Title
APPROXIMATION OF REGGE POTENTIALS THROUGH FORM FACTORS

Author
Chew, Geoffrey F.

Publication Date
1965-04-30

UCRL-16101

University of California

Ernest O. Lawrence Radiation Laboratory

APPROXIMATION OF REGGE POTENTIALS THROUGH FORM FACTORS

Berkeley, California

Submitted to Progress of Theoretical Physics UCRL-16101

UNIVERSITY OF CALIFORNIA
Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

APPROXIMATION OF REGGE POTENTIALS THROUGH FORM FACTORS

Geoffrey F. Chew

April 30, 1965

UCRL-16101

APPROXIMATION OF REGGE POTENTIALS THROUGH FORM FACTORS*

Geoffrey F. Chew

Department of Physics and Lawrence Radiation Laboratory
University of California, Berkeley, California

April 30, 1965

ABSTRACT

A discussion is given of certain consequences of employing Regge poles rather than fixed-J poles as the source of two-particle generalized potentials. Three qualitative aspects are emphasized:

(a) For the purposes of strip approximation dynamics, contrary to common belief, the Regge potential has roughly the same energy dependence as that due to exchange of a fixed-spin particle.

(b) The Regge potential, relative to that produced by a corresponding fixed-spin particle, is damped by an exponential "form factor," roughly estimated as $\approx e^{2\alpha'(t-m^2)}$, where α' is the trajectory slope, t the negative square of momentum transfer, and m the particle mass.

(c) Ambiguous zero-range components in the fixed-J potential become replaced by unambiguous short-range components in the Regge potential.

I. INTRODUCTION

Within a nuclear democracy governed by bootstrap dynamics all poles are of the Regge type, but partial bootstrap calculations for practical reasons more often than not compute the two-particle generalized potentials as if they were generated by fixed-J poles communicating with crossed reactions. Sometimes this practice gives reasonably accurate results, but sometimes it is totally erroneous. Also, there are ambiguities associated with the zero-range components of fixed-J potentials. It is the purpose of this paper to elucidate the qualitative conditions under which the use of fixed-J potentials is legitimate and to explain how the zero-range ambiguity is removed if one understands the asymptotic behavior of Regge parameters. We also attempt to dispel a myth concerning the energy dependence of the Regge potential.

Our entire discussion is carried out within the framework of the new form of strip approximation, in which the dynamics requiring a potential is confined to the low-energy interval where bound states and resonances are prominent.[1] As explained in reference 1, one cannot extend potential dynamics to the high-energy region without double-counting; but if high energies are dominated by Regge poles--as suggested by most experiments to date[2]--there may be no need for a detailed dynamics outside the low-energy strip.

To formulate the strip approximation, it is supposed that the four-line connected part may be broken into two separately analytic parts:

$$A(s,t) = V^s(t,s) + A^s(s,t), \qquad \text{(I:1)}$$

the first term $V^s(t,s)$ not containing any poles in the channel invariant s and the second term $A^s(s,t)$ not containing any poles in the crossed-channel invariants t and u. Conversely, $A^s(s,t)$ is supposed to contain all the s poles while $V^s(t,s)$ contains all the crossed poles. Regge asymptotic behavior prescribes that

$$A^s(s,t) \underset{\substack{t \text{ or } u \to \infty \\ s \text{ fixed}}}{\propto} t^{\alpha_j(s)} \pm u^{\alpha_j(s)} , \qquad \text{(I:2)}$$

while

$$V^s(t,s) \underset{\substack{s \text{ or } u \to \infty \\ t \text{ fixed}}}{\propto} s^{\alpha_i(t)} \pm u^{\alpha_i(t)} , \qquad \text{(I:3)}$$

with a corresponding behavior for V^s as s or $t \to \infty$ with u fixed. The power $\alpha_j(s)$ is the leading Regge trajectory communicating with the s reaction, while $\alpha_i(t)$ is the corresponding trajectory for the t reaction. The sign (±) is determined by the trajectory signature.

The possibility of a strip approximation depends on the further assumption that, as a function of s, $A^s(s,t)$ is large only within a strip $-s_1 \lesssim s \lesssim s_1$. The experimental basis for such an assumption has been discussed in reference 1, leading to the conclusion that $s_1 \gtrsim 4\ \text{GeV}^2$. The strip width s_1 must be chosen large enough so that (a) all significant s resonances are included inside the strip, and (b) at all energies above the strip the Regge asymptotic expansion in terms of a finite number of crossed poles is a reasonable approximation. It follows that

$$A(s,t) \approx V^s(t,s), \qquad \text{for } s > s_1. \qquad \text{(I:4)}$$

These requirements do not place an upper limit on s_1; in practice, however, one usually chooses s_1 as low as possible so as to minimize the number of channels that must be included in the strip dynamics.

