment we treat the more general case discussed at the end of the last section. Without any loss of generality we may take $r = 0$ and $r' = \rho$.

Now

$$S^{-1}(0,\lambda A) = S^{-1}(0,0) + \lambda\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \tag{5.3}$$

and

$$S^{-1}(0,\lambda A)S(0,0) = 1 + \lambda\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} S(0,0)\,. \tag{5.4}$$

It would be tempting to differentiate (5.2) with respect to λ and thus obtain

$$\frac{\partial}{\partial\lambda}\ln\langle\sigma_0\sigma_\rho\rangle = \tfrac{1}{2}\,\mathrm{Tr}\left[\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} S(0,\lambda A)\right]. \tag{5.5}$$

However, the quantities on the right-hand side of (5.5) are too singular for these manipulations and thus we have to isolate these singular parts first. These difficulties occur only in the first few terms in the expansion of (5.2) in power of λ. We deal with the identity

$$\frac{\partial}{\partial\lambda}\left\{\ln\langle\sigma_0\sigma_\rho\rangle - \tfrac{1}{2}\,\mathrm{Tr}\left[\lambda\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} S(0,0)\right] + \lambda^2\,\tfrac{1}{4}\,\mathrm{Tr}\left[\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} S(0,0)\right]^2\right\}$$
$$= -\tfrac{1}{2}\,\mathrm{Tr}\left\{\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}\left[S(0,\lambda A) - S(0,0) + \lambda S(0,0)\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} S(0,0)\right]\right\}. \tag{5.6}$$

A little manipulation using (4.22) shows that the right-hand side of (5.6) vanishes as well as does the term linear in λ on the left-hand side. We are left with ($\lambda = 1$)

$$\ln\langle\sigma_0\sigma_\rho\rangle = -\tfrac{1}{4}\operatorname{Tr}\left\{\left[\begin{pmatrix}0 & A\\ A & 0\end{pmatrix}S(0,0)\right]^2\right\}. \tag{5.7}$$

Parenthetically, we may remark that this result is analogous to the observation that in two-dimensional QED only the one-loop contributions to the propagator are nonvanishing.

Returning to (5.7) and remembering the discussion at the beginning of the last section regarding the regularization of the free propagator, we obtain

$$\ln\langle\sigma_0\sigma_\rho\rangle = \tfrac{1}{8}\int_0^\rho dx\,dy\,\frac{1}{(x-y)^2+(\eta a)^2}, \tag{5.8}$$

with η of order unity and a the lattice constant. The evaluation of (5.8) is straightforward yielding ($\rho \gg a$)

$$\langle\sigma_0\sigma_\rho\rangle = \text{const}\times\left(\frac{a}{\rho}\right)^{1/4}. \tag{5.9}$$

This result agrees with Ref. 2 and Ref. 6.[11]

VI. CORRELATION FUNCTION—PERTURBATION ABOUT $m=0$

Having obtained, in the last section, the correlation function at $T=T_c$, we take this into account and write

$$\langle\sigma_0\sigma_\rho\rangle \xrightarrow[\rho\gg a]{} \text{const}\times\left(\frac{a}{\rho}\right)^{1/4}F(m\rho). \tag{6.1}$$

We shall now derive an expression suitable for a perturbation expansion in powers and logarithms of $m\rho$.

Combining (3.7) and (5.2) we obtain

$$\ln F(m\rho) = \tfrac{1}{2}\operatorname{Tr}\ln[S^{-1}(m,A)S(m,0)S^{-1}(0,0)\times S(0,A)]. \tag{6.2}$$

Noting that

$$S^{-1}(m,A) = S^{-1}(0,A) - S^{-1}(0,0) + S^{-1}(m,0), \tag{6.3}$$

we find[12] that

$$\ln F = \tfrac{1}{2}\operatorname{Tr}\ln\{1 - S^{-1}(0,0)[S(0,A)-S(0,0)]\times S^{-1}(0,0)[S(m,0)-S(0,0)]\}. \tag{6.4}$$

We remind ourselves that

$$S^{-1}(0,0) = \begin{pmatrix}2\partial & 0\\ 0 & 2\bar\partial\end{pmatrix}, \tag{6.5}$$

while $S(m,0)-S(0,0)$ may be brought to the form

$$S(m,0)-S(0,0) = \frac{m}{2\pi}G^{-1}\begin{bmatrix}\dfrac{|z-z'|K_1(m|z-z'|)-1/m}{z-z'} - K_0(m|z-z'|) & 0\\ 0 & \dfrac{|z-z'|K_1(m|z-z'|)-1/m}{z-z'} + K_0(m|z-z'|)\end{bmatrix}G, \tag{6.6}$$

where G has been defined in (4.12).

