What does the body do, when the body is doing mathematics?

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Abstract

Mathematical concepts are paragons of abstraction. The actual practice of mathematics, though, is decidedly concrete, material, one might even say fleshy. This raises the question, *What is the body doing, when the body is doing mathematics?* We answer by analyzing a video corpus of mathematicians in their natural habitat: at the blackboard, chalk in hand. Mixing qualitative and quantitative analyses, we describe systematically the ways mathematicians use their bodies to write, move, and gesture. Some surprises arise, such as the observation that mathematicians point nearly constantly but seldom produce representational gestures; and that they spend most of their time away from the blackboard at which they are writing. We discuss implications for creativity, mathematical cognition, and theories of embodied and distributed cognition more broadly.

Keywords: creativity; embodiment; mathematics; insight; active perception; gesture

Introduction

Mathematical concepts are paragons of abstraction. And yet the actual *practice* of mathematics is decidedly concrete, material, one might even say *fleshy*. Mathematicians pace back and forth. Their hands become covered with chalk. They scribble, sketch, erase. They gesture, pointing to an equation or tracing shapes in the air. Despite the abstraction of mathematical concepts, the actual work of mathematical understanding and discovery is undeniably embodied. But what, exactly, are these mathematical bodies doing? *What is the body doing, when the body is doing mathematics?*

Here, we address this question using a video corpus of mathematicians in their natural habitat: at the blackboard, chalk in hand. Using a combination of qualitative and quantitative analysis, we describe what mathematicians are doing with their bodies as they struggle to prove vexing conjectures.

The body during mathematical reasoning

We know from past research that the body is busy during mathematical reasoning.

One of the body's contributions is to transform the environment through writing — sketching diagrams, writing equations, and creating other inscriptions (Barany & MacKenzie, 2014; Greiffenhagen, 2014; Marghetis, Samson, & Landy, 2019). Mathematicians report that they rely on these external representations (Johansen & Misfeldt, 2020; Poincaré, 1913), and many mathematicians are so serious about chalk and blackboards that they import one brand, Hagoromo Full-touch Chalk, from Japan or South Korea (Weisberger, 2019). Over the course of a single session, a mathematician may create dozens of new equations, diagrams, or other inscriptions, thus populating the blackboard with objects with which they can then interact (Marghetis et al., 2019). All that chalkboard scribbling can pay off. The clever choice of a good visual representation is often the key to solving an impenetrable problem (Pólya, 1990). External representations, moreover, can transform abstract concepts and procedures into much simpler visuo-spatial patterns or motor routines (Hutchins, 1995; Goldstone, Marghetis, Weitnauer, Ottmar, & Landy, 2017; Rumelhart, Smolensky, McClelland, & Hinton, 1986). Real-world mathematical cognition involves a lot of writing.

A second contribution of the body is to *gesture*. Many scholars of mathematical learning and reasoning have focused on representational gestures (Edwards, 2009; Perry, Church, & Goldin-Meadow, 1988; Núñez, 2006; Marghetis & Núñez, 2013). Representational gestures use resemblance to communicate their meaning (McNeill, 2008). For instance, when a child explains how they solved an equivalence problem in arithmetic (e.g., $5 + 3 + 4 = [\quad] + 4$), they might produce a "grouping" gesture that represents the combination of 5 and 3 to create 8 (Perry et al., 1988). Explicitly teaching these gestures to students helps them learn the ideas they represent (Novack, Congdon, Hemani-Lopez, & Goldin-Meadow, 2014). Representational gestures, moreover, are not limited to novices. Mathematicians use metaphorical gestures to teach concepts such as infinity (Núñez, 2006). And mathematical experts working alone, without an audience of eager students, still produce representational gestures that reveal their understanding of highly abstract concepts—for instance by tracing a dynamic trajectory through the air to evoke the concept of a continuous function (Marghetis & Núñez, 2013). From elementary school to PhD-level research, the work of mathematics is accompanied by gestures that evoke abstract concepts through shape and movement.

The current study

When it comes to the role of the body during mathematical reasoning, much of what we know so far is focused on novices (Perry et al., 1988) or pedagogical settings

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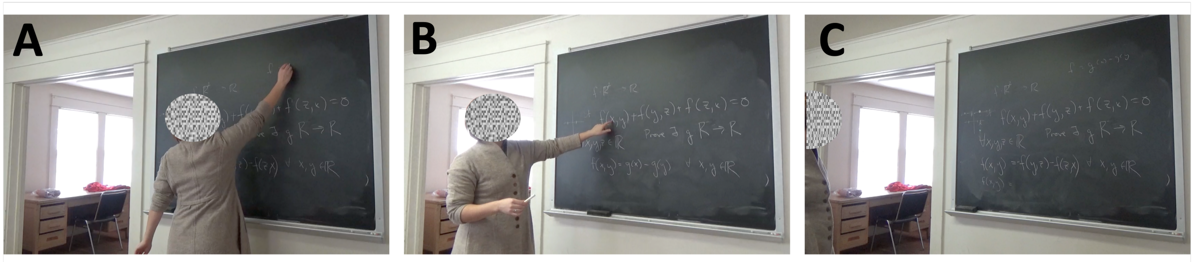

Figure 1: While copiously writing and gesturing, mathematicians moved from within writing distance (A), to medium distance where writing is difficult but pointing is easy (B), to a far distance where they could see the entire blackboard (C).

