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A GEOMETRIC REALIZATION OF PARTIALLY-SYMMETRIC MACDONALD POLYNOMIALS

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Abstract. We formulate a precise conjecture relating integral form partially-symmetric Macdonald polynomials and the parabolic flag Hilbert schemes of Carlsson, Gorsky, and Mellit. This extends, in an explicit fashion, Haiman's realization of modified Macdonald symmetric functions via Hilbert schemes of points in the plane. As evidence for our conjecture we prove that it is compatible with the action of certain elements in Carlsson and Mellit's algebra $\mathbb{A}_{t,q}$, including degree 1 Pieri formulas.

Keywords. Macdonald polynomials, Hilbert schemes, Hecke algebras

Mathematics Subject Classifications. 05E10, 05E05

1. Introduction

Let Λ be the algebra of symmetric functions in $x_1, x_2, \ldots$ over $\mathbb{K} = \mathbb{Q}(q,t)$. A key structure in Haiman's geometric realization of Macdonald symmetric functions [Hai01] (see also [Hai03]) is a vector space isomorphism

$$\bigoplus_{n\geqslant 0} K_{\mathbb{T}}(\mathrm{Hilb}_n)_{\mathrm{loc}} \xrightarrow{\Phi_0} \Lambda \tag{1.1}$$

between Λ and the localized equivariant K-groups of the Hilbert schemes Hilb_n of points in the plane $\mathbb{C}^2$, where $\mathbb{T}$ is the natural two-dimensional torus acting on the plane. Under this isomorphism, the modified Macdonald symmetric functions $\tilde{H}_\mu$ are realized as the images $\Phi_0([I_\mu])$ of the K-theory classes corresponding to $\mathbb{T}$-fixed points I_μ in $\sqcup_{n\geqslant 0}\mathrm{Hilb}_n$. It is of great significance

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that the map Φ_0 has a genuinely geometric origin—namely, the Procesi bundle—which exhibits Macdonald positivity.

Work of Schiffmann and Vasserot [SV13] offers another point of view on the map Φ_0: it is an intertwining map between two irreducible representations of the elliptic Hall algebra $\mathcal{E}$. While this point of view is not strong enough to establish Macdonald positivity, it does provide a way to uniquely characterize the map Φ_0, up to a scalar.

By methods akin to those of [SV13], Carlsson, Gorsky, and Mellit [CGM20] construct an extension of the map Φ_0, with Carlsson and Mellit's algebra $\mathbb{A}_{t,q}$ [CM18] in place of the elliptic Hall algebra $\mathcal{E}$. The extension $\Phi_\bullet = (\Phi_k)_{k\geqslant 0}$ is a sequence of $\mathbb{K}$-linear isomorphisms between the following spaces

$$\bigoplus_{n\geqslant k} K_{\mathbb{T}}(\mathrm{PFH}_{n,n-k})_{\mathrm{loc}} =: U_k \xrightarrow{\Phi_k} V_k := \Lambda \otimes \mathbb{K}[y_1,\dots,y_k] \tag{1.2}$$

The collections $U_\bullet = (U_k)_{k\geqslant 0}$ and $V_\bullet = (V_k)_{k\geqslant 0}$ of $\mathbb{K}$-vector spaces afford a quiver representation of the algebra $\mathbb{A}_{t,q}$, under which each is irreducible and admits a distinguished generator. Here $\mathrm{PFH}_{n,n-k}$ is the parabolic flag Hilbert scheme, also introduced in [CGM20], which parametrizes flags of ideals $I_n \subset I_{n-1} \subset \cdots \subset I_{n-k}$ in $\mathbb{C}[x,y]$ with the property that $\dim \mathbb{C}[x,y]/I_s = s$ and $yI_{n-k} \subset I_n$. The action of $\mathbb{A}_{t,q}$ on the direct sums of K-groups of these spaces above is constructed in [CGM20] by geometrically-defined operators, while the right-hand side of (1.2) affords the polynomial representation of [CM18]. Of course, the requirement that $\Phi_\bullet$ intertwine these two actions is sufficient to uniquely characterize it, up to a scalar.

Our work is motivated by the following question: *what are the images of $\mathbb{T}$-fixed point classes in $\mathrm{PFH}_{n,n-k}$ under the isomorphisms $\Phi_\bullet$?* One knows that Φ_0 recovers Haiman's isomorphism (1.1), where the $\mathbb{T}$-fixed point classes are sent to modified Macdonald symmetric functions $\tilde{H}_\mu$ (up to explicit signs and monomials in q,t). We propose an explicit answer for all k involving a new family of modified partially-symmetric Macdonald functions $\tilde{H}_{(\lambda|\gamma)}$.

Conjecture (see Conjecture 4.2). *Under the isomorphisms $\Phi_\bullet$, the $\mathbb{T}$-fixed point classes are sent to modified partially-symmetric Macdonald polynomials, as follows:*

$$H_{\mu,w} \mapsto t^{n(\mathrm{sort}(\lambda,\gamma))+|(\lambda|\gamma)|} \mathcal{J}^{w_0}_{(\lambda|\gamma)}\left(\frac{X}{t^{-1}-1}\,\Big|\, y\right)^* =: \tilde{H}_{(\lambda|\gamma)}.$$

For the sake of orientation, let us describe a few pieces of the notation used above, leaving the rest for the main body of the paper. The $\mathbb{T}$-fixed points $I_{\mu,w} \in \mathrm{PFH}_{n,n-k}$ are indexed by pairs (μ, w) such that μ is a partition of size n and w is an ordered horizontal strip of size k in μ; the classes $H_{\mu,w}$ are explicit scalar multiples of the $[I_{\mu,w}]$. The partially-symmetric integral form Macdonald functions $\mathcal{J}^{w_0}_{(\lambda|\gamma)}(X \,|\, y_1,\dots,y_k) \in \Lambda\otimes\mathbb{K}[y_1,\dots,y_k]$ are defined in (2.8) by partially-symmetrizing the (stable) nonsymmetric Macdonald polynomials, multiplying by explicit scalars introduced in [Goo24] and [Lap22], and then performing a twist by the long element of S_k. The $\mathcal{J}^{w_0}_{(\lambda|\gamma)}(X\,|\,y)$—and their cousins $\tilde{H}_{(\lambda|\gamma)}$—are indexed by pairs $(\lambda \mid \gamma)$ consisting of a partition λ and a composition $\gamma \in (\mathbb{Z}_{\geqslant 0})^k$ such that the total size $|\lambda|+|\gamma|$ is $n-k$. An explicit bijection ϕ between these indexing sets, so that $\phi(\mu,w) = (\lambda \mid \gamma)$, is recalled in Section 4.1

following [CGM20]. Finally, $X \mapsto X/(t^{-1}-1)$ denotes the familiar plethystic transformation on Λ from Macdonald theory, and $f(X \mid y)^* = f(X \mid y)|_{t \mapsto t^{-1}}$.

Thus our conjecture closely resembles—and clearly extends—the construction of modified Macdonald functions $\tilde{H}_\mu$ from the usual integral form Macdonald functions.

In this paper, we make substantial progress toward a proof of Conjecture 4.2 by showing that it is compatible with the action of certain $\mathbb{A}_{t,q}$-generators; see Theorem 4.3. We will complete the proof in forthcoming work with M. Bechtloff Weising by means of the limit Cherednik operators [IW24, BW24].

It is important to mention that the maps $\Phi_\bullet$ of [CGM20], while uniquely characterized as $\mathbb{A}_{t,q}$-intertwiners, have no genuinely geometric construction at present. In light of our conjecture and Lapointe's positivity conjectures [Lap22] for partially-symmetric Macdonald polynomials, it is natural to hope for a geometric realization of the $\Phi_\bullet$ exhibiting positivity.

2. Partially symmetric Macdonald polynomials

2.1. Diagrams

In constructions related to compositions and their diagrams, we mostly follow the conventions of [HHL08] (except that we write dg instead of dg' for their diagrams). Compositions are tuples $\nu = (\nu_1, \dots, \nu_n) \in (\mathbb{Z}_{\geqslant 0})^n$. A composition ν is a partition (with at most n parts) if its entries are weakly decreasing. Let $\mathbb{Y}_n \subset (\mathbb{Z}_{\geqslant 0})^n$ be the set of partitions with at most n parts. For any composition $\nu \in (\mathbb{Z}_{\geqslant 0})^n$, let $\mathrm{sort}(\nu)$ be the unique partition in $\mathbb{Y}_n$ obtained by rearranging the parts of ν. For $\nu \in (\mathbb{Z}_{\geqslant 0})^n$, let $|\nu| = \nu_1 + \cdots + \nu_n$.

The diagram of a composition $\nu \in (\mathbb{Z}_{\geqslant 0})^n$ is the subset $\mathrm{dg}(\nu) \subset (\mathbb{Z}_{\geqslant 0})^2$ given by

$$\mathrm{dg}(\nu) = \{(i,j) \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant \nu_i\}.$$

We view the parts ν_i of ν as columns in $\mathrm{dg}(\nu)$. Elements of $\mathrm{dg}(\nu)$ are called boxes of ν. We identify a composition with its diagram and write $(i,j) \in \nu$ to mean $(i,j) \in \mathrm{dg}(\nu)$. We also define the following subsets of $\mathrm{dg}(\nu)$:

$$\mathrm{dg}_r(\nu) = \{(i,j) \in \nu \mid j = r\}$$

The leg and arm lengths of a box $\square = (i,j) \in \nu$ are defined by

$$\begin{aligned}
&\ell_\nu(\square) = \nu_i - j, \\
&a_\nu(\square) = \#\{1 \leqslant r < i \mid j \leqslant \nu_r \leqslant \nu_i\} + \#\{i < r \leqslant n \mid j-1 \leqslant \nu_r < \nu_i\}.
\end{aligned}$$

We will also use the following alternate version of arm length (see Example 2.1):

$$\widetilde{a}_\nu(\square) = \#\{1 \leqslant r < i \mid j \leqslant \nu_r \leqslant \nu_i\} + \#\{i < r \leqslant n \mid j \leqslant \nu_r < \nu_i\}.$$

Let $\mathbb{Y} = \cup_{n \geqslant 0} \mathbb{Y}_n$ be the set of all partitions. For any $\nu \in \mathbb{Y}$, denote by $\nu' \in \mathbb{Y}$ the transposed partition and let $\mathrm{dg}'(\nu) = \mathrm{dg}(\nu')$. Thus $\mathrm{dg}'(\nu)$ is the usual (French) diagram of partition ν given by rows of boxes of lengths $\nu_1, \nu_2, \dots$ in the first quadrant. Let leg_ν and arm_ν denote the usual leg and arm functions on $\mathrm{dg}'(\nu)$ as defined in, e.g., [Hai01]. (For a box $\square \in \mathrm{dg}'(\nu)$, we have $\mathrm{leg}_\nu(\square) = \ell_{\nu'}(\square)$, and $\mathrm{arm}_\nu(\square)$ is related to the function a_{ν_-} for the weakly increasing rearrangement ν_- of ν.)

For any $\nu \in \mathbb{Y}$, define $n(\nu) = \sum_i (i-1)\nu_i$.

2.2. Split diagrams

We write $\nu = (\lambda|\gamma)$ to indicate a splitting of a composition ν into two compositions λ and γ, i.e., $\nu = (\nu_1, \ldots, \nu_n) = (\lambda_1, \ldots, \lambda_m, \gamma_1, \ldots, \gamma_k)$ where $\lambda = (\lambda_1, \ldots, \lambda_m) \in (\mathbb{Z}_{\geqslant 0})^m$ and $\gamma = (\gamma_1, \ldots, \gamma_k) \in (\mathbb{Z}_{\geqslant 0})^k$ and $n = m + k$. In this situation, we will often assume in addition that λ is a partition or the reverse of a partition.

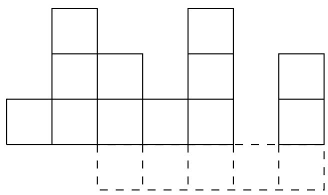
Starting with $\nu = (\lambda|\gamma)$ as above and λ a partition, we construct an augmented diagram $\widehat{\mathrm{dg}}(\nu)$ as follows. First, we form $\nu^- := (\lambda^-|\gamma)$, where λ^- is the weakly increasing rearrangement of λ. The augmented diagram associated with ν is then defined as follows

$$\widehat{\mathrm{dg}}(\nu) = \mathrm{dg}(\nu^-) \cup \{(m+1, 0), \ldots, (n, 0)\}.$$

Note that the additional boxes $(m+1, 0), \ldots, (n, 0)$ lie directly below the columns of $\mathrm{dg}(\nu^-)$ corresponding to γ.

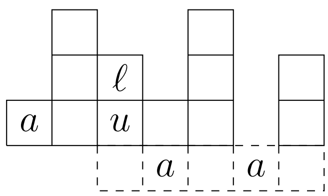
We call the subsets of $\mathrm{dg}(\nu^-)$ corresponding to λ^- and γ the symmetric and nonsymmetric parts of the diagram, respectively. We will use different arm functions for boxes in these two parts of $\mathrm{dg}(\nu^-)$, as follows. For boxes in the symmetric part of $\mathrm{dg}(\nu^-)$, we will use $\widetilde{a}_{\nu^-}$. For boxes in the nonsymmetric part, we will use the arm function a_{ν^-}. These arm functions have interpretations as counting certain boxes in $\widehat{\mathrm{dg}}(\nu^-)$, as we illustrate in the example below.

Example 2.1. Let $\nu = (3, 1 \mid 2, 1, 3, 0, 1)$. Then $\nu^- = (1, 3 \mid 2, 1, 3, 0, 1)$ and the augmented diagram $\widehat{\mathrm{dg}}(\nu^-)$ is

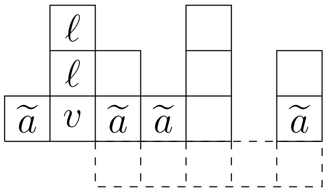


where the boxes added to $\mathrm{dg}(\nu^-)$ are dashed.

Let us illustrate the boxes that are counted for each of the arms and legs. For $u = (3, 1)$, a box in the nonsymmetric part of $\widehat{\mathrm{dg}}(\nu^-)$, the following boxes marked ℓ and a contribute to $\ell_{\nu^-}(u) = 1$ and $a_{\nu^-}(u) = 1 + 2 = 3$:



For $v = (2,1)$, in the symmetric part of the diagram, the boxes that contribute to $\ell_{\nu^-}(u) = 2$ and $\widetilde{a}_{\nu^-}(v) = 1 + 3 = 4$ are:



2.3. Nonsymmetric Macdonald polynomials

Let $\mathbb{K} = \mathbb{Q}(q,t)$ and $n > 0$. For any composition $\nu \in (\mathbb{Z}_{\geqslant 0})^n$, denote by $E_\nu = E_\nu(x;q,t) \in \mathbb{K}[x_1,\ldots,x_n]$ the nonsymmetric Macdonald polynomial of type GL_n as defined in [HHL08].

