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Scale Covariance of Fractal Sets and Measures, A Differential Approach to the Box-Counting Function of a Fractal, with Applications

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Scale Covariance of Fractal Sets and Measures, A Differential Approach to the Box-Counting Function of a Fractal, with Applications

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

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by

John Roosevelt Quinn

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To my sons, Patrick, Nicholas and Joshua, whose energy lights up my world, and to all the world's children, who shall inherit a world alight with applications of the mathematics that many of them will help to uncover.

ABSTRACT OF THE DISSERTATION

Scale Covariance of Fractal Sets and Measures, A Differential Approach to the Box-Counting Function of a Fractal, with Applications

by

John Roosevelt Quinn

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2013
Dr. Michel L. Lapidus, Chairperson

Abstract: The scale symmetry of self-similarity is a fundamental one in geometry and in physics. We develop a calculus of the scale space evolution of self-similar fractal sets via an analysis of box-counting functions on these structures utilizing the theories of distributions and hyperfunctions. A differential study of the box-counting function can account for the oscillations in the local geometry of some examples of such structures, paralleling the theory of the complex dimensions of fractal strings. The algebraic structure on the iterates of the unit interval under an Iterated Function System admits a tensor product representation we develop to define an intrinsic geometry of fractals and an integral calculus on these objects.

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1 Introduction: a Brief History of Fractals

Since the discovery of the first fractals by analysts seeking to understand the limitations of the Fourier Transform in the 19^{th} century, the science of fractals proceeded slowly at first, then steadily gathered momentum, after Benoit Mandelbrot coined and popularized the term.

In the early days of fractals, a search for the limits of the uniqueness of Fourier series motivated Cantor's definition of his infamous set, upon which a cumulative density function was defined, dubbed the devil's staircase, which was rife with paradoxes. While it was differentiable almost everywhere, with derivative 0, it was monotone increasing. While the set of points on which it increases have zero measure in $[0, 1]$, the range of the staircase function being $[0, 1]$ puts that set of points into surjection onto $[0, 1]$, implying the uncountability of Cantor's set.

Analysts Gaston Julia and Pierre Fatou seemed to have first used formal iterative process to construct fractals, but the iteration of polynomials in $\mathbb{C}$, will not be our setting. We shall work with fractals that induce simple, easily recognizable structures in $\mathbb{R}^n$. Geometers such as Poincare and Minkowski, topologists like Felix Hausdorff, and mathematicians such as Waclaw Sierpinski, and Helge von Koch, ushered the fractal through its infancy, but our framework for the exploration of fractal theory was provided much later by John Hutchinson.

Benoit Mandelbrot is owed a great debt of gratitude by many of us who first experienced the fractal due to his books [Man2]. It was his eye that seems to have first caught the similarity of many fractal shapes to the rough objects observed in the natural world all around us, especially the living world. It seems to have been his idea to automate the process of iteration of mathematical functions using the electronic computer, and in so doing to unleash the beautiful images of fractals that still captivate the world today, most specifically images of his iconic "Mandelbrot set", and the Julia sets associated to the family of polynomials $z^2 + c$ in the complex plane, thus becoming that rarest of heroes, the pop-culture mathematician.

The fractal, with all its infinite ruggedness greatly generalized calculus, daring authors to solve differential equations on them [Strich4], or on regions with fractal boundaries [Harr],[LapVanF]. Such a calculus turns out to be a happy, if unexpected, marriage of the theories of measures and distributions, Fourier analysis and analytic number theory, algebra and analysis. It is to this generalized calculus that this work belongs.

1.1 Scope of this Work, Outline of Topics, and Research Program

This work is the current implementation of the author's research program to develop a natural calculus of fractals. We choose a simple class of fractals which we find to be easy to understand as emblematic of fractal processes involving iteration.

We draw inspiration from the statistical mechanics of critical systems, which evolve toward a scale invariant power-law state as a parameter we deem analogous to the scale of observation of a self-similar object such as a fractal is varied. Such systems display the property of *universality*, governed by the critical exponents (analogous to fractal dimension), of critical systems, along with connectivity and other, debated properties of such complex systems. This work has not yet been able to offer concrete predictions regarding such systems, but we hope to develop some techniques that will have such applications.

Central to the influence of this field on our work has been the notions of log-periodicity, periodic corrections to scaling, discrete and continuous scale invariance, and oscillations. Our guiding principle has been that such notions would precipitate from the notions of complex dimensions of fractal strings, and the non-Minkowski measurability of lattice strings. We derive periodic correction to scaling from a different approach, however, that of the evolution, over scales of the box-counting function associated to a fractal set. We call such evolution the scale-covariance associated to the set.

In the subsequent chapter, we encounter the notion of contractivity, and the existence of fractal attractors of Iterated Function Systems via Banach's Contraction Mapping Principle. We define and differentiate the concepts of scale invariance and self-similarity in the third chapter, observing that these symmetries of scale allow for dimensional analysis, and are nothing less than a fundamental symmetry of nature. The fourth chapter offers a review of theories of fractal cosmology and offers a description of space-time under the families of contractions induced by multiple observers of events, whose observations must disagree slightly.

The fifth chapter invokes one of our primary characters, the box-counting function. It is this measure of a set's evolution through alteration of the scale of observation, that we track and whose (log-)asymptotics we associate with the notion of fractal dimension. We find that deterministic fractals' evolution in scale space is almost-everywhere differentiable (with zero derivative).

In the spirit of the discovery of Cantor sets to explore the uniqueness properties of Fourier analysis, a representation theory, we seek tools of representation theory to describe fractal sets. In Chapter 6, we review a representation theory of fractals by coalgebras, then show that IFS and their images form the morhpisms and objects of a tensor category.

In chapter 7 we find a tensor (monoidal) representation for IFS fractals, and explore the theoretical description entailed by that representation. In ongoing work (see appendices) on this theme we characterize the titular property of scale covariance as the dual of the representation contravector, and solving the fixed flow defined by this vector reveals this object as a functor from the category of IFS (pre)fractals to that of pointed topological spaces, preparing such objects for further algebraic understanding.

In chapter 8 we put the representation to work to help compute measures defined on fractals, and in the appendix, we begin to consider some obvious analogies to theories of physics.

Finally, in chapter 9, Appendices containing ongoing research, are presented. We also characterize important functions relating to Fourier analysis of fractals as fixed points of the Fourier transform.

Our developments of an almost-everywhere scale derivative and of an approximate integral make a contribution to the generalization of calculus to fractals getting our program of the study and application of fractal mathematics to a start. Our future work, discussed further in chapter 10, includes further defining the applications of our scale derivative to non-deterministic fractals and almost self-similar sets, and our representation is poised for use to compute correlation integrals and connectivities for fractals.

Throughout the development of physical theories, the tools of integration and differentiation have played roles in duality. A system is governed by laws presented as differential equations, which are satisfied by integral relations.

Often the integral relations are much more succinct or hold in a more general context. In this work we contrast the log-periodicity of the complex dimensions of fractal strings with the log-periodicity of the scale evolution of the box-counting function of a fractal. The first object manifests as the singularities of the Mellin transform of a measure associated with the complement to the fractal set, i.e. as an integral theory, and the second as a distributional (double-logarithmic) derivative.

When we observe the graphs of systems in Nature deemed to obey some kind of power-law, we often notice periodic oscillations in the log-log-plot. Herein we describe such oscillations in this plot for deterministic fractals of (log-)box counting function vs (log-) of scale, at a given scale r.

1.2 Measures and Transforms

Define a sigma-algebra of sets to be a family of sets closed under countable unions and complementations, and observe that the set of countable intersections and unions of subsets of a given set is the smallest such sigma-algebra defined on a particular set. On this family, we call a function $\mu : \mathcal{M} \to \mathbb{R}$ on the sigma-algebra of measurable subsets $\mathcal{M}$ of a topological space X, a measure if it satisfies:

1. $\mu(\emptyset) = 0$.

2. $\mu(A \cup B) = \mu(A) + \mu(B)$, if $A \cap B = \emptyset$.

Integral transforms, particularly of measures, becomes an irreplaceable tool at our disposable.

Definition 1. *The Fourier transform $\hat{\mu}(k)$ of $\mu(x)$ is given by $\int_{\mathbb{R}} e^{2\pi ikx}\, \mu(d(x))$,*

Among many useful properties, in fractal geometry, this transform explains the frequency information contained in a measure by its asymptotics.

Definition 2. *The Mellin transform of a measure η is given, $\int_{\mathbb{R}} x^{s-1}\eta(dx)$.*

In the well-established, rigorous theory of fractal strings, [Lap-vanF] this object's abscissa of convergence is shown to be the fractal dimension of a fractal boundary described by η. The singularities of this transform, known as the complex dimensions of the fractal string, then give the (often periodic) oscillations of the asymptotic behavior of the volume of the neighborhood of the complement of a fractal with the scale of observation. Indeed it is this property of complex dimension that allows for the first mathematically rigorous definition of fractality.

1.3 Fractals and Prefractals in Nature and the Human World

For many years, mankind's mathematics struggled to describe such shapes as clouds or mountains. The irregular, inexact patterns were just not amenable to treatment with the calculus, or other smooth mathematics generated by polynomial algebra or special functions.Fractals now supply us with computer images of such objects, and many more. Yet each image and indeed, each such natural structure is limited to construction through a finite number of iterations, at a finite energy level corresponding to a cut-off of the lower size permitted by the naturally occurring approximation.

While we have not treated almost self-similar sets explicitly in the sequel, we define an $\epsilon-$almost self-similar set F_ϵ to be one within ϵ of a set that is self-similar (and obeys a generalization of the functional relation obeyed by F_ϵ to a greater range of scales), with respect to the Hausdorff metric that measures the difference between sets A and B as $d_H(A,B) = \inf\{\epsilon \geq 0\,;\; A \subseteq B_\epsilon \text{ and } Y \subseteq A_\epsilon\}$, where $A_\epsilon := \bigcup_{x\in X}\{z \in M\,;\; d(z,x) \leq \epsilon\}$.

Other naturally occuring (pre-)fractals appear in living organisms, the brain, communications and other networks, etc. Interestingly the scaling of the brain is fractal in the zero-asymptotic, but for larger scales, the connectivity of the brain increases exponentially rather than by a power law. This is one application that has been held firmly in mind to test the descriptive power of the concept of scale -invariance, in future works.

Notably many human structures such as transportation networks and cities have (pre-)fractal appearances or distributions that exhibit fractal scaling known as power laws [Bro-Lie]. Yet the human hand is also seen at work unravelling nature's fractality [Pad-Sal], smoothing over deserts and forest, and erecting regular polygons in place of trees. Interestingly, fractal designs such as the fractal miniaturized antenna, have become commonplace in some of our more advanced technologies, a nod to the ancient power of nature's spontaneous organization.

Even man's organizations such as armies and corporations bear hierarchical resemblances to pre-fractal constructions. Particularly notable in the keeping of these hierarchical symmetries have been the game worlds of Norman Bel-Geddes, whose tiny mechanized worlds foreshadowed his "Futurama" exhibit at the New York world's fair in 1939, an exhibit and design outlook which still influences design today[Szer]. So realistic and interesting were Mr. Bel-Geddes diversions that notables of the time wagered huge sums, decided policy, and forged enlarged networks of personal connections at the Bel-Geddes home during his extravagant games [Szer]. Ultimately, Bel-Geddes was a proponent of smooth design however, most notably we can cite his Chrysler Airflow [Szer].

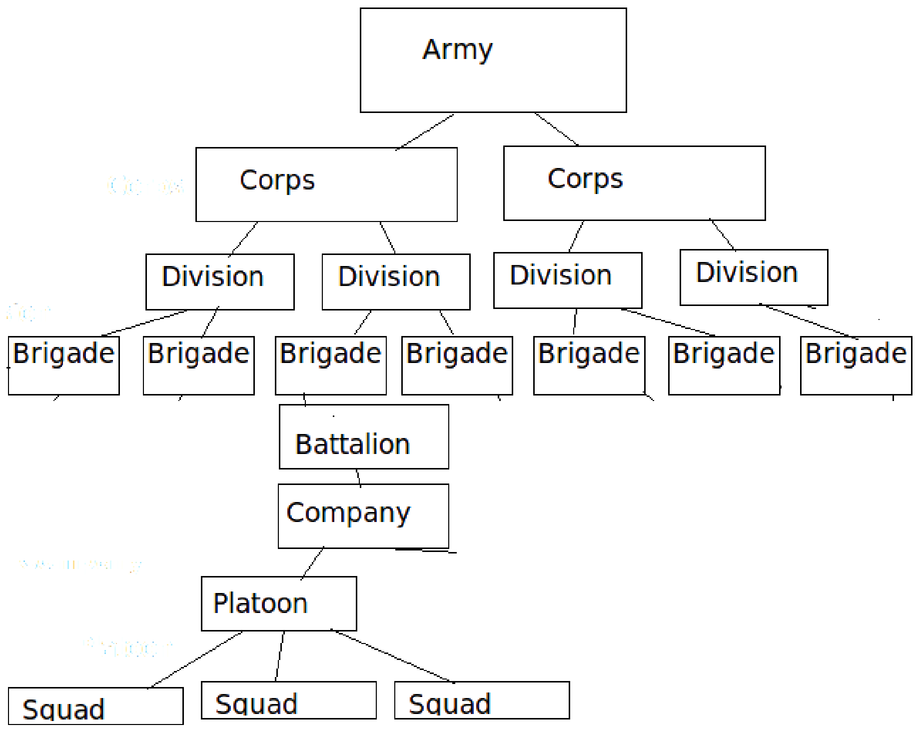

Figure 1: The hierarchy of military units.

1.4 The Theory of Fractal Strings

For the full theory of fractal strings, we refer the reader to [Lap-vanF]. In the most basic conception, we consider the set of deleted intervals involved in the construction of a cantor dust on the real line. We call this collection of lengths of one-dimensional continua the ordinary fractal string $\mathcal{L}$. Since these lengths form a bounded, open subset, the collection is at most countable, and the sum of the lengths is finite. In fact there is a $d \leq 1$ such that $d = inf_s\{\zeta_F(s) = \sum_{\ell \in \mathcal{L}} \ell^{-s} < \infty\}$, this abscissa of convergence corresponds to the fractal dimension of the boundary of the string, when embedded in $\mathbb{R}$. Furthermore, the poles of the analytic continuation of $\zeta_F(s)$ extend above and below real line in the complex plane, and their periodicity or lack thereof determines if the micro-local geometry of the string is too oscillatory to allow for the Minkowski measurability of the string, except in the case of $s = \frac{1}{2}$, assuming that the Riemann Hypothesis holds (see [Lap-vanF]).

1.4.1 Ordinary Fractal Strings

Definition 3. *An* ordinary fractal string Ω *is a bounded open subset of the real line. Every such set can be written as a countable union of connected open intervals with associated lengths* $\mathcal{L} := \{l_1, l_2, ...\}$ *written in non-increasing order. We allow* Ω *to consist of finitely many open intervals, in which case* $\mathcal{L}$ *would consist of finitely many lengths.*

Remark 1. *We refer to* $\mathcal{L}$ *as a fractal string.*

Definition 4. *We define the* Geometric Zeta Function *of the fractal string* $\mathcal{L}$ *to be the Dirichlet series* $\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s = \sum_l w_l l^s$ *where* w_l *represents to multiplicity of a given length* l*.*

Remark 2. $\zeta_{\mathcal{L}}(0)$ *is just the number of connected open intervals and* $\zeta_{\mathcal{L}}(1)$ *is just the total length of the fractal string.*

Theorem 1 (Lap-vanF)**.** *Suppose $\mathcal{L}$ has infinitely many lengths. Then the abscissa of convergence of the geometric zeta function of $\mathcal{L}$ coincides with D, the Minkowski dimension of $\partial\mathcal{L}$.*

Definition 5. *If $\zeta_{\mathcal{L}}$ has a meromorphic extension to all of $\mathbb{C}$, we call $\mathcal{D}_{\mathcal{L}}(\mathbb{C}) = \{\omega \in \mathbb{C} : \zeta_{\mathcal{L}}$ has a pole at $\omega\}$ the set of* complex dimensions *of $\mathcal{L}$.*

Example 1 (The Cantor String)**.** *The geometric zeta function of the Cantor string is $\zeta_{CS}(s) = \sum_{n=0}^{\infty} 2^n \cdot 3^{-(n+1)s} = \frac{3^{-s}}{1-2\cdot 3^{-s}}$.*

The complex dimensions are found by setting the denominator equal to zero and solving for s. We get $\mathcal{D}_{CS}(s) = \{D + in\mathbf{p} : n \in \mathbb{Z}\}$ where $D = \log_3 2$ is the Minkowski dimension (abscissa of convergence of the gzf) and $\mathbf{p} = \frac{2\pi}{\log 3}$. We call $\mathbf{p}$ the oscillatory period*.*

1.4.2 Generalized Fractal Strings

"Besides ordinary fractal strings, generalized fractal strings enable us to deal with strings whose lengths vary continuously or whose multiplicities are non-integral or even infinitesimal." [Lap-vanF]

Definition 6.

- *For a measure η, we denote $|\eta|$ the total variation associated with η, $|\eta|(A) := sup\{\sum_{k=1}^{m} |\eta(A_k)|\}$, where $m \geq 1$ and $\{A_k\}_{k=1}^{m}$ ranges over all finite partitions of A into disjoint measurable subsets of $(0, \infty)$.*

- *A* generalized fractal string *is either a locally complex or a locally positive measure η on $(0, \infty)$, such that $|\eta|(0, x_0) = 0$ for some $x_0 \in (0, \infty)$.*

- *The* dimension *of η, denoted $D = D_\eta$, is the abscissa of convergence of the Dirichlet integral $\zeta_{|\eta|}(\sigma) = \int_0^\infty x^{-\sigma} |\eta|(dx)$.*

- *The* geometric zeta function *is defined as the Mellin transform of η, $\zeta_\eta(s) = \int_0^\infty x^{-s} \eta(dx)$, for $Re(s) > D_\eta$.*

Remark 3. *The term* generalized fractal strings *is fitting because this new definition contains ordinary fractal strings as a special case. In particular, we associate the discrete measure $\eta = \sum_l w_l \delta_{\{l^{-1}\}}$ with the fractal string consisting of all lengths in the sum.*

Example 2. *The* harmonic string *coincides with the positive measure $h = \sum_{j=1}^\infty \delta_{\{j\}}$. This associated geometric zeta function is equal to the Riemann zeta function.*

1.4.3 Minkowski Measurability of Fractal Strings

For $\epsilon \geq 0$, define the ϵ-neighborhood of an OFS $\mathfrak{L}$, as

$$\{x \in \mathfrak{L} \text{ such that } d(x, \partial L) < \epsilon\},$$

then we define the volume of the ϵ-neighborhood of $\mathfrak{L}$ as

$$V(\epsilon) = vol_1\{x \in \mathfrak{L} \text{ such that } d(x, \partial L) < \epsilon\}.$$

The dimension of a fractal string $\mathfrak{L}$ is defined as:

$$D = D_{\mathfrak{L}} = \inf\{\alpha \geq 0 \text{ such that } v(\epsilon) = O(\epsilon^{1-\alpha})\}$$

(Check) We say that $\mathfrak{L}$ has oscillations of order D in its geometry, if $V(\epsilon)$ is periodic along the line $Re(s) = D$.

We define the upper Minkowski content of a fractal string $\mathfrak{L}$ as

$$\mathbb{M}^* = \mathbb{M}^*(D, \mathfrak{L}) = \limsup_{\epsilon \to 0^+} V(\epsilon)\epsilon^{-(1-D)},$$

and the lower Minkowski content of a fractal string $\mathfrak{L}$ as

$$\mathbb{M}_* = \mathbb{M}_*(D, \mathfrak{L}) = \liminf_{\epsilon \to 0^+} V(\epsilon)\epsilon^{-(1-D)},$$

.

Iff $\mathbb{M}^* = \mathbb{M}_*$, we call the resulting

$$\lim_{\epsilon \to 0^+} V(\epsilon)\epsilon^{-1(1-D)} =: \mathbb{M} = \mathbb{M}(D, \mathfrak{L}),$$

the Minkowski content of $\mathfrak{L}$, and we say that $\mathfrak{L}$ is Minkowski measurable.

Theorem 2. *An OFS $\mathfrak{L}$ is Minkowski measurable iff D is a simple pole of $\zeta_{\mathfrak{L}}$, and is the only pole with $Re(s) = D$.*

1.4.4 A Rigorous Definition of Fractality

The theory of fractal strings and their complex dimensions allows us to attempt to define the notion of fractality rigorously. We recall that Mandelbrot's original definition of fractals [Fal] as being sets whose topological dimension strictly exceeds their Hausdorff, box-counting, or other fractal dimension, was proven inadequate due to the existence of fractal curves such as the "Devil's Staircase", and other sets with fractal dimension equal to their topological dimension. All such constructions have been found to have at least one complex dimension with non-zero real part, prompting the following:

Definition 7. *A set F is fractal if it has at least one non-real complex dimension s with real part $\mathfrak{Re}(s) > 0$.*

2 Contractivity and the Existence of Fractal Sets

2.1 The Contraction Mapping Principle

In the setting of complete metric spaces, the contraction mapping principle is the crucial tool used to prove the existence of self-similar fractal sets and measures. We discuss this principle and its applications. We will show that self-similarity is fundamental to much of applied science. We also investigate the role of Banach's theorem in the proofs of many fundamental results in mathematics.

