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On the Combinatorics of LLT Polynomials in Classical Lie Type

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by

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Abstract

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LLT polynomials were first introduced by Lascoux, Leclerc, and Thibon using the action of an affine Hecke algebra for S_n, and can be viewed as a q-generating function for both ribbon tableaux and tuples of semistandard Young tableaux. This definition has since been expanded to arbitrary Lie type, although with no combinatorial definition. We establish a combinatorial model for LLT polynomials in particular cases for Sp_{2n} and further conjecture a similar model for the orthogonal Lie types. Our definition uses a new object we call an out-in tableau as well as a correspondence between oscillating tableaux and symplectic tableaux that we use to give a proof of a Cauchy identity for Sp_{2n}.

Dedicated to IO and AO.

Contents

Acknowledgments

This work is the culmination of countless meetings with my advisor, Mark Haiman. So first and foremost, I would like to acknowledge Mark, not only for providing this area of research, but for his indefatigable patience at the local cafe. His willingness to explain the same concept n times over, with unending enjoyment, never ceased to amaze me.

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Lastly, thank you Mom and Dad for supporting me throughout my entire life, and to my favorite sister Rachel for always believing in me. The answer to your question "so what is your research on?" awaits.

Chapter 1

Introduction

LLT polynomials are a class of symmetric polynomials originally introduced by Lascoux, Leclerc and Thibon (and for whom the polynomials are eponymously named) in [57]. Although the original motivation for LLT polynomials was to study certain plethysm coefficients, they have since then enjoyed a wide range of applications, been given several equivalent definitions, and been shown to possess many astonishing properties. Of particular interest in this work is their extension to arbitrary Lie types, although before delving into this, we take the time to expound on the details of two attributes of LLT polynomials, namely their positivity and S_n-invariance.

The original definition of LLT polynomials comes via a relationship with the Fock space representation of the quantum affine Lie algebra $\mathcal{U}_q(\widehat{\mathfrak{sl}}_n)$. They are expressed as a sum over k-ribbon tableaux, weighted with a spin statistic which arises naturally in this representation [41, 57]. In particular, there are natural vertex operators on this space whose action on certain basis elements is captured by the LLT polynomials. The fact that LLT polynomials are symmetric follows from the commutativity of these vertex operators.

Later, Bylund and Haiman discovered an alternative way to model LLT polynomials, instead indexed by k-tuples of skew Young diagrams, weighted with an inversion statistic. The Bylund-Haiman model is described in [30], and the relationship between the two definitions uses the Stanton-White correspondence [85], which sends ribbon tableaux to tuples of semistandard Young tableaux. A purely combinatorial proof of the symmetry of LLT polynomials was given in [31] using these inversion variants, and recently another proof given using the integrability of a vertex model [1, 21].

As innocuous as it seems, the fact that LLT polynomials are symmetric has led to several important uses in combinatorics. In [30], the authors conjectured a combinatorial formula for the Frobenius character of the ring of diagonal coinvariants (known later as the shuffle conjecture). This Frobenius character is inherently symmetric, owing to a natural S_n-module structure for the diagonal coinvariant ring. The combinatorial formula was shown to expand into LLT polynomials, thus witnessing its symmetricity. LLT polynomials played a crucial part in the subsequent proof of the shuffle conjecture by Carlsson and Mellit [16] and also recently in another proof by Blasiak, Morse, Haiman and Seelinger [10]. A similar argument was used later in [31] to show that a proposed monomial expansion for Macdonald polynomials was indeed symmetric, and again in [6] and [27] for other related polynomials.

From the inversion definition of LLT polynomials (Definition 2.2.1), one can easily see that at $q = 1$, the polynomial devolves into a product of Schur polynomials, and hence the coefficient of a Schur polynomial s_λ in the LLT polynomial indexed by a tuple of partitions $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(k)})$ gives a q-analog $c^{\lambda}_{\mu^{(1)},\dots,\mu^{(k)}}(q)$ of the classical Littlewood-Richardson coefficients. It was shown in [60] that the polynomials $c^{\lambda}_{\mu^{(1)},\dots,\mu^{(k)}}(q)$ are equal to certain parabolic Kazhdan-Lusztig polynomials for an affine symmetric group. As it is known that these Kazhdan-Lusztig polynomials have non-negative coefficients, the result implies that LLT polynomials in this case are Schur-positive, that is they expand in the Schur basis with coefficients in $\mathbb{N}[q]$. This argument was extended by Grojnowski and Haiman [28] to arbitrary skew partitions, and moreover generalized to any complex reductive Lie group. Together with the combinatorial expansion of Macdonald polynomials into LLT polynomials, this gives another proof of Macdonald's famed positivity conjecture, which states that the Macdonald polynomials $H_\mu(x; q, t)$ are Schur-positive. The original proof is due to Haiman [34] by means of the geometry of Hilbert schemes.