(Greiffenhagen, 2014; Núñez, 2006). When the actual activity of mathematical experts has been studied, the focus has often specifically targeted certain kinds of gestures—for instance, metaphorical representations of concepts (Núñez, 2006; Marghetis & Núñez, 2013)—or has been impressionistic, self-reported, or anecdotal (Barany & MacKenzie, 2014; Johansen & Misfeldt, 2020). This raises the question: What are experts typically doing with their bodies when they are engaged in creative mathematical reasoning?

Here, we leverage a video corpus of mathematicians working alone to explore, systematically, the uses of the body during mathematical reasoning. We focus on three dimensions of bodily activity: writing, gesture, and movement. We identify when mathematicians create new inscriptions, when and how they gesture, and how they move around in relation to the blackboard. To foreshadow our results: We find that they start by creating new inscriptions, but that this activity drops off over time. Experts do gesture abundantly, but few of these gesture are representational; instead, the overwhelming majority of gestures are pointing or 'deictic' gestures that are meaningful in virtue of what they are pointing at. Unexpectedly, mathematicians spend a lot of their time moving—stepping back and forth from the blackboard. By the end of a work session, our experts have spent more time standing *away* from the blackboard, looking at the mess of inscriptions that they have created, than actually creating new inscriptions. We end by discussing implications for our understanding of creativity and mathematical cognition, and for theories of embodied and embedded cognition more broadly.

Methods

Video Corpus of Mathematicians

We used a video corpus of PhD-level mathematical experts generating proofs of mathematical conjectures (total corpus duration: 4 hours and 40 minutes). This corpus was collected by Marghetis et al. (2019) to investigate mathematical reasoning in ecologically valid settings. Mathematicians worked at the blackboard either in their own office or a seminar room in their mathematics department. Participants were presented with up to three mathematical conjectures and asked to generate proofs while speaking aloud. Conjectures were selected from the William Lowell Putnam Mathematics Competition, a major North American mathematics competition, and encompassed multiple topics, including Set Theory and Geometry; see (Marghetis et al., 2019) for details.

The entire corpus was coded previously for the creation of inscriptions (e.g., diagrams, equations) and subsequent interactions with them (e.g., through gesture, gaze, adding annotations, etc.) (Marghetis et al., 2019).

Here we analyze a subset of the corpus that includes $N = 6$ mathematicians (3 women and 3 men) trying to prove two conjectures, for a total of 12 proof sessions lasting 4 hours and 5 minutes. The time series of inscription creation and interaction included $N = 4375$ events.

We augmented this dataset with additional coding of two aspects of the body's involvement in mathematical reasoning: movement around the room and hand gestures. To quantify movement and gesture, for each proof session we selected random minute-long segments for moment-to-moment video coding ($N = 5$ from each session, for a total duration of 60 minutes). If mathematicians were distracted during a randomly selected section (e.g., talking to passing colleagues or upset by a loud noise outside the room), we replaced it with a new minute-long segment from that session.

Movement Coding

Mathematicians typically stood at one of three distances: the distance at which they would typically write on the blackboard; a full arm's length away from the blackboard; and even farther away, such that they couldn't interact with the blackboard (Figs. 1, 8A). Two independent coders coded location. Interrater reliability was substantial (Cohen's $\kappa = .70$, $p < .001$); disagreements were resolved by discussion.

Gesture Coding

We identified all gestures, defined as non-instrumental movements of the hands. Gestures were coded along two dimensions: source of meaning and handshape. Beat gestures —- meaningless gestures that are entrained with the prosody of speech — were excluded.