For one example, we recall that the contractivity of the Picard operator, used to show the existence and uniqueness of the solutions to initial-value problems via Banach's theorem can be applied to the solution of certain inverse problems of ordinary differential equations. We review also the use of the contraction mapping theorem to show that the final coalgebra carried by the set of streams of symbols representing a fractal in a coalgebraic representation theory is a fixed point of a contractive functor and that thus fractality is categorical.

As a novel application, we present a scenario in which position uncertainty of locations and the geometric contractivity of the causal history of an interval of spacetime imply that past events have the structure of spacelike fractals.

Contractivity appears to be an organizing principle in our universe, with the attractive forces in the physical world drawing particles closer together over time, in effect performing a contraction mapping on the configuration spaces of systems, thus allowing gravity to organize the universe on the largest scales. Analogously, in the universe of mathematics, we have the existence of attractors of iterated function systems due to a powerful theorem of Stephen Banach: the contraction mapping theorem. Loosely speaking, this theorem states that if a metric space is complete we should expect the process of "shrinking" in that space to terminate somewhere in that space. We need the metric so we can know what shrinking is, and we need completeness so that there are no "holes" into which the sequence of contracted images could disappear. Seeing this as such a fundamental organizing principle, we dub this mighty theorem "Banach's contraction mapping principle".

2.2 The Contraction Mapping Theorem

Definition 8. *For a subset D of a metric space (X, d), a mapping $S : D \to D$ is called a* contraction mapping *on D if there is a number c with $0 < c < 1$, such that $d(S(x), S(y)) \leq cd(x, y)$ for all $x, y \in D$. We call the number c the* scaling ratio*,* contraction ratio*, or* Lipschitz constant *of S.*

Banach's contraction mapping theorem, known variously as Banach's fixed point theorem, or Banach's contraction mapping principle, is the most widely applied of the class of fixed point theorems [Smart], which are considered among the most useful in mathematics. These theorems tell us that a function satisfying some general hypotheses has a value in its domain which is fixed under evaluation by the function. In the case of Banach's theorem, the condition on our function is that the function be a contraction mapping; i.e., that the distance between points in the image of the function will be less than the distance between the corresponding points in the preimage. We require also that the domain and range of this contraction be a metric space in which Cauchy sequences converge, that is, we need our space to be complete.

It is this theorem which implies the existence of attractors of iterated function systems (IFS), which are families of contraction mappings on complete metric spaces [Hut],[Fal], whose attractors form an important class of fractals. Since contraction mappings are automatically continuous, they conserve compactness of the preimage, so that we expect the limiting attractor to preserve compactness. But what is perhaps unexpected is that the limit of the images of the IFS as the number of iterations tends to infinity will be interesting. After all, the theorem tells us of the existence of a fixed *point*, not of a fixed *space*. Perhaps even stranger, we find that the same attractor results when we iterate the IFS starting with any nonempty compact set in the domain of the IFS. By envisioning these attractors as points in a space of compact sets, endowed with the Hausdorff metric under which this space is complete, we will see that the existence of attractors of the IFS is a direct result of Banach's celebrated theorem.

We now state and prove Banach's theorem and discuss some of its implications in section 3.

Theorem 3. (Banach's Contraction Mapping Theorem) *For a complete metric space (X,d) and a contraction mapping* $S : X \to X$*, there exists a unique* $\xi \in X$ *such that* $S(\xi) = \xi$ *and for all* $x \in X$ *the sequence*

$$\{x_n\}_{n=0}^{\infty} := \{x, S(x), S^{(2)}(x), ..., S^{(n)}(x), ...\}_{n=0}^{\infty},$$

converges to ξ *(where we define* $S^{(n)}(x) = S(S^{(n-1)}(x))$ *and* $S^0(x) = x$*).*

This celebrated result of Steven Banach, (possibly going back at least as far as Emile Picard in the case of nonlinear contractions in complete metric spaces) has very many well known applications and implications. A simple and immediate corollary, proved in the discussion of inverse problems, is the collage theorem which is often used in fractal image processing and bounds the distance between the preimage of a contraction mapping and the fixed point of that mapping.

Proof. First we show uniqueness: If $x \neq y$ and $S(x) = x$ and $S(y) = y$, then since $c < 1$ we have

$$d(x, y) = d(S(x), S(y)) \leq cd(x, y) < d(x, y) \neq 0.$$

Thus we reach a contradiction, $d(x, y) < d(x, y)$, so therefore, the fixed point must be unique.

For convergence and existence of a fixed point, we show that for any x, the sequence of iterates under $S(x)$, i.e. the sequence

$$\{x, S(x), S^{(2)}(x), ..., S^{(n)}(x), ...\}_{n=0}^{\infty}$$

is Cauchy, then by completeness, we will have convergence. Observe that for all

$n \geq 1$ we have $d(S^{(n)}(x), S^{(n+1)}(x)) \leq cd(S^{(n-1)}(x), S^{(n)}(x))$, so that

$$d(S^{(n)}(x), S^{(n+1)}(x)) \leq c^2 d(S^{(n-2)}(x), S^{(n-1)}(x)) \leq ... \leq c^n d(x, S(x)).$$

Then, for some $m > n$, by the triangle inequality we have

$$d(S^{(n)}(x), S^{(m)}(x)) \leq \sum_{i=n}^{m-1} d(S^{(i)}(x), S^{(i+1)}(x)),$$

so that

$$d(S^{(n)}(x), S^{(m)}(x)) \leq c^n d(x, S(x)) + c^{n+1} d(x, S(x)) + ... + c^{m-1} d(x, S(x))$$

$$= (\sum_{r=n}^{m-1} c^r) d(x, S(x)) \leq \sum_{r=n}^{\infty} c^r d(x, S(x)) = \frac{c^n}{1-c} d(x, S(x)).$$

Since $c < 1$, for any $\epsilon > 0$ we can find $N \geq 1$ such that $\frac{c^N}{1-c} d(x, S(x)) < \epsilon$. Then if $m > n \geq N$ we have $d(S^n(x), S^m(x)) \leq \frac{c^n}{1-c} d(x, S(x)) \leq \frac{c^N}{1-c} d(x, S(x)) < \epsilon$. So we see that $\{x, S(x), S^{(2)}(x), ..., S^{(n)}(x), ...\}_{n=0}^{\infty}$ is a Cauchy sequence. Since (X, d) is complete there exists a unique ξ such that $x_n \to \xi$ as $n \to \infty$. Hence, by the continuity of S, ξ is clearly a fixed point of S.

□

2.3 Corollaries, and Applications, Existence of Fractals

Here we see how this theorem is applied in some classic cases:

Definition 9. *We call a finite family S of contraction mappings $\{S_i\}_{i=1}^N$ (with $N \geq 2$), an* iterated function system *or* IFS*. An IFS acts on a set A by*

$$S(A) := \cup_{i=1}^N S_i(A),$$

for any subset A of X [Hut].

We call a compact set F invariant under the IFS $S = \{S_i\}_{i=1}^N$*, if $F = \cup_{i=1}^N S_i(F)$. We then refer to F as* the attractor of S*, and we write $F = S(F)$, to denote that F is fixed under S.*

Theorem 4. (Existence of attractors of IFS.) *For any iterated function system S on a complete metric space (X, d), there exists a unique invariant set F fixed under S, and for any nonempty compact subset $E \subset X$, such that $S_i(E) \subset E$ for all i, the iterates $S^{(n)}(E) \to F$ as $n \to \infty$.*

Sketch. An IFS defined on a complete metric space (X, d), naturally induces a contraction mapping in the complete metric space of nonempty compact subsets of X, equipped with the Hausdorff metric [Fal]. Thus, by the contraction mapping theorem (Theorem 1), there exists a unique invariant set F. Iteration of S applied to $E \subset X$ as above, results in a decreasing sequence $S^{(n)}(E)$ of non-empty compact sets containing F. Therefore, the intersection

$\cap_{n=1}^{\infty} S^{(n)}(E) = F$. □

Let us recall that a vector $\rho = (\rho_1, \rho_2, ..., \rho_N)$ is called a probability vector when we have $\rho_i \in [0, 1]$, for all i and $\sum_{i=1}^{N} \rho_i = 1$.

Definition 10. *Let $S = \{S_i\}_{i=1}^{N}$ be an IFS, and let $\rho = (\rho_1, \rho_2, ..., \rho_N)$ be a probability vector with $\rho_i \in (0, 1)$ for all i. We call (S, ρ) the* IFS weighted by ρ. *It acts on measures acting on sets by*

$$(S, \rho)\mu(E) = \sum_{i=1}^{N} \rho_i \mu(S_i^{-1}(E)).$$

We call a measure μ such that $(S, \rho)\mu = \mu$ an invariant measure *under (S, ρ).*

Theorem 5. (Existence of invariant measures.) *For an IFS $S = \{S_i\}_{i=1}^{N}$, weighted by a probability vector ρ, there exists μ, a unique Borel regular, unit mass measure with bounded support, such that μ is fixed under (S, ρ).*

Sketch. (S, ρ) is a contraction in the complete metric space of Borel-regular probability measures, under the L-metric [Hut]. Thus, existence and uniqueness follow, by Theorem 1. □

Analogous theorems for random self-similar fractals [Fal], and measures [Hut-Ru], may also be obtained, but under the weaker conditions of almost sure convergence and equality as distributions respectively.

Theorem 6. (Existence and uniqueness of solutions to a first order initial value problem of ordinary differential equations) *Suppose $g(t, x)$ and $\frac{\partial g}{\partial x}$ are* *continuous functions on some rectangle $a < t < b, c < x < d$ containing the point (t_0, x_0). Then there is an interval $t_0 - h < t < t_0 + h$ contained in $a < t < b$ on which there is a unique solution to the initial value problem $\dot{x}(t) = g(t, x(t))$, with $x(t_0) = x_0$.*

Sketch. The Picard integral operator $P(g(t, x(t)) = \int_{t_0}^{t} g(s, x(s))\, ds + x_0$ is a contraction mappping on the interval $t_0 - h < t < t_0 + h$, and clearly solves the initial value problem. The result follows from the completeness of $\mathbb{R}^2$ and Theorem 1. □

This method is referred to as *fractal-based* in [K-L-M-V] and is suggestive enough that we will consider an example later.

Theorem 7. (Newton's method) *For a function $f(x) \in C^2([a, b])$, with a simple zero, $\hat{x} \in [a, b]$, there exists a neighborhood $N_\alpha(\hat{x}) \subset [a, b]$, of $\hat{x}$, such that for $G(x) = x - \frac{f(x)}{f'(x)}$, the sequence $\{x_n\}_{n=0}^{\infty} := \{x, G(x), G^{(2)}(x), ...\}$, converges to $\hat{x}$. We call the neighborhood $N_\alpha(\hat{x})$, a* basin of attraction*.*

Sketch. For x close enough to $\hat{x}$, $f(\hat{x}) \approx f(x) + f'(x)(\hat{x} - x)$. Solving for $\hat{x}$ gives us a formula for $G(x)$. $G'(\hat{x}) = 0$ and $G(x) \in C^2([a, b])$, so there is a neighborhood $N_\alpha(x)$ such that $G'(x) < 1$. Then $G(x)$ is a contraction mapping within $N_\alpha(\hat{x}) \subset \mathbb{R}$. Observe that $\hat{x}$ is a fixed point of G(x). Therefore, by completeness of $\mathbb{R}$, by Theorem 1, for any $x \in N_\alpha(\hat{x})$ the sequence $\{x_n\}_{n=0}^{\infty}$ of iterates of G(x), converges to $\hat{x}$. □

Interestingly, for many functions in higher dimensions, we find these basins to have intricate fractal boundaries [Pei-Ric].

Theorem 8. (Inverse Function Theorem) *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be a continuously differentiable function in an open set containing* a*, and* $det(f'(a)) \neq 0$*. Then there is an open set* V *containing* a *and an open set* W *containing* $f(a)$ *such that* $f : V \to W$ *has a continuous inverse* $f^{-1} : W \to V$ *which is differentiable and for all* $y \in W$*,*

$$(f^{-1})'(y) = [f'(f^{-1}(y))]^{-1}$$

Sketch. Inverting the Taylor expansion

$$f(x) = f(a) + f'(a)(x - a) + o(\|x - a\|)$$

we get that the local inverse $f^{-1} : W \to V$ is differentiable at a with

$$(f^{-1})'(f(a)) = [f'(a)]^{-1}$$

Then, normalizing $a = f(a) = 0$ and $f'(0) = I_n$, the identity matrix on $\mathbb{R}^n$, so that continuity of $f'(x)$ shows that $f'(x)$ is close to I_n for x close to 0. Then with the fundamental theorem of calculus this implies that $x \mapsto x - f(x) + y$ is a contraction mapping on a small ball around the origin for small y. Thus, by the completeness of $\mathbb{R}^n$ and by Theorem 1, the inverse exists, and by uniqueness of the attractor, it is given by the above formula. □

Recall also that the proof of the implicit function theorem relies on the inverse function theorem.

2.3.1 Fractal Method of Solutions to Inverse Problems of ODEs

The following result is a well-known and immediate corollary of the Banach fixed point theorem.

Theorem 9. (Collage Theorem) [K,L,M,V], [Barn]*For a complete metric space (X, d) and a contraction mapping $S : X \to X$, with contraction constant c, if ξ is the fixed point of S, i.e., if $S(\xi) = \xi$, then for any $x \in X$,*

$$d(x, \xi) \leq \frac{1}{1-c} d(x, S(x)).$$

Proof. $d(x, \xi) \leq \sum_{i=1}^{\infty} d(S^{(i-1)}, S^{(i)}) \leq d(x, S(x)) \sum_{i=1}^{\infty} c^i$, by the triangle inequality and contractivity of S. □

This theorem, a simple and well-known consequence of the contraction mapping principle (Theorem 1), is a key ingredient in the solutions to many inverse problems of fractals and contraction mapping techniques. It is sometimes known as the "Collage Theorem" in textbooks on fractals (see e.g. [Barn] or [K-L-M-V]). Perhaps its best known use is to find an IFS that adequately fits a given fractal. While this possibility guides our treatment of self-similarity throughout this paper, i.e., that self-similar fractal sets are attractors of IFS or can at least be closely approximated by such, here we concentrate on the contraction mapping theorem's use in certain inverse problems of ordinary differential equations.

These inverse problems are viewed as a process of approximating a target element in a complete metric space by the fixed point of a contraction mapping, accomplished by minimizing the distance between the target element and its image under a suitable contraction mapping, so that the collage theorem then would bound the distance between the target element and the fixed point of the contraction mapping [K-L-M-V].

For an initial value problem, $\dot{x}(t) = f(t, x(t))$ with $x(0) = x_0$, we may approximate solutions by use of a contractive Picard integral operator $P(f)(t) = \int_0^t f(s, x(s))\, ds + x_0$, for Lipschitz continuous functions $f(t, x(t))$ (recall Theorem 4). For the first n elements $\{\phi_i(t, x)\}_{i=1}^n$ of an orthonormal basis of an appropriate L^2 space, we approximate $f(t, x(t))$ by $\sum_{i=1}^n a_i \phi_i(t, x)$, creating a Picard operator P_a for each $a = (a_1, a_2, ..., a_n) \in \mathbb{R}^n$. Then we seek to minimize, by classical methods the L^2 distance squared of the difference between $x(t)$ and $P_a(x)$: we have

$$|x - P_a x|_2^2 = \int_{t \in \mathbb{I}} |x(t) - \int_0^t \sum_{i=1}^n a_i \phi_i(s, x(s))\, ds|^2\, dt.$$

3 Self-Similarity

Self-similarity comes in many flavors, including algebraic, analytic, geometric and stochastic. We will concern ourselves chiefly with the geometric and analytic notions, therefore, we will define a set as being self-similar when it is composed of scaled isometric copies of itself, we see that this describes the unique attractor of an IFS e.g.,

$$F = \cup_{i=1}^{N} S_i(F) = S_1(F) \cup S_2(F) \cup ... \cup S_N(F).$$

An example is the unique compact set F which is fixed by the contraction mapping S in the complete metric space of compact sets under the Hausdorff metric. Even for self-similar sets that are not formed by iterated function systems, a contraction mapping based algorithm exists to find an IFS fractal arbitrarily close to F [Barn]. Thus self-similarity and its implications are very closely related to contraction mappings and Banach's fixed-point theorem.

Definition 11. *We say that the IFS* $S = \{S_i\}_{i=1}^{N}$ *satisfies the* open set condition *if there is an open set* U *such that* $\cup_{i=1}^{N} S(U) \subset U$*, and if* $i \neq j$ *then* $S_i(U) \cap S_j(U) = \emptyset$*. For a self-similar set* F *that is the attractor of an iterated function system* $S = \{S_i\}_{i=1}^{N}$*, satisfying the open set condition, with* $N \geq 2$*, and with scaling ratios* $\{r_i\}_{i=1}^{N}$*, we define the* similarity dimension *of* F *to be the unique real solution* d_S *to the equation* $\sum_{i=1}^{N} r_i^{d_S} = 1$.

Example 3. *Let* $S = \{S_i\}_{i=1}^{N}$ *be an IFS satisfying the open set condition and with scaling ratios* $\{r_i\}_{i=1}^{N}$ *such that* $r_i = r$ *for all* $i = 1, ..., N$*. Then* d_S *solves* $N \cdot r^d = 1$*, so that we obtain* $d_S = -\frac{logN}{logr}$.

We may informally derive the equation in definition 4, above, by reasoning that the d-dimensional volume of F scaled by a factor a, $vol(aF) = \sum_{i=1}^{N} vol(a \cdot r_i \cdot F)$, by the disjointness provided by the open set condition, and by the assumption that $vol(aF)$ scales as $vol(aF) = a^{d_S} vol(F)$; where d_S is the similarity dimension. This scaling behavior is at the heart of our notion of self-similarity for measures and functions.

Definition 12. *We say that a measure μ or function f is* scale invariant with exponent β *if there is a number β such that $\mu(aE) = a^{\beta}\mu(E)$ or if $f(ax) = a^{\beta} f(x)$, respectively [Sor].*

We will see that this notion of self-similarity is fundamental to applied science, since scale invariance implies that physical laws are independent of the units used to measure them.

3.1 Symmetry of Scale and Conservation of Physical Quantities

Fractal research often concerns one particular symmetry in nature: the symmetry of scale. Often we will invoke the beauty of fractal images or the amazing complexity of chaos, but we may not mention that this particular symmetry is of fundamental importance in science, especially in the field theories of physics [D-M-S]. Transformations of scale, together with translations, the special conformal transformations and the Lorentz transformations, form the group of conformal symmetries, the global symmetry group of a non-supersymmetric interacting field theory. Loosely speaking then, these symmetries imply that physical laws should be the same no matter where we find ourselves (translation subgroup), no matter what speed we are travelling at (Lorentz subgroup), no matter if we exchange the roles of the very far and very near (inversions in the special conformal transformations), and no matter the size of the scale of observation (scale symmetry subgroup).

We recall an impressive theorem of Emmy Noether [Arn].

Theorem 10. (Noether's Theorem) *If an action admits a one parameter family of diffeomorphisms, it has a first integral.*

This theorem is paraphrased in [Sor] as saying that "for every continuous symmetry of the laws of physics, there must exist a conservation law. For every conservation law, there must exist a continuous symmetry," referring to the vanishing derivative of the first integral as a conservation law. This theorem is stated in terms of an action e.g. a Lagrangian $L = T - V$. If the potential $V(\vec{r})$ has the property that for any scalar α, $V(\alpha\vec{r}) = \alpha^k V(\vec{r})$ for some k, i.e. if V is scale invariant, then under the transformations $\vec{r} \mapsto \alpha\vec{r}$ and $t \mapsto \beta t$ we have $\dot{r} \mapsto \frac{\alpha}{\beta}\dot{r}$ and $T \mapsto \frac{\alpha^2}{\beta^2}T$, so that $\beta = \alpha^{1-\frac{k}{2}}$ means that

$$L(\alpha\vec{r}) = T(\alpha\vec{r}) - V(\alpha\vec{r}) = \frac{\alpha^2}{\beta^2}T(\vec{r}) - \alpha^k V(\alpha\vec{r}) = \alpha^k L(\vec{r}),$$

so that the Lagrangian is invariant when we assume a symmetry of scale.

This calculation demonstrates that the action has a scale symmetry, and implies the existence of a conserved quantity. The change in scale of the parameters of the action corresponds to a change in units of meaurement,and the conservation of the relative quantities then results in the validity of the science of dimensional analysis. Dimensional analysis has proven useful in the study of difficult nonlinear problems, once a suitable choice of a "similarity variable" has been made [Sor].