It is worth noting that all of the mentioned proofs of positivity build upon heavy geometric machinery, and these statements still lack combinatorial proofs. Some work has been made along these lines, to name a few [38, 64, 73, 90]. There have also been several lines of work towards positivity when LLT polynomials are expanded into elementary symmetric polynomials [2, 4, 5].

The primary concern in this work is to extend these combinatorial descriptions of LLT polynomials to the other classical Lie types, with a particular focus on the case of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$. We mention that independently from [28], another definition of LLT polynomials in classical Lie type was given by Lecouvey in [62], although the two definitions were shown to be equivalent in [55].

To elaborate on the work in [28], the authors associate to any complex reductive Lie group G its *LLT series* (Definition 4.6.1), which depends on a Levi subgroup L and two strictly dominant weights β, γ for L. The LLT series is an infinite formal series of irreducible characters of G, with coefficients they show to be in $\mathbb{N}[q]$. When $G = \mathrm{GL}_n$, they show that a truncation of their LLT series to polynomial characters coincides with the combinatorial LLT polynomials. We detail this in Proposition 5.3.1, although we use here a new formulation of LLT polynomials which we call a *coinversion* LLT polynomial (Definition 2.2.3). These serve as generating functions for k-tuples of semistandard Young tableaux, weighted with a coinversion statistic. The definition is easily seen to be equivalent to the inversion definition after inverting q and multiplying by a suitable power of q. This new formulation was detailed to the author in personal correspondence with M. Haiman, and is also developed in the first of a recent series of publications [10]. In the sections below, we strive to follow the notation therein, but deviate slightly in order to give a self-contained treatment of the material.

We can summarize our problem at hand as follows.

Problem 1. *Let G be a complex reductive Lie group with a Levi subgroup L and associated*

LLT series $\mathcal{L}^G_{L,\beta,\gamma}(x;q)$ for β,γ strictly dominant weights for L. We wish to describe a generating set of combinatorial objects T, along with a statistic $\operatorname{stat} T$*, for which we have an identity*

$$\mathcal{L}^G_{L,\beta,\gamma}(x;q)\big|_{\mathrm{pol}} = \sum_T q^{\operatorname{stat} T} x^T \tag{1.1}$$

where pol *denotes some truncation of a formal series to finitely many terms.*

We have the following two main results:

1. When $G = \mathrm{Sp}_{2n}$, $L = T$ is a maximal torus, and $|\beta_i| + |\gamma_i|$ is sufficiently far from 0 for all i, the sum is over a new set of objects we call *out-in tableaux* (Definition 5.3.1), which extends the notion of an oscillating tableau. The statistic in question is an inversion of an out-in tableau (Definition 5.3.2) and we define symplectic polynomial truncation (Definition 5.3.1) in this case.

2. When $G = \mathrm{Sp}_{2n}$ and $q = 1$, the sum is over skew symplectic tableaux, as defined in [53].

Let us describe briefly how these results are obtained. For G a Lie group of classical type, we let e_k denote the character of the k^{th} exterior power $\Lambda^k(V)$ of the defining representation V of G. After making the usual identifications of the weight ring of G with a Laurent polynomial ring in the variables $x = \{x_1, \ldots, x_n\}$ and their inverses, e_k is the k^{th} elementary symmetric polynomial in a subset of the variables x, x^{-1}, and 1.

The desired expression (1.1) for when the Levi is the torus hinges on two key results. We start with deriving a combinatorial expression for the matrix coefficients $c^{(k)}_{\beta,\gamma}(q)$ in the expansion

$$e_k E_\gamma(x;q) = \sum_\beta c^{(k)}_{\beta,\gamma}(q) E_\beta(x;q) \tag{1.2}$$

where the polynomials $E_\gamma(x;q)$ are non-symmetric Hall-Littlewood polynomials, appearing in [42] as specializations of non-symmetric Macdonald polynomials. We accomplish this for Sp_{2n} (Proposition 5.3.2) and SO_{2n+1} (Proposition 5.4.1).

