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Financial Markets with Delay

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by

Seyyed Mostafa Mousavi

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I would like to dedicate this dissertation to my parents, Ezat Mousavi and Masoumeh Shokouhi, and my brothers, Saeed and Yahya Mousavi.

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Curriculum Vitæ

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Abstract

Financial Markets with Delay

by

Seyyed Mostafa Mousavi

We propose two models to study different aspects of delay in financial markets. In the first model, we discuss option pricing with delayed information. We study super replication with delayed information in a discrete model and derive its continuous limit. In the second model, we discuss systemic risk using a finite-player linear-quadratic stochastic differential game with delay, where the evolution of log-monetary reserves of banks is described by coupled diffusions driven by controls with delay in their drifts, and banks are minimizing their finite-horizon objective functions.

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Chapter 1

Introduction

In various areas of financial mathematics, a universal assumption is that there does not exist any type of delay. This is mainly because delay adds memory to the system, and in other words, it makes the corresponding stochastic processes non-Markov. As [28] mentions, one of building blocks in the theory of stochastic processes is the Markov property. Therefore, it is difficult to study Non-Markov processes as we no longer have access to the well-developed machinery of Markov stochastic processes.

On the other hand, delay is a common element in many natural systems, and as a result, it is more realistic to consider it in our models. In this endeavor, we study delay in financial markets in this dissertation. Different types of delay can be considered in financial markets. For example, we can have delay in the our flow of information, or we can have control with delay in the state of our system.

We propose two models to study delay in financial markets. In the first model, we attempt to model delay in the flow of information in option pricing. A common assumption is that a trader makes his decisions with full access to the prices of the assets, but there is always a lag between the times of order decision and execution. we start with the binomial model proposed by [10] and consider constant known periods of delay in the information flow. Then, We take the worst case scenario approach (i.e, super replication) to price and replicate contingent claims with convex payoff functions. Also, we discuss the continuous time limit as the time-step and delay length tend to zero. A very important consequence of our model is that it verifies the intuition of traders that delayed information would exaggerate the volatility smile, but it does not cause it.

In the second model, we propose a stochastic game model of inter-bank lending and borrowing where banks can borrow from or lend to a central bank, but they need to take responsibility for their past lending or borrowing. This model generalizes the one in [6]. The evolution of the log-monetary reserves of N banks is described by a system of coupled delayed stochastic differential equations. Each bank try to minimize their finite time objective function by controlling the rate of borrowing or lending. The banks are coupled through the average capitalization. We derive open-loop and closed-loop Nash equilibria. Our results show that the delay reduces the liquidity in the market. However, the central bank still acts as a clearing house.

The dissertation is organized as follows. In chapter 2, we discuss the first model, that is option pricing with delayed information. In chapter 3, we present the model of systemic risk and stochastic games with delay.

Chapter 2

Option Pricing with Delayed Information

2.1 Abstract

We propose a model to study the effects of delayed information on option pricing. We first talk about the absence of arbitrage in our model,and then discuss super replication with delayed information in a binomial model, notably, we present a closed form formula for the price of convex contingent claims. Also, we address the convergence problem as the time-step and delay length tend to zero and introduce analogous results in the continuous time framework. Finally, we explore how delayed information exaggerates the volatility smile. This chapter is a joint work with *Tomoyuki Ichiba*.

2.2 Introduction

All participants in financial markets have access only to delayed information. Delay adds more uncertainty to the market, and it is of great importance to study it. A universal assumption in options pricing literature is that a trader makes his decisions with full access to the prices of the assets (i.e, no delayed information). However, in practice, there is a lag between when the order is decided and its execution time. In particular, there are two important types of delays in financial markets. First is the delay in order execution, that is, the order would be executed with some delay after the trader places it. For example, if the order is made in the morning, it would be executed in the afternoon. Second is the delay in receiving information, that is, the trader observes the prices and other important information with some delay, usually because of the technological barriers, exacerbated by having long physical distance from the exchange.

In the view of traders, these two types of delays act similarly. In both cases, orders are executed with prices which are unknown at the time they are made. In other words, the source of the delayed information does not change the decisions of the trader. For example, let $\{0, 1, \dots\}$ be a discrete trading horizon. If there exists a delay with length of 1 period, then regardless of what the source of delay is, no trade happens at time 0, and in later times trades happen based on the information available up until the previous period. The reason is that if the delay is only in receiving information, then, at time 0 the trader does not have any information, so he waits till time 1 to get time-0 prices to make a trade and those trades would of course be executed with time-1 prices. If the delay is only in order execution, then at time-0 and based on time-0 prices, the trader makes an order, but that order would be executed with time-1 prices.

In this work, we start with the binomial model proposed by [10] and consider fixed periods of delay in the flow of information. Therefore, agents have an information stream smaller than the information flow of the traded asset. We show that the market with delayed information is incomplete, and it is not possible to perfectly replicate most contingent claims. Incomplete markets pose various challenges and for a review of different approaches, we refer to [41]. We take the worst case scenario approach, that is super replication, to price and replicate convex contingent claims. This approach is first suggested by [12] in their seminal paper. We derive recursive and closed-form formulas for pricing convex contingent claims in the discrete time model. Later, we study the continuous time limit as the time-step and delay length tend to zero. We show that the price process under our pricing measure converges to the Black-Scholes price process, but with enlarged volatility.

A very interesting aspect of our model is the way it shows how delayed information affects the volatility smile. Our model confirms the intuition of traders that delayed information would exaggerate the volatility smile, but it does not cause it. We show that in the continuous limit, volatility is constant and there is no smile, but in the discrete model, we can observe volatility smile. In other words, it suggests the idea that the smile observed in the market might not all be by the market itself, and it could have been exaggerated because of the way we interact with delayed information.

Our model with delayed information has some similarities with the models with transaction costs, notably in both models, we encounter similar limit theorems and both risky asset price processes converge to the Black-Scholes price process with some enlarged volatility. In other words, enlarging volatility can be considered as the way to take into account both transaction costs and delayed information. [34] is first to discuss transaction costs in option pricing models. [4] studies transaction costs in binomial models, and [31] provides rigorous limit theorems for such models. For extensive literature on option pricing with transaction costs, we refer to [26].

[27] provides an absence of arbitrage condition in discrete time models with delayed information. [29] studies market viability in scenarios that the agent has delayed or limited information. In the literature, in markets with delayed information, risk-minimizing hedging strategies, which is another hedging approach in incomplete markets, have been studied. Using this approach, [11] models lack of information by letting the assets to be observed only at discrete times, and [40] presents the general case of restricted information. Some other works in this direction are [16], [35], [30] and [7].

The paper is organized as follows. In section 2.3, we set up the discrete time model with delayed information and define the super-replication price. We discuss the super-replicating strategy in an N-period binomial model with $H = N-1$ periods of delay in subsection 2.3.4, and we generalize the results to an N-period binomial model with H periods of delay in subsection 2.3.5 using both dynamic programming and direct approaches. A geometrical representation of the strategy is presented in subsection 2.3.6. In section 2.4, we study the asymptotic behavior of the model as the time step and delay length tend to zero. In particular, subsection 2.4.2 is devoted to the discussion of how delayed information affects the volatility smile.

2.3 Discrete Time Model

Before introducing delays, let us recall the N-period binomial tree model of [10] for a financial market with a single risky asset and a single risk-free asset (e.g., stock). Given $N \in \mathbb{N}$, let us denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space for the canonical space

$\Omega := \{0,1\}^N$ of N-period binomial tree with the Borel σ-algebra $\mathcal{F}$ generated by Ω. For every $\omega := (\omega_1, \ldots, \omega_N) \in \Omega$ we define a coordinate map by $Z_k(\omega) = \omega_k$ for each $k = 1, \ldots, N$. Let $\mathbb{P}$ be the probability measure under which Z_k, $k = 1, \ldots, N$ are independent, Bernoulli random variables with $\mathbb{P}(Z_k = 1) = \mathbb{P}(Z_k = 0) = 1/2$, $k = 1, \ldots, N$. Define the filtration $\mathfrak{F} := \{\mathcal{F}_k, k = 0, \ldots, N\}$, where $\mathcal{F}_k$ is the σ-field $\sigma(Z_1, \ldots, Z_k)$ generated by the first k variables for $k = 1, \ldots, N$ and $\mathcal{F}_0$ is the trivial σ-field, i.e., $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

In the N-period binomial tree model, the risky asset price $S_k : \Omega \to \mathbb{R}$ and its discounted price $\widetilde{S}_k : \Omega \to \mathbb{R}$, discounted by instantaneous rate $r > 0$, at time k, are defined by

$$S_k(\omega) := S_0\, u^{I_k(\omega)}\, d^{k-I_k(\omega)}, \quad I_k(\omega) := \sum_{l=1}^{k} Z_l(\omega), \quad \widetilde{S}_k(\omega) := e^{-rk}\, S_k(\omega) \qquad k = 1, \ldots, N, \tag{2.1}$$

where S_0 is a given initial price of risky asset at time 0, and u (or d) is a fixed ratio by which the price process goes up (or down) in one period with $u > 1 + r > d > 0$. The price processes are adapted to the filtration $\mathfrak{F}$.

2.3.1 Delayed Filtration

We shall introduce delays in the flow of information in the N-period binomial model. For simplicity, let us consider the situation where an investor sends buy or sell orders to the market at time t, but her orders are not executed until time $t + H$ with $H \in \{0, \ldots, N-1\}$ *delay periods*. The investor herself knows that she has H delay periods when she is sending orders. Then we define the *delayed filtration*

$\mathfrak{G} := \{\mathcal{G}_k, k = 0, 1, \ldots, N\}$, where $\mathcal{G}_k := \mathcal{F}_0$, for $k = 0, \ldots, H-1$, and

$$\mathcal{G}_k := \mathcal{F}_{k-H}, \quad k = H, \ldots, N, \tag{2.2}$$

In other words, $\mathcal{G}_k$ is the information set of the price process until time $\min(k-H, 0)$, rather than time k. In the following, we shall consider investments based on this delayed information.

Let $\mathcal{A}_{\mathfrak{G}}$ be the set of all $\mathfrak{G}$-adapted stochastic processes $\Delta := \{\Delta_k, k = 0, \ldots, N-1\}$ with $\Delta_k \equiv 0$, $k = 0, \ldots, H-1$. Here, each $\Delta \in \mathcal{A}_{\mathfrak{G}}$ represents a strategy for this investor based on the delayed information, that is, the positive $\Delta_k > 0$ (the negative $\Delta_k < 0$, respectively) corresponds to the total number of shares of the risky asset that the investor decides to own (to owe, respectively) at time k, given information $\mathcal{G}_k$. In other words, the order made at time $k-H$ to buy or sell $(\Delta_k - \Delta_{k-1})$ shares of the risky asset, gets executed at time k with price S_k (not S_{k-H}), because of H periods of delay. Thus the investor has to deal with the risk of price changes between the time of order submission and execution.

For an initial investment of x_0 in the risk free asset and a strategy $\Delta \in \mathcal{A}_{\mathfrak{G}}$, we shall consider the portfolio value process $V_k(x_0, \Delta)(\omega)$, $k = 0, \ldots, N$, $\omega \in \Omega$. The first order Δ_H submitted at time 0 is executed at time H, and the portfolio value process is not observed until time H. Thus we define

$$\begin{aligned} V_H(x_0, \Delta)(\omega) &:= x_0 \cdot e^{rH} + \Delta_H \cdot S_H(\omega), \\ V_0(x_0, \Delta)(\omega) &:= e^{-rH} \cdot V_H(x_0, \Delta)(\omega) = x_0 + \Delta_H \cdot \widetilde{S}_H(\omega), \end{aligned} \tag{2.3}$$

and in general

$$V_k(x_0,\Delta)(\omega) := \begin{cases} e^{-r(H-k)} \cdot V_H(x_0,\Delta)(\omega)\,, \quad k = 0,\ldots,H-1\,, \\ \\ e^{rk}x_0 + \sum_{l=H}^{k-1} S_l(\omega)\cdot\big(\Delta_{(l-1)\vee H} - \Delta_l\big) + S_k(\omega)\cdot\Delta_{(k-1)\vee H}, \\ \hfill k = H,\ldots,N\,. \end{cases} \tag{2.4}$$

For $k = H,\ldots,N$, the first term in the portfolio value process ($e^{rk}x_0$) in (2.4) corresponds to the initial investment in the risk free asset. The second term ($\sum_{l=H}^{k-1} S_l(\omega)\cdot(\Delta_{(l-1)\vee H} - \Delta_l)$) is due to the cash flow in the risk free asset up until time k, and the third term ($S_k(\omega)\cdot\Delta_{(k-1)\vee H}$) relates to the investment in the risky asset at time k. We call $V_k(x_0,\Delta)$, $k = 0,\ldots,N$ the value process from the strategy $(x_0,\Delta)\in(\mathbb{R},\mathcal{A}_{\mathfrak{G}})$.

By construction, the changes in the portfolio value process ($V_k(x_0,\Delta)$) in (2.4) starting from its first realization at time H, are only due to the variation in asset prices. In other words, no money is added to or withdrawn from the portfolio.

Note that the initial portfolio value $V_0(x_0,\Delta)(\omega)$ in (2.3) is a random variable, not a constant. This is because it is defined by discounting the time-H portfolio value $V_H(x_0,\Delta)(\omega)$, which is the first time the portfolio value is observed due to the existence of delay.

For $k = H,\ldots,N$, Δ_k is $\mathcal{G}_k$-measurable, but $V_k(x_0,\Delta)$ is $\mathcal{F}_k$-measurable. Thus $V_k(x_0,\Delta)$ is $\mathcal{F}_{k\vee H}$-measurable for $k = 0,\ldots,N$. In this sense, the portfolio is constructed based on the delayed information.

2.3.2 Absence of Arbitrage

We shall first introduce the notion of arbitrage in our model. In general, arbitrage means that one cannot reap any benefit for free, that is without taking any risk. In our model with delayed information, as it is shown in (2.3), the initial portfolio value $V_0(x_0, \Delta)$ is a random variable, because of the existence of delay. Therefore, we need to adjust the classical notion of arbitrage in the domain of $(\mathbb{R}, \mathcal{A}_{\mathfrak{G}})$ strategies, to take this into account.

Definition 1 (Arbitrage)**.** *An arbitrage opportunity is the strategy* $(x_0, \Delta) \in (\mathbb{R}, \mathcal{A}_{\mathfrak{G}})$ *such that*

$$\begin{aligned} \max_{\omega\in\Omega}\{V_0(x_0,\Delta)\,(\omega)\} &= 0, \\ \mathbb{P}(V_N(x_0,\Delta) \geq 0) &= 1, \\ \mathbb{P}(V_N(x_0,\Delta) > 0) &> 0. \end{aligned} \tag{2.5}$$

The primary difference with the classical definition of arbitrage is the condition that the *maximum* of time-0 portfolio value needs to be zero ($\max_{\omega\in\Omega}\{V_0(x_0,\Delta)\,(\omega)\} = 0$). It is obvious that in the case of complete information (i.e, $H = 0$), this definition boils down to the classical definition of arbitrage opportunity.

We need to show that there is no arbitrage in our discrete time model with delayed information. [27] proves that in a general discrete time model with restricted information, there does not exist classical arbitrage, if and only if there exists a probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ such that the optional projection under $\widetilde{\mathbb{P}}$ of the discounted stock price on the delayed filtration, is a $\widetilde{\mathbb{P}}$-martingale The setup of our model is a bit different than that in [27], given that our first order to buy/ sell the risky asset is executed at time H, rather than at time 0 (i.e. $\Delta_k = 0$, $k = 0, \dots, H-1$). This makes the initial portfolio value ($V_0(x_0,\Delta)$) a random variable, rather than always a constant. Theorem 1 shows that still in our model, there does not exists arbitrage, in the sense of Definition 1.

Theorem 1. *There does not exists any arbitrage opportunity in our discrete time model, in the domain of* $(\mathbb{R},\mathcal{A}_{\mathfrak{G}})$ *strategies.*

Proof. According to Definition 1, absence of arbitrage means that for any strategy $(x_0,\Delta)\in(\mathbb{R},\mathcal{A}_{\mathfrak{G}})$ such that $\max_{\omega\in\Omega}\{V_0(x_0,\Delta)(\omega)\}=0$, the condition $\mathbb{P}(V_N(x_0,\Delta)\geq 0)=1$ implies that $\mathbb{P}(V_N(x_0,\Delta)=0)=1$.

In the domain of $(\mathbb{R},\mathcal{A}_{\mathfrak{G}})$, according to (2.3), the condition $\max_{\omega\in\Omega}\{V_0(x_0,\Delta)(\omega)\}=0$ is equivalent to

$$\max_{\omega\in\Omega}\{V_H(x_0,\Delta)(\omega)\} \;=\; 0.$$

Which means that in all $(N-H)$-period binomial models starting from time H, the initial values for the (x_0,Δ) strategy is non-positive.

If we consider all these $(N-H)$-period binomial models individually, they lie in the general discrete time model framework in [27]. Therefore, in each of these models, even if we consider the initial values of the strategy to be zero, the condition $\mathbb{P}(V_N(x_0,\Delta)\geq 0)=1$ implies $\mathbb{P}(V_N(x_0,\Delta)=0)=1$, given that we show that there exists a probability measure $\widetilde{\mathbb{P}}\sim\mathbb{P}$ such that the $\widetilde{\mathbb{P}}$-optional projection of the discounted stock price on the delayed filtration, is a $\widetilde{\mathbb{P}}$-martingale That is

$$\mathbb{E}^{\widetilde{\mathbb{P}}}\left(\widetilde{S}_{k+1}|\mathcal{G}_k\right)=\mathbb{E}^{\widetilde{\mathbb{P}}}\left(\widetilde{S}_k|\mathcal{G}_k\right),\quad k=H,\dots,N-1. \tag{2.6}$$

Define the probability measure $\widetilde{\mathbb{P}}$ such that the coordinate maps $Z_k, k=1,\dots,N$ are still independent Bernoulli random variables, but with parameters

$$\widetilde{\mathbb{P}}(Z_k = 1) = \frac{ue^r - d}{u - d} = 1 - \widetilde{\mathbb{P}}(Z_k = 0), \quad k \in \{1, \dots, N\}.$$

which are the risk-neutral probabilities in the usual binomial model without any delay.

Given that the discounted stock price $(\widetilde{S}_k)$ is $(\mathcal{F}_k)$-martingale under $\widetilde{\mathbb{P}}$, it follows that condition (2.6) holds, which shows that there is not any arbitrage opportunity from time H to N. Consequently, given (2.3), we conclude that there is not arbitrage in the model in the domain of $(\mathbb{R}, \mathcal{A}_{\mathfrak{G}})$ strategies. □

Remark 1. *The domain of* $(\mathbb{R}, \mathcal{A}_{\mathfrak{G}})$ *strategies in Theorem (2.3.2) does not include all* $\mathfrak{F}$*-adapted strategies, but only those which are* $\mathfrak{G}$*-adapted. In other words, we are excluding the case that an agent with full information come and exploit the market with delayed information. If we include all* $\mathfrak{F}$*-adapted strategies, it is likely to have arbitrage opportunities.*

2.3.3 Super-Replication Price

Given that there is no arbitrage in the market, it now makes sense to discuss about pricing.

Definition 2 (Super-replication price and the value process of super-replicating portfolio)**.** *For any contingent claim with payoff function* $\varphi : \Omega \to \mathbb{R}$ *and expiration time* N *, its super-replication price* $\bar{\pi}(\varphi)$ *is defined as the minimal initial value of portfolio which exceeds the value* φ *at time* N *, i.e.,*

$$\bar{\pi}(\varphi) := \inf_{(x_0, \Delta) \in \Gamma} \max_{\omega \in \Omega} \left\{ V_0(x_0, \Delta)(\omega) = x_0 + \Delta_H \widetilde{S}_H(\omega) \right\}, \tag{2.7}$$

where

$$\Gamma := \{(x_0, \Delta) \in \mathbb{R} \times \mathcal{A}_{\mathfrak{G}} : V_N(x_0, \Delta) \geq \varphi \;\; \mathbb{P} - a.s.\} \,. \tag{2.8}$$

If there exists a pair (x_0^*, Δ^*) *that attains the infinitum in (2.7), i.e.,* $\bar{\pi}(\varphi) = \max_{\omega\in\Omega} V_0(x_0^*, \Delta^*)(\omega)$ *, then the time-*k *super replicating portfolio value* $\mathcal{V}_k(\omega)$ *is defined as*

$$\mathcal{V}_k(\omega) := V_k(x_0^*, \Delta^*)(\omega)\,, \quad k = 0, \ldots, N\,, \tag{2.9}$$

and consequently, $\bar{\pi}(\varphi) = \max_{\omega\in\Omega} \mathcal{V}_0(\omega)$ *.*

Remark 2. *The super-replication price is the most conservative pricing approach for the seller of the option, considering the worst-case scenario. In other words, it is straightforward to show that any price greater than the super-replication price causes arbitrage in the market.*

Remark 3. *It is remarkable to note that call-put parity does not hold anymore. The reason is that the super-replication price* $\bar{\pi}$ *is a coherent risk measure on the space* $\mathbb{L}^\infty(\Omega, \mathfrak{F}, \mathbb{P})$ *of payoff functions, and therefore it is sub additive*

All of the results in this paper are for European-style contingent claims with convex payoff functions. In section 2.3.4, we consider first the case $H = N-1$ and determine the super-replication price and the corresponding strategy. This would make the building block for the general case discussed in section 2.3.5. The case for non-convex payoff functions is computationally more demanding as we do not have access to all the machinery developed for convex functions.

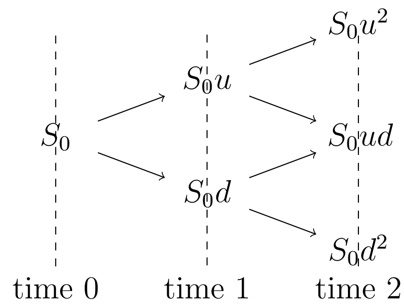

Figure 2.1: Asset price process S_k in a 2-period binomial model

2.3.4 An N-period binomial model with $H = N - 1$ periods of delay

We determine the super-replication price and the corresponding strategy for the European contingent claims when $H = N - 1$. Having $H = N - 1$ periods of delayed information means that at time 0 the risky asset price S_0 is observed, but the order Δ_H, sent by the investor at time 0, would be executed at time H. For example, when $N = 2$ and $H = 1$, the order Δ_1 sent at time 0 is executed at time 1 with two possible prices $S_1 = S_0 d$ or $S_1 = S_0 u$ (see Figure 2.1).

Let us observe that in the case of $H = N - 1$, the terminal value $V_N(x_0, \Delta)$ in (2.4) is simplified to

$$V_N(x_0, \Delta)(\omega) = e^{rN} x_0 + S_N(\omega) \cdot \Delta_{N-1} . \tag{2.10}$$

There are $(N + 1)$ possible values of $S_N(\omega)$, $\omega \in \Omega$ in (2.1) and there are only two controls (x_0, Δ_{N-1}) in the terminal value. Since there are $(N + 1)$ constraints and only two controls, the minimization problem in (2.7) has possibly infinitely many solutions. In other words, in an economic sense, the market is not complete. To learn more about pricing in incomplete markets, we refer to [41].

