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Onsager's Conjecture with Physical Boundaries and an Application to the Vanishing Viscosity Limit

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Abstract

We consider the incompressible Euler equations in a bounded domain in three space dimensions. Recently, the first two authors proved Onsager's conjecture for bounded domains, i.e., that the energy of a solution to these equations is conserved provided the solution is Hölder continuous with exponent greater than 1/3, uniformly up to the boundary. In this contribution we relax this assumption, requiring only interior Hölder regularity and continuity of the normal component of the energy flux near the boundary. The significance of this improvement is given by the fact that our new condition is consistent with the possible formation of a Prandtl-type boundary layer in the vanishing viscosity limit.

1 Introduction

As early as in 1949, L. Onsager [25] conjectured that an ideal incompressible flow will conserve energy if it is Hölder continuous with exponent greater than 1/3. His conjecture, which was based on Kolmogorov's 1941 theory of turbulence and was taken up by mathematicians only in the 1990s, when Eyink [14] and Constantin-E-Titi [10] independently proved respective versions of this conjecture. These results were later sharpened by Cheskidov et al. [8].

More recently, new interest has arisen in the relation between regularity and energy conservation as studied by Onsager. One direction of research has established the "other direction" of Onsager's Conjecture, that is the optimality of the exponent 1/3. In other words, the aim has been to exhibit, for every $\alpha < 1/3$, a weak solution of the Euler equations in $C^{0,\alpha}$ which does *not* conserve energy. This has been achieved, as the culmination of a series of works by De Lellis-Székelyhidi and others throughout several years, by Isett [20]

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and Buckmaster-De Lellis-Székelyhidi-Vicol [5]. However one should keep in mind that the existence of solutions which belong to $C^{0,\alpha}$ with $\alpha < 1/3$ and which dissipate the energy does not imply that all solutions that do not belong to $C^{0,\alpha}$ with $\alpha > 1/3$ dissipate the energy. In fact, the authors of [1] provide simple examples of weak solutions of the Euler equations which conserve the energy and which are not more regular than L^2, in particular they are not even bounded. Eventually, it is mostly in the presence of boundary effects that one can establish some type of complete equivalence between loss of regularity and non-conservation of energy, cf. Theorem 4.1 in [2], following a theorem of Kato [21].

Another recent line of research has focused on the extension of the classical results [14, 10, 8] to other systems of fluid dynamics, such as the inhomogeneous incompressible Euler and Navier-Stokes equations [24, 15], the isentropic compressible Euler equations [15], the full Euler system [12], the compressible Navier-Stokes equations [29], and a general class of hyperbolic conservation laws [18].

All these results are proved only in the absence of physical boundaries, i.e. on the whole space or the torus. Except for the case of the half-space [26], Onsager's Conjecture had not been studied in domains with boundaries until the recent work [3] of the first two authors, who proved energy conservation of weak solutions of the incompressible Euler equations in (smooth) bounded domains $\Omega \subset \mathbb{R}^n$ under the assumption that the solution be in $C^{0,\alpha}(\overline{\Omega})$ for some $\alpha > 1/3$.

The aim of the present note is to give a less restrictive assumption on the regularity of the velocity. More precisely, we show that the energy is conserved if the weak solution (u, p) of the Euler equations possesses the following properties (cf. Theorem 4.1, below):

- At least for some $\beta < \infty$ and some $V_\gamma \subset \Omega$, where $\gamma > 0$ and $\{x \in \Omega : d(x, \partial\Omega) < \gamma\} \subset V_\gamma$, one has $p \in L^{3/2}((0,T); H^{-\beta}(V_\gamma))$.
- $u \in L^3((0,T); C^{0,\alpha}(\tilde{\Omega}))$ for any $\tilde{\Omega} \subset\subset \Omega$, with an exponent $\alpha > 1/3$ that may depend on $\tilde{\Omega}$;
- the energy flux
$$\left(\frac{|u|^2}{2} + p\right) u$$
has a continuous normal component near the boundary of Ω.

This may seem at first glance like a merely technical improvement, but, unlike the hypothesis of [3], our assumptions are consistent with the formation of a boundary layer in the vanishing viscosity limit. Indeed, consider a sequence of Leray-Hopf weak solutions of the Navier-Stokes equations, with no-slip boundary conditions, and viscosity tending to zero. Then the discrepancy with the no-normal flow boundary condition for the Euler equations may lead to the formation of a boundary layer, where the normal directional derivative of the *tangential* velocity component, and hence the $C^{0,\alpha}-$norm of the velocity, will blow up as the viscosity goes to zero. Note that this is not in contradiction with our regularity assumptions. The precise statement on the viscosity limit is contained in Theorem 5.1.

As in [3], our argument relies on commutator estimates as introduced in [10], but we pay special attention to clearly separate the *local* and the *global* arguments. In section 3, we follow the work of Duchon-Robert [13] to establish the local conservation of energy (Theorem 3.1). However, since we no longer work in the whole space, or in the case of periodic boundary conditions, we have to study more carefully the regularity of the pressure (Proposition 3.3). Section 4 presents the passage from local to global energy conservation. This is the only place where the regularity of the boundary of our domain comes into play. Finally, in section 5 we present the above-mentioned application concerning the vanishing viscosity limit.

Let us add a final remark concerning our assumptions: The hypothesis on the behavior of the pressure, near the boundary, is very weak and we don't see any way to remove it. This is because in bounded domains the interior Hölder regularity of the pressure no longer automatically follows from that of the velocity. The Hölder spaces in the regularity assumption on the velocity, however, can easily be replaced, e.g., by the critical Besov space from [8] or the averaged Besov-type condition from [16] without significant changes. We prefer here to use Hölder spaces in order to keep the presentation simple.

2 Weak solutions of the Euler equations defined on $(0,T)\times\Omega$

We recall that with Ω denoting an open subset of $\mathbb{R}^n$ a weak solution of the Euler equations is a pair of distributions $(t,x)\mapsto(u(t,x),p(t,x))\in(\mathcal{D}'((0,T)\times\Omega))^n\times\mathcal{D}'((0,T)\times\Omega)$ with $u\in(C_{weak}((0,T);L^2(\Omega)))^n$ which satisfies, in the sense of distributions, the divergence free condition and the momentum equation:

$$\nabla\cdot u=0 \quad\text{in}\quad D'((0,T)\times\Omega) \quad\text{and}\quad \partial_t u+\nabla_x\cdot(u\otimes u)+\nabla_x p=0 \quad\text{in } (\mathcal{D}'((0,T)\times\Omega))^n \tag{2.1}$$

meaning in particular that

$$\forall\Psi\in(\mathcal{D}((0,T)\times\Omega))^n \quad \langle\langle u,\partial_t\Psi\rangle\rangle+\langle\langle u\otimes u,\nabla_x\Psi\rangle\rangle+\langle\langle p,\nabla_x\cdot\Psi\rangle\rangle=0 \tag{2.2}$$

with $\langle\langle\cdot,\cdot\rangle\rangle$ denoting the spatial duality between more general spaces in particular between $\mathcal{D}'((0,T)\times\Omega)$ and $\mathcal{D}((0,T)\times\Omega)$.

Remark 2.1.

- *As usual the condition $u\in C_{weak}((0,T);L^2(\Omega))$ implies that $u\otimes u$ is well defined in $D'((0,T)\times\Omega))$.*
- *Since $\mathcal{D}((0,T)\times\Omega)$ is the closure for the topology of test functions of the tensor product $\mathcal{D}(0,T)\otimes\mathcal{D}(\Omega)$, the relation (2.1) is equivalent to the relation:*

$$\forall\phi\in\mathcal{D}(\Omega)\,,\quad \partial_t\langle u,\phi\rangle+\langle\nabla\cdot(u\otimes u),\phi\rangle+\langle\nabla p,\phi\rangle=0 \tag{2.3}$$

in $\mathcal{D}'(0,T)$. with $\langle\cdot,\cdot\rangle$ denoting the duality between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$.