Secondly, when $G = \mathrm{Sp}_{2n}$, we relate the coefficients $c^{(k)}_{\beta,\gamma}(q)$ to the coefficient of a monomial in $\mathcal{L}^G_{T,\beta,\gamma}(x;q)$ by way of a Cauchy identity (Corollary 3.3.2). We prove the Cauchy identity combinatorially with the use of a new bijection between semistandard oscillating tableaux and King symplectic tableaux (Theorems 3.2.1, 3.2.2). In [7], Berele modified the RSK insertion algorithm to give a combinatorial proof of Schur-Weyl duality for Sp_{2n}; we use our bijection to extend Berele's insertion algorithm and prove the Cauchy identity, much in the same way RSK extends the Robinson-Schensted insertion scheme. We mention that this bijection was also independently discovered in [63] to exhibit a crystal structure on King symplectic tableaux.

We note that we lack such a Cauchy identity for the orthogonal Lie groups, which is why our results above only hold for Sp_{2n}. However, as a corollary to (1.2) that holds in both the symplectic and orthogonal types, we arrive at a new Pieri rule for Demazure characters (Corollary 5.3.4), owing to the fact that the non-symmetric Hall-Littlewood polynomials specialize to Demazure characters.

For the case when $q = 1$ and the Levi is arbitrary, we compute the coefficient in (1.2) by a different means. We recast the product on the left hand side as a certain action on the *abacus* of γ, a tool used in GL_n to visualize the combinatorics of k-cores and k-quotients. Here, we employ a Lie-theoretic perspective of a k-core and k-quotient, deriving from the work in [28].

We take the time now to outline more precisely the contents of this work. In Chapter 2, after reviewing some preliminary tableau combinatorics, we define the combinatorial LLT polynomials in all their guises (spin, inversion, and coinversion) and record their relationships to each other and to other symmetric functions. In Chapter 3 we introduce our bijection between symplectic tableaux and semistandard oscillating tableaux, and use it to prove character identities for Sp_{2n}. In Chapter 4 we first review the theory of affine root systems, extended affine Weyl groups, and extended affine Hecke algebras, the last of which will play an essential role in the definition of the LLT series in general Lie type. We also take the time here to give an overview of how the combinatorics of k-cores and k-quotients relate to the action of the extended affine Weyl group of GL_n. In Chapter 5 we prove the results stated prior, and also detail work towards the orthogonal Lie types. We end with concluding remarks and conjectures in Chapter 6. We remark that much of the preliminary material throughout this work can be found in various classical and recent texts, for example our presentation of general type LLT polynomials follows closely that of [28] and [10]. We apologize to the original authors for any undue repetition, and we hope that our verbosity here is offset with a more illuminating explication.

Chapter 2

Tableau Combinatorics

This chapter provides the reader with the necessary background on the combinatorics of partitions, tableaux, and symmetric functions, along with their relationships to the representation theory of GL_n. Most of this material can be found e.g. in either of the comprehensive resources [72, 84]. We also define combinatorial LLT polynomials, the objects at the heart of our work. We provide the definition of these ever-important polynomials in all their guises, namely their spin, inversion, and newly presented coinversion formulations. As the literature on LLT polynomials is much too vast to cover in this chapter, our presentation will necessarily be an abridged form. We refer the reader to the original source [57] for a more thorough treatment of the spin definition, and to [30] for the inversion definition.

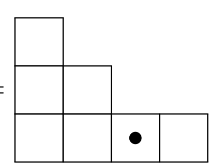
2.1 Partitions and tableaux

Fix n and let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ be a partition with n parts. Note that we consider our partitions to have a fixed number of parts, but allow for the possibility of parts of zero. We let the **length** $\ell(\lambda)$ be the number of non-zero parts of λ. The **size** of λ is $|\lambda| = \sum_i \lambda_i$ and if $|\lambda| = m$, we write $\lambda \vdash m$ to mean λ is a partition of m. We associate to λ its Young (or Ferrers) diagram $D(\lambda) \subseteq \mathbb{Z} \times \mathbb{Z}$, given as

$$D(\lambda) = \{(i, j) \mid 1 \leq i \leq \ell(\lambda),\ 1 \leq j \leq \lambda_i\}$$

We draw our diagrams in French notation, in the first quadrant, such as below

$\lambda = (4, 2, 1), \qquad D(\lambda) =$



We refer to the elements in $D(\lambda)$ as **cells** or **boxes**. The cell labelled above has coordinates (1,3). Given partitions λ, μ, the **skew diagram** $D(\lambda/\mu)$ is the set of cells contained in $D(\lambda)$ but not in $D(\mu)$.

In what follows we will use λ and $D(\lambda)$ interchangeably, when it will not cause confusion. We will also make frequent use of the staircase partition $\rho_n := (n-1, \ldots, 1, 0)$, which has the property that $\lambda + \rho_n$ has distinct parts, for any partition λ with n non-negative parts. For ease of notation, we drop the subscript when it is clear from context.