Theorem 2. *For a European-style contingent claim with payoff* $\varphi := \Phi(S_N)$ *for some convex function* $\Phi(\cdot)$ *in the* N*-period binomial model with* $H = N-1$ *periods of delay, the super-replication price is*

$$\bar{\pi}(\varphi) = \max\left(x_0^* + e^{-rH}\,\Delta_H^* \cdot S_0\,u^H\,,\, x_0^* + e^{-rH}\,\Delta_H^* \cdot S_0\,d^H\right), \tag{2.11}$$

where the corresponding strategy (x_0^*, Δ^*) *is given by* $\Delta_j^* \equiv 0\,,\ j = 0, 1, \ldots, H-1\,,$

$$\Delta_H^* = \Delta_{N-1}^* = \frac{\Phi(S_0u^N) - \Phi(S_0d^N)}{S_0 \cdot (u^N - d^N)} \quad \textit{and} \quad x_0^* = e^{-rN} \cdot \frac{u^N\Phi(S_0d^N) - d^N\Phi(S_0u^N)}{u^N - d^N}\,. \tag{2.12}$$

Proof. First, we shall prove that for any $\omega \in \Omega\,,\ (x_0^*, \Delta^*)$ in (2.12) satisfies

$$\inf_{(x_0,\Delta)\in\Gamma} \left\{V_0(x_0,\Delta)(\omega) = x_0 + \Delta_H \widetilde{S}_H(\omega)\right\} = V_0(x_0^*, \Delta^*)(\omega) = x_0^* + \Delta_H^* \cdot \widetilde{S}_H(\omega). \tag{2.13}$$

Here the infimum is taken over the set Γ in (2.8), that is, $x_0 \in \mathbb{R}$ and $\Delta \in \mathcal{A}_{\mathfrak{G}}$ must satisfy $V_N(x_0, \Delta) \geq \varphi(S_N)$ almost surely. Note that $V_N(x_0, \Delta) = (e^{rN}x_0 + x \cdot \Delta_{N-1})|_{x=S_N}$ in (2.10) is realized as the value at $x = S_N$ of linear function $y = e^{rN}\,x_0 + x\cdot\Delta_{N-1}$ with the slope Δ_H and the y-intercept $e^{rN}x_0$ in the (x, y) coordinates. Moreover, since the payoff function $\Phi(\cdot)$ is convex, by Jensen's inequality, one can verify

$$\begin{aligned}\Gamma = \{(x_0, \Delta) \in \mathbb{R} \times \mathcal{A}_{\mathfrak{G}} \,:\, & e^{rN}x_0 + S_0u^N \cdot \Delta_{N-1} \geq \Phi(S_0u^N), \\ & e^{rN}x_0 + S_0d^N \cdot \Delta_{N-1} \geq \Phi(S_0d^N)\}\,.\end{aligned} \tag{2.14}$$

That is, in order to check whether the inequality $V_N(x_0, \Delta) \geq \Phi(S_N)$ holds with probability one, it suffices to check it just at the extreme cases, in which the asset price S_N at time N is the minimum $S_0 d^N$ or the maximum $S_0 u^N$ in the binomial tree model. Then it is easy to check that the choice (x_0^*, Δ_H^*) in (2.12) belongs to the set Γ as we have $e^{rN} x_0^* + \Delta_H^* S_0 u^N = \varphi(S_0 u^N)$, $e^{rN} x_0^* + \Delta_H^* S_0 d^N = \varphi(S_0 d^N)$. In other words, the minimization problem is reduced to a linear programming problem

$$\begin{aligned} &\underset{(x_0,\Delta_H)\in\mathbb{R}^2}{\text{minimize}}\ x_0 + \Delta_H \cdot \widetilde{S}_H(\omega) \\ &\text{subject to} \quad e^{rN} x_0 + S_0 u^N \cdot \Delta_H \ \geq\ \Phi(S_0 u^N)\,, \\ &\ \text{and} \quad e^{rN} x_0 + S_0 d^N \cdot \Delta_H \ \geq\ \Phi(S_0 d^N)\,. \end{aligned}$$

Define the Lagrangian as

$$\begin{aligned} \mathcal{L} \ :=\ & x_0 + \Delta_H \widetilde{S}_H(\omega) + \lambda_1[\Phi(S_0 u^N) - \big(e^{rN} x_0 + S_0 u^N \Delta_H\big)] \\ & + \lambda_2[\Phi(S_0 d^N) - \big(e^{rN} x_0 + S_0 d^N \Delta_H\big)]\,, \end{aligned}$$

where λ_1 and λ_2 are the Lagrangian multipliers. Then, it is easy to check that the quantities

$$\begin{aligned} x_0^* \ &=\ e^{-rN} \cdot \frac{u^N \Phi(S_0 d^N) - d^N \Phi(S_0 u^N)}{u^N - d^N}\,, \quad \Delta_H^* \ =\ \frac{\Phi(S_0 u^N) - \Phi(S_0 d^N)}{S_0 u^N - S_0 d^N}\,, \\ \lambda_1^* \ &=\ \frac{\widetilde{S}_H(\omega) - e^{-rN} S_0 d^N}{S_0 \cdot (u^N - d^N)}\,, \quad \lambda_2^* \ =\ \frac{e^{-rN} S_0 u^N - \widetilde{S}_H(\omega)}{S_0 u^N - S_0 d^N} \end{aligned}$$

satisfy the Karush-Kuhn-Tucker conditions for the minimization. Hence, (2.13) follows, and it is the key to prove that

$$\inf_{(x_o,\Delta)\in\Gamma} \max_{\omega\in\Omega} V_0(x_0,\Delta)(\omega) \ =\ \max_{\omega\in\Omega} \inf_{(x_o,\Delta)\in\Gamma} V_0(x_0,\Delta)(\omega)\,. \tag{2.15}$$

Thus, we get

$$\bar{\pi}(\varphi) \;=\; \max_{\omega\in\Omega}\ \inf_{(x_o,\Delta)\in\Gamma} V_0(x_0,\Delta)(\omega) \;=\; \max_{\omega\in\Omega} V_0(x_0^*,\Delta^*)(\omega)\,. \tag{2.16}$$

Then, the proof is completed by the following observation

$$\max_{\omega\in\Omega}\{V_0(x_0^*,\Delta^*)(\omega) = x_0^* + \Delta_H^*\widetilde{S}_H(\omega)\} = \\ \max\left(x_0^* + e^{-rH}\Delta_H^*\cdot S_0u^H, x_0^* + e^{-rH}\Delta_H^*\cdot S_0d^H\right). \tag{2.17}$$

□

By using Theorem 2, the portfolio value $\mathcal{V}_H \in \mathfrak{F}_H$ in (2.9) at time H of the super-replicating strategy can be calculated as

$$\begin{aligned}\mathcal{V}_H \;=\; e^{rH}x_0^* + \Delta_H^*\cdot S_H \;&=\; \sum_{j=0}^{H} e^{-r(N-H)}\,\mathbb{E}^{\mathbb{Q}_j}[\Phi(S_N)]\cdot \mathbb{1}_{\{S_H = S_0\,u^j\,d^{H-j}\}} \\ &=\; \sum_{j=0}^{H} e^{-r(N-H)}\left[\,\mathfrak{p}_j\,\Phi(S_0u^N) + \mathfrak{q}_j\,\Phi(S_0d^N)\,\right]\cdot \mathbb{1}_{\{S_H = S_0\,u^j\,d^{H-j}\}}.\end{aligned} \tag{2.18}$$

Here $\{\mathbb{Q}_j\}_{j=0}^{H}$ are probability measures on $(\Omega,\mathfrak{F})$ defined by

$$\begin{aligned}&\mathbb{Q}_j(I_N \;=\; N) \;:=\; \mathfrak{p}_j \;=\; 1-\mathbb{Q}_j(I_N\;=\;0) \;=\; 1-\mathfrak{q}_j\,, \\ &\mathfrak{p}_j \;:=\; \frac{u^jd^{H-j}e^{r(N-H)} - d^{H+1}}{u^{H+1}-d^{H+1}}\,,\;\; j \;=\; 0,\ldots,H\,.\end{aligned} \tag{2.19}$$

Remark 4. *We can conclude from the form in (2.18) that $\mathcal{V}_H$, the value of the* *super-replicating portfolio at time H, is a function of S_0 and S_H. In other words*

$$\mathcal{V}_H \equiv \mathcal{V}_H(S_0, S_H),$$

Therefore, the value process for the super-replicating portfolio is path dependent*, due to the existence of H periods of lag between the times of order submission and execution.*

Thus, the super-replication price $\bar{\pi}(\varphi)$ can be calculated as

$$\begin{aligned}
\bar{\pi}(\varphi) &= \max_{\omega\in\Omega} \mathcal{V}_0(\omega) = \max_{j\in\{0,\dots,H\}} e^{-rN}\,\mathbb{E}^{\mathbb{Q}_j}[\Phi(S_N)] = \max_{j\in\{0,H\}} e^{-rN}\,\mathbb{E}^{\mathbb{Q}_j}[\Phi(S_N)] \\
&= \max_{j\in\{0,H\}} e^{-rN}\left[\mathfrak{p}_j\Phi(S_0u^N) + \mathfrak{q}_j\Phi(S_0d^N)\right] \\
&= e^{-rN}\max\left(\mathfrak{p}_u\Phi(S_0u^N) + \mathfrak{q}_u\Phi(S_0d^N)\,,\ \mathfrak{p}_d\Phi(S_0u^N) + \mathfrak{q}_d\Phi(S_0d^N)\right)\ . \qquad (2.20)
\end{aligned}$$

where the third equality follows similarly as in (2.17).

Notation: From now on, we use $(\mathfrak{p}_u, \mathfrak{q}_u)$ as $(\mathfrak{p}_H, \mathfrak{q}_H)$ and $(\mathfrak{p}_d, \mathfrak{q}_d)$ as $(\mathfrak{p}_0, \mathfrak{q}_0)$, since $(\mathfrak{p}_H, \mathfrak{q}_H)$ and $(\mathfrak{p}_0, \mathfrak{q}_0)$ correspond to the measures at the extreme points $S_H = S_0u^H$ and $S_H = S_0d^H$ respectively.

2.3.5 An N-Period binomial Model with H Periods of Delay

We extend our considerations from section 2.3.4 and generalize the model to the N-period binomial model with $H(\leq N-1)$ periods of delay. We determine the super-replication price and the corresponding strategy for European style contingent claims with convex payoff functions. Here we shall solve the problem from both a dynamic programming (or backward induction) approach and a direct approach.

Dynamic Programming Approach

First, let us define the tree $\mathcal{T}_N(0,0)$ of length N as the set of nodes (i,j), such that there are i ups and j downs from the node $(0,0)$ with $0 \le i+j \le N$, i.e.,

$$\mathcal{T}_N(0,0) \,:=\, \{(i,j) \in \mathbb{N}_0^2 \,:\, 0 \le i+j \le N\}$$

Then define its $(H+1)$-period subtree $\mathcal{T}_{H+1}(a,b)$ starting from the node (a,b) at time $a+b$ by

$$\mathcal{T}_{H+1}(a,b) \,:=\, \{(i,j) \in \mathbb{N}_0^2 \,:\, a+b \le i+j \le a+b+H+1\,,\, i \ge a\,,\, j \ge b\}$$

for every $(a,b) \in \mathcal{T}_N(0,0)$ such that $a+l \le N-(H+1)$.

We shall identify all $N-H$ subtrees $\mathcal{T}_{H+1}(a,b)$ starting from the nodes (a,b) at time $N-(H+1)$ (i.e, $a+b = N-(H+1)$). We use the results in section 2.3.4 and consider the value process of the super-replicating portfolio at time $N-1$ as the new payoffs for the next round of $(H+1)$-period subtrees starting from the nodes at time $N-(H+2)$. Then, we keep super-replicating backwards in the same manner.

Remark 5. *Given that in the dynamic programming approach, we are using the results in section (2.3.4) in each step, and Remark (4), we can conclude that the value process at level* $k \in \{H,\ldots,N\}$ *for the super-replicating strategy in the general model is also path dependent, that is*

$$\mathcal{V}_k \,\equiv\, \mathcal{V}_k(S_{k-H}, S_k), \quad k \,=\, H,\ldots,N\,. \tag{2.21}$$

Therefore, let us define the payoff for the subtree $\mathcal{T}_{H+1}(a,b)$ starting from the node (a,b) at time $a+b$ at its leaf node (p,q) (i.e, $p+q=a+b+H+1$) by

$$\Phi_{\mathcal{T}_{H+1}(a,b)}\,(p,q) \;:=\; \begin{cases} \mathcal{V}_{p+q}\left(S_{a+b}d\,,\; S_{a+b}d^{H+1} \right) & \text{if } p \,=\, a\,; \\[2ex] \max\Big\{ \mathcal{V}_{p+q}\left(S_{a+b}d\,,\; S_{a+b}u^i d^{H+1-i} \right), \\ \qquad\qquad \mathcal{V}_{p+q}\left(S_{a+b}u\,,\; S_{a+b}u^i d^{H+1-i} \right) \Big\} \\ \qquad\qquad\qquad \text{if } p \,=\, a+i\,,\; i \,=\, 1,\ldots,H\,; \\[2ex] \mathcal{V}_{p+q}\left(S_{a+b}u\,,\; S_{a+b}u^{H+1} \right) & \text{if } p \,=\, a+H+1\,; \end{cases} \tag{2.22}$$

for $p+q \,\leq\, N-1$, and $\Phi_{\mathcal{T}_{H+1}(a,b)}\,(p,q) := \Phi(S_N)\,,\; p+q \,=\, N$ where $S_N \,=\, S_0u^pd^q$.

Intuitively, for the subtree $\mathcal{T}_{H+1}\,(a,b)$ starting at time $a+b$, there are only two $(H+1)$-period subtrees, $\mathcal{T}_{H+1}\,(a+1,b)$ and $\mathcal{T}_{H+1}\,(a,b+1)$, starting at time $a+b+1$ that can induce payoff at time $p+q$. So, we need to take the maximum of the two possible value process as the new payoff because we always consider worst case scenario in super replication. Note that at the edge points, there exists only one value process.

Example 1. *In the* 4*-period binomial tree model (as in Figure 2.2) with* $H=1$*, what new payoff we need to consider on the node* $S_3 \,=\, S_0u^2d$ *depends on whether we are considering this node as part of the subtree* $\mathcal{T}_2\,(1,0)$ *or* $\mathcal{T}_2\,(0,1)$*. As part of the subtree* $\mathcal{T}_2\,(1,0)$*, the payoff* $\left(\Phi_{\mathcal{T}_2(1,0)}\,(2,1)\right)$ *would be the maximum of the corresponding value processes of the subtrees* $\mathcal{T}_2\,(1,1)$ *and* $\mathcal{T}_2\,(2,0)$*, while as part of the subtree* $\mathcal{T}_2\,(0,1)$*, the payoff* $\left(\Phi_{\mathcal{T}_2(0,1)}\,(2,1)\right)$ *would be the corresponding value processes of the subtrees* $\mathcal{T}_2\,(1,1)$*.*

One important ingredient in the dynamic programming approach is that when we start from a convex payoff function, the payoff in (2.22) for all the intermediary $(H+1)$-period subtrees needs to be convex with respect to the corresponding risky asset prices, in order to be able to use Theorem (2) in each step and keep super-replicating backwards. Theorem (3) formalizes this relation.

Theorem 3. *For a European-style contingent claim with payoff $\varphi := \Phi(S_N)$ for some convex function $\Phi(\cdot)$ in the N-period binomial model with $H \le N-1$ periods of delay, the payoff function $\Phi_{\mathcal{T}_{H+1}(a,b)}(.,.)$, $a+b = 0,\dots,N-(H+1)$ in (2.22) for all the intermediary $(H+1)$-period subtrees are convex with respect to the corresponding risky asset prices.*

Proof. Note that for $a+b = N-(H+1)$, the payoff functions $\Phi_{\mathcal{T}_{H+1}(a,b)}(.,.)$ for all $(N-H)$ intermediary $(H+1)$-period subtrees are convex, since the final payoff function $\Phi(S_N)$ is convex.

Now we show that all the payoff functions $\Phi_{\mathcal{T}_{H+1}(a',b')}(.,.)$, $a'+b' = a+b-1$ will be convex, if all the payoff functions $\Phi_{\mathcal{T}_{H+1}(a,b)}(.,.)$, $a+b \in \{0,\dots,N-(H+1)\}$ are convex. By induction this completes the proof.

Given that the payoff function $\Phi_{\mathcal{T}_{H+1}(a,b)}(.,.)$ is convex, by Theorem (2), there exists x_1^* and Δ_1^* such that we define

$$h_1(t) := \begin{cases} \mathcal{V}_{a+b+H}(\, S_{a'+b'}u \,,\, S_{a+b+H}\,) = e^{rH}x_1^* + \Delta_1^* t, \\ \hfill t \in \{S_{a'+b'}ud^H,\dots,S_{a'+b'}u^{H+1}\}; \\ \\ e^{rH}x_1^* + \Delta_1^* t, \quad t = S_{a'+b'}d^{H+1}, \end{cases}$$

Similarly, there exists x_2^* and Δ_2^* such that we define

$$h_2(t) := \begin{cases} \mathcal{V}_{a+b+H}(\, S_{a'+b'}d \,,\, S_{a+b+H}\,) = e^{rH}x_2^* + \Delta_2^* t, \\ \qquad\qquad\qquad\qquad t \in \{S_{a'+b'}d^{H+1}, \ldots, S_{a'+b'}u^H d\}; \\ \\ e^{rH}x_2^* + \Delta_2^* t, \quad t = S_{a'+b'}u^{H+1}, \end{cases}$$

we can define

$$h(t) \;:=\; \max\left(h_1(t), h_2(t)\right), \quad t \in \{S_{a'+b'}d^{H+1}, \ldots, S_{a'+b'}u^{H+1}\}; \tag{2.23}$$

Note that $h(t) = \Phi_{\mathcal{T}_{H+1}(a',b')}\,(p,q)$ where $t := S_0 u^p d^q$, given that $\Phi_{\mathcal{T}_{H+1}(a,b)}\,(.,.)$ is convex, and (2.22).

The discrete function $h(.)$ is convex if for any v and w such that $S_0 u^v d^w \in \{S_{a'+b'}ud^H, S_{a'+b'}u^H d\}$, we have

$$h\left(t_p\right) + h\left(t_n\right) \geq 2h\left(t_m\right). \tag{2.24}$$

where $t_p := S_0 u^{v-1} d^{w+1}$, $t_n := S_0 u^{v+1} d^{w-1}$ and $t_m := S_0 u^v d^w$.

Depending on the choice of v and w, there are 4 cases:

Case 1: $h(t_p) = h_1(t_p)$ and $h(t_n) = h_1(t_n)$. Then, given the form in (2.23), we have $h(t_m) = h_1(t_m)$. Then, it is straightforward to show that (2.24) follows by linearity of the function $h_1(t_m)$.

Case 2: $h(t_p) = h_2(t_p)$ and $h(t_n) = h_2(t_n)$. This case follows similar to that of case 1.

Case 3: $h(t_p) = h_1(t_p)$ and $h(t_n) = h_2(t_n)$. Then, $h(t_m)$ would equal to either $h_1(t_m)$ or $h_2(t_m)$. Without loss of generality assume that $h(t_m) = h_1(t_m)$. Then given that

$h(t_n) = h_2(t_n)$, we conclude by the form in (2.23) that $h_2(t_n) \geq h_1(t_n)$. So, we derive

$$h\left(t_p\right) + h\left(t_n\right) \;=\; h_1\left(t_p\right) + h_2\left(t_n\right) \;\geq\; h_1\left(t_p\right) + h_1\left(t_n\right) \;\geq\; 2h_1\left(t_m\right) \;=\; 2h\left(t_m\right).$$

Where the last inequality follows by the linearity of the $h_1(.)$ function.

Case 4: $h(t_p) = h_2(t_p)$ and $h(t_n) = h_1(t_n)$. This case follows similar to that of case 3. □

Therefore, given Theorem (3), we can apply the dynamic programming approach, and derive the portfolio value $\mathcal{V}_k(S_{k-H}, S_k)$ in (2.21) at level $k = a+b+H$, $k \in \{H, \ldots, N-1\}$ of the super-replicating strategy, using representation (2.18), as

$$\begin{aligned}\mathcal{V}_k(S_0 u^a d^b, S_k) \;=\; & \sum_{j=0}^{H} e^{-r}\Big[\mathfrak{p}_j \Phi_{\mathcal{T}_{H+1}(a,b)}\left(a+H+1, b\right) \\ & +\mathfrak{q}_j \Phi_{\mathcal{T}_{H+1}(a,b)}\left(a, b+H+1\right)\Big]\mathbb{1}_{\{S_k = S_{k-H}u^j d^{H-j}\}}, \quad k = H, \ldots, N-1.\end{aligned} \tag{2.25}$$

where $\mathfrak{p}_j$ and $\mathfrak{q}_j$, $j = 0, \ldots, H$ are defined as in (2.19).

Plugging in (2.22) for $k = H, \ldots, N-2$, we obtain the key recursive formula

$$\begin{aligned}\mathcal{V}_k(S_{k-H}, S_k) \;=\; & \sum_{j=0}^{H} e^{-r}\Big[\mathfrak{p}_j \mathcal{V}_{k+1}(\, S_{k-H}u \,,\, S_{k-H}u^{H+1}) \\ & +\mathfrak{q}_j \mathcal{V}_{k+1}(\, S_{k-H}d \,,\, S_{k-H}d^{H+1}\,)\Big]\mathbb{1}_{\{S_k = S_{k-H}u^j d^{H-j}\}}, \quad k = H, \ldots, N-2.\end{aligned} \tag{2.26}$$

Remark 6. *We can conclude that, when we are super-replicating backwards, the value process $\mathcal{V}_k(S_{k-H}, S_k)$ in (2.21) is only required at the two extreme points $S_k = S_{k-H}u^H$*

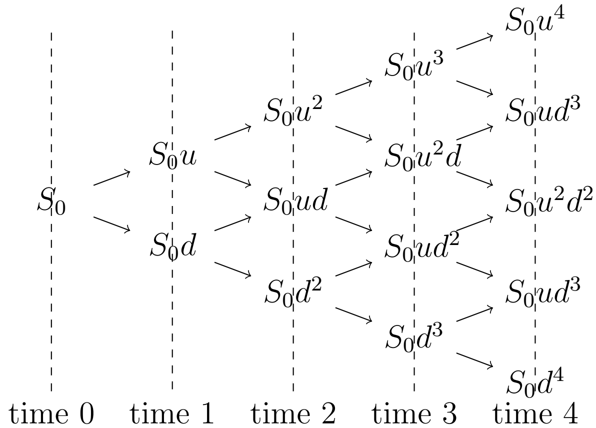

Figure 2.2: Asset price process S_k in a 4-period binomial model

and $S_k = S_{k-H}d^H$*, because of the form on the right hand side of the recursive formula (2.26). In other words, we just use* $(\mathfrak{p}_u, \mathfrak{q}_u) = (\mathfrak{p}_H, \mathfrak{q}_H)$ *and* $(\mathfrak{p}_d, \mathfrak{q}_d) = (\mathfrak{p}_0, \mathfrak{q}_0)$.

Therefore, similar to (2.20), the super-replication price $\bar{\pi}(\varphi)$ can be finally calculated as

$$\bar{\pi}(\varphi) = e^{-rH} \max\left(\mathcal{V}_H(S_0, S_0u^H), \mathcal{V}_H(S_0, S_0d^H)\right). \tag{2.27}$$

Direct Approach

In this section, we solve the recursive equation (2.26) and obtain the value process $\mathcal{V}_k(S_{k-H}, S_k)$ for the super-replicating strategy explicitly. As Remark (6) suggests, when we super-replicate backwards, we just need the value process at the extreme points, that is $\mathcal{V}_k(S_{k-H}, S_{k-H}u^H)$ and $\mathcal{V}_k(S_{k-H}, S_{k-H}d^H)$, $k = H, \ldots, N-1$.

Define probability spaces $(\Omega_k, \mathcal{F}_k, \mathbb{Q}_k)$ for $k = H, \ldots, N-1$ with $\Omega_k = \{0,1\}^{\widetilde{N}+H}$, the Borel σ-algebra $\mathcal{F}_k$ on Ω_k, and $\widetilde{N} = N - k$. For every $\omega_k = (\omega_{k,1}, \ldots, \omega_{k,\widetilde{N}+H}) \in \Omega_k$, we define a coordinate map by $Z_{k,m}(\omega_k) = \omega_{k,m}$ for each $m \in \{1, \ldots, \widetilde{N}+H\}$.