Gesture handshape was coded as one of three possible categories: *index* gestures using a canonical pointing handshape,

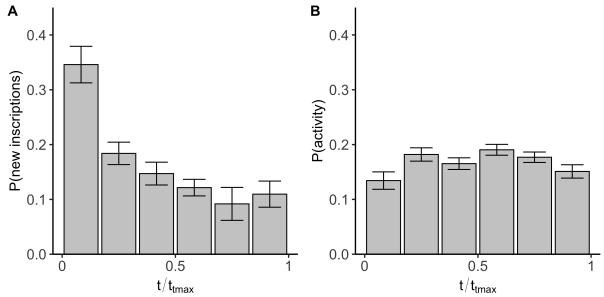

Figure 2: Mathematicians first created new inscriptions and then interacted with them. (A) Time course of inscription creation. Most inscriptions were created in the first third of a session. (B) Time course of interactions with inscriptions (e.g., gesturing, adding annotations). (Bars show mean proportion of total events within each time period ± SEM. Time was normalized to range from 0 to 1.)

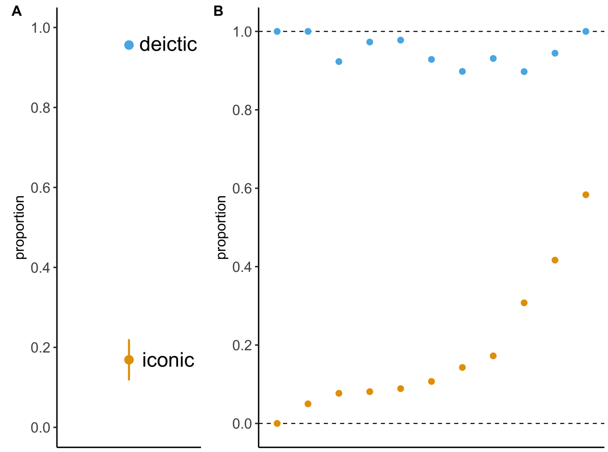

Figure 3: Gestures were mostly deictic and seldom representational. (A) Nearly every gesture had a deictic component; few were representational. (B) Sessions varied in the use of representational gesture. Some sessions had no representational gestures; in others, most were both deictic and representational.

with index finger extended; *extended* gestures with multiple extended fingers; and *other* gestures with other handshapes (e.g., a fist while holding chalk). Interrater reliability was substantial (Cohen's $\kappa = .67$, $p < .001$)

Gesture meaning was coded along three dimensions: deictic, representational, and emblematic (McNeill, 2008). A 'deictic' gesture is meaningful in virtue of reference to—or spatial co-location with—its referent. A canonical example is a pointing gesture with index finger extended. A 'representational' gesture is meaningful in virtue of resemblance to its referent, which can be a concrete object (i.e., iconic gesture) or an abstract concept (i.e., metaphoric gesture). Finally, an 'emblematic' gesture is meaningful in virtue of convention, such as the "thumbs-up" gesture that indicates agreement in many cultures. For instance, a tracing gesture that outlines a triangle drawn on the blackboard is both deictic (pointing to the inscription) and representational (creating the shape of a triangle). Two independent coders coded each dimension of meaning for every gesture. No gestures were identified as emblematic. Interrater agreement was high (presence of deictic meaning: 97% agreement; presence of iconic meaning: 88% agreement). In cases of disagreement, analyses followed the more experienced coder.

Results

Creating and interacting with inscriptions

Every mathematician wrote extensively, often filling the entire blackboard. Mathematicians front-loaded the creation of new inscriptions, creating most inscriptions in the first third of the session (Fig. 2A). Most of the heavy-lifting of filling the blackboard, therefore, occurred toward the start of mathematicians' efforts. This was followed by more subtle blackboard engagement: glancing at equations, gesturing toward diagrams, adding annotations that elaborated and extended previous inscriptions. These subsequent interactions were distributed more uniformly over time (Fig. 2B).

Gesture

What were the mathematicians doing when they were not interacting with the blackboard? Sometimes they stood still. But often their hands were active still—but instead of writing, *gesturing*. All mathematicians in the corpus gestured regularly while working on their proofs. During the sixty minute-long segments selected randomly for gesture coding, mathematicians made a total of 421 gestures (including 12 beat gestures); only four minutes did not include gesture. No gestures were emblematic.

Nearly every gesture was deictic ($M = .96 \pm .01$ *SE*). Few were representational ($M = .17 \pm .05$ *SE*; Fig. 3A, Fig. 4), though this varied across sessions (Fig. 3B). Indeed, the most common handshape was some form of pointing ($M = .77 \pm .04$ *SE*), primarily the canonical handshape with index finger extended ($M = .45 \pm .08$ *SE*) or sometimes multiple fingers extended ($M = .31 \pm .07$ *SE*; Fig. 5). When gestures were representational, most were simultaneously deictic (71%) — "environmentally coupled gestures" (Goodwin, 2007) that were meaningful in virtue of their relation to blackboard inscriptions (Fig. 6).

Movement

Mathematicians started working on a proof by populating the blackboard with inscriptions. Subsequently, however, their bodies adopted a variety of stances: standing slightly away from the blackboard so they can point easily, or even farther so the entire blackboard could be surveyed at once (Fig. 7).