The **transpose** of λ, denoted λ', is the partition whose diagram is the reflection of $D(\lambda)$ across the line $y = x$. The **dominance order** is the partial order on partitions of a fixed size defined by

$$\lambda \leq \mu \iff \lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i, \quad \text{for all } i \tag{2.1}$$

We have that $\lambda \leq \mu$ iff $\mu' \leq \lambda'$.

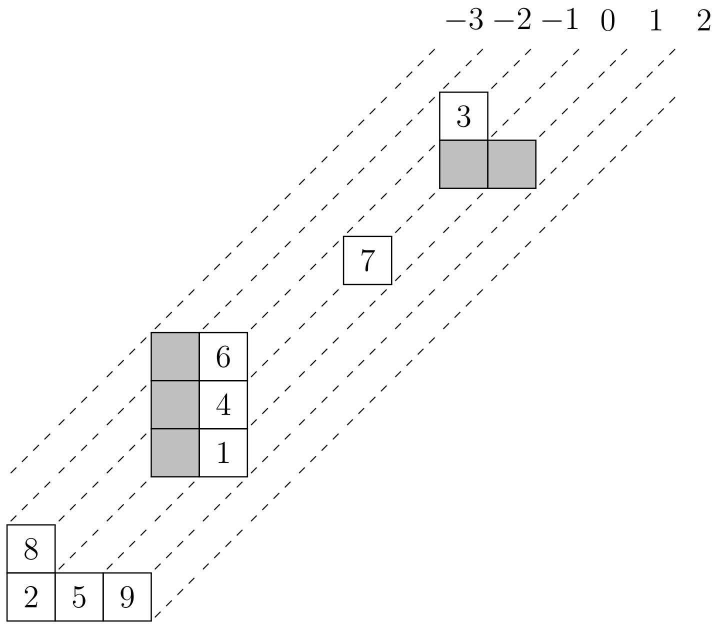
A **Young tableau** T is a diagram $D(\lambda)$ together with a **filling** $\sigma : D(\lambda) \to \mathcal{A}$ for some totally ordered alphabet $\mathcal{A}$. We think of T as placing the value $\sigma(u)$ in each cell $u \in D(\lambda)$. We will almost always take $\mathcal{A}$ to be $\mathbb{Z}$ or $\mathbb{Z}_+$, perhaps though with a non-standard ordering. A **semistandard Young tableau** is a tableau in which values along each row are weakly increasing from left to right, and the values along each column strictly increasing from bottom to top (all in French notation). When the alphabet is taken to be $\mathbb{Z}_+$, the **weight** of a tableau T is the tuple $(\mu_1, \mu_2, \ldots)$, where μ_i is the number of cells with value i. We let $\mathrm{SSYT}(\lambda)$ denote the set of semistandard Young tableaux of shape λ and $\mathrm{SSYT}(\lambda, \mu)$ denote the subset with weight μ.

The **content** of a cell $u = (i, j)$ in row i and column j of any Young diagram is $c(u) = j - i$. Viewing a Young diagram as a subset of $\mathbb{Z} \times \mathbb{Z}$, we define a **(skew) shape with contents** to be an equivalence class of a (skew) Young diagram up to content-preserving translations. Given a tuple $\boldsymbol{\beta}/\boldsymbol{\gamma} = (\beta^{(1)}/\gamma^{(1)}, \ldots, \beta^{(k)}/\gamma^{(k)})$ of skew partitions, define a semistandard Young tableau T of shape $\boldsymbol{\beta}/\boldsymbol{\gamma}$ to be a semistandard Young tableau on each $\beta^{(j)}/\gamma^{(j)}$, that is,

$$\mathrm{SSYT}(\boldsymbol{\beta}/\boldsymbol{\gamma}) = \mathrm{SSYT}(\beta^{(1)}/\gamma^{(1)}) \times \cdots \times \mathrm{SSYT}(\beta^{(k)}/\gamma^{(k)})$$

We can picture this as placing the Young diagrams diagonally "on content lines" with the first shape in the South-West direction and the last shape in the North-East direction. See Example 2.1.1 below.

Example 2.1.1. Let $\boldsymbol{\beta}/\boldsymbol{\gamma} = ((3,1)/\varnothing, (2,2,2)/(1,1,1), (1)/\varnothing, (2,1)/(2))$. The top row labels the contents of each line.