Let $\mathbb{Q}_k$ be the probability measure under which $Z_{k,m}, m = 1, \ldots, \widetilde{N}+H$ with initial position $Z_{k,0}$ is a Markov chain, and for $l = 1, \dots, \widetilde{N} - 1$, it has transition matrix

$$Q = \begin{pmatrix} \mathfrak{q}_d & \mathfrak{p}_d \\ \mathfrak{q}_u & \mathfrak{p}_u \end{pmatrix} \quad \text{on } \{0, 1\}, \tag{2.28}$$

Besides, for $l = \widetilde{N}, \dots, \widetilde{N} + H$,

$$\begin{aligned} \mathbb{Q}_k \left(Z_{k,\widetilde{N}+H} = \cdots = Z_{k,\widetilde{N}} = 1 | Z_{k,\widetilde{N}-1} = 1 \right) &= \mathfrak{p}_u, \\ \mathbb{Q}_k \left(Z_{k,\widetilde{N}+H} = \cdots = Z_{k,\widetilde{N}} = -1 | Z_{k,\widetilde{N}-1} = 1 \right) &= \mathfrak{q}_u, \\ \mathbb{Q}_k \left(Z_{k,\widetilde{N}+H} = \cdots = Z_{k,\widetilde{N}} = 1 | Z_{k,\widetilde{N}-1} = 0 \right) &= \mathfrak{p}_d, \\ \mathbb{Q}_k \left(Z_{k,\widetilde{N}+H} = \cdots = Z_{k,\widetilde{N}} = -1 | Z_{k,\widetilde{N}-1} = 0 \right) &= \mathfrak{q}_d. \end{aligned} \tag{2.29}$$

The risky asset price S_{k-H+m} satisfies

$$S_{k-H+m} := S_{k-H} u^{I_{k,m}} d^{m-I_{k,m}}, \quad I_{k,m} = \sum_{l=1}^{m} Z_{k,l}, \quad m = 1, \dots, \widetilde{N} + H\,, \tag{2.30}$$

Remark 7. *Under measures $\mathbb{Q}_k$, $k = H, \dots, N-1$, $\mathfrak{p}_u$ is the probability of an upward move preceded with an upward move, $\mathfrak{q}_u$ is the probability of a downward move preceded with an upward move, $\mathfrak{p}_d$ is the probability of an upward move preceded with a downward move, and $\mathfrak{q}_d$ is the probability of a downward move preceded with a downward move. Besides, equations (2.29) are to ensure that the last $H+1$ moves are all either upward or downward.*

Remark 8. *Under measures $\mathbb{Q}_k$, $k = H, \dots, N-1$, probability of a downward move preceded by a downward move ($\mathfrak{q}_d$) is higher than the probability of a downward move preceded by an upward move ($\mathfrak{q}_d$). Similar is also true for upward moves. So, the vari-*

ance of the risky asset price is higher under these measures than the initial measure $\mathbb{P}$.

Remark 9. *If we put* $H = 0$*, the transition matrix (2.28) would have duplicate rows (i.e.* $\mathfrak{p}_u = \mathfrak{p}_d$ *and* $\mathfrak{q}_u = \mathfrak{q}_d$*). Therefore, in this case, the model boils down to the binomial tree model of [10], and all the equations get significantly simplified accordingly.*

Theorem (4) expresses $\mathcal{V}_k(S_{k-H}, S_{k-H}u^H)$ and $\mathcal{V}_k(S_{k-H}, S_{k-H}d^H)$, $k = H, \ldots, N-1$ as expectations under the measure $\mathbb{Q}_k$.

Theorem 4. *For a European-style contingent claim with payoff* $\varphi := \Phi(S_N)$ *for some convex function* $\Phi(S_N) \in \mathbb{L}^\infty(\Omega_k, \mathfrak{F}_k, \mathbb{Q}_k)$*,* $k = H, \ldots, N-1$*, the value process* $\mathcal{V}_k(S_{k-H}, S_{k-H}u^H)$ *and* $\mathcal{V}_k(S_{k-H}, S_{k-H}d^H)$*,* $k = H, \ldots, N-1$ *for the super-replicating strategy, in an* N*-period binomial model with* H *periods of delay, can be calculated as*

$$\mathcal{V}_k(S_{k-H}, S_{k-H}u^H) = e^{-r\widetilde{N}}\mathbb{E}^{\mathbb{Q}_k}\left(\Phi\left(S_N\right)|Z_{k,0} = 1\right), \tag{2.31}$$

$$\mathcal{V}_k(S_{k-H}, S_{k-H}d^H) = e^{-r\widetilde{N}}\mathbb{E}^{\mathbb{Q}_k}\left(\Phi\left(S_N\right)|Z_{k,0} = 0\right). \tag{2.32}$$

Proof. We need to show that (2.31) and (2.32) satisfy the recursive equation (2.26) for $k = H, \ldots, N-2$, and equation (2.25) for $k = N-1$. For $k = N-1$, it is already shown in (2.18), and for $k = H, \ldots, N-2$, by conditioning on $Z_{k,1}$, (2.31) satisfies

$$\begin{aligned}\mathcal{V}_k(S_{k-H}, S_{k-H}u^H) &= e^{-r\widetilde{N}}\mathbb{E}^{\mathbb{Q}_k}\left(\Phi\left(S_N\right)|Z_{k,0} = 1\right), \\ &= e^{-r\widetilde{N}}\Bigg[\mathbb{E}^{\mathbb{Q}_k}\left(\Phi\left(S_N\right)|Z_{k,0} = 1, Z_{k,1} = 0\right)\mathbb{Q}_k(Z_{k,1} = 0|Z_{k,0} = 1) \\ &\quad +\mathbb{E}^{\mathbb{Q}_k}\left(\Phi\left(S_N\right)|Z_{k,0} = 1, Z_{k,1} = 1\right)\mathbb{Q}_k(Z_{k,1} = 1|Z_{k,0} = 1)\Bigg].\end{aligned}$$

Note that by the way the spaces $(\Omega_k, \mathcal{F}_k, \mathbb{Q}_k)$ and $(\Omega_{k+1}, \mathcal{F}_{k+1}, \mathbb{Q}_{k+1})$ are constructed,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_k}\left(\Phi\left(S_N\right)|Z_{k,0}=1, Z_{k,1}=0\right) &= e^{-r}\mathbb{E}^{\mathbb{Q}_{k+1}}\left(\Phi\left(S_N\right)|Z_{k+1,0}=0\right), \\ \mathbb{E}^{\mathbb{Q}_k}\left(\Phi\left(S_N\right)|Z_{k,0}=1, Z_{k,1}=1\right) &= e^{-r}\mathbb{E}^{\mathbb{Q}_{k+1}}\left(\Phi\left(S_N\right)|Z_{k+1,0}=1\right). \end{aligned}$$

Also, $\mathbb{Q}_k(Z_{k,1}=0|Z_{k,0}=1)=\mathfrak{q}_u$ and $\mathbb{Q}_k(Z_{k,1}=1|Z_{k,0}=1)=\mathfrak{p}_u$. Therefore,

$$\begin{aligned} \mathcal{V}_k(S_{k-H}, S_{k-H}u^H) &= e^{-r\widetilde{N}}\Big[\mathfrak{p}_u\mathbb{E}^{\mathbb{Q}_{k+1}}\left(\varphi\left(S_N\right)|Z_{k+1,0}=1\right) \\ &\qquad +\mathfrak{q}_u\mathbb{E}^{\mathbb{Q}_{k+1}}\left(\varphi\left(S_N\right)|Z_{k+1,0}=0\right)\Big], \\ &= e^{-r}\Big[\mathfrak{p}_u\mathcal{V}_{k+1}(S_{k-H+1}=S_{k-H}u, S_{k+1}=S_{k-H}u^{H+1}) \\ &\qquad +\mathfrak{q}_u\mathcal{V}_{k+1}(S_{k-H+1}=S_{k-H}d, S_{k+1}=S_{k-H}d^{H+1})\Big]. \end{aligned}$$

which completes the proof. Similarly, it can also be shown for (2.32). □

Remark 10. *If we are interested just to find out the time-*0 *super-replication price* $\bar{\pi}(\varphi)$*, we only need the probability space* $(\Omega_H, \mathcal{F}_H, \mathbb{Q}_H)$ *where* $\Omega_H=\{0,1\}^N$*. Then, we would have*

$$\bar{\pi}(\varphi)=e^{-rN}\max\left\{\mathbb{E}^{\mathbb{Q}_H}\left(\Phi\left(S_N\right)|Z_{H,0}=1\right), \mathbb{E}^{\mathbb{Q}_H}\left(\Phi\left(S_N\right)|Z_{H,0}=0\right)\right\}. \qquad (2.33)$$

Lemma 1 calculates $\mathbb{E}^{\mathbb{Q}_k}\left(\Phi\left(S_N\right)|Z_{k,0}=1\right)$, $k=H,\ldots,N-1$.
For $H+1\le i\le \widetilde{N}+H-1, 1\le j\le \min(i-H, \widetilde{N}+H-i)$. Define

$$f(i,j) := \binom{\widetilde{N}+H-i-1}{j-1}\binom{i-H}{j}\mathfrak{q}_u^{(j)}\mathfrak{q}_d^{(\widetilde{N}+H-i-j)}\mathfrak{p}_u^{(i-j-H)}\mathfrak{p}_d^{(j)}. \qquad (2.34)$$

Also for $0 \le i \le \widetilde{N}-2, 1 \le j \le \min(i+1, \widetilde{N}-i-1)$, define

$$\begin{aligned} h(i,j) \quad := \quad & \binom{\widetilde{N}-i-2}{j-1}\binom{i}{j-1}\mathfrak{q}_u^{(j)}\mathfrak{q}_d^{(\widetilde{N}-i-j)}\mathfrak{p}_u^{(i-j+1)}\mathfrak{p}_d^{(j-1)} \\ & +\binom{\widetilde{N}-i-2}{j-1}\left[\binom{i+1}{j}-\binom{i}{j-1}\right]\mathfrak{q}_u^{(j+1)}\mathfrak{q}_d^{(\widetilde{N}-i-j-1)}\mathfrak{p}_u^{(i-j)}\mathfrak{p}_d^{(j)}. \end{aligned} \tag{2.35}$$

Lemma 1. *For a function $\Phi(S_N) \in \mathbb{L}^\infty(\Omega_k, \mathfrak{F}_k, \mathbb{Q}_k)$, $k = H, \ldots, N-1$, the conditional expectation $\mathbb{E}^{\mathbb{Q}_k}\left(\Phi\left(S_N\right)|Z_{k,0}=1\right)$ can be explicitly calculated as*

$$\mathbb{E}^{\mathbb{Q}_k}\left(\Phi\left(S_N\right)|Z_{k,0}=1\right) = \sum_{i=0}^{\widetilde{N}+H} \mathbb{Q}_k\left(S_N = S_{k-H}u^i d^{\widetilde{N}+H-i}|Z_{k,0}=1\right)\Phi(S_{k-H}u^i d^{\widetilde{N}+H-i}), \tag{2.36}$$

where $\mathbb{Q}_k\left(S_N = S_{k-H}u^i d^{\widetilde{N}+H-i}\,|\,Z_{k,0}=1\right)$ is given by

$$\begin{cases} \displaystyle\sum_{j=1}^{\min(i+1,\widetilde{N}-i-1)} h(i,j) & 0 \le i \le H; \\ \displaystyle\sum_{j=1}^{\min(i+1,\widetilde{N}-i-1)} h(i,j) + \sum_{j=1}^{\min(i-H,\widetilde{N}+H-i)} f(i,j) & H+1 \le i \le \widetilde{N}-2; \\ \displaystyle\mathfrak{p}_u^{(\widetilde{N}-1)}\mathfrak{q}_u + \sum_{j=1}^{\min(\widetilde{N}-H-1,H+1)} f(i,j) & i = \widetilde{N}-1; \\ \displaystyle\sum_{j=1}^{\min(i-H,\widetilde{N}+H-i)} f(i,j) & \widetilde{N} \le i \le \widetilde{N}+H-1; \\ \mathfrak{p}_u^{(\widetilde{N})} & i = \widetilde{N}+H, \end{cases} \tag{2.37}$$

Proof. Note that $\mathbb{Q}_k\left(S_N = S_{k-H}u^i d^{\widetilde{N}+H-i} | Z_{k,0} = 1\right)$, $i = 0, \ldots, \widetilde{N}+H$ is the sum of several products of $\widetilde{N}$ elements chosen out of $\{\mathfrak{p}_u, \mathfrak{p}_d, \mathfrak{q}_u, \mathfrak{q}_d\}$, and each product term corresponds to a path in the tree starting from the node S_{k-H}, and ending in the node $S_N = S_{k-H}u^i d^{\widetilde{N}+H-i}$.

Given equations (2.29), the last $H+1$ moves need to be either upward or downward, and they contribute to as just one single move. Since it is conditioned on $Z_{k,0} = 1$, according to Remark (7), the first element in all of the product terms is either $\mathfrak{q}_u$ or $\mathfrak{p}_u$. For $H+1 \leq i \leq \widetilde{N}-2$, the last $(H+1)$-period move to $S_N = S_{k-H}u^i d^{\widetilde{N}+H-i}$ can be both downward and upward.

In the case that it is upward, we need to consider all the paths starting from S_{k-H} to $S_{N-2} = S_{k-H}u^{i-2}d^{\widetilde{N}+H-i}$ which consist of $i-2$ upward moves and $\widetilde{N}+H-i$ downward ones. There are $\binom{\widetilde{N}+H-2}{\imath-2}$ of such paths, but these paths are not all equivalent and result in different product terms of $\widetilde{N}$ elements chosen out of $\{\mathfrak{p}_u, \mathfrak{p}_d, \mathfrak{q}_u, \mathfrak{q}_d\}$, based on the location of the $\widetilde{N}+H-i$ downward moves in the path.

Note that all paths which have the same number of downward groups result in the same product terms, where a downward group is any number of consecutive downward moves preceded (if any) by an upward move and also succeeded (if any) by an upward move. For example, both of the sequences $\nearrow\searrow\searrow\searrow\nearrow\searrow$ and $\searrow\searrow\nearrow\nearrow\searrow\searrow$ have two groups of $\searrow$ moves. The reason for studying downward groups is that the starting element in all of them is $\mathfrak{q}_u$.

In this notation, j corresponds to the number of groups which starts from 1 (assuming that there exists at least one downward move) and can reach to $\min(i-H, \widetilde{N}+H-i)$. Notice that there are $\binom{\widetilde{N}+H-i-1}{j-1}\binom{i-H}{j}$ paths which have exactly j groups. Therefore, along those path the power of both $\mathfrak{q}_u$ and $\mathfrak{p}_d$ is j and consequently, the powers of $\mathfrak{q}_d$ and $\mathfrak{p}_u$ are respectively $\widetilde{N}+H-i-j$ and $i-j-H$. Here $f(i,j)$ in equation (2.34) corresponds to these paths.

The second case is that the last $(H+1)$-period move is downward. Then, we need to consider all the paths starting from the node S_{k-H} to $S_{N-2} = S_{k-H}u^i d^{\widetilde{N}+H-i-2}$ which consist of i upward moves and $\widetilde{N} + H - i - 2$ downward ones. Here not only the number of downward moves is important, but also the direction (upward or downward) of the move from time $N-3$ to $N-2$ is also relevant.

Note that there are $\binom{\widetilde{N}-i-2}{j-1}\binom{i}{j-1}$ paths which have exactly j groups such that the last 1-period move from $N-3$ to $N-2$ is downward, so the corresponding product term is $\mathfrak{q}_u^{(j)}\mathfrak{q}_d^{(\widetilde{N}-i-j)}\mathfrak{p}_u^{(i-j+1)}\mathfrak{p}_d^{(j-1)}$, and there are $\binom{\widetilde{N}-i-2}{j-1}[\binom{i+1}{j}-\binom{i}{j-1}]$ paths which have exactly j groups such that the last 1-period move from $N-3$ to $N-2$ is upward. The function $h(i,j)$ in equation (2.35) takes all these paths into account.

For $H+1 \le i \le \widetilde{N}-2$, it is necessary to use both $f(i,j)$ and $h(i,j)$ to take into account that the last $(H+1)$-period move can be both upward and downward. The same reasoning works for $0 \le i \le H$ and $\widetilde{N} \le i \le \widetilde{N}+H-1$, but here the last $(H+1)$-period move can only be downward for $0 \le i \le H$ and upward for $\widetilde{N} \le i \le \widetilde{N}+H-1$. For $i = \widetilde{N}-1$ when the last $(H+1)$-period move is downward and $i = \widetilde{N}+H$, the fuctions $f(i,j)$ and $h(i,j)$ cannot be used because in all of the paths from S_{k-H} to $S_{N-2} = S_{k-H}u^{\widetilde{N}+H-2}$, there is not any downward move at all to make a downward group (i.e, $j=0$). $\square$

Similarly, Lemma 2 calculates $\mathbb{E}^{\mathbb{Q}_k}\left(\Phi\left(S_N\right)|Z_{k,0}=0\right)$, $k = H, \ldots, N-1$. Also for

$H+2 \le i \le \widetilde{N}+H, 1 \le j \le \min(i-H-1, \widetilde{N}+H-i+1)$, define

$$\widetilde{f}(i,j) := \binom{i-H-2}{j-1}\binom{\widetilde{N}+H-i}{j-1}\mathfrak{q}_u^{(j-1)}\mathfrak{q}_d^{(\widetilde{N}+H-i-j+1)}\mathfrak{p}_u^{(i-j-H)}\mathfrak{p}_d^{(j)} + \binom{i-H-2}{j-1}\left[\binom{\widetilde{N}+H-i+1}{j} - \binom{\widetilde{N}+H-i}{j-1}\right]\mathfrak{q}_u^{(j)}\mathfrak{q}_d^{(\widetilde{N}+H-i-j)}\mathfrak{p}_u^{(i-j-H-1)}\mathfrak{p}_d^{(j+1)}. \tag{2.38}$$

For $1 \le i \le \widetilde{N}-1, 1 \le j \le \min(i, \widetilde{N}-i)$, define

$$\widetilde{h}(i,j) := \binom{i-1}{j-1}\binom{\widetilde{N}-i}{j}\mathfrak{q}_u^{(j)}\mathfrak{q}_d^{(\widetilde{N}-i-j)}\mathfrak{p}_u^{(i-j)}\mathfrak{p}_d^{(j)}. \tag{2.39}$$

Lemma 2. *For a function $\Phi(S_N) \in \mathbb{L}^\infty(\Omega_k, \mathfrak{F}_k, \mathbb{Q}_k)$, $k = H, \ldots, N-1$, the conditional expectation $\mathbb{E}^{\mathbb{Q}_k}(\Phi(S_N)\,|Z_{k,0}=1)$ can be explicitly calculated as*

$$\mathbb{E}^{\mathbb{Q}_k}(\Phi(S_N)\,|Z_{k,0}=1) = \sum_{i=0}^{\widetilde{N}+H} \mathbb{Q}_k\left(S_N = S_{k-H}u^i d^{\widetilde{N}+H-i}|Z_{k,0}=1\right)\Phi(S_{k-H}u^i d^{\widetilde{N}+H-i}), \tag{2.40}$$

where $\mathbb{Q}_k\left(S_N = S_{k-H}u^i d^{\widetilde{N}+H-i}\,|\,Z_{k,0}=1\right)$ *is given by*

$$\begin{cases} \mathfrak{q}_d^{(\widetilde{N})} & i = 0 \\ \sum_{j=1}^{\min(i,\widetilde{N}-i)} \widetilde{h}(i,j) & 1 \le i \le H; \\ \mathfrak{q}_d^{(\widetilde{N}-1)}\mathfrak{p}_d + \sum_{j=1}^{\min(H+1,\widetilde{N}-H-1)} \widetilde{h}(i,j) & i = H+1; \\ \sum_{j=1}^{\min(i,\widetilde{N}-i)} \widetilde{h}(i,j) + \sum_{j=1}^{\min(i-H-1,\widetilde{N}+H-i+1)} \widetilde{f}(i,j) & H+2 \le i \le \widetilde{N}-1; \\ \sum_{j=1}^{\min(i-H-1,\widetilde{N}+H-i+1)} \widetilde{f}(i,j) & \widetilde{N} \le i \le \widetilde{N}+H. \end{cases} \tag{2.41}$$

Proof. The proof follows very similarly as that of Lemma 1 with only this difference that since $Z_{k,0} = 0$, we look for upward groups instead of downward groups. □

2.3.6 Geometrical Representation

In this subsection, we first discuss Theorem 2 from a geometrical perspective Then, we represent the dynamic programming approach in subsection 2.3.5 geometrically For convenience, assume that interest rate $r = 0$, and $H = 1$ in this subsection.

In Theorem 2, we discussed that in an N-period binomial model with $H = N - 1$ periods of delay, for a European-style convex contingent claim with payoff function $\varphi := \Phi(S_N) \in \mathbb{L}^\infty(\Omega, \mathfrak{F}, \mathbb{P})$, there exist Δ_H^* and x_0^* such that

$$\mathcal{V}_H(S_0, S_H) = x_0^* + \Delta_H^* S_H. \tag{2.42}$$

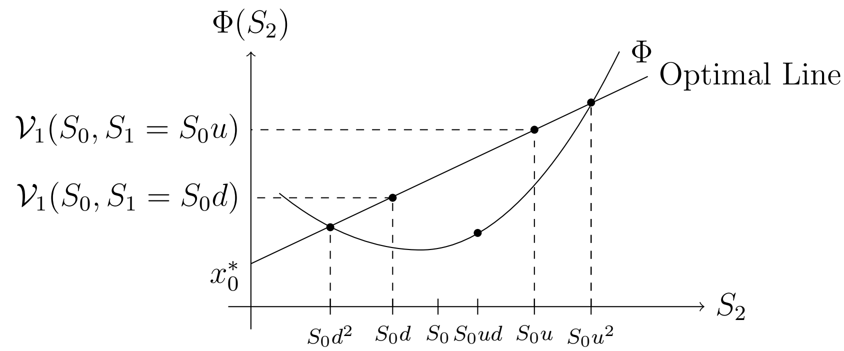

Figure 2.3: Super-replicating Strategy in a 2-period binomial Model with a 1-period Delay. The optimal line characterizes the super-replicating strategy. The slope of it is Δ_1^* and its intercept is x_0^*. The super-replication price is $\bar{\pi}(\varphi) = \max\{\mathcal{V}_1(S_0, S_1 = S_0d), \mathcal{V}_1(S_0, S_1 = S_0u)\}$.

This suggests that there exists a line with slope Δ_H^* and intercept x_0^* such that the super-replicating value function $\mathcal{V}_H(S_0, S_H)$ lie on that line. Figure 2.3 shows this optimal line, the super-replication price, and the super-replicating value functions in a 2-period binomial model with 1 period of delay.

It is more intuitive to demonstrate the dynamic programming approach in subsection 2.3.5 geometrically Figure 2.2 shows a 4-period binomial model with 1-period delay. Figure 2.4 shows how to geometrically find the super-replication price for a contingent claim with convex payoff function ($\Phi(.)$). For convenience and to avoid a clutter of points on the x-axis, suppose $ud = 1$, so some of the points in the model lie on each other.