Let the symmetric group S_n act in the natural way on $\mathbb{Z}^n$ and on $\mathbb{K}[x_1,\ldots,x_n]$. For $1 \leqslant i < n$, denote by s_i the simple transposition $(i,i+1) \in S_n$. Define (see, e.g., Section 4 in [Kno07]) T_i the Demazure–Lusztig operators on $\mathbb{K}[x_1,\ldots,x_n]$ for all $i = 1,\ldots,n-1$ by

$$T_i = ts_i + \frac{(t-1)x_{i+1}}{x_{i+1}-x_i}(1-s_i) = \frac{x_{i+1}-tx_i}{x_{i+1}-x_i}s_i + \frac{(t-1)x_{i+1}}{x_{i+1}-x_i}. \tag{2.1}$$

These operators satisfy the braid relations

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad \text{for all } i < n-1, \tag{2.2}$$

$$T_iT_j = T_jT_i, \quad \text{for all } i \neq j \pm 1, \tag{2.3}$$

and the quadratic relations

$$(T_i - t)(T_i + 1) = 0, \quad \text{for all } i < n. \tag{2.4}$$

For any reduced expression $w = s_{i_1}\cdots s_{i_\ell} \in S_n$, we unambiguously define the operator $T_w = T_{i_1}\cdots T_{i_\ell}$.

Lemma 2.2 ([HHL08, (17)]). *If $\nu_i > \nu_{i+1}$ for some $1 \leqslant i < n$, then*

$$T_iE_\nu = E_{s_i(\nu)} - \frac{1-t}{1-q^{\ell_\nu(\square)+1}t^{a_\nu(\square)}}E_\nu \tag{2.5}$$

where $\square = (i, \nu_{i+1}+1) \in \mathrm{dg}(\nu)$.

2.4. Partially symmetric Macdonald polynomials

Suppose $m,k \geqslant 0$ and $n = m + k > 0$. In this situation, we regard S_m as the subgroup of S_n fixing the elements $\{m+1,\ldots,n\}$. Define the partial Hecke symmetrizer

$$P_m^+ = \sum_{w \in S_m} T_w.$$

For $\lambda \in (\mathbb{Z}_{\geqslant 0})^m$, let $(S_m)_\lambda \subset S_m$ be its stabilizer and

$$S_\lambda(t) = \sum_{w \in (S_m)_\lambda} t^{\ell(w)}.$$

Definition 2.3. For a partition $\lambda \in \mathbb{Y}_m$ and a composition $\gamma \in (\mathbb{Z}_{\geqslant 0})^k$, the partially symmetric Macdonald polynomial $P_{(\lambda|\gamma)} = P_{(\lambda|\gamma)}(x; q, t)$ is defined by

$$P_{(\lambda|\gamma)} = \frac{P_m^+ \cdot E_{(\lambda|\gamma)}}{S_\lambda(t)}.$$

The $P_{(\lambda|\gamma)}$ form a basis for the space $\mathbb{K}[x_1, \ldots, x_n]^{S_m}$ of partially-symmetric polynomials (as proved in [Goo22, Proposition 5.5]).

Definition 2.4. The integral form of the partially symmetric Macdonald polynomial $J_{(\lambda|\gamma)} = J_{(\lambda|\gamma)}(x; q, t)$ is the scalar multiple

$$\mathcal{J}_{(\lambda|\gamma)} = j_{(\lambda|\gamma)} P_{(\lambda|\gamma)}$$

where

$$j_{(\lambda|\gamma)} := \prod_{\square \in \lambda^-} (1 - q^{\ell_{\nu^-}(\square)} t^{\tilde{a}_{\nu^-}(\square)+1}) \prod_{\square \in \gamma} (1 - q^{\ell_{\nu^-}(\square)+1} t^{a_{\nu^-}(\square)+1})$$

with $\nu^- = (\lambda^- \mid \gamma)$.

It is shown in [Goo24] (and asserted in [Lap22]) that $\mathcal{J}_{(\lambda|\gamma)} \in \mathbb{Z}[q,t][x_1, \ldots, x_n]$.

For later use, we record the action of T_{m+i} for $i = 1, \ldots, k$ on the integral forms. In the case where $\gamma_i > \gamma_{i+1}$, by using [HHL08, eqs. (17) and (18)] and the fact that T_{m+i} commutes with P_m^+ we have:

$$\begin{aligned} T_{m+i}\mathcal{J}_{(\lambda|\gamma)} &= \frac{j_{(\lambda|\gamma)}}{j_{(\lambda|s_i(\gamma))}} \mathcal{J}_{(\lambda|s_i(\gamma))} - \frac{1-t}{1 - q^{l_{(\lambda|\gamma)}(u)+1} t^{a_{(\lambda|\gamma)}(u)}} \mathcal{J}_{(\lambda|\gamma)} \\ &= \frac{1 - q^{l_{(\lambda|\gamma)}(u)+1} t^{a_{(\lambda|\gamma)}(u)+1}}{1 - q^{l_{(\lambda|\gamma)}(u)+1} t^{a_{(\lambda|\gamma)}(u)}} \mathcal{J}_{(\lambda|s_i(\gamma))} - \frac{1-t}{1 - q^{l_{(\lambda|\gamma)}(u)+1} t^{a_{(\lambda|\gamma)}(u)}} \mathcal{J}_{(\lambda|\gamma)}, \end{aligned} \tag{2.6}$$

where $u = (m + i, \gamma_{i+1} + 1)$.

2.5. Stability

For any $n = m + k$ with $m, k \geqslant 0$, we make the identification

$$\mathbb{K}[x_1, \ldots, x_n]^{S_m} \cong \mathbb{K}[x_1, \ldots, x_m]^{S_m} \otimes \mathbb{K}[y_1, \ldots, y_k] \tag{2.7}$$

by mapping $x_1, \ldots, x_m$ to themselves and $x_{m+1}, \ldots, x_n$ to $y_1, \ldots, y_k$, respectively. We will write $f(x \mid y)$ for the image of $f \in \mathbb{K}[x_1, \ldots, x_n]^{S_m}$ under this identification.

With this notation in place, we can formulate the stability of partially symmetric Macdonald polynomials, which is proved in [Lap22] and [Goo24].

Proposition 2.5. *For any $\lambda \in \mathbb{Y}_m$ and $\gamma \in (\mathbb{Z}_{\geqslant 0})^k$, we have*

$$P_{(\lambda,0\,|\,\gamma)}(x, 0 \mid y) = P_{(\lambda|\gamma)}(x \mid y).$$

Since $j_{(\lambda,0\,|\,\gamma)} = j_{(\lambda|\gamma)}$, the integral forms $\mathcal{J}_{(\lambda|\gamma)}$ have the same stability property. Taking $m \to \infty$, we write $P_{(\lambda|\gamma)} = P_{(\lambda|\gamma)}(X \,|\, y)$ and $\mathcal{J}_{(\lambda|\gamma)} = \mathcal{J}_{(\lambda|\gamma)}(X \,|\, y)$ for the corresponding elements of $\Lambda \otimes \mathbb{K}[y_1, \dots, y_k]$, where Λ is the $\mathbb{K}$-algebra of symmetric functions in $x_1, x_2, \dots$. As λ ranges over all partitions and $\gamma \in (\mathbb{Z}_{\geqslant 0})^k$, these form bases of $\Lambda \otimes \mathbb{K}[y_1, \dots, y_k]$. This comes from combining the fact that the $P_{(\lambda|\gamma)}$ form a basis for $\mathbb{K}[x_1, \dots, x_n]^{S_m}$ with Proposition 2.5.

By slight abuse of notation, we now denote by T_i for $i = 1, \dots, k-1$ the Demazure–Lusztig operators (2.1) acting on the variables $y_1, \dots, y_k$ in $\Lambda \otimes \mathbb{K}[y_1, \dots, y_k]$. Under the identification (2.7), this T_i corresponds to T_{m+i} in the variables $x_{m+i}, \dots, x_n$. The following lemma shows the action of T_i is well-defined on $\Lambda \otimes \mathbb{K}[y_1, \dots, y_k]$.

Lemma 2.6. *Let the projection,*

$$\pi_{m+1} : \mathbb{K}[x_1, \dots, x_{m+1}]^{S_{m+1}} \otimes \mathbb{K}[y_1, \dots, y_k] \to \mathbb{K}[x_1, \dots, x_m]^{S_m} \otimes \mathbb{K}[y_1, \dots, y_k],$$

be given by $\pi_{m+1}(x_{m+1}) = 0$, $\pi_{m+1}(x_j) = x_j$ *for* $1 \leqslant j < m$, *and* $\pi_{m+1}(y_j) = y_j$ *for all* $1 \leqslant j \leqslant k$. *For any* $f = f(x \mid y) \in \mathbb{K}[x_1, \dots, x_{m+1}]^{S_{m+1}} \otimes \mathbb{K}[y_1, \dots, y_k]$, *we have* $\pi_{m+1} T_i f = T_i \pi_{m+1} f$.

Proof. It suffices to consider a simple tensor $f = c(x_1, \dots, x_m, x_{m+1}) \otimes d(y_1, \dots, y_k)$ in $\mathbb{K}[x_1, \dots, x_{m+1}]^{S_{m+1}} \otimes \mathbb{K}[y_1, \dots, y_k]$.

We have

$$\begin{aligned} \pi_{m+1} T_i \left(c(x_1, \dots, x_{m+1}) \otimes d(y_1, \dots, y_k) \right) &= c(x_1, \dots, x_m, 0) \otimes T_i(d(y_1, \dots, y_k)) \\ &= c(x_1, \dots, x_m, 0) \otimes T_i(d(y_1, \dots, y_k)) \\ &= T_i \left(c(x_1, \dots, x_m, 0) \otimes d(y_1, \dots, y_k) \right) \\ &= T_i \pi_{m+1} \left(c(x_1, \dots, x_m, x_{m+1}) \otimes d(y_1, \dots, y_k) \right). \square \end{aligned}$$

We define the variant of the integral form Macdonald functions in $\Lambda \otimes \mathbb{K}[y_1, \dots, y_k]$:

$$\mathcal{J}^{w_0}_{(\lambda|\gamma)}(X \,|\, y) = t^{-\ell(w_0)} w_0 T_{w_0} \mathcal{J}_{(\lambda|\gamma)}(X \,|\, y) \tag{2.8}$$

where $w_0 \in S_k$ is the long element.

3. Carlsson–Mellit algebra

All material from this section is due to [CM18] and [CGM20].

3.1. The algebra

Let $\mathbb{A}_{t,q}$ be the Carlsson–Mellit algebra as defined in [CGM20]; note that compared to [CGM20] we swap the roles of q and t. By definition, $\mathbb{A}_{t,q}$ is the quotient $\mathbb{K}Q/\!\sim$ of the path algebra $\mathbb{K}Q$ of a quiver Q by certain relations $\sim$.

The quiver $Q = (Q_0, Q_1)$ has vertex set $Q_0 = \mathbb{Z}_{\geqslant 0}$ and arrows Q_1 given by the following for all $k \in \mathbb{Z}_{\geqslant 0}$, all $1 \leqslant i < k$, and all $1 \leqslant j \leqslant k$:

$$k \xrightarrow{\mathsf{T}_i} k, \quad k \xrightarrow{\mathsf{y}_j} k, \quad k \xrightarrow{\mathsf{z}_j} k, \tag{3.1}$$

$$k \xrightarrow{\mathsf{d}_+} k+1, \quad k \xrightarrow{\mathsf{d}_+^*} k+1, \quad k+1 \xrightarrow{\mathsf{d}_-} k, \tag{3.2}$$

Note that the arrows in (3.1) are loops, while those in (3.2) join consecutive vertices. We note also that the arrow labels are overloaded. For instance, T_i may denote an arrow from vertex k to itself for any $k > i$.

The product on $\mathbb{K}Q$ is given by concatenation, with the convention that $p_1 p_2$ is the path obtained by first following p_2 and then p_1, provided the head of p_2 is the tail of p_1 (and $p_1 p_2 = 0$ otherwise). Thus, for instance, we can multiply $k+1 \xrightarrow{\mathsf{z}_1} k$ and $k \xrightarrow{\mathsf{d}_+} k$ for any $k \in \mathbb{Z}_{\geqslant 0}$ to form $\mathsf{z}_1 \mathsf{d}_+$, which is the path

$$k \xrightarrow{\mathsf{d}_+} k+1 \xrightarrow{\mathsf{z}_1} k+1.$$

Since the arrows are overloaded, it is more precise to denote this product using a notation such as $k \xrightarrow{\mathsf{z}_1 \mathsf{d}_+} k+1$ which remembers the tail and head of the path (or at least one of them). On the other hand, such notation can get cumbersome and is not ideal for complicated expressions, so we will often specify the tail and/or head of a product in writing, e.g., $\mathsf{z}_1 \mathsf{d}_+$ with tail k for our current example.

The relations $\sim$ defining $\mathbb{A}_{t,q} = \mathbb{K}Q/\!\sim$ are given in Table 3.1.

Let $\mathbb{A}_t \subset \mathbb{A}_{t,q}$ be the subalgebra of $\mathbb{A}_{t,q}$ generated by the elements $\mathsf{d}_-, \mathsf{d}_+, \mathsf{T}_1, \mathsf{T}_2, \ldots$ and $\mathsf{y}_1, \mathsf{y}_2, \ldots$. (In fact, the $\mathsf{y}_1, \mathsf{y}_2, \cdots$ are expressible in terms of the other generators.)

3.2. Polynomial representation

The polynomial representation of $\mathbb{A}_{t,q}$ was introduced in [CM18]. We use the conventions of [CGM20], where it was shown that the polynomial representation is isomorphic as an $\mathbb{A}_t$-representation to the geometric representation described in the next subsection.

Recall that Λ is the $\mathbb{K}$-algebra of symmetric functions in $x_1, x_2, \ldots$. The polynomial representation is a quiver representation of $\mathbb{A}_{t,q}$ on the collection $V_\bullet = (V_k)_{k \geqslant 0}$ of $\mathbb{K}$-vector spaces $V_k = \Lambda \otimes \mathbb{K}[y_1, \ldots, y_k]$.