3.2 Dimensional Analysis

As a consequence of the scale symmetry of the Lagrangian of a system, we can justify the use of a tool for forming hypotheses, checking solutions and determining units of relevant quantities, often greatly simplifying the analysis of nonlinear problems. Units of measurements are thought of as measuring what are called the dimensions of a system. In mechanics, these are the fundamental quantities mass (M), length (L), and time (T).

The independence of the scale of units demonstrated above implies that meaningful physical laws must be homogenous in terms of physical dimensions, so that the same dimensions appear on both sides of an equal sign, and only quantities in the same dimensions can be added or subtracted. Quantities in differing dimensions are combined by multiplication, so that monomials $M^{\mu}L^{\lambda}T^{\tau}$ represent elements $<\mu,\lambda,\tau>$ in a 3-dimensional vector space over $\mathbb{Q}$ with rational powers $(M^{\mu}L^{\lambda}T^{\tau})^{q}$ of those monomials corresponding to scalar multiplication of these vectors. In light of this structure, we can view the choice of fundamental dimensions as a basis of $\mathbb{Q}^3$, with the basis $\{M, L, T\}$ corresponding to $\{M, L, T\}$=$\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ but a basis consisting of the dimensions force(F), length (L), and time (T) corresponds to the basis $\{F, L, T\}$=$\{(1, 1, -2), (0, 1, 0), (0, 0, 1)\}$ (with respect to the basis $\{M, L, T\}$) since $[F] = [MLT^{-2}]$.

Example 4. *In his 1941 theory of turbulence, A. N. Kolmogorov determined that the velocity* u_l *of the flow in an eddy of size* l *should be a function of the energy transfer rate* $\epsilon = \frac{d(u^2)}{dt}$. *The relevant quantities have the dimensions* $[l] = L$, $[u_l] = LT^{-1}$ *and* $[\epsilon] = L^2T^{-3}$, *corresponding to the vectors* $(0, 1, 0), (0, 1, -1)$ *and* $(0, 2, -3)$ *respectively. Then we solve* $(0, 1, -1) = a[b(0, 1, 0) + c(0, 2, -3)]$, *since we want to express* u_l *in terms of* l *and* ϵ. *The solution* $a = \frac{1}{3}, b = 1, c = 1$ *corresponds to multiplying the two variables and taking the cube root, to obtain* $u_l = c(l\epsilon)^{1/3}$. *The dimensionless constant* c *is a result of Buckingham's Pi theorem, which implies that since there is one dimension in our basis unused in our formula, there is one dimensionless constant in the solution [Bar].*

Anyone who has tried to determine the dependency of a unit of measurement on other units, has wanted for a systematic way to determine the sought exponents. The formalism just introduced, of course, reduces this to a problem in linear algebra.

Proposition 1. *To determine the dependency* $U = (U_1^{a_1} U_2^{a_2} ... U_N^{a_N})^q$ *of a unit* U *in dimensions* $M_1, M_2, ... M_N$ *with*

$$[U] = M_1^{\mu_1} M_2^{\mu_2} ... M_N^{\mu_N}$$

on N *independent units* $U_1, U_2, ..., U_N$, *with*

$$[U_j] = M_1^{\mu_{1j}} M_2^{\mu_{2j}} ... M_N^{\mu_{Nj}},$$

we solve the matrix equation

$$[\mu_1, \mu_2, ..., \mu_N] = q[a_1, a_2, ..., a_N] \begin{bmatrix} \mu_{11} & \mu_{21} & \cdots & \mu_{N1} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{N2} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{N1} & \mu_{N2} & \cdots & \mu_{NN} \end{bmatrix}.$$

Thus, we obtain

$$q[a_1, a_2, ..., a_N] = [\mu_1, \mu_2, ..., \mu_N] \begin{bmatrix} \mu_{11} & \mu_{21} & \cdots & \mu_{N1} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{N2} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{N1} & \mu_{N2} & \cdots & \mu_{NN} \end{bmatrix}^{-1}$$

Proof. Since the units U_j are independent, the vectors of the exponents $[\mu_{1j}, \mu_{2j}, ..., \mu_{Nj}]$ of the dimensions $M_1, M_2, ..., M_N$, are linearly independent. Then, since we have N independent vectors with N components, the matrix $[\mu_{ij}]$ is invertible. □

3.3 Self-Similarity and Scale Invariance

We will define the notion of self-similarity for sets (such as fractals) as well as for time developing phenomena (such as solutions to differential equations). We will refer to this second notion of self-similarity as scale invariance. We will see that scale invariance occurs in discrete as well as in continuous forms.

A time developing phenomenon is called *self-similar* if the spatial distributions of its properties at various differing moments of time can be obtained from one another by means of a $similarity\ transformation$ (we call a composition of an isometry and a dilation a similarity transformation). A compact set F is called $self\text{-}similar$ if there are N similarity transformations $\{\Phi_j\}_{j=1}^N$ on $\mathbb{R}^d$ with scaling ratios $\{r_j\}_{j=1}^N \in (0,1)^N$ such that $F = \cup_{j=1}^N \Phi_j(F)$.

A function f of a parameter x is called *scale invariant* under the transformation of scale $x \mapsto \lambda x$ if there is a function μ of λ such that $f(x) = \mu(\lambda) f(\lambda x)$, which we verify is solved by the power law $f(x) = cx^\alpha$ since these suppositions imply, since f is scale invariant, $cx^\alpha = \mu f(\lambda x) = \mu \lambda^\alpha x^\alpha$, so that $c = \mu\lambda^\alpha$ and $\alpha = \frac{log(c) - log(\mu)}{log(\lambda)}$, so that the values of f at two different scales depends on the ratio of the two scales. This definition matches with that of a self-similar set for $N = 1$, without the dilatory requirement. We call this scale invariance *discrete* or *continuous* if the defining relationship $f(x) = \mu(\lambda) f(\lambda x)$ holds for a discrete or respectively, a continuous set of values λ. In the discrete case, we call the value (or values) λ the $preferred$ $scales$ of f.

As Gottfried Liebniz famously noticed, the real number line displays a continuous scale invariance since dilating or contracting $\mathbb{R}$ by any finite, non-zero quantity leaves it fixed. Similarly for $\mathbb{R}^N$ for any $N \in \mathbb{N}$, as for any line through the origin in $\mathbb{R}^N$, thus for the entire set of such lines, so for the real projective planes $\mathbb{RP}^N$. We see that this scale invariance is geometric and not a topological property since we notice that in any compactified metric space, the point at infinity, denoted ∞, is fixed under all dilations and contractions, then so will be the compactified $\mathbb{R}^N \cup \{\infty\}$. But $\mathbb{R}^N \cup \{\infty\} = S^N$ topologically, yet we can find homeomorphisms from S^N to regular polygons, whose curvature, concentrated distributionally in the vertices and edges (as in the Regge calculus), is non-zero, contrasting that of $\mathbb{R}^N \cup \{\infty\}$.

3.4 Examples of Scale Invariance

First examples would include lines ($y = cx$) and monomials ($y = cx^n$). The slope fields of the solutions to homogenous differential equations are another important class of examples, these are equations in e.g. two variables which can be reduced to an equation in the ratio of those variables. Included here are the rational linear transformations of the conformal symmetry group (e.g. $y' = \frac{px+qy}{rx+sy}$), which depending on the number of eigenvalues of the associated matrix to the transformation ($A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$), yield a phase portrait of lines through the origin, concentric circles or infinite spirals.

3.5 Symmetry of Scale and Conservation of Physical Quantities

As fractal researchers we are often called upon to explain why we are so interested in one particular symmetry above all others in nature: the symmetry of scale. Often we will invoke the beauty of fractal images or the amazing complexity of chaos, but we do not mention frequently enough that this particular symmetry is of fundamental importance in science and in the universe. Together with the translations, the special conformal transformations and the Lorentz transformations, the transformations of scale factor into the largest possible global symmetry group of a non-supersymmetric interacting field theory. Loosely speaking then, these symmetries imply that physical laws should be the same no matter where we find ourselves (translation subgroup), no matter what speed we are travelling at (Lorentz subgroup), no matter the size of the scale of observation (scale symmetry subgroup), and no matter if we exchange the roles of the very far and very near (inversions in the special conformal transformations). We recall the impressive theorem of Emmy Noether, which we paraphrase as saying that for any symmetry in a system, there is a conserved quantity (i.e. a conservation law). This theorem is formally stated in terms of an action e.g. a Lagrangian $L = T - V$, so that if the potential $V(\vec{r})$ is scale invariant so that $V(\alpha\vec{r}) = \alpha^k V(\vec{r})$ for some k, under the transformations $\vec{r} \mapsto \alpha\vec{r}$ and $t \mapsto \beta t$ then $\dot{r} \mapsto \frac{\alpha}{\beta}\dot{r}$ and $T \mapsto \frac{\alpha^2}{\beta^2}T$, so that $\beta = \alpha^{1-\frac{k}{2}}$ means that $L(\alpha\vec{r}) = T(\alpha\vec{r}) - V(\alpha\vec{r}) = \frac{\alpha^2}{\beta^2}T(\vec{r}) - \alpha^k V(\alpha\vec{r}) = \alpha^k L(\vec{r})$, so that the Lagrangian is invariant when we assume a symmetry of scale.

This calculation formalizes our intuition that a physical quantity is independent of the scale at which it is measured, and is codified in the science of dimensional analysis. While it seems unsurprising that a tool derived under the assumption of scale invariance should be useful for tackling problems involving a symmetry of scale, the technique of dimensional analysis has proven amazingly useful in the study of difficult non-linear problems, once a suitable choice of a “similarity variable” has been made.

3.6 Dimensional Analysis and Similarity Solutions

Among all of the permissible symmetries of a physical theory, the symmetry of scales has given us the most practial tool for attacking difficult physical problems: dimensional analysis. Every given physical question involves physical quantities measured in units that measure what are called physical dimensions: quantities such as length (L), time (t), temperature (T), and mass (M). Their products like (n-)volume (L^n), speed (LT^{-1}), and energy (MLT^{-2}) result in integer powers of these dimensions. For a suitable given problem, there will be a unique combination of the input data that will result in a meaningful solution, a fact that allows us to quickly outline the solutions to difficult problems up to dimensionless constants.

For example, in his 1941 theory of turbulence A. N. Kolmogorov determined that the velocity u_l of the flow in an eddy of size l should be a function of the energy transfer rate $\epsilon = \frac{d(u^2)}{dt}$. The relevant quantities have the dimensions $[l] = L$, $[u_l] = LT^{-1}$ and $[\epsilon] = L^2T^{-3}$. Since we want u_l in terms of l and ϵ we have only the choice of multiplying the two variables and taking the cube root, to obtain $u_l = c(l\epsilon)^{1/3}$.

As an example of the use of similarity variables we may consider the diffusion equation $\partial_t u = \partial_{xx} u$, and seek a $u(x, t)$ such that $u = t^\alpha f(xt^\beta)$ for some α and β. Making the choice of similarity variable $\xi = tx^\beta$ the equation reduces to $f'' + \xi/2f' - \alpha f = 0$ using $\beta = -\alpha/2$ and the constant mass property of diffusion we get $f(\xi) = Ae^{-\xi^2/4}$ so that $u = \frac{A}{\sqrt{t}}e^{-x^2/4t}$.

4 Applications of Self Similarity and Scale Invariance

4.1 Self-Similarity in Cosmology

In effect, the work of great astronomers and physicists, such as Luciano Pietronero and Bob Oldershaw, suggests that for (a model of) our universe, U, we have $U = SU$ for S a family of random scaling laws, so that U is a statistically self-similar fractal. While interesting, this similarity holds only for a certain range of scales approximately between tens and hundreds of millions of light years [Old], [Pie]. Like virtually all fractality that is observed in the natural world, it is approximate [Sta-Sta]. This leads us to wonder about the breaking of the scale symmetry, an aspect of the phenomenon of scale covariance.

If the family of scaling laws S should be found describing the distribution of matter and energy in our universe, and should the equation $U = SU$ hold over many or all scales then a long-sought, one-line "equation of the universe" will have been obtained, perfect to print on t-shirts popularizing science and mathematics. We recall the comment of the Marquis de Laplace that knowing the initial arrangement of particles in the universe would grant him knowledge of the motion of all particles in the universe, so we would expect $U = SU$ to govern the dynamics of the universe as well, and yield testable conclusions about the state, composition, size and age of the universe.

Figure 2: A t-shirt depicting a fractal 'equation of the universe'.

Similar catchy slogans have established this link between form and function (structure and dynamics) previously. Sir Isaac Newton's second law of motion $F = mA$, a dynamical expression to be sure, also expresses structure in its implication of conical section trajectories for objects in the universe, moving under the influence of gravitation. Albert Einstein's equations $E = mc^2$ and $G_{\mu\nu} = 8\pi T_{\mu\nu}$, respectively inform us as to the structure and dynamics of the microscopic and macroscopic worlds, explicitly dealing with both aspects.

4.2 Application of Contractivity to Causality

The following construction is presented as an example of how simple geometric assumptions can lead to fractality. It was intended as a counter to the cosmological assumption that the expansion of the universe can be reversed to a single point. It reveals fractals to be a type of multilateral perspective. The introduction of perspective into the consciousness of western civilization had huge effects, leading to classical mechanics and non-Euclidean geometry and relativity [Shlain]. Indeed projective geometry is still a thriving research field of mathematics. The generalization of geometry offered by the theory of fractals promises nothing less.

4.2.1 Spacelike Cantor Sets in a Toy Model

The hypothesis that the contents of the universe may be arranged in fractal patterns [Old],[Pie], seems to suggest that space itself may have an underlying self-similarity. The goal of this section is to explore a simplified scenario in which spacelike fractal sets can occur. The main ingredients are the domain of causal contact viewed as a contraction mapping on intervals of space as time is reversed, the completeness of Euclidean space, and the almost sure discrepancy between measurements of positions by differing observers, when the measurement of the position of the source is taken as a continuous random variable. This classical scenario blissfully ignores quantum uncertainty, (other) spacetime singularities, and assumes an absolute time and an impartial observer.

Recall that the domain of influence at time t of an initial condition at space-time coordinates $(x, t) = (x_0, 0)$ is the interval $[x_0 - ct, x_0 + ct]$ in one dimension of space, as given by solutions to the wave equation $u_{xx} - cu_{tt} = 0$ [Stra]. We first consider discrete increments in time to study the fractality of the origin of a signal in our scenario. Then, this interval, for a given time of measurement $t > 1$, with t fixed, in steps of Δt is contracted by a contraction factor $2c(t - \Delta t)/2ct = (t - \Delta t)/t < 1$. Thus, we observe the contractivity of causally connected regions when looking into the past.

In our scenario, the emission from a spacelike fractal source, results from the physical reality that two measurements of our source, with space coordinate in a non-trivial complete metric space, will almost surely yield two different positions of that source. The idealization of our model suggests that each such front is independent of which observer will measure it, therefore, compositions of contraction mappings of each front will be taken with respect to each of the measurements of the initial position. Let us imagine the signal observed by the two observers at time t_0, with t_0 fixed, whose positions are located in the space-interval $(\xi_1 - tc, \xi_2 + tc)$, and who detect the source at spacetime coordinates $(\xi_1, 0)$ and $(\xi_2, 0)$, respectively, with $\xi_1 = -\xi_2$ and $\xi_2 > 0$ for definiteness. Then we define contraction mappings on the interval $(\xi_1 - tc, \xi_2 + tc)$, with time parameterizing the steps of the composition of the resulting IFS. Taking steps in time at the negative integer powers of the initial time allows for an infinite number of steps so that the resulting IFS can converge to a true fractal attractor at the time of emission of the signal. The result is a spacelike Cantor set in this simplified case of two observations.

Example 5. *We embed our images of the interval $I = [\xi_1 - ct_0, \xi_2 + ct_0]$ in $I \times n$ at the times $n = 1 - \frac{\log t}{\log t_0} \in [0, \infty]$ (for $t \in [0, t_0]$), in the space of scales of the initial time t_0. We have $t = t_0^{\frac{\log t}{\log t_0}} \in (0, t_0]$, then $t = t_0^{1-n}$ so that*

$x = ct \mapsto x = ct_0^{1-n}$.

Iterating the IFS $S = \{S_i\}_{i=1}^2$, *in steps of* $n = 1 - \log_{t_0} t \in \mathbb{N}$, *we obtain a space interval for each contraction mapping at every finite* n. *We define* $S_1(x) = r_1 x + \xi_1$ *and* $S_2(x) = r_2 x + \xi_2$, *for contraction ratios* r_1 *and* r_2 *and for* ξ_1, ξ_2, *the initial source locations as measured by the observers.*

Taking the ratio of the lengths of the successive intervals, at times $n+1$ *and* n, *we compute* $r_i := \frac{2ct_0^{1-(n+1)}}{2ct_0^{1-n}} = \frac{1}{t_0} < 1$, *thus establishing contractivity (under the assumption above that* $t_0 > 1$, *noting that a similar argument works for small* t_0*) and providing* $r_1 = r_2 = t_0^{-1}$ *so that* $S_1(x) = xt_0^{-1} + \xi_1$ *and* $S_2(x) = xt_0^{-1} + \xi_2$. *Then the invariant set* $F = \cup_{i=1}^2 S_i(F)$ *will define a fractal attractor on the surface* $t = 0$ *or* $n = \infty$, *by contractivity of* S *applied to* $I = [\xi_1 - ct_0, \xi_2 + ct_0]$. *A symmetrical argument supplies a fractal attractor for the future dependecy of present events.*

We can easily calculate the fractal dimension of the resulting invariant set F. For equicontractive self-similar fractals (eventually) without overlap of the images of the contraction mappings (as e.g. for $\xi_2 = \frac{1}{3}$ and $t_0 = 3$), the box-counting dimension, and Hausdorff fractal dimensions are both equal to the similarity dimension, by Moran's Theorem [Hut], the exponent d_S that solves the Moran equation with $r_i = r$ for all i. Thus we compute $d_H(F) = d_B(F) = d_S(F) = \frac{\log 2}{\log t_0}$. We note that the time of observation is in the denominator of this expression, suggesting that the dimension of the fractal source is seen to diminish with the passage of time, and our distance from the source renders it more pointlike in appearance.

5 Box Counting Functions of Compact Sets

Definition 13. *Let X be a complete metric space under the metric $d : X \times X \to \mathbb{R}$. A Box-Counting Function of a set $E \subset (X, d)$ is a function $N_E(r) : E \to \mathbb{N}$ such that $N_E(r) = |\mathfrak{U}|$, where $\mathfrak{U}$ is a minimal covering of E by sets of diameter not more than r.*

Definition 14. *Let X be a complete metric space under the metric $d : X \times X \to \mathbb{R}$. A compact subset $K \subset X$ is a set such that for any covering $|\mathfrak{U}|$, there exists a finite sub-collection $U = \cup_{n=1}^{N} U_n$, such that $K \subset U$.*

Proposition 2. *Let (X, d) be a complete metric space. For any compact subset $K \subset X$, a box-counting function $N(r) := f_r(K)$ is defined.*

Proof. Let $\mathfrak{U}_r$ be a covering of K by sets of diameter at most r. Since K is compact, there is a finite sub-collection U_r that covers K. The class of all sub-collections U_r has a member with minimal cardinality, since the cardinality of the U_r is finite and bounded below by one. The cardinality $|U_r|$ of this minimal element U_r is the value of the Box-Counting function $N(r)$. □

For general IFS fractals, the box counting function may be somewhat difficult quantify with precision. We study first the example of the Cantor set as a simple example that is well-behaved: it has singularities only at the powers of the preferred scale.

A simple example when this happy condition does not hold is furnished by the fractal whose complement in $[0, 1]$ is the famous Fibonacci string [Lap-vanF].

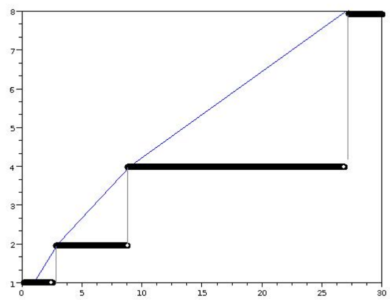

Figure 3: The box-counting function of the Cantor set.

Example 6. *Let F be the fixed-point attractor of $S = \{S_i(x)\}_{i=1}^{2}$, with $S_1(x) = \frac{1}{2}x$ and $S_2(x) = \frac{1}{4}x + \frac{3}{4}$. Observe that for inverse scale r, $N_F(4^n) = 3^n$, for $n > 0$. Yet we see that $N_F(r) = 2$ for $r \in (1, 2]$. Furthermore, we might guess that this cycle will repeat, multiplied by 3 through each period of the scale variable through a magnification of 4, so that $N_F(8)$ would equal 6, but we see we can cover F at the scale $\frac{1}{8}$ using only 5 boxes.*

We can, however, find some things that are usually true about the box counting function:

- It is non-decreasing with the inverse scale.
- Taking values on the natural numbers, it can have only finitely many singularities on a "fundamental half-open period" $(r_0^n, r_0^{n+1}]$, for a preferred inverse scale r_0, the smallest of the contraction ratios of S.
- $N_F(r_0^n) = N_F(r_0)^n$.