Now in order to find the super-replication prices at time 3, it is necessary to consider the three 2-period binomial models with 1-period delay $\mathcal{T}_2(2,0)$, $\mathcal{T}_2(1,1)$ and $\mathcal{T}_2(0,2)$. In Figure 2.4, the lines (om), (nl) and (mk) show the optimal super-replication lines for each of these models respectively. As it can be seen, there are two payoffs at either of the nodes $S_3 = S_0u^2d$ and $S_3 = S_0ud^2$ depending on which subtree is used for pricing (i.e. depending on what S_1 is). Now, we go one period further back

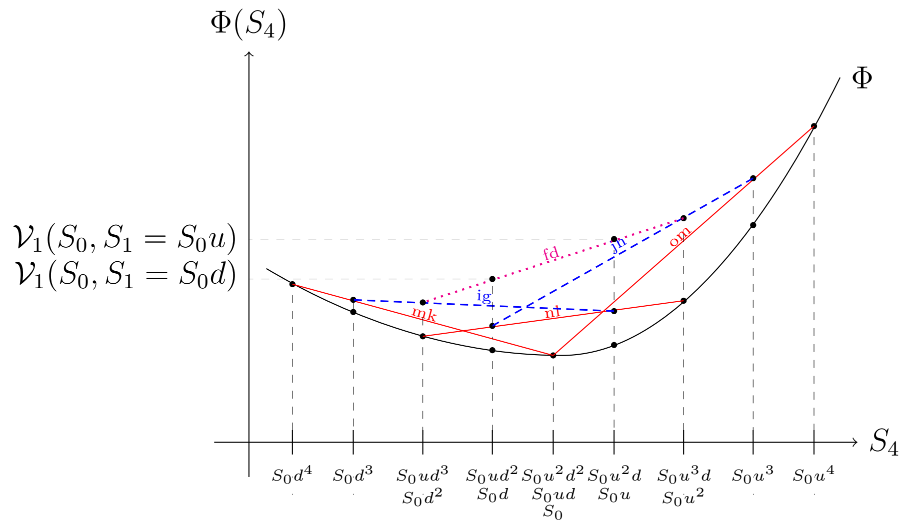

Figure 2.4: Geometrical Representation of the Super replicating Strategy in a 4-period binomial Model with 1-period Delay using a Dynamic Programming Approach

to find out the payoffs at time 2. We need to consider two 2-period models $\mathcal{T}_2(1,0)$ and $\mathcal{T}_2(0,1)$. Note that in each of these two models, the corresponding payoff at nodes $S_2 = S_0u^2d$ (out of two payoffs $\Phi_{\mathcal{T}_2(1,0)}(2,1)$ and $\Phi_{\mathcal{T}_2(0,1)}(2,1)$) and $S_2 = S_0ud^2$ (out of two payoffs $\Phi_{\mathcal{T}_2(1,0)}(1,2)$ and $\Phi_{\mathcal{T}_2(1,0)}(1,2)$) needs to be chosen. As Theorem (3) suggests, the payoff functions for both of these models are convex. The lines (jh) and (ig) demonstrate the optimal lines for these models. Similarly, to calculate the payoff at time 1, the 2-period model $\mathcal{T}_2(0,0)$ needs to be used and the line (fd) shows the optimal line for this model. Finally, we have the super-replication price $\bar{\pi}(\varphi) = \max\{\mathcal{V}_1(S_0, S_1 = S_0d), \mathcal{V}_1(S_0, S_1 = S_0u)\}$.

2.4 Continuous Time Model

In this section, we discuss the asymptotic behavior of the model. We define the probability spaces $(\Omega^n, \mathfrak{F}^n, \mathbb{Q}^n)$, $n \in \mathbb{N}$ such that $\Omega^n = \{0,1\}^n$, and $\mathfrak{F}^n$ is the Borel

σ-algebra on Ω^n. For every $\omega^n = (\omega_1^n, \dots, \omega_n^n) \in \Omega^n$, we define a coordinate map by $Z_\ell^n(\omega^n) = \omega_\ell^n$ for each $\ell \in \{1, \dots, n\}$. Define the filtration $\{\mathcal{F}_\ell^n, \ell = 0, \dots, n\}$, where $\mathcal{F}_\ell^n$ is the σ-field $\sigma(Z_1^n, \dots, Z_\ell^n)$ generated by the first ℓ variables for $\ell = 1, \dots, n$ and $\mathcal{F}_0$ is the trivial σ-field.

Let $\mu, \sigma, r \in [0, \infty)$, $H \in \mathbb{N}$, and $T > 0$ (fixed time horizon). Then, define the sequences

$$\begin{aligned} \mu_n &= \mu T \delta_n^2, \quad \sigma_n = \sigma\sqrt{T}\delta_n, \quad u_n = \exp(\mu_n + \sigma_n), \\ d_n &= \exp(\mu_n - \sigma_n), \quad r_n = rT\delta_n^2, \quad H_n = HT\delta_n^2. \end{aligned} \tag{2.43}$$

Where the order $\delta_n = {}^1/\sqrt{n}$, as in Donsker's theorem.

Remark 11. *H characterizes the number of periods we have delayed information, which is constant in the asymptotic analysis. However, H_n is the amount of time we have delayed information, which should vanish in the limit. Otherwise, the super-replication price would explode and converge to the maximum of the contingent claim payoff function.*

2.4.1 Price Process Asymptotic

Define the probability measures $\mathbb{Q}^n$, similar to (2.28) and (2.29), such that $Z_\ell^n, \ell = 1, \dots, n$ with initial position Z_0^n is a Markov chain, and for $\ell = 1, \dots, n - H - 1$, it has transition matrix

$$Q_n = \begin{pmatrix} \mathfrak{q}_{n,d} & \mathfrak{p}_{n,d} \\ \mathfrak{q}_{n,u} & \mathfrak{p}_{n,u} \end{pmatrix} \quad \text{on } \{0, 1\}, \tag{2.44}$$

Besides, for $m = n - H, \dots, n$,

$$\begin{aligned}
\mathbb{Q}^n\left(Z_n^n = \cdots = Z_{n-H}^n = 1 | Z_{n-H-1}^n = 1\right) &= \mathfrak{p}_{n,u}, \\
\mathbb{Q}^n\left(Z_n^n = \cdots = Z_{n-H}^n = -1 | Z_{n-H-1}^n = 1\right) &= \mathfrak{q}_{n,u}, \\
\mathbb{Q}^n\left(Z_n^n = \cdots = Z_{n-H}^n = 1 | Z_{n-H-1}^n = 0\right) &= \mathfrak{p}_{n,d}, \\
\mathbb{Q}^n\left(Z_n^n = \cdots = Z_{n-H}^n = -1 | Z_{n-H-1}^n = 0\right) &= \mathfrak{q}_{n,d}.
\end{aligned} \tag{2.45}$$

Where $\mathfrak{p}_{n,u}, \mathfrak{q}_{n,u}, \mathfrak{p}_{n,d}$ and $\mathfrak{q}_{n,d}$ are defined, similar to (2.19), as

$$\mathfrak{p}_{n,u} := \frac{u_n^{(H)} e^{r_n} - d_n^{(H+1)}}{u_n^{(H+1)} - d_n^{(H+1)}} = 1 - \mathfrak{q}_{n,u}, \quad \mathfrak{p}_{n,d} := \frac{d_n^{(H)} e^{r_n} - d_n^{(H+1)}}{u_n^{(H+1)} - d_n^{(H+1)}} = 1 - \mathfrak{q}_{n,d}, \tag{2.46}$$

Then, the risky asset price S_ℓ^n, similar to (2.1), satisfies

$$S_\ell^n = S_0 \exp\left[\ell \mu_n + \sigma_n \sum_{i=1}^{\ell} X_i^n\right], \quad \ell = 0, \dots, n. \tag{2.47}$$

Where $X_i^n = 2Z_i^n - 1$.

Lemma 3 provides asymptotic for $\mathfrak{p}_{n,u}$ and $\mathfrak{p}_{n,d}$.

Lemma 3. *We have*

$$\mathfrak{p}_{n,u} = \frac{2H+1}{2(H+1)} - \left(\frac{\mu - r}{2\,(H+1)\,\sigma} + \frac{2H+1}{4(H+1)}\sigma\right)\sqrt{T}\delta_n + \mathcal{O}\left(\delta_n^2\right), \tag{2.48}$$

$$\mathfrak{p}_{n,d} = \frac{1}{2(H+1)} - \left(\frac{\mu - r}{2\,(H+1)\,\sigma} + \frac{2H+1}{4(H+1)}\sigma\right)\sqrt{T}\delta_n + \mathcal{O}\left(\delta_n^2\right). \tag{2.49}$$

Proof. The proof simply follows by using Taylor's expansion on u_n, d_n and r_n, and plugging them in (2.46). □

Discretize the time interval by setting $t_\ell^n := T\ell/n$. By interpolating the risky asset price process (S_ℓ^n) on the intervals $[t_{\ell-1}^n, t_\ell^n)$ constant piecewisely, we get the process $S^{(n)} = (S_t^{(n)})_{0\le t\le T}$

$$S_t^{(n)} := S_{\lfloor nt\rfloor/T}^n, \quad 0 \le t \le T, \tag{2.50}$$

Where $\lfloor . \rfloor$ is the floor function.

The process $S^{(n)}$ has trajectories which are right continuous with left limits. Note that in particular

$$S_{t_\ell^n}^{(n)} = S_\ell^n, \quad \ell = 0, \dots, n.$$

Here, $S^{(n)}$ under measure $\mathbb{Q}^n$ is distributed according to a probability measure ρ_n on the Skorokhod space $\mathbb{D}[0,T]$ of right continuous functions with left limits. Theorem 5 provides a weak convergence for the sequence $(\rho_n)_{n\in\mathbb{N}}$.

Theorem 5. *The sequence of processes $(S^{(n)})_{n\in\mathbb{N}}$ converges in distribution to the process $(S_t)_{0\le t\le T}$ with dynamics*

$$dS_t = rS_t dt + \widetilde{\sigma} dW_t, \quad 0 \le t \le T. \tag{2.51}$$

Where $(W_t)_{0\le t\le T}$ is a Brownian motion, and we have the enlarged volatility

$$\widetilde{\sigma} = \sqrt{2H+1}\sigma. \tag{2.52}$$

Proof. First, note that

$$\mathbb{Q}^n\left(X_\ell^n = 1 | X_{\ell-1}^n\right) = \frac{1}{2}\left[\mathfrak{p}_{n,u} + \mathfrak{p}_{n,d} + X_{\ell-1}^n\left(\mathfrak{p}_{n,u} - \mathfrak{p}_{n,d}\right)\right], \quad \ell = 1, \dots, n-H-1,$$

According to Lemma 3, we conclude that

$$\mathbb{Q}^n\left(X_\ell^n = 1 | \mathcal{F}_{\ell-1}^n\right) = \mathfrak{p}_\ell\left(\ell, X_{\ell-1}^n\right), \quad \ell = 1, \ldots, n-H-1,$$

where in the notation of [24]

$$\begin{aligned}
\mathfrak{p}_n(\ell, x) &= \frac{1}{2}\left[1 + \phi\delta_n + \lambda_n x\right] + \mathcal{O}(\delta_n^2), \\
\phi &= -2\left[\frac{\mu - r}{2(H+1)\sigma} + \frac{2H+1}{4(H+1)}\sigma\right]\sqrt{T}, \\
\lambda_n &= \frac{H}{H+1} + \mathcal{O}(\delta_n^2).
\end{aligned}$$

[24] provides a functional central limit theorem for generalized correlated random walks. If we define functions $a_n(t, y) := \lambda_n$ and $b_n(t, y) := \phi$, theorem 1 and remark 3 in [24] and continuous mapping theorem show that $S^{(n)}$, regardless of the initial distribution of X_0^n, converges in distribution to $(S_t)_{0 \le t \le T}$ in (2.51). Note that here $\lambda_n = \mathfrak{p}_{n,u} - \mathfrak{p}_{n,d}$ corresponds to the gap caused by the delay in the flow of information, and this is the main source making the price process under the pricing measure more volatile. A brief summary of the asymptotic results in [24] are provided in (A). $\square$

2.4.2 Exaggerated Volatility Smile

In this subsection, we discuss the volatility smile of the model, and how it evolves with the number of periods (n). Volatility smile is the graph of Black-Scholes implied volatility with respect to the strike price. Implied volatility is the value of the volatility in the Black-Scholes pricing model which generates a price equal to that of our model. Several market features, such as crash phobia, have been attributed as the culprits of the market smile. The volatility smile has been one of the central topics in option pricing literature, and many models have been developed to capture it. We refer to [19] for more discussion in this regard.

Our model with delayed information shows that delayed information exaggerates the smile. Figure 2.5 plots the volatility smiles for call and put options in the model with and without delayed information when $n = 100$. In the model with delayed information ($H_n = \frac{1}{100}$ year ≈ 2.52 days), we observe volatility smile, on the contrary with the model without delayed information where we get an almost flat smile, which is expected according to the Remark 9. Note that in the model with delayed information, we have different smiles for call and put option, and that is because there is not any call-put parity, as discussed in Remark 3.

Figure 2.5 plots the volatility smiles for call and put options when the number of periods is very big ($n = 250,000$) for the model with delayed information ($H_n = \frac{1}{250{,}000}$ year ≈ 30 seconds). We observe almost the same flat volatility smiles for both call and put options, which can be also calculated by the theoretical results in 2.52.

These volatility smiles in Figures 2.5 and 2.6 confirm the intuition of traders that delayed information would exaggerate the volatility smile, but it is not its culprit. This is because in the continuous limit, volatility is constant and there is no smile, but in the discrete model, we can observe volatility smile. Therefore, it conveys that the smile observed in the market might have been exaggerated by the way we interact with delayed information, and the smile might not be caused all by the market itself.

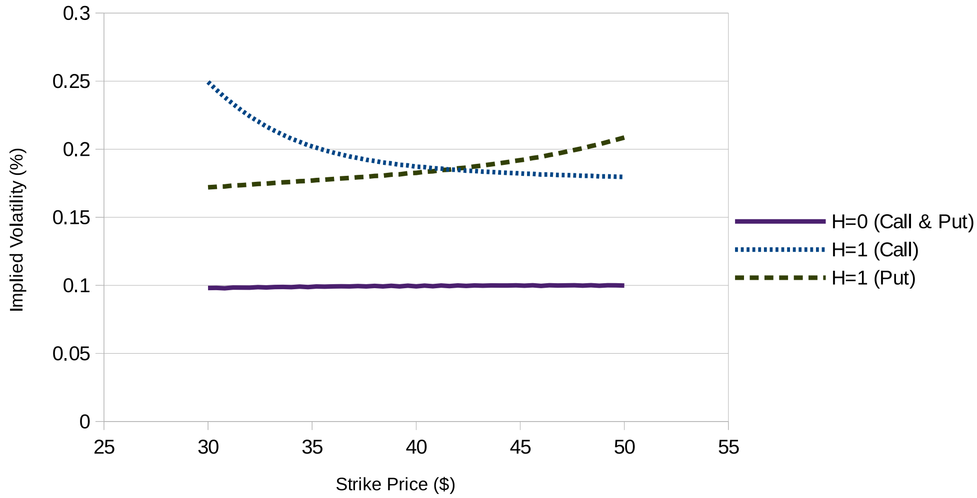

Figure 2.5: Volatility smile for the Call and Put options in the binomial model with and without delayed information ($H_n = \frac{1}{100}$ year $\approx$ 2.52 days and 0 day respectively). The parameters are $\sigma = 0.1$, $T = 1$, $r = 0$, $S_0 = 40$, and $n = 100$

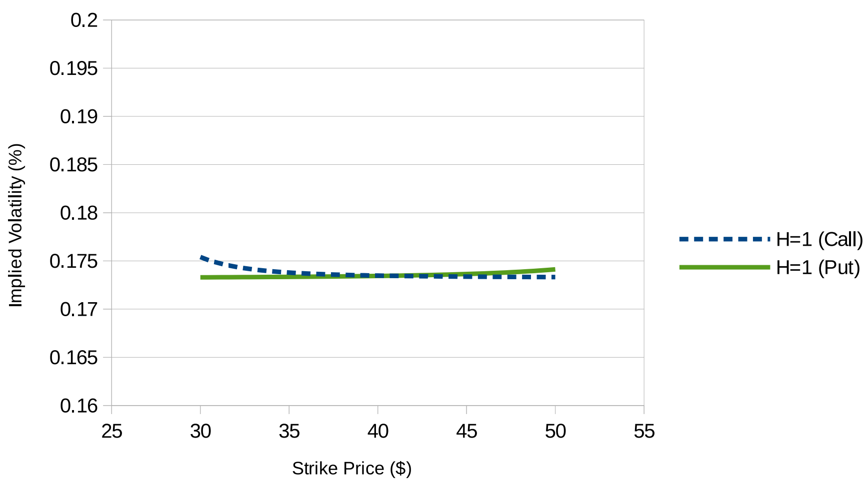

Figure 2.6: Volatility smile for the Call and Put options in the binomial model with delayed information ($H_n = \frac{1}{250{,}000}$ year $\approx$ 30 seconds). The parameters are $\sigma = 0.1$, $T = 1$, $r = 0$, $S_0 = 40$, and $n = 250,000$

Chapter 3

Systemic Risk and Stochastic Games with Delay

3.1 Abstract

We propose a model of inter-bank lending and borrowing which takes into account clearing debt obligations. The evolution of log-monetary reserves of N banks is described by coupled diffusions driven by controls with delay in their drifts. Banks are minimizing their finite-horizon objective functions which take into account a quadratic cost for lending or borrowing and a linear incentive to borrow if the reserve is low or lend if the reserve is high relative to the average capitalization of the system. As such, our problem is an N-player linear-quadratic stochastic differential game with delay. An open-loop Nash equilibrium is obtained using a system of fully coupled forward and advanced backward stochastic differential equations. We then describe how the delay affects liquidity and systemic risk characterized by a large number of defaults. We also derive a close-loop Nash equilibrium using an HJB approach. This

chapter is a joint work with *René Carmona*, *Jean-Pierre Fouque* and *Li-Hsien Sun*.

3.2 Introduction

In [6], we proposed a stochastic game model of inter-bank lending and borrowing where banks borrow from or lend to a central bank with no obligation to pay back their loans and no gain from lending. The main finding was that in equilibrium, the central bank is acting as a clearing house, liquidity is created, thus leading to a more stable system. Systemic risk was analyzed as in [15] in the case of a linear model without control. Systemic risk being characterized as the rare event of a large number of defaults occurring when the average capitalization reaches a prescribed level, the conclusion was that inter-bank lending and borrowing leads to stability through a flocking effect. For this type of interaction without control, we also refer to [14, 17] and [18].

In order to make the toy model of [6] more realistic, we introduce delay in the controls. This forces banks to take responsibility for past lending and borrowing. In this paper, the evolution of the log-monetary reserves of N banks is described by a system of delayed stochastic differential equations, and banks try to minimize their costs or maximize their profits by controlling the rate of borrowing or lending. They interact via the average capitalization meaning that banks consider this average as a critical level to determine borrowing from or lending to the central bank.

We identify open-loop Nash equilibria by solving fully coupled forward and *advanced* backward stochastic differential equations (FABSDEs) introduced by [39]. Our conclusion is that the new effect created by the need to *pay back* or *receive refunds* due to the presence of the delay in the controls, reduces the liquidity observed in the case without delay. However, despite these quantitative differences, the central bank is still acting as a clearing house. A closed-loop Nash equilibrium to this stochastic game with delay is derived from the Hamilton-Jacobi-Bellman (HJB) equation approach using the results in [22] and we provide a verification Theorem.

For a general introduction to BSDEs, stochastic control and stochastic differential games without delay, we refer to the recent monograph [5]. Stochastic control problems with delay have been studied from various points of view. When the delay only appears in the state variable, solutions to delayed optimal control problems were derived from variants of the Pontryagin-Bismut-Bensoussan stochastic maximum principle. See for instance [36] and [37]. Alternatively, in order to use dynamic programming, [32] and [33] reduce the system with delay to a finite-dimension problem, but still the delay does not appear in the control like in the case we want to study.

The general case of stochastic optimal control of stochastic differential equations with delay both in the state and the control is studied using an infinite-dimensional HJB equation in [21], and [22]. The case with pointwise delayed control is studied in [23]. The general stochastic control problem in the case of delayed states and controls both appearing in the forward equation is studied in [8], [9] and [43] by using the forward and advanced backward stochastic equations. Linear-Quadratic mean field Stackelberg games with delay and with a major player and many small players are studied in [2].

The typical problem studied in this paper can be described as follows. The dynamics of the log-monetary reserves of N banks are given by the following coupled diffusion processes X_t^i, $i = 1, \cdots, N$,

$$dX_t^i \;=\; \left(\alpha_t^i - \alpha_{t-\tau}^i\right) dt + \sigma dW_t^i, \quad 0 \le t \le T, \tag{3.1}$$

where W_t^i, $i = 1, \cdots, N$ are independent standard Brownian motions, and the rate of borrowing or lending α_t^i represents the control exerted by bank i on the system. In this example, we use the simplest possible form of delay, the delayed control $\alpha_{t-\tau}^i$ corresponding to repayments after a fixed time τ such that $0 \leq \tau \leq T$. We shall use deterministic initial conditions given by

$$X_0^i = \xi^i, \qquad \text{and} \qquad \alpha_t^i = 0, \quad t \in [-\tau, 0). \tag{3.2}$$

For simplicity, we assume that the banks have the same volatility $\sigma > 0$. In what follows we use the notations $X = (X^1, \cdots, X^N)$, $x = (x^1, \cdots, x^N)$, $\alpha = (\alpha^1, \cdots, \alpha^N)$, and $\overline{x} = \frac{1}{N}\sum_{i=1}^N x^i$.

Before concentrating on the specific case (3.1), we prove a dedicated version of the sufficient condition of the Pontryagin stochastic maximum principle for a more general class of models for which the dynamics of the states are given by stochastic differential equations of the form:

$$dX_t^i \;=\; \left(\int_0^\tau \alpha_{t-s}^i \theta(ds)\right) dt + \sigma dW_t^i, \quad 0 \leq t \leq T, \tag{3.3}$$

where θ is a nonnegative measure on $[0, \tau]$. The special case (3.1) corresponds to $\theta = \delta_0 - \delta_\tau$.

Bank i chooses its own strategy α^i in order to minimize its objective function of the form:

$$J^i(\alpha) \;=\; \mathbb{E}\left\{ \int_0^T f_i(X_t, \alpha_t^i) dt + g_i(X_T) \right\}. \tag{3.4}$$

In this paper, we concentrate on the running and terminal cost functions used in [6], namely:

$$f_i(x, \alpha^i) = \frac{(\alpha^i)^2}{2} - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2, \quad q \geq 0, \quad \epsilon > 0, \tag{3.5}$$

and

$$g_i(x) = \frac{c}{2}\left(\overline{x} - x^i\right)^2, \quad c \geq 0, \tag{3.6}$$

with $q^2 < \epsilon$ so that $f_i(x, \alpha)$ is convex in (x, α). Note that the case $\tau > T$ corresponds to no repayment and therefore no delay in the equations,. The case $\tau = 0$ corresponds to the case with no control and therefore no lending or borrowing. The term $q\alpha^i(\overline{x} - x^i)$ in the objective function (3.5) is an incentive to lend or borrow from a central bank which in this model does not make any decision and simply provides liquidity. However, we know that in the case with no delay ([6]), in equilibrium, the central bank acts as a clearing house. We will see in Section 3.7 that this is still the case with delay.

The paper is organized as follows. In Section 3.3, we briefly review the model without delay presented in [6]. The analysis of the stochastic differential games with delay is presented in Section 3.4 where we derive an exact open-loop Nash equilibrium using the FABSDE approach. In the process, we derive the *clearing house* role of the central bank in Remark 12. Section 3.5 is devoted to the derivation of a closed-loop equilibrium using an infinite-dimensional HJB equation approach with pointwise delayed control presented in [23]. In Section 3.6, we provide a verification Theorem. The effect of delay in term of financial implication is discussed in Section 3.7 where the main finding is that the introduction of delay in the model does not change the fact that in equilibrium, the central bank acts as a clearing house. However, liquidity is affected by the delay time.

3.3 Stochatic Games and Systemic Risk

The aim of this section is to briefly review the model of inter-bank lending or borrowing without delay studied in [6]. It is described by the model presented in the previous section but with $\tau > T$ so that the delay term $\alpha^i_{t-\tau}$ in (3.1) is simply zero. The setup (3.4,3.5,3.6) of the stochastic game remains the same.

The open-loop problem consists in searching for an equilibrium among strategies $\{\alpha^i_t, i = 1, \cdots, N\}$ which are adapted processes satisfying some integrability property such as $\mathbb{E}\left(\int_0^T |\alpha^i_t| dt\right) < \infty$. The Hamiltonian for bank i is given by

$$H^i(x, y^i, \alpha) = \sum_{k=1}^{N} \alpha^k y^{i,k} + \frac{(\alpha^i)^2}{2} - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2, \tag{3.7}$$

where $y^i = (y^{i,1}, \cdots, y^{i,N})$, $i = 1, \cdots, N$ are the adjoint variables.