- *Since p is a distribution, it is locally of finite order (in (x,t)) and therefore can be written as*
$$p(x,t) = (\frac{\partial}{\partial t})^k \nabla_x^l P(t,x)$$
with k (resp. l) a finite integer (resp. finite multi-integer) and $P(t,x) \in L^\infty((0,T)\times\Omega)$. For both the local result (cf. Section 3) and the global result, some extra regularity hypothesis of the pressure is required. With this assumption the impermeability boundary condition is not required for the local result. On the other hand, for global energy conservation (cf. Section 4), as expected, the impermeability boundary condition
$$u \cdot \vec{n} = 0 \quad on \quad (0,T) \times \partial\Omega \tag{2.4}$$
is compulsory and, as usual, one observes that since $\nabla \cdot u = 0$ in $(0,T) \times \Omega$, the relation (2.4) is well defined at least in $C_{weak}((0,T); H^{-1/2}(\partial\Omega))$.

3 The local version of the Duchon-Robert Theorem

This section is devoted to the proof of a local energy conservation law in some time/space cylindrical domain
$$\tilde{Q} = (t_1, t_2) \times \tilde{\Omega} \subset\subset (0,T) \times \Omega.$$
No regularity hypothesis on Ω is requested in this section other than being open and bounded. However it is assumed (without loss of generality) that $\partial\tilde{\Omega}$ is a C^1 manifold.

Theorem 3.1. *Let $(u,p) \in L^\infty(0,T; L^2(\Omega)) \times \mathcal{D}'((0,T)\times\Omega)$ be a weak solution of the Euler equations*
$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0\,, \quad \nabla \cdot u = 0, \tag{3.1}$$
which in an open subset $\tilde{Q} = (t_1,t_2) \times \tilde{\Omega}$ satisfies the following two conditions:

1. *"Local in time regularity of the pressure near the boundary of $\tilde{\Omega}$". For some $\gamma > 0$ and for $V_\gamma = \{x \in \tilde{\Omega} : d(x, \partial\tilde{\Omega}) < \gamma\}$ there exist $M_0(V_\gamma) > 0$ and $\beta(V_\gamma) > 0$ such that*
$$p \in L^{3/2}((t_1,t_2); H^{-\beta(V_\gamma)}(V_\gamma)) \le M_0(V_\gamma) < \infty; \tag{3.2}$$

2. *"Interior $\frac{1}{3}$ Hölder regularity": For some $\alpha(\tilde{Q}) > \frac{1}{3}$ and $M(\tilde{Q}) > 0$ one has*
$$\int_{t_1}^{t_2} \|u(.,t)\|^3_{C^{0,\alpha(\tilde{Q})}(\overline{\tilde{\Omega}})} dt \le M(\tilde{Q}) < \infty\,. \tag{3.3}$$

Then (u,p) satisfies in $\tilde{Q} = (t_1,t_2) \times \tilde{\Omega}$ the local energy conservation:
$$\partial_t \frac{|u|^2}{2} + \nabla_x \cdot \left(\left(\frac{|u|^2}{2} + p \right) u \right) = 0 \quad in \ \mathcal{D}'((t_1,t_2) \times \tilde{\Omega})\,. \tag{3.4}$$

The proof of the above theorem is divided into three subsections. First, standard notations are introduced and an extension-restriction Proposition is proven. Then an "interior estimate" for the pressure is deduced from hypothesis (3.2). Eventually, the proof is accomplished by showing formula (3.4) for test functions of the form $\chi(t)\phi(x)$ with $\chi \in \mathcal{D}(t_1, t_2)$ and $\phi \in \mathcal{D}(\tilde{\Omega})$, and then the proof is extended by the density of finitely many combinations of such test functions in $\mathcal{D}((t_1, t_2) \times \tilde{\Omega})$.

3.1 Notations and extension-restriction construction

For present convenience in this section, and for further treatments, we consider in the space $\mathbb{R}^m_z$ standard mollifiers, a family of well adapted open sets and restriction or regularization of distributions. Then the notations and properties will be adapted to spatial domains or "time-space" cylindrical domains, i.e. $\mathbb{R}^n_x$ or $\mathbb{R}^{n+1}_{t,x} = \mathbb{R}_t \times \mathbb{R}^n_x$. With a smooth non-negative function $(s \mapsto \rho(s)) \in \mathcal{D}(\mathbb{R})$ with support in $|s| < 1$ and of total integral 1, i.e.

$$(s \mapsto \rho(s) \geq 0) \in \mathcal{D}(\mathbb{R}), \quad \int_{\mathbb{R}} \rho(s) ds = 1, \tag{3.5}$$

one denotes by $z \mapsto \rho_\sigma(z)$ the mollifier in $\mathbb{R}^m_z$ given by

$$z \mapsto \rho_\sigma(z) = \frac{1}{\sigma^m} \rho\left(\frac{|z|}{\sigma}\right). \tag{3.6}$$

For an open set $\tilde{Q} \subset \mathbb{R}^m_z$ and a distribution $T \in \mathcal{D}'(\tilde{Q})$, the relation $T = 0$ is equivalent to the property that for any given test function $\Psi \in \mathcal{D}(\mathbb{R}^m_z)$ (fixed for the rest of the argument) one has

$$\langle\langle T, \Psi \rangle\rangle = 0. \tag{3.7}$$

By definition Ψ is compactly supported in $\tilde{Q}$. Its support will be denoted by S_Ψ and with $\eta > 0$ small enough one can introduce three open sets with the following properties:

$$\begin{aligned} &S_\Psi \subset\subset Q_3 \subset\subset Q_2 \subset\subset Q_1 \subset\subset \tilde{Q}, \\ &d(S_\Psi, \mathbb{R}^m \backslash Q_3) > \eta,\ d(Q_3, \mathbb{R}^m \backslash Q_2) > \eta,\ d(Q_2, \mathbb{R}^m \backslash Q_1) > \eta,\ \text{and}\, d(Q_1, \mathbb{R}^m \backslash \tilde{Q}) > \eta. \end{aligned} \tag{3.8}$$

Next, in order to extend to $\mathcal{D}'(\mathbb{R}^m)$ distributions defined as elements of $\mathcal{D}'(\tilde{Q})$, one introduces a smooth function $I_{2,\eta} \in \mathcal{D}(\mathbb{R}^m_z)$ with the following properties:

$$z \in Q_2 \Rightarrow I_{2,\eta}(z) = 1 \quad \text{and} \quad z \notin Q_1 \Rightarrow I_{2,\eta}(z) = 0. \tag{3.9}$$

As a consequence the support of $I_{2,\eta}$ is contained in $\overline{Q_1} \backslash \overline{Q_2}$ and any distribution $T \in \mathcal{D}'(\tilde{Q})$ generates a distribution $\mathcal{D}'(\mathbb{R}^m_z)$ denoted $I_{2,\eta}T$ or $\overline{T}$ according to the formula:

$$\langle\langle \overline{T}, \Psi \rangle\rangle = \langle\langle I_{2,\eta}T, \Psi \rangle\rangle = \langle\langle T, I_{2,\eta}\Psi \rangle\rangle. \tag{3.10}$$

For such a construction one has the following:

Proposition 3.2. *For any scalar or tensor valued functions $z \mapsto w(z) \in L^p(Q_1)$ (with $1 \le p \le \infty$), for any C^∞ function such that the function $z \mapsto f(w(z))$ is well defined for $w \in L^p(Q_1)$, one has the following properties: $\overline{f} := I_{2,\eta} f$ given by the formula*

$$\langle\langle \overline{f}, \Psi \rangle\rangle = \langle\langle I_{2,\eta} f, \Psi \rangle\rangle = \int_{\mathbb{R}^m} f(w(z)) I_{2,\eta}(z) \Psi(z) dz \tag{3.11}$$

is a well defined distribution and, when applied to test functions $\Psi \in \mathcal{D}(\mathbb{R}^m_z)$ with support in Q_3, it satisfies the following relation:

$$\langle\langle \overline{f}(w), \Psi \rangle\rangle = \int_{Q_3} f(w(z)) I_{2,\eta}(z) \Psi(z) dz = \langle\langle f(\overline{w}), \Psi \rangle\rangle \,. \tag{3.12}$$

For any multi-order derivative D^α and any $\Psi \in \mathcal{D}(\mathbb{R}^m)$ with support in Q_3, it satisfies also the following relation:

$$\langle\langle D^\alpha \overline{f(w)}, \Psi \rangle\rangle = \langle\langle D^\alpha f(\overline{w})), \Psi \rangle\rangle \,. \tag{3.13}$$

Finally, for $0 < \sigma$ small enough, i.e.

$$0 < \sigma < \frac{\eta}{2} < \frac{d(Q_3, \mathbb{R}^m_z \backslash \overline{Q}_2)}{2} \,, \tag{3.14}$$

one has:

$$\langle\langle \rho_\sigma \star \overline{f(w)}, \Psi \rangle\rangle = \langle\langle \rho_\sigma \star f(\overline{w}), \Psi \rangle\rangle \,. \tag{3.15}$$

Proof. The formula (3.12) is a direct consequence of the fact that on the support of Ψ one has $I_{2,\eta} = 1$. By the same token, for the formula (3.13) one returns to the definition of derivatives in the sense of distributions and writes:

$$\begin{aligned} &\langle\langle D^\alpha \overline{f(w)}, \Psi \rangle\rangle = (-1)^{|\alpha|} \langle\langle \overline{f(w)}, D^\alpha \Psi \rangle\rangle = (-1)^{|\alpha|} \langle\langle f(w), I_{2,\eta} D^\alpha \Psi \rangle\rangle \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^m_z} f(I_{2,\eta}(z) w(z)) D^\alpha \Psi(z) dz = (-1)^{|\alpha|} \langle\langle f(\overline{w}), D^\alpha \Psi \rangle\rangle = \langle\langle D^\alpha f(\overline{w}), \Psi \rangle\rangle. \end{aligned} \tag{3.16}$$

Eventually, since

$$d(z, Q_3) < \eta \Rightarrow z \in Q_2 \Rightarrow I_{2,\eta}(z) = 1 \,, \tag{3.17}$$

one has:

$$\begin{aligned} \langle\langle \rho_\sigma \star \overline{f(w)}, \Psi \rangle\rangle &= \int_{\mathbb{R}^m_z} \overline{f(w(z))} \left(I_{2,\eta}(z) \int_{\mathbb{R}^m_y} \rho_\sigma(z-y)(y) \Psi(y) dy \right) dz \\ &= \int_{\mathbb{R}^m_z} \overline{f(w(z))} \left(\int_{\mathbb{R}^m_y} \rho_\sigma(z-y) \Psi(y) dy \right) dz \\ &= \int_{\mathbb{R}^m_z} f(w(z)) \left(\int_{\mathbb{R}^m_y} \rho_\sigma(z-y) \Psi(y) dy \right) dz \\ &= \int_{\mathbb{R}^m_z} \left(f(I_{2,\eta}(z) w(z)) \int_{\mathbb{R}^m_y} \rho_\sigma(z-y) \Psi(y) \right) dy = \langle\langle \rho_\sigma \star f(\overline{w}), \Psi \rangle\rangle \,. \end{aligned} \tag{3.18}$$

$\square$

Below time-space cylindrical domains are considered. For a given function $\psi \in \mathcal{D}(\tilde{Q})$ with support contained in $(t_a, t_b) \times S_\psi$ we introduce sets

$$S_\psi \subset\subset \Omega_3 \subset\subset \Omega_2 \subset\subset \Omega_1 \subset\subset \tilde{\Omega} \subset \mathbb{R}^n_x$$

and $\eta > 0$ such that:

$$d(S_\Psi, \mathbb{R}^m \backslash Q_3) > \eta,\ d(Q_3, \mathbb{R}^m \backslash Q_2) > \eta,\ d(Q_2, \mathbb{R}^m \backslash Q_1) > \eta\,,\ \text{and}\, d(Q_1, \mathbb{R}^m \backslash \tilde{Q}) > \eta\,. \quad (3.19)$$

For a time interval $0 < t_1 < t_2 \le T$ with τ small enough, Proposition 3.2 will be applied to the open sets:

$$\begin{aligned} &(t_a, t_b) \times S_\phi \subset\subset Q_3 = (t_1 + 3\tau, t_2 - 3\tau) \times \Omega_3 \subset\subset Q_2 = (t_1 + 2\tau, t_2 - 2\tau) \times \Omega_2 \\ &\subset\subset Q_1 = (t_1 + \tau, t_2 - \tau) \times \Omega_1 \subset\subset \tilde{Q} = (t_1, t_2) \times \tilde{\Omega}\,. \end{aligned} \quad (3.20)$$

In the same way the extension process and notation are adapted as follows: One uses functions $I_{2,\tau} \in \mathcal{D}(\mathbb{R}_t)$, $I_{2,\eta} \in \mathcal{D}(\mathbb{R}^n_x)$ and $I_{2,\sigma}$ with the following properties:

$$\begin{aligned} &t \in (t_1 + 2\tau, t_2 - 2\tau) \Rightarrow I_{2,\tau}(z) = 1 \quad \text{and} \quad t \notin (t_1 + \tau, t_2 - \tau) \Rightarrow I_{2,\tau}(t) = 0\,. \\ &x \in \Omega_2 \Rightarrow I_{2,\eta}(x) = 1 \quad \text{and} \quad x \notin \Omega_1 \Rightarrow I_{2,\eta}(x) = 0\,, \\ &I_{2,\sigma}(x,t) = I_{2,\tau}(t) I_{2,\eta}(x)\,. \end{aligned} \quad (3.21)$$

As above any distribution $T \in \mathcal{D}'(\tilde{Q})$ is extended as a distribution in $\mathcal{D}'(\mathbb{R}_t \times \mathbb{R}^n_x)$ to

$$\overline{T} = I_{2,\sigma} T \quad (3.22)$$

and the same way the mollifiers ρ_σ are replaced by the mollifiers:

$$\forall \left(\kappa < \frac{\tau}{2}, \epsilon < \frac{\eta}{2}\right) \qquad \rho_\sigma(t,x) = \rho_{\kappa,\eta}(t,x) = \frac{1}{\kappa}\rho\left(\frac{|t|}{\kappa}\right)\frac{1}{\epsilon^n}\rho\left(\frac{|x|}{\epsilon}\right) \quad (3.23)$$

Eventually for $w_x \in \mathcal{D}'(\mathbb{R}^n_x)$ and for $w_{(x,t)} \in \mathcal{D}'(\mathbb{R}_t \times \mathbb{R}_x)$ we use the following notation:

$$(w_x)^\epsilon = \rho_\epsilon \star_x w_x \quad \text{and} \quad (w)^{\epsilon,\kappa} = \rho_\sigma \star w = \rho_\kappa \star_t \rho_\epsilon \star_x w. \quad (3.24)$$