Overall, mathematicians were most commonly close to the blackboard, ready to write ($M_{close} = .47 \pm .05\,SE$). Nonetheless, half the time was equally divided between a medium and a far distance from the board ($M_{medium} = .26 \pm .04\,SE$, $M_{far} = .26 \pm .05\,SE$) (Fig. 8B). Interestingly, this pattern var-

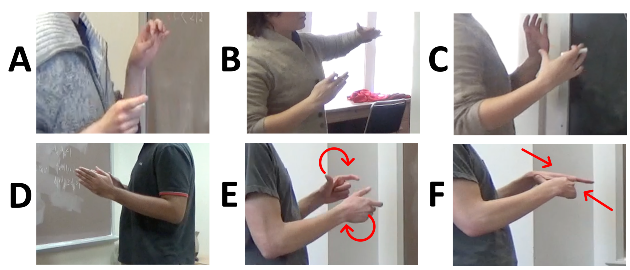

Figure 4: Examples of representational gestures. (A) ‘Small number’ represented metaphorically by a tiny circle of index finger and thumb. (B) ‘Geometric coordinates’ represented by crossed arms. (C) Relative location of two triangles represented by both hands. (D) ‘Numerical interval’ represented metaphorically by both hands indicating endpoints. (E) The spatial transformation of a triangle enacted by the rotation of thumbs and index fingers. (F) “Pieces have to lock together to form a triangle” represented by both hands moving away and toward each other. (Red arrows indicate movement.)

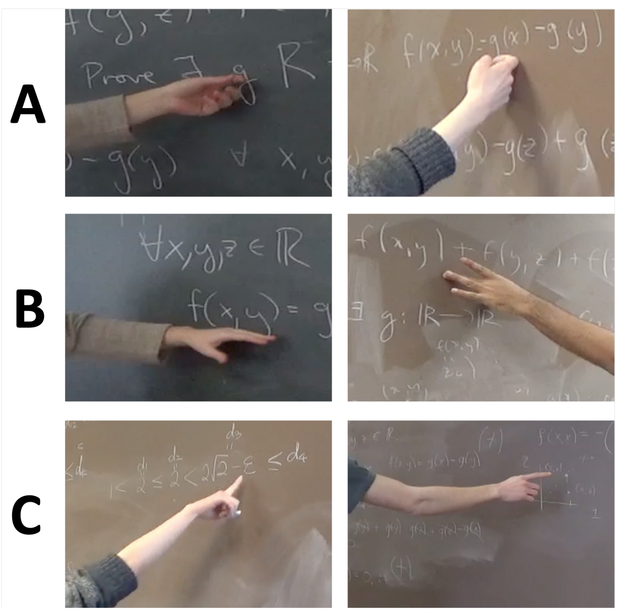

Figure 5: Deictic gestures used a variety of handshapes: (A) pointing with fists or pent figures, often while holding chalk; (B) pointing with multiple extended fingers; (C) canonical pointing handshape with extended index finger.

ied over time, with earlier activity largely dominated by at-the-blackboard activities (i.e., close), and later activity dominated by away-from-the-blackboard activity (i.e., medium and far) (Fig. 8C). The variation across individuals was even greater (Fig. 8D). Some proof sessions were spent almost entirely at the blackboard. In others, the mathematician spent most of their time too far from the blackboard to write, whether just an arm’s length away or standing even farther.

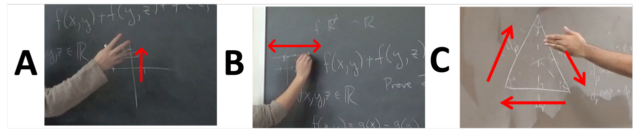

Figure 6: Representational gestures were often simultaneously deictic. (A) Mathematician moves her hand away from the board while saying, “I’m imagining [a triangle] coming out of the chalkboard.” Handshape represents the geometric figure; initial location indexes a relevant inscription. (B) A back-and-forth gesture points to an inscribed line (deictic) while also enacting the line’s shape (representational). (C) Gesture points to a triangle (deictic) while enacting its borders (representational). (Red arrows indicate movement.)