For a given $\alpha = (\alpha^i)_{i=1,\cdots,n}$, the controlled forward dynamics of the states X^i_t are given by (3.1) without the delay term and with initial conditions $X^i_0 = \xi^i$. The adjoint processes $Y^i_t = (Y^{i,j}_t;\ j = 1, \cdots, N)$ and $Z^i_t = (Z^{i,j,k}_t;\ j = 1, \cdots, N,\ k = 1, \cdots, N)$ for $i = 1, \cdots, N$ are defined as the solutions of the backward stochastic differential equations (BSDEs):

$$dY^{i,j}_t = -\partial_{x^j} H^i(X_t, Y^i_t, \alpha_t) dt + \sum_{k=1}^{N} Z^{i,j,k}_t dW^k_t \tag{3.8}$$

with terminal conditions $Y^{i,j}_T = \partial_{x^j} g_i(X_T)$ for $i, j = 1, \cdots, N$ where g_i is given by (3.6). For each admissible strategy profile $\alpha = (\alpha^i)_{i=1,\cdots,n}$, standard existence and uniqueness results for BSDEs apply and the existence of the adjoint processes is guaranteed. Note that from (3.7), we have

$$\partial_{x^j} H^i = -q\alpha^i(\frac{1}{N} - \delta_{i,j}) + \epsilon(\overline{x} - x^i)(\frac{1}{N} - \delta_{i,j}).$$

The necessary condition of the Pontryagin stochastic maximum principle suggests that one minimizes the Hamiltonian H^i with respect to α^i which gives:

$$\hat{\alpha}^i = -y^{i,i} + q(\overline{x} - x^i). \tag{3.9}$$

With this choice for the controls α^i, the forward equation becomes coupled with the backward equation (3.8) to form a forward-backward coupled system. In the present linear-quadratic case, we make the ansatz

$$Y_t^{i,j} = \phi_t(\frac{1}{N} - \delta_{i,j})(\overline{X}_t - X_t^i), \tag{3.10}$$

for some deterministic scalar function ϕ_t satisfying the terminal condition $\phi_T = c$. Using this ansatz, the backward equations (3.8) become

$$dY_t^{i,j} = (\frac{1}{N} - \delta_{i,j})(\overline{X}_t - X_t^i)\left[q(1 - \frac{1}{N})\phi_t - (\epsilon - q^2)\right] dt + \sum_{k=1}^{N} Z_t^{i,j,k} dW_t^k. \tag{3.11}$$

Using (3.9) and (3.10), the forward equation becomes

$$dX_t^i = \left[q + (1 - \frac{1}{N})\phi_t\right](\overline{X}_t - X_t^i)dt + \sigma dW_t^i. \tag{3.12}$$

Differentiating the ansatz (3.10) and identifying with the Ito's representation (3.11), one obtains from the martingale terms the deterministic adjoint variables

$$Z_t^{i,j,k} = \phi_t \sigma(\frac{1}{N} - \delta_{i,j})(\frac{1}{N} - \delta_{i,k}) \text{ for } \ k = 1, \cdots, N,$$

and from the drift terms that the function ϕ_t must satisfy the scalar Riccati equation

$$\dot{\phi}_t = 2q(1 - \frac{1}{2N})\phi_t + (1 - \frac{1}{N})\phi_t^2 - (\epsilon - q^2), \tag{3.13}$$

with the terminal condition $\phi_T = c$. The explicit solution is given in [6]. Note that the form (3.9) of the control α_t^i, and the ansatz (3.10) combine to give:

$$\alpha_t^i = \left[q + (1 - \frac{1}{N})\phi_t \right] (\overline{X}_t - X_t^i), \tag{3.14}$$

so that, in this equilibrium, the forward equations become

$$dX_t^i = \left(q + (1 - \frac{1}{N})\phi_t \right) (\overline{X}_t - X_t^i)dt + \sigma dW_t^i. \tag{3.15}$$

Rewriting $(\overline{X}_t - X_t^i)$ as $\frac{1}{N}\sum_{j=1}^N (X_t^j - X_t^i)$, we see that the central bank is simply acting as a clearing house. From the form (3.15), we observe that the X^i's are mean-reverting to the average capitalization given by

$$d\overline{X}_t = \frac{\sigma}{N}\sum_{j=1}^N dW_t^j, \quad \overline{X}_0 = \frac{1}{N}\sum_{j=1}^N \xi^j.$$

In [15], we identified the systemic event as

$$\left\{ \min_{0 \le t \le T} (\overline{X}_t - \overline{X}_0) \le D \right\}$$

and we computed its probability

$$\mathbb{P}\left(\min_{0\leq t\leq T}(\overline{X}_t - \overline{X}_0) \leq D\right) \quad = \quad 2\Phi\left(\frac{D\sqrt{N}}{\sigma\sqrt{T}}\right), \tag{3.16}$$

where Φ is the $\mathcal{N}(0,1)$-cdf. This systemic risk probability is exponentially small of order $\exp(-D^2N/(2\sigma^2T))$ as in the large deviation estimate.

3.4 Stochastic Games with Delay

Most often, a tailor made version of the stochastic maximum principle is used as a workhorse to construct open loop Nash equilibria for stochastic differential games. Here, we provide such a tool in a more general set up than used in the paper because we believe that this result is of independent interest on its own. We then specialize it to the model considered for systemic risk in Section 3.4.3.

3.4.1 The Model

We work with a finite horizon $T > 0$. Recall that we denote by $\tau > 0$ the delay length. As explained in the introduction, the delay is implemented with a (signed) measure θ on $[0,\tau]$, and in the case of interest, we shall use the particular case $\theta = \delta_0 - \delta_\tau$. All the stochastic processes are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a right continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$. The state and control processes are denoted by $\mathbf{X} = (X_t)_{0\leq t\leq T}$ and $\mathbf{ff} = (\alpha_t)_{0\leq t\leq T}$. They are progressively measurable processes with values in $(\mathbb{R}^d)^N$ and a closed convex subset A of $(\mathbb{R}^d)^N$

respectively. They are linked by the dynamical equation:

$$dX_t =< \alpha_{[t]}, \theta > dt + \sigma dW_t \tag{3.17}$$

where $\mathbf{W} = (W_t)_{0\leq t\leq T}$ is a $(d \times N)$-dimensional $\mathbb{F}$-Brownian motion, σ is a positive constant or a matrix. We use the notation $\alpha_{[t]} = \alpha_{[t-\tau,t]}$ for the restriction of the path of α to the interval $[t-\tau, t]$. By convention, and unless specified otherwise, we extend functions defined on the interval $[0, T]$ to functions on $[-\tau, T + \tau]$ by setting them equal to 0 outside the interval $[0, T]$. Also, we use the bracket notation $< f, \theta >$ to denote the integral $\int_0^\tau f(s)\theta(ds)$.

We assume that the dynamics of the state X_t of the system are given by a stochastic differential equation (3.17) which we can rewrite in coordinate form if we denote by X_t^i the N components of X_t, in which case we can interpret X_t^i as the private state of player i:

$$dX_t^i = \left(\int_0^\tau \alpha_{t-s}^i \theta(ds)\right) dt + \sigma dW_t^i, \quad 0 \leq t \leq T, \tag{3.18}$$

where the components W_t^i, $i = 1, \cdots, N$ of W_t are independent standard Wiener processes, and the component processes $(\alpha_t^i)_{t\geq 0}$ can be interpreted as the strategies used by the individual players. As explained in the introduction, θ is a nonnegative measure on $[0, \tau]$ implementing the impact of the delay on the dynamics. Recall that the special case of interest corresponds to $\theta = \delta_0 - \delta_\tau$. We assume the initial conditions:

$$X_0^i = \xi^i, \qquad \text{and} \qquad \alpha_t^i = 0, \quad t \in [-\tau, 0). \tag{3.19}$$

The assumptions that the various states have the same volatility $\sigma > 0$ and the delay measure θ is the same for all the players are only made for convenience. These symmetry properties are important to derive mean field limits, but they are not really needed when we deal with finitely many players. The objective function of player i is given by (3.4) which we repeat here:

$$J^i(\alpha) \;=\; \mathbb{E}\bigg\{\int_0^T f_i(X_t,\alpha^i_t)dt + g_i(X_T)\bigg\}.$$

For the sake of simplicity, we assume that the cost f_i to player i depends only upon the control α^i_t of player i, and not on the controls α^j_t for $j\neq i$ of the other players. In the case of games with mean field interactions, the cost functions are often of the form $f_i(x,\alpha) = f(x^i,\overline{x},\alpha)$ and $g_i(x) = g(x^i,\overline{x})$, as in the particular case of the systemic risk model studied in this paper where:

$$f_i(x,\alpha^i) = f(x^i,\overline{x},\alpha^i) = \frac{(\alpha^i)^2}{2} - q\alpha^i(\overline{x}-x^i) + \frac{\epsilon}{2}(\overline{x}-x^i)^2,$$

for $q\geq 0$ and $\epsilon > 0$ as in (3.5), and:

$$g_i(x) = g(x^i,\overline{x}) = \frac{c}{2}\left(\overline{x}-x^i\right)^2, \quad c\geq 0,$$

as in (3.6) and with $q^2 < \epsilon$ to make sure that $f_i(x,\alpha)$ is convex in (x,α). Next, we introduce the system of adjoint equations.

3.4.2 The Adjoint Equations

For each player i and each given admissible control $\boldsymbol{\alpha}^i = (\alpha^i_t)_{0\leq t\leq T}$ for player i, we define the adjoint equation for player i as the Backward Stochastic Differential

Equation (BSDE):

$$dY_t^i = -\partial_x f_i(X_t, \alpha_t^i)dt + Z_t^i dW_t, \qquad 0 \le t \le T \tag{3.20}$$

with terminal condition $Y_T^i = \partial_x g_i(X_T)$, and we call the processes $\mathbf{Y}^i = (Y_t^i)_{0\le t\le T}$ and $\mathbf{Z}^i = (Z_t^i)_{0\le t\le T}$ the adjoint processes corresponding to the strategy $\boldsymbol{\alpha}^i = (\alpha_t^i)_{0\le t\le T}$ of player i. Notice that each $\mathbf{Y}^i$ has the same dimension as $\mathbf{X}$, namely $N \times d$ if d is the dimension of each individual player private state X_t^i, while each $\mathbf{Z}^i$ has dimension $N^2 \times d$. Accordingly, we shall use the notation $Y_t^i = (Y_t^{i,j})_{j=1,\cdots,N}$ where each $Y_t^{i,j}$ has the same dimension d as each of the private states X_t^j, and similarly, $Z_t^i = (Z_t^{i,j,k})_{j,k=1,\cdots,N}$. In the application of interest to us in this paper we have $d = 1$.

As before, the following notation will turn out to be helpful. If $\mathbf{Y} = (Y_t)_{0\le t\le T}$ is a progressively measurable process (scalar or multivariate) with continuous sample paths, we denote by $\tilde{\mathbf{Y}} = (\tilde{Y}_t)_{0\le t\le T}$ the process defined by:

$$\tilde{Y}_t = \mathbb{E}\left[\int_0^\tau Y_{t+s}\theta(ds) \,|\mathcal{F}_t\right] = \int_0^\tau \mathbb{E}[Y_{t+s}|\mathcal{F}_t]\,\theta(ds), \qquad 0 \le t \le T.$$

Moreover, for each $t \in [0,T]$, $x \in (\mathbb{R}^d)^N$ and $y \in \mathbb{R}^d$, we denote by $\hat{\alpha}^i(x,y)$ any $\alpha \in \mathbb{R}^d$ satisfying:

$$\partial_\alpha f_i(x,\alpha) = -y. \tag{3.21}$$

Under specific assumptions the implicit function theorem will provide existence of $\hat{\alpha}_i$, and regularity properties of this function with respect to the variables x and y.

3.4.3 Sufficient Condition for Optimality

Theorem 6. *Let us assume that the cost functions f_i are continuously differentiable in $(x, \alpha) \in (\mathbb{R}^d)^N \times \mathbb{R}^d$, and g_i are continuously differentiable on $(\mathbb{R}^d)^N$ with partial derivatives of (at most) linear growth, and that:*

(i) the functions g_i are convex;

(ii) the functions $(x, \alpha) \mapsto f_i(x, \alpha)$ are convex.

If $\boldsymbol{\alpha} = (\alpha_t^1, \cdots, \alpha_t^N)_{0\le t\le T}$ is an admissible adapted (open loop) strategy profile, and $(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = ((X_t^1, \cdots, X_t^N), (Y_t^1, \cdots, Y_t^N), (Z_t^1, \cdots, Z_t^N))$ are adapted process such that the dynamical equation (3.17) *and the adjoint equations* (3.20) *are satisfied for the controls $\alpha_t^i = \hat{\alpha}^i(X_t, \tilde{Y}_t^{i,i})$, then the strategy profile $\boldsymbol{\alpha} = (\alpha_t^1, \cdots, \alpha_t^N)_{0\le t\le T}$ is an open loop Nash equilibrium.*

Proof. We follow the proof given in [5] in the case without delay. We fix $i \in \{1, \cdots, N\}$, a generic admissible control strategy $(\beta_t)_{0\le t\le T}$ for player i, and for the sake of simplicity, we denote by X' the state $X_t^{(\hat{\alpha}^{-i},\beta)}$ controlled by the strategies $(\hat{\alpha}^{-i}, \beta)$. The function g_i being convex, almost surely, we have:

$$
\begin{aligned}
&g_i(X_T) - g_i(X'_T) \\
&\quad \le (X_T - X'_T) \cdot \partial_x g_i(X_T) \\
&\quad = (X_T - X'_T) \cdot Y_T^i \\
&\quad = \int_0^T (X_t - X'_t)\, dY_t^i + \int_0^T Y_t^i\, d(X_t - X'_t) \\
&\quad = -\int_0^T (X_t - X'_t) \cdot \partial_x f_i(X_t, \alpha_t^i)\, dt + \int_0^T Y_t^i \cdot < \alpha_{[t]} - (\hat{\alpha}^{-i}, \beta)_{[t]}, \theta > \; dt \; + \text{martingale} \\
&\quad = -\int_0^T (X_t - X'_t) \cdot \partial_x f_i(X_t, \alpha_t^i)\, dt + \int_0^T Y_t^{i,i} \cdot < \alpha_{[t]}^i - \beta_{[t]}, \theta > \; dt \; + \text{martingale.}
\end{aligned}
$$

Notice that we can use the classical form of integration by parts is due to the fact that the volatilities of all the states are the same constant σ. Taking expectations of both sides and plugging the result into

$$J^i(\boldsymbol{\alpha}) - J^i((\boldsymbol{\alpha}^{-i}, \boldsymbol{\beta})) = \mathbb{E}\bigg\{ \int_0^T [f_i(X_t, \alpha_t^i) - f_i(X_t', \beta_t)]dt \bigg\} + \mathbb{E}\{g_i(X_T) - g_i(X_T')\},$$

we get:

$$\begin{aligned} & J^i(\boldsymbol{\alpha}) - J^i((\boldsymbol{\alpha}^{-i}, \boldsymbol{\beta})) \\ & \quad \leq \mathbb{E}\left\{ \int_0^T [f_i(X_t, \alpha_t^i) - f_i(X_t', \beta_t)]dt - \int_0^T (X_t - X_t') \cdot \partial_x f_i(X_t, \alpha_t^i)\, dt \right\} \\ & \qquad\qquad + \mathbb{E}\left\{ \int_0^T Y_t^{i,i} \cdot < \alpha_{[t]}^i - \beta_{[t]}, \theta > \; dt \right\} \\ & \quad \leq \mathbb{E}\left\{ \int_0^T [\alpha_t^i - \beta_t]\partial_\alpha f_i(X_t, \alpha_t^i) + Y_t^{i,i} \cdot < \alpha_{[t]}^i - \beta_{[t]}, \theta > \; dt \right\}. \end{aligned} \tag{3.22}$$

Notice that:

$$\begin{aligned} \mathbb{E}\bigg[\int_0^T Y_t^{i,i} \cdot < \alpha_{[t]}^i - \beta_{[t]}, \theta > dt \bigg] &= \mathbb{E}\bigg[\int_0^\tau \bigg(\int_{-s}^{T-s} Y_{t+s}^{i,i}[\alpha_t^i - \alpha_t^i]dt \bigg) \theta(ds) \bigg] \\ &= \int_0^\tau \int_0^T \mathbb{E}[Y_{t+s}^{i,i}[\alpha_t^i - \beta_t]dt\, \theta(ds)] \\ &= \int_0^\tau \int_0^T \mathbb{E}[\mathbb{E}[Y_{t+s}^{i,i}|\mathcal{F}_t][\alpha_t^i - \beta_t]dt\, \theta(ds)] \\ &= \mathbb{E}\bigg[\int_0^\tau \int_0^T \bigg(\int_0^\tau \mathbb{E}[Y_{t+s}^{i,i}|\mathcal{F}_t]\theta(ds) \bigg) [\alpha_t^i - \beta_t]dt \bigg] \\ &= \mathbb{E}\bigg[\int_0^T \widetilde{Y}_t^{i,i} \cdot [\alpha_t^i - \beta_t]dt \bigg]. \end{aligned}$$

Consequently:

$$\begin{aligned} J^i(\boldsymbol{\alpha}) - J^i((\boldsymbol{\alpha}^{-i}, \boldsymbol{\beta})) &\leq \mathbb{E}\left\{ \int_0^T \Big([\alpha_t^i - \beta_t]\partial_\alpha f_i(X_t, \alpha_t^i) + \widetilde{Y}_t^{i,i} \cdot [\alpha_t^i - \beta_t] \Big)\, dt \right\} \\ &= 0 \end{aligned}$$

by definition (3.21) of $\hat{\alpha}(t, \hat{X}_t, \widetilde{Y}_t^{i,i})$. $\square$

Example

We shall use the above result when $d = 1$, $\theta = \delta_0 - \delta_{-\tau}$ so that $< \alpha_{[t]}, \theta >= \int_0^\delta \alpha_{t-\tau}\, \theta(d\tau) = \alpha_t - \alpha_{t-\delta}$, and the cost functions are given by (3.5) and (3.6), namely:

$$f_i(x, \alpha) = \frac{1}{2}\alpha^2 - q\alpha(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2$$

for some positive constants q and ϵ satisfying $q < \epsilon^2$ which guarantees that the functions f_i are convex. Notice that relation (3.21) gives $\hat{\alpha}^i(x, y) = -y - q(x^i - \overline{x})$. To derive the adjoint equations we compute:

$$\partial_{x^i} f_i(x, \alpha) = (1 - \frac{1}{N})[q\alpha + \epsilon(x^i - \overline{x})], \qquad \text{and} \qquad \partial_{x^j} f_i(x, \alpha) = -\frac{1}{N}[q\alpha + \epsilon(x^i - \overline{x})],$$

for $j \neq i$. Accordingly, the system of forward and advanced backward equations identified in the above theorem reads:

$$\begin{cases} dX_t^i = - < \widetilde{Y}_{[t]}^{i,i} + q(X_{[t]}^i - \overline{X}_{[t]}), \theta > dt + \sigma dW_t^i, \qquad i = 1, \cdots, N \\ dY_t^{i,j} = (\delta_{i,j} - \frac{1}{N})[q\widetilde{Y}_t^{i,j} + (q^2 - \epsilon)(X_t^i - \overline{X}_t)]dt + \sum_{k=1}^N Z_t^{i,j,k} dW_t^k \qquad i, j = 1, \cdots, N \end{cases} \tag{3.23}$$

where we used the Kronecker symbol $\delta_{i,j}$ which is equal to 1 if $i = j$ and 0 if $i \neq j$. If we specialize this system to the case $\theta = \delta_0 - \delta_\tau$, we have $\widetilde{Y}_t^{i,j} = Y_t^{i,j} - \mathbb{E}[Y_{t+\tau}^{i,j} | \mathcal{F}_t]$, so that the forward advanced-backward system reads:

$$
\begin{cases}
dX_t^i = (-Y_t^{i,i} + Y_{t-\tau}^{i,i} + \mathbb{E}[Y_{t+\tau}^{i,i}|\mathcal{F}_t] - \mathbb{E}[Y_t^{i,i}|\mathcal{F}_{t-\tau}] \\
\qquad\qquad -q[X_t^i - X_{t-\tau}^i - \overline{X}_t + \overline{X}_{t-\tau}])dt + \sigma dW_t^i, \qquad i = 1, \cdots, N \\
dY_t^{i,j} = (\delta_{i,j} - \frac{1}{N})[qY_t^{i,j} - q\mathbb{E}[Y_{t+\tau}^{i,j}|\mathcal{F}_t] + (q^2 - \epsilon)(X_t^j - \overline{X}_t)]dt + \sum_{k=1}^N Z_t^{i,j,k} dW_t^k \\
\qquad\qquad i, j = 1, \cdots, N.
\end{cases}
\tag{3.24}
$$

The version of the stochastic maximum principle proved in Theorem 6 reduces the problem of the existence of Nash equilibria for the system, to the solution of forward anticipated-backward stochastic differential equation. The following result can be used to resolve the existence issue but first we make the following remark which is key in term of financial interpretation.

Remark 12 (**Clearing House Property**). *In the present situation, in contrast with the case without delay presented in Section 3.3, we will not be able to derive explicit formulas for the equilibrium optimal strategies such as (3.14). However, it is remarkable to see that the clearing house property $\sum \alpha^i = 0$ still holds. Indeed, setting $i = j$ in (3.23) and summing over N to derive an equation for $\overline{Y}_t = \frac{1}{N}\sum_{i=1}^N Y_t^{i,i}$ and $\overline{Z}_t^k = \frac{1}{N}\sum_{i=1}^N Z_t^{i,i,k}$, we find:*

$$
d\overline{Y}_t = -\left(\frac{1}{N} - 1\right) q\widetilde{\overline{Y}}_t dt + \sum_{k=1}^N \overline{Z}_t^k dW_t^k, \quad t \in [0, T],
$$

with terminal condition $\overline{Y}_t = 0$ for $t \in [T, T + \tau]$. This equation admits the unique solution:

$$
\overline{Y}_t = 0, \quad t \in [0, T + \tau], \quad \textit{and} \quad \overline{Z}_t^k = 0, \; k = 1, \cdots, N, \; t \in [0, T].
$$

and as a result,

$$\overline{\hat{\alpha}}_t = -\widetilde{\overline{Y}}_t = 0. \tag{3.25}$$

In what follows, on the top of $q^2 < \epsilon$, we further assume that

$$q^2(1 - \frac{1}{2N})^2 \leq \epsilon(1 - \frac{1}{N}), \tag{3.26}$$

which is satisfied for N large enough, or q small enough.

Theorem 7. *The FABSDE* (3.24) *has a unique solution.*

Proof. Assuming that $(\check{X}, \check{Y}, (\check{Z}^k)_{k=1,\cdots,N})$ is given as an input, we solve the system (3.37) for $\lambda = \lambda_0$ and the processes ϕ_t, ψ_t^k, r_t and the random variable ζ replaced according to the prescriptions:

$$\begin{aligned}
\phi_t &\leftarrow \phi_t + \kappa[\check{Y}_t - < \widetilde{\check{Y}}_{[t]} + q\check{X}_{[t]}, \theta >] \\
\psi_t^k &\leftarrow \psi_t^k + \kappa[\check{Z}_t^k + \sigma(\frac{1}{N} - \delta_{i,k})], \quad k = 1, \cdots, N \\
r_t &\leftarrow r_t + \kappa[\check{X}_t + (1 - \frac{1}{N})[q\widetilde{\check{Y}}_t + \left(q^2 - \epsilon\right) \check{X}_t]] \\
\zeta &\leftarrow \zeta + \kappa[-\check{X}_T + c(1 - \frac{1}{N})\check{X}_T],
\end{aligned}$$

and denote the solution by $(X, Y, (Z^k)_{k=1,\cdots,N})$. In this way, we defined a mapping

$$\Phi : (\check{X}, \check{Y}, (\check{Z}^k)_{k=1,\cdots,N}) \to \Phi(\check{X}, \check{Y}, (\check{Z}^k)_{k=1,\cdots,N}) = (X, Y, (Z^k)_{k=1,\cdots,N}),$$

and the proof consists in proving that the latter is a contraction for small enough $\kappa > 0$.