The function $V^s(t,s)$ is our generalized potential, to be used with multichannel two-particle s-discontinuity formulas inside the strip, $s < s_1$, in order dynamically to generate the function $A^s(s,t)$ which contains the s poles. The dynamical equations may be of the N/D type or equivalently of the Mandelstam iterative type. We are not here concerned with further approximations, often made in N/D equations, that lead to violation of (I:4) by causing $A^s(s,t)$ to diverge as $s \to \infty$.* We confine our attention to the potential itself.

II. CONTRIBUTION TO THE POTENTIAL FROM AN INDIVIDUAL CROSSED POLE

For reactions without spin in which particle masses (m_a, m_b) do not change, a fairly simple formula has been given for the contribution to the potential from an individual Regge pole in the t reaction:[1]

$$V_i^s(t,s) = -\frac{1}{2}\,\beta_i\,\Gamma_i(t)\int_{z_t(s_2,t)}^{-\infty} dz'\,P_{\alpha_i(t)}(-z')$$

$$\times\left[\frac{1}{z'-z_t} \pm \frac{1}{z'+z_t}\right], \qquad \text{(II:1)}$$

*For example, the approximation of left-hand discontinuities by those of the potential, ignoring the oscillations required to maintain consistency between threshold and asymptotic behavior.

where

$$\Gamma_i(t) = (2\alpha_i(t) + 1)\ \gamma_i(t) \left[-q_a(t) q_b(t)\right]^{\alpha_i(t)} , \qquad \text{(II:2)}$$

and

$$z_t(s,t) = \frac{s + q_a^2(t) + q_b^2(t)}{2q_a(t)\ q_b(t)} , \qquad \text{(II:3)}$$

$$q_a^2(t) = \frac{t}{4} - m_a^2 , \qquad q_b^2 = \frac{t}{4} - m_b^2 . \qquad \text{(II:4)}$$

Here $\alpha_i(t)$ is the Regge trajectory and $\gamma_i(t)$ the reduced residue, while β_i is the appropriate element of the crossing matrix. The (±) sign in (II:1) is determined by the signature of the trajectory. The lower limit of the integral in (II:1) has been chosen to make the potential real for $s < s_2$, so s_2 in principle might be set as low as the leading multiparticle threshold, well inside the strip. Multiparticle channels inside the strip, however, are better represented by unstable two-particle channels than through the Regge expansion, so it seems doubtful that one would ever want to choose s_2 below about $s_1/2$. We shall set $s_2 = s_1$ for the purposes of the present discussion, thereby achieving a potential that is real (nonabsorptive) throughout the strip.*

For regions of t where Re $\alpha_i(t) > 0$, Formula (II:1) needs to be defined by analytic continuation. The result is

---

*The potential is a matrix connecting all important two-particle channels; we are considering one (not necessarily diagonal) element of the matrix.

$$V_i^s(t, s) = -\frac{1}{2}\beta_i \Gamma_i(t) \left\{ \pi \frac{P_{a_i(t)}(z_t) \pm P_{a_i(t)}(-z_t)}{\sin \pi a_i(t)} + \int_{-1}^{z_1(t) = z_t(s_1, t)} dz' \, P_{a_i(t)}(-z') \left[\frac{1}{z' - z_t} \pm \frac{1}{z' + z_t} \right] \right\} . \qquad \text{(II:1')}$$

III. THE CROSSED REACTION PARTIAL-WAVE EXPANSION OF THE POTENTIAL

We have defined the potential associated with a particular Regge pole so that at fixed t it is analytic within an ellipse in z_t passing through $\pm z_t(s_1, t)$, a region that includes the entire strip interval. Everywhere inside the strip, therefore, we may express the potential through a Legendre polynomial expansion in z_t:

$$V_i^s(t, s) = \sum_{\substack{J = \text{even} \\ \text{or odd integers}}} (2J+1)\, V_J^i(t)\, P_J(z_t), \qquad \text{(III:1)}$$

where

$$V_J^i(t) = \frac{1}{2}\int_{-1}^{+1} dz_t\, P_J(z_t)\, V_i^s(t,s) \tag{III:2}$$

$$= (-1)^J \beta_i \Gamma_i(t) \left\{ \frac{1}{[J-\alpha_i(t)][J+\alpha_i(t)+1]} - \int_{+1}^{-z_1(t)} dz'\, P_{\alpha_i(t)}(z')\, Q_J(z') \right\} .$$