Let us now evaluate

$$\Delta = S^{-1}(0,0)[S(0,A)-S(0,0)]S^{-1}(0,0). \tag{6.7}$$

With S' defined in (4.15) and (4.20) we have

$$G^{-1}\Delta G = \begin{pmatrix}2\bar\partial & 0\\ 0 & 2\bar\partial\end{pmatrix}S'\begin{pmatrix}2\bar\partial & 0\\ 0 & 2\bar\partial\end{pmatrix} - \begin{pmatrix}2\bar\partial & 0\\ 0 & 2\bar\partial\end{pmatrix}. \tag{6.8}$$

A double dispersion relation may be written for the terms occurring in S',

$$\frac{1}{z-z'}\left(\frac{\rho-z}{z}\right)^{1/2}\left(\frac{z'}{\rho-z'}\right)^{1/2} = \frac{1}{z-z'} + \frac{P}{\pi}\int_0^\rho dx\,\frac{P}{\pi}\int_0^\rho dx'\left[\frac{1}{z-x}\,\frac{1}{x-x'}\,\frac{1}{x'-z'}\left(\frac{\rho-x}{x}\right)^{1/2}\left(\frac{x'}{\rho-x'}\right)^{1/2}\right], \tag{6.9}$$

and an analogous expression for the other term. An application of (4.10) to (6.9) yields

$$2\bar\partial\,\frac{1}{2\pi}\,\frac{1}{z-z'}\left(\frac{\rho-z}{z}\right)^{1/2}\left(\frac{z'}{\rho-z'}\right)^{1/2}2\bar\partial-2\bar\partial=\frac{2}{\pi}\,\delta(t)\theta(x)\theta(\rho-x)\,\mathrm{P}\,\frac{1}{x-x'}\left(\frac{\rho-x}{x}\right)^{1/2}\left(\frac{x'}{\rho-x'}\right)^{1/2}\delta(t')\theta(x')\theta(\rho-x')\,. \tag{6.10}$$

The singularity at $x=x'$ is to be intepreted as the Cauchy principal value in both x and x'. Returning to (6.8) and applying (6.10) we find that

$$G^{-1}\Delta G=\frac{2}{\pi}\,\delta(t)\theta(x)\theta(\rho-x)\delta(t')\theta(x')\theta(\rho-x')\begin{pmatrix}\mathrm{P}\frac{1}{x-x'}\left(\frac{\rho-x}{x}\right)^{1/2}\left(\frac{x'}{\rho-x'}\right)^{1/2} & 0\\ 0 & \mathrm{P}\,\frac{1}{x-x'}\left(\frac{x}{\rho-x}\right)^{1/2}\left(\frac{\rho-x'}{x'}\right)^{1/2}\end{pmatrix}. \tag{6.11}$$

The δ and θ functions imply that the evaluation of (6.2) may be restricted to operators acting on the one-dimensional interval $t=t'=0,\ 0<x,\ x'<\rho$.

Combining all the above we find[13] that

$$\ln F=\tfrac{1}{2}\,\mathrm{Tr}\ln\left[1-\frac{m}{\pi^2}\begin{pmatrix}\mathrm{P}\,\frac{1}{x-x'}\left(\frac{\rho-x}{x}\right)^{1/2}\left(\frac{x'}{\rho-x'}\right)^{1/2} & 0\\ 0 & \mathrm{P}\,\frac{1}{x-x'}\left(\frac{x}{\rho-x}\right)^{1/2}\left(\frac{\rho-x'}{x'}\right)^{1/2}\end{pmatrix}\right.$$
$$\left.\times\begin{pmatrix}\frac{|x'-x''|K_1(m|x'-x''|)-1/m}{x'-x''}-K_0(m|x'-x''|) & 0\\ 0 & \frac{-|x'-x''|K_1(m|x'-x''|)+1/m}{x'-x''}-K_0(m|x'-x''|)\end{pmatrix}\right]. \tag{6.12}$$