Consider $(\widehat{X}, \widehat{Y}, (\widehat{Z}^k)_{k=1,\cdots,N}) = (X - X', Y - Y', (Z^k - Z^{k\prime})_{k=1,\cdots,N})$ where $(X, Y, (Z^k)_{k=1,\cdots,N})$ and $(X', Y', (Z^{k'})_{k=1,\cdots,N})$ are the corresponding image using inputs $(\check{X}, \check{Y}, (\check{Z}^k)_{k=1,\cdots,N})$ and $(\check{X}', \check{Y}', (\check{Z}^{k\prime})_{k=1,\cdots,N})$. We obtain

$$
\begin{aligned}
d\widehat{X}_t &= [-(1-\lambda_0)\widehat{Y}_t - \lambda_0 < \widetilde{\widehat{Y}}_{[t]} + q\widehat{X}_{[t]}, \theta > +\kappa[\widehat{\check{Y}}_t - < \widetilde{\widehat{\check{Y}}}_{[t]} + q\widehat{\check{X}}_{[t]}, \theta >]]dt \\
&\quad + \sum_{k=1}^{N}[-(1-\lambda_0)\widehat{Z}_t^k + \kappa\widehat{\check{Z}}_t^k]dW_t^k \\
d\widehat{Y}_t &= [-(1-\lambda_0)\widehat{X}_t + \lambda_0(1-\frac{1}{N})[q\widetilde{\widehat{Y}}_t + (q^2-\epsilon)\widehat{X}_t] \\
&\quad +\kappa[\widehat{\check{X}}_t + (1-\frac{1}{N})[q\widetilde{\widehat{\check{Y}}}_t + (q^2-\epsilon)\widehat{\check{X}}_t]]]dt + \sum_{k=1}^{N}\widehat{Z}_t^k dW_t^k, \qquad (3.27)
\end{aligned}
$$

with initial condition $\widehat{X}_0 = 0$ and terminal conditions $\widehat{Y}_T = (1-\lambda_0)\widehat{X}_T + \lambda_0 c(1-\frac{1}{N})\widehat{X}_T - \kappa\widehat{\check{X}}_T + \kappa c(1-\frac{1}{N})\widehat{\check{X}}_T$ and $\widehat{Y}_t = 0$ for $t \in (T, T+\tau]$ in the case of $c > 0$, and $\widehat{Y}_T = 0$ and $\widehat{Y}_t = 0$ for $t \in (T, T+\tau]$ in the case of $c = 0$. As we stated in the text, we only give the proof in the case $c = 0$ to simplify the notation. The proof of the case $c > 0$ is a easy modification. Using the form of the terminal condition and Itô's formula, we get

$$
\begin{aligned}
&0 = \mathbb{E}[\widehat{Y}_T\widehat{X}_T] \\
&= \mathbb{E}\int_0^T \Big\{\widehat{Y}_t\Big[-(1-\lambda_0)\widehat{Y}_t - \lambda_0 < \widetilde{\widehat{Y}}_{[t]} + q\widehat{X}_{[t]}, \theta > +\kappa[\widehat{\check{Y}}_t - < \widetilde{\widehat{\check{Y}}}_{[t]} + q\widehat{\check{X}}_{[t]}, \theta >]\Big] \\
&\quad +\widehat{X}_t\Big[-(1-\lambda_0)\widehat{X}_t + \lambda_0(1-\frac{1}{N})[q\widetilde{\widehat{Y}}_t + (q^2-\epsilon)\widehat{X}_t] \\
&\quad +\kappa[\widehat{\check{X}}_t + (1-\frac{1}{N})[q\widetilde{\widehat{\check{Y}}}_t + (q^2-\epsilon)\widehat{\check{X}}_t]]\Big] - (1-\lambda_0)\sum_{k=1}^{N}|\widehat{Z}_t^k|^2 + \kappa\sum_{k=1}^{N}\widehat{Z}_t^k\widehat{\check{Z}}_t^k\Big\}dt
\end{aligned}
\qquad (3.28)
$$

$$
\begin{aligned}
&= -(1-\lambda_0)\mathbb{E}\int_0^T |\widehat{Y}_t|^2 dt - \lambda_0 \mathbb{E}\int_0^T \widehat{Y}_t < \widetilde{\widehat{Y}}_{[t]} + q\widehat{X}_{[t]}, \theta > dt \\
&\quad +\kappa\mathbb{E}\int_0^T \widehat{Y}_t[\widehat{\check{Y}}_t - < \widetilde{\widehat{\check{Y}}}_{[t]} + q\widehat{\check{X}}_{[t]}, \theta >]dt - (1-\lambda_0)\mathbb{E}\int_0^T |\widehat{X}_t|^2 dt \\
&\qquad +\lambda_0(1-\frac{1}{N})\mathbb{E}\int_0^T \widehat{X}_t[q\widetilde{\widehat{Y}}_t + (q^2-\epsilon)\widehat{X}_t]dt \\
&\qquad +\kappa\mathbb{E}\int_0^T \widehat{X}_t[\widehat{\check{X}}_t + (1-\frac{1}{N})[q\widetilde{\widehat{\check{Y}}}_t + (q^2-\epsilon)\widehat{\check{X}}_t]]dt \\
&\qquad -(1-\lambda_0)\mathbb{E}\int_0^T \sum_{k=1}^N |\widehat{Z}_t^k|^2 dt + \kappa \sum_{k=1}^N \widehat{Z}_t^k \widehat{\check{Z}}_t^k dt \qquad (3.29)
\end{aligned}
$$

and rearranging the terms we find:

$$
\begin{aligned}
&(1-\lambda_0)[\mathbb{E}\int_0^T |\widehat{X}_t|^2 dt + \mathbb{E}\int_0^T |\widehat{Y}_t|^2 dt + \mathbb{E}\int_0^T \sum_{k=1}^N |\widehat{Z}_t^k|^2 \; dt] \\
&= \kappa\mathbb{E}\int_0^T \widehat{X}_t \widehat{\check{X}}_t dt - \lambda_0\mathbb{E}\int_0^T \widehat{Y}_t < \widetilde{\widehat{Y}}_{[t]} + q\widehat{X}_{[t]}, \theta > dt \\
&\quad +\kappa\mathbb{E}\int_0^T \widehat{Y}_t[\widehat{\check{Y}}_t - < \widetilde{\widehat{\check{Y}}}_{[t]} + q\widehat{\check{X}}_{[t]}, \theta >]dt \\
&\quad +\lambda_0(1-\frac{1}{N})\mathbb{E}\int_0^T \widehat{X}_t[q\widetilde{\widehat{Y}}_t + (q^2-\epsilon)\widehat{X}_t]dt \\
&\quad +\kappa(1-\frac{1}{N})\mathbb{E}\int_0^T \widehat{X}_t[q\widetilde{\widehat{\check{Y}}}_t + (q^2-\epsilon)\widehat{\check{X}}_t]]dt + \kappa\mathbb{E}\int_0^T \sum_{k=1}^N \widehat{Z}_t^k \widehat{\check{Z}}_t^k \; dt \qquad (3.30)
\end{aligned}
$$

Letting $\mu = \epsilon(1-\frac{1}{N}) - q^2(1-\frac{1}{2N})^2 > 0$, we obtain:

$$
\begin{aligned}
&(1-\lambda_0+\lambda_0\mu)\mathbb{E}\int_0^T |\widehat{X}_t|^2 dt + (1-\lambda_0)\mathbb{E}\int_0^T |\widehat{Y}_t|^2 dt + (1-\lambda_0)\mathbb{E}\int_0^T \sum_{k=1}^N |\widehat{Z}_t^k|^2 dt \\
&\leq \kappa\mathbb{E}\int_0^T \widehat{Y}_t[\widehat{\check{Y}}_t - < \widetilde{\widehat{\check{Y}}}_{[t]} + q\widehat{\check{X}}_{[t]}, \theta >]dt \\
&\quad +\kappa(1-\frac{1}{N})\mathbb{E}\int_0^T \Big(\big(q^2-\epsilon\big)\,\widehat{\check{X}}_t + q\widetilde{\widehat{\check{Y}}}_t)\widehat{X}_t dt + \kappa\mathbb{E}\int_0^T \sum_{k=1}^N \widehat{Z}_t^k \widehat{\check{Z}}_t^k dt,
\end{aligned}
$$

and a straightforward computation using repeatedly Cauchy–Schwarz and Jensen's inequalities leads to the existence of a positive constant K_1 such that

$$
\begin{aligned}
&(1-\lambda_0+\lambda_0\mu)\mathbb{E}\int_0^T|\widehat{X}_t|^2dt+(1-\lambda_0)\mathbb{E}\int_0^T|\widehat{Y}_t|^2dt+(1-\lambda_0)\mathbb{E}\int_0^T\sum_{k=1}^N|\widehat{Z}_t^k|^2dt\\
&\quad\leq \kappa K_1\Big\{\mathbb{E}\int_0^T|\widehat{X}_t|^2dt+\mathbb{E}\int_0^T|\widehat{Y}_t|^2dt+\mathbb{E}\int_0^T\sum_{k=1}^N|\widehat{Z}_t^k|^2dt\\
&\qquad+\mathbb{E}\int_0^T|\widehat{\widehat{X}}_t|^2dt+\mathbb{E}\int_0^T|\widehat{\widehat{Y}}_t|^2dt+\mathbb{E}\int_0^T\sum_{k=1}^N|\widehat{\widehat{Z}}_t^k|^2dt\Big\}.
\end{aligned}
$$

Referring to [3], applying Itô's formula to $|\widehat{X}_t|^2$ and $|\widehat{Y}_t|^2$, Gronwall's inequality, and again Cauchy-Schwarz and Jensen's inequalities, owing to $0\leq\lambda_0\leq 1$, we obtain a constant $K_2>0$ independent of λ_0 so that

$$
\begin{aligned}
\sup_{0\leq t\leq T}\mathbb{E}|\widehat{X}_t|^2\leq \kappa K_2&\left\{\mathbb{E}\int_0^T|\widehat{\widehat{X}}_t|^2+|\widehat{\widehat{Y}}_t|^2+\sum_{k=1}^N|\widehat{\widehat{Z}}_t^k|^2dt\right\}\\
&+K_2\left\{\mathbb{E}\int_0^T|\widehat{Y}_t|^2+\sum_{k=1}^N|\widehat{Z}_t^k|^2dt\right\},
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\int_0^T|\widehat{X}_t|^2dt\leq \kappa K_2T&\left\{\mathbb{E}\int_0^T|\widehat{\widehat{X}}_t|^2+|\widehat{\widehat{Y}}_t|^2+\sum_{k=1}^N|\widehat{\widehat{Z}}_t^k|^2dt\right\}\\
&+K_2T\left\{\mathbb{E}\int_0^T|\widehat{Y}_t|^2+\sum_{k=1}^N|\widehat{Z}_t^k|^2dt\right\},
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\int_0^T|\widehat{Y}_t|^2+\sum_{k=1}^N|\widehat{Z}_t^k|^2dt \quad\leq\quad & \kappa K_2\left\{\mathbb{E}\int_0^T|\widehat{\widehat{X}}_t|^2+|\widehat{\widehat{Y}}_t|^2+\sum_{k=1}^N|\widehat{\widehat{Z}}_t^k|^2dt\right\}\\
&+K_2\mathbb{E}\int_0^T|\widehat{X}_t|^2dt. \qquad (3.31)
\end{aligned}
$$

By using (3.31), there exists $0 < \mu' < \mu/K_2$ such that

$$\begin{aligned}
&\lambda_0\mu' K_2 \mathbb{E}\int_0^T |\widehat{X}_t|^2 dt \\
&\geq \lambda_0\mu' \left(\mathbb{E}\int_0^T |\widehat{Y}_t|^2 + \sum_{k=1}^N |\widehat{Z}_t^k|^2 dt\right) \\
&\qquad -\lambda_0\mu'\kappa K_2 \left\{\mathbb{E}\int_0^T |\widehat{\tilde{X}}_t|^2 + |\widehat{\tilde{Y}}_t|^2 + \sum_{k=1}^N |\widehat{\tilde{Z}}_t^k|^2 dt\right\} \\
&\geq \lambda_0\mu' \left(\mathbb{E}\int_0^T |\widehat{Y}_t|^2 + \sum_{k=1}^N |\widehat{Z}_t^k|^2 dt\right) \\
&\qquad -\mu'\kappa K_2 \left\{\mathbb{E}\int_0^T |\widehat{\tilde{X}}_t|^2 + |\widehat{\tilde{Y}}_t|^2 + \sum_{k=1}^N |\widehat{\tilde{Z}}_t^k|^2 dt\right\}
\end{aligned} \tag{3.32}$$

Therefore, we have

$$\begin{aligned}
&\Big(1 - \lambda_0 + \lambda_0(\mu - K_2\mu')\Big)\mathbb{E}\int_0^T |\widehat{X}_t|^2 dt \\
&+(1 - \lambda_0 + \lambda_0\mu')\mathbb{E}\int_0^T |\widehat{Y}_t|^2 dt + (1 - \lambda_0 + \lambda_0\mu')\mathbb{E}\int_0^T \sum_{k=1}^N |\widehat{Z}_t^k|^2 dt \\
\leq\ & \kappa K_1 \Big\{\mathbb{E}\int_0^T |\widehat{X}_t|^2 dt + \mathbb{E}\int_0^T |\widehat{Y}_t|^2 dt + \mathbb{E}\int_0^T \sum_{k=1}^N |\widehat{Z}_t^k|^2 dt \\
&\quad + \mathbb{E}\int_0^T |\widehat{\tilde{X}}_t|^2 dt + \mathbb{E}\int_0^T |\widehat{\tilde{Y}}_t|^2 dt + \mathbb{E}\int_0^T \sum_{k=1}^N |\widehat{\tilde{Z}}_t^k|^2 \Big\} \\
&+\kappa K_2\mu' \left\{\mathbb{E}\int_0^T |\widehat{\tilde{X}}_t|^2 dt + \mathbb{E}\int_0^T |\widehat{\tilde{Y}}_t|^2 dt + \mathbb{E}\int_0^T \sum_{k=1}^N |\widehat{\tilde{Z}}_t^k|^2 dt\right\}.
\end{aligned} \tag{3.33}$$

Note that since $\mu - K_2\mu'$ and μ' stay in positive, we have $(1-\lambda_0+\lambda_0(\mu - K_2\mu')) \geq \mu''$ and $(1 - \lambda_0 + \lambda_0\mu') \geq \mu''$ where for some $\mu'' > 0$. Combining the inequalities (3.31-

3.33), we obtain

$$\mathbb{E}\int_0^T |\widehat{X}_t|^2 dt + \mathbb{E}\int_0^T |\widehat{Y}_t|^2 dt + \mathbb{E}\int_0^T \sum_{k=1}^N |\widehat{Z}_t^k|^2 dt$$
$$\leq \kappa K \left(\mathbb{E}\int_0^T |\widehat{\widehat{X}}_t|^2 dt + \mathbb{E}\int_0^T |\widehat{\widehat{Y}}_t|^2 dt + \mathbb{E}\int_0^T \sum_{k=1}^N |\widehat{\widehat{Z}}_t^k|^2 dt \right), \quad (3.34)$$

where the constant K depends upon μ', μ'', K_1, K_2, and T. Hence, Φ is a strict contraction for sufficiently small κ. □

Remark 13. *While this theorem gives existence of open loop Nash equilibria for the model, it is unlikely that uniqueness holds. However, the cost functions f_i and g_i depending only upon x^i and $\overline{x}$, one could consider the mean field game problem corresponding to the limit $N \to \infty$, and in this limiting regime, it is likely that the strict convexity of the cost functions could be used to prove some form of uniqueness of the solution of the equilibrium problem.*

Proof. We first solve the system considering only the case $j = i$. Once this is done, we should be able to inject the process $X_t = (X_t^1, \cdots, X_t^N)$ so obtained into the equation for $dY_t^{i,j}$ for $j \neq i$, and solve this advanced equation with random coefficients.

Summing over $i = 1, \cdots, N$ the equations for X^i in (3.23), using the clearing house property of Remark 12, and denoting $\overline{\xi} = \frac{1}{N}\sum_{i=1}^N \xi^i$ give

$$\overline{X}_t = \overline{\xi} + \frac{\sigma}{N}\sum_{i=1}^N W_t^i, \quad t \in [0, T]. \quad (3.35)$$

Therefore, without loss of generality, we can work with the "centered" variables $X_t^{i,c} = X_t^i - \overline{X}_t$, $Y_t^{i,i,c} = Y_t^{i,i} - \overline{Y}_t = Y_t^{i,i}$, and $Z_t^{i,i,k,c} = Z_t^{i,i,k} - \overline{Z}_t^k = Z_t^{i,i,k}$ which must satisfy the system:

$$\begin{cases} dX_t^{i,c} = - < \widetilde{Y}_{[t]}^{i,i} + qX_{[t]}^{i,c}, \theta > dt + \sigma \sum_{k=1}^N (\delta_{i,k} - \frac{1}{N}) dW_t^k, \\ dY_t^{i,i} = (1 - \frac{1}{N})[q\widetilde{Y}_t^{i,i} + (q^2 - \epsilon)X_t^{i,c}]dt + \sum_{k=1}^N Z_t^{i,i,k} dW_t^k \end{cases} \tag{3.36}$$

with $X_0^{i,c} = \xi^{i,c} := \xi^i - \overline{\xi}$, $Y_T^{i,i} = -c\left(\frac{1}{N} - 1\right) X_T^{i,c}$, and $Y_t^{i,i} = 0$ for $t \in (T, T+\tau]$ for $i = 1, \cdots, N$. We solve this system by extending the continuation method (see for example [38] and [39]) to the case of stochastic games. We consider a system which is written as a perturbation of the previous one without delay. Since we now work with $i \in \{1, \cdots, N\}$ fixed, we drop the exponent i from the notation for the sake of readability of the formulas.

$$\begin{cases} dX_t^\lambda = [-(1-\lambda)Y_t^\lambda - \lambda < \widetilde{Y}_{[t]}^\lambda + qX_{[t]}^\lambda, \theta > + \phi_t]dt \\ \qquad\qquad + \sum_{k=1}^N [-(1-\lambda)Z_t^{k,\lambda} + \lambda\sigma(\delta_{i,k} - \frac{1}{N}) + \psi_t^k] dW_t^k, \\ dY_t^\lambda = [-(1-\lambda)X_t^\lambda + \lambda(1-\frac{1}{N})[q\widetilde{Y}_t^\lambda + (q^2-\epsilon)X_t^\lambda] + r_t]dt + \sum_{k=1}^N Z_t^{k,\lambda} dW_t^k \end{cases} \tag{3.37}$$

with initial condition $X_0^\lambda = \xi^{i,c}$ and terminal condition $Y_T^\lambda = (1-\lambda)X_T^\lambda - \lambda c\left(\frac{1}{N} - 1\right) X_T^\lambda + \zeta^{i,i}$ and $Y_t^\lambda = 0$ for $t \in (T, T+\tau]$ in the case of $c > 0$, and $Y_T^\lambda = \zeta^{i,i}$ and $Y_t^\lambda = 0$ for $t \in (T, T+\tau]$ in the case of $c = 0$.

Here (recall that i is now fixed), ϕ_t, ψ_t^k, r_t are for $k = 1, \cdots, N$, square integrable processes which will be chosen at each single step of the induction procedure. Also ζ is a $L^2(\Omega, \mathcal{F}_T)$ random variable. Observe that if $\lambda = 0$, the system (3.37) is a particular case of the system in Lemma 2.5 in [38] for which existence and uniqueness is established, and it becomes the system (3.36) when setting $\lambda = 1$, $\zeta^{i,i} = 0$, $\phi_t^i = 0$, $\psi_t^{i,i,k} = 0$, $r_t^{i,i} = 0$, $i = 1, \cdots, N$ and $k = 1, \cdots, N$, for $0 \le t \le T$. We only give the proof of existence and uniqueness for the solution of the system (3.36) in the case of $c = 0$. The same arguments can be used to treat the case $c > 0$.

The proof relies on the following technical result which we prove in the appendix.

Lemma 4. *If there exists* $\lambda_0 \in [0, 1)$ *such that for any* ζ *and* ϕ_t, r_t, ψ_t^k, $k = 1, \cdots, N$ *for* $0 \leq t \leq T$ *the system (3.37) admits a unique solution for* $\lambda = \lambda_0$*, then there exists* $\kappa_0 > 0$*, such that for all* $\kappa \in [0, \kappa_0)$*, (3.37) admits a unique solution for any* $\lambda \in [\lambda_0, \lambda_0 + \kappa)$.

Taking for granted the result of this lemma, we can prove existence and uniqueness for (3.37). Indeed, for $\lambda = 0$, the result is known. Using Lemma 4, there exists $\kappa_0 > 0$ such that (3.37) admits a unique solution for $\lambda = 0 + \kappa$ where $\kappa \in [0, \kappa_0)$. Repeating the inductive argument n times for $1 \leq n\kappa_0 < 1 + \kappa_0$ gives the result for $\lambda = 1$ and, therefore, the existence of the unique solution for (3.36). Since $X_t^{i,c} = X_t^i - \overline{X}_t$, $Y_t^{i,i,c} = Y_t^{i,i}$ and $Z_t^{i,i,k,c} = Z_t^{i,i,k}$, and $\overline{X}_t$ is given by (3.35), we obtain a unique solution $(X_t^i, Y_t^{i,i}, Z_t^{i,i,k})$ to the system (3.23). □

3.5 Hamilton-Jacobi-Bellman (HJB) Approach

In this section, we return to the particular case $\theta = \delta_0 - \delta_\tau$ of the drift given by the delayed control $\alpha_t - \alpha_{t-\tau}$. The HJB approach for delayed systems has been applied by [42] to a deterministic linear quadratic control problem. Later, [20] followed a similar approach for stochastic control problems. Here, we generalize the approach [20] based on an infinite dimensional representation and functional derivatives. We extend this approach to our stochastic game model with delay in order to identify a closed-loop Nash equilibrium.

Note that two specific features of our discussion require additional work for our argument to be fully rigorous at the mathematical level. First, the delayed control in the state equation appears as a mass at time $t-\tau$ and a smoothing argument as in [23] is needed. Second, we are using functional derivatives and proper function spaces should be introduced for our computations to be fully justified. However, since most of the functions we manipulate are linear or quadratic, we refrain from giving the details. In that sense, and for these two reasons, what follows is merely heuristic. A rigorous proof of the fact that the equilibrium identified in this section is actually a Nash equilibrium will be given in Section 3.6.

3.5.1 Infinite Dimensional Representation

Let $\mathbb{H}^N$ be the Hilbert space defined by

$$\mathbb{H}^N = \mathbb{R}^N \times L^2([-\tau, 0]; \mathbb{R}^N),$$

with the inner product

$$\langle z, \tilde{z} \rangle = z_0 \tilde{z}_0 + \int_{-\tau}^{0} z_1(\xi) \tilde{z}_1(\xi)\, d\xi,$$

where $z, \tilde{z} \in \mathbb{H}^N$, and z_0 and $z_1(.)$ correspond respectively to the $\mathbb{R}^N$-valued and $L^2([-\tau, 0]; \mathbb{R}^N)$-valued components.

By reformulating the system of coupled diffusions (3.1) in the Hilbert space $\mathbb{H}^N$, the system of coupled Abstract Stochastic Differential Equations (ASDE) for $Z = (Z^1, \cdots, Z^N) \in \mathbb{H}^N$ appears as

$$\begin{aligned} dZ_t &= (AZ_t + B\alpha_t)\, dt + G dW_t, \quad 0 \le t \le T, \\ Z_0 &= (\xi, 0) \in \mathbb{H}^N. \end{aligned} \tag{3.38}$$

where $W_t = (W_t^1, \cdots, W_t^N)$ is a standard N-dimensional Brownian motion and $\xi = (\xi^1, \cdots, \xi^N)$.

Here $Z_t = (Z_{0,t}, Z_{1,t,r})$, $r \in [-\tau, 0]$ corresponds to $(X_t, \alpha_{t-\tau-r})$ in the system of diffusions (3.1). In other words, for each time t, in order to find the dynamics of the states X_t, it is necessary to have X_t itself, and the past of the control $\alpha_{t-\tau-r}$, $r \in [-\tau, 0]$.