3.2 Local estimate on the pressure

With α and β denoting the numbers $\alpha(\tilde{Q}) > \frac{1}{3}$ and $\beta(V_\gamma)$ one has the following:

Proposition 3.3. *Let (u,p) be a weak solution of the Euler equations which satisfies in $\tilde{Q} = (t_1, t_2) \times \tilde{\Omega}$ the hypothesis of Theorem 3.1. Then the restriction of the pressure p to $Q_2 = (t_1, t_2) \times \Omega_2$ belongs to the space $L^{3/2}((t_1, t_2); C^{0,\alpha}(\overline{\Omega_2}))$ and satisfies the estimate:*

$$\|p\|_{L^{3/2}((t_1,t_2);C^{0,\alpha}(\overline{\Omega_2}))} \le C(\|u\|_{L^3((t_1,t_2);C^\alpha(\tilde{\Omega}))}, \|p\|_{L^{3/2}((t_1,t_2);H^{-\beta}(V_\gamma))})\,. \quad (3.25)$$

Proof. Taking the divergence of (2.1), one deduces that p satisfies in $\mathcal{D}'((0,T)\times\Omega)$ the relation

$$-\Delta p = \sum_{i,j=1}^{n} \partial^2_{x_i,x_j}(u_i u_j)\,. \tag{3.26}$$

Then one introduces a function $x \mapsto \tilde\eta(x) \in \mathcal{D}(\mathbb{R}^n)$ with support in $\tilde\Omega$ and which is equal to 1 in $\tilde\Omega \setminus V_\gamma$ to obtain:

$$-\Delta(\tilde\eta p) = \tilde\eta \sum \partial^2_{x_i,x_j}(u_i u_j) + R \tag{3.27}$$

with

$$R = -2\nabla_x \tilde\eta \cdot \nabla_x p - p\Delta\tilde\eta. \tag{3.28}$$

In equation (3.27) all the terms ($\tilde\eta p$ and R) are compactly supported in $\tilde\Omega$ and therefore they can be extended by 0 as distributions in $\mathbb{R}^n$. With $K_n(|x|)$ being the fundamental solution of the equation $-\Delta K_n = \delta_0$ in $\mathbb{R}^n$, one has:

$$\tilde\eta p = K_n \star \left[\tilde\eta \sum \partial^2_{x_i,x_j}(u_i u_j)\right] + K_n \star R. \tag{3.29}$$

By virtue of standard Hölder regularity estimates on the expression

$$K_n \star \left[\tilde\eta \sum \partial^2_{x_i,x_j}(u_i u_j)\right](x) = \int_{\mathbb{R}^n} K_n(|x-y|)(\tilde\eta \sum \partial^2_{y_i,y_j} u_i u_j)dy \tag{3.30}$$

(cf. [22]), one deduces from (3.30) and (3.3) the relation:

$$\|K_n \star \left[\tilde\eta \sum \partial^2_{x_i,x_j}(u_i u_j)\right]\|_{L^{3/2}((t_1,t_2);C^{0,\alpha}(\mathbb{R}^n))} \le C(\|u\|_{L^3((t_1,t_2);C^\alpha(\overline{\tilde\Omega}))})\,. \tag{3.31}$$

One the other hand, concerning the term

$$(K_n \star R)(x) = \int_{\mathbb{R}^n} K_n(|x-y|)R(y)dy$$

one observes that for $x \in \Omega_2$ and $y \in \operatorname{supp} R$ one has $|x-y| \ge \eta$ and therefore, for $K_n \star R$ restricted to Ω_2 and for any $s > 0$ one deduces from (3.2), and from the choice of $\tilde\eta$ the estimate:

$$\|K_n \star R\|_{H^s(\Omega_2)} \le C\|p\|_{H^{-\beta}(V_\gamma)}. \tag{3.32}$$

Eventually on Q_2, p coincides with $\tilde\eta p$ hence the estimate (3.25) follows from (3.31) and (3.32).

□

From this lemma one deduces the following:

Corollary 3.4. *Under the hypotheses of Proposition 3.3, the restriction of $\partial_t u$ to $Q_2 = (t_1,t_2)\times\Omega_2$ is bounded in*

$$L^{3/2}((t_1,t_2);H^{-1}(\Omega_2))\,.$$

Proof. Thanks to the relation

$$\partial_t u = -\nabla\cdot((u\otimes u) + pI) \tag{3.33}$$

the proof follows from estimate (3.25). □

3.3 Completion of the proof of Theorem 3.1

Let us start by considering a test function $\Psi = \chi(t)\phi(x)$ with compact support $(t_a, t_b) \times S_\phi \subset\subset (t_1, t_2) \times \tilde{\Omega}$ and introduce for $1 \le i \le 3$ the sufficiently small numbers (τ, η), the open sets Q_i satisfying relation (3.20) and the corresponding mollifiers which satisfy (3.23). The function $\chi(t)\phi(x)u(x,t)$ belongs to $L^\infty(Q_1)$ and has support in Q_3. Therefore one can introduce its extension by 0 outside Q_1 :

$$(t,x) \mapsto \overline{(\chi(t)\phi(x)u(x,t))} = \chi(t)\phi(x)\overline{(u(x,t))} \in L^\infty(\mathbb{R}_t \times \mathbb{R}^n_x)\,. \tag{3.34}$$

This extension is regularized according to the notation and formula:

$$\Psi_{\epsilon,\kappa} = \rho_{\epsilon,\kappa} \star (\chi(t)\phi(x)\,(\rho_{\epsilon,\kappa} \star \overline{u})\,(x,t)) =: \left(\chi(t)\phi(x)\overline{u(x,t)}^{\epsilon,\kappa}\right)^{\epsilon,\kappa} \in \mathcal{D}(\mathbb{R}_t \times \mathbb{R}^n_x)\,. \tag{3.35}$$

With $\kappa < \frac{\tau}{2}$ and $\epsilon < \frac{\eta}{2}$, the support of $\Psi_{\epsilon,\kappa}$ is contained in $Q_1 \subset\subset \tilde{Q}$ .Therefore the formula

$$0 = \langle\langle(\partial_t u + \nabla_x \cdot (u \otimes u) + \nabla_x p), \Psi_{\epsilon,\kappa}\rangle\rangle \tag{3.36}$$

makes sense and is the sum of three well defined terms:

$$I_1^{\epsilon,\kappa} = \langle\langle \partial_t u, \Psi_{\epsilon,\kappa}\rangle\rangle\,, \quad I_2^{\epsilon,\kappa} = \langle\langle \nabla_x \cdot (u \otimes u), \Psi_{\epsilon,\kappa}\rangle\rangle \quad \text{and} \quad I_3^{\epsilon,\kappa} = \langle\langle \nabla_x p, \Psi_{\epsilon,\kappa}\rangle\rangle\,. \tag{3.37}$$