For our purposes, it will be sufficient to know the action of the subalgebra $\mathbb{A}_t \subset \mathbb{A}_{t,q}$ on the polynomial representation. The operators $\mathsf{T}_1, \ldots, \mathsf{T}_{k-1}$ and $\mathsf{y}_1, \ldots, \mathsf{y}_k$, mapping $V_k \to V_k$ for all $k \geqslant 0$, are given by

$$\begin{aligned} \mathsf{T}_i \cdot f(X \,|\, y) &= \frac{(t-1)y_i}{y_{i+1} - y_i} f + \frac{y_{i+1} - t y_i}{y_{i+1} - y_i} s_i f \\ \mathsf{y}_i \cdot f &= y_i f \end{aligned}$$

where $f = f(X \,|\, y) \in V_k$ and s_i permutes y_i and y_{i+1}. Note that T_i are closely related to (but not the same as) the Demazure–Lusztig operators T_i; we will make the relation between these operators precise in Section 5 below.

relation	*tail* $\to$ *head*	*for all*
$(\mathsf{T}_i - 1)(\mathsf{T}_i + t) = 0$	$k \to k$	$i < k$
$\mathsf{T}_i\mathsf{T}_{i+1}\mathsf{T}_i = \mathsf{T}_{i+1}\mathsf{T}_i\mathsf{T}_{i+1}$	$k \to k$	$i < k-1$
$\mathsf{T}_i\mathsf{T}_j = \mathsf{T}_j\mathsf{T}_i$	$k \to k$	$i, j < k$ with $\lvert i-j\rvert > 1$
$\mathsf{y}_i\mathsf{y}_j = \mathsf{y}_j\mathsf{y}_i$	$k \to k$	$i, j \leqslant k$
$\mathsf{z}_i\mathsf{z}_j = \mathsf{z}_j\mathsf{z}_i$	$k \to k$	$i, j \leqslant k$
$\mathsf{y}_i\mathsf{T}_j = \mathsf{T}_j\mathsf{y}_i$	$k \to k$	$i \leqslant k$ and $j < k$ with $i \neq j, j+1$
$\mathsf{z}_i\mathsf{T}_j = \mathsf{T}_j\mathsf{z}_i$	$k \to k$	$i \leqslant k$ and $j < k$ with $i \neq j, j+1$
$\mathsf{T}_i\mathsf{y}_{i+1}\mathsf{T}_i = t\mathsf{y}_i$	$k \to k$	$i < k$
$\mathsf{T}_i^{-1}\mathsf{z}_{i+1}\mathsf{T}_i^{-1} = t^{-1}\mathsf{z}_i$	$k \to k$	$i < k$
$\mathsf{d}_-^2\mathsf{T}_{k-1} = \mathsf{d}_-^2$	$k \to k-2$	$k \geqslant 2$
$\mathsf{d}_-\mathsf{T}_i = \mathsf{T}_i\mathsf{d}_-$	$k \to k-1$	$k > 0$ and $i < k-1$
$\mathsf{d}_-\mathsf{y}_i = \mathsf{y}_i\mathsf{d}_-$	$k \to k-1$	$k > 0$ and $i < k$
$\mathsf{d}_-\mathsf{z}_i = \mathsf{z}_i\mathsf{d}_-$	$k \to k-1$	$k > 0$ and $i < k$
$\mathsf{T}_1\mathsf{d}_+^2 = \mathsf{d}_+^2$	$k \to k+2$	
$\mathsf{T}_1^{-1}(\mathsf{d}_+^*)^2 = (\mathsf{d}_+^*)^2$	$k \to k+2$	
$\mathsf{d}_+\mathsf{T}_i = \mathsf{T}_{i+1}\mathsf{d}_+$	$k \to k+1$	$i < k$
$\mathsf{d}_+^*\mathsf{T}_i^{-1} = \mathsf{T}_{i+1}^{-1}\mathsf{d}_+^*$	$k \to k+1$	$i < k$
$\mathsf{d}_+\mathsf{y}_i = \mathsf{T}_1\mathsf{T}_2\cdots\mathsf{T}_i\mathsf{y}_i\mathsf{T}_i^{-1}\cdots\mathsf{T}_1^{-1}\mathsf{d}_+$	$k \to k+1$	$i \leqslant k$
$\mathsf{d}_+^*\mathsf{z}_i = \mathsf{T}_1^{-1}\mathsf{T}_2^{-1}\cdots\mathsf{T}_i^{-1}\mathsf{z}_i\mathsf{T}_i\cdots\mathsf{T}_1\mathsf{d}_+$	$k \to k+1$	$i \leqslant k$
$\mathsf{d}_+\mathsf{z}_i = \mathsf{z}_{i+1}\mathsf{d}_+$	$k \to k+1$	$i \leqslant k$
$\mathsf{d}_+^*\mathsf{y}_i = \mathsf{y}_{i+1}\mathsf{d}_+^*$	$k \to k+1$	$i \leqslant k$
$\mathsf{z}_1\mathsf{d}_+ = -qt^{k+1}\mathsf{y}_1\mathsf{d}_+^*$	$k \to k+1$	
$[\mathsf{d}_+, \mathsf{d}_-] = (t-1)\mathsf{T}_1\mathsf{T}_2\cdots\mathsf{T}_{k-1}\mathsf{y}_k$	$k \to k$	$k > 0$
$[\mathsf{d}_+^*, \mathsf{d}_-] = (t^{-1}-1)\mathsf{T}_1^{-1}\mathsf{T}_2^{-1}\cdots\mathsf{T}_{k-1}^{-1}\mathsf{z}_k$	$k \to k$	$k > 0$

Table 3.1: Relations $\sim$ defining $\mathbb{A}_{t,q}$. Here, $k \in \mathbb{Z}_{\geqslant 0}$ is arbitrary unless quantified further, and $[X, Y] = XY - YX$ denotes the commutator.

The operators $\mathsf{d}_- : V_{k+1} \to V_k$ and $\mathsf{d}_+ : V_k \to V_{k+1}$ are given by

$$\begin{aligned} \mathsf{d}_- \cdot f &= -[y_{k+1}^{-1}]\Big(f(X-(t-1)y_{k+1})\Omega(-y_{k+1}^{-1}X)\Big) \\ \mathsf{d}_+ \cdot f &= \mathsf{T}_1 \cdots \mathsf{T}_k \cdot f(X+(t-1)y_{k+1}), \end{aligned}$$

where $[y^s]A(y) = A_s$ for a formal series $\sum_{s\in\mathbb{Z}} A_s y^s$, and $\Omega(-yX) = \sum_{i\geqslant 0}(-y)^i e_i(X)$ in terms of the elementary symmetric functions e_i. We use plethystic notation for the $\mathbb{K}[y_1,\ldots,y_k]$-linear automorphism $f \mapsto f(X \pm (t-1)y_{k+1})$ of V_k, which is determined by its values on power sums p_i:

$$p_i(X \pm (a-b)) = p_i(X) \pm (a^i - b^i).$$

These formulas define $V_\bullet$ as an $\mathbb{A}_t$-representation; in fact, it is proved in [CM18] that $V_\bullet$ is isomorphic to the left $\mathbb{A}_t$-submodule of $\mathbb{A}_t$ spanned by paths with tail 0.

Lemma 3.1. *The operator of multiplication by the symmetric function $e_1(X)$ on $f = f(X\,|\,y) \in V_k$ is expressed in terms of the generators of $\mathbb{A}_t$ as follows:*

$$e_1(X)f = \mathsf{d}_-\mathsf{T}_k^{-1}\cdots\mathsf{T}_1^{-1}\mathsf{d}_+ \cdot f.$$

Proof. Given $f = f(X\,|\,y) \in V_k$,

$$\begin{aligned} \mathsf{d}_-\mathsf{T}_k^{-1}\cdots\mathsf{T}_1^{-1}\mathsf{d}_+ \cdot f &= \mathsf{d}_-\mathsf{T}_k^{-1}\cdots\mathsf{T}_1^{-1}\mathsf{T}_1\cdots\mathsf{T}_k f(X+(t-1)y_{k+1}) \\ &= \mathsf{d}_- f(X+(t-1)y_{k+1}) \\ &= -[y_{k+1}^{-1}]\left(f(X\,|\,y)\cdot\Omega(-y_{k+1}^{-1}X)\right) \\ &= -f(X\,|\,y)\cdot[y_{k+1}^{-1}]\left(\sum_{i\geqslant 0}(-y_{k+1}^{-1})^i e_i(X)\right) \\ &= f(X\,|\,y)\cdot e_1(X). \end{aligned}$$

□

Remark 3.2. We have corrected a minor typo in the action of T_i on V made in [CGM20].

3.3. Parabolic flag Hilbert schemes

For any integers k and n such that $0 \leqslant k \leqslant n$, let $\mathrm{PFH}_{n,n-k}$ denote the parabolic flag Hilbert scheme of points in $\mathbb{C}^2$ introduced in [CGM20]. The points of $\mathrm{PFH}_{n,n-k}$ are flags of ideals

$$I_n \subset I_{n-1} \subset \cdots \subset I_{n-k} \subset \mathbb{C}[x,y],$$

such that I_s has codimension s in $\mathbb{C}[x,y]$ for all s, and

$$yI_{n-k} \subset I_n.$$

The space $\mathrm{PFH}_{n,n-k}$ has the structure of a smooth algebraic variety of dimension $2n-k$. We note that $\mathrm{PFH}_{n,n}$ is the usual Hilbert scheme Hilb_n of n points in $\mathbb{C}^2$, while $\mathrm{PFH}_{n,0}$ is isomorphic to $\mathbb{C}^n$.

The group $\mathbb{T} = \mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathbb{C}[x, y]$, Hilb_n, and $\mathrm{PFH}_{n,n-k}$ via its natural action on $\mathbb{C}^2$. Let $q, t \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$ be the $\mathbb{T}$-weights of y and x, respectively; a monomial $y^r x^c$ has $\mathbb{T}$-weight $q^r t^c$. We visualize $y^r x^c$ as being located in row r and column c of an infinite array of unit boxes in the first quadrant, with indices beginning at 0. We also refer to $q^r t^c$ as the $\mathbb{T}$-weight of the box $\square = (c + 1, r + 1)$, denoted $\mathrm{wt}(\square) = q^r t^c$.

The $\mathbb{T}$-fixed points of Hilb_n are precisely the monomial ideals

$$I_\mu = (x^{\mu_1} y^0, x^{\mu_2} y^1, \dots),$$

where $\mu = (\mu_1, \mu_2, \dots) \in \mathbb{Y}$ is a partition of size $|\mu| = n$. The $\mathbb{T}$-fixed points of $\mathrm{PFH}_{n,n-k}$ are given by flags of monomial ideals

$$I_{\mu^{(n)}} \subset I_{\mu^{(n-1)}} \subset \cdots \subset I_{\mu^{(n-k)}} \subset \mathbb{C}[x, y]$$

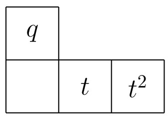
where $\mu^{(s)}$ is a partition of size s for each s and $yI_{\mu^{(n-k)}} \subset I_{\mu^{(n)}}$. In order to satisfy these conditions, the indexing partitions must satisfy

$$\mathrm{dg}'(\mu^{(n)}) \supset \mathrm{dg}'(\mu^{(n-1)}) \supset \cdots \supset \mathrm{dg}'(\mu^{(n-k)}), \tag{3.3}$$

with $\square_s = \mathrm{dg}'(\mu^{(s)}) \setminus \mathrm{dg}'(\mu^{(s-1)})$ having size 1 for each s, and with $\mathrm{dg}'(\mu^{(n)}) \setminus \mathrm{dg}'(\mu^{(n-k)})$ being a horizontal strip (i.e., having at most one box in each column.)

Following [CGM20], we will denote by $I_{\mu,w}$ the $\mathbb{T}$-fixed point of $\mathrm{PFH}_{n,n-k}$ determined by a chain of partitions (3.3), where $\mu = \mu^{(n)}$ and $w = (w_1, \dots, w_k)$ with $w_t = \mathrm{wt}(\square_{n-t+1})$ for $1 \leqslant t \leqslant k$. Thus μ is the biggest partition in the chain, and the w_t are the weights of the boxes whose removal determines the rest of the flag, in the order of removal.

Example 3.3. The $\mathbb{T}$-fixed point $I_{(3,1),(t^2,q,t)}$ in $\mathrm{PFH}_{4,1}$ is given by the flag $I_{(3,1)} \subset I_{(2,1)} \subset I_{(2)} \subset I_{(1)}$.



3.4. Geometric representation

The localized $\mathbb{T}$-equivariant K-theory

$$K_{\mathbb{T}}(\mathrm{PFH}_{n,n-k})_{\mathrm{loc}} = K_{\mathbb{T}}(\mathrm{PFH}_{n,n-k}) \otimes_{\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]} \mathbb{K}$$

is a $\mathbb{K}$-vector space with basis given by the classes $[I_{\mu,w}]$ of skyscraper sheaves at the $\mathbb{T}$-fixed points $I_{\mu,w} \in \mathrm{PFH}_{n,n-k}$.

The family $U_\bullet = (U_k)_{k\geqslant 0}$ of $\mathbb{K}$-vector spaces $U_k = \bigoplus_{n\geqslant k} K_{\mathbb{T}}(\mathrm{PFH}_{n,n-k})_{\mathrm{loc}}$ affords a quiver representation of the algebra $\mathbb{A}_{t,q}$ by geometrically-defined operators [CGM20]. More precisely, in [CGM20], a geometric action of a closely related algebra $\mathbb{B}_{t,q}$ is constructed first, and then this action is pulled back along a homomorphism $\beta : \mathbb{A}_{t,q} \to \mathbb{B}_{t,q}$.