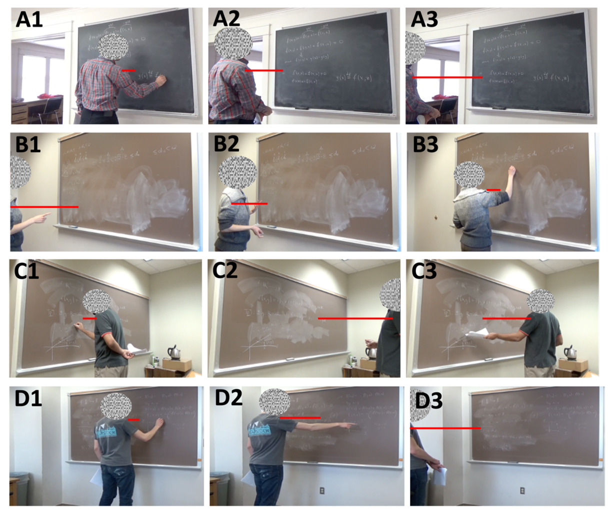

Figure 7: Rapid changes in distance within 20-seconds of activity for four mathematicians (rows). First panel in each row shows the starting position; third panel shows the final position. Red lines illustrate distance to the blackboard.

Discussion

In our corpus of real-world mathematical activity, mathematicians’ bodies interacted constantly with the surrounding world: they created an abundance of inscriptions on the board, then interacted with them through gesture, speech, and gaze. This was accompanied by an unexpected amount of meandering. Mathematicians repeatedly stepped away from the blackboard, out of reach, then returned to stand close enough to write comfortably. Much back-and-forth occurred within seconds.

The mathematicians’ physical labor followed distinct temporal patterns. The creation of inscriptions was front-loaded — that is, they created most inscriptions at the start and then interacting with them regularly throughout the session. This was reflected in patterns of movement: Mathematicians spent more time standing close to the board earlier in the session and more time farther from the board as the session continued. Despite the abstract nature of mathematical concepts, actual mathematical practice appears to be a species of physical labor.

The body is thus active during creative mathematical reasoning. But does all this physical activity help? Or is it a distraction, perhaps a sign of frustration and failure? We discuss these questions next.

Why the abundance of inscriptions?

Mathematicians themselves claim outright that inscriptions are an important part of their creative process (Barany & MacKenzie, 2014; Johansen & Misfeldt, 2020; Greiffenhagen, 2014). For example, one mathematician described his use of inscriptions as a necessity: “I have something in my head, but I need to write it down in order for it to be concrete and correct... What you have in your head is an attempt to structure information. Or the beginning of it. And then you start writing it down, and it might not be exactly what you had expected. You need to change it before it works, or it might not work” (Johansen & Misfeldt, 2020, p. 7). It makes sense, then, that the mathematicians in our corpus created so many inscriptions toward the start of their efforts — exactly when they are first trying to make sense of the problem. By transforming their inner thoughts into concrete external artifacts, they begin the process of seeing if their understanding makes sense.

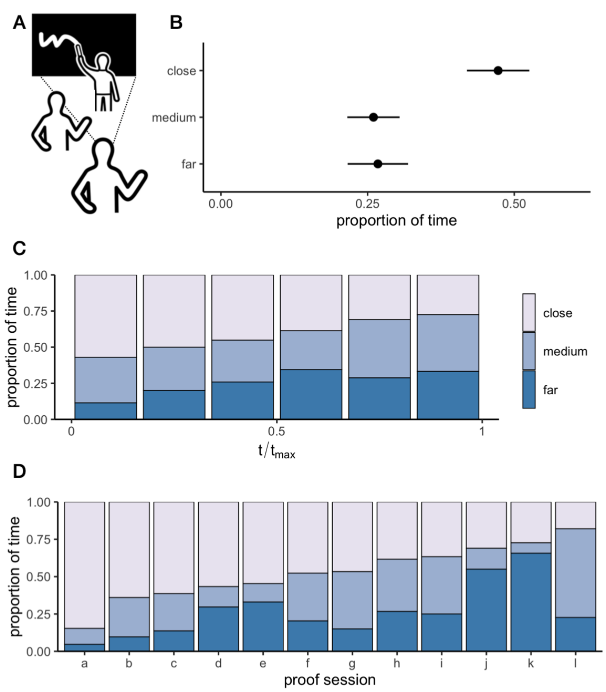

Figure 8: Mathematicians’ changing proximity to the board. (A) Three categories of distance from the blackboard: close enough to write; arm’s length away; beyond arm’s length. (B) Mean proportion of time spent at each distance. (Error bars = SEM). (C) Mean proportion of time spent at each distance (y-axis) in periods ranging from start to end of proof sessions. Horizontal axis shows normalized time within each session. (D) Individual differences in proportion of time spent at each distance. Each stack represents one proof session, ordered by proportion of time spent close to the blackboard.

Once created, these inscriptions can become an integral part of reasoning. The stability of external representations can offer structure to an otherwise nebulous process of abstract reasoning. Inscriptions can function as material anchors for abstract concepts, thus facilitating understanding and affording further manipulation and reasoning (Hutchins, 1995, 2005; Fauconnier & Turner, 2002). Mathematicians fill blackboards not just to communicate, therefore, but to understand, reason, and discover.

Why the abundance of gestures?