The first item fails in the case of "negative scaling". This should relate to the concept of negative dimension in [Man]. We might like to say that there can only be $m-1$ singularities in each such period, but this would rest upon the perfect log-periodicity of $N_F(r)$, which fails by the counterexample above. This is another example of the log-periodic corrections to scale. The third item can be defeated by IFS with construction rules that depend on the scale or stage of the construction.

We will consider the third item a condition on many constructions we will study. This will characterize an intrinsic geometry of the prefractal at each appropriate scale of observation, when we allow boxes to fit only within the prefractal construction at a given scale. We shall see in our study of representations of fractals that this condition holds for the IFS fractals that we can represent with vectors and matrices.

5.1 An Equivalent Definition

Mesh Coverings: Covering $K \subset \mathbb{R}^m$ with a δ-coordinate mesh of cubes of the form:

$$[a_1\delta, (a_1+1)\delta] \times \cdots \times [a_m\delta, (a_m+1)\delta]$$

(with the $a_i \in \mathbb{Z}$), we can define $N^*_\delta(K)$ to be the number of δ-mesh cubes that intersect K. If $N_{\delta\sqrt{m}}(K)$ is the minimal cardinality of coverings of K by sets of diameter $\delta\sqrt{m}$, then $N_{\delta\sqrt{n}}(K) \leq N^*_\delta(K)$. Since we can contain any set Δ of diameter at most δ in at most 3^m mesh cubes of diameter δ, by choosing any cube intersecting Δ and its $3^m - 1$ neighbors in $\mathbb{R}^m$, we see that $N^*_\delta(K) \leq 3^m N_{\delta\sqrt{m}}(K)$. Then since the number of K-intersecting $\delta\sqrt{n}$-mesh cubes is within a constant multiple of the minimal number of sets of diameter δ needed to cover K, we can take $N^*_\delta(K)$ to be the function $N_\delta(K)$.

Remark 4. *The bounds we have established on the equivalent form of the counting function leave the fractal dimension of K invariant. See [Fal] for a detailed proof.*

It is this definition of the box counting function that gives us hope of an algorithmic approach to finding a box counting function of a fractal. We will fix a grid of unit boxes at the origin, then continuously decrease the scale of that grin, while counting the number of boxes with nonempty intersection with $F_n = S^n(I)$. While this $N_{Grid(F)}(r)$ need not take the same values everywhere as a minimal covering of F by boxes of inverse scale r, we know that the ratio of its logarithm to that of r converges to the same limit as for $N_F(r)$.

5.2 Box Counting Dimension as Slope of regression line

Given finite, randomly distributed sample data $Y = \{y_1, ..., y_n\}$, of an underlying parameter $X = \{x_1, ..., x_n\}$, we might make a logarithmic regression of the changes in the sample data over the changes in the parameter times the sample mean of the parameter over the sample mean of the data.

Curiously, these quantities will fail to exist for an infinite, power law distributed data set. Yet, the prescribed ratio will approach a limit, see the next chapter.

To compute this dependence at the means, since it may change discontinuously at each point, we can compute an ordinary least squares regression

$$\eta_{XY} = \frac{n\sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \frac{\frac{1}{n}\sum_{i=1}^n x_i}{\frac{1}{n}\sum_{i=1}^n y_i}.$$

On the other hand, if we wish to compute this dependence at a point, we may compute the following quantity, which requires differentiability of the data's dependence on the parameter, or we can work with generalized derivatives (see next chapter). We can compute the relative rate of change of $Y = y(x)$ with $X = x$ as

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \frac{x}{y} = \frac{dy}{dx} \frac{y}{x} = \frac{d \ln y}{d \ln x}.$$

5.2.1 Application to economics

This formula is both an estimator of the slope of a double-logarithmic plot of two variables, as well as the economic price elasticity of demand for X representing price of a good, or the income elasticity of consumption for X representing customer (mean) income, with Y indicating the quantity demanded of the good in question.

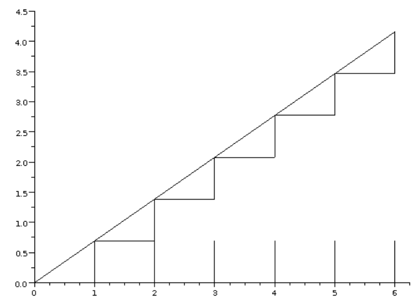

Figure 4: A plot of $\log N_r(\mathcal{C})$ against $\log r$, its linear interpolation and singularities. Note delta masses are truncated to the height of their coefficients.

Elasticity determines the relative change of a quantity demanded, under change in the price of a good, or the income of the consumer. We observe that goods which are strictly needed will have much lower response of quantity demanded to changes in price, while consumption of luxuries will will respond markedly to changes in price.

Since economics is the distribution of goods in the presence of scarcity, this elasticity could be deemed a principle of singular importance in organizing the entire economy by goods and industries. In recognition that this quantity is akin to a local fractal dimension (see next chapter), we could see the importance of (multi-)fractal dimension as an organizing principle of mathematical functions, at least for those modelling phenomena within the human world.

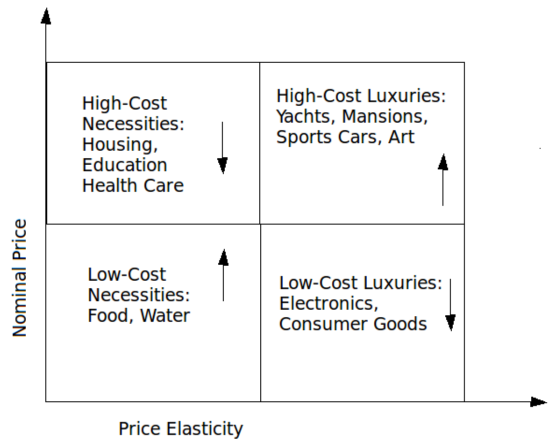

Figure 5: Goods and industries plotted on the axes of price versus elasticity. The arrows show the author's hypothesized directions of change in price during times of economic contraction. During economic expansion, the arrows would be reversed.

5.3 Box-Counting Dimension as a Derivative

In effect, the work of scientists who propose fractal geometries to describe natural structures, suggests that self-similarity holds only for a certain ranges of scales [Old], [Pie]. Like virtually all fractality that is observed in the natural world, it is approximate [Sta-Sta]. This leads us to attempt to develop fractal analysis tools that can estimate the fractal dimension of a set at a particular scale. Scale dependence of structures is called scale covariance, or dependence of a phenomenon on the scale of observation, which includes self-similarity as a special case.

According to B. B. Mandelbrot, "the familiar box dimension D_B simply measures the rate of increase of $N(b)$ with b", [Man2], see also [Sta]. Thus motivated, we begin our study of scale covariance by considering the slope of a log-log plot of the box-counting function against the scale of measurement [Bro-Lie], as the generalized derivative, $\frac{d\log N(r)}{d\log(r)}$. Using this derivative, we can find ODE's to describe fractals as well as prefractal, almost self-similar structures. The box-counting dimension is the limit as the scale vanishes of this derivative, essentially a "boundary condition", for ODE's describing scale covariance. In the remainder of this section, we will employ the convention of using the variable r, to represent the *inverse* scale.

Definition 15. *The* box-counting derivative*, is the dependence of the logarithm of the box counting function on the logarithm of the inverse scale, a generalized function on the space of inverse scales* $r \in (0, \infty)$*,*

$$\partial_{Box} N(r) := r \frac{\frac{dN(r)}{dr}}{N(r)},$$

in the sense of generalized derivatives of distributions.

Remark 5.

1. *For* $N(r)$ *a smooth, non-vanishing, mass distribution,* $\partial_{Box} N(r)$ *is the corresponding double logarithmic derivative:*

$$\frac{d\log N(r)}{d\log r} = \lim_{r_0 \to r} \frac{\log N(r) - \log N(r_0)}{(\log r - \log r_0)}$$

2. *Whenever the limit of this quantity exists as* $r \to \infty$*, for a given* F*, we see that* $\partial_{Box} N_F(r)$ *approaches the box-counting dimension* $d_B(F)$*.*

3. *For a given* F *with* $N(r)$ *a discrete counting function,* $\partial_{Box} N(r)$ *is*

a sum of Dirac measures. This singular measure has support at the singularities of $N(r)$, and we recover the "slope of the log-log- plot" by evaluating this singular measure at its singularities. This quantity estimates the box-counting dimension of F, and may vary over changes in inverse scale r.

Example 7. *For $N(r)$ smooth, we can verify that we get the expected power law solution $N(r) = r^k$, for a fixed constant k, for the equation $\frac{d\log N(r)}{d\log r} = \frac{\log N(r)}{\log r}$ by a simple separation of variables and exponentiation. If we suppose that scale invariance holds only in a range of scales $[a, b]$, and that F scales as do points outside of $[a, b]$, then separating and integrating we find that $N(r) = e^{\frac{b}{a}^k}$, that is, N remains constant on $[a, b]$.*

Example 8. *Let F be the middle thirds Cantor set. Observing that each increment of (inverse) scale increases the number of "boxes" (line segments in $\mathbb{R}$) needed to cover F by a factor of 2. Then we can compute the counting function of F in terms of the inverse scale (or magnification factor) as $N_{\mathfrak{F}}(r) = 2^{\lceil \log_3 r \rceil_0}$, where we define $\lceil r \rceil_0 := max(0, \lceil r \rceil)$, and $\lceil r \rceil$ is the least integer greater than or equal to r. Then*

$$\partial_{Box} N_{\mathfrak{F}}(r) = \frac{r}{2^{\lceil \log_3 r \rceil_0}} 2^{\lceil \log_3 r \rceil_0} \frac{\log 2 \sum_{n=0}^{\infty} \delta(r - 3^n)}{r \log 3} = \frac{\log 2 \sum_{n=0}^{\infty} \delta(r - 3^n)}{\log 3}.$$

We recover the pointwise "slope of the log-log- plot" by computing

$$\partial_{Box} N_{\mathfrak{F}}(r) = \begin{cases} 0 & \textit{if } \log_3 r \notin \mathbb{Z} \\ \frac{\log 2}{\log 3} & \textit{if } \log_3 r \in \mathbb{Z}. \end{cases}$$

We see that $\partial_{Box} N_{\mathfrak{F}}(r) = \partial_{Box} N_{\mathfrak{F}}(3r)$*, and it has multiplicative period of* 3*.*

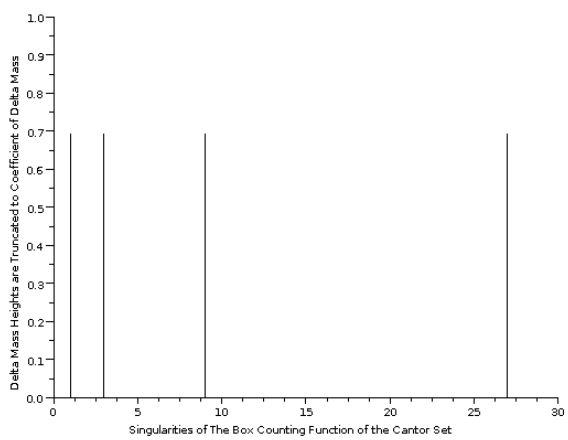

Figure 6: A plot of $\partial_{Box} N_{\mathcal{C}}(r)$.

In the concluding example above, we observed the log-periodicity of the measure associated to the estimated box-dimension for a given deterministic fractal. Indeed, for integers n and k, and for $\log r = n(1+\frac{2\pi ik}{\log 3})$, we see that r is in the support of the measure $\partial_{Box} N_{\mathfrak{F}}(r)$. This logarithmic period of $\frac{2\pi i}{\log 3}$, has been observed, in the study of the Cantor string in [Lap-vanF] . This is one example of log-periodic scaling observed in both fractal geometry and in the study of critical systems [Sor]. Since this periodicity may be different for fractals that share the same box-counting dimension, we propose that this technique can be useful in evaluating geometrical models of critical systems.

6 Algebra and the Category of Fractals

6.1 Categorical Nature of Self-Similarity

It seems natural to ask, "in what sense are self-similarity and fractality categorical concepts?" That is, "When can we expect these notions to have precise meanings in other sub-fields, or even throughout mathematics? What implications will follow?"

In the work of F. Lawvere we find that (suitably generalized) metric spaces X are $[0, \infty]$-*enriched* categories [Law], whose objects are the elements $x \in X$ and with (an object of) morphisms $X(x, y) = d(x, y)$[Stub-Bo]. Furthermore non-expansive maps, including contraction mappings, are the functors between these categories [Law]. In a generalized, topological view of self-similarity [Lei], we find that self-similarity and compact metrizability are equivalent for topological spaces, giving us a whole category of self-similar objects, whose objects are the non-negative real numbers $a \in [0, \infty]$ and with morphisms $Hom(a, b) = \begin{cases} b - a & b \geq a \\ 0 & a > b \end{cases}$.

6.2 Coalgebraic Representation Theory of Fractal Sets

Extending the theory of self-similar fractals to that of self-similar measures, as in [Hut], has been a natural step toward understanding the algebra of self-similarity, since we may define integral transforms of these measures [Stri], giving us a type of representation theory for these objects [Loom].

Recent work in algebra ([H-J-N]) has developed a representation theory of the streams of characters comprising the words on the alphabet of indices of the contraction maps of the iterated function systems that give rise to self-similar fractals. Indeed, the contraction mapping principle is key to establishing a bijection between the fractal set itself and the representation by streams of characters.

Definition 16. *An* infinite stream of characters *is a word* $a_0a_1a_2... \in \{0, ..., N-1\}^\omega$*, the space of infinite words on* $\{0, ..., N-1\}$*. For each element* x *in a self-similar fractal* F *given by an IFS,* $S = \{S_i\}_{i=0}^{N-1}$*, a stream can be chosen so that for all* i*,* $x \in S_{(a_i)} \circ S_{a_i-1} \circ .... \circ S_{(a_0)}(I)$*, where, without loss of generality,* $I = [0, 1]$*.*

A self-similar fractal $F \subset I$ resulting from an IFS (here we work without overlaps, see [H-J-N] and [Lei] for generalizations) can be given a symbolic representation in terms of words $\sigma = a_0a_1... \in \mathbf{N}^\omega$, the space of infinite words on the alphabet $\mathbf{N} := \{1, ..., N\}$, we shall call each such word a stream, after [H-J-N]. Given this stream σ we assign a point $[[\sigma]]$ in an interval $\mathbb{I}$, calling the assigment $[[\_]] : \mathbf{N}^\omega \to \mathbb{I}$ the *denotation map* and see that its restriction to equivalence classes determined by the elements in F which the streams indicate, $\mathbf{N}^\omega \xrightarrow{\sim} F$ is bijective by construction, with inverse $F \xrightarrow{\sim} N^\omega$ called the *representation map*.

Again following [H-J-N], we see that the set of symbolic representatives of F, $\mathbf{N}^\omega$ carries the final coalgebra $\iota : \mathbf{N}^\omega \xrightarrow{\sim} \mathbf{N} \cdot \mathbf{N}^\omega$ for the combinatorial specification of F, the functor $\mathbf{N} \cdot (\_) : \mathbf{Sets} \to \mathbf{Sets}$, reflecting the N-fold recursive construction of F. We call this final coalgebra the *symbolic fractal* for F, and note its recursive structure. It is this view of the alphabet $\mathbf{N}$ as an IFS which we identify with an algebra $\chi : \mathbf{N} \cdot F \to F$. Then the (restricted) denotation map makes the following diagram commute:

$$\begin{array}{ccc} N \cdot N^\omega & \xrightarrow{N \cdot [[\_]]_\chi} & N \cdot F \\ \cong \Big\uparrow \iota & & \Big\downarrow \chi \\ N^\omega & \xrightarrow{[[\_]]_\chi} & F \end{array} \tag{1}$$

In the absence of any overlap between the images of I under the functions in the IFS, Theorem 1 is used to show uniqueness of the denotation map:

Theorem 11. *There exists a unique denotation map* $[[\_]]_\chi$ *that makes the diagram (1) commute.*

The following sketch highlights the use of Banach's theorem:

Sketch. The set of morphisms $\mathbf{Sets}(\mathbf{N}^\omega, \mathbb{I})$ is a complete metric space under $d(f,g) = sup_{\sigma\in\mathbf{N}^\omega}\{d(f\sigma, g\sigma)\}$. On $\mathbf{Sets}(\mathbf{N}^\omega, \mathbb{I})$ the map $\Phi : F \mapsto \chi \circ (\mathbf{N} \cdot F) \circ \iota$ is a contraction map. Therefore, by the Banach fixed point theorem (Theorem 1), it has a unique fixed point.

□

6.3 Algebra of IFS Fractals

6.3.1 The Category of IFS fractals

Denote by $\mathfrak{C}$ the compact subsets of $[0,1]^n \subset \mathbb{R}^n$. Any contractive family of similarities, or IFS S takes an object in $\mathfrak{C}$ to another object in $\mathfrak{C}$, so it is a morphism in $\mathbf{mor}_{\mathfrak{C}}$. If we allow the "empty composition" of S to be denoted by S^0, then S^0 is an identity on $\mathfrak{C}$.We may consider also a small category of images of I under composition powers of an IFS $S^{(n)}$. Then the powers of S commute, suggesting the structure of a fusion category. We would like to show that the small category of approximations to an IFS fractal is an object in a monoidal category in which the tensor product is the application of an IFS, which we show has a tensor product representation thoery.

Theorem 12. *The images of $I = [0,1]^n \subset \mathbb{R}^n$ under finite families of contractive similarities is a category $\mathbb{IFS}$, with objects $\boldsymbol{Ob}_{\mathbb{IFS}} = S^j(I) : S \in \mathfrak{S}$, and morphisms $\boldsymbol{Mor}_{\mathbb{IFS}} = \{S^j : S \in \mathfrak{S}\}$.*

Proof. For the class $\mathfrak{S} = \{S :$ S is a finite family of contraction maps on I$\}$ of families of contractions on $I \subset \mathbb{R}^N$, let $S^{-1} \circ S^j = S^{j-1}$, $S^{-j} = S^{-1} \circ$

$S^{-1} \circ ... \circ S^{-1}$ and $S^0(I) = I$, then for any $S, T \in \mathfrak{S}$, we have $S^{(j)}(I) = S^{(j)}(T^{-k}(T^k(I))$, so that $S^j T^{-k} : T^k(I) \to S^j(I)$ is a morphism from $T^k(I)$ to $S^j(I)$, then $\mathbf{hom}_{\mathbb{IFS}}(T^k(I), S^j(I))$ contains at least one element, $S^j T^{-k}$.

Associativity of these morphisms under composition comes from associativity of functions under composition. □

We can consider the class $\{S^n(I)\}_{n=0}^{\infty}$ to be the objects of a small category, with morphisms the $\{S^n(\cdot)\}_{n=0}^{\infty}$, and their inverses. Topologically, we can identify I and $S^{-j}(I)$, for any $j > 1$, so we see the natural monoid structure, since not every element can have an inverse operation applied to it, much like for the natural numbers.

Proposition 3. *For any given contractive family of similarities $S = \{S_i : [0,1]^n \to [0,1]^n\}_{i=1}^n$, the set of images $S^{(k)}([0,1]^n)$ of (closed) subsets of $[0,1]^n$ under iterated application of S is a monoid with operation the application of S, when we define $S^{(0)}([0,1]^n) = [0,1]^n$.*

Proof. Define $[0,1]^n = S^{(0)}([0,1]^n)$ to give that $S^{(0)}([0,1]^n)$, is the identity for the monoid. For any two iterated compositions $S^{(a)}$ and $S^{(b)}$, we have that $S^{(a)} \circ S^{(b)} = S^{(a+b)}$, so that the set of images is closed under compositions. Since composition is associative, we have that $S^{(a)} \circ (S^{(b)} \circ S^{(c)}) = S^{(a+b+c)} = (S^{(a)} \circ S^{(b)}) \circ S^{(c)}$. □

We can show that the small category with objects $\{S^n(I)\}_{n=0}^{\infty}$ and morphisms $\{S^n(\cdot)\}_{n=0}^{\infty}$ is a tensor category.

Theorem 13. *The small category with objects* $\{S^n(I)\}_{n=0}^{\infty}$ *and morhpisms* $\{S^n(\cdot)\}_{n=0}^{\infty}$ *is a tensor category.*

Proof. Within our small category, the unit object will be $\mathbf{1} = I = S^0(I)$ with map $\iota : \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$, when we define the tensor product to be $\otimes : (S^n(I), S^m(I)) \to S^{n+m}(I)$ with the identification $S^{-k}(I) = I$, for positive integers k, the associativity diagram commutes since

$$(S^n(I) \otimes S^m(I)) \otimes S^k(I) = S^{n+m}(I) \otimes S^k(I) = S^{n+m+k}(I)$$

$$= S^n(I) \otimes (S^{m+k}(I)) = S^n(I) \otimes (S^m(I) \otimes S^k(I)).$$

Of course, $S^0(I) \otimes S^n(I) = S^n(I)$. One can easily verify that the pentagon and unit axioms hold.