For our purposes, it will be sufficient to recall the action of generators of the subalgebra $\mathbb{A}_t \subset \mathbb{A}_{t,q}$ on the $\mathbb{T}$-fixed point basis:

$$\mathsf{d}_+ \cdot [I_{\mu,w}] = -t^k \sum_{\substack{\square\in A(\mu)\\ x:=\mathrm{wt}(\square)}} x d_{\mu+x,\mu} \prod_{i=1}^{k} \frac{x-qw_i}{x-qtw_i}[I_{\mu+x,xw}] \tag{3.4}$$

$$\mathsf{d}_- \cdot [I_{\mu,wx}] = [I_{\mu,w}] \tag{3.5}$$

$$\mathsf{T}_i \cdot [I_{\mu,w}] = \frac{(t-1)w_{i+1}}{w_i - w_{i+1}}[I_{\mu,w}] + \frac{w_i - tw_{i+1}}{w_i - w_{i+1}}[I_{\mu,s_i(w)}] \tag{3.6}$$

where $A(\mu)$ is the set of μ-addable boxes (relative to $\mathrm{dg}'(\mu)$), $\mu + x = \mu + \square$ for $x = \mathrm{wt}(\square)$ with $\square \in A(\mu)$ is the partition obtained by adding $\square$ to μ, and $d_{\mu+x,\mu}$ is the Macdonald–Pieri coefficient

$$d_{\mu+x,\mu} = \prod_{\square\in R_{\mu+x,\mu}} \frac{t^{\mathrm{arm}_\mu(\square)} - q^{\mathrm{leg}_\mu(\square)+1}}{t^{\mathrm{arm}_\mu(\square)+1} - q^{\mathrm{leg}_\mu(\square)+1}} \prod_{\square\in C_{\mu+x,\mu}} \frac{t^{\mathrm{arm}_\mu(\square)+1} - q^{\mathrm{leg}_\mu(\square)}}{t^{\mathrm{arm}_\mu(\square)+1} - q^{\mathrm{leg}_\mu(\square)+1}}$$

where $R_{\mu+x,\mu}$ resp. $C_{\mu+x,\mu}$ is the set of boxes in the row resp. column of $\mathrm{dg}'(\mu)$ corresponding to $\square \in A(\mu)$ such that $x = \mathrm{wt}(\square)$. By rewriting the quadratic relation $(\mathsf{T}_i - 1)(\mathsf{T}_i + t) = 0$ in the form

$$\mathsf{T}_i(t^{-1}\mathsf{T}_i + (1-t^{-1})\mathrm{id}_{U_k}) = \mathrm{id}_{U_k},$$

and using the above formula for the action of $\mathsf{T}_i \cdot [I_{\mu,w}]$, we find:

$$\mathsf{T}_i^{-1}[I_{\mu,w}] = \frac{(1-t^{-1})w_i}{w_i - w_{i+1}}[I_{\mu,w}] + \frac{t^{-1}w_i - w_{i+1}}{w_i - w_{i+1}}[I_{\mu,s_i(w)}]. \tag{3.7}$$

3.5. Isomorphism

As shown in [CGM20], there exists a unique isomorphism $\Phi_\bullet = (\Phi_k)_{k\geqslant 0}$ of $\mathbb{A}_t$-quiver representations $U_\bullet \xrightarrow{\Phi_\bullet} V_\bullet$ sending $[I_{\varnothing,()}] \in K_{\mathbb{T}}(\mathrm{PFH}_{0,0})_{\mathrm{loc}} \subset U_0$ to $1 \in \Lambda = V_0$. Moreover, it is shown that $\Phi_0([I_\mu])$ is an explicit scalar multiple (recalled below) of $\tilde{H}_\mu$ for any $I_\mu \in \mathrm{Hilb}_n = \mathrm{PFH}_{n,n}$, where $\tilde{H}_\mu$ is the modified Macdonald symmetric function. We note that the $\mathbb{A}_t$-equivariance of $\Phi_\bullet$ can be upgraded to $\mathbb{A}_{t,q}$ by means of an involution $\mathcal{N}$ on $\mathbb{A}_{t,q}$ (see [CGM20] for details).

4. Conjecture

4.1. Bijections between indexing sets

We recall a bijection introduced in [CGM20, Proof of Theorem 7.0.1].

Proposition 4.1. *For a pair (μ, w) indexing a $\mathbb{T}$-fixed point in $\mathrm{PFH}_{n,n-k}$, define $\phi(\mu, w) = (\lambda \,|\, \gamma)$ with $\lambda \in \mathbb{Y}$ and $\gamma \in (\mathbb{Z}_{\geqslant 0})^k$ as follows:*

- *Let $\gamma_i = l'(\square_i)$ for $1 \leqslant i \leqslant k$, where $\square_i$ is the box of μ with weight w_i; note that $l'(\square_i) + 1$ is the height of the column in μ containing $\square_i$.*
- *Form a partition κ by removing all columns from μ which contain a box $\square_i$ for $1 \leqslant i \leqslant k$. Let $\lambda = \kappa^t$.*

Then the map ϕ is a bijection.

Proof. The inverse of ϕ is given is as follows. Starting with $(\lambda \mid \gamma)$, we reconstruct (μ, w) by the following steps:

- Let $\kappa = \lambda^t$.

  To form μ, insert k labeled columns into κ, with heights $\gamma_1 + 1, \ldots, \gamma_k + 1$ and labels $1, \ldots, k$, respectively, so that each labeled column lies to the right of any unlabeled column in μ which has the same height and the labeled columns with the same height have increasing labels from right to left.
- For each $1 \leqslant i \leqslant k$, define w_i to be the weight of the box at the top (French) or bottom (English) of the column labeled i in μ. That is

  $$w_i = q^{\gamma_i} t^{c_i}$$

  where

  $$c_i = |\{j : \lambda_j \geqslant \gamma_i + 1\}| + |\{j > i : \gamma_j = \gamma_i\}| + |\{j : \gamma_j > \gamma_i\}|. \tag{4.1}$$

The only remaining condition to check is that the constructed (μ, w) corresponds to a valid chain of partitions. Because the columns are sorted with labeled columns to the right of unlabeled ones, and with labels increasing right to left, each diagram in the chain remains a partition diagram. And as there is at most one labeled box at the top of each column, the labeled boxes form a horizontal strip. This guarantees that the inverse image of ϕ corresponds to a unique $\mathbb{T}$-fixed point in $\mathrm{PFH}_{n,n-k}$. $\square$

4.2. Conjecture

We are now ready to formulate:

Conjecture 4.2. Under the isomorphism

$$U_\bullet \xrightarrow{\Phi_\bullet} V_\bullet$$

the $\mathbb{T}$-fixed point classes are sent to modified partially-symmetric Macdonald polynomials, as follows:

$$(-1)^{|\mu|}q^{n(\mu)}t^{n(\mu')}[I_{\mu,w}] \mapsto t^{n(\mathrm{sort}(\lambda,\gamma))+|(\lambda|\gamma)|}\mathcal{J}^{w_0}_{(\lambda|\gamma)}\Big(\frac{X}{t^{-1}-1}\,\Big|\,y\Big)^* \tag{4.2}$$

where $\phi(\mu,w)=(\lambda\mid\gamma)$, $\mathcal{J}^{w_0}_{(\lambda|\gamma)}(X\,|\,y)$ is defined in (2.8), and $f(X\,|\,y)^*=f(X\,|\,y)|_{t\mapsto t^{-1}}$.

Here $f(X\,|\,y)\mapsto f\big(\frac{X}{t^{-1}-1}\,|\,y\big)$ denotes the $\mathbb{K}[y_1,\dots,y_k]$-linear automorphism of V_k given by the plethysm

$$p_i\Big(\frac{X}{t^{-1}-1}\Big)=\frac{p_i(X)}{t^{-i}-1}$$

on the first tensor factor Λ. Conjecture 4.2 is known to hold on the $k=0$ summands, as shown in [CGM20]; the right-hand side of (4.2) is the modified Macdonald symmetric function $\tilde{H}_\lambda$ in this case.

Following [CGM20], let us introduce the notation

$$H_{\mu,w}=(-1)^{|\mu|}q^{n(\mu)}t^{n(\mu')}[I_{\mu,w}]. \tag{4.3}$$

We also define

$$\tilde{H}_{(\lambda|\gamma)}=\tilde{H}_{(\lambda|\gamma)}(X\,|\,y)=t^{n(\mathrm{sort}(\lambda,\gamma))+|(\lambda|\gamma)|}\mathcal{J}^{w_0}_{(\lambda|\gamma)}\Big(\frac{X}{t^{-1}-1}\,\Big|\,y\Big)^*$$

and call these the *modified partially-symmetric Macdonald functions*. The assertion of (4.2) is then simply that $\Phi_k(H_{\mu,w})=\tilde{H}_{(\lambda|\gamma)}$ for all $k\geqslant 0$ and all (μ,w) indexing a $\mathbb{T}$-fixed point in $\mathrm{PFH}_{n,n-k}$.

The goal of this paper is to begin a proof of this conjecture by matching matrix elements for the action of $\mathbb{A}_{t,q}$ on both sides. More precisely, our main result is as follows:

Theorem 4.3. *Let $\Phi'_\bullet : U_\bullet \to V_\bullet$ be the collection $\mathbb{K}$-linear maps defined by assignment (4.2) on basis elements. Then $\Phi'_\bullet$ respects the action of the following elements of $\mathbb{A}_t$:*

1. $\mathsf{T}_1,\mathsf{T}_2,\dots,$
2. *for any $k\geqslant 0$, the element $\mathsf{d}_-\mathsf{T}_k^{-1}\cdots\mathsf{T}_1^{-1}\mathsf{d}_+$, which acts on V_k as the operator of multiplication by the elementary symmetric function $e_1(X)$.*

Since $\Phi_0 = \Phi'_0$, and in particular on the element $I_{\varnothing,()}$, it suffices to verify that $\Phi'_\bullet$ is fully $\mathbb{A}_t$-equivariant in order to establish $\Phi_\bullet = \Phi'_\bullet$. However, it is difficult to compute the action of $\mathsf{d}_\pm$ directly in terms our elements on the right-hand side of (4.2). We will complete the proof of Conjecture 4.2 in forthcoming work with M. Bechtloff Weising using a variant of this idea—namely, by combining our Theorem 4.3 with his results on limit Cherednik operators [BW24].

An explicit example is given in the next subsection. We also show for this example that $\Phi'_\bullet$ respects the action of the y_i. It should be possible to prove this in general using the same type of argument as the one we give for $e_1(X)$, based on results of [Goo24], though we will not need it in the sequel.

Let us comment on the normalization given on the right-hand side of (4.2). The factor $t^{n(\mathrm{sort}(\lambda|\gamma))+|(\lambda|\gamma)|}$ is required for the $e_1(X)$-equivariance, as our proof will show. The factor $t^{-\ell(w_0)}$ in (2.8) is also forced upon us, as we explain below (see §4.4).

4.3. Example

One may compute directly that

$$e_1(X)\mathcal{J}_{(\varnothing|0,1)}(X\,|\,y) = \frac{1}{1-qt}\mathcal{J}_{(2|0,0)}(X\,|\,y) + \frac{1-q}{(1-t)(1-qt)}\mathcal{J}_{(1|0,1)}(X\,|\,y).$$

Since the operator $w_0T_{w_0}$ from the definition (2.8) of $\mathcal{J}^{w_0}_{(\lambda|\gamma)}$ commutes with multiplication by $e_1(X)$, which we will abbreviate to e_1, this implies

$$e_1\cdot\mathcal{J}^{w_0}_{(\varnothing|0,1)}\Big(\frac{X}{t^{-1}-1}\,\Big|\,y\Big)^* = \frac{t-1}{1-qt^{-1}}\mathcal{J}^{w_0}_{(2|0,0)}\Big(\frac{X}{t^{-1}-1}\,\Big|\,y\Big)^* + \frac{1-q}{1-qt^{-1}}t\mathcal{J}^{w_0}_{(1|0,1)}\Big(\frac{X}{t^{-1}-1}\,\Big|\,y\Big)^*$$

and hence

$$e_1\cdot\tilde{H}_{(\varnothing|0,1)} = \frac{t-1}{t-q}\tilde{H}_{(2|0,0)} + \frac{1-q}{t-q}\tilde{H}_{(1|0,1)}.$$

For $\mu=(2,1)$ and $w=(t,q)$, which satisfies $\phi(\mu,w)=(\varnothing|0,1)$, we have

$$\begin{aligned}
\mathsf{d}_+\cdot[I_{(2,1),(t,q)}] &= -t^2t^2\frac{(t-q^2)(1-q)}{(t^2-q^2)(t-q)}\frac{(t^2-qt)(t^2-q^2)}{(t^2-qt^2)(t^2-q^2t)}[I_{(3,1),(t^2,t,q)}]\\
&= -t^2[I_{(3,1),(t^2,t,q)}]\\
\mathsf{T}_2^{-1}\mathsf{T}_1^{-1}\mathsf{d}_+\cdot[I_{(2,1),(t,q)}] &= -t^2\frac{t-1}{t-q}[I_{(3,1),(t^2,t,q)}] - t^2\frac{1-q}{t-q}[I_{(3,1),(t^2,q,t)}]
\end{aligned}$$

and hence

$$\mathsf{d}_-\mathsf{T}_2^{-1}\mathsf{T}_1^{-1}\mathsf{d}_+\cdot H_{(2,1),(t,q)} = \frac{t-1}{t-q}H_{(3,1),(t^2,t)} + \frac{1-q}{t-q}H_{(3,1),(t^2,q)}$$

where

$$\begin{aligned}
H_{(2,1),(t,q)} &= (-1)^3qt[I_{(2,1),(t,q)}],\\
H_{(3,1),(t^2,t)} &= (-1)^4qt^3[I_{(3,1),(t^2,t)}],\\
H_{(3,1),(t^2,q)} &= (-1)^4qt^3[I_{(3,1),(t^2,q)}].
\end{aligned}$$

We note that if we had started with $\mathcal{J}_{(\varnothing|1,0)}$, the Pieri formula would have three terms; the index $(\varnothing|1,0)$ corresponds to $\lambda=(2,1)$, $w=(q,t)$.