The mathematicians gestured a lot. Much of the literature on gesture in mathematical reasoning have focused on representational gestures — iconic gestures that represent geometric shapes, for instance, or metaphorical ones that represent more abstract concepts such as infinity (Marghetis & Núñez, 2013; Edwards, 2009; McNeill, 2008; Walkington, Woods, Nathan, Chelule, & Wang, 2019; Núñez, 2006; Nathan & Walkington, 2017). The majority of the mathematicians’ gestures, by contrast, were deictic gestures that pointed to the blackboard or made connections between inscriptions.

Deictic gestures are associated with successful mathematical reasoning, from basic counting to more advanced high school mathematics. When children are first learning to count, deictic gestures help them anchor their attention to a sequence of objects (Alibali & DiRusso, 1999). As students progress to more complicated mathematical concepts, their teachers’ deictic gestures help them “link” different representations to create a deeper understanding (Richland, 2015; Alibali & Nathan, 2007). Here we show that deictic gestures are ubiquitous even at the highest levels of mathematics, among research mathematicians engaged in creative thought. We suspect these expert gestures play a similar role to the one they play for mathematical neophytes: self-regimenting one’s own attention by directing the whole body toward an inscription, and sometimes helping to establish and explore links between inscriptions.

Why the abundance of movement?

Perhaps our most surprising observation was how frequently and expansively mathematicians moved around the room. Every mathematician spent time at each of the three distances that we studied — within writing distance, slightly farther but within reach of the board, and too far to reach – often moving through all three distances in a matter of seconds. One deflationary account of this movement is that it is aimless wandering, perhaps driven by boredom or fatigue. The fact that every mathematician moved so much, however, makes us suspect that something else is at play.

There is an active literature on the impacts of movement and physical posture on creativity (Frith, Miller, & Loprinzi, 2020; Matheson & Kenett, 2020; Sargent, LePage, Kenett, & Matheson, 2023; Slepian & Ambady, 2012; Leung et al., 2012; Andolfi, Di Nuzzo, & Antonietti, 2017; Hao, Yuan, Hu, & Grabner, 2014)). Some of these benefits reflect the role of movement in transforming the environment, since transformations of the environment can suggest new ideas or helpfully manipulate the available information (Kirsh, 2014; Kirsh & Maglio, 1994; Vallée-Tourangeau, Ross, Ruffatto Rech, & Vallée-Tourangeau, 2021; Weller, Villejoubert, & Vallée-Tourangeau, 2011; Glucksberg, 1964; Duncker, 1945). The movements performed by the mathematicians here, however, do not have this instrumental character. Mathematicians moved back and forth without actually interacting with their surroundings. Unlike the scrabble player who shuffles his tiles in the hope of stumbling upon a suggestive combination (Kirsh, 2014), these mathematicians wandered without leaving a trace. Why?

We have a speculation, one that we are currently testing in the lab: By adjusting their distance from the board, mathematicians actively manipulate the visual information that is available or salient. When they step towards the board, they enforce a kind of visual focus on whatever inscriptions are immediately in front of them. When they step away, they adopt a holistic view of the whole blackboard, thus juxtaposing inscriptions that are scattered across the board.

This visual access to multiple inscriptions can facilitate the discovery of unexpected connections. Discovering a novel connection can instigate a change in perception of a puzzle or concept. Indeed, discovering unexpected connections is known to be integral to high-level creativity in general and to mathematical creativity in particular (Hadamard, 1954; Simonton, 2012, 2021; Tabatabaeian, Deluna, Landy, & Marghetis, 2022; Pólya, 1990). The mathematician Henri Poincaré (1913) described the process of mathematical creation as the creation of combinations of mathematical facts, combinations which hopefully reveal "unsuspected kinship between other facts, long known, but wrongly believed to be strangers to one another"(p. 386). We suspect that the mathematicians' wandering is a strategy for cultivating such unexpected discoveries.

This hypothesis is supported by the observations that when away from the board, mathematicians often stare intently at the board, moving their gaze from one inscription to the next as if trying to uncover hidden connections. In fact, one mathematician actually mumbled, while far from the board, "I'm scanning the board for a piece of information that I overlooked." By moving around, therefore, mathematicians can surprise themselves unexpected information and connections. We are testing this hypothesis, derived from naturalistic observation, in a series of controlled lab experiments. If our speculation bears out, then mathematicians' "aimless" wandering may not be so aimless.