□

Conjecture 1. *The small category with objects* $\{S^n(I)\}_{n=0}^{\infty}$ *and morphisms* $\{S^n(\cdot)\}_{n=0}^{\infty}$ *is a braided fusion category.*

7 Tensor Representations of IFS Fractals

This technique of representing approximations to fractals and their measures is inspired by graph theoretic adjacency matrix techniques, and, superficially, by the notion of schemes in Khinchin's theory of information [Khin]. In graph theory we are interested in adjancency properties of graphs, so we

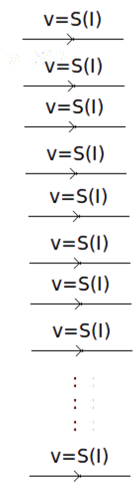

Figure 7: The graphical calculus of the construction of a fractal attractor F to an IFS S.

need a matrix to encode this information, but fractals on the line are totally disconnected. Yet, the connectedness properties of the approximations to the fractal give differential geometric and homological algebraic information about the fractal and its approximants, and we can use a simple vector to represent the connected components of these approximations. Schemes in Khinchin's work relate probabilities to states, and they are multiplied in a row-wise format. We will put this to use when the geometry of the IFS images plays an important role that the topological information cannot communicate by itself. Similar representations are mentioned [FouSmiSpei], in the context of computing correlation dimensions and their oscillations.

The core idea is to simplify the computation of composition of contraction mappings by representing them as a type of multiplication, something jokingly referred to as "complication", the conflation of multiplication and composition [unpublished remark due to Erin Pearse]. Exposing iterated function systems as a tensor product allows simplified computation of box-counting functions and measures, representations and illustrations of fractal sets, duality pairings between the self-similar measures (and the approximations thereof) supported on the fractal and the flows on the approximations and its limits, additional connections to topology, geometry and physics, and exposes the categorical nature of fractals defined as the attractors of Iterated Function Systems.

7.1 Representations

Definition 17. *We define a (set theoretic) representation of F by an algebraic object G to be a function ρ, such that for all $x \in F$ and $n_1, n_2 \in G$,*

$$\rho(1)[x] = x$$

$$\rho(n_1 n_2)[x] = \rho(n_1)[\rho(n_2)[x]].$$

Much as groups are commonly represented by linear groups, we will be representing monoids with monoids under tensor products (bi-linear monoids). While the representation we construct is fairly simplistic, its value lies in its utility for numerical approximation, and in placing IFS fractals in a context of measure theoretic, geometrical and algebraic techniques.

7.2 Tensor Representation Theory of IFS Fractal Sets and Measures

Our first theorem gives a representation of the connected components of the $n-$times iterated image of I under S by the components of $\vec{v}^{\otimes^n}$, with deleted intervals represented by $\{0\}$, and remaining segments by $\{1\}$.

Definition 18. *We call an IFS S* equicontractive *if all the contraction ratios are identical, for each contractive similarity $S_i(x)$. We call S* (translationally) dependent *if the translations in each $S_i(x)$ are a multiple of the contraction ratios.*

In the sequel, be advised that we will not be working with rotations, so that the transformations engendered by the IFS in our study will be dependent merely upon the contraction ratio and the translation preformed by each function in an IFS.

Definition 19. *(Representation vector, equicontractive case:) A representation vector $\vec{v} = (v_1, ..., v_N)$ with elements in $\mathbb{F}_2$ represents a set $S(I) : I = [0, 1]$, with*

$$v_i \approx \begin{cases} \frac{1}{N}\lambda^{-1}(1) + \frac{i-1}{N} \text{ if } v_i = 1 \\ \frac{1}{N}\lambda^{-1}(1)^* \text{ if } v_i = 0 \end{cases} ,$$

as a representation, where by λ we mean Lebesgue measure restricted to the unit interval $I = [0, 1]$, and by $\lambda^{-1}(1)$, we identify all sets under almost everywhere equivalence, so that $\lambda^{-1}(1)$ refers unambiguously to [0,1] *(at least as an equivalence class). By $\lambda^{-1}(1)^*$, we refer to the deleted intervals as elements of a class of intervals, for treatment of the deleted intervals as an ordinary fractal string (see [LapVanF]). Optionally, the positional information of the deleted lengths can be stored, in which case we would have $v_i = \frac{1}{N}\lambda^{-1}(1)^* + \frac{i-1}{N}$ if $v_i = 0$. As another alternative, we could have $v_i = \frac{1}{N}\lambda^{-1}(0)^* = \emptyset$ if $v_i = 0$, (as an equivalence class) if we truly wish to delete these intervals.*

Remark 6. *Once we introduce the tensor product of the representation vectors we shall use the representation induced by the tensor product considering the k^{th} tensor power of $\vec{v}$ as a vector in its own right and applying the above definition.*

Remark 7. *If we consider the $\lambda^{-1}(1)$ and $\lambda^{-1}(0)$'s not as equivalence classes, we could create more general constructions, but uniqueness would be sacrificed. This would open the door to fractal preimages of the IFS.*

Remark 8. *The following theorem establishes that the tensor representation is a valid representation for the sequence of applications of the IFS to an underlying set. Contrast this with the more ad-hoc definitions above that characterize the intervals in the successive images of the IFS. We believe that the tensor representation of measures (in an upcoming section) will have more inevitability.*

Definition 20. *(Kronecker Product of Tensors) Let* $A = \begin{matrix} a_{11} & ... & a_{1N} \\ \vdots & ... & \vdots \\ a_{M1} & ... & a_{MN} \end{matrix}$ *and* $B = \begin{matrix} b_{11} & ... & b_{1Q} \\ \vdots & ... & \vdots \\ b_{P1} & ... & b_{PQ} \end{matrix}$ *be tensors of rank one or two, represented by an array, such as a vector or matrix. The Kronecker Product* $A \otimes B$ *of* A *and* B *is defined as* $A \otimes B = \begin{matrix} a_{11}B & ... & a_{1N}B \\ \vdots & ... & \vdots \\ a_{M1}B & ... & a_{MN}B \end{matrix}$. *Note the dimension of this tensor is* $MP \times PQ$.

Lemma 1. *Representation lemma: The tensor category* $\{S^{(n)}(I), \otimes\}_{n=0}^{\infty}$ *can be represented by the monoid of tensor powers* $(\{\vec{v}^{\otimes^n}\}_{i=1}^{\infty}, \otimes),$.

Proof. Let $\rho(S^n(I)) = \vec{v}^{\otimes^n}$, that is let $S^n(I)$ be represented by the tensor $\vec{v}^{\otimes^n}$. we compute

$$\rho(S^{n_1}(S^{n_2}(I)) = \vec{v}^{\otimes^{n_1+n_2}},$$

$$= \vec{v}^{\otimes^{n_1}} \otimes \vec{v}^{\otimes^{n_2}}$$

$$= \rho(S^{n_1}(I))\rho(S^{n_2}(I)).$$

□

Thus we have a monoid representation for the $\vec{v}^{\otimes^i}$, which we consider, by bijection, as a representation of the $S^i(I)$.

Theorem 14. *Tensor Representation Theorem, equicontractive case: Let an IFS on* $\mathbb{R}$, $S = \{\frac{v_i}{N}x + \frac{i-1}{N}\}_{i\in\mathcal{I}\subset\{1,2,...,N\}}$, *be represented by a vector* $\vec{v} = [v_1, v_2, ..., v_N]$, *with* $v_i \in \{0,1\}$, *with* 0 *representing any deleted subintervals. The* m^{th} *iterate,* $S^m(I)$, *is represented as* $S^m(I) \cong \otimes^m \vec{v}$, *where we take the Kronecker product of the vectors as row vectors.*

Proof. Using induction on m, for a base case, we use that

$$S = \{\frac{v_i}{N}x + \frac{i-1}{N}\}_{i\in\mathcal{I}\subset\{1,2,...,N\}},$$

for $x \in I$, exactly what is represented by $\vec{v}$, by construction. Now if we suppose that, for some k, $S^{k-1}(I)$ is represented by $\vec{v}^{\otimes^{k-1}}$, then

$$S^k(I) = S(S^{k-1}(I)) = \cup_{i\in\mathcal{I}\subset\{1,2,...,N\}} S_i(S^{k-1}(I))$$

$$= \cup_{i\in\mathcal{I}\subset\{1,2,...,N\}} \frac{v_i}{N}(S^{k-1}(I)) + \frac{i-1}{N}$$

which, by hypothesis, is represented by

$$[v_1\vec{v}^{\otimes^{k-1}}, v_2\vec{v}^{\otimes^{k-1}}, ..., v_N\vec{v}^{\otimes^{k-1}}] = \vec{v} \otimes \vec{v}^{\otimes^{k-1}}$$

$$= \vec{v}^{\otimes^k}.$$

□

7.3 The Space of Representation Tensors

Here we further define the representation tensors. Characteristic functions of the sets represented by the representation tensors reside in $L^2([0,1])$. We find a diffeomorphism between our vectors and the usual vectors in $\mathbb{R}^N$.

This use of vectors is inspired by the notation and usage of vectors in array programming languages, such as scilab, for which we can write simple programs using these notions to approximate density functions of self-similar measures. In such applications, a partition of an interval will be denoted by a vector with components equal to the endpoints of the subintervals. In our usage, the indices of the vector, normalized by its length, represent those endpoints. Heretofore we use a convention that the 0^{th} index of any tensor is 0.

Of course, to view the line segment $I \subset [0,1]$ as an $N-$dimensional space, we need an independent x_i in each of the N subintervals. This is justified, since we define a weighted uniform mass distribution of each subinterval, independent of the others except for their overall sum (a degree of freedom we can restore by normalizing over the sum of the weights).

Proposition 4. *Diffeomorphism Between Vectors over* $\mathbb{R}^N$ *and Partition Valued Vectors:*
For each $\{x\}_{i=1}^{N} \in I_N = \cup_{i=1}^{N} I_i = \cup_{i=1}^{N} \frac{1}{N}[i-1, i]$ *with each* $x_i \in I_i^{\circ}$*,(*I_N *being the interval I=[0,1] partitioned into N equal subintervals (which could be represented by a vector* $\vec{v} = [1, ..., 1]$*), there is a diffeomorphism* $\Phi(x)$:

$I_N \to \mathbb{R}^N$.

Proof. We define a function $\Phi : \mathbb{R} \to \mathbb{R}^N$ as follows. Let $\vec{w} = \Phi(\{x\}_{i=1}^N)$. Let $\Phi_i(x) = w_i(x_i) = \tan[\pi(Nx_i - i) + \frac{\pi}{2}]$, for $x_i \in (\frac{i-1}{N}, \frac{i}{N})$. Then $\frac{\partial \Phi_i}{\partial x_i} = \pi N(\sec[\pi(Nx_i - i) + \frac{\pi}{2}])^2 > 0$, and for $x_i = w_i \in \mathbb{R}$, $\Phi^{-1}(x_i) = v_i(x) = \frac{\frac{1}{\pi}(\tan^{-1}(x) - \frac{1}{2}) + i}{N}$, so that $\frac{\partial \Phi^{-1}}{\partial x_i} = \frac{1}{N\pi(1+x^2)} > 0$. Then we see there is a differentiable function with differentiable inverse from the partition of $I = [0,1]$ by $\vec{v}$ to $\mathbb{R}^N$, each with non-vanishing derivative. □

We see that the endpoints of each subinterval are sent to $\infty := \{\pm\infty\}$, which is the natural identification for the endpoints of the sub-intervals, as well as the topological compactification of $\mathbb{R}^N$.

Proposition 5. *Characteristic functions on the sets represented by the sequence of vectors* $\vec{v}, \vec{v}^{\otimes^2}, \vec{v}^{\otimes^3}, \ldots,$ *when suitably normalized, are* $L^2([0,1],\lambda)$ *functions, and converge to a limit in* $L^2([0,1],\lambda)$*, which we define as*

$$\lim_{k\to\infty} \vec{v}^{\otimes^k} =: \nu$$

.

Proof. Let $m = \sum_{i=1}^N v_i$ be the number of components of $\vec{v}$ that are different than zero. (This foreshadows our application to self-similar measures). We determine $\|\vec{v}\| = (\int_{[0,1]} (\sum_{i=1}^N \frac{a_i}{N} \chi_{[\frac{i-1}{N}, \frac{i}{N}]}(x))^2 \, dx)^{\frac{1}{2}} = \frac{m}{N}$, and

$\|\vec{v}^{\otimes^k}\| = (\int_{[0,1]}(\sum_{i=1}^{N^k}\frac{(\vec{v}^{\otimes^k})_i}{N^k}\chi_{[\frac{i-1}{N^k},\frac{i}{N^k}]}(x))^2\,dx)^{\frac{1}{2}} = \frac{m^k}{N^k}$. Thus,

$$\lim_{k\to\infty}\|v^{\otimes^k}\| = \lim_{k\to\infty}((\sum_{i=1}^{N^k}\frac{m^k}{N^{2k}})^2)^{\frac{1}{2}} = \begin{cases} 1 \text{ if } m = N \\ 0 \text{ if } m < N \end{cases} .$$

.

□

Remark 9. *The above result shows that representation vectors converge to the equivalence class in $L^2([0,1],\lambda)$ corresponding to $\chi_{[0,1]}(x)$, for $\vec{v}$ consisting entirely of 1's, and to the equivalence class of 0, for $\vec{v}$ with at least one zero.*

Remark 10. *$\vec{v}$ with components in $\mathbb{F}_2$ is in the Hilbert space $H = \mathbb{R}^N$, with the normalized inner product and induced norm: $(\vec{w},\vec{v}) = \sum_{i=1}^{N}\frac{v_i w_i}{N^2}$. The Kronecker powers of $\vec{v}$ are elements of a Fock space [Reed-Si] $\mathcal{H} = \oplus_{n=0}^{\infty}H^{\otimes^n}$. We have $\|\lim_{n\to\infty}\otimes^n\vec{w}\| = \Pi_{i=1}^{\infty}(\vec{w},\vec{w})^{\frac{1}{2}}$.*

Definition 21. *We define the intrinsic Fock space of $\lim_{n\to\infty}\vec{v}^{\otimes^n}$ to be the Fock space $\mathcal{H}_i = \oplus_{n=0}^{\infty}H_i^{\otimes^n}$, where H_i is the Hilbert subspace of $\mathbb{R}$ consisting of the real dimensions in which $\vec{v} = (v_1,...,v_N)$ has $v_i = 1$.*

Thus $\vec{v}$ has norm one on its intrinsic Fock space.

8 Generalizations and Applications of Tensor Representations

In the following we generalize the types of fractals to be represented from the equicontractive fractals on $\mathbb{R}$ to fractals with differing contraction ratios and corresponding lengths. Then we seek to utilise the representation to approximate self-similar measures on the line. Higher dimensional generalizations are also easily within reach. Dual spaces of the representation vectors will provide geometrical information and lead to a physical interpretation of these fractals and their representations.

8.1 Application to Box-Counting

In the equicontractive case, we obtain a simple formula for the number $N_F(N^k)$ of boxes of inverse scale N^k required to cover the fractal $F = S(F)$. (Here let the dimension of the representation vector $\vec{v}$ be N, and let $r_0 = \frac{1}{N}$ be the preferred scale, also let $m = \sum_{i=1}^{N} v_i$ be the number of non-zero components of $\vec{v}$).

Proposition 6. *Box-Counting Formula, equicontractive case*

$$N_F(N^k) = \sum_{i=1}^{N^k} (\vec{v}^{\otimes^k})_i = m^k.$$

Proof. Since $\vec{v}^{\otimes^k}$ represents $S^k(I)$, and each component of $\vec{v}^{\otimes^k}$ represents a sub-interval of length N^{-k}, the number of 1's in $\vec{v}^{\otimes^k}$ gives the number of "boxes" of size N^k needed to cover $S^k(I)$, and hence F. Since the other entries are just 0, the number of ones is $\sum_{i=1}^{N^k}(\vec{v}^{\otimes^k})_i$.

Claim:

$$\sum_{i=1}^{N^k}(\vec{v}^{\otimes^k})_i = m^k.$$

Proof. (Proof of Claim) Let $\sum_{i=1}^{M}\vec{v}_i = m$, the number of non-zero components of $\vec{v}$. Then suppose that $\sum_{i=1}^{N^{k-1}}(\vec{v}^{\otimes^k})_i = m^{k-1}$. Applying the definition of the Kronecker product, $\sum_{i=1}^{N^K}\vec{v}\otimes\vec{v}^{\otimes^{k-1}} = \sum_{i=1}^{N} v_i \sum_{i=1}^{N^{k-1}} v_i^{\otimes^k} = mm^{k-1}$ for a total of $m(m^{k-1}) = m^k$ boxes. □

We see then that away from integer powers of r, we have

$$N_F(r) = m^{max\{\lceil \log_N r \rceil, 0\}}.$$

For our equicontractive representations we consider an intrinsic scaling in which the discontinuities of $N(r)$ occur only at inverse scales m^k. That is to say that the counting function is constant at scales in between those represented by the iterated tensor powers.

8.2 $\partial_{box}N_F(r)$, Similarity and Box-Counting Dimensions

For an (equicontractive) IFS represented by a tensor $\vec{v}$ with $dim(\vec{v}) = N$, and $\sum_{i=1}^{N} v_i = m$, we can easily determine the similarity dimension: $d_s = \frac{\log m}{\log N}$. Then by Moran's Theorem [Hut], we have that $\frac{\log m}{\log N} = d_H = d_{box}$.

With the intrinsic geometry of the IFS fractal (meaning that we consider only boxes within the [pre-]fractal image at the scale of interest), we can compute the box counting derivative as well: we take $N_F(r^k) = \sum_{i=1}^{N^k} (\vec{v}^{\otimes^k})_i = m^k$, to see that

$$N_F(r) = m^{\sum_{i=0}^{\max(0,\log_N r)} \theta(i)},$$

$$= m^{\lceil \log_N r \rceil_0}$$

where $\theta(x)$ denotes the Heaviside step function, and where we define $\lceil r \rceil_0 := max(0, \lceil r \rceil)$, and $\lceil r \rceil$ is the least integer greater than or equal to r. Then

$$\partial_{Box} N_{\mathfrak{F}}(r) := r \frac{\frac{dN(r)}{dr}}{N(r)}$$

$$= \frac{r}{m^{\lceil \log_N r \rceil_0}} m^{\lceil \log_N r \rceil_0} \frac{\log m \sum_{n=0}^{\infty} \delta(r - N^n)}{r \log N} = \frac{\log m \sum_{n=0}^{\infty} \delta(r - N^n)}{\log N}.$$

We recover the pointwise "slope of the log-log- plot" by computing

$$\partial_{Box} N_F(r) = \begin{cases} 0 & \text{if } \log_N r \notin \mathbb{Z} \\ \frac{\log m}{\log N} & \text{if } \log_N r \in \mathbb{Z}. \end{cases}$$

We see that $\partial_{Box} N_{\mathfrak{F}}(r) = \partial_{Box} N_{\mathfrak{F}}(Nr)$, and it has multiplicative period of N.

Now observe that, as a hyperfunction, the expression

$$\partial_{Box} N_F(r) = \frac{\log m}{\log N} \sum_{n=0}^{\infty} \delta(r - N^n) = \frac{\log m}{\log N} [\sum_{n=0}^{\infty} \frac{(2\pi i)^{-1}}{r - N^n}],$$

has poles for all k, since $\partial_{Box} N_F(\log r) = \partial_{Box} N_F(\log r + \frac{2\pi i k}{\log N})$, for $k \in \mathbb{Z}$.

We see thus that the oscillatory period of $\partial_{Box} N_F(r)$ is $\frac{2\pi}{\log N}$, in accordance with the ordinary fractal string correspdonding to the deleted lengths in the construction of F.

8.3 Higher Dimensions

If we wish, we can represent certain equicontractive fractals $F = S(F)$ in $\mathbb{R}^2$, with $F \subset I^2$, such that $S(I) = \{\frac{1}{M}x + \frac{(i-1) \mod M}{M}, \frac{M - \lceil \frac{i}{M} \rceil)}{M}\}_{i \in \mathcal{I} \subset \{1,2,\ldots,M^2\}}$.

Here we would label the boxes: $\begin{matrix} 1 & 2 & \ldots & M \\ M+1 & \ldots & \ldots & \vdots \\ \vdots & \ldots & \ldots & \vdots \\ M^2 - M + 1 & \ldots & \ldots & M^2 \end{matrix}$.

If $F = S(F) \subset I^N \subset \mathbb{R}^N$, is an equicontractive fractal with translations a multiple of the contraction factor, we can use a tensor of rank N, with dimension M^N for some M, such that in each of the N directions, the index is counted away from the origin, then $S(I) = \{\frac{1}{M}x + (\frac{(i_1-1) \mod M}{M}, \ldots, \frac{(i_N-1) \mod M}{M})\}_{(i_1,i_2,\ldots,i_M) \in \mathcal{I}^M \subset \{1,2,\ldots,N\}^M}$.

8.4 The non-Equicontractive Case

In the following we will discuss two methods of computing tensor representations for attractors of non-equicontractive IFS. We will use this representation to approximate more general self-similar measures on $[0,1] \subset \mathbb{R}$.