Let us also consider the operator y_2 acting on $H_{(2,1),(q,t)}$. Using $\mathsf{y}_2=\mathsf{T}_1^{-1}\varphi$ where $(t-1)\varphi=[\mathsf{d}_+,\mathsf{d}_-]$, we find by [CGM20, (6.2.1)] that

$$y_2\cdot H_{(2,1),(q,t)}=\frac{1}{t-q}H_{(2,2),(qt,q)}+\frac{(t-1)t}{(q-t)(t^2-q)}H_{(3,1),(t^2,q)}+\frac{1}{q-t^2}H_{(3,1),(q,t^2)}.$$

On the other hand, we compute $T_{w_0}^{-1}w_0y_2w_0T_{w_0}=T_1^{-1}y_1T_1$ on $\mathcal{J}_{(\varnothing|1,0)}$ as

$$T_1^{-1}y_1T_1\mathcal{J}_{(\varnothing|1,0)}=\frac{t^{-1}}{1-qt}\mathcal{J}_{(\varnothing|1,1)}-\frac{1-t}{(1-qt)(1-qt^2)}t^{-1}\mathcal{J}_{(1|0,1)}-\frac{1}{1-qt^2}\mathcal{J}_{(1|1,0)}.$$

This implies that $y_2\cdot\mathcal{J}^{w_0}_{(\varnothing|1,0)}\left(\frac{X}{t^{-1}-1}\,|\,y\right)^*$ is the sum of the terms

$$\frac{1}{t-q}\cdot t^2\mathcal{J}^{w_0}_{(\varnothing|1,1)}\left(\frac{X}{t^{-1}-1}\,|\,y\right)^*$$
$$\frac{(t-1)t}{(q-t)(t^2-q)}\cdot t^2\mathcal{J}^{w_0}_{(1|0,1)}\left(\frac{X}{t^{-1}-1}\,|\,y\right)^*$$
$$\frac{1}{q-t^2}\cdot t^2\mathcal{J}^{w_0}_{(1|1,0)}\left(\frac{X}{t^{-1}-1}\,|\,y\right)^*$$

and therefore that

$$y_2\cdot\tilde{H}_{(\varnothing|1,0)}=\frac{1}{t-q}\tilde{H}_{(\varnothing|1,1)}+\frac{(t-1)t}{(q-t)(t^2-q)}\tilde{H}_{(1|0,1)}+\frac{1}{q-t^2}\tilde{H}_{(1|1,0)},$$

in exact agreement with the computation of $y_2\cdot H_{(2,1),(q,t)}$, according to the assignment in our conjecture. (We note that multiplication by y_2 on $\mathcal{J}_{(\varnothing|1,0)}$ does *not* give matching coefficients; the twist by $w_0T_{w_0}$ is essential.)

4.4. Normalization

The reader may wonder if the operator $t^{-\ell(w_0)}w_0T_{w_0}$ from (2.8) is correctly normalized, since one could multiply this by any nonzero scalar (depending on k) to achieve the same effect on operators via conjugation. Let us briefly justify this choice, by considering the case of $\mu=(k)$ and $w=(t^{k-1},\cdots,t,1)$, which satisfies $\phi(\mu,w)=(\varnothing|0,\ldots,0)$.

We have $\mathcal{J}_{(\varnothing|0,\ldots,0)}=1$ and

$$\tilde{H}_{(\varnothing|0,\ldots,0)}=\mathcal{J}^{w_0}_{(\varnothing|0,\ldots,0)}=1,$$

thanks in particular to the factor $t^{-\ell(w_0)}$ in (2.8).

A necessary condition for Conjecture 4.2 to be true is that

$$\mathsf{d}_-\cdot\tilde{H}_{(\varnothing|0,\ldots,0)}=\tilde{H}_{(1|0,\ldots,0)}\in V_{k-1},$$

since $\mathsf{d}_-\cdot H_{(k),(t^{k-1},\ldots,1)}=H_{(k),(t^{k-1},\ldots,t)}$.

We clearly have $\mathsf{d}_-\cdot\tilde{H}_{(\varnothing|0,\ldots,0)}=\mathsf{d}_-\cdot 1=e_1(X)$. Also, $\mathcal{J}^{w_0}_{(1|0,\ldots,0)}$ is easily seen to be $(1-t)e_1(X)$. Hence $\tilde{H}_{(1|0,\ldots,0)}=e_1(X)$, and this shows that our normalization is consistent with Conjecture 4.2.

5. Demazure–Lusztig operators

Our goal in this section is to prove assertion (1) from Theorem 4.3. We assume throughout this section that (μ, w) indexes a $\mathbb{T}$-fixed point in $\mathrm{PFH}_{n,n-k}$ and that $\phi(\mu, w) = (\lambda \mid \gamma)$. Also let $m = n - k$ and regard λ as an element of $\mathbb{Y}_m \subset (\mathbb{Z}_{\geqslant 0})^m$.

Lemma 5.1. *Suppose $\gamma_i > \gamma_{i+1}$ and let $u = (m + i, \gamma_{i+1} + 1) \in \mathrm{dg}(\lambda^- | \gamma)$. Then*

$$\frac{w_i}{w_{i+1}} = q^{l(u)+1} t^{-a(u)}. \tag{5.1}$$

Proof. Comparing the column heights of γ_i and γ_{i+1}, we have $\gamma_i - \gamma_{i+1} = l(u) + 1$ and, using (4.1),

$$\begin{aligned} c_{i+1} - c_i = |\{j : \gamma_i \geqslant \lambda_j \geqslant \gamma_{i+1} + 1\}| + |\{j < i : \gamma_i \geqslant \gamma_j \geqslant \gamma_{i+1} + 1\}| \\ + |\{j > i + 1 : \gamma_i > \gamma_j \geqslant \gamma_{i+1}\}| + 1 \\ = a(u) \end{aligned}$$

with arms and legs taken in $\mathrm{dg}(\lambda^- | \gamma)$. □

We now have enough to prove Theorem 4.3 (1), which asserts that $\Phi'_\bullet$ respects the actions of each T_i.

Proof of Theorem 4.3 (1). Let $\Phi'_\bullet$ be the collection of maps defined in (4.2). We will use the notation established in Conjecture 4.2 throughout this proof. We must show the actions of $\Phi'_k \circ \mathsf{T}_i$ and $\mathsf{T}_i \circ \Phi'_k$ agree on all $\mathbb{T}$-fixed point classes $[I_{\mu,w}] \in K_{\mathbb{T}}(\mathrm{PFH}_{n,n-k})$ for all $i < k \leqslant n$. Write $\phi(\mu, w) = (\lambda \mid \gamma)$ as above.

Assuming first that $\gamma_i > \gamma_{i+1}$ and letting u be as defined above and $\nu = (\lambda^- \mid \gamma)$, we now appeal to (2.6) to see that

$$T_i^* \mathcal{J}_{(\lambda|\gamma)}(X \mid y)^* = \frac{q^{-(l_\nu(u)+1)} t^{a_\nu(u)+1} - 1}{q^{-(l_\nu(u)+1)} t^{a_\nu(u)} - 1} \mathcal{J}_{(\lambda|s_i(\gamma))}(X \mid y)^* + \frac{t - 1}{q^{l_\nu(u)+1} t^{-a_\nu(u)} - 1} \mathcal{J}_{(\lambda|\gamma)}(X \mid y)^*$$

where

$$T_i^* = s_i + \frac{(1-t) y_{i+1}}{y_{i+1} - y_i}(1 - s_i) = \frac{y_i - t y_{i+1}}{y_i - y_{i+1}} s_i + \frac{(1-t) y_{i+1}}{y_{i+1} - y_i}.$$

More generally, we let $T_w^* = t^{\ell(w)}(* \circ T_w \circ *)$ where $*$ is the automorphism sending $t \mapsto t^{-1}$ and fixing everything else. Then

$$w_0 T_{w_0}^* T_i^* (w_0 T_{w_0}^*)^{-1}$$

agrees with the operator T_i from the polynomial representation of $\mathbb{A}_{t,q}$; this follows from the relation $T_{w_0} T_i T_{w_0}^{-1} = T_{k-i}$ (see, e.g., [Mac03, (3.1.8)]). Hence we have

$$\mathsf{T}_i \cdot \tilde{H}_{(\lambda|\gamma)} = \frac{q^{-(l_\nu(u)+1)} t^{a_\nu(u)+1} - 1}{q^{-(l_\nu(u)+1)} t^{a_\nu(u)} - 1} \tilde{H}_{(\lambda|s_i(\gamma))} + \frac{t - 1}{q^{l_\nu(u)+1} t^{-a_\nu(u)} - 1} \tilde{H}_{(\lambda|\gamma)}. \tag{5.2}$$

On the other hand, by (5.1), we see that

$$\begin{aligned}\mathsf{T}_i \cdot [I_{\mu,w}] &= \frac{(t-1)w_{i+1}}{w_i - w_{i+1}}[I_{\mu,w}] + \frac{w_i - tw_{i+1}}{w_i - w_{i+1}}[I_{\mu,s_i(w)}] \\ &= \frac{t-1}{q^{l_\nu(u)+1}t^{-a_\nu(u)} - 1}[I_{\mu,w}] + \frac{q^{-(l_\nu(u)+1)}t^{a_\nu(u)+1} - 1}{q^{-(l_\nu(u)+1)}t^{a_\nu(u)} - 1}[I_{\mu,s_i(w)}]. \end{aligned} \tag{5.3}$$

This completes the proof of Theorem 4.3(1) in this case when $\gamma_i > \gamma_{i+1}$.

In the case where $\gamma_i < \gamma_{i+1}$, note that $\phi(\mu, s_i(w)) = (\lambda, s_i(\gamma))$ and $(s_i(\gamma))_i > s_i(\gamma))_{i+1}$. The new box whose coordinates are used in the equation is $u' = (m+i, (s_i(\gamma))_{i+1} + 1) \in \mathrm{dg}(\lambda^- \mid s_i(\gamma))$, and we let $\nu' = (\lambda^- \mid s_i(\gamma))$. Then we can use,

$$\mathsf{T}_i \cdot \tilde{H}_{(\lambda|s_i(\gamma))} = \frac{q^{-(l_{\nu'}(u')+1)}t^{a_{\nu'}(u')+1} - 1}{q^{-(l_{\nu'}(u')+1)}t^{a_{\nu'}(u')} - 1}\tilde{H}_{(\lambda|\gamma)} + \frac{t-1}{q^{l_{\nu'}(u')+1}t^{-a_{\nu'}(u')} - 1}\tilde{H}_{(\lambda|s_i(\gamma))}.$$

From there, we can apply $\mathsf{T}_i^{-1} = t^{-1}\mathsf{T}_i + (1 - t^{-1}) \cdot \mathrm{id}$ to both sides and solve for $\mathsf{T}_i(\tilde{H}_{(\lambda|\gamma)})$. By making use of linearity of Φ_k' and the agreement of (5.2) with (5.3), we obtain an identical formula by applying T_i^{-1} to

$$\mathsf{T}_i \cdot [I_{\mu,s_i(w)}] = \frac{t-1}{q^{l_{\nu'}(u)+1}t^{-a_{\nu'}(u)} - 1}[I_{\nu,s_i(w)}] + \frac{q^{-(l_{\nu'}(u)+1)}t^{a_{\nu'}(u)+1} - 1}{q^{-(l_{\nu'}(u)+1)}t^{a_{\nu'}(u)} - 1}[I_{\nu,w}].$$

The equations agree at every step, and hence this completes the case where $\gamma_i < \gamma_{i+1}$.

When $\gamma_i = \gamma_{i+1}$, the assertion follows immediately from $T_i\mathcal{J}_{(\lambda|\gamma)} = t\mathcal{J}_{(\lambda|\gamma)}$ (see, e.g., [Che05, (3.3.40)]) and $w_i = tw_{i+1}$, so we are done. □

6. Pieri formulas

In this section, we prove assertion (2) from Theorem 4.3, which is significantly more difficult than assertion (1). We use a Pieri formula for the partially-symmetric integral forms $\mathcal{J}_{(\lambda|\gamma)}$ established in [Goo24] using interpolation polynomials.

We make the following notational adjustments in order to match [Goo24] as closely as possible: we work in $\mathrm{PFH}_{N,N-k}$ and use (ξ, w) instead of (μ, w) to index a typical $\mathbb{T}$-fixed point in this space, so $|\xi| = N$ and the length of w is k. Also, when $\phi(\xi, w) = (\lambda \mid \gamma)$, we may regard λ as an element of $\mathbb{Y}_m \subset (\mathbb{Z}_{\geqslant 0})^m$ and $(\lambda \mid \gamma)$ as an element of $(\mathbb{Z}_{\geqslant 0})^n$, where $m = n - k$ and $n > N$ is arbitrary. Our computations will show that the Pieri formula for $\mathcal{J}_{(\lambda|\gamma)}$ in n total variables does not depend on n in this range.

To state formulas for $e_1(X)\mathcal{J}_{(\lambda|\gamma)}$, we need several definitions. Starting with the eigenvalues[1], we have

$$\overline{\nu}_i = q^{\nu_i}t^{-l_\nu'(i)}, 1 \leqslant i \leqslant n, \tag{6.1}$$
$$l_\nu'(i) = \#\{j < i \mid \nu_j > \nu_i\} + \#\{j > i \mid \nu_j \geqslant \nu_i\}. \tag{6.2}$$

[1]These are the eigenvalues of Cherednik–Dunkl operators on the nonsymmetric Macdonald polynomials. Even though we will not use these operators directly in this paper, we still refer to the $\overline{\nu}_i$ as eigenvalues.

We now take $(\lambda \mid \gamma)$ and build several new compositions, which we call η, $\widetilde{\lambda}$, $\widetilde{\mu}$, and μ. Let $I := \{t_1, \ldots, t_r\}$ with $0 = t_0 < t_1 < \cdots < t_r < t_{r+1} = k+1$. Choose one entry of λ to become $\widetilde{\lambda}_m$ and then set η following,

$$\begin{aligned} \eta_i &= \gamma_i \text{ if } i \notin I \\ \eta_{t_j} &= \gamma_{t_{j-1}} \text{ for } 1 \leqslant j \leqslant r \text{ where } \gamma_{t_0} = \widetilde{\lambda}_m. \end{aligned} \tag{6.3}$$

Next we build $\widetilde{\mu}$, which will be a rearrangement of the entries of λ, except with the entry we chose as $\widetilde{\lambda}_m$ replaced with $\gamma_{t_r} + 1$. For the ordering, all columns in $\widetilde{\mu}$ of height $\gamma_{t_r} + 1$ are sorted to the right, and we call h the index of the leftmost column of height $\gamma_{t_r} + 1$, so $\widetilde{\mu}_h = \cdots = \widetilde{\mu}_m = \gamma_{t_r} + 1$. The columns of other heights are sorted to be weakly increasing. We build the rest of $\widetilde{\lambda}$ by setting $(\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_{m-1}) := (\widetilde{\mu}_1, \ldots, \widetilde{\mu}_{m-1})$. Finally, define μ to be the weakly-decreasing rearrangement of $\widetilde{\mu}$.

We also require that I satisfy the following two properties:

- $\gamma_j \neq \gamma_{t_u}$ for any $u \in \{1, \ldots, r\}$ and $j \in \{t_u + 1, \ldots, t_{u+1} - 1\}$, and
- $\gamma_j \neq \widetilde{\lambda}_m$ for any $j \in \{1, \ldots, t_1 - 1\}$.