We see that, as expected, the potential component $V_J^i(t)$ has a pole at $t = t_J^i$, where $\alpha_i(t_J^i) = J$. That is,

$$V_J^i(t) \xrightarrow[t \to t_J^i]{} \beta_i\, [q_a(t)q_b(t)]^J \frac{R_J^i}{t_J^i - t} , \tag{III:3}$$

where

$$R_J^i = \gamma_i(t_J^i) \Big/ \left(\frac{d\alpha_i}{dt}\right)_{t=t_J^i} . \tag{III:4}$$

Let us now identify the standard approximation of the potential by a fixed J component. First one singles out of the partial-wave expansion (III:1) a particular term, usually the lowest J value J_0; then one approximates the coefficient $V_{J_0}^i(t)$ by (III:3), keeping only the t dependence associated with the pole and the crossed-channel thresholds. One then has

$$V_i^s(t,s) \approx \beta_i (2J_0 + 1)\,(R_{J_0}^{\ i}/t_{J_0}^{\ i} - t)$$

$$\times \left[q_a(t) q_b(t)\right]^{J_0} \; P_{J_0}\left(\frac{s + q_a^2(t) + q_b^2(t)}{2 q_a(t) q_b(t)}\right) . \qquad \text{(III:5)}$$

The task of this paper is to compare Formula (III:5) to the more complete expression (II:1), (II:1'), or equivalently (III:1) together with (III:2).

IV. COMPARISON OF FIXED-J AND REGGE POTENTIALS FOR $|t| \ll s_1$.

Our first remark is that for $0 < s < s_1$ and $t \leq 0$, the only region in which the potential is needed for dynamics, it is not unreasonable to keep only the first term of the partial-wave expansion (III:1). The expansion is rapidly convergent, with an exponential behavior for large J determined by the z_t singularity at $z_1(t)$:

$$V_J^{\ i}(t) \underset{J \to \infty}{<} C(t)\, e^{-J \log\left[z_1(t) + (z_1^2(t) - 1)^{1/2}\right]} . \qquad \text{(IV:1)}$$

This exponential cutoff starts becoming effective for $J \gtrsim 1$ because of the Froissart prohibition on Regge poles for $J > 1$ when $t < 0$.[3] It is easy to verify that the real part of $\log[z_1 + (z_1^2 - 1)^{1/2}]$ is not only positive but of the order of magnitude unity over most of the strip.

Thus any serious complaint about the fixed-J approximation (III:5) to the potential ought not be with respect to the s dependence, which within most of the strip is given adequately by $P_{J_0}[z(s,t)]$. Of course as one approaches the upper boundary of the strip, more than one Legendre polynomial should be kept, but the characteristic Regge s dependence (I:3) will never appear inside the strip. We could make it appear inside by choosing $s_2 \ll s_1$ in Formula (II:1), but as explained above such a procedure raises problems of double counting.

Evidently Formula (III:5) is accurate when t is sufficiently near $t_{J_0}{}^i$, but often the poles of interest, like the ρ and ω, lie a substantial distance from $t = 0$. We thus require a critical comparison for $t \leq 0$ of Formula (III:2) with the approximation (III:3). An essential feature of (III:2), not apparent from the expression given, is the exponential dependence on J shown in (IV:1). We have not been able to carry out the integration in (III:2) so as to exhibit explicitly this exponential behavior, but Khuri and Jones found a slightly different Regge formula[4] for which the partial-wave projection yields

$$V_J{}^i(t) = \beta_i \left[q_a(t) q_b(t)\right]^{\alpha_i(t)} \frac{\gamma_i(t)}{J - \alpha_i(t)}$$

$$\exp\left(-[J-\alpha_i(t)] \log\{z_1(t) + [z_1{}^2(t)-1]^{1/2}\}\right). \qquad \text{(IV:2)}$$

This formula shares with Formula (III:2) the properties of chief interest to us and has the advantage of being much more transparent. Our problem then becomes a comparison of Formula (IV:2) with (III:3).