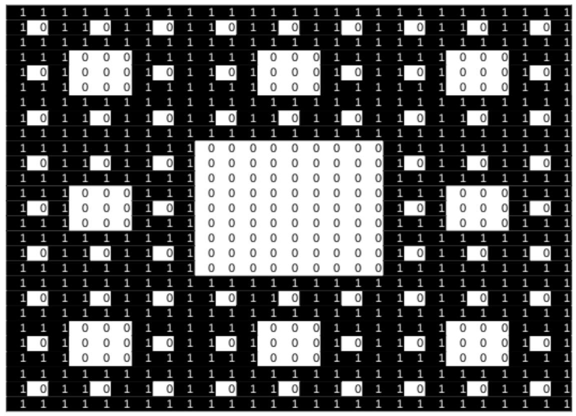

Figure 8: The third tensor power of a matrix generating the Sierpinski Carpet.

8.4.1 Method of Blocks

A first attempt to generalize our construction beyond the equicontractive case might try to assign multiple 1's to a subinterval of I with length a multiple of $\frac{1}{N}$. Quickly we notice we have a problem: consider the IFS

$$S = \{S_1(x), S_2(x)\},$$

with $S_1(x) = \frac{1}{4}x$ and $S_2(x) = \frac{1}{2}x + 1/2$. We might try the vector $\vec{v} = [1, 0, 1, 1]$ to represent $S(I)$, but we know that $S^2(I)$ should have only four connected components, yet $\vec{v}^{\otimes 2} =$

$$[1, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1]$$

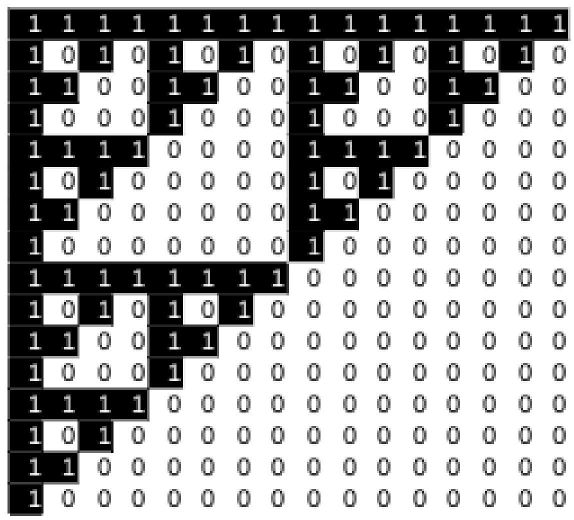

Figure 9: The fourth tensor power of a $2x2$ matrix generating a right-angled Sierpinski triangle. Matrix courtesy of Leo Vu, UCR '14.

would have 5 connected components under identification of the endpoints of each subinterval represented by the $(\vec{v}^{\otimes^2})_i$.

For such rationally dependent IFS, we could correct this problem by use of a reversal of the order of the tensor product for the indices with repeated ones. For example:

$$[1,0,1,1] \otimes_\sigma [1,0,1,1] = [1,0,1,1,0,0,0,0,1,1,0,0,1,1,1,1].$$

Here the operation $\otimes_\sigma$, refers to the permutation of the indices:

For n-factors of 1

$[...,1,1,1,...(n - \text{total ones})...,1,...] \otimes_\sigma \vec{v} = [...,\vec{v} \otimes (1,1,1,...,1),...]$.

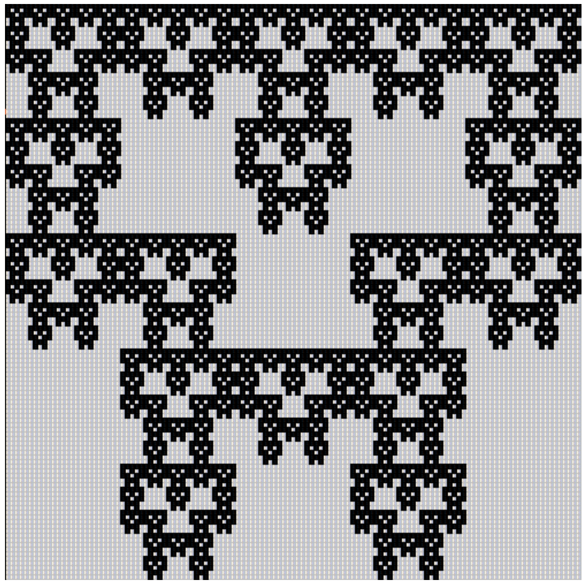

Figure 10: The third level iteration of a fractal dubbed ‘the skull of doom’ by contributors Nicholas and Joshua Quinn.

When working with this representation, repeated ones within the same block can be distinguished from other ones by roman numerals. The the permuted product becomes repetition by the number of I’s. Thus, our example becomes $(1, 0, \mathrm{II}) \otimes_\sigma (1, 0, \mathrm{II}) = (1, 0, \mathrm{II}, 0, 0, 0, 0, \mathrm{II}, 0, 0, \mathrm{IIII})$. Although the I’s are components of $\vec{v}^{\otimes_\sigma^k}$, they are not separated by commas, to indicate their connection at a given stage of the construction.

Since this technique restricts us to rational contraction ratios, and shifts dependent on their common denominator, we employ a more powerful technique, still using the tensor product formalism.

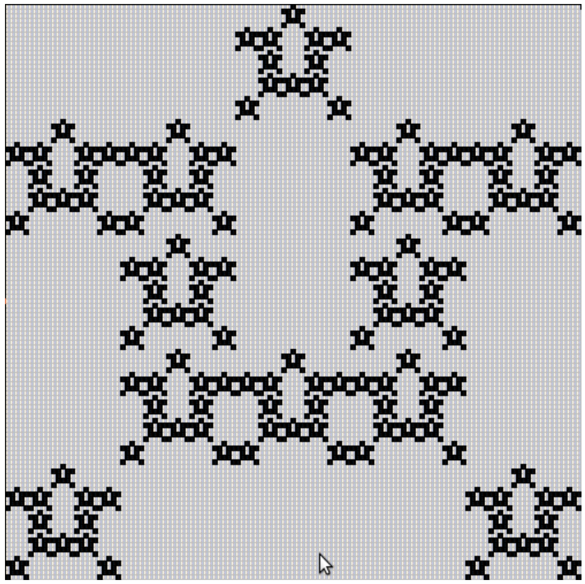

Figure 11: A star-shaped fractal.

8.4.2 Augmented Tensors

Here, we use a pair of vectors for the representation tensor, one vector containing the topological information (with connected components at a given stage of construction represented by ones [in the case of no adjacent ones]) and another containing the geometric information (lengths). We write $\vec{v} = [\frac{\vec{T}}{\vec{\ell}}]$ to denote the augmentation of the tensors (row vectors) $\vec{\ell}$, corresponding to the sequence of (included *and* excluded) lengths, and $\vec{T}$ with 1's representing the topologically connected components of $S(I)$ and the deleted segments as 0's. The corresponding modified tensor product denoted $\otimes_{\frac{\vec{T}}{\vec{\ell}}}$, is just the pairwise tensor product of the adjoined pairs of vectors, that is, if the representation tensors $\vec{v} = [\frac{\vec{v}_\ell}{\vec{v}_T}]$ and $\vec{w} = [\frac{\vec{w}_\ell}{\vec{w}_T}]$, then $\vec{v} \otimes_{\frac{\vec{T}}{\vec{\ell}}} \vec{w} = [\frac{\vec{v}_\ell \otimes \vec{w}_\ell}{\vec{v}_T \otimes \vec{w}_T}]$.

This modification is reminiscent of the "schemes" of Khinchin's *Information Theory*, and if somewhat ad-hoc, it will provide the power to allow us to represent arbitrary self-similar fractals on the line, and to approximate measures on them.

Definition 22. *(Representation vector, general case:) A representation vector* $\vec{v} = [\frac{\vec{T}}{\vec{\ell}}]$ *over* $\mathbb{R}$ *represents a set* $S(I)$ *with* $I = [0,1]$*, and with*

$$[\frac{T_i}{\ell_i}] \approx \begin{cases} \ell_i \lambda^{-1}(1) + \sum_{j=0}^{i-1} \ell_i \ \textit{if}\ T_i \neq 0 \\ \\ \ell_i \lambda^{-1}(1)^* \ \textit{if}\ T_i = 0 \end{cases} \qquad ,$$

where by λ *we mean Lebesgue measure restricted to the unit interval* $I = [0,1]$*, and by* $\lambda^{-1}(1)$*, we identify all sets under almost everywhere equivalence, so that* $\lambda^{-1}(1)$ *refers unambiguously to [0,1] (at least as an equivalence class). By* $\lambda^{-1}(1)^*$*, we refer to the deleted intervals as elements of a class of intervals, for treatment of the deleted intervals as an ordinary fractal string (see [LapVanF]). Optionally, the positional information of the deleted lengths can be stored, in which case we would have* $[\frac{T_i}{\ell_i}] = \ell_i \lambda^{-1}(1)^* + \sum_{j=0}^{i-1} \ell_i$ *if* $T_i = 0$*. As another alternative, we could have* $[\frac{T_i}{\ell_i}] = \lambda^{-1}(0)^* = \emptyset$ *if* $T_i = 0$*, (as an equivalence class) if we truly wish to delete these intervals.*

Theorem 15. *Augmented Tensor Representation Theorem: For an IFS* $S = \{\ell_i x + \sum_{j=0}^{i-1} \ell_i\}_{i \in \mathcal{I} \subset \{1,2,\ldots,N\}}$*, (letting* $\ell_0 = 0$*) with* $m = \sum_{i=1}^{N} T_i$ *represented by an augmented tensor* $\vec{v} = [\frac{\vec{T}}{\vec{\ell}}]$*, with* $T_i \in \{0,1\}$ *and* $0 < \ell_i < 1$ *for all i, and* $\sum_{i=1}^{N} \ell_i = 1$*, the* m^{th} *iterate of* $I = [0,1]$ *under S,* $S^m(I)$*, is represented* *as* $S^m(I) \cong \otimes^m_{\frac{\vec{T}}{\vec{\ell}}}\vec{v}$*, where we take the tensor product as represented by the augmented Kronecker product of the vectors* $\vec{T}$*, and* $\vec{\ell}$ *as row (contravariant) vectors, which we denote* $\otimes^k_{\frac{\vec{T}}{\vec{\ell}}}\vec{v} = [\frac{\vec{T}^{\otimes^k}}{\vec{\ell}^{\otimes^k}}]$*.*

Proof. $\vec{T}$ represents an equicontractive fractal, with the same number of connected components at each stage of iteration of the fractal F with $S = S(F)$. So we see that each iteration of $\frac{T_i}{\ell_i}$ has the correct connectedness. Since $\vec{\ell}$ scales the $S^k(I)$, if $\vec{v}^{\otimes^{k-1}} \approx S^{k-1}(I)$, then $S^k(I) = \cup_{i\in\mathcal{I}}\ell_i S^{k-1}(I) + \sum_{j=0}^{i-1}\ell_i$, which is represented by $\vec{v}\otimes\vec{v}^{\otimes^{k-1}}$.

□

Remark 11. *If we wish to determine the sequence of mappings of* I *that is represented by each term* $(\ell^{\otimes^k})_i$*, we notice that there are* N^k *such terms, representing the images of each of the contractions* S_i*, here considering those subintervals on which* $T_i = 0$ *to have contraction mappings fitting* I *into the appropriate deleted length. Then we see that the first* N *of these subintervals result from the last composition being with* S_1*, and the next* N *with* S_2*, and so forth. We recover the formula that* $(\ell^{\otimes^k})_i = \rho(S_{i_1\ldots i_k}(I))$*, where the* $(i_j)_{j=1}^k$ *can be gotten by the formula:*

$$i_j = \lfloor (i-1)N^{j-k} \rfloor \pmod{N} + 1,$$

in other words, the index of the j^{th} *function in the sequence of mappings in* $S^k(I)$ *represented by* $(\ell^{\otimes^k})_i$ *is given by* 1 *more than the* j^{th} *digit of the base* N *expansion of* $i-1$*.*

Since we have constructed our fractals on a grid of lengths of the images of I under the mappings and the gaps, we observe that our construction satisfies the Open Set Condition [Hut], allowing us to use simple counting techniques to approximate measures.

We may wish to show that we can represent any augmented tensor by the method of blocks. We would show that any rational sequence of lengths has a common denominator and could be represented by blocks of the appropriate lengths, and any non-rational augmented tensor could be approximated to arbitrary precision by a sequence of such blocks.Now we make explicit the representation of composition by multiplication:

Proposition 7. *Composition Law: For Families of contractions, V and W, represented by $\vec{v}$ and $\vec{w}$, respectively, their composition $V \circ W$ is represented by $\vec{v} \otimes \vec{w}$.*

Proof. Sketch: The action of W on I is $W(I) = \cup_{i=1}^{N} W_i(I) = \cup_{i=1}^{N} w_i x + \omega_i$, which is represented by $\vec{w}$, let $\vec{w}$ have sequence of lengths $\vec{\ell}_w$, and let V, acting on I by $V(I) = \cup_{i=1}^{M} v_i x + \nu_i$ have sequence of lengths $\vec{\ell}_v$. The composition $V \circ W = \cup_{i=1}^{M} \cup_{j=1}^{N} v_i(w_i x + \omega_i) + \nu$ has lengths and positions given by $\{\vec{\ell}_v \otimes \vec{\ell}_w)_i\}_{i=1}^{MN}$ and $\sum_i^{MN} (\vec{\ell}_v \otimes \vec{\ell}_w)_i$ respectively. □

8.5 Application to Self-Similar Measures

Recall Definition 3 of a self similar measure μ, invariant under (ρ, S). We represent this measure as follows:

Definition 23. *We define the term-wise product of* $\vec{v} \in \mathbb{R}^N$ *and* $\vec{r} \in \mathbb{R}^N$, *the vector* $\vec{v} * .\vec{r} \in \mathbb{R}^N := (v_1 r_1, ..., v_N r_N)$ *and the (augmented) representation tensor for* $\mu = (\rho, S)\mu$ *as* $\vec{v} = [\frac{\vec{r}}{\vec{\ell}}]$, *where* $\vec{r} := \vec{\rho} * .\vec{T}$.

We may place probability measures on the interval that approximate IFS self-similar measures to arbitrary accuracy by means of this representation.

Theorem 16. ***Approximation of self-similar measures by augmented tensor representations:*** *Let* S *be a family of similarities*

$$S = \{S_i(x) = \ell_i x + \sum_{j=0}^{i-1} \ell_j\}_{i \in \mathcal{I} \subset \{1,2,...,N\}},$$

and let ρ *be a vector in* $[0,1]^N$, *with zeros in all entries excluded from the indices of* S, *with* $\sum_{i=1}^N \rho_i = 1$, *and* $\rho_i \in (0,1)$ *for all* $I \in \mathcal{I}$ *(so that* ρ *is a probability vector in the included indices). Suppose that* $\vec{v} = \frac{\vec{T}}{\vec{\ell}}$ *represents the gaps and contraction ratios of (I under)* S, *and that* $\vec{r} = (\rho_i)_{i=1}^N$ *represents the applied weight. Then* $(\rho, S)\mu$ *is represented by* $\frac{\vec{r}\vec{T}}{\vec{\ell}}$, *with* $\vec{r}\vec{T} = (\rho_i T_i)_{i=1}^N$. *Then, for* $x = \sum_{i=1}^p (\ell^{\otimes^k})_i$,

$$\mu([0,x]) = \sum_{i=0}^p (\vec{r}^{\otimes^k})_i.$$

Proof. Partition $[0,x]$ into subintervals with measure zero overlap as

$$[0,x] = [0, (\vec{\ell}^{\otimes^k})_1] \cup \cdots \cup [\sum_{i=0}^{p_j - 1} (\vec{\ell}^{\otimes^k})_i, \sum_{i=0}^{p_j} (\vec{\ell}^{\otimes^k})_i] \cdots \cup [\sum_{i=0}^{p-1} (\vec{\ell}^{\otimes^k})_i, \sum_{i=0}^{p} (\vec{\ell}^{\otimes^k})_i],$$

so that

$$[0,x] = \cup_{p_j=1}^{p}[\sum_{i=0}^{p_j-1}(\vec{\ell}^{\otimes k})_i, \sum_{i=0}^{p_j}(\vec{\ell}^{\otimes k})_i],$$

so that $\mu([0,x]) = \sum_{p_j=1}^{p}\mu([\sum_{i=0}^{p_j-1}(\vec{\ell}^{\otimes k})_i, \sum_{i=0}^{p_j}(\vec{\ell}^{\otimes k})_i])$. Then,

$$\mu([\sum_{i=0}^{p_j-1}(\vec{\ell}^{\otimes k})_i, \sum_{i=0}^{p_j}(\vec{\ell}^{\otimes k})_i]) = (\rho, S)\mu([\sum_{i=0}^{p_j-1}(\vec{\ell}^{\otimes k})_i, \sum_{i=0}^{p_j}(\vec{\ell}^{\otimes k})_i])$$

$$= (\rho, S)^k\mu([\sum_{i=0}^{p_j-1}(\vec{\ell}^{\otimes k})_i, \sum_{i=0}^{p_j}(\vec{\ell}^{\otimes k})_i])$$

$$= \sum_{i_1,\ldots,i_k=0}^{N}\rho_{i_1}\ldots\rho_{i_k}\mu((S_{i_1,\ldots,i_k})^{-1}([\sum_{i=0}^{p_j-1}(\vec{\ell}^{\otimes k})_i, \sum_{i=0}^{p_j}(\vec{\ell}^{\otimes k})_i])$$

$$= \sum_{i_1,\ldots,i_k=0}^{N}\rho_{i_1}\ldots\rho_{i_k}\mu(\frac{[\sum_{i=0}^{p_j-1}(\vec{\ell}^{\otimes k})_i, \sum_{i=0}^{p_j}(\vec{\ell}^{\otimes k})_i] - \sum_{j=0}^{i_j-1}(\ell^{\otimes k})_j}{(\ell^{\otimes k})_{i_j}})$$

$$= \sum_{i_1,\ldots,i_k=0}^{N}\rho_{i_1}\ldots\rho_{i_k}\begin{cases} 0=\mu(\emptyset) & \text{if } i_j \neq p_j \\ 1=\mu([0,1]) & \text{if } i_j = p_j \end{cases} = \sum_{i_1,\ldots,i_k=0}^{N}(r^{\otimes k})_{p_j}\delta_{i_j,p_j}$$

$$= (r^{\otimes k})_{p_j}$$

Then, summing over the p_j,

$$\mu([0,x]) = \sum_{p_j=1}^{p}\mu([\sum_{i=0}^{p_j-1}(\vec{\ell}^{\otimes k})_i, \sum_{i=0}^{p_j}(\vec{\ell}^{\otimes k})_i]) = \sum_{p_j=1}^{p}(\vec{r}^{\otimes k})_{p_j}.$$

$$= \sum_{i=0}^{p}(\vec{r}^{\otimes k})_i.$$

□

Formula 1. *Linear Interpolation Formula: For* $x \in [0,1]$, *with* $x \neq \sum_{i=0}^{p}(\vec{v}^{\otimes^k})_i$, *for any* k, *we approximate*

$$\mu([0,x]) \approx \sum_{i=0}^{p-1}(r^{\otimes^k})_i + \frac{[x - \sum_{i=0}^{p-1}(\ell^{\otimes^k})_i](r^{\otimes^k})_p}{(\ell^{\otimes^k})_p},$$

where p *is such that* $x \in [\sum_{i=0}^{p-1}(\ell^{\otimes^k})_i, \sum_{i=0}^{p}(\ell^{\otimes^k})_i)$.

Proposition 8. *For two measures* μ *and* μ_k *that agree on the right end-points and deleted intervals of* $[0,1]$ *in the* $k^{th}-$*stage of the construction of* F, $sup_x|g(x) - g_k(x)| \leq sup_{i\in\{1,\ldots,N^k\}}(r^{\otimes^k})_i$, *where* $g(x)$ *and* $g_k(x)$ *are the cumulative distribution functions of* μ *and* μ_k, *respectively.*

Proof. By the theorem on approximation of self-similar measures by augmented tensor representations, if x is in a deleted interval, or on a right endpoint of a subinterval, μ and μ_1 agree, so the difference in their ramp functions will be zero. Suppose $x \in [\sum_{i=0}^{p-1}(\ell^{\otimes^k})_i, \sum_{i=0}^{p}(\ell^{\otimes^k})_i]$, a so-called "island of the k-th level" [Strich3], i.e. a non-deleted interval.

$$|g(x) - g_k(x)| = |\mu([0,x]) - \mu_1([0,x])|$$
$$= |\mu([\sum_{i=0}^{p-1}(\ell^{\otimes^k})_i, x]) - \mu_1([\sum_{i=0}^{p-1}(\ell^{\otimes^k})_i, x])|,$$

for some j, since μ and μ_1 agree on deleted intervals. Since

$$\mu([\sum_{i=0}^{p-1}(\ell^{\otimes^k})_i, \sum_{i=0}^{p}(\ell^{\otimes^k})_i]) = \mu_1([\sum_{i=0}^{p-1}(\ell^{\otimes^k})_i, \sum_{i=0}^{p}(\ell^{\otimes^k})_i]) = (r^{\otimes^k})_p,$$

$|g(x) - g_k(x)| \leq (r^{\otimes^k})_p$. □

Note that in the last line we have used that μ is a non-atomic measure. Indeed, if μ were atomic, let x_0 be the atom of largest measure under μ, then we would have $\mu(\{x_0\}) = \sum_{i=1}^{N} \rho_i \mu(\{S_i^{-1}(x_0)\}) = k > 0$. Then we would have $\mu(\{x_0\}) = \mu(\{S_i^{-1}(x_0)\})$ for all i, then that would imply $N = 2$, and $S_1([0,1]) \cap S_2([0,1]) = \{x_0\}$, their common endpoint, a contradiction for $N \neq 2$. Also, $S_i^{-1}(x_0) = x_0$ implies $S_i(x_0) = x_0$ for all i, but that would contradict that $Lip(S_i) < 1$ is constant.