The limit of these three terms for $(\epsilon, \kappa) \to 0$ is evaluated below, observing that, since the support of $\chi(t)\phi(x)$ is compactly contained in Q_3, the support of $\Psi_{\epsilon,\kappa}$ is in fact contained in Q_2 . Hence for the first term one has:

$$\begin{aligned}
I_1^{\epsilon,\kappa} &= -\langle\langle u, \partial_t \Psi_{\epsilon,\kappa}\rangle\rangle = -\int_{Q_2} u \cdot \partial_t \Psi_{\epsilon,\kappa} dx dt = -\int_{\mathbb{R}_t \times \mathbb{R}^n_x} \overline{u} \cdot \partial_t \Psi_{\epsilon,\kappa} dx dt \\
&= -\int_{\mathbb{R}_t \times \mathbb{R}^n_x} \overline{u} \cdot \partial_t \left(\chi(t)\phi(x)(\overline{u(x,t)})^{\epsilon,\kappa}\right)^{\epsilon,\kappa} dx dt \\
&= \int_{\mathbb{R}_t \times \mathbb{R}^n_x} (\partial_t \overline{u})^{\epsilon,\kappa} \cdot \left(\chi(t)\phi(x)(\overline{u(x,t)})^{\epsilon,\kappa}\right) dx dt\,.
\end{aligned} \tag{3.38}$$

Since the support of $(\chi(t)\phi(x)\overline{u(x,t)})^{\epsilon,\kappa}$ is strictly contained in Q_2 , then by virtue of the formula (3.13) of Proposition 3.2 we have:

$$I_1^{\epsilon,\kappa} = \int_{\mathbb{R}_t \times \mathbb{R}^n_x} (\partial_t (\overline{u})^{\epsilon,\kappa}) \cdot (\overline{u(x,t)})^{\epsilon,\kappa} \cdot \chi(t)\phi(x) dx dt = -\langle\langle \frac{((\overline{u})^{\epsilon,\kappa})^2}{2}, \partial_t(\chi(t)\phi(x))\rangle\rangle\,. \tag{3.39}$$

By the same token, for the second term, one has:

$$\begin{aligned}
I_2^{\epsilon,\kappa} &= \langle\langle \nabla_x (u \otimes u), \Psi_{\epsilon,\kappa}\rangle\rangle = -\int_{\mathbb{R}_t}\int_{\mathbb{R}^n_x} \left[(\overline{u} \otimes \overline{u})^{\epsilon,\kappa} : \nabla_x \left(\chi(t)\phi(x)(\overline{u(x,t)})^{\epsilon,\kappa}\right)\right] dx dt \\
&= -\int_{\mathbb{R}_t} \chi(t) \int_{\mathbb{R}^n_x} \left[((\overline{u} \otimes \overline{u})^{\epsilon,\kappa} - (\overline{u})^{\epsilon,\kappa} \otimes (\overline{u})^{\epsilon,\kappa}) : \nabla_x(\phi(x)(\overline{u(x,t)})^{\epsilon,\kappa})\right] dx dt \\
&- \int_{\mathbb{R}_t} \chi(t) \int_{\mathbb{R}^n_x} \left[(\overline{u})^{\epsilon,\kappa} \otimes (\overline{u})^{\epsilon,\kappa} : \nabla_x(\phi(x)(\overline{u(x,t)})^{\epsilon,\kappa})\right] dx dt\,.
\end{aligned} \tag{3.40}$$

On the other hand, by a classical computation one has

$$\begin{aligned}
&\int_{\mathbb{R}_t} dt \int_{\mathbb{R}^n_x} \left((\overline{u})^{\epsilon,\kappa} \otimes (\overline{u})^{\epsilon,\kappa} \right) : \nabla_x (\chi(t)\phi(x)(\overline{u(x,t)})^{\epsilon,\kappa} \Big) \, dx \\
&= \int_{\mathbb{R}_t} dt \int_{\mathbb{R}^n_x} \frac{|(\overline{u})^{\epsilon,\kappa}|^2}{2} (\overline{u})^{\epsilon,\kappa} \cdot \nabla_x (\chi(t)\phi(x)) dx \\
&- \int_{\mathbb{R}_t} \chi(t) dt \int_{\mathbb{R}^n_x} \nabla_x \cdot (\overline{u(x,t)})^{\epsilon,\kappa} \frac{|(\overline{u})^{\epsilon,\kappa}|^2}{2} \phi(x) dx \,.
\end{aligned} \tag{3.41}$$

Since $\chi(t) \frac{|(\overline{u})^{\epsilon,\kappa}|^2}{2} \phi(x)$ is a smooth function, with support contained in Q_3, as above (with formula (3.13) of Proposition 3.2) one also has:

$$\begin{aligned}
&\int_{\mathbb{R}_t} dt \int_{\mathbb{R}^n_x} \nabla_x \cdot (\overline{u(x,t)})^{\epsilon,\kappa} \chi(t) \frac{|(\overline{u})^{\epsilon,\kappa}|^2}{2} (\phi(x)) dx = \\
&\int_{\mathbb{R}_t} dt \int_{\mathbb{R}^n_x} \left(\overline{\nabla_x \cdot u(x,t)} \right)^{\epsilon,\kappa} \chi(t) \frac{|(\overline{u})^{\epsilon,\kappa}|^2}{2} (\phi(x)) dx = 0 \,.
\end{aligned} \tag{3.42}$$

Thus, one eventually has:

$$\begin{aligned}
I_2^{\epsilon,\kappa} &= - \int_{\mathbb{R}_t} \chi(t) \int_{\mathbb{R}^n_x} \left[((\overline{u} \otimes \overline{u})^{\epsilon,\kappa} - (\overline{u})^{\epsilon,\kappa} \otimes (\overline{u})^{\epsilon,\kappa}) : \nabla_x (\phi(x)(\overline{u(x,t)})^{\epsilon,\kappa}) \right] dxdt \\
&- \int_{\mathbb{R}_t} \chi(t) \int_{\mathbb{R}^n_x} \frac{|(\overline{u})^{\epsilon,\kappa}|^2}{2} (\overline{u})^{\epsilon,\kappa} \cdot \nabla_x \phi(x) dxdt \,.
\end{aligned} \tag{3.43}$$

For the term $I_3^{\epsilon,\kappa}$ one uses the fact that (according to Proposition 3.3 and Corollary 3.4)

$$p \in L^{3/2}((t_1,t_2); C^{0,\alpha}(\overline{\Omega_2})) \tag{3.44}$$

and therefore one can also write:

$$\begin{aligned}
I_3^{\epsilon,\kappa} &= \langle\langle \nabla p, \Psi^{\epsilon,\kappa} \rangle\rangle = - \int_{\mathbb{R}_t \times \mathbb{R}^n_x} p \nabla_x \cdot \left(\chi(t)\phi(x)((\overline{u(x,t)})^{\epsilon,\kappa})^{\epsilon,\kappa} \right) dxdt \\
&= - \int_{\mathbb{R}_t} \chi(t) \int_{\mathbb{R}^n_x} (\overline{p})^{\epsilon,\kappa} (\overline{u(x,t)})^{\epsilon,\kappa} \cdot \nabla_x \phi(x) dxdt \\
&- \int_{\mathbb{R}_t} \chi(t) \int_{\mathbb{R}^n_x} (\overline{p})^{\epsilon,\kappa} \phi(x) \nabla_x \cdot (\overline{u(x,t)})^{\epsilon,\kappa} dxdt \,.
\end{aligned} \tag{3.45}$$

Eventually, as in the previous two derivations,

$$\begin{aligned}
&\int_{\mathbb{R}_t} \chi(t) \int_{\mathbb{R}^n_x} (\overline{p})^{\epsilon,\kappa} \phi(x) \nabla_x (\overline{u(x,t)})^{\epsilon,\kappa} dxdt \\
&= \int_{\mathbb{R}_t} \chi(t) \int_{\mathbb{R}^n_x} (\overline{p})^{\epsilon,\kappa} \phi(x) (\overline{\nabla_x \cdot u(x,t)})^{\epsilon,\kappa} dxdt = 0 \,.
\end{aligned} \tag{3.46}$$