Such an I will be called maximal with respect to $(\lambda \mid \gamma)$. This can be used to define the support of the expansion,

$$\mathbb{M}_{(\lambda|\gamma)} := \{(\mu \mid \eta) \mid \mu \in S_m(\widetilde{\mu}),\ \mu_1 \geqslant \cdots \geqslant \mu_m,\ \eta \text{ is as defined above for some maximal } I\}.$$

Example 6.1. Consider the product $e_1(X)\mathcal{J}_{(5,4,4,1,0|2,1,3)}$. We can choose an entry 1 from λ to be $\widetilde{\lambda}_m$, and a possible maximal set I is $\{2, 3\}$. This gives us,

$$(\lambda \mid \gamma) = (5, 4, 4, \underline{1}, 0 \mid 2, \underline{1}, \underline{3}) \to (5, 4, 4, \underline{4}, 0 \mid 2, \underline{1}, \underline{1}) = (\mu \mid \eta).$$

This is best visualized as cycling the underlined terms to the right, and the rightmost entry is increased by 1 as it wraps back to the first underlined position. We then identify $\gamma_{t_r} = 3$, so $\widetilde{\mu}_3 = \widetilde{\mu}_4 = \widetilde{\mu}_5 = 4$, making $h = 3$. We use that to find $\widetilde{\lambda}$ as well, so

$$\widetilde{\mu} = (0, 5, 4, 4, 4), \quad \widetilde{\lambda} = (0, 5, 4, 4, 1).$$

Note that $I = \{3\}$ would not be maximal, since this would make $\gamma_2 = \widetilde{\lambda}_m = 1$, violating the second maximality condition. The notion of maximality can be understood as I being the maximal subset of $[1, k]$ which creates a given $(\mu \mid \eta)$. In this example, both $I = \{3\}$ and $I = \{2, 3\}$ lead to $(\mu \mid \eta) = (4, 4, 0, 2 \mid 2, 1, 1)$, hence $\{3\}$ is not maximal.

Theorem 6.2 ([Goo24], Theorem 5.7)**.** *One has an expansion*

$$e_1(X)\mathcal{J}_{(\lambda|\gamma)} = \sum_{(\mu|\eta)\in\mathbb{M}_{(\lambda|\gamma)}} \mathcal{A}^{(\lambda|\gamma)}_{(\mu|\eta)}\mathcal{J}_{(\mu|\eta)}$$

with coefficients given as follows:

$$\mathcal{A}^{(\lambda|\gamma)}_{(\mu|\eta)} = \prod_{\square\in\mathrm{dg}_{(\widetilde{\lambda}_m+1)}(\lambda^-)} \frac{t-q^{\ell_{(\lambda^-|\gamma)}(\square)+1}t^{a_{(\lambda^-|\gamma)}(\square)+1}}{1-q^{\ell_{(\lambda^-|\gamma)}(\square)+1}t^{a_{(\lambda^-|\gamma)}(\square)+1}}\cdot j_{\mathrm{C}}\cdot p'_I\cdot\left(\frac{1}{1-t}\right)\cdot(q^{-\widetilde{\mu}_h+1})\cdot\overline{(\widetilde{\lambda}\mid\gamma)}_m \tag{6.4}$$

where,

$$p'_I = \left(\frac{(t-1)q^{-1}\overline{\widetilde{\mu}}_h}{q^{-1}\overline{\widetilde{\mu}}_h-\overline{\eta}_{t_r}}\right)\prod_{u=1}^{r-1}\left(\frac{(t-1)\overline{\eta}_{t_{u+1}}}{\overline{\eta}_{t_{u+1}}-\overline{\eta}_{t_u}}\right)\prod_{j=t_r+1}^{k}\left(\frac{(q^{-1}t)\overline{\widetilde{\mu}}_h-\overline{\eta}_j}{q^{-1}\overline{\widetilde{\mu}}_h-\overline{\eta}_j}\right)\prod_{u=1}^{r}\prod_{j=t_{u-1}+1}^{t_u-1}\left(\frac{t\overline{\eta}_{t_u}-\overline{\eta}_j}{\overline{\eta}_{t_u}-\overline{\eta}_j}\right)$$

for $0=t_0<t_1<\cdots<t_r<t_{r+1}=k+1$, *and*

$$j_{\mathrm{C}} = \prod_{\square\in\mathrm{C}_{\lambda^-}} \frac{1-q^{\ell_{(\lambda^-|\gamma)}(\square)}t^{\widetilde{a}_{(\lambda^-|\gamma)}(\square)+1}}{1-q^{\ell_{(\lambda^-|\gamma)}(\square)+1}t^{\widetilde{a}_{(\lambda^-|\gamma)}(\square)+1+m_{j-1}(\eta)}},$$

where C_{λ^-} *is the rightmost column in* $\mathrm{dg}(\lambda^-)$ *of height* $\widetilde{\lambda}_m$ *and* $m_{j-1}(\eta)$ *is the number of columns in* η *of height* $j-1$.

6.1. p'_I vs. coefficient of $\mathsf{T}_k^{-1}\cdots\mathsf{T}_1^{-1}$

To compare p'_I and the coefficient resulting from $\mathsf{T}_k^{-1}\cdots\mathsf{T}_1^{-1}$, we first rewrite p'_I using the eigenvalues $\overline{\gamma}_i$ replacing their respective $\overline{\eta}_i$.

Lemma 6.3. *The* p'_I *defined above can be rewritten in the form,*

$$p'_I = \frac{(t-1)\overline{\gamma}_{t_1}}{\overline{\gamma}_{t_1}-\overline{\eta}_{t_1}}\cdot\prod_{u=1}^{r-1}\frac{(t-1)\overline{\gamma}_{t_{u+1}}}{\overline{\gamma}_{t_{u+1}}-\overline{\gamma}_{t_u}}\cdot\prod_{j=1}^{t_1-1}\frac{t\overline{\eta}_{t_1}-\overline{\gamma}_j}{\overline{\eta}_{t_1}-\overline{\gamma}_j}\cdot\prod_{u=1}^{r}\prod_{j=t_u+1}^{t_{u+1}-1}\frac{t\overline{\gamma}_{t_u}-\overline{\gamma}_j}{\overline{\gamma}_{t_u}-\overline{\gamma}_j}. \tag{6.5}$$

Proof. Comparing the formulas for the eigenvalues in (6.1) and (6.2) to the weights in (4.1) gives a simple conversion,

$$(\overline{\gamma}_i)_{t\mapsto t^{-1}} = w_i \qquad \text{and} \qquad (\overline{\eta}_{t_1})_{t\mapsto t^{-1}} = x.$$

This allows us to rewrite,

$$[p'_I]^* = \frac{(t^{-1}-1)w_{t_1}}{w_{t_1}-x}\cdot\prod_{u=1}^{r-1}\frac{(t^{-1}-1)w_{t_{u+1}}}{w_{t_{u+1}}-w_{t_u}}\cdot\prod_{j=1}^{t_1-1}\frac{t^{-1}x-w_j}{x-w_j}\cdot\prod_{u=1}^{r}\prod_{j=t_u+1}^{t_{u+1}-1}\frac{t^{-1}w_{t_u}-w_j}{w_{t_u}-w_j}, \tag{6.6}$$

where $*$ is the automorphism of $\mathbb{K}$ sending $q\mapsto q$ and $t\mapsto t^{-1}$. $\square$

We want to match the coefficients in the expansion of $e_1(X)$ with $\mathsf{d}_-\mathsf{T}_k^{-1}\cdots\mathsf{T}_1^{-1}\mathsf{d}_+$. Since $\mathsf{d}_+H_{\xi,w}$ is a linear combination of some $H_{\xi+x,xw}$, the product $\mathsf{T}_k^{-1}\cdots\mathsf{T}_1^{-1}$ acts on $H_{\xi+x,xw}$ where $xw=(x,w_1,\dots,w_k)$. The following lemma shows the result of those actions.

Lemma 6.4. *Given $I=\{t_1,\dots,t_r\}$, let $v_I=s_k\cdots\widehat{s}_{t_r}\cdots\widehat{s}_{t_1}\cdots s_1$, where hats indicate omission of that term from the product. Then the expansion of $\mathsf{T}_k^{-1}\cdots\mathsf{T}_1^{-1}H_{\xi+x,xw}$ is given by,*

$$\mathsf{T}_k^{-1}\cdots\mathsf{T}_1^{-1}H_{\xi+x,xw}=\sum_{I\subseteq[1,k]}\widetilde{p}_I\cdot H_{\xi+x,v_I(xw)} \tag{6.7}$$

where

$$\widetilde{p}_I=\frac{(1-t^{-1})x}{x-w_{t_1}}\prod_{u=1}^{r-1}\frac{(1-t^{-1})w_{t_u}}{w_{t_u}-w_{t_{u+1}}}\cdot\prod_{j=1}^{t_1-1}\frac{t^{-1}x-w_j}{x-w_j}\cdot\prod_{u=1}^{r}\prod_{j=t_u+1}^{t_{u+1}-1}\frac{t^{-1}w_{t_u}-w_j}{w_{t_u}-w_j}. \tag{6.8}$$

Proof. We will use the formula for the action of T_i^{-1} on $H_{\xi,w}$ by combining (3.7) with Theorem 4.3,

$$\mathsf{T}_i^{-1}(H_{\xi,w})=\frac{(1-t^{-1})w_i}{w_i-w_{i+1}}H_{\xi,w}+\frac{t^{-1}w_i-w_{i+1}}{w_i-w_{i+1}}H_{\xi,s_i(w)}. \tag{6.9}$$

For a fixed set $I=\{t_1,\dots,t_k\}$, define a function f_ℓ^I on a sequence $w=(w_1,\dots,w_k)$ by:

$$f_\ell^I(w)=\begin{cases}\frac{(1-t^{-1})w_\ell}{w_\ell-w_{\ell+1}} & \text{if } \ell\in I\\ \frac{t^{-1}w_\ell-w_{\ell+1}}{w_\ell-w_{\ell+1}} & \text{if } \ell\notin I\end{cases} \tag{6.10}$$

Now define σ_j to be the subword of $s_j\cdots s_1$ with the same simple transpositions removed as v_I. Finally, we introduce the provisional notation $a\ni b$, when a admits a binomial expansion, to mean that the binomial expansion of a contains b as a summand. In the expansion of $\mathsf{T}_k^{-1}\cdots\mathsf{T}_1^{-1}H_{\xi+x,xw}$, a summand can be identified by a choice for each $i\in[1,k]$ of either the left rational function in (6.9), which does not include the transposition s_i, or the right function which does. This allows us to isolate a particular summand and its coefficient as follows:

$$\begin{aligned}\mathsf{T}_k^{-1}\cdots\mathsf{T}_1^{-1}H_{\xi+x,xw}&\ni(f_k^I(\sigma_{k-1}(xw))\cdots f_2^I(\sigma_1(xw))\cdot f_1^I(xw))s_k\cdots\widehat{s}_{t_r}\cdots\widehat{s}_{t_1}\cdots s_1H_{\xi+x,xw} \qquad (6.11)\\&=(f_k^I(\sigma_{k-1}(xw))\cdots f_2^I(\sigma_1(xw))\cdot f_1^I(xw))H_{\xi+x,v_I(xw)}\end{aligned}$$

The following are computations of the pieces $f_\ell^I(\sigma_{\ell-1}(w))$ of the coefficient for a summand appearing in the right side of (6.7) for a given I.

Case 1: Let $\ell<t_1$.

$$\begin{aligned}&\mathsf{T}_\ell^{-1}\left(\mathsf{T}_{\ell-1}^{-1}\cdots\mathsf{T}_1^{-1}H_{\xi+x,xw}\right)\\&\qquad\ni\frac{t^{-1}(\sigma_{\ell-1}(xw))_\ell-(\sigma_{\ell-1}(xw))_{\ell+1}}{(\sigma_{\ell-1}(xw))_\ell-(\sigma_{\ell-1}(xw))_{\ell+1}}s_\ell\left(f_{\ell-1}^I(\sigma_{\ell-2}(xw))\cdots f_1^I(xw)H_{\xi+x,\sigma_{\ell-1}(xw)}\right).\end{aligned}$$

The permutation $\sigma_{\ell-1}$ moves x into position ℓ, so $((\sigma_{\ell-1}(xw))_\ell=x$, and $(\sigma_{\ell-1}(xw))_{\ell+1}=w_\ell$. So explicitly,

$$f_\ell^I(\sigma_{\ell-1}(xw))=\frac{t^{-1}x-w_\ell}{x-w_\ell}.$$

Case 2: Let $\ell = t_1$. Then like in Case 1,

$$\begin{aligned}&\mathsf{T}_{t_1}^{-1}\left(\mathsf{T}_{t_1-1}^{-1}\cdots\mathsf{T}_1^{-1}H_{\xi+x,xw}\right)\\ &\qquad\ni \frac{(1-t^{-1})\cdot(\sigma_{t_1-1}(xw))_{t_1}}{(\sigma_{t_1-1}(xw))_{t_1}-(\sigma_{t_1-1}(xw))_{t_1+1}}\left(f_{t_1-1}^I(\sigma_{t_1-2}(xw))\cdots f_1^I(xw)H_{\xi+x,\sigma_{(t_1-1)}(xw)}\right).\end{aligned}$$

From this we find

$$f_{t_1}^I(\sigma_{t_1-1}(xw)) = \frac{(1-t^{-1})x}{x-w_{t_1}}$$

Case 3: Let $\ell = t_i$ with $i > 1$. Once again:

$$\begin{aligned}&\mathsf{T}_{t_i}^{-1}\left(\mathsf{T}_{t_i-1}^{-1}\cdots\mathsf{T}_1^{-1}H_{\xi+x,xw}\right)\\ &\qquad\ni \frac{(1-t^{-1})\cdot(\sigma_{t_i-1}(xw))_{t_i}}{(\sigma_{t_i-1}(xw))_{t_i}-(\sigma_{t_i-1}(xw))_{t_i+1}}\left(f_{t_i-1}^I(\sigma_{t_i-2}(xw))\cdots f_1^I(xw)H_{\xi+x,\sigma_{(t_i-1)}(xw)}\right).\end{aligned}$$

The missing transposition $\widehat{s}_{t_{i-1}}$ in σ_{t_i-1} leaves $w_{t_{i-1}}$ in the $(t_{i-1}+1)$st position, and the product $s_{t_i-1}\cdots s_{t_{i-1}+1}$ moves $w_{t_{i-1}}$ into the (t_i)th position. As a result, $(\sigma_{t_i-1}(xw))_{t_i} = w_{t_{i-1}}$. And the (t_i+1)st position is untouched, so $(\sigma_{t_i-1}(xw))_{\ell+1} = w_{t_i}$. Altogether we obtain,

$$f_{t_i}^I(\sigma_{t_i-1}(xw)) = \frac{(1-t^{-1})w_{t_{i-1}}}{w_{t_{i-1}}-w_{t_i}}$$