The previous theorems then establish that we can approximate the measure of the interval $[0, x]$ under any self-similar measure μ defined by constant contraction ratios and constant shifts by collapsing the representation tensor $\vec{r}^{\otimes^k}$, of the weights ρ_i, along the appropriate indices i, to an error within $(\max r_i)^k$. Thus given $\epsilon > 0$, we can pick k such that the error is less than ϵ, therefore our construction is a faithful approximation of μ.

Theorem 17. *For any self-similar measure μ, on $[0,1]$ represented by $\vec{v}$, given any $\epsilon > 0$, we can find a power $k \in \mathbb{N}$ such that $|\mu([0,x]) - \sum_{i=0}^{p}(r^{\otimes^k})_i| < \epsilon$, where $x \in [\sum_{i=0}^{p-1}(\ell^{\otimes^k})_i, \sum_{i=0}^{p}(\ell^{\otimes^k})_i]$.*

Proof. Pick $k > \frac{\log \epsilon}{-\log max_i\{r_i\}_{i \in \mathcal{I}}}$, and use [] Theorem 16, Formula 1 and Proposition 8 above. □

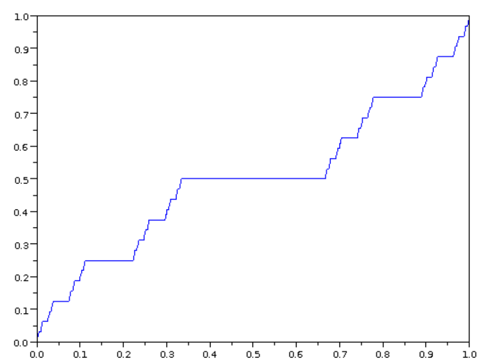

Figure 12: The ramp function of the natural weight measure on the Cantor set.

8.5.1 Examples

Example 9. ***Ordinary Fractal Strings:*** *We may wish to determine the sequence of lengths of deleted intervals in the construction of F, in order to utilize the theory of ordinary fractal strings. We see that the deleted lengths (called "Lakes" by [Strich2]), are represented by zeros in the topological component of the representation tensor. Since multiplying by zero always results in zero, we see that while the intervals (called "islands" in [Strich2]) comprising the k^{th} step in the construction of F are vanishing with increasing k, the deleted lengths gain in numbers until their lengths sum to 1 (exception: see next example). We notice that each zero results in as many new zeros as there are non-zero components of $\vec{T}$ (or $\vec{v}$, for the equicontractive fractals* *represented by the method of blocks). Thus we see that decomposing $\vec{\ell}$ (or just $\vec{v}$) into two vectors of smaller dimension: $\vec{d} = (v_i)_{\vec{T}_i=0}$ for the deleted lengths, and $\vec{F} = (v_i)_{\vec{T}_i \neq 0}$ (with the obvious adjustment in the case of the method of blocks).*

Then the elements in the sequence of lengths corresponding to F represented by $\vec{v}$, are given by the components of the vectors $\{\vec{d} \otimes F^{\otimes^i}\}_{i=0}^{\infty}$, which just states that we multiply each original gap length by each length of an interval in each stage of approximation of F. Observe that this is really just a permutation of the standard product giving the approximations to F. Thus we see the complementarity of the fractal set and the fractal string at work.

Thus we derive an explicit formula in the case of the method of blocks, in the absence of repeated zeros in the original $\vec{v}$. Given that each subinterval represented by a component of $\vec{v}$ has length $\frac{1}{N}$, whether it will be assigned to $\vec{d}$ or to $\vec{F}$, the above set becomes $\cup_{n=0}^{\infty} \frac{1}{N}^{n+1}$, with a multiplicity of $|\vec{d}|(N - |\vec{d}|)^n$, for each length corresponding to n from 0 to ∞.

Example 10. ***Fat Cantor Set**: We represent this important fractal with an augmented vector that depends on the level of the approximation. Recalling that, in its most common guise, at each stage n of construction, we delete 2^{n-1} intervals of length $\frac{1}{2^{2n}}$. Letting $\vec{v} = \frac{\vec{T}}{\vec{\ell}}$, Here $T = [1, 0, 1]$ and $\ell = \ell_n$ now depends on n. Specifically, $\ell_n = (\ell_{n,1}, \ell_{n,2}, \ell_{n,3}) = [\frac{2^n+1}{2^{n+1}+4} \; \frac{1}{2^n+2} \; \frac{2^n+1}{2^{n+1}+4}]$.*

Example 11. ***Fracas**: This is a contractive construction F in which the family of functions and hence its representation tensor changes at each stage of the iteration. We would represent $F = \rho(\otimes_{n=1}^{\infty} \vec{v}_n)$. Here $\vec{\ell}$ and $\vec{T}$ depend* *on n in general. These represent the most general states in the category. There is a dichotomy: fractal or not, for fracas represented by infinite products of tensors. Fracas with the last infinitude of tensors in the product equal to* $[1] = \rho^{-1}(I) = id_{\rho(\mathbb{IFS}),\otimes}$*, are defined as the union of subintervals* $S^k(I) = \rho(\otimes_{n=1}^{k}\vec{v}_n)$ *of I represented by the products of the finitely many tensors different than* $[1]$*, (or just I if* $F = \rho(\otimes_{i=1}^{\infty}[1])$*).*

Example 12. ***Closed forms***: *While no closed form sums are known for general partial multinomial row sums , we can find closed forms for measures of ϵ balls for fractals with natural weights and dependent gaps using change of base formulas.*

Change of Base formulas: *If F is fixed under an equicontrative family with dependent shifts*

$$S = \{S_i(x) = \frac{1}{N}x + \frac{i-1}{N}\}_{i\in\mathcal{I}\subset\{1,2,\ldots,\ldots,N\}}$$

$$= \rho(\vec{v}) : \vec{v} \in \mathbb{F}_2^N \text{ and } \Sigma_{i=0}^{N} v_i = m.$$

i.e. F is a generalized cantor set with one or more gaps of length $\frac{1}{N}$, represented by a vector $\vec{v}$ of length N. Then if $x \in [0,1]$, with base N expansion $x = 0.x_1x_2...x_k = \sum_{i=1}^{k} x_i N^{-i}$*, then we can find* $\mu([0,x]) = \sum_{i=1}^{k} x_i^* m^{-i}$*, where*

$$x_i^* = \begin{cases} 0 & \text{\textit{if there is a } } j < i \text{ \textit{such that} } x_j \text{ \textit{is one less than the index of a gap.}} \\ x_i & \text{\textit{if} } x_i + 1 \leq g_1 \text{ \textit{the index of the first gap, and the above does not hold.}} \\ x_i - s & \text{\textit{if} } x_i + 1 \geq g_s \text{ \textit{the index of the} } s^{th} \text{ \textit{gap, and the first condition does not hold}} \end{cases}.$$

Proof. Sketch: Observe that, for $x = \sum_{i=1}^{k} x_i N^{-i}$ not in a deleted length,

$$S_{x_1+1}(...(S_{x_k+1}(I)...) = \frac{1}{N}(...(\frac{1}{N}(\frac{1}{N}[0,1] + \frac{x_k}{N}) + \frac{x_{k-1}}{N})...) + \frac{x_1}{N}$$

$$= [\sum_{i=1}^{k} x_i N^{-i}, \sum_{i=1}^{k} x_i N^{-i} + \frac{1}{N^k}],$$

which has weight $(\frac{1}{m})^k$. If x is in a deleted length, then there is an $i_0 \in \{1, ..., k\}$ such that $x_{i_0} + 1 = j \in \mathcal{I}^c$, that is the first such N-ary digit of x, then all $x \in [0, \sum_{i=1}^{i_0} x_i]$ are not in a deleted length, so that $\mu[0, \sum_{i=1}^{i_0} x_i] = \sum_{i=1}^{i_0} x_i m^{-i}$. The further digits of x will make no further contribution, so that case 1 holds. If no base-N digit of x exceeds (one less than) any index of any gap then let $i_0 = k$ in the above and the second case holds. If x is not in a deleted interval, and a base-N digit x_i exceeds one less than the index of s gaps, then $\frac{s}{m^i}$ will be subtracted from $\mu([0, x])$, for the deleted intervals, so that the third case holds.

□

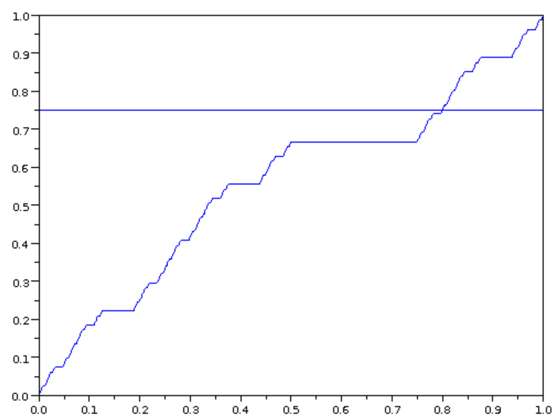

Figure 13: The ramp function of a single-gap equicontractive fractal. One can compute the density function $\mu([0,x])$ using a change of base formula.

9 Appendices

9.1 Work in progress

We have used the row tensor representation in an effort to capture the covariance of scale associated with each fractal's construction. We seek additional methods from geometry to assist in the computation of scale derivatives and counting functions. To every finite-dimensional row vector $\vec{v}$, we can assign a dual column vector $\vec{v}^*$.

For representation vectors we compute the outer product $A = \vec{v} \otimes \vec{v} = \vec{v}^* \vec{v}$. Deriving with respect to the representation at each stage, tells us that the scale derivative is the outer product.

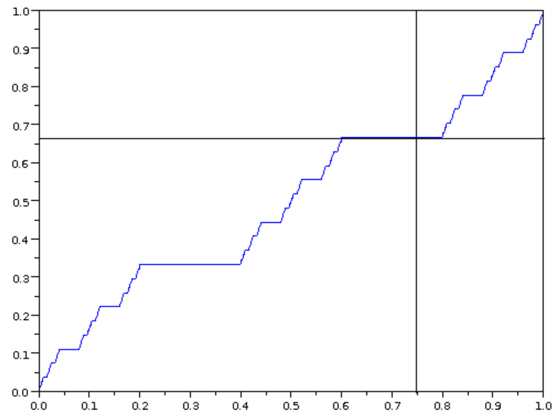

Figure 14: The ramp function of an equicontractive fractal, whose density function can be computed using a change of base formula

9.1.1 Duality of Representation Vectors

A representation vector $\rho(S) = \vec{v} = [v_1, v_2, ..., v_N]$ is defined to be the contravariant vector $[v_1 dx_1, v_2 dx_2, ..., v_N dx_N]$, a linear functional for the covariant vector

$$\vec{v}^* = \begin{bmatrix} v_1 \frac{\partial}{\partial x_1} \\ \vdots \\ v_N \frac{\partial}{\partial x_N} \end{bmatrix}.$$

9.1.2 Scale *Co*variance

The dual vector $\vec{v}^*$ gives the space of flows in the scale space of the self-similar system, and we solve the system of DE's

$$v_i \frac{\partial x^i}{\partial t} = x^i$$

to obtain "the self similar fixed flow" of the self-similar system, which exists by duality to the fixed measure under application of the measure represented by $\vec{v}$. This action is the infinitesimal generator for the evolution semigroup of the quantum graph induced by the IFS S.

Solving the above, with $t \in \mathbb{R}^m$,

$$x_i = e^{v_i^{-1} t}$$

so that $x_i(t) = x_i(t+2\pi i v_i)$, so that we can think of each branch of the fractal evolution as having a period relating to the relative size of each branch [see the next chapter for non-uniform contraction ratios]. As the approximating tensor approaches the limit tensor, the contravariant representation tensor of the underlying representation represents an uncountable set of points, then by duality the covariant flow represents an uncountable continuum of loops. Identifying this group of flows with the homology group of this Cantor-type dust on $[0, 1]$, an uncountable free product of factors of $\mathbb{Z}$, we conjecture that the covariant representation tensor is a functor from the category of IFS fractals (images of compact intervals under IFS,), to the homology groups of associated pointed topological spaces (wedges $\Lambda_{i=1}^{N}(S^1)$), requiring very little effort to be taken to the category of free abelian groups.

Conjecture 2. *"Fixed Flow" Group structure: We can obtain the fundamental group of the prefractal $S^n(I)$, under the identification of the endpoints of the intervals, with imaginary time $t \in i\mathbb{R}^m$ as the space of complete revolutions in the solution space to the "fixed flow equation". We recognize the n-fold wedge of circles given in Scale Covariance above.*

We conjecture that this group provides us with the scale space evolution of F. Ignoring differing scaling ratios, the resulting Cayley graphs appear isomorphic to the covering spaces of free groups, and since the free group on two generators has free groups of any number of generators as subgroups, we see that the space $\mathfrak{C}$, the cantor space, has the scale evolutions of all other IFS within it's scale evolution. This is just the approximation by binary streams of arbitrary data. This result should be an algebraic corollary to the result that all non-empty, perfect, compact, metrizable, totally disconnected spaces are homeomorphic to the Cantor set.

Theorem 18. *Periodic sequences of contractive families: A periodic sequence $\{\mathcal{S}_i\}_{i=1}^{P}$, with each $\mathcal{S}_i$ being a contractive family $\{\mathcal{S}_{i_{j_i}}\}_{i=1}^{P}$, with $j_i \in \{1, 2, ..., N_i\}$ of mappings is itself a finite family of contraction mappings, and hence posesses an invariant set F. F is represented by $\lim_{n\to\infty}(\otimes_{i=1}^{P}\vec{v}_i)^{\otimes^n}$*

Proof. Apply the representations $S_i \approx \vec{v}_i$, to get $S_{i_1} \circ S_{i_2} \circ ... \circ S_{i_M} = \vec{v}_{i_m} \otimes$

$... \otimes \vec{v}_{i_2} \otimes \vec{v}_{i_1}$, as a new vector $\vec{v}$. □

9.1.3 Toward Applications to Physics

It is well known that coupled states in classical and quantum mechanics can be represented as Kronecker products of the state vectors.

By the norm calculations on the normalized representation vectors, we see that our fractals and their prefractal images are located on the unit ball of a particular vector subspace of the Fock space that contains the infinite tensor product representation. This is precisely the subspace of all powers of tensor products of the subspace of $\mathbb{R}^N$ containing the representation vector $\vec{v}$, on which the entries of $\vec{v}$ are non-zero.

Conjecture 3. *Special States of Free Quantum Fields are Fractals: Observe that many particle interactions in quantum mechanics are given by n-fold tensor products of state spaces and their vectors. States represented by tensor powers of IFS representation vectors $\vec{v}$describe free fields of many particles in scale invariant states.*

Proposed applications then would be for macroscopic observations of quantum phenomena of free fields of uniform states due to the scale-invariance, and for coherent states represented by powers of simple tensors. Within the confines of current physics research, following [Rodr-Mig-Berg-Lew-Sier], we can hope to quantify the exact nature of the self-similarity of the quantum observables in their work, estimate dimensions, and perhaps, predict new phenomena due to scaling corrections in the appropriate box-counting derivatives.

Conjecture 4. *Continuous Iteration Counts for IFS:*

Recall that the coefficients of $v_1^{k-r_1} v_2^{r_1-r_2} \dots v_n^{r_{n-1}}$ *in*

$$\sum_{i=1}^{N^k} (\vec{v}^{\otimes^k})_i = (v_1 + ... + v_N)^k$$

are those of the multinomial coefficients for $(v_1 + ... + v_n)^k$*, Then a Newton formula can sum over indices for real-valued iteration counts* k *using a generalized multinomial formula:*

$$(v_1 + ... + v_n)^k = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{r_1} \cdots \sum_{r_1=1}^{r_{n-2}} \binom{k}{r_1} \binom{r_1}{r_2} \dots \binom{r_{n-2}}{r_{n-1}} v_1^{k-r_1} v_2^{r_1-r_2} \dots v_n^{r^{n-1}}.$$

Such a formula would allow us to integrate [see section on integration with respect to self-similar measures] over regions closed under permutations of indices of representation of intervals.

9.2 Operator Form of the Representation Tensor

We have the dual pairing $vv^T = \|v\|^2$, of the representation vector v, giving its euclidean norm squared, but we also have the outer product $A = v^T v$, a matrix form of the representation. In physics, I propose that this would be analogous to the operator corresponding to the observable of a particle in a state composed of the superpositions of states, i.e. for $v = [1, 0, 1]$, a particles is in a superposition of two states. Notice that, A is real and symmetric, but singular, in fact rank 1. The matrix A is an element of a group that gives the continuous functions that transform representation tensors of fractals with similar topologies and geometries into one another. So, for our example v,

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \in G = \left\{ \begin{bmatrix} a & 0 & a \\ 0 & 0 & 0 \\ a & 0 & a \end{bmatrix}, \cdot \right\},$$

with identity $\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$.

In the present example, for $A \in G$, we calculate $A^{-1} = \begin{bmatrix} \frac{1}{4a} & 0 & \frac{1}{4a} \\ 0 & 0 & 0 \\ \frac{1}{4a} & 0 & \frac{1}{4a} \end{bmatrix}$.

More generally, $A = \vec{v}^T \vec{v}$ generates a group G under matrix multiplication This group has identity $\frac{1}{\sum_{i=1}^{N} v_i} A$, and a given matrix $C = cA \in G$ has inverse $\frac{1}{c(\sum_{i=1}^{N} v_i)^2} A$. In our formalism, concerned with (the normalization of) $\vec{v}$, as representing a probability vector, this is not a meaningful result, but in [Strich3], self-similar measures are allowed to be weighted by positive component vectors. In this setting, such a group can effect coordinate changes between scalar multiples of probability vectors.

The operator A is not unitary, but it preserves the vector $\vec{v}^T$ as an eigenvector with eigenvalue $\|\vec{v}\|^2$, since

$$\|A\vec{v}\| = \|(\vec{v}^T\vec{v})\vec{v}^T\| = \|\vec{v}^T(\vec{v}\vec{v}^T)\| = \|\vec{v}\|^2\|\vec{v}^T\|.$$

The operator has 0 as an eigenvalue with multiplicity $N-1$, these eigenvectors consist of the vectors with non-zero values only in entries corresponding to the rows of zeros in A, and other vectors sent to 0 by A, vectors $(a_i v_i)_{i=1}^N$, with $\sum_{i:v_i\neq 0} a_i = 0$.

For an IFS fractal with representation vector in $\mathbb{F}_2^N$, (which we can use with the $\otimes_\sigma$ technique for any IFS fractal with rational contraction ratios), we observe that we can recover our counting function from the operator form $A = \vec{v}^T\vec{v}$.

Proposition 9. *Let $A = \vec{v}^T\vec{v}$, for a representation vector $\vec{v} \in \mathbb{F}_2^N$. For $n \in \mathbb{Z}$, at the scale N^{-n}, we can find the box-counting function for the attractor F of the IFS represented by $\vec{v}$ by the following means:*

1. $N_F(N^{-n}) = (\sum_{i=1}^N v_i)^n$.

2. $N_F(N^{-n}) = \lambda^n$, *where* $A\vec{v}^T = \lambda\vec{v}^T$.

3. $N_F(N^{-n}) = (tr(A))^{-n}$.

4. $N_F(N^{-n}) = m$, *where* $A^n\vec{v}^T = m\vec{v}^T$.

5. $N_F(N^{-n}) = (U^{-1}A(U^T)^{-1})_{NN}^n$, *the lower right hand entry of the diagonalization of A, where $U = [\vec{v}_{0,1}|...|\vec{v}_{0,N-1}|\frac{\vec{v}}{\|\vec{v}\|}]$, where the $\vec{v}_{0,1}$ are the*

normalized eigenvectors of A with eigenvalue 0.