Hence:

$$I_3^{\epsilon,\kappa} = -\int_{\mathbb{R}_t} \chi(t) \int_{\mathbb{R}^n_x} (\overline{p})^{\epsilon,\kappa} (\overline{u(x,t)})^{\epsilon,\kappa} \cdot \nabla_x \phi(x) dxdt \,. \tag{3.47}$$

With formulas (3.39), (3.43) and (3.47) for $I_i^{\epsilon,\kappa}$, with $i = 1,2$ and 3, one obtains that:

$$\begin{aligned}&\int_{Q_2} \left[\frac{((\overline{u})^{\epsilon,\kappa})^2}{2} \partial_t(\phi(x)\chi(t)) + \left(\frac{|(\overline{u})^{\epsilon,\kappa}|^2}{2} (\overline{u})^{\epsilon,\kappa} + (\overline{p})^{\epsilon,\kappa} (\overline{u(x,t)})^{\epsilon,\kappa} \right) \cdot \nabla_x(\phi(x)\chi(t)) \right] dxdt \\ &= \int_{\mathbb{R}_t} \chi(t) \int_{\mathbb{R}^n_x} \left[((\overline{u} \otimes \overline{u})^{\epsilon,\kappa} - (\overline{u})^{\epsilon,\kappa} \otimes (\overline{u})^{\epsilon,\kappa}) : \nabla_x(\phi(x) \overline{u(x,t)})^{\epsilon,\kappa} \right] dxdt\end{aligned} \tag{3.48}$$

Now use the fact that the support of $(t,x) \mapsto (\chi(t)\phi(x))$ is contained in $Q_3 \subset\subset (t_1,t_2) \times \Omega_2$ in conjunction with the following facts: With $C_k < \infty$ $(1 \le k \le 3)$,

$$\begin{aligned}&\text{by hypothesis} \quad \|u\|_{L^3((t_1,t_2);C^\alpha(\overline{\Omega_2}))} \le C_1 \,, \\ &\text{by Proposition 3.3} \quad \|p\|_{L^{3/2}((t_1,t_2);C^{0,\alpha}(\overline{\Omega_2}))} \le C_2 \,, \\ &\text{by Corollary 3.4} \quad \|\partial_t u\|_{L^{3/2}((t_1,t_2);C^{0,\alpha}(\overline{\Omega_2}))} \le C_3 \,;\end{aligned} \tag{3.49}$$

thus we can show, with the Aubin-Lions Theorem, first that, letting $\kappa \to 0$, in (3.48) one obtains the relation:

$$\begin{aligned}&\int_{Q_2} \frac{((\overline{u})^{\epsilon})^2}{2} \partial_t(\phi(x)\chi(t)) + \left(\frac{|(\overline{u})^{\epsilon}|^2}{2} (\overline{u})^{\epsilon} + (\overline{p})^{\epsilon} (\overline{u(x,t)})^{\epsilon} \right) \cdot \nabla_x(\phi(x)\chi(t)) dxdt \\ &= \int_{\mathbb{R}_t} \chi(t) \int_{\mathbb{R}^n_x} \left[((\overline{u} \otimes \overline{u})^{\epsilon} - (\overline{u})^{\epsilon} \otimes (\overline{u})^{\epsilon}) : \nabla(\phi(x) (\overline{u(x,t)})^{\epsilon}) \right] dxdt \,.\end{aligned} \tag{3.50}$$

Second, using again the Aubin-Lions Theorem on the left hand side of (3.50), one has:

$$\begin{aligned}&\langle\langle \frac{|u|^2}{2}, \partial_t \chi(t)\phi(x) \rangle\rangle + \langle\langle \left(\frac{|u|^2}{2} + p \right) u \cdot \nabla_x(\chi(t)\phi(x)) \rangle\rangle \\ &= \lim_{\epsilon \to 0} \int_{\mathbb{R}_t} \chi(t) \int_{\mathbb{R}^n_x} \left[((\overline{u} \otimes \overline{u})^{\epsilon} - (\overline{u})^{\epsilon} \otimes (\overline{u})^{\epsilon}) : \nabla_x(\phi(x) (\overline{u(x,t)})^{\epsilon}) \right] dxdt\end{aligned} \tag{3.51}$$

For the right-hand side of (3.51), one observes that it satisfies the estimate

$$\begin{aligned}&|\langle\langle \chi(t), ((\overline{u})^{\epsilon} \otimes (\overline{u})^{\epsilon} - (\overline{u \otimes u})^{\epsilon}) : \nabla(\phi(x)(\overline{u}^{\epsilon})) \rangle\rangle| \\ &\le \int_{t_1}^{t_2} \chi(t) |\langle ((\overline{u})^{\epsilon} \otimes (\overline{u})^{\epsilon} - (\overline{u \otimes u})^{\epsilon}), \nabla(\phi(x)(\overline{u}^{\epsilon})) \rangle| dt.\end{aligned} \tag{3.52}$$

For the right-hand side of (3.52), following ideas that by now have become classical (cf. [3], [10] or [13]), we obtain the estimate:

$$\begin{aligned}&\int_{t_1}^{t_2} \chi(t) |\langle ((\overline{u})^{\epsilon} \otimes (\overline{u})^{\epsilon} - (\overline{u \otimes u})^{\epsilon}), \nabla(\phi(x)(\overline{u}^{\epsilon})) \rangle| dt \\ &\qquad \le C(\chi,\phi) \epsilon^{3\alpha(\tilde{Q})-1} \int_{t_1}^{t_2} (\|u(\,.\,,t\|_{C^{0,\alpha(\tilde{Q})}(\tilde{\Omega})})^3 dt.\end{aligned} \tag{3.53}$$

Letting $\epsilon \to 0$ and using (3.53) we complete the proof of the theorem. The proof of (3.53) is based on a time independent estimate which, for the sake of completeness, is given below as the object of the following

Proposition 3.5. *Let $u \in C^{\alpha(\tilde\Omega)}(\overline{\tilde\Omega})$ with $\tilde\Omega \subset\subset \mathbb{R}^n_x$ and $\alpha > \frac{1}{3}$, $\phi \in \mathcal{D}(\tilde\Omega)$ a test function with support S_ϕ, and for $1 \le i \le 3$ select open sets*

$$S_\phi \subset\subset \Omega_3 \subset\subset \Omega_2 \subset\subset \Omega_1 \subset\subset \tilde\Omega, \tag{3.54}$$

a number $0 < \eta$, and a function $I_2 \in \mathcal{D}(\mathbb{R}^n_x)$ with the properties (3.8) and (3.9), so that in particular I_2 is equal to 1 in Ω_2 and equal to 0 outside Ω_3. If ρ_ϵ is a space mollifier, then for the functions

$$(\overline{u})^\epsilon = \rho_\epsilon \star \overline{u} = \rho_\epsilon \star (I_2 u) \quad \textit{and} \quad (\overline{u \otimes u})^\epsilon = \rho_\epsilon \star \overline{u \otimes u} = \rho_\epsilon \star (I_2(u \otimes u)) \tag{3.55}$$

one has, for $\epsilon \in (0, \frac{\eta}{2})$, the estimate:

$$|\langle((\overline{u})^\epsilon \otimes (\overline{u})^\epsilon - (\overline{u \otimes u})^\epsilon) : \nabla_x(\phi(x)(\overline{u})^\epsilon)\rangle| \le C(\phi)\epsilon^{3\alpha(\tilde\Omega)-1}(\|u\|_{C^{0,\alpha(\tilde\Omega)}(\overline{\tilde\Omega})})^3. \tag{3.56}$$