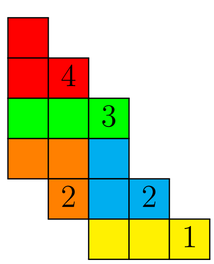


A **ribbon** is a connected skew diagram with no 2×2 box. If the ribbon has k boxes, we say it is a k-ribbon. We label a ribbon in a shape by its **tail**, which is the cell in the ribbon with maximal content (i.e. the bottom-right box in French notation). The content of a ribbon will be the content of its tail, and the **residue** of a k-ribbon will be its content modulo k. We define a **horizontal k-ribbon strip** to be a shape tileable by k-ribbons such that the tail of every ribbon is in the bottom of its column (this latter condition is what makes the ribbon strip "horizontal"). A **semistandard ribbon tableau** of shape λ/μ is a sequence

$$\mu = \nu^{(0)} \subseteq \nu^{(1)} \subseteq \nu^{(2)} \cdots \subseteq \nu^{(r)} = \lambda$$

such that each $\nu^{(i)}/\nu^{(i-1)}$ is a horizontal ribbon strip. The **weight** of such a semistandard ribbon tableau is the composition $(\alpha_1, \dots, \alpha_r)$ where α_i is the number of k-ribbons in $\nu^{(i)}/\nu^{(i-1)}$. Often we think of the ribbons in the i^{th} horizontal strip as being labelled i, as in Figure 2.1.2. We will let $\mathrm{SSRT}_k(\lambda)$ denote set of semistandard k-ribbon tableaux of shape λ, and similarly for $\mathrm{SSRT}_k(\lambda, \mu)$.

Example 2.1.2. Below is a skew semistandard 3-ribbon tableau of shape $(5, 4, 3, 3, 2, 1)/(2, 1)$ with the ribbons colored and labelled.



k-cores and k-quotients

The notion of a k-core and k-quotient of a partition was introduced long ago, as a generalization of integer quotients and remainders. Perhaps it is because the concepts are so old that in the present day it seems as if every combinatorialist has a different way to construct and visualize these objects, each with their own set of conventions and choices. Following [28], we add to this milieu of constructions by employing a nonstandard definition of a k-quotient; however we still provide the historical definitions for completeness, because we will see later in Section 4.3 how these concepts can be realized in a Lie-theoretic manner. The uninitiated reader is welcome to refer to [72, Ch. I.1] for a thorough primer on the combinatorics of k-cores, k-quotients, and abaci.

A diagram λ is a **k-core** if there is no diagram μ such that λ/μ is a k-ribbon. If we start removing k-ribbons from a partition λ until it is no longer possible, at each step maintaining a partition shape, what remains is the unique partition, called the k-core of λ, denoted $\mathrm{core}_k(\lambda)$. For example, in Figure 2.1 below, we see 2 ways to remove k-ribbons, both resulting in the same k-core. The fact that $\mathrm{core}_k(\lambda)$ does not depend on the order of

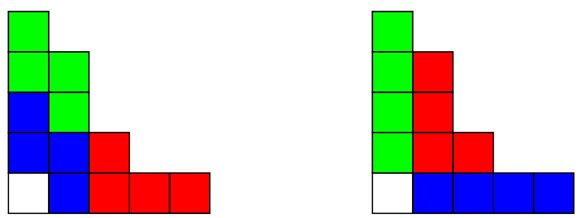


Figure 2.1: Two ways to remove 4-ribbons from $\lambda = (5, 3, 2, 2, 1)$. The order is given by first removing red, then green, then blue. In both cases, what remains is $\mathrm{core}_4(\lambda) = (1)$.

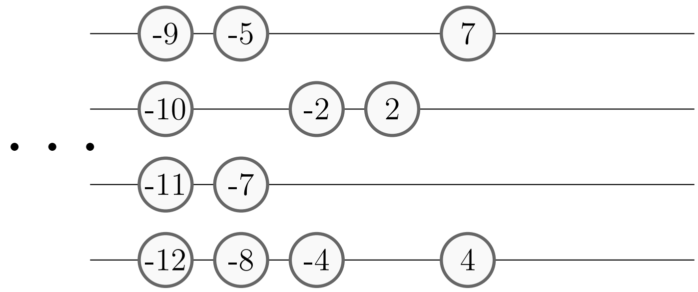
the ribbons removed can be seen more easily with the aid of an **abacus**, also known as a Maya diagram. An abacus for us will be drawn with beads on k horizontal lines, known as **rungs**, that represent residue classes mod k.[1] We adopt certain conventions, among those being:

- We will read left to right, bottom to top, so that the bottom rung will correspond to numbers with residue 0 mod k.
- We will always pad our abaci with infinitely many negative beads and will neglect drawing large enough negative beads.