Case 4: Let $t_{i-1} < \ell < t_i$ for $i > 1$. Like in Case 1, we have,

$$\begin{aligned}&\mathsf{T}_{\ell}^{-1}\left(\mathsf{T}_{\ell-1}^{-1}\cdots\mathsf{T}_1^{-1}H_{\xi+x,xw}\right)\\ &\qquad\ni \frac{t^{-1}(\sigma_{\ell-1}(xw))_{\ell}-(\sigma_{\ell-1}(xw))_{\ell+1}}{(\sigma_{\ell-1}(xw))_{\ell}-(\sigma_{\ell-1}(xw))_{\ell+1}}s_\ell\left(f_{\ell-1}^I(\sigma_{\ell-2}(xw))\cdots f_1^I(xw)H_{\xi+x,\sigma_{\ell-1}(xw)}\right).\end{aligned}$$

The relevant portion of σ is the product $s_{\ell-1}\cdots s_{t_{i-1}+1}$, which moves w_{t_i} to the (ℓ)th position, making $(\sigma_{\ell-1}(xw))_\ell = w_{t_{i-1}}$. And the simpler part is $(\sigma_{\ell-1}(xw))_{\ell+1} = w_\ell$, as it has not yet permuted in the T_ℓ^{-1} coefficient formula. So we obtain

$$f_\ell^I(\sigma_{\ell-1}(xw)) = \frac{t^{-1}w_{t_{i-1}}-w_\ell}{w_{t_{i-1}}-w_\ell}.$$

Combining all the coefficients, $\widetilde{p}_I = f_k^I(\sigma_{k-1}(w))\cdots f_2^I(\sigma_1(w))\cdot f_1^I(w)$, gives the desired formula (6.8). □

Corollary 6.5.

$$\widetilde{p}_I = [p_I']^*\cdot x\cdot(w_{t_r})^{-1} \tag{6.12}$$

Proof. This comes immediately from comparing terms in (6.6) with (6.8), which only differ in the numerators in the leftmost two rational functions. □

Remark 6.6. A priori, the formula (6.7) includes all subsets $I \subseteq [1,k]$, which would include those which swap labels of boxes in the same row. That cannot be allowed, as $(\xi, v_I(xw))$ would no longer correspond to a flag of ideals. However, any such term will vanish. This is seen by comparing coordinates of the boxes, where if adjacent boxes are swapped, it causes one of the terms $(t^{-1}x - w_j)$ or $(t^{-1}w_{t_u} - w_j)$ to be 0. For the same reason, if I is maximal, the associated term has a nonzero coefficient. The corresponding arguments for the expansion of $e_1(X)\mathcal{J}_{(\lambda|\gamma)}$ are proven very similarly in [Goo24, Lemma 4.10, Cor. 4.11]. Further, the maximality condition prevents terms in the expansion from overlapping, as can be seen in Example 6.1, and due to Proposition 6.7, the same will be true in the $\mathsf{d}_-\mathsf{T}_k^{-1}\cdots\mathsf{T}_1^{-1}\mathsf{d}_+H_{\xi,w}$ setting.

The following proposition allows us to match the supports of the expansions of $e_1(X)\mathcal{J}_{(\lambda|\gamma)}$ and $\mathsf{d}_-\mathsf{T}_k^{-1}\cdots\mathsf{T}_1^{-1}\mathsf{d}_+H_{\xi,w}$.

Proposition 6.7. *Let ϕ be the defined in Proposition 4.1, $v_I = s_k\cdots\widehat{s}_{t_r}\cdots\widehat{s}_{t_1}\cdots s_1$, and suppose $\phi(\xi,w) = (\lambda \mid \gamma)$. Then ϕ restricts to a bijection from the support of $\mathsf{d}_-\mathsf{T}_k^{-1}\cdots\mathsf{T}_1^{-1}\mathsf{d}_+H_{\xi,w}$ to $\mathbb{M}_{(\lambda|\gamma)}$, the support of $e_1(X)\mathcal{J}_{(\lambda|\gamma)}$. In particular, when $I \subseteq [1,k]$ is maximal, $\widetilde{\lambda}_m$ is the height of the column that the box x is added to, and $(\eta \mid \mu)$ is as defined earlier, this map is given explicitly by,*

$$\phi(\xi + x, (v_I(xw))') = (\eta \mid \mu),$$

Proof. We will justify this by tracking changes in the chain of ideals through several steps,

$$(\xi, w) \to (\xi + x, xw) \to (\xi + x, v_I(xw)) \to (\xi + x, (v_I(xw))'),$$

and how the corresponding images under ϕ end with $(\eta \mid \mu)$ as described at the beginning of this section. Note that these three steps in order from left to right are the results of the actions of d_+, $\mathsf{T}_k^{-1}\cdots\mathsf{T}_1^{-1}$, and d_- respectively. As the first two of those actions result in linear combinations of fixed point classes, we will track a single summand, for a particular choice of x and I.

First, as in (3.4), the action $\mathsf{d}_+ \cdot H_{\xi,w}$ gives a sum over ξ-addable boxes with basis elements $H_{\xi+x,xw}$. For x to be a ξ-addable box, the column x is added to must not have a box with coordinates appearing in w. Further, there is at most one column of a given height the box can be added to (the leftmost) for $\xi + x$ to be a partition diagram. This choice of added box x corresponds to the choice of $\widetilde{\lambda}_m$ from $(\lambda|\gamma)$, and so because the column with x becomes a 'labeled' column. This gives us,

$$\phi(\xi + x, xw) = (\lambda_1, \ldots, \widetilde{\lambda}_{m-1}, \widetilde{\lambda}_{m+1}, \ldots, \lambda_m \mid \widetilde{\lambda}_m, \gamma_1, \ldots, \gamma_k)$$

Next, using (6.11), and comparing to the construction of η in $(\mu \mid \eta)$, we find,

$$\phi(\xi + x, v_I(xw)) = (\lambda_1, \ldots, \widetilde{\lambda}_{m-1}, \widetilde{\lambda}_{m+1}, \ldots, \lambda_m \mid \eta_1, \ldots, \eta_k\gamma_{t_r})$$

Last, the action d_- simply removes the last entry in $v_I(xw)$. This removes the label from the box labeled $k+1$, which is equivalent to making the column with label $k+1$ a symmetric column with 1 more box, so if we let sort be the operation that sorts a composition to be in decreasing order,

$$\begin{aligned}\phi(\xi + x, (v_I(xw))' &= (\mathrm{sort}(\lambda_1, \ldots, \widetilde{\lambda}_{m-1}, \widetilde{\lambda}_{m+1}, \ldots, \lambda_m, \gamma_{t_r} + 1) \mid \eta_1, \ldots, \eta_k) \\ &= (\mu \mid \eta).\end{aligned}$$

Finally, we note by looking at the numerators of (6.8) that $\widetilde{p}_I \neq 0$ if and only if

$$\begin{aligned} w_j &\neq t^{-1}x \quad \forall j \in \{1,\dots,t_1-1\}, \text{ and} \\ w_j &\neq t^{-1}w_{t_u} \quad \forall u \in \{1,\dots,r\}, \forall j \in \{t_u+1,\dots,t_{u+1}-1\}. \end{aligned}$$

Inductively, these conditions imply that if some box has a label belonging to I, all labeled boxes which are adjacent to that box and which have consecutive labels must also belong to I (as otherwise, the first one that is not in the set would violate one of the conditions). This is identical to the maximality condition on a choice of I with respect to γ. $\square$

6.2. Remaining pieces

First we must address some internal cancellation from terms in [CGM20]. We recall the formula for $\mathsf{d}_+ H_{\xi,w}$ and Pieri coefficients $d_{\xi+x,\xi}$ from [CGM20] here:

$$\mathsf{d}_+ H_{\xi,w} = t^k \sum_x d_{\xi+x,\xi} \prod_{i=1}^{k} \frac{x-qw_i}{x-qtw_i} H_{\xi+x,xw}$$

$$d_{\xi+x,\xi} = \prod_{\square \in R_{\xi+x,\xi}} \frac{t^{\mathrm{arm}_\xi(\square)} - q^{\mathrm{leg}_\xi(\square)+1}}{t^{\mathrm{arm}_\xi(\square)+1} - q^{\mathrm{leg}_\xi(\square)+1}} \prod_{\square \in C_{\xi+x,\xi}} \frac{t^{\mathrm{arm}_\xi(\square)+1} - q^{\mathrm{leg}_\xi(\square)}}{t^{\mathrm{arm}_\xi(\square)+1} - q^{\mathrm{leg}_\xi(\square)+1}}$$

We split $R_{\xi+x,\xi}$ into two groups,

$$\mathrm{R_S}(x) := \{\square \in R_{\xi+x,\xi} \mid \square \text{ is not in a labeled column}\} \tag{6.13}$$

$$\mathrm{R_{NS}}(x) := \{\square \in R_{\xi+x,\xi} \mid \square \text{ is in a labeled column}\}, \tag{6.14}$$

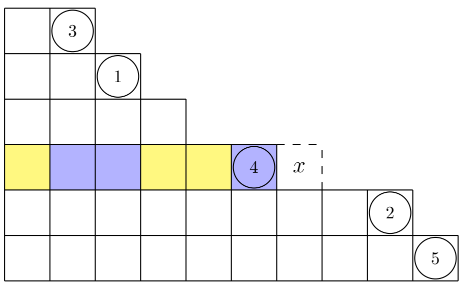
which are the boxes in the same row as x that are in 'symmetric' and 'nonsymmetric' columns respectively. We break up the column labels $[1,k]$ as well,

$$\mathrm{LL}(x) := \{i \in [1,k] \mid \text{ The column labeled } i \text{ is to the left of } x\} \tag{6.15}$$

$$\mathrm{RL}(x) := \{i \in [1,k] \mid \text{ The column labeled } i \text{ is to the right of } x\}, \tag{6.16}$$

the 'left-labeled' and 'right-labeled' columns respectively.

Example 6.8. The following is the diagram for the pair $H_{\xi,w} = H_{(10,9,6,4,3,2),(q^4t^2,qt^8,q^5t,q^2t^5,t^9)}$, and $H_{\xi+x,xw} = H_{(10,9,7,4,3,2),(q^2t^6,q^4t^2,qt^8,q^5t,q^2t^5,t^9)}$. Yellow boxes belong to $\mathrm{R_S}(x)$, and blue belong to $\mathrm{R_{NS}}(x)$. The labeled columns are $\mathrm{LL}(x) = \{1,3,4\}$ and $\mathrm{RL}(x) = \{2,5\}$.



Lemma 6.9. *If $i \in \mathrm{LL}(x)$ and $\square$ is the unique box in $R_{\xi+x,\xi}$ in the column labeled i, then*

$$\left(\frac{t^{\mathrm{arm}_\xi(\square)} - q^{\mathrm{leg}_\xi(\square)+1}}{t^{\mathrm{arm}_\xi(\square)+1} - q^{\mathrm{leg}_\xi(\square)+1}}\right) \cdot \left(\frac{x - qw_i}{x - qtw_i}\right) = t^{-1}.$$

Proof. We will write u to indicate the coordinate of $\square$, $u = q^r t^c$. Then we have

$$\frac{x}{u} = t^{\mathrm{arm}_{\xi+x}(\square)}, \qquad \frac{w_i}{u} = q^{\mathrm{leg}_{\xi+x}(\square)} \tag{6.17}$$

and hence

$$\frac{x - qw_i}{x - qtw_i} = \frac{\frac{x}{u} - q\frac{w_i}{u}}{\frac{x}{u} - qt\frac{w_i}{u}} = \frac{t^{\mathrm{arm}_{\xi+x}(\square)} - q^{\mathrm{leg}_{\xi+x}(\square)+1}}{t^{\mathrm{arm}_{\xi+x}(\square)} - tq^{\mathrm{leg}_{\xi+x}(\square)+1}} = t^{-1}\frac{t^{\mathrm{arm}_\xi(\square)+1} - q^{\mathrm{leg}_\xi(\square)+1}}{t^{\mathrm{arm}_\xi(\square)} - q^{\mathrm{leg}_\xi(\square)+1}} \tag{6.18}$$

since $\mathrm{arm}_{\xi+x}(\square) = \mathrm{arm}_\xi(\square) + 1$ and $\mathrm{leg}_{\xi+x}(\square) = \mathrm{leg}_\xi(\square)$. $\square$

Corollary 6.10.

$$\prod_{\square \in R_{\xi+x,\xi}} \frac{t^{\mathrm{arm}_\xi(\square)} - q^{\mathrm{leg}_\xi(\square)+1}}{t^{\mathrm{arm}_\xi(\square)+1} - q^{\mathrm{leg}_\xi(\square)+1}} \prod_{i \in \mathrm{LL}(x)} \frac{x - qw_i}{x - qtw_i} = t^{-|\mathrm{LL}(x)|} \prod_{\square \in \mathrm{R_S}(x)} \frac{t^{\mathrm{arm}_\xi(\square)} - q^{\mathrm{leg}_\xi(\square)+1}}{t^{\mathrm{arm}_\xi(\square)+1} - q^{\mathrm{leg}_\xi(\square)+1}}. \tag{6.19}$$

Proof. Apply Lemma 6.19 for each nonsymmetric column to cancel with terms appearing in the product over $R_{\xi+x,\xi}$. $\square$

We now do something similar with the column terms, which behave differently since a box in the same column as x may have multiple labeled boxes to its right.