Evidently some knowledge of the trajectory and reduced residue functions, $\alpha_i(t)$ and $\gamma_i(t)$, is required. Experimental indications are that for $|t| \lesssim t_{J_0}{}^i$ a linear approximation to the trajectory is not misleading, so we take

$$\alpha_i(t) \approx J_0^{\,i} + \alpha_i'(t_{J_0}^{\,i})(t - t_{J_0}^{\,i}). \qquad \text{(IV:3)}$$

Let us also assume $s_1 \gg m_a^2, m_b^2, |t|$, so that the exponential factor in (IV:2) becomes

$$\left[\frac{s_1}{q_a(t)q_b(t)}\right]^{\alpha_i(t)-J} . \qquad \text{(IV:4)}$$

With these simplications, we reduce (IV:2) to

$$V_{J_0}^{\,i}(t) \approx \beta_i \left[q_a(t)q_b(t)\right]^{J_0} \frac{R_{J_0}^{\,i}(t)}{t_{J_0}^{\,i} - t} , \qquad \text{(IV:2')}$$

where

$$R_J^{\,i}(t) = \frac{\gamma_i(t)}{\alpha_i'(t_J^{\,i})} \; s_1^{\,\alpha_i(t)-J} . \qquad \text{(IV:5)}$$

The adequacy of the fixed-J approximation (III:3) has thus finally reduced to the adequacy of ignoring the t dependence of the function $R_J^{\,i}(t)$ defined by (IV:5). To make further progress the reduced-residue function requires attention.

A rough formula has been given by Chew and Teplitz for the residue function:[5]

$$\frac{\gamma_i(t)}{\alpha_i'(t)} \approx (\bar{t} - t) \frac{\mathcal{V}^{\alpha_i(t)}(\bar{t})}{[q_a(\bar{t})q_b(\bar{t})]^{\alpha_i(t)}} , \qquad \text{(IV:6)}$$

where $\mathcal{V}^J(t)$ is the projection of the <u>t-reaction</u> potential onto (complex) angular momentum J. The quantity $\bar{t}$ is a characteristic energy (squared) presumed to be somewhere in the middle of the t strip. Since the widths of all strips should be similar, we estimate $\bar{t}$ as $\approx s_1/2$. In deriving Formula (IV:2') it was assumed that $|t| \ll s_1$, so variation of the factor $(\bar{t} - t)$ in (IV:6) is weak and the most important t dependence is that of the function

$$F_i(t) = \left[\frac{s_1}{q_a(\bar{t})q_b(\bar{t})}\right]^{\alpha_i(t)-J_0} \frac{\mathcal{V}^{\alpha_i(t)}(\bar{t})}{\mathcal{V}^{J_0}(\bar{t})}, \tag{IV:7}$$

which for convenience has been normalized to unity at $t = t_{J_0}{}^i$. The function $F_i(t)$ evidently may be regarded as a kind of form factor.

The first factor in (IV:7), if we estimate $q_a(\bar{t})$ as

$$q_a(\bar{t} = s_1/2) \approx (s_1/8)^{1/2},$$

and similarly for $q_b(\bar{t})$, becomes

$$e^{-[J_0-\alpha_i(t)]\ln 8} \approx e^{(t-t_{J_0}{}^i)/t_c}, \tag{IV:8}$$

where

$$t_c{}^{-1} = \alpha_i{}'(t_{J_0}{}^i) \ln 8. \tag{IV:9}$$

The second factor is harder to estimate and may in some cases be important, but in simple models its variation with t for $|t| < \bar{t}$ is no stronger than that of factors already neglected.[5] The essential point is that the t reaction potential has singularities in z_t much closer than z_1, arising from the s poles--which tend to occur toward the lower side of the s strip. Thus the systematic exponential decrease of $\mathcal{V}^J(\bar{t})$ for increasing

J is much slower than (IV:1). Of course one cannot rule out a strong variation with J for t-reaction potentials with complex structure.

A crude estimate of the Regge effect for small $|t|$, therefore, is to reduce the fixed-J potential by a "form factor"

$$F_i(t) \approx \exp[(t - t_{J_0}^{\ i})/t_c]$$

with

$$t_c^{-1} \approx 2\, \alpha_i'(t_{J_0}^{\ i}). \qquad \text{(IV:10)}$$

Using the ρ trajectory as an example, $\alpha_\rho(0) \approx 0.5,$[2,6] so

$$2\,\alpha_\rho' \approx 2\, \frac{1-0.5}{m_\rho^{\ 2}} = m_\rho^{\ 2},$$

and we have

$$F_\rho(t) \approx \exp[(t - m_\rho^{\ 2})/m_\rho^{\ 2}] \quad . \qquad \text{(IV:11)}$$

Thus the "range" of the Reggeized ρ potential, as measured by its logarithmic derivative with respect to t at t = 0, is $\sqrt{2}$ times as great as that of a conventional ρ potential. Also it is damped in strength by a factor $\approx 1/e$.