Fix a partition λ and integer $k \geq 1$. Let $\delta = (-1, -2, \ldots)$ and consider λ padded with infinitely many zeros. We associate to λ the abacus with k rungs and beads at positions $\lambda + \delta$ as in Example 2.1.3 below.

[1] Others might draw an abacus with vertical and horizontal lines, which are placed by tracing the boundary of the partition.

Example 2.1.3. Below is the abacus for the partition $\lambda = (8, 6, 5, 2, 1, 1, 0)$ with $k = 4$.



We make the following observation, without proof.

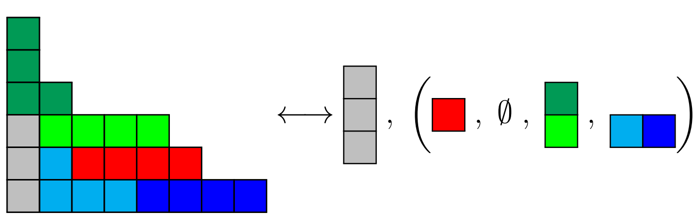
Observation 2.1.1. Removing a k-ribbon of residue r from λ is equivalent to subtracting k from some part of $\lambda + \delta$ with residue r, and then rearranging in decreasing order. On the abacus, this results in moving a bead on the r^{th} rung one position left on its rung. Consequently, removing all k-ribbons from λ is equivalent to left-justifying its abacus.

Remark 2.1.1. We note that effect of adding δ_n is to make λ a strict partition. Any choice of such a δ would suffice. We've chosen this specific δ_n so that the r^{th} rung aligns with the ribbons of residue r in any ribbon tiling of λ.

Now, notice that given a k-core $\nu = \mathrm{core}_k(\lambda)$, we can recover λ if we know the number of times each bead on a rung was moved. Encoding this information on each rung amounts to constructing what is known as the k-quotient. Historically, the k-quotient of a partition λ is a k-tuple of partitions $(\lambda^{(1)}, \ldots, \lambda^{(k)})$ in which $\lambda^{(r)}$ is read off from the abacus by looking at the $(r-1)^{th}$ rung and for each bead, counting the number of *gaps* to the left of that bead, i.e. the number of times one must move each bead to left-justify the abacus. In this way, one has the correspondence

$$\{\text{Partitions}\} \longleftrightarrow \{k\text{-cores}\} \times \{k\text{-tuples of partitions}\} \tag{2.2}$$

Example 2.1.4. Continuing with our choice of λ in Example 2.1.3, we see that its k-core is (1,1,1) and its k-quotient in the terminology of [72] is $(1, \emptyset, (1, 1), (2))$, since e.g. in the third rung, one has to move the two beads left one unit to get a left-justified abacus, whence the third partition in the quotient is (1,1). We pick a ribbon tiling of λ and color the ribbons so that the ribbons with residue r correspond to boxes in the r^{th} partition in the quotient.



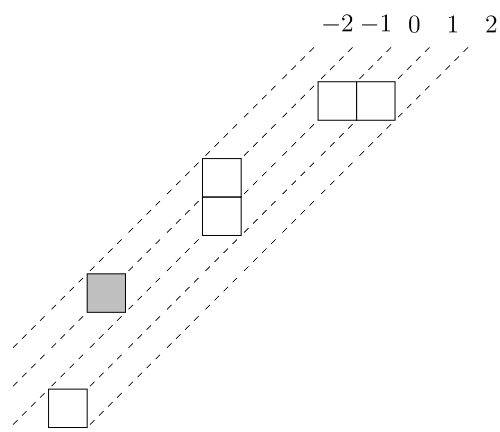
We can make this a bit cleaner by modifying the construction of the k-quotient so that it absorbs the data from the k-core. More precisely, note that given a k-core ν, there are exactly k partitions μ such that μ/ν is a k-ribbon, each with a distinct content modulo k. These partitions μ are exactly the partitions one gets by selecting a rung on the abacus representing ν and moving the rightmost bead one unit to the right. We will encode the k-core by considering the k-quotient as a tuple of skew shapes with contents, whose origins are placed on specific content lines determined by the contents of these k partitions.