Lemma 6.11. *Let $\square = (i, j) \in C_{\xi+x,\xi}$, and let $\ell_1 < \cdots < \ell_b \in \mathrm{RL}(x)$ be the labels of all labeled boxes in row j, the same row as $\square$. Then we have*

$$\frac{t^{\mathrm{arm}_\xi(\square)+1} - q^{\mathrm{leg}_\xi(\square)}}{t^{\mathrm{arm}_\xi(\square)+1} - q^{\mathrm{leg}_\xi(\square)+1}} \cdot \prod_{i=1}^{b} \frac{x - qw_{\ell_i}}{x - qtw_{\ell_i}} = \frac{t^{\mathrm{arm}_{\xi+x}(\square)+1-b} - q^{\mathrm{leg}_{\xi+x}(\square)-1}}{t^{\mathrm{arm}_{\xi+x}(\square)+1} - q^{\mathrm{leg}_{\xi+x}(\square)}}.$$

Proof. Since the labels $\ell_1, \ldots, \ell_b$ must appear in adjacent columns, with ℓ_1 the farthest to the right, we can write $w_{\ell_1} = tw_{\ell_2} = \cdots = t^{b-1}w_{\ell_b}$. Therefore we can simplify,

$$\prod_{i=1}^{b} \frac{x - qw_{\ell_i}}{x - qtw_{\ell_i}} = \prod_{i=1}^{b} \frac{x - t^{-i+1}qw_{\ell_1}}{x - t^{-i+2}qw_{\ell_1}} = \frac{x - t^{-b+1}qw_{\ell_1}}{x - tqw_{\ell_1}}$$

Once again writing u for the coordinate of $\square$, we compare arm and leg counts,

$$\frac{x}{u} = q^{\mathrm{leg}_{\xi+x}(\square)}, \qquad \frac{w_{\ell_1}}{u} = t^{\mathrm{arm}_{\xi+x}(\square)},$$

which allows us to simplify,

$$\prod_{i=1}^{b} \frac{x - qw_{\ell_i}}{x - qtw_{\ell_i}} = \frac{\frac{x}{u} - t^{-b+1} q^{\frac{w_{\ell_1}}{u}}}{\frac{x}{u} - tq^{\frac{w_{\ell_1}}{u}}} = \frac{q^{\mathrm{leg}_{\xi+x}(\square)} - qt^{\mathrm{arm}_{\xi+x}(\square)+1-b}}{q^{\mathrm{leg}_{\xi+x}(\square)} - qt^{\mathrm{arm}_{\xi+x}(\square)+1}} = \frac{t^{\mathrm{arm}_{\xi+x}(\square)+1-b} - q^{\mathrm{leg}_{\xi+x}(\square)-1}}{t^{\mathrm{arm}_{\xi}(\square)+1} - q^{\mathrm{leg}_{\xi}(\square)}}$$

where in the last equality we use,

$$\mathrm{arm}_{\xi+x}(\square) = \mathrm{arm}_{\xi}(\square), \qquad \mathrm{leg}_{\xi+x}(\square) = \mathrm{leg}_{\xi}(\square) + 1.$$

□

Corollary 6.12. *We have the following identity:*

$$\prod_{\square \in C_{\xi+x,\xi}} \frac{t^{\mathrm{arm}_{\xi}(\square)+1} - q^{\mathrm{leg}_{\xi}(\square)}}{t^{\mathrm{arm}_{\xi}(\square)+1} - q^{\mathrm{leg}_{\xi}(\square)+1}} \cdot \prod_{i \in \mathrm{RL}(x)} \frac{x - qw_i}{x - qtw_i} = \prod_{\square \in C_{\xi+x,\xi}} \frac{t^{\mathrm{arm}_{\xi+x}(\square)+1-b(\square)} - q^{\mathrm{leg}_{\xi+x}(\square)-1}}{t^{\mathrm{arm}_{\xi+x}(\square)+1} - q^{\mathrm{leg}_{\xi+x}(\square)}}, \tag{6.20}$$

where $b(\square)$ is the number of labeled boxes in the same row as $\square$.

Proof. The products on the left side of the equation can be split into individual boxes in $C_{\xi+x,\xi}$ along with their boxes in $\mathrm{RL}(x)$ in the corresponding rows. Each of these can then be converted according to Lemma 6.11, giving us the product on the right. □

Theorem 6.13. *Consider the expansion*

$$\left(\mathsf{d}_- \mathsf{T}_k^{-1} \cdots \mathsf{T}_1^{-1} \mathsf{d}_+\right) H_{\xi,w} = \sum_{x,v_I} C^{\xi,w}_{\xi+x,(v_I(xw))'} H_{\xi+x,(v_I(xw))'}, \tag{6.21}$$

where v_I is the permutation $v_I = s_k \cdots \widehat{s}_{t_r} \cdots \widehat{s}_{t_1} \cdots s_1$ associated to $I = \{t_1, \dots, t_r\}$, and $(v_I(xw))'$ indicates the last entry of $v(wx)$ is removed. The coefficients satisfy

$$C^{\xi,w}_{\xi+x,(v_I(xw))'} = t^{c_r} \cdot \left[\mathcal{A}^{(\lambda|\gamma)}_{(\mu|\eta)} \cdot (1-t)\right]^*,$$

where ξ, w corresponds to $(\lambda \mid \gamma)$ and $\xi + x, (v_I(xw))'$ corresponds to $(\mu \mid \eta)$ as in Section 4.1, the column index of w_{t_r} is c_r, and $*$ is the $\mathbb{K}$-automorphism fixing q and sending $t \mapsto t^{-1}$.

Proof. Applying (6.19) and (6.20), we have a simplified formula for $\mathsf{d}_+ H_{\xi,w}$,

$$\sum_x t^{|\mathrm{RL}(x)|} \prod_{\square \in \mathrm{R_S}(x)} \frac{t^{\mathrm{arm}_{\xi}(\square)} - q^{\mathrm{leg}_{\xi}(\square)+1}}{t^{\mathrm{arm}_{\xi}(\square)+1} - q^{\mathrm{leg}_{\xi}(\square)+1}} \prod_{\square \in C_{\xi+x,\xi}} \frac{t^{\mathrm{arm}_{\xi+x}(\square)+1-b(\square)} - q^{\mathrm{leg}_{\xi+x}(\square)-1}}{t^{\mathrm{arm}_{\xi+x}(\square)+1} - q^{\mathrm{leg}_{\xi+x}(\square)}} H_{\xi+x,xw}.$$

Combining this form with (6.8) and (3.5) gives a formula for $C^{\xi,w}_{\xi+x,(v(xw))'}$,

$$\widetilde{p}_I \cdot t^{|\mathrm{RL}(x)|} \prod_{\square \in \mathrm{R_S}(x)} \frac{t^{\mathrm{arm}_\xi(\square)} - q^{\mathrm{leg}_\xi(\square)+1}}{t^{\mathrm{arm}_\xi(\square)+1} - q^{\mathrm{leg}_\xi(\square)+1}} \prod_{\square \in C_{\xi+x,\xi}} \frac{t^{\mathrm{arm}_{\xi+x}(\square)+1-b(\square)} - q^{\mathrm{leg}_{\xi+x}(\square)-1}}{t^{\mathrm{arm}_{\xi+x}(\square)+1} - q^{\mathrm{leg}_{\xi+x}(\square)}}$$

On the other side, from (6.4),

$$\mathcal{A}^{(\lambda|\gamma)}_{(\mu|\eta)} \cdot (1-t) = \prod_{\square \in \mathrm{dg}_{(\widetilde{\lambda}_{n-k}+1)}(\lambda^-)} \frac{t - q^{\ell_{(\lambda^-|\gamma)}(\square)+1} t^{a_{(\lambda^-|\gamma)}(\square)+1}}{1 - q^{\ell_{(\lambda^-|\gamma)}(\square)+1} t^{a_{(\lambda^-|\gamma)}(\square)+1}} \cdot j_{\mathrm{C}} \cdot p'_I \cdot (q^{-\widetilde{\mu}_h+1}) \cdot \overline{(\widetilde{\lambda} \mid \gamma)}_{n-k}.$$

It therefore remains to prove the following correspondences, where $*$ is again the $\mathbb{K}$-automorphism fixing q and sending $t \mapsto t^{-1}$.

1. $$\left[\prod_{\square \in \mathrm{dg}_{(\widetilde{\lambda}_{n-k}+1)}(\lambda^-)} \frac{t - q^{\ell_{(\lambda^-|\gamma)}(\square)+1} t^{a_{(\lambda^-|\gamma)}(\square)+1}}{1 - q^{\ell_{(\lambda^-|\gamma)}(\square)+1} t^{a_{(\lambda^-|\gamma)}(\square)+1}}\right]^* = \prod_{\square \in \mathrm{R_S}(x)} \frac{t^{\mathrm{arm}_\xi(\square)} - q^{\mathrm{leg}_\xi(\square)+1}}{t^{\mathrm{arm}_\xi(\square)+1} - q^{\mathrm{leg}_\xi(\square)+1}}$$

2. $$[j_{\mathrm{C}}]^* = t^{|\mathrm{RL}(x)|} \cdot \prod_{\square \in C_{\xi+x,\xi}} \frac{t^{\mathrm{arm}_{\xi+x}(\square)+1-b(\square)} - q^{\mathrm{leg}_{\xi+x}(\square)-1}}{t^{\mathrm{arm}_{\xi+x}(\square)+1} - q^{\mathrm{leg}_{\xi+x}(\square)}}$$

3. $$\left[p'_I \cdot (q^{-\widetilde{\mu}_h+1}) \cdot \overline{(\widetilde{\lambda} \mid \gamma)}_{n-k}\right]^* = \widetilde{p}_I \cdot t^{c_r}$$

Starting with (1), begin with the following:

$$\left[\prod_{\square \in \mathrm{dg}_{(\widetilde{\lambda}_{n-k}+1)}(\lambda^-)} \frac{t - q^{\ell_{(\lambda^-|\gamma)}(\square)+1} t^{a_{(\lambda^-|\gamma)}(\square)+1}}{1 - q^{\ell_{(\lambda^-|\gamma)}(\square)+1} t^{a_{(\lambda^-|\gamma)}(\square)+1}}\right]^* = \prod_{\square \in \mathrm{dg}_{(\widetilde{\lambda}_{n-k}+1)}(\lambda^-)} \frac{t^{a_{(\lambda^-|\gamma)}(\square)} - q^{\ell_{(\lambda^-|\gamma)}(\square)+1}}{t^{a_{(\lambda^-|\gamma)}(\square)+1} - q^{\ell_{(\lambda^-|\gamma)}(\square)+1}}$$

For a box $\square = (i, \widetilde{\lambda}_{n-k}+1)$ in $d_{(\widetilde{\lambda}_{n-k}+1)}(\lambda)$, notice that $\widetilde{\lambda}_{n-k}+1$ is the height of the added box x. So this product is over all boxes in $\widetilde{\lambda}$, the symmetric part of the diagram, of the same height as x, each of which corresponds to a box $\square' \in \mathrm{R_S}(x)$. The arm and leg counts for the corresponding boxes are easily compared using the respective definitions to get,

$$\mathrm{leg}_\xi(\square') = \ell_{(\lambda^-|\gamma)}(\square), \qquad \mathrm{arm}_\xi(\square') = a_{(\lambda^-|\gamma)}(\square).$$

The conversion of the products to the form on the right side of (1) follows immediately.

For (2), we begin in the same way,

$$[j_{\mathrm{C}}]^* = \left[\prod_{\square \in \mathrm{C}_{\lambda^-}} \frac{1 - q^{\ell_{(\lambda^-|\gamma)}(\square)} t^{\widetilde{a}_{(\lambda^-|\gamma)}(\square)+1}}{1 - q^{\ell_{(\lambda^-|\gamma)}(\square)+1} t^{\widetilde{a}_{(\lambda^-|\gamma)}(\square)+1+m_{j-1}(\eta)}}\right]^*$$
$$= \prod_{\square \in \mathrm{C}_{\lambda^-}} \frac{t^{\widetilde{a}_{(\lambda^-|\gamma)}(\square)+1} - q^{\ell_{(\lambda^-|\gamma)}(\square)}}{t^{\widetilde{a}_{(\lambda^-|\gamma)}(\square)+1+m_{j-1}(\eta)} - q^{\ell_{(\lambda^-|\gamma)}(\square)+1}} \cdot \prod_{j=1}^{(\lambda|\gamma)_{n-k}+1} t^{m_{j-1}(\eta)} \tag{6.22}$$

The t-product counts the columns in η which are lower than the newly-added box x, which are equivalently those boxes in $\mathrm{RL}(x)$, so the product can be simplified to $t^{|\mathrm{RL}(x)|}$. Consider $\square = (i,j)$ in C_{λ^-} and the corresponding $\square' = (h,j)$ in $C_{\xi+x,\xi}$. Before we compare arm and leg counts, we note that $m_{j-1}(\eta) = b(\square')$, since the labeled boxes in row j are in the same columns that are nonsymmetric and have height exactly $j-1$ when considered in $(\lambda^- \mid \gamma)$. It is straightforward to see that

$$\mathrm{leg}_{\xi+x}(\square') = \ell_{(\lambda^-|\gamma)}(\square) + 1,$$

since $\mathrm{leg}_{\xi+x}(\square')$ also counts the box x. For the arms,

$$\widetilde{a}_{(\lambda^-|\gamma)}(\square) = \mathrm{arm}_{\xi+x}(\square') - m_{j-1}(\eta),$$

as both arms count symmetric columns of height $j, \ldots, (\widetilde{\lambda} \mid \gamma)_{n-k}$ and nonsymmetric columns of height $j, \ldots, (\lambda^- \mid \gamma)_{n-k}-1$, but only $\mathrm{arm}_{\xi+x}(\square')$ counts nonsymmetric columns of height $j-1$. Thus (6.22) can be converted to

$$t^{|\mathrm{RL}(x)|} \prod_{\square' \in C_{\xi+x,\xi}} \frac{t^{\mathrm{arm}_{\xi+x}(\square)+1-b(\square)} - q^{\mathrm{leg}_{\xi+x}(\square)-1}}{t^{\mathrm{arm}_{\xi+x}(\square)+1} - q^{\mathrm{leg}_{\xi+x}(\square)}},$$

which completes (2).

For (3), we need to account for the remaining missing monomials. According to (6.12), $\widetilde{p}_I = [p_I']^* \cdot x \cdot (w_{t_r})^{-1}$. We also use $\left[\overline{(\widetilde{\lambda} \mid \gamma)_{n-k}}\right]^* = x$. Then comparing everything,

$$\begin{aligned}\left[p_I' \cdot q^{-\widetilde{\mu}_h+1} \cdot \overline{(\widetilde{\lambda} \mid \gamma)_{n-k}}\right]^* &= w_{t_r} \cdot q^{-\widetilde{\mu}_h+1} \cdot \widetilde{p}_I \\ &= t^{c_r} \cdot \widetilde{p}_I,\end{aligned}$$

as $-\widetilde{\mu}_h + 1$ is equal to the row index of w_{t_r}. □

Finally, we observe that in the setting of Theorem 6.13,

$$c_r = n(\mathrm{sort}(\mu, \eta)) - n(\mathrm{sort}(\lambda, \gamma)),$$

and this completes the proof of part (2) of Theorem 4.3.

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