Proof. First use the formula

$$\begin{aligned}\nabla\cdot(\phi(x)(\overline{u}^\epsilon)) &= \nabla_x \int_{\mathbb{R}^n_x} \rho_\epsilon(x-y)\phi(x)I_2(y)u(y)dy \\ &= \nabla_x \int_{\mathbb{R}^n_x} (\rho_\epsilon(x-y)\phi(x) - \rho_\epsilon(y)\phi(y))I_2(y)u(y)dy\end{aligned} \tag{3.57}$$

to show that

$$|\nabla\cdot(\phi(x)(\overline{u}^\epsilon))| \le C(\phi)\epsilon^{\alpha(\tilde\Omega)-1}\|u\|_{C^{0,\alpha(\tilde\Omega)}(\overline{\tilde\Omega})}. \tag{3.58}$$

Second, thanks to the formula (3.18) of Proposition 3.2, in Ω_2 one has $(\overline{u\otimes u})^\epsilon = (\overline{u}\otimes\overline{u})^\epsilon$ and therefore with the elementary identity

$$\begin{aligned}&((\overline{u})^\epsilon \otimes (\overline{u})^\epsilon) - (\overline{u}\otimes\overline{u})^\epsilon = (\overline{u} - (\overline{u})^\epsilon) \otimes (\overline{u} - (\overline{u})^\epsilon) - \int (\delta_y\overline{u} \otimes \delta_y\overline{u})\rho_\epsilon(y)dy \\ &\text{with} \quad \delta_y\overline{u} = \overline{u}(x-y) - \overline{u}(x),\end{aligned} \tag{3.59}$$

one obtains

$$\|(\overline{u\otimes u})^\epsilon - ((\overline{u})^\epsilon \otimes (\overline{u})^\epsilon)\|_{L^\infty(\Omega_2)} \le C(\|u\|_{C^{0,\alpha(\tilde\Omega)}(\overline{\tilde\Omega})})^2\epsilon^{2\alpha(\tilde\Omega)}. \tag{3.60}$$

With (3.58) and (3.60) the proof is completed. □

Remark 3.6. *The formula (3.59) is an illustration of the similitude and difference existing between weak convergence, statistical theory of turbulence and regularization. With $\overline{u}_\epsilon$ denoting weak convergence or statistical theory one has*

$$((\overline{u})^\epsilon \otimes (\overline{u})^\epsilon) - (\overline{u}\otimes\overline{u})^\epsilon = (\overline{u} - (\overline{u})^\epsilon) \otimes (\overline{u} - (\overline{u})^\epsilon)\,, \tag{3.61}$$

where the right-hand side of (3.61) is the Reynolds stress tensor. On the other hand the presence of the term

$$\int (\delta_y \overline{u} \otimes \delta_y \overline{u}) \rho_\epsilon(y) dy,$$

in the formula (3.59) is due to the fact that instead of a weak limit of a family of solutions, it is an average of the same function $\overline{u}$ which is involved. This type of regularization is present in the original proof of Leray [23], in several type of α-models [6, 7, 9, 19], or eventually in turbulence modelling, for instance, in the contributions of Germano [17].

4 From local to global energy conservation

To consider the global conservation of energy, the impermeability condition will be used. Hence we assume that the boundary $\partial\Omega$ is a C^1 manifold with $\vec{n}(x)$ denoting the outward normal at any point of $\partial\Omega$ and we introduce the function and the set

$$d(x) = d(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y| \geq 0 \,, V_{\eta_0} = \{x \in \Omega \,, d(x) < \eta_0\}, \tag{4.1}$$

which have the following properties:

For $0 < \eta_0$ small enough $d(x)_{|V_{\eta_0}} \in C^1(\overline{V_{\eta_0}})$, for any $x \in V_{\eta_0}$ there exists a unique $\sigma(x) \in \partial\Omega$ such that $d(x)) = |x - \sigma(x)|$ and moreover one has:

$$\forall x \in V_{\eta_0} \qquad \nabla_x d(x) = -\vec{n}(\sigma(x)) \,. \tag{4.2}$$

Theorem 4.1. *Let (u, p) be a weak solution of the Euler equations in $(0, T) \times \Omega$, satisfying the following hypotheses:*

1. *For some $\eta_0 > 0$,*

$$p \in L^{3/2}((0, T) : H^{-\beta}(V_{\eta_0})) \quad \textit{with} \quad \beta < \infty \,, \tag{4.3a}$$

$$\lim_{\eta \to 0} \sup_{t \in (0,T)\, d(x) < \eta < \eta_0)} \left| \left(\left(\frac{|u|^2}{2} + p \right) u(x, t) \right) \cdot \vec{n}(\sigma(x)) \right| = 0 \,; \tag{4.3b}$$

2. *For every open set $\tilde{Q} = (t_1, t_2) \times \tilde{\Omega} \subset\subset (0, T) \times \Omega$ there exists $\alpha(\tilde{Q}) > 1/3$ such that u satisfies Hypothesis (3.3) of Theorem 3.1:*

$$\int_{t_1}^{t_2} \|u(., t)\|^3_{C^{0,\alpha(\tilde{Q})}(\overline{\tilde{\Omega}})} dt \leq M(\tilde{Q}) < \infty \,. \tag{4.4}$$

Then, (u, p) globally conserves the energy, i.e., it satisfies for any $0 \leq t_1 < t_2 \leq T$ the relation:

$$\|u(t_2)\|_{L^2(\Omega)} = \|u(t_1)\|_{L^2(\Omega)}.$$

Proof. Start with any open subset $\tilde{\Omega} \subset\subset \Omega$ such that

$$\Omega \setminus \tilde{\Omega} \subset\subset V_{\eta_0}, \tag{4.5}$$

and then introduce a smooth function $x \mapsto \theta(x) \in \mathcal{D}(\Omega)$ equal to 1 for $d(x) \geq \frac{\eta_0}{2}$. If Ω' is a domain with $\tilde{\Omega} \subset\subset \Omega' \subset\subset \Omega$ and $\Omega \setminus \Omega' \subset\subset V_{\eta_0}$, then from the property

$$u \otimes u \in L^{3/2}((t_1, t_2); C^{0,\alpha((t_1 t_2)\times\Omega')}(\overline{\Omega'})), \tag{4.6}$$

by Hypothesis (4.3a), (4.5), and Proposition 3.3 we deduce that

$$p \in L^{3/2}((0,T); C^{0,\alpha}(\overline{\tilde{\Omega}})). \tag{4.7}$$

Then, thanks to Theorem 3.1, one concludes that the relation

$$\frac{d}{dt}\langle\frac{|u|^2}{2}, \psi\rangle - \langle\left(\frac{|u|^2}{2} + p\right) u, \nabla_x \psi\rangle = 0 \tag{4.8}$$

holds for any $\psi \in \mathcal{D}(\tilde{\Omega})$ in the sense of $\mathcal{D}'(0,T)$. The estimates (4.6) and (4.7) show that the formula (4.8) remains also valid for test functions $\psi \in C^1_c(\tilde{\Omega})$ (i.e. with compact support in $\tilde{\Omega}$). Eventually, introduce a function $s \mapsto \phi(s)$ equal to 1 for $s > \frac{1}{2}$ and equal to 0 for $s < \frac{1}{4}$. With $0 < \tilde{\eta} < \eta_0$ one has:

$$\begin{aligned} &\psi_{\tilde{\eta}}(x) = \phi(\frac{d(x)}{\tilde{\eta}}) \in C^1(\Omega) \\ &\nabla_x \psi_{\tilde{\eta}}(x) = -\frac{1}{\tilde{\eta}}\phi'\left(\frac{d(x)}{\tilde{\eta}}\right)\vec{n}(\sigma(x)) \quad \text{for } \frac{\tilde{\eta}}{4} < d(x) < \frac{\tilde{\eta}}{2}\,;\ \text{otherwise} = 0\,. \end{aligned} \tag{4.9}$$