Definition 2.1.1. Let λ be a partition with $\mathrm{core}_k(\lambda) = \nu$. Let $\{c_1, \ldots, c_k\}$ be the contents of the distinct k-ribbons that can be added to ν, ordered so that $c_r = q_r k + r - 1$. The **k-quotient** of λ, denoted $\mathrm{quot}_k(\lambda)$ is the tuple of skew shapes with contents $\boldsymbol{\beta} = (\beta^{(1)}, \ldots, \beta^{(k)})$ such that

(i) each $\beta^{(r)}$ is a partition diagram, translated so that the box at the origin has content q_r,

(ii) the multiset of integers $c(x)k + r - 1$ for $x \in \boldsymbol{\beta}$ and $1 \le r \le k$ is equal to the multiset of contents of the ribbons in any ribbon tiling of λ/ν.

Example 2.1.5. Continuing again with λ in the previous example, the k-quotient $\mathrm{quot}_k(\lambda)$ is drawn below



where we have shaded the empty partition only to denote on which content line it resides.

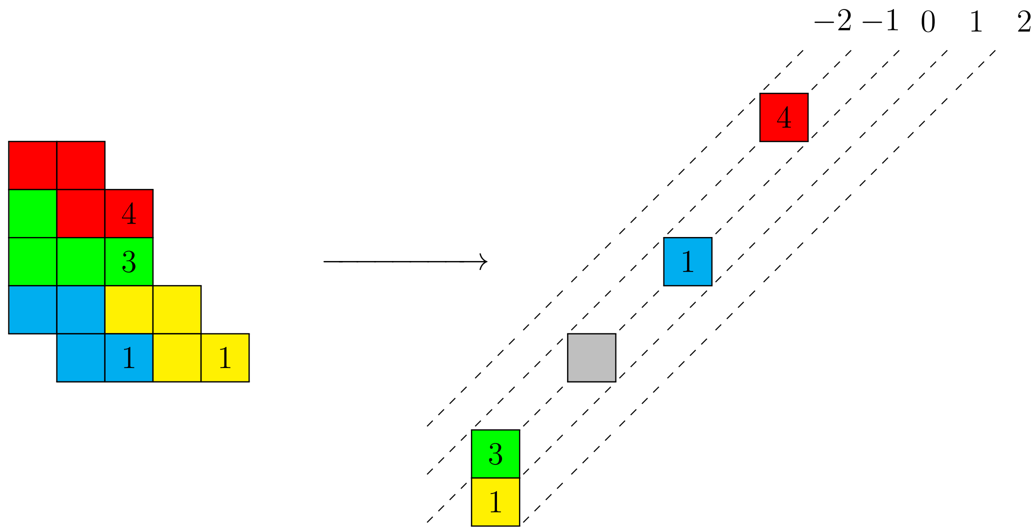
Definition 2.1.1 can be extended to any skew shape with contents. Indeed, if λ/μ can be tiled by k-ribbons, then $\mathrm{core}_k(\lambda) = \mathrm{core}_k(\mu)$ and $\mathrm{quot}_k(\mu) \subseteq \mathrm{quot}_k(\lambda)$. As such, we can define $\mathrm{quot}_k(\lambda/\mu) = (\beta^{(1)}/\gamma^{(1)}, \ldots, \beta^{(k)}/\gamma^{(k)})$, where $\boldsymbol{\beta} = \mathrm{quot}_k(\lambda)$ and $\boldsymbol{\gamma} = \mathrm{quot}_k(\mu)$.

We are now in a position to spruce up (2.2) from partitions to tableaux. The Stanton-White correspondence [85] extends this correspondence to a weight-preserving bijection

$$\mathrm{quot}_k : \mathrm{SSRT}_k(\lambda/\mu) \mapsto \mathrm{SSYT}(\mathrm{quot}_k(\lambda/\mu)) \tag{2.3}$$

defined so that if T is a semistandard k-ribbon tableau, and $\mathrm{quot}_k(T) = (T^{(1)}, \ldots, T^{(k)})$, then a ribbon in T with content $pk + r - 1$ and label i will correspond to a box in $T^{(r)}$ with content p and label i.

Example 2.1.6. On the left we give a semistandard 4−ribbon tableau T of shape $\lambda = (5,4,3,3,2)/(1)$. The first horizontal strip consists of the blue and yellow ribbons labelled 1, the second is empty, the third consists of the green ribbon labelled 3, and the fourth consists of the red ribbon labelled 4. The corresponding 4-quotient is given on the right. In the quotient, the shaded square denotes the empty partition.



2.2 Combinatorial LLT polynomials

In this section we review the theory of LLT polynomials and set notation.

Let $T = (T^{(1)}, \dots, T^{(k)})$ be a SSYT on a tuple of skew shapes with contents. Given a cell u in $T^{(r)}$, we define the **adjusted content** to be $\tilde{c}(u) = c(u)k + r - 1$. We choose the reading order on cells so that their adjusted contents increase. In other words, we read from smallest to largest content line, moving along a fixed content line from the SW to NE direction.