Toward a full description of situated reasoning

The current study was limited by the existing frameworks of gesture studies and methods of human-annotated analysis. Every theoretical framework makes some phenomena pop while others fade from awareness. Traditional taxonomies of gesture, for instance, focus on binary relations between form and meaning — referring by pointing, referring in virtue of resemblance, etc. The mathematicians' gestures, however, often figured in complex webs of meaning, acting as intermediaries between multiple external referents or ideas, much like the gestures used by teachers to connect multiple representations of the same concept(Richland, 2015; Alibali & Nathan, 2007). While these 'linking' gestures may resemble traditional pointing gestures, they play a more complicated role than merely directing attention to a single target. must consider the role of gestures within these larger webs of meaning.

The description of situated reasoning offered here was limited by its methods: careful annotation by humans. Recent developments in computer vision and machine learning have unlocked new horizons in the study of real-world situated activity (Cao, Simon, Wei, & Sheikh, 2017). Machine learning tools for human pose estimation offer a complementary approach to quantifying the range of physical labor involved in situated reasoning — writing, wandering, gesturing — with the possibility of extracting a moment-by-moment record of where and what a reasoner is doing while interacting in their natural habitat, be it a mathematician's office, a scientist's laboratory, or an artist's studio. In combination with traditional human annotation, these tools for movement analysis promise a future in which even the most elaborate forms of situated reasoning may be described both in great detail and as part of an integrated theory of abstract thought.

Conclusion

Mathematics is manual labor (Marghetis, Edwards, & Núñez, 2014). Much as a spider spins a web to increase its chances of snagging a meal, the creative mathematician constructs a notational niche — a web of equations, diagrams, and other inscriptions. The mathematician can then explore this world of ideas, quite literally moving from one idea to the next by wandering about. To fully understand this kind of distributed cognition ecosystem (Hutchins, 2010), we must look beyond the confines of the mathematician's skull. The most abstract insights, it turns out, emerge from physical labor in a material world.

References

Alibali, M. W., & DiRusso, A. A. (1999). The function of gesture in learning to count: More than keeping track. *Cognitive development*, *14*(1), 37–56.

Alibali, M. W., & Nathan, M. J. (2007). Teachers' Gestures as a Means of Scaffolding Students' Understanding: Evidence from an Early Algebra Lesson. In R. Goldman, R. Pea, B. Barron, & S. J. Derry (Eds.), *Video Research in the Learning Sciences* (p. 349–365). Routledge.

Andolfi, V. R., Di Nuzzo, C., & Antonietti, A. (2017). Opening the mind through the body: The effects of posture on creative processes. *Thinking Skills and Creativity*, *24*, 20–28.

Barany, M. J., & MacKenzie, D. (2014). Chalk: Materials and concepts in mathematics research. In C. Coopmans, J. Vertesi, M. Lynch, & S. Woolgar (Eds.), *Representation in scientific practice revisited* (pp. 107–130). Inside Technology, MIT Press.

Cao, Z., Simon, T., Wei, S.-E., & Sheikh, Y. (2017). Realtime multi-person 2d pose estimation using part affinity fields. In *Proceedings of the ieee conference on computer vision and pattern recognition* (pp. 7291–7299).

Duncker, K. (1945). On problem-solving. *Psychological monographs*, *58*(5), 1–113.

Edwards, L. D. (2009). Gestures and conceptual integration in mathematical talk. *Educational studies in mathematics*, *70*(2), 127–141.

Fauconnier, G., & Turner, M. (2002). *The way we think: Conceptual blending and the mind's hidden complexities*. Basic Books.

Frith, E., Miller, S., & Loprinzi, P. D. (2020). A review of experimental research on embodied creativity: revisiting the mind–body connection. *The Journal of Creative Behavior*, *54*(4), 767–798.

Glucksberg, S. (1964). Problem solving: Response competition and the influence of drive. *Psychological Reports*, *15*(3), 939–942.

Goldstone, R. L., Marghetis, T., Weitnauer, E., Ottmar, E. R., & Landy, D. (2017). Adapting perception, action, and technology for mathematical reasoning. *Current Directions in Psychological Science*, *26*(5), 434–441.

Goodwin, C. (2007). Environmentally coupled gestures. *Gesture and the dynamic dimensions of language*, 195–212.

Greiffenhagen, C. (2014). The materiality of mathematics: presenting mathematics at the blackboard. *The British Journal of Sociology*, *65*(3), 502–528.

Hadamard, J. (1954). *An essay on the psychology of invention in the mathematical field*. Dover Publications.

Hao, N., Yuan, H., Hu, Y., & Grabner, R. H. (2014). Interaction effect of body position and arm posture on creative thinking. *Learning and Individual Differences*, *32*, 261–265.

Hutchins, E. (1995). *Cognition in the wild* (No. 1995). MIT press.

Hutchins, E. (2005). Material anchors for conceptual blends. *Journal of pragmatics*, *37*(10), 1555–1577.

Hutchins, E. (2010). Cognitive ecology. *Topics in cognitive science*, *2*(4), 705–715.