Setting $\psi = \psi_{\tilde{\eta}}$ in (4.8), one has

$$\begin{aligned} &\int_\Omega \frac{|u(t_2,x)|^2}{2}\phi\left(\frac{d(x)}{\tilde{\eta}}\right)dx - \int_\Omega \frac{|u(t_1,x)|^2}{2}\phi\left(\frac{d(x)}{\tilde{\eta}}\right)dx \\ &\qquad = -\int_{t_1}^{t_2}\int_\Omega \left(\frac{|u|^2}{2} + p\right) u(x,t)\cdot\vec{n}(\sigma(x))\frac{1}{\tilde{\eta}}\phi'(\frac{d(x)}{\tilde{\eta}})dxdt\,. \end{aligned} \tag{4.10}$$

With Hypothesis (4.3b), the result now follows from the Lebesgue Theorem by letting $\tilde{\eta} \to 0$. □

Remark 4.2. *The hypotheses (4.3a) and (4.3b) are satisfied if for some η_0 one has*

$$\begin{aligned} &(u,p) \in L^\infty((0,T)\times V_{\eta_0})) \quad \textit{and} \\ &u(x,t)\cdot\vec{n}(\hat{x}) \in C^0(\overline{(0,T)\times V_{\eta_0}}). \end{aligned}$$

In particular, all hypotheses of Theorem 4.1 are satisfied if $u \in L^\infty((0,T); C^{0,\alpha}(\overline{\Omega}))$ for some $\alpha > \frac{1}{3}$, so we re-obtain the result of [3].

Remark 4.3. *Local versions of energy/entropy conservation for compressible models were proved e.g. in [15, 12, 18]. With an argument similar as above, we expect it to be possible to show also the respective global versions of these results in bounded domains.*

5 An application to the vanishing viscosity limit

The above improvement is of interest as it gives a sufficient condition for non-anomalous energy dissipation in the zero viscosity limit which is not in contradiction with the presence of a Prandtl type boundary layer. This is the object of the very easy, but essential theorem below:

Theorem 5.1. *Let $u_\nu(x,t)$ be a family of Leray-Hopf weak solutions of the Navier-Stokes equations in $\mathbb{R}_t^+ \times \Omega$:*

$$\begin{aligned}&\partial_t u_\nu + (u_\nu \cdot \nabla_x) u_\nu - \nu \Delta u_\nu + \nabla p_\nu = 0, \qquad \nabla \cdot u_\nu = 0,\\ &u_\nu(t,x) = 0 \ \textit{on} \quad \mathbb{R}_t^+ \times \partial\Omega \quad \textit{and} \quad u_\nu(\cdot, x) = u_0 \in L^2(\Omega)\,.\end{aligned}$$

Assume that on $(0,T)\times\Omega$ the family u_ν satisfies the hypotheses of Theorem 4.1 uniformly in ν; more precisely:

1. *There exists, for some $\eta_0 > 0$, a neighborhood of $\partial\Omega$, $V_{\eta_0} = \{x \in \Omega\,, d(x) < \eta_0\}$, and $\beta < \infty$ (all being independent of ν) such that one has:*

$$\sup_\nu \|p_\nu\|_{L^{3/2}((0,T);H^{-\beta}(V_{\eta_0}))} < \infty;$$

2. *For any $\tilde{\Omega} \subset\subset \Omega$ there exists $\alpha = \alpha(\tilde{\Omega}) > \frac{1}{3}$ and a constant $M(\tilde{\Omega})$ such that for any $\nu > 0$ one has:*

$$\|u_\nu\|_{L^3((0,T);C^{0,\alpha}(\overline{\tilde{\Omega}}))} \le M(\tilde{\Omega}); \tag{5.1}$$

3. *For some $\gamma > 0$ one has, with a constant M_2 independent of ν,*

$$\forall (t,x) \in (0,T)\times V_\gamma : \quad |u_\nu(t,x)| + |p_\nu(t,x)| \le M_2,$$

where $V_\gamma = \{x \in \Omega : d(x,\partial\Omega) < \gamma\}$;

4. *There exist a neighborhood $\tilde{V}$ of $\partial\Omega$ and a ν-independent modulus of continuity $s \mapsto \omega(s)$, with $\lim_{s\to 0}\omega(s) = 0$, such that one has:*

$$|u_\nu(t,x)\cdot \vec{n}(\hat{x})| \le \omega(d(x))\,, \quad \textit{for every} \quad x \in \Omega \cap \tilde{V}\,. \tag{5.2}$$

Then (extracting a subsequence $\overline{\nu}$ if necessary) $u_{\overline{\nu}}$ converges weakly in $L^\infty((0,T);L^2(\Omega))$ to a function $\overline{u_{\overline{\nu}}}$ which is a weak solution of the Euler equations, and which also satisfies the hypotheses of Theorem 4.1, so that, in particular, there is no anomalous energy dissipation at the vanishing viscosity limit:*

$$\lim_{\overline{\nu}\to 0} \overline{\nu} \int_0^T \int_\Omega |\nabla_x u_{\overline{\nu}}(t,x)|^2 dx dt = 0. \tag{5.3}$$

Proof. Taking into account the above derivation, the proof is almost trivial. To show (5.3) one introduces

$$m = \limsup_{\nu \to 0} \int_0^T \int_\Omega |\nabla_x u_\nu(t,x)|^2 dxdt$$

and a sequence ν_i such that

$$m = \lim_{i\to\infty} \nu_i \int_0^T \int_\Omega |\nabla_x u_{\nu_i}(t,x)|^2 dxdt.$$

From such a sequence we extract yet another subsequence ν_j, such that $u_j = u_{\nu_j}$ converges to a limit $\overline{u_j}$ weakly* in $L^\infty(0,T); L^2(\Omega))$. From the Leray-Hopf energy inequality one now has:

$$\|u_j(T)\|_{L^2(\Omega)} - \|u_0\|_{L^2(\Omega)} + 2\nu_j \int_0^T \int_\Omega |\nabla_x u_j(t,x)|^2 dxdt \le 0$$

and weak convergence gives

$$\|\overline{u_j}(T)\|_{L^2(\Omega)} - \|u_0\|_{L^2(\Omega)} + 2m \le 0.$$

Then the above results concerning energy conservation give: $\|\overline{u_j}(T)\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)}$ and hence $m = 0$. ☐

Remark 5.2. *In the absence of boundaries the estimate on the pressure follows directly from the equation*

$$-\Delta p = \sum_{ij} \partial_{x_i}\left(u_j \partial_{x_j} u_i\right) = \sum_{ij} \partial_{x_i}\partial_{x_j}\left(u_i u_j\right). \tag{5.4}$$

This is no more the case in the presence of boundaries and some (very weak) hypothesis as in [3] or [26] seems to be both natural and compulsory. In [3] this was the object of Proposition 1.2. The necessity of this hypothesis is eventually confirmed by the analysis made in [11], where instead of such a hypothesis on the pressure, some uniform (with respect to the viscosity $\nu \to 0$) regularity of the flow u_ν is assumed (cf. for instance formula (3.2) in [11]). Then the authors obtain the convergence to an admissible weak solution. On the other hand they do not show that such solution may conserve the total energy or may coincide with a Lipschitz solution with the same initial data. This confirms the necessity of the hypothesis (5.2) as discussed in [4].

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