Note that we are giving no justification for the "Born approximation" (or peripheral model)

$$A(s,t) \approx V^s(t,s)$$

for $s < s_1$. What we are suggesting is that a simple form factor may correct the most serious trouble with the fixed-J potential for small $|t|$. The potential is to be inserted into dynamical equations--which still have to be solved before the low-energy amplitude is achieved.

The above arguments could be extended to Regge recurrences, that is, to J values in the expansion (III:1) beyond the first. The exponential damping rapidly suppresses these, however, by successive factors of the order $e^{-2\Delta J} = e^{-4}$. There will rarely be any need, therefore, to go beyond the leading physical value of J. The Pomeranchuk trajectory presents a special problem because the first physical J value, J = 0, has no t pole associated with it. We shall deal elsewhere with this extremely important special case.

V. BEHAVIOR OF THE POTENTIAL AT LARGE |t|

We have proposed adding a particular form factor to the fixed-J potential (III:5). Our estimation of this factor employed many approximations that required t to be less than s_1, but it is plausible that the Regge parameters $\alpha_i(t)$ and $\gamma_i(t)$ will have a behavior as $t \to \infty$ that will cause the actual form factor to decrease strongly. At first sight, the asymptotic behavior for large t might seem irrelevant, since in the physical region of the s reaction $|t| < s$ and we are confining ourselves to $s < s_1$. If, however, an analytic continuation is attempted in the angular momentum of the s reaction one must, in the projection of the potential, integrate to $-\infty$ in t. The continuation is defined only for Re $J_s > N$, if the potential behaves like t^N as $t \to \infty$.

Now, the fixed-J potential (III:5) behaves like $t^{J_0 - 1}$ as $t \to \infty$, so one is in difficulty for s-reaction angular momenta $\leq J_0 - 1$. For example, the ρ potential gives trouble for J = 0 in the $\pi\pi$ system. In this case Formula (III:5) becomes

$$V_\rho^{\,s}(t,s) \approx \beta_\rho\, 3\, \frac{R_1^{\,\rho}}{m_\rho^{\,2} - t}\; \frac{2s + t - 4m_\pi^{\,2}}{4}$$

$$= \frac{3}{4}\, \beta_\rho\, R_1^{\,\rho} \left[\frac{2s + m_\rho^{\,2} - 4m_\pi^{\,2}}{m_\rho^{\,2} - t} - 1 \right] . \qquad \text{(V:1)}$$

Obviously the -1 inside the bracket is effective for no angular momenta in the s reaction except the S wave, and there is always uncertainty about whether this singular ("zero-range") component in the potential should be taken seriously.

The Reggeized potential eliminates all such ambiguities if the effective form factor vanishes more rapidly than $t^{-(J_0-1)}$ as $t \to \infty$. Suppose, for example, that we add the simple exponential (IV:11) to (V:1). There is now no difficulty. Both terms in (V:1) are to be taken seriously. The first is attractive (if β_ρ is positive) and has a "range" $\approx \sqrt{2}/m_\rho$ as explained above. The second is repulsive and of a shorter "range" $\approx 1/m_\rho$ that arises entirely from the form factor. Both terms are effective for all s-reaction angular momenta, although the first term is relatively more important for high angular momentum and the second term for low.

ACKNOWLEDGMENTS

I am very grateful to Dr. P. D. B. Collins and Dr. V. C. Teplitz for numerous discussions concerning the points covered in this paper. The motivation for the qualitative analysis presented here arose from quantitative numerical calculations performed by Collins and Teplitz.

FOOTNOTES AND REFERENCES

*Work performed under auspices of the U. S. Atomic Energy Commission.

1. G. F. Chew, Phys. Rev. 129, 2363 (1963); G. F. Chew and C. E. Jones, Phys. Rev. 135, B208 (1964).

2. See R. Phillips and W. Rarita, Lawrence Radiation Laboratory Report, UCRL-16033, April 1965, for a recent review of all high-energy πN and KN experiments and for additional references.

3. M. Froissart, Phys. Rev. 123, 1053 (1961).

4. N. Khuri, Phys. Rev. 130, 429 (1963); C. E. Jones, private communication (1962).

5. G. F. Chew and V. Teplitz, Phys. Rev. 136, B1154 (1964).

6. R. K. Logan, Phys. Rev. Letters 14, 414 (1965).