Johansen, M. W., & Misfeldt, M. (2020). Material representations in mathematical research practice. *Synthese*, *197*(9), 3721–3741.

Kirsh, D. (2014). The importance of chance and interactivity in creativity. *Pragmatics & Cognition*, *22*(1), 5–26.

Kirsh, D., & Maglio, P. (1994). On distinguishing epistemic from pragmatic action. *Cognitive science*, *18*(4), 513–549.

Leung, A. K.-y., Kim, S., Polman, E., Ong, L. S., Qiu, L., Goncalo, J. A., & Sanchez-Burks, J. (2012). Embodied metaphors and creative "acts". *Psychological science*, *23*(5), 502–509.

Marghetis, T., Edwards, L. D., & Núñez, R. (2014). More than mere handwaving. *Emerging perspectives on gesture and embodiment in mathematics*, 227–246.

Marghetis, T., & Núñez, R. (2013). The motion behind the symbols: A vital role for dynamism in the conceptualization of limits and continuity in expert mathematics. *Topics in cognitive science*, *5*(2), 299–316.

Marghetis, T., Samson, K., & Landy, D. (2019). The complex system of mathematical creativity: Modularity, burstiness, and the network structure of how experts use inscriptions. *CogSci*, 763–769.

Matheson, H. E., & Kenett, Y. N. (2020). The role of the motor system in generating creative thoughts. *NeuroImage*, *213*, 116697.

McNeill, D. (2008). *Gesture and thought*. University of Chicago press.

Nathan, M. J., & Walkington, C. (2017). Grounded and embodied mathematical cognition: Promoting mathematical insight and proof using action and language. *Cognitive research: principles and implications*, *2*(1), 1–20.

Novack, M. A., Congdon, E. L., Hemani-Lopez, N., & Goldin-Meadow, S. (2014). From action to abstraction: Using the hands to learn math. *Psychological Science*, *25*(4), 903–910.

Núñez, R. (2006). Do real numbers really move? language, thought, and gesture: The embodied cognitive foundations of mathematics. In *18 unconventional essays on the nature of mathematics* (pp. 160–181). Springer.

Perry, M., Church, R. B., & Goldin-Meadow, S. (1988). Transitional knowledge in the acquisition of concepts. *Cognitive Development*, *3*(4), 359–400.

Poincaré, H. (1913). *The foundations of science: Science and hypothesis, the value of science, science and method*. The Science Press.

Pólya, G. (1990). *Mathematics and plausible reasoning.* Princeton University Press.

Richland, L. E. (2015). Linking gestures: Cross-cultural variation during instructional analogies. *Cognition and Instruction*, *33*(4), 295–321.

Rumelhart, D. E., Smolensky, P., McClelland, J. L., & Hinton, G. (1986). Schemata and sequential thought processes in pdp models. In D. E. Rumelhart, J. L. McClelland, & PDP Research Group (Eds.), *Parallel distributed processing: explorations in the microstructures of cognition* (chap. 14).

Sargent, M., LePage, A., Kenett, Y. N., & Matheson, H. E. (2023). The effects of environmental scene and body posture on embodied strategies in creative thinking. *Creativity Research Journal*, 1–15.

Simonton, D. K. (2012). Combinational creativity and sightedness: Monte carlo simulations using three-criterion definitions. *The International Journal of Creativity & Problem*

Solving, *22*(2), 5–17.
Simonton, D. K. (2021). Scientific creativity: Discovery and invention as combinatorial. *Frontiers in Psychology*, *12*, 1–11.
Slepian, M. L., & Ambady, N. (2012). Fluid movement and creativity. *Journal of Experimental Psychology: General*, *141*(4), 625.
Tabatabaeian, S., Deluna, A., Landy, D., & Marghetis, T. (2022). Mathematical insights as novel connections: Evidence from expert mathematicians. In *Proceedings of the annual meeting of the cognitive science society* (Vol. 44).
Vallée-Tourangeau, F., Ross, W., Ruffatto Rech, R., & Vallée-Tourangeau, G. (2021). Insight as discovery. *Journal of Cognitive Psychology*, *33*(6-7), 718–737.
Walkington, C., Woods, D., Nathan, M. J., Chelule, G., & Wang, M. (2019). Does restricting hand gestures impair mathematical reasoning? *Learning and Instruction*, *64*, 101225.
Weisberger, M. (2019). *Why mathematicians are obsessed with (and hoarding) this chalk.* Retrieved 11.01.2023, from `https://www.livescience.com/65419-chalk-hoarded-by-mathematicians.html`
Weller, A., Villejoubert, G., & Vallée-Tourangeau, F. (2011). Interactive insight problem solving. *Thinking & Reasoning*, *17*(4), 424–439.