Again, the above operators are restricted to the real interval $(0,\rho)$. The lower line in (6.12) is the transpose of the upper one and contributes the same value to the trace removing the factor $\frac{1}{2}$ in front of (6.12),

$$\ln F(m\rho)=\mathrm{Tr}\ln(1-\sigma t)\,, \tag{6.13}$$

where σ, t are one-dimensional operators

$$\sigma(x,x')=\frac{1}{\pi}\,\mathrm{P}\,\frac{1}{x-x'}\left(\frac{\rho-x}{x}\right)^{1/2}\left(\frac{x'}{\rho-x'}\right)^{1/2}, \tag{6.14a}$$

$$t(x,x')=\frac{m}{\pi}\left(\frac{|x-x'|K_1(m|x-x'|)-1/m}{x-x'}-K_0(m|x-x'|)\right). \tag{6.14b}$$

The expansion of the logarithm in (6.13) yields the desired perturbation in powers of m and $\ln m$ agreeing with that of Ref. 6.

We may also note that the calculations for $T<T_c$ of the two-point connected correlation functions (i.e., with the subtraction of the square of the magnetization, a quantity vanishing in the scaling region) are given by the same formulas as above, with m treated as a negative quantity.

As an example we shall calculate the first term in this expansion:

$$\begin{aligned}\ln F_1(m\rho)&=-\frac{m}{\pi^2}\,\mathrm{P}\int_0^\rho dx\,dx'\,\frac{1}{x-x'}\left(\frac{\rho-x}{x}\right)^{1/2}\left(\frac{x'}{\rho-x'}\right)^{1/2}\left(\ln\frac{|m||x-x'|}{2}+\gamma\right)\\&=\frac{m\rho}{\pi^2}\,\mathrm{P}\int_0^1 dx\,dy\,\frac{1}{x-y}\left(\frac{x}{1-x}\right)^{1/2}\left(\frac{1-y}{y}\right)^{1/2}(\ln|x-y|+\gamma+\ln|m\rho|)\,.\\&=A\,m\rho\ln|m\rho|+Bm\rho\,.\end{aligned} \tag{6.15}$$

The evaluation of the integrals is tedious and gives

$$A=\tfrac{1}{2},\quad B=\tfrac{1}{2}(\gamma-3\ln 2)\,. \tag{6.16}$$

As remarked by the authors of Ref. 6, it is easy to see that the leading term of order n is $(A\,m\rho\ln|m|\rho)^n$, which implies that

$$F(m\rho)=1+A\,m\rho\ln|m|\rho+Bm\rho + O(m^2\rho^2\ln|m|\rho)\,. \tag{6.17}$$

Further terms in the series require skill in analytic integration.[14]

VII. CONCLUSION

Equation (6.13) provides for a systematic evaluation of the correlation function in the scaling region. In Ref. 4 a closed expression for this correlation function was obtained in terms of Painlevé functions. We have not found a way to do this directly in the continuous limit. However, suggestive results may be obtained. Note that

$$\sigma^{-1}=-\frac{\mathrm{P}}{\pi}\,\frac{1}{x-x'}\,, \tag{7.1}$$

where the inverse is taken in the functional sense over the interval $(0,\rho)$. Thus, returning to (6.13), we find that

$$\ln F(m\rho)=\mathrm{Tr}\ln(\sigma^{-1}-t)+\mathrm{Tr}\ln\sigma\,. \tag{7.2}$$

Likewise,

$$\sigma^{-1}-t=-\frac{1}{\pi}\left(\frac{\partial}{\partial x}+m\right)K_0(m|x-x'|)\,. \tag{7.3}$$

Separately the two terms on the right-hand side of (7.2) are singular and some regularization scheme has to be introduced to give them individual meaning. If we were to reintroduce a lattice for this purpose the evaluation of (7.2) would parallel the discussion of Ref. 6. Furthermore, it seems reasonable to expect that the methods of this paper can be extended to higher-order correlation functions, yielding a systematic series expansion.

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[14]Further terms are explicitly presented in Ref. 6.