Proof. *1.* was proved when the representation was first introduced. *2.* holds since $A\vec{v}^T = \vec{v}^T\vec{v}\vec{v}^T = \|\vec{v}\|^2\vec{v}^T = (\sum_{i=1}^N v_i^2)\vec{v}^T = (\sum_{i=1}^N v_i)\vec{v}^T$, since $v_i \in \{0,1\}$, then *1.* implies *2.* *3.* follows from *1.* as well since $tr(A) = \sum_{i=1}^N (\vec{v}^T\vec{v})_{ii} = (\sum_{i=1}^N v_i^2) = \sum_{i=1}^N v_i$. In *4.* we have $m = (\sum_{i=1}^N v_i)^n$, since $A^n\vec{v}^T = (\vec{v}^T\vec{v})^n\vec{v}^T = (\sum_{i=1}^N v_i)^{n-1}A\vec{v}^T = (\sum_{i=1}^N v_i)^n$. For *5.*, we know that $Rank(A) = 1$, and that A is real and symmetric. so that A has N independent eigenvectors. Since $\vec{v}^T$ is an eigenvector of A with eigenvalue $N_F(\frac{1}{N})$,, the other eigenvectors must have eigenvalue 0. Normalizing these vectors and placing them as columns in a matrix U, in ascending order of their corresponding eigenvalue, diagonalizes the matrix A^n with the eigenvalue $\sum_{i=1}^N v_i$, in the bottom corner element, since $(U^{-1}A(U^T)^{-1})^n = U^{-1}A^n(U^T)^{-1}$.

□

The free (fundamental, i.e. homotopy) group structure of the fixed flow in the dual geometry of $\vec{v}$ has the homotopy groups of the fixed flows of all the tensor powers of $\vec{v}$ as subgroups [Munk]. Thus, the first power of the operator form contains all the algebraic information of the geometry on F. For this reason, we might try to also consider the matrix A to be a metric tensor on $S(I)$, or even on F. The inner product $\vec{v}\cdot\vec{w}$ should then relate to the Hausdorff distance between the prefractals represented by $\vec{v}$, and $\vec{w}$. Then in the Maxwell equation formulation of gravity, we can use an augmented tensor $\vec{v} = \frac{\vec{T}}{\vec{\ell}}$, with $\sum_{i=1}^N \ell_i = 1$ and $T_i \in \mathbb{R}$, with at least one i such that

T_i=0, to obtain analogs of the fields B_G and E_G. The contraction of the images of I under $\vec{T}$ is analogous to the divergence of the E field, while the vector $\vec{\ell}$ preserves the space I, analogous to the divergencelessness of the B field.

Then since we are constructing a measure on a space of zero Lebesgue measure, we see the kinship of this construction to that of John Wheeler's concept of "mass without mass", or "charge without charge". This is reasonable since a mathematical fractal involves structure at all scales, even below the Planck scale and corresponding Planck energy, thus the field described by the energy in the coherent states depicted by the infinite tensor product representing a fractal, would describe a geon field.

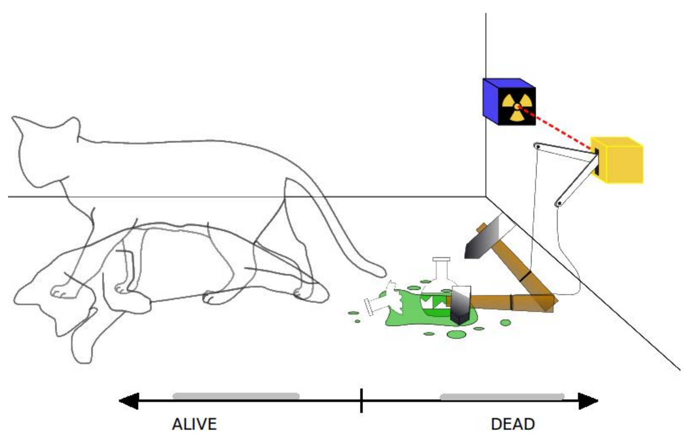

Figure 15: A scenario in which the first stage approximation to a Cantor set represents the disjoint outcomes of life or death for Schrodinger's cat. Then the Cantor set will represent a continuum of such cats.

9.3 Appendix: The fixed points of the Fourier Transform-Fractal Functions and Scale Evolutions

We define the Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ to be the function

$$\mathfrak{F}(f(x))(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x}\, dx.$$

Since $|\int_{\mathbb{R}^n} f(x)e^{-2\pi\xi\cdot x}\, dx| \leq \int_{\mathbb{R}^n} |f(x)e^{-2\pi\xi\cdot x}\, |dx = \int_{\mathbb{R}^n} |f(x)| = \|f\|_{L^1(\mathbb{R}^n)}$, $\|\hat{f}(\xi)\|_\infty \leq \|f(x)\|_{L^1(\mathbb{R}^n)}$. By the continuity of the Fourier transform, then we have that $\mathfrak{F} : L^1(\mathbb{R}^n) \to BC(\mathbb{R}^n)$. We refer the reader to [Fol] for the many useful and well known properties of the Fourier transform, but mention in brief the most useful of those properties.

The Fourier transform takes the input function from the underlying space we call the time domain to its dual space, the frequency domain. Similar dualities can be used between configuration and momentum space in quantum mechanics, and others. In the transformed space, differential equations (for functions with L^1 derivatives) are transformed into algebraic equations, which are then solved and Fourier inverted to find the solutions to the original differential equation.

Recalling the theory of Pontryagin duality of compact Abelian groups, if $f(x)$ is periodic in $\mathbb{R}^n$ we regard $f(x)$ as a function on $\mathbb{T}^n$, the torus of n dimensions. Since $\mathbb{T}^n$ is compact for all n, $\hat{\mathbb{T}^n} = \mathbb{Z}$ is discrete. The proper translation invariant measure on $\mathbb{Z}$ is the counting measure, hence for such functions the Fourier transform becomes a series over $\mathbb{Z}$, and we define the

Fourier Series of $f(x) \in L^2(\mathbb{R}^n)$ as $\sum_{-\infty}^{\infty}(\int_{\mathbb{T}^n} f(x)e^{-2\pi in\cdot x}\,dx)\,e^{2\pi in\cdot x}$.

The famous theorem of Plancherel shows that $\mathfrak{F}$ extends to a unitary isomorphism on L^2 uniquely. Then since the space $\mathfrak{S}$ of functions that are rapidly decreasing at infinity are dense in L^p, we can define the Fourier transform on the dual space $\mathfrak{S}'$ of continuous linear functionals on $\mathfrak{S}$, the space of distributions that increase slowly at infinity. For such objects we define their Fourier transform in terms of the test functions in $\phi \in \mathfrak{S}$. Let $T \in \mathfrak{S}'$, then $(\hat{T}, \phi) = (T, \hat{\phi})$.

Also we recall that the Fourier transforms of measures μ on measure spaces X are defined as $\hat{\mu}(\xi) = \int_X e^{-2\pi i\xi\cdot x}\,d\mu(x)$.

In our efforts of understanding self-similarity and its implications, we see that the Fourier transform is tightly interwoven into the theory of fractal geometry. While fractals are fixed points of contractive similarities, we find that some fractals, or their scale evolutions are themselves fixed under Fourier transformation.

9.3.1 Familiar Fixed Points of $\mathfrak{F}$

We recall that for a Gaussian, $f(x) = e^{-\pi a|x|^2}$, the Fourier Transform is $\hat{f}(\xi) = a^{-n/2}e^{-\pi|\xi|^2/a}$, so that $\widehat{e^{-\pi|x|^2}} = e^{-\pi|\xi|^2}$. This fixed point is key in the theory of Fourier transforms, since this fact is used in the proof of the Fourier Inversion Theorem.

A famous periodic fixed point of the Fourier Transform is the "Shah function" $\Delta(x) = \sum_{n=-\infty}^{\infty}\delta(x-n)$, which can be defined as periodic on $\mathbb{T}$. This property gives the Fourier transform useful sampling properties. Computing its Fourier series, we obtain $\sum_{-\infty}^{\infty}(\int_{-1}^{1}\delta(x)e^{-2\pi inx}\,dx)\,e^{2\pi inx} = \sum_{-\infty}^{\infty} e^{2\pi inx} = \delta(x)$, on $\mathbb{T}$, hence $\hat{\Delta}(\xi) = \Delta(\xi)$. We have seen that $\partial_{Box}N_F(r) = d_s\sum_{n=0}^{\infty}\delta(r-n\log N)$, is a "half-Shah" function and we will use the fact that the shah function is fixed under $\mathfrak{F}$, to compute the Fourier transform of the scale evolution $\partial_{Box}N_F(r)$.

9.3.2 A distributional fixed point of the Fourier Transform

The Dirac-δ function has the Fourier transform 1, and its derivatives have multiples of powers of ξ. Thus, by duality, we see that the Fourier transforms of monomials are the derivatives of the δ-distribution. Yet, the negative half-power distribution is a true distributional fixed point of the Fourier transform of distributions on $\mathbb{R}$.

Proposition 10. *The tempered distribution, $f(x) = |x|^{-\frac{1}{2}}$, is a fixed point of the (extended) Fourier transform.*

We compute the (extended) Fourier transform of $f(x) = |x|^{-\frac{1}{2}}$,

$$\mathfrak{F}_E[|x|^{-\frac{1}{2}}](k) = \int_{-\infty}^{\infty} |x|^{-\frac{1}{2}} e^{-2\pi ixk}\,dx$$

$$= 2\int_{0}^{\infty} x^{-\frac{1}{2}} \cos 2\pi xk\,dx$$

.

Substituting $x = \frac{1}{2k\pi}y^2$, and $dx = \frac{1}{k\pi}y\,dy$, then,

$$\mathfrak{F}_E[|x|^{-\frac{1}{2}}](k) = 2\int_0^\infty \frac{\sqrt{2\pi k}}{y} cos(y^2)\frac{1}{k\pi}y\,dy$$

$$= \frac{2\sqrt{2}}{\sqrt{|k|\pi}}\int_0^\infty \cos y^2\,dy$$

$$= \frac{2\sqrt{2}}{\sqrt{|k|\pi}}\sqrt{\frac{\pi}{8}} = \frac{1}{\sqrt{|k|}}.$$

Interestingly, this fact hints that the (non-smooth) scaling law of a power-law distribution in dimension $\frac{1}{2}$ is a perturbation of one that is fixed under Fourier transformation.

9.3.3 Fractal functions as Fixed Points of the Fourier Transform

We find that Lacunary Fourier series are identified with their Fourier transforms, when the sequence of coefficients is in $\ell^2(\mathbb{R})$. Thus, we regard these important fractals as fixed points of the Fourier Transform.

We recall a theorem due to Kolmogorov, which we paraphrase as stating that if $\{\lambda_k\}$ contains *Hadamard gaps*, then $S(\lambda_k, \theta) = \sum_{k=1}^{\infty} a_k \cos(\lambda_k\theta)$ converges almost everywhere if and only if $\sum_{k=1}^{\infty} a_k^2 < \infty$, where we say that $\{\lambda_k\}$ has Hadamard gaps if there exists $\delta > 0$ such that $\lim_{k\to\infty} \frac{\lambda_k}{\lambda_{k-1}} > 1 + \delta$. We then say that such a series is a Fourier series, hence, S is its own Fourier series, by construction. A famous example is the Weierstrass function $f(x) = \sum_{k=0}^{\infty} a^k cos(b^k\pi x)$. We see that $\{b^k\}$ has Hadamard gaps, evaluating $\lim_{k\to\infty} \frac{b^k}{b^{k-1}} = b$, so that for $\{a_k\} \in \ell^2(\mathbb{R})$, and $b > 1$, $f(x)$ is its own Fourier series, hence we regard it as a fixed point of the Fourier transform.

9.3.4 The "Half-Shah" Function and $\partial_{Box} N_F(r)$

Let us call $\sum_{n=0}^{\infty} \delta(r-n)$ a "Half-Shah" function, in recognition that it is the Dirac Comb on the non-negative integers $\mathbb{N} \cup \{0\}$. For some equicontractive deterministic fractals, we have $\partial_{Box} N_F(r) = d_s \sum_{n=0}^{\infty} \delta(r - n \log N)$. Let us change variables so that we have a function $f(r) = k \sum_{n=0}^{\infty} \delta(r-n)$. Taking $\int_{\mathbb{R}} \sum_{n=0}^{\infty} \delta(r-n) e^{-2\pi rix}\, dx = \sum_{n=1}^{\infty} e^{2\pi inx}$, we compute:

$$\sum_{n=1}^{\infty} e^{inx} = \frac{1}{2} \sum_{n=-\infty}^{\infty} e^{inx} + \frac{1}{2} \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n) e^{inx} - \frac{1}{2}.$$

$$= \sum_{-\infty}^{\infty} \delta(x-n) + i \sum_{n=1}^{\infty} \sin(nx).$$

This formal sum is non-convergent almost everywhere, yet, one term is the full Dirac comb. We see that we must normalize our half comb with its mirror image before transforming it, then we observe this pair will be fixed under the transformation to the frequency domain. This puzzling behavior hints to a relationship between fractals and tilings, as the negative integers correspond to "contraction" ratios larger than one. Perhaps we can imagine 'tiling' the real line with expanding Cantor sets to satisfy the weak differential equation given by the full Dirac comb.

9.3.5 Approximate Integration via Tensor Representations and Fourier Transforms of Fractal Measures

The Riemann integral of a function $f(x)$ defined on $[0,1]$ is $\int_0^1 f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^n f(x_k^*)\,\Delta x_k$, for sample points $x_k^* \in [x_{k-1}, x_k]$. We might write $\int_0^1 f(x)\,dx = \lim_{n\to\infty}\vec{f}\cdot\vec{\Delta x}$, where $\vec{f} = (f(x_k^*))_{k=1}^n$, and $\vec{\Delta x} = (x_k - x_{k-1})_{k=1}^N$, with $x_0 = 0$.

We can approximate the integral of a continuous function over an IFS fractal on the line similarly. The augmentation vector $\vec{\ell}^{\otimes^k} = ((\ell^{\otimes^k})_j)_{j=1}^N$ serves the role of the $\vec{\Delta x}$, and using the *right endpoint rule* of numerical integration, we evaluate f at the $x_i = \sum_{j=0}^i (\ell^{\otimes^k})_j$, multiplying $\vec{f}$ termwise by T_i in the augmented representation.

Since the continuous functions are uniformly continuous on compact sets, (since $(\vec{v}^{\otimes^k})_{k=1}^\infty$ is a convergent sequence), then we might approximate

$$\int_{F\cap[0,\epsilon]} f\,dx = \lim_{k\to\infty}\sum_{i=0}^{j=\lceil \epsilon N^k\rceil} f(x_i)(\ell^{\otimes^k})_i =: \lim_{k\to\infty}\vec{f_k}\cdot\vec{\ell}^{\otimes^k} * .\vec{\chi}(\{1,2,...,j\}).$$

Here, we define $\vec{f_k} := (f(\sum_{i=0}^j (\ell^{\otimes^k})_j))_{j=1}^{N^k}$, $\vec{\chi}(\{1,2,...,j\}) \in \mathbb{F}^{N^k} = [1,...,1,0,...0]$, and the x_i are as defined above.

Similarly, we can approximate the integral of $f(x)$ with respect to $\mu_F = (\rho, S)\mu_F$, as

$$\int_{F\cap[0,\epsilon]} f\,d\mu_F(x) = \lim_{k\to\infty}\sum_{i=0}^{j=\lceil \epsilon N^k\rceil} f(x_i)(r^{\otimes^k})_i = \lim_{k\to\infty}\vec{f_k}\cdot\vec{r}^{\otimes^k} * .\vec{\chi}(\{1,2,...,j\}).$$

The convergence of all such Riemann sums for all $f \in C([0,1])$ establishes the following:

Theorem 19. *The (weighted) invariant measure* $\mu = (\rho, S)\mu$ *supported on the attractor of* S *is the (weak) limit* $\lim_{k\to\infty}(\vec{v} * .\vec{r})^{\otimes^k} = [\frac{(\vec{T}*.\vec{r})^{\otimes^k}}{\vec{\ell}^{\otimes^k}}]$.

Proof. (Sketch)By the uniform continuity of $f(x) \in C([0,1])$, and the convergence of the measure representation, the result follows by the discussion above. ☐

Now armed with an approximate integration technique, we may numerically estimate the Fourier transform of the measure μ_F:

$$\hat{\mu}_F(\xi) = \int_F e^{-2\pi i \xi x}\, d\mu_F(x) = \lim_{k\to\infty} \sum_{i=0}^{N^k} e^{-2\pi i x_i} (r^{\otimes^k})_i.$$

9.4 The Functorial Hypothesis

We might consider the fixed flow $\mathfrak{F}$ on the dual geometry $\vec{v}^T$, to be a functor from $\mathbf{Ob}_{\mathbb{IFS}}$ to $\mathbf{Ob}_{AB\,Grp}$, (specifically to a countable product of $\mathbb{Z}$), similarly, with an indepenedent choice of pole of the hyperfunction representation of $\partial_{box}(F)(r)$ at each inverse scale $\{N^n\}_{n=0}^{\infty}$, we can regard that object as an isomorphic functor. Then by the duality isomorphism between $\vec{v}$ and $\vec{v}^T$, we can estimate the measure represented by $\vec{v}$ by integrating ∂_{box} over a selection P of its poles .

Select elements P such that, for any $k \in \mathbb{Z}$,

$$\mu([0, \sum_{i=1}^{k} (\ell^{\otimes^n})_i]) = \int_P \partial_{Box}(F)(r)\,\frac{dr}{r}.$$

For example, let P be the first x_i poles of the hyperfunction representation of $\partial_{Box}(F)(r)$, at each inverse scale N^n, for $x = \sum_{i=1}^{n} x_i N^{-i}$.

We now find the heretofore perhaps questionable "intrinsic geometry hypothesis" to be a valuable tool in simplifying the above formula, since the singularities in the measure occur only at powers of the preferred scale, for the Functorial Hypothesis to hold as formulated, the same should be true of the box-counting function. This is guaranteed when we count only using boxes within the approximating prefractal at the given scale.

9.5 Appendix: Very Simple Fractal Code

In the following are some brief scilab codes for computing various quantities using tensor representations of fractals.

While Scilab includes the Kronecker product as a standard operation, denoted ".*.", iterating this operation could become tedious without a little code:

```
function vn=tensorpower(v,n)
k=v
```

```
for i=1:n-1
k=k.*.v
end
vn=k
endfunction
```

For fractals in the interval, with general real contraction ratios and gap lengths, we can compute the self-similar measure quite simply.

plotnussm.sce plots non-equicontractive self-similar measures using tensor representation approximation to n^{th} tensor power

```
clf;
plot(cumsum(tensorpower(l,n)),cumsum(tensorpower(r,n)))\\
```

By contrast, systems without a built-in function for Kronecker multiplication complicate these matters somewhat. Included here is a brief TI calculator program for Kronecker products of vectors:

```
vkron(v,w)
Prgm
dim(w)[2]->m
dim(v)[2]*dim(w)[2]->k
newMat(1,k)->kron
```

```
0-> b
For i,1,k
mod(i,m)->b
If b=0 Then
m -> b
EndIf
v[1,ceiling(i/m)]*w[1,b]->kron[1,i]
EndFor
Disp kron
EndPrgm
```

And for square matrices, we can call vkron, above:

```
mkron2(v,w)
Prgm
ClrIO
dim(v)[1]->m
dim(v)[2]->n
dim(w)[1]->p
dim(w)[1]->q
For ii,1,m
For jj,1,p
vkron(v[ii],w[jj])
EndFor
```

EndFor

EndPrgm

10 Concluding Remarks

We see that fractal concepts and methods stemming from the properties of self-similarity and contractivity [K-L-M-V],[Law], are fundamental to much of analysis and physics. By generalizing our notion of fractal dimension into a notion of scale covariance, we find algebraic structure within the evolution of fractals.

We have seen that IFS fractal constructions satisfy many algebraic structures, and we can use this fact to develop simple algebraic representations of IFS fractals, allowing for computations of measures and their transforms, and illuminating possible avenues for the applications of IFS fractals to the physics of particles and spacetime.

Not all our work has been conceptual. Concrete, original contributions have been made, to wit:

- Log-periodicity of fractals' scale evolution can be derived from the fractal itself, via $\partial_{box}(F)(r)$.
- A tensor representation allows us to approximate the measure of sets under self-similar measures.

Yet, much more work lies ahead:

- An explicit theory of integration on IFS fractals is at hand, perhaps ready for numerical solutions of differential equations on fractals, and for computation of integral transforms on fractal measures.

- Experimenting with computer code, particularly, with generalizing the concept of self-similar measure to the settings of signed measures opens the door to an entire class of fractal curves and surfaces. Such a model has been proposed to

- We conjecture that the dual to the representation vector acts as a functor to the category of abelian groups, as does the box-derivative. By the isomorphism to the representation vector of a self-similar measure, we can compute measures by integrating over singularities of the box-derivative (complex dimensions). Graph theoretic techniques can advance this approach further, or provide more explicit means to obtain rigorous proofs.

- State vectors of many particle systems in coherent states are convergent infinite tensor products. What kinds of states are described by the representations of fractals by convergent infinite tensor products?

The infinite regress demanded by fractal constructions describe scales that invoke energy levels beyond all bound. Surely the explorers of this landscape will harness these powerful forces for the enlightenment of all.

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