The operator $A : D(A) \subset \mathbb{H}^N \to \mathbb{H}^N$ is defined as

$$A : (z_0, z_1(r)) \to (z_1(0), -\frac{dz_1(r)}{dr}) \;\; a.e., \;\; r \in [-\tau, 0],$$

and its domain is

$$D(A) = \{(z_0, z_1(.)) \in \mathbb{H}^N : z_1(.) \in W^{1,2}([-\varnothing, 0]; \mathbb{R}^N), z_1(-\varnothing) = 0\}.$$

The adjoint operator of A is $A^* : D(A^*) \subset \mathbb{H}^N \to \mathbb{H}^N$ and is defined by

$$A^* : (z_0, z_1(r)) \to (0, \frac{dz_1(r)}{dr}) \;\; a.e., \;\; r \in [-\tau, 0],$$

with domain

$$D(A^*) = \{(w_0, w_1(.)) \in \mathbb{H}^N : w_1(.) \in W^{1,2}([-\varnothing, 0]); \mathbb{R}^N), w_0 = w_1(0)\}.$$

The operator $B : \mathbb{R}^N \to \mathbb{H}^N$ is defined by

$$B : u \to (u, -\delta_{-\tau}(r)u), \;\; r \in [-\tau, 0],$$

where $\delta_{-\tau}(.)$ is the Dirac measure at $-\tau$.

Remark 14. *Note that in [20], the case of pointwise delay is not considered as the above operator B becomes unbounded because of the dirac measure. Here, we still use the unbounded operator B (in a heuristic sense!) and for a rigorous treatment, we refer to [23] where they use partial smoothing to accommodate the case of pointwise delay.*

Finally, the operator $G : \mathbb{R}^N \to \mathbb{H}^N$ is defined by

$$G : z_0 \to (\sigma z_0, 0).$$

Remark 15. *Let Z_t be a weak solution of the system of coupled ASDEs (3.38) and X_t be a continuous solution of the system of diffusions (3.1), then, with a similar line of reasoning as in Proposition 2 in [20], it can be proved that $X_t = Z_{0,t}$, a.s. for all $t \in [0, T]$.*

3.5.2 System of Coupled HJB Equations

In order to use the dynamic programming principle for stochastic games (we refer to [5]) in search of closed-loop Nash equilibrium, the initial time is varied. At time $t \in [0, T]$, given initial state $Z_t = z$ (whose second component is the past of the control), bank i chooses the control α^i to minimize its objective function $J^i(t, z, \alpha)$.

$$J^i(t, z, \alpha) = \mathbb{E}\left\{ \int_t^T f_i(Z_{0,s}, \alpha_s^i)dt + g_i(Z_{0,T}) \mid Z_t = z \right\}, \tag{3.39}$$

In equilibrium, that is all other banks $j \neq i$ have optimized their objective function, bank i's value function $V^i(t,z)$ is

$$V^i(t,z) = \inf_{\alpha^i} J^i(t,z,\alpha). \tag{3.40}$$

The set of value functions $V^i(t,z)$, $i = 1, \cdots, N$ is a solution (in a suitable sense) of the following system of coupled HJB equations:

$$\begin{aligned}
&\partial_t V^i + \frac{1}{2} Tr(Q\partial_{zz}V^i) + \langle Az, \partial_z V^i\rangle + H_0^i(\partial_z V^i) = 0, \\
&V^i(T) = g_i,
\end{aligned} \tag{3.41}$$

where $Q = G * G$, and the Hamiltonian function $H_0^i(p^i) : \mathbb{H}^N \to \mathbb{R}$ is defined by

$$H_0^i(p^i) = \inf_{\alpha^i} [\langle B\alpha, p^i\rangle + f_i(z_0, \alpha^i)]. \tag{3.42}$$

Here, $p^i \in \mathbb{H}^N$ and can be written as $p^i = (p^{i,1}, \cdots, p^{i,N})$ where $p^{i,k} \in \mathbb{H}^1$, $k = 1, \cdots, N$. Given that $f_i(z_0, \alpha^i)$ is convex in (z_0, α^i),

$$\hat{\alpha}^i = -\langle B, p^{i,i}\rangle - q(z_0^i - \bar{z}_0). \tag{3.43}$$

Therefore,

$$\begin{aligned}
H_0^i(p) \quad &= \langle B\hat{\alpha}, p^i\rangle + f_i(z_0, \hat{\alpha}^i), \\
&= \sum_{k=1}^{N} \langle B, p^{i,k}\rangle \left(-\langle B, p^{k,k}\rangle - q(z_0^k - \bar{z}_0)\right) \\
&+ \frac{1}{2}\langle B, p^{i,i}\rangle^2 + \frac{1}{2}(\epsilon - q^2)(\bar{z}_0 - z_0^i)^2.
\end{aligned} \tag{3.44}$$

We then make the ansatz

$$
\begin{aligned}
V^i(t,z) &= E_0(t)(\bar z_0 - z_0^i)^2 - 2(\bar z_0 - z_0^i)\int_{-\tau}^{0} E_1(t,-\tau-s)(\bar z_{1,s} - z_{1,s}^i)ds \\
&+ \int_{-\tau}^{0}\int_{-\tau}^{0} E_2(t,-\tau-s,-\tau-r)(\bar z_{1,s} - z_{1,s}^i)(\bar z_{1,r} - z_{1,r}^i)dsdr + E_3(t),
\end{aligned}
\tag{3.45}
$$

where $E_0(t)$, $E_1(t,s)$, $E_2(t,s,r)$ and $E_3(t)$ are some deterministic functions to be determined. It is assumed that $E_2(t,s,r) = E_2(t,r,s)$.

Remark 16. *Note that the ansatz (3.45) depends on $z \in \mathbb{H}^N$ whose second component is the past of all banks' controls α. In other words, the value function $V^i(t,z)$ is an explicit function of the past of all banks' controls $\alpha_{t-\tau-r}$, $r \in [-\tau, 0]$.*

The derivatives of the ansatz (3.45) are as follows

$$
\begin{aligned}
\partial_t V^i &= \frac{dE_0(t)}{dt}(\bar z_0 - z_0^i)^2 - 2(\bar z_0 - z_0^i)\int_{-\tau}^{0} \frac{\partial E_1(t,-\tau-s)}{\partial t}(\bar z_{1,s} - z_{1,s}^i)ds \\
&+ \int_{-\tau}^{0}\int_{-\tau}^{0} \frac{\partial E_2(t,-\tau-s,-\tau-r)}{\partial t}(\bar z_{1,s} - z_{1,s}^i)(\bar z_{1,r} - z_{1,r}^i)dsdr + \frac{dE_3(t)}{dt},
\end{aligned}
\tag{3.46}
$$

$$
\partial_{z^j} V^i = \begin{bmatrix} 2E_0(t)(\bar z_0 - z_0^i) - 2\int_{-\tau}^{0} E_1(t,-\tau-s)(\bar z_{1,s} - z_{1,s}^i)ds \\ -2(\bar z_0 - z_0^i)E_1(t,s) + 2\int_{-\tau}^{0} E_2(t,-\tau-s,-\tau-r)(\bar z_{1,r} - z_{1,r}^i)dr \end{bmatrix} \left(\frac{1}{N} - \delta_{i,j}\right),
\tag{3.47}
$$

$$
\partial_{z^j z^k} V^i = \begin{bmatrix} 2E_0(t) & -2E_1(t,-\tau-s) \\ -2E_1(t,-\tau-s) & 2E_2(t,-\tau-s,-\tau-r) \end{bmatrix} \left(\frac{1}{N} - \delta_{i,j}\right)\left(\frac{1}{N} - \delta_{i,k}\right).
\tag{3.48}
$$

By plugging the ansatz (3.45) into the HJB equation (3.41), and collecting all the corresponding terms, the following set of equations are derived for $t \in [0,T]$ and $s, r \in [-\tau, 0]$.

The equation corresponding to the *constant* terms is

$$\frac{dE_3(t)}{dt} + (1 - \frac{1}{N})\sigma^2 E_0(t) = 0, \tag{3.49}$$

The equation corresponding to the $(\bar{z}_0 - z_0^i)^2$ terms is

$$\tfrac{dE_0(t)}{dt} + \tfrac{\epsilon}{2} = 2(1 - \tfrac{1}{N^2})(E_1(t,0) + E_0(t))^2 + 2q(E_1(t,0) + E_0(t)) + \tfrac{q^2}{2}. \tag{3.50}$$

The equation corresponding to the $(\bar{z}_0 - z_0^i)(\bar{z}_1 - z_1^i)$ terms is

$$\tfrac{\partial E_1(t,s)}{\partial t} - \tfrac{\partial E_1(t,s)}{\partial s} = 2(1 - \tfrac{1}{N^2})\left(E_1(t,0) + E_0(t) + \tfrac{q}{2(1-\frac{1}{N^2})}\right)\left(E_2(t,s,0) + E_1(t,s)\right). \tag{3.51}$$

The equation corresponding to the $(\bar{z}_1 - z_1^i)(\bar{z}_1 - z_1^i)$ terms is

$$\begin{aligned} &\tfrac{\partial E_2(t,s,r)}{\partial t} - \tfrac{\partial E_2(t,s,r)}{\partial s} - \tfrac{\partial E_2(t,s,r)}{\partial r} = \\ &2(1 - \tfrac{1}{N^2})\left(E_2(t,s,0) + E_1(t,s)\right)\left(E_2(t,r,0) + E_1(t,r)\right). \end{aligned} \tag{3.52}$$

The boundary conditions are

$$
\begin{aligned}
&E_0(T) = \frac{c}{2},\\
&E_1(T,s) = 0,\\
&E_2(T,s,r) = 0,\\
&E_2(t,s,r) = E_2(t,r,s),\\
&E_1(t,-\tau) = -E_0(t), \quad \forall t \in [0,T),\\
&E_2(t,s,-\tau) = -E_1(t,s), \quad \forall t \in [0,T),\\
&E_3(T) = 0.
\end{aligned}
\tag{3.53}
$$

Note that with these boundary conditions (at $t = T$), we have $V^i(T,z) = g_i(z_0) = \frac{c}{2}(\bar{z}_0 - z_0^i)^2$, as desired.

Remark 17. *The set of equations (3.49–3.52) on the domain $t \in [0,T]$, $s,r \in [-\tau,0]$, and with boundary conditions (3.53) admits a unique solution. This can be shown by following the steps of the proof of Theorem 6 in [1] and using a fixed point argument (see also [8]).*

If all the other banks choose their optimal controls, then the bank i's optimal strategy $\hat{\alpha}^i$, $i = 1,\cdots,N$ follows

$$
\begin{aligned}
\hat{\alpha}_t^i \quad &= -\langle B, \partial_{z^i} V^i \rangle - q(z_0^i - \bar{z}_0),\\
&= 2\left(1 - \frac{1}{N}\right)\left[\left(E_1(t,0) + E_0(t) + \frac{q}{2\left(1 - \frac{1}{N}\right)}\right)(\bar{z}_0 - z_0^i)\right.\\
&\left. - \int_{-\tau}^{0} \left(E_2(t,-\tau-s,0) + E_1(t,-\tau-s)\right)(\bar{z}_{1,s} - z_{1,s}^i) ds\right].
\end{aligned}
\tag{3.54}
$$

In terms of the original system of coupled diffusions (3.1), the closed-loop Nash equilibrium corresponds to

$$\begin{aligned}\hat{\alpha}_t^i &= 2\left(1-\frac{1}{N}\right)\left[\left(E_1(t,0)+E_0(t)+\frac{q}{2\left(1-\frac{1}{N}\right)}\right)(\bar{X}_t-X_t^i)\right.\\ &\left.+\int_{t-\tau}^{t}\left[E_2(t,s-t,0)+E_1(t,s-t)\right](\bar{\hat{\alpha}}_s-\hat{\alpha}_s^i)ds\right], \quad i=1,\cdots,N.\end{aligned} \tag{3.55}$$

Remark 18. *As pointed out in Remark 12 of Section 3.4, in the present situation we still have* $\sum_{i=1}^N \hat{\alpha}_t^i = 0$ *and therefore, in this equilibrium, the central bank serves as a clearing house (see also the discussion of Section 3.7).*

3.6 A Verification Theorem

In this section, we provide a verification theorem establishing that the strategies given by (3.55) correspond to a Nash equilibrium. Our solution is only *almost explicit* because the equilibrium strategies are given by the solution of a system of integral equations. This approach has been used by [1] to find the optimal control in a deterministic delayed linear quadratic control problem. Recently, [8] and [25] have applied this approach to delayed linear quadratic stochastic control problems. In this section, we generalize it to delayed linear -quadratic stochastic games differential games.

We recall that at time $t \in [0,T]$, given $x = (x^1, \cdots, x^N)$, which should be viewed as the state of the N banks at time t, and an A-valued function α on $[0,\tau)$, which should be viewed as their collective controls over the time interval $[t-\tau, t)$, bank i

chooses the strategy α^i to minimize its objective function

$$J^i(t,x,\alpha,\alpha^t)=\mathbb{E}\left\{\int_t^T f_i(X_s,\alpha_s^i)ds+g_i(X_T)\mid X_t=x,\alpha_{[t)}=\alpha\right\}. \qquad (3.56)$$

Here $\alpha_{[t)}$ is defined as the restriction of the path $s\mapsto\alpha_s$ to the interval $[t-\tau,t)$ and α^t is an admissible control strategy for the N banks over the time interval $[t,T]$. We denote by $\mathbb{A}^t$ this set of admissible strategies.

In the search for Nash equilibria, for each bank i, we assume that the banks $j\neq i$ chose their strategies $\alpha^{-i,t}$ for the *future* $[t,T]$, in which case, bank i's should choose a strategy $\alpha^{i,t}\in\mathbb{A}^{i,t}$ in order to try to minimize its objective function $J^i(t,x,\alpha,(\alpha^{i,t},\alpha^{-i,t}))$. As a result we define the value function $V^i(t,x,\alpha,\alpha^{-i,t})$ of bank i by:

$$V^i(t,x,\alpha,\alpha^{-i,t})=\inf_{\alpha^{i,t}\in\mathbb{A}^{i,t}}J^i(t,x,\alpha,(\alpha^{i,t},\alpha^{-i,t})). \qquad (3.57)$$

Because of the linear nature of the dynamics of the states, together with the quadratic nature of the costs, we expect that in equilibrium, the functions J^i and V^i to be quadratic functions of the state x and the past α of the control. This is consistent with the choices we made in the previous section. Accordingly, we write the functions V^i as

$$\begin{aligned}V^i(t,x,\alpha) &= E_0(t)(\bar{x}-x^i)^2+2(\bar{x}-x^i)\int_{t-\tau}^{t}E_1(t,s-t)(\bar{\alpha}_s-\alpha_s^i)ds\\ &+\int_{t-\tau}^{t}\int_{t-\tau}^{t}E_2(t,s-t,r-t)(\bar{\alpha}_s-\alpha_s^i)(\bar{\alpha}_r-\alpha_r^i)dsdr+E_3(t), \qquad (3.58)\end{aligned}$$

where the deterministic functions $E_i\,(i=0,\cdots,3)$, are the solutions of the system

(3.49–3.52) with the boundary conditions (3.53). We dropped the dependence of V^i upon its fourth parameter $\alpha^{-i,t}$ because the right hand side of (3.58) does not depend upon $\alpha^{-i,t}$.

The main result of this section is Proposition 2 below which says that any solution of the system (3.55) of integral equations provides a Nash equilibrium. For that reason, we first prove existence and uniqueness of solutions of these integral equations when they are recast as a fixed point problem in classical spaces of adapted processes. This is done in Lemma 5 below. We simplify the notation and we rewrite equation (3.55) for the purpose of the proof of the lemma. We set:

$$\varphi(t) = 2\left(1 - \frac{1}{N}\right)\left(E_1(t,0) + E_0(t) + \frac{q}{2\left(1-\frac{1}{N}\right)}\right)$$

and

$$\bar{\psi}(t,s) = [E_2(t, s-t, 0) + E_1(t, s-t)]\mathbf{1}_{[t-\tau,t]}(s)$$

so that equation (3.55) can be rewritten as:

$$\begin{aligned}
\hat{\alpha}^i_t &= \varphi(t)(\bar{X}_t - X^i_t) + \int_0^t \bar{\psi}(t,s)(\bar{\hat{\alpha}}_s - \hat{\alpha}^i_s)ds \\
&= \varphi(t)\left((\bar{\xi} - \xi^i) - \int_0^t [(\bar{\hat{\alpha}}_s - \hat{\alpha}^i_s) - (\bar{\hat{\alpha}}_{s-\tau} - \hat{\alpha}^i_{s-\tau})]ds + \sigma[\bar{W}_t - W^i_t]\right) \\
&\qquad + \int_0^t \bar{\psi}(t,s)(\bar{\hat{\alpha}}_s - \hat{\alpha}^i_s)ds. \qquad (3.59)
\end{aligned}$$

Summing these equations for $i = 1, \cdots, N$, we see that any solution should necessarily satisfy $\sum_{1\le i\le N} \hat{\alpha}^i = 0$, so that if we look for a solution of the system (3.55), we might as well restrict our search to processes satisfying $\bar{\hat{\alpha}}_t = 0$ for all $t \in [0,T]$.

So we denote by $\mathbb{R}^N_0$ the set of elements $x = (x^1, \cdots, x^N)$ of $\mathbb{R}^N$ satisfying

$\sum_{1\leq i\leq N} x^i = 0$, and by $\mathcal{H}_0^{2,N}$ the space of $\mathbb{R}_0^N$-valued adapted processes $a = (a_t)_{0\leq t\leq T}$ satisfying

$$\|a\|_0^2 := \mathbb{E}\left[\int_0^T |a_t|^2 dt\right] < \infty.$$

Clearly, $\mathcal{H}_0^{2,N}$ is a real separable Hilbert space for the scalar product derived from the norm $\|\cdot\|_0$ by polarization. For $a \in \mathcal{H}_0^{2,N}$ we define the $\mathbb{R}_0^N$-valued process $\Psi(a)$ by:

$$\Psi(a)_t^i = \varphi(t)(\bar{\xi} - \xi^i) + \sigma\varphi(t)[\bar{W}_t - W_t^i] + \int_0^t \psi(t,s)a_s^i ds, \qquad 0 \leq t \leq T,\ i = 1, \cdots, N. \tag{3.60}$$

where the function ψ is defined by $\psi(t,s) = 1 - \mathbf{1}_{[0,0\vee(t-\tau)]}(s) - \bar{\psi}(t,s)$. We shall use the fact that the functions φ and ψ are bounded.

Given the above set-up, existence and uniqueness of a solution to (3.55) is given by the following lemma whose proof mimics the standard proofs of existence and uniqueness of solutions of stochastic differential equations.

Lemma 5. *The map Ψ defined by* (3.60) *has a unique fixed point in $\mathcal{H}_0^{2,N}$.*

We now prove existence of Nash equilibria for the system.

Proposition 1. *The strategies $(\hat{\alpha}_t^i)_{0\leq t\leq T,\, i=1,\cdots,N}$ given by the solution of the system of integral equations (3.55) form a Nash equilibrium, and the corresponding value functions are given by (3.58).*

In other words, we prove that

$$V^i(0, \xi^i, \alpha_{[0)}) \leq J^i(0, \xi^i, \alpha_{[0)}, (\alpha^i, \hat{\alpha}^{-i})),$$

for any α^i, and choosing $\alpha^i = \hat{\alpha}^i$ gives:

$$V^i(0, \xi^i, \alpha_{[0)}) = J^i(0, \xi^i, \alpha_{[0)}, (\hat{\alpha}^i, \hat{\alpha}^{-i})).$$

Notice that the equilibrium strategies which we identified are in feedback form in the sense that each $\hat{\alpha}^i_t$ is a deterministic function of the trajectory $X_{[0,t]}$ of the past of the state. Notice also that there is absolutely nothing special with the time $t = 0$ and the initial condition $X_0 = \xi, \alpha_{[0)} = 0$. Indeed for any $t \in [0, T]$ and $\mathbb{R}^N$-valued square integrable random variable ζ, the same proof can be used to construct a Nash equilibrium for the game over the interval $[t, T]$ and any initial condition $(X_t = \zeta, \alpha_{[t)})$.

Proof. We fix an arbitrary $i \in \{1, \cdots, N\}$, an admissible control $\alpha^i \in \mathbb{A}^{-i}$ for player i, and we assume that the state process $(X_t)_{0 \leq t \leq T}$ for the N banks is controlled by $(\alpha^i_t, \hat{\alpha}^i_t)_{0 \leq t \leq T}$ where $(\hat{\alpha}^k_t)_{0 \leq t \leq T, k=1,\cdots,N}$ solves the system of integral equations (3.55). Next, we apply Itô's formula to $V^i(t, X_t, \alpha_{[t)})$ where the function V^i is defined by (3.58). We obtain

$$
\begin{aligned}
&dV^i(t, X_t, \alpha_{[t]}) = \\
&\Bigg\{ \frac{dE_0(t)}{dt}(\bar{X}_t - X_t^i)^2 + 2E_0(t)(\bar{X}_t - X_t^i)\left(\bar{\alpha}_t - \alpha_t^i - (\bar{\alpha}_{t-\tau} - \alpha_{t-\tau}^i)\right) \\
&+ \sum_{j=1}^N \sigma^2 E_0(t)(\frac{1}{N} - \delta_{i,j})^2 + 2\left(\bar{\alpha}_t - \alpha_t^i - (\bar{\alpha}_{t-\tau} - \alpha_{t-\tau}^i)\right) \int_{t-\tau}^t E_1(t, s-t)(\bar{\alpha}_s - \alpha_s^i) ds \\
&+2(\bar{X}_t - X_t^i) \int_{t-\tau}^t \left[\frac{\partial E_1(t, s-t)}{\partial t} - \frac{\partial E_1(t, s-t)}{\partial s} \right] (\bar{\alpha}_s - \alpha_s^i) ds \\
&+2(\bar{X}_t - X_t^i) E_1(t, 0)(\bar{\alpha}_t - \alpha_t^i) - 2(\bar{X}_t - X_t^i) E_1(t, -\tau)(\bar{\alpha}_{t-\tau} - \alpha_{t-\tau}^i) \\
&+ \int_{t-\tau}^t \int_{t-\tau}^t \Bigg[\frac{\partial E_2(t, s-t, r-t)}{\partial t} - \frac{\partial E_2(t, s-t, r-t)}{\partial s} \\
&- \frac{\partial E_2(t, s-t, r-t)}{\partial r} \Bigg] (\bar{\alpha}_s - \alpha_s^i)(\bar{\alpha}_r - \alpha_r^i) ds dr \\
&+(\bar{\alpha}_t - \alpha_t^i) \left(\int_{t-\tau}^t E_2(t, s-t, 0)(\bar{\alpha}_s - \alpha_s^i) ds + \int_{t-\tau}^t E_2(t, 0, r-t)(\bar{\alpha}_r - \alpha_r^i) dr \right) \\
&-(\bar{\alpha}_{t-\tau} - \alpha_{t-\tau}^i) \left(\int_{t-\tau}^t E_2(t, s-t, -\tau)(\bar{\alpha}_s - \alpha_s^i) ds + \int_{t-\tau}^t E_2(t, -\tau, r-t)(\bar{\alpha}_r - \alpha_r^i) dr \right) \\
&+ \frac{dE_3(t)}{dt} \Bigg\} dt \\
&+ \sum_{j=1}^N \Bigg\{ + 2E_0(t)(\bar{X}_t - X_t^i)(\frac{1}{N} - \delta_{i,j}) \\
&+2(\frac{1}{N} - \delta_{i,j}) \int_{t-\tau}^t E_1(t, s-t)(\bar{\alpha}_s - \alpha_s^i) ds \Bigg\} \sigma dW_t^j.
\end{aligned}
\tag{3.61}
$$

Then, integrating between 0 and T, using $V^i(T, X_T) = g_i(X_T)$ (ensured by the boundary conditions at $t = T$ for $E_k,\ k = 0, 1, 2, 3$), taking expectation, using the differential equations (3.49-3.52), using the short notation $A_1 = 1 - \frac{1}{N}$, $A_2 = 1 - \frac{1}{N^2}$, and adding

$\mathbb{E}\int_0^T f_i(X_s,\alpha_s^i)dt$ on both sides, one obtains:

$$
\begin{aligned}
&-V^i(0,\xi^i,\alpha_{[0)})+\mathbb{E}(g_i(X_T))+\mathbb{E}\int_0^T f_i(X_s,\alpha_s^i)dt = -V^i(0,\xi^i,\alpha_{[0)})+J^i(0,\xi^i,\alpha_{[0)},\alpha)\\
&=\mathbb{E}\int_0^T\Bigg\{\left[-\frac{\epsilon}{2}+2A_2(E_1(t,0)+E_0(t))^2+2q(E_1(t,0)+E_0(t))+\frac{q^2}{2}\right](\bar{X}_t-X_t^i)^2\\
&+2E_0(t)(\bar{X}_t-X_t^i)\big((\bar{\alpha}_t-\alpha_t^i)-(\bar{\alpha}_{t-\tau}-\alpha_{t-\tau}^i)\big)+\sigma^2E_0(t)\sum_{j=1}^N(\frac{1}{N}-\delta_{i,j})^2\\
&+2\left(\bar{\alpha}_t-\alpha_t^i-(\bar{\alpha}_{t-\tau}-\alpha_{t-\tau}^i)\right)\int_{t-\tau}^t E_1(t,s-t)(\bar{\alpha}_s-\alpha_s^i)ds\\
&+2(\bar{X}_t-X_t^i)\int_{t-\tau}^t\Bigg[2A_2\left(E_1(t,0)+E_0(t)+\frac{q}{2A_2}\right)\Bigg(E_2(t,s-t,0)\\
&\qquad\qquad+E_1(t,s-t)\Bigg)\Bigg](\bar{\alpha}_s-\alpha_s^i)ds\\
&+2(\bar{X}_t-X_t^i)E_1(t,0)(\bar{\alpha}_t-\alpha_t^i)-2(\bar{X}_t-X_t^i)E_1(t,-\tau)(\bar{\alpha}_{t-\tau}-\alpha_{t-\tau}^i)\\
&+\int_{t-\tau}^t\int_{t-\tau}^t\Bigg[2A_2\left(E_2(t,s-t,0)+E_1(t,s-t)\right)\Bigg(E_2(t,r-t,0)\\
&\qquad\qquad+E_1(t,r-t)\Bigg)\Bigg](\bar{\alpha}_s-\alpha_s^i)(\bar{\alpha}_r-\alpha_r^i)dsdr\\
&+(\bar{\alpha}_t-\alpha_t^i)\left(\int_{t-\tau}^t E_2(t,s-t,0)(\bar{\alpha}_s-\alpha_s^i)ds+\int_{t-\tau}^t E_2(t,0,r-t)(\bar{\alpha}_r-\alpha_r^i)dr\right)\\
&-(\bar{\alpha}_{t-\tau}-\alpha_{t-\tau}^i)\left(\int_{t-\tau}^t E_2(t,s-t,-\tau)(\bar{\alpha}_s-\alpha_s^i)ds+\int_{t-\tau}^t E_2(t,-\tau,r-t)(\bar{\alpha}_r-\alpha_r^i)dr\right)\\
&-A_1\sigma^2E_0(t)+\frac{1}{2}(\alpha_t^i)^2-q\alpha_t^i(\bar{X}_t-X_t^i)+\frac{\epsilon}{2}(\bar{X}_t-X_t^i)^2\Bigg\}dt. \qquad (3.62)
\end{aligned}
$$

Observe that the terms in ϵ cancel, the terms in σ^2 cancel, and the terms involving delayed controls cancel using symmetries and boundary conditions (3.53) for the functions E_k's.

Next, motivated by (3.55), we rearrange the terms left in (3.62) so that the square of

$$\alpha_t^i - 2A_1\Big[\Big(E_1(t,0)+E_0(t)+\frac{q}{2A_1}\Big)(\bar X_t - X_t^i) + \int_{t-\tau}^t \Big[E_2(t,s-t,0) + E_1(t,s-t)\Big](\bar{\hat\alpha}_s - \hat\alpha_s^i)ds\Big]$$

appears first. We obtain

$$\begin{aligned}
&-V^i(0,\xi^i,\alpha_{[0)}) + J^i(0,\xi^i,\alpha_{[0)},\alpha) = \\
&\mathbb{E}\int_0^T \Bigg\{\frac{1}{2}\Bigg(\alpha_t^i - 2A_1\Bigg[\Bigg(E_1(t,0)+E_0(t)+\frac{q}{2A_1}\Bigg)(\bar X_t - X_t^i) \\
&\qquad + \int_{t-\tau}^t \left[E_2(t,s-t,0)+E_1(t,s-t)\right](\bar{\hat\alpha}_s - \hat\alpha_s^i)ds\Bigg]\Bigg)^2 \\
&+(\bar X_t - X_t^i)^2\Bigg[-2[A_1(E_1(t,0)+E_0(t)+\frac{q}{2}]^2 + 2A_2(E_1(t,0)+E_0(t))^2 \\
&\qquad +2q(E_1(t,0)+E_0(t)) + \frac{q^2}{2}\Bigg] \\
&+(\bar X_t - X_t^i)\left[2\alpha_t^i[A_1(E_1(t,0)+E_0(t)] + 2(E_1(t,0)+E_0(t))(\bar\alpha_t - \alpha_t^i)\right] \\
&+(\bar X_t - X_t^i)\left(\int_{t-\tau}^t (E_2(t,s-t,0)+E_1(t,s-t)(\bar\alpha_s - \alpha_s^i)ds\right)\Bigg[-4A_1\Big(A_1(E_1(t,0) \\
&\qquad +E_0(t)+\frac{q}{2}\Big) + 4A_2\left(E_1(t,0)+E_0(t)+\frac{q}{2A_2}\right)\Bigg] \\
&+\left(\int_{t-\tau}^t (E_2(t,s-t,0)+E_1(t,s-t)(\bar\alpha_s - \alpha_s^i)ds\right)\left[2A_1\alpha_t^i + 2(\bar\alpha_t - \alpha_t^i)\right] \\
&+\left(\int_{t-\tau}^t (E_2(t,s-t,0)+E_1(t,s-t)(\bar\alpha_s - \alpha_s^i)ds\right)^2\left[-2A_1^2 + 2A_2\right]\Bigg\}dt. \qquad (3.63)
\end{aligned}$$

Using $A_2 = A_1^2 + \frac{2}{N}A_1$ and the relation $\bar\alpha_t - \alpha_t^i = \frac{1}{N}\sum_{j\neq i}\alpha_t^j - A_1\alpha_t^i$, we simplify (3.63) to obtain:

$$
\begin{aligned}
&-V^i(0,\xi^i,\alpha_{[0)}) + J^i(0,\xi^i,\alpha_{[0)},\alpha) = \\
&\mathbb{E}\int_0^T \Bigg\{ \frac{1}{2}\Bigg(\alpha_t^i - 2A_1\Bigg[\Bigg(E_1(t,0)+E_0(t)+\frac{q}{2A_1}\Bigg)(\bar{X}_t - X_t^i) \\
&\qquad\qquad + \int_{t-\tau}^t \left[E_2(t,s-t,0)+E_1(t,s-t)\right](\bar{\hat{\alpha}}_s - \hat{\alpha}_s^i)ds\Bigg]\Bigg)^2 \\
&+(\bar{X}_t - X_t^i)^2\left[\frac{4}{N}A_1(E_1(t,0)+E_0(t))^2 + \frac{2q}{N}(E_1(t,0)+E_0(t))\right] \\
&+(\bar{X}_t - X_t^i)\left[\frac{2}{N}\sum_{j\neq i}\alpha_t^j(E_1(t,0)+E_0(t))\right] \\
&+(\bar{X}_t - X_t^i)\left(\int_{t-\tau}^t (E_2(t,s-t,0)+E_1(t,s-t)(\bar{\alpha}_s - \alpha_s^i)ds\right)\Bigg[\frac{8}{N}A_1(E_1(t,0) \\
&\qquad\qquad + E_0(t)) + \frac{2q}{N}\Bigg] \\
&+\left(\int_{t-\tau}^t (E_2(t,s-t,0)+E_1(t,s-t)(\bar{\alpha}_s - \alpha_s^i)ds\right)\left[\frac{2}{N}\sum_{j\neq i}\alpha_t^j\right] \\
&+\left(\int_{t-\tau}^t (E_2(t,s-t,0)+E_1(t,s-t)(\bar{\alpha}_s - \alpha_s^i)ds\right)^2\left[\frac{4}{N}A_1\right]\Bigg\}dt. \qquad (3.64)
\end{aligned}
$$

Now, assuming that the players $j \neq i$ are using the strategies $\hat{\alpha}_t^j$ given by (3.55), the quantity $\sum_{j\neq i}\alpha_t^j$ becomes

$$
\begin{aligned}
\sum_{j\neq i}\hat{\alpha}_t^j = -2A_1\Bigg[&\left(E_1(t,0)+E_0(t)+\frac{q}{2A_1}\right)(\bar{X}_t - X_t^i) \\
&+\int_{t-\tau}^t \left[E_2(t,s-t,0)+E_1(t,s-t)\right](\bar{\hat{\alpha}}_s - \hat{\alpha}_s^i)ds\Bigg].
\end{aligned}
$$

Plugging this last expression in (3.64), one sees that the terms after the square cancel and we get

$$
\begin{aligned}
&-V^i(0,\xi^i,\alpha_{[0]}) + J^i(0,\xi^i,\alpha_{[0]},(\alpha^i,\hat{\alpha}^{-i})) = \\
&\mathbb{E}\int_0^T \left\{ \frac{1}{2}\left(\alpha_t^i - 2A_1\left[\left(E_1(t,0)+E_0(t)+\frac{q}{2A_1}\right)(\bar{X}_t - X_t^i)\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. + \int_{t-\tau}^{t}\left[E_2(t,s-t,0)+E_1(t,s-t)\right](\bar{\hat{\alpha}}_s - \hat{\alpha}_s^i)ds\right]\right)^2\right\}dt.
\end{aligned}
\tag{3.65}
$$

Consequently $V^i(0,\xi^i,\alpha_{[0]}) \leq J^i(0,\xi^i,\alpha_{[0]},(\alpha^i,\hat{\alpha}^{-i}))$, and choosing $\alpha^i = \hat{\alpha}^i$ leads to $V^i(0,\xi^i,\alpha_{[0]}) = J^i(0,\xi^i,\alpha_{[0]},(\hat{\alpha}^i,\hat{\alpha}^{-i}))$. □

3.7 Financial Implications and Numerical Illustration

The main finding is that taking into account repayment with delay does not change the fact that the central bank providing liquidity is acting as a *clearing house* in all the Nash equilibria we identified (open-loop in Section 3.4 or closed-loop in Sections 3.5 and 3.6).

The delay time, that is the single repayment maturity τ that we considered in this paper, controls the liquidity provided by borrowing and lending. The two extreme case are:

1. No borrowing/lending: $\tau = 0$:

   In that case, no liquidity is provided and the log-reserves X_t^i follow independent Brownian motions.

2. No repayment: $\tau \geq T$:

   This is the case studied previously in [6] and summarized in Section 3.3. The rate of liquidity (the speed at which money is flowing through the system) is given by $\left[q + (1 - \frac{1}{N})\phi_t\right]$ as shown in equation (3.12).

3. Intermediate regime $0 < \tau < T$:

   We conjecture that the rate of liquidity is monotone in τ. For instance, in the case of the close-loop equilibrium obtained in Section 3.6 given by (3.55), the rate of liquidity is $[2E_1(t, 0) + 2E_0(t) + q]$ where the function E_1 and E_0 are solutions to the system (3.49–3.51). These solutions are not given by closed form formulas. We computed them numerically. We show in Figure 3.1 that as expected, liquidity increases as τ increases. This is clear for values of τ which are small relative to the time horizon T. For values of τ which are large and comparable with T, the boundary effect becomes more important as oscillations propagate backward.

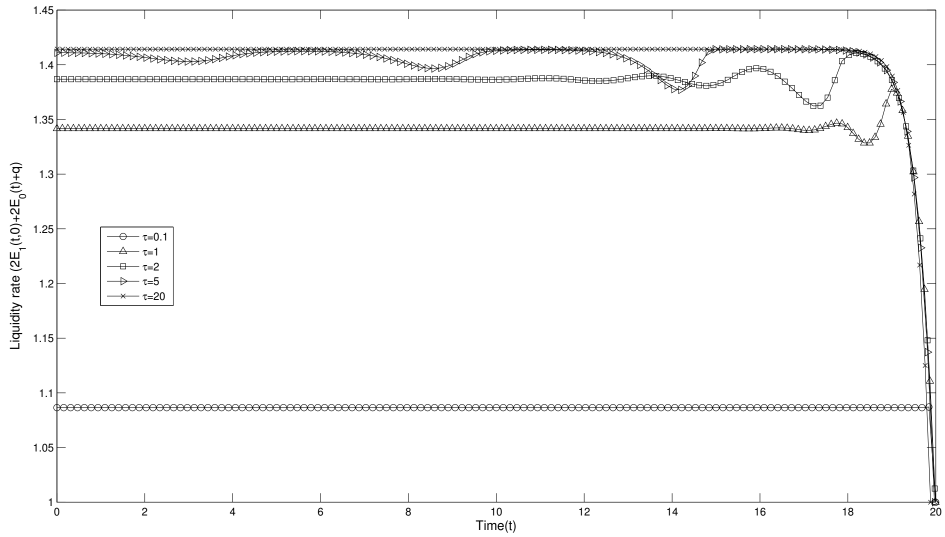

Figure 3.1: Liquidity as a function of the delay time τ. The parameters are $T = 20$, $q = 1$, $\varepsilon = 2$, and $c = 0$.

Appendix A

Generalized Correlated Random Walks Diffusion Limit

This section provides a diffusion limit for generalized correlated random walks. It is a brief version of [24], up until some notational changes.

Consider probability spaces $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$, $n \in \mathbb{N}$ where we define binary random variables X_k^n taking -1 and 1. Then define Y^n by

$$Y_k^n = Y_0^n + \sum_{j=1}^{k} \left(\mu_n + \sigma_n X_j^n\right), \quad k = 0, \ldots, n$$

Where μ_n and σ_n are defined as in A.1.

Define $t_k^n := Tk/n$, and then by the constant interpolation on the intervals $[t_{k-1}^n, t_k^n)$, we get the Right Continuous with Left Limits (RCLL) process $Y^{(n)}$ by

$$Y_t^{(n)} := Y_{\lfloor nt \rfloor/T}^n, \quad 0 \leq t \leq T,$$

Let $\mathbb{F}^{(n)} = \left(\mathcal{F}_t^{(n)}\right)_{0\leq t\leq T}$. Therefore

$$\mathcal{F}_t^{(n)} = \sigma\left(Y_0^n, \ldots, Y_{k-1}^n\right) := \mathcal{F}_{k-1}^n, \;\; t \in [t_{k-1}^n, t_k^n), k = 1, \ldots, n.$$

Here, $Y^{(n)}$ under measure $\mathbb{P}^n$ is distributed according to a probability measure ρ_n on the Skorokhod space $\mathbb{D}[0,T]$ of RCLL functions. The goal is to provide a weak convergence result for (ρ_n), $n \in \mathbb{N}$ under some regularity conditions.

We impose three conditions:

- There exists constants $\mu \in \mathbb{R}$, $\sigma > 0$ and $\beta \in (0,1)$ such that

$$\begin{aligned} \mu_n &= \mu\delta_n^2 + \mathcal{O}\left(\delta_n^{1+\beta}\right), \\ \sigma_n &= \sigma\delta_n + \mathcal{O}\left(\delta_n^{2+\beta}\right). \end{aligned} \tag{A.1}$$

- We have

$$\mathbb{P}^n\left(X_k^n = 1|\mathcal{F}_{k-1}^n\right) = p_n\left(k, Y_{k-1}^n, X_{k-1}^n\right), \;\; k = 1, \ldots, n.$$

  Where

$$p_n\left(k,y,x\right) = \frac{1}{2}\left(1 + xa\left(t_k^n, y\right) + \delta_n b\left(t_k^n, y\right)\right) + \mathcal{O}\left(\delta_n^{1+\beta}\right).$$
$$k = 1, \ldots, n, \;\; y \in \mathbb{R}, \;\; x = -1, 1$$

- Functions $a(.,.)$ and $b(.,.)$ are regular enough such that we can define the operator L on C^2 functions $f(y)$ via

$$(Lf)(t,y) := \frac{1}{2}\sigma^2 \frac{1+a(t,y)}{1-a(t,y)} f''(y) + \left(\mu + \frac{\sigma b(t,y)}{1-a(t,y)} + \frac{\sigma^2 a'(t,y)}{(1-a(t,y))^2}\right) f'(y).$$

Theorem 8. *If* (Y_0^n) *converges in distribution to* Y_0*, and the martingale problem for* L *is well-posed on* $C[0,T]$*, then* $\left(Y^{(n)}\right)$ *converges in distribution to* Y *with dynamics*

$$dY_t = \left(\mu + \frac{\sigma b(t,Y_t)}{1-a(t,Y_t)} + \frac{\sigma^2 a'(t,Y_t)}{\left(1-a(t,Y_t)\right)^2}\right) dt + \sigma \sqrt{\frac{1+a(t,Y_t)}{1-a(t,Y_t)}} dW_t \tag{A.2}$$

and initial value Y_0*.*

The starting point for the proof is the following proposition (2) from [13].

Proposition 2. *Define the operator* G *on* C^∞ *functions* f *with compact support by* $Gf := \frac{1}{2}cf'' + \gamma f'$ *where* $c : \mathbb{R} \to [0,\infty)$ *and* $\gamma : \mathbb{R} \to \mathbb{R}$ *are continuous functions. Assume that the martingale problem for* G *is well-posed on* $C[0,T]$*. Consider* $\Gamma^{(n)}$ *and* $C^{(n)}$ *as* $\mathbb{F}^{(n)}$*-adapted processes such that* $N^{(n)} := Y^{(n)} - Y_0^{(n)} - \Gamma^n$ *and* $\left(N^{(n)}\right)^2 - C^{(n)}$ *are* $(\mathbb{F}^{(n)}, \mathbb{P}^n)$ *local martingales for all* n*, and* $C^{(n)}$ *is increasing. Define the stopping time*

$$\tau_n^r := \inf\left\{t \geq 0 \middle| |Y_t^{(n)}| \geq r \text{ or } |Y_{t-}^{(n)}| \geq r\right\} \wedge 1.$$

Also, assume that

$$\lim_{n\to\infty} \mathbb{E}^n \left[\sup_{0\le t\le \tau_n^r} |Y_t^{(n)} - Y_{t-}^{(n)}|^2 \right] = 0,$$

$$\lim_{n\to\infty} \mathbb{E}^n \left[\sup_{0\le t\le \tau_n^r} |\Gamma_t^{(n)} - \Gamma_{t-}^{(n)}|^2 \right] = 0,$$

$$\lim_{n\to\infty} \mathbb{E}^n \left[\sup_{0\le t\le \tau_n^r} |C_t^{(n)} - C_{t-}^{(n)}| \right] = 0,$$

$$\sup_{0\le t\le \tau_n^r} \left| \Gamma_t^{(n)} - \int_0^t \gamma(Y_s^{(n)}) ds \right| \to 0, \quad \textit{in probability}$$

$$\sup_{0\le t\le \tau_n^r} \left| C_t^{(n)} - \int_0^t c(Y_s^{(n)}) ds \right| \to 0, \quad \textit{in probability}$$

Then $(Y^{(n)})$ *converges in distribution to the solution for the martingale problem* (G, ν)*, where* ν *is the probability measure to which* $Y_0^{(n)}$ *under* $\mathbb{P}^n$ *converge weakly.*

From now onwards, we drop all subscripts and superscripts n, except for μ_n and σ_n where it might cause confusion.

We first use Doob decomposition and write $Y = Y_0 + M + A$. Then we get

$$\Delta A_k := \mathbb{E}\left(\Delta Y_k | \mathcal{F}_{k-1}\right) = \mu_n + \sigma_n \mathbb{E}\left(X_k | \mathcal{F}_{k-1}\right) = \mu_n + \sigma_n \left(2p(Y_{k-1}, X_{k-1}) - 1\right).$$

By using our first two conditions,

$$\Delta A_k = \mu\delta^2 + \sigma b(Y_{k-1})\delta^2 + \sigma_n X_{k-1} a(Y_{k-1}) + \mathcal{O}(\delta^{2+\beta}).$$

We use Taylor expansion on ${a(Y_{k-1})}/{1-a(Y_{k-1})}$ to get

$$\frac{a(Y_{k-1})}{1-a(Y_{k-1})} = \frac{a(Y_{k-2})}{1-a(Y_{k-2})} + \frac{a'(Y_{k-2})}{\left(1-a(Y_{k-2})\right)^2}\Delta Y_{k-1} + \mathcal{O}(\delta^{1+\beta}).$$

Multiplying both sides by $\sigma_n X_{k-1} = \Delta Y_{k-1} - \mu_n$, we obtain

$$\sigma_n X_{k-1} \frac{a(Y_{k-1})}{1-a(Y_{k-1})} = \frac{a(Y_{k-2})}{1-a(Y_{k-2})}\left(\Delta Y_{k-1} - \mu\delta^2\right) + \frac{a'(Y_{k-2})}{\left(1-a(Y_{k-2})\right)^2}\sigma^2\delta^2 + \mathcal{O}(\delta^{2+\beta}).$$

Using the identity $a/1-a = 1/1-a - 1$ and $\Delta Y = \Delta M + \Delta A$, we derive

$$\begin{aligned}\frac{\Delta Y_k}{1-a(Y_{k-1})} \quad = \quad & \frac{\Delta M_k}{1-a(Y_{k-1})} + \frac{1}{1-a(Y_{k-2})}\Delta Y_{k-1} - \Delta Y_{k-1} + \frac{\mu + \sigma b(Y_{k-1})}{1-a(Y_{k-1})}\delta^2 \\ & + \mu\delta^2\left(1 - \frac{1}{1-a(Y_{k-2})}\right) + \frac{a'(Y_{k-2})}{\left(1-a(Y_{k-2})\right)^2}\sigma^2\delta^2 + \mathcal{O}(\delta^{2+\beta}).\end{aligned}$$

We now provide three lemmas. For their proofs, we refer to [24].

Lemma 6. *Let N be a martingale defined as*

$$N_m := \sum_{k=1}^{m} \frac{\Delta M_k}{1-a(Y_{k-1})}, \quad m = 0, \ldots, n$$

Then, define $\Gamma := Y - Y_0 - N$. We get

$$\Gamma_m = \sum_{k=1}^{m}\left(\mu + \frac{\sigma b(Y_{k-1})}{1-a(Y_{k-1})} + \frac{\sigma^2 a'(Y_{k-1})}{(1-a(Y_{k-1}))^2}\right)\delta^2 + \mathcal{O}(\delta^\beta), \quad m = 0, \ldots n.$$

Lemma 7. *Define the process C by*

$$C_m := \sum_{k=1}^{m} \frac{Var\left[\Delta M_k | \mathcal{F}_{k-1}\right]}{\left(1-a(Y_{k-1})\right)^2}, \quad m = 0, \ldots n.$$

Then,

$$C_m = \sum_{k=1}^{m} \sigma^2 \frac{1+a(Y_{k-1})}{1-a(Y_{k-1})}\delta^2 + \mathcal{O}(\delta^\beta), \quad m = 0, \ldots, n,$$

and $N^2 - C$ is a martingale.

We now show that the processes N, Γ and C satisfy the assumptions of our proposition. Define functions

$$\gamma(x) := \mu + \frac{\sigma b(x)}{1-a(x)} + \frac{\sigma^2 a'(x)}{(1-a(x))^2}, \;\; c(x) := \sigma^2 \frac{1+a(x)}{1-a(x)}.$$

Using our first condition, we get

$$\sup_{0\le t\le T} |Y_t^{(n)} - Y_{t-}^{(n)}| = \max_{k=1,\ldots,n} |\Delta Y_k^n| = \mathcal{O}(\delta_n).$$

Since $Y^{(n)}$ is the piecewise constant interpolation of Y^n, we obtain

$$\sup_{0\le t\le T} |\Gamma_t^{(n)} - \Gamma_{t-}^{(n)}| + \sup_{0\le t\le T} |C_t^{(n)} - C_{t-}^{(n)}| = \mathcal{O}(\delta_n^\beta).$$

Lemma 8. *For our γ, c, $\Gamma^{(n)}$ and $C^{(n)}$, the last two conditions regarding convergence in probability in (2) hold.*

The above three lemmas show that all the assumptions of Proposition (2) are satisfied. So, $Y^{(n)}$ converges in distribution to the solution of the martingale problem (G, ν), which is the same as Y in Theorem (8). This completes the proof.

If we allow functions a and b to depend on either t or n, similar arguments still go through with some modifications. For more details in this regard, we again refer to [24].

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