We say two cells **attack** each other if their adjusted contents differ by less than k and are not equal. In other words, two cells attack each other if either (1) they are on the same content line in different shapes, or (2) they are on adjacent content lines, with the cell on the larger content line in an earlier shape. We define an **attacking inversion** of T to be a pair of attacking boxes with different entries in which the larger entry precedes the smaller in reading order.

Definition 2.2.1. Let $\boldsymbol{\beta}/\boldsymbol{\gamma}$ be a tuple of skew partitions. The **inversion LLT polynomial** is the generating function

$$\mathcal{G}_{\boldsymbol{\beta}/\boldsymbol{\gamma}}(x;q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\beta}/\boldsymbol{\gamma})} q^{\mathrm{inv}(T)} x^T \tag{2.4}$$

where $\mathrm{inv}(T)$ is the number of attacking inversions of T.

As is the case for Macdonald polynomials, the number of attacking inversions can be reformulated as the number of inversion triples, which we now define. Given a tuple $\boldsymbol{\beta}/\boldsymbol{\gamma}$ of skew partitions, we say that three cells $u, v, w \in \mathbb{Z} \times \mathbb{Z}$ form a **triple** of $\boldsymbol{\beta}/\boldsymbol{\gamma}$ if (i) $v \in \boldsymbol{\beta}/\boldsymbol{\gamma}$, (ii) they are situated as below

$$\begin{array}{ccc} & \boxed{u} & \boxed{w} \\ & & \\ \boxed{v} & & \end{array} \tag{2.5}$$

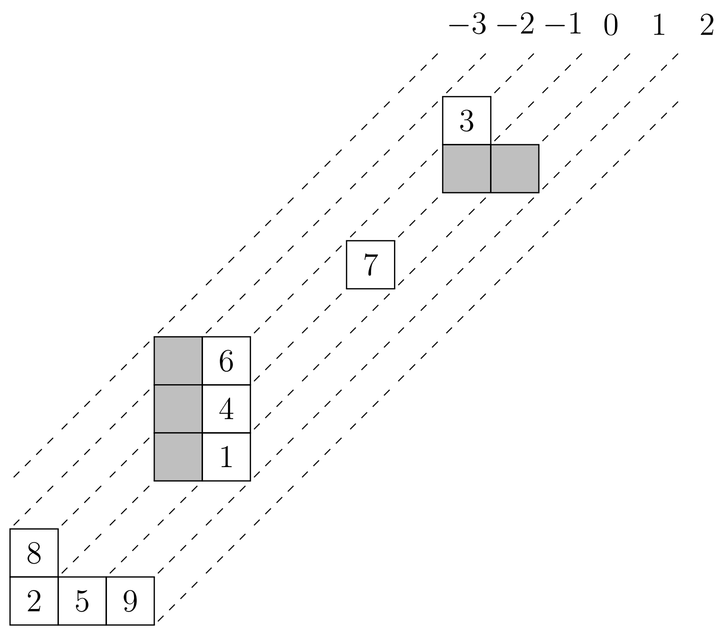
namely with v and w on the same content line and w in a later shape, and u on a content line one smaller, in the same row as w, and (iii) if u, w are in row r of $\beta^{(j)}/\gamma^{(j)}$, then u and w must be between the cells $(r, \gamma_r^{(j)} - 1), (r, \beta_r^{(j)} + 1)$, inclusive. It is important to note that while v must be a cell in $\boldsymbol{\beta}/\boldsymbol{\gamma}$, we allow the cells u and w to not be in any of the skew shapes, in which case u must be at the end of some row in $\boldsymbol{\gamma}$ and w must be the cell directly to the right of the end of some row in $\boldsymbol{\beta}$.

Definition 2.2.2. Let $\boldsymbol{\beta}/\boldsymbol{\gamma}$ be a tuple of skew partitions and let $T \in \mathrm{SSYT}(\boldsymbol{\beta}/\boldsymbol{\gamma})$. Let a, b, c be the entries in the cells of a triple (u, v, w), where we set $a = 0$ and $c = \infty$ if the respective cell is not in $\boldsymbol{\beta}/\boldsymbol{\gamma}$. Given the triple of entries

$$\begin{array}{ccc} & \boxed{a} & \boxed{c} \\ & & \\ \boxed{b} & & \end{array} \tag{2.6}$$

we say this is a **coinversion triple** of T if $a \leq b \leq c$. Otherwise, we have $b < a \leq c$ or $a \leq c < b$, and we say the triple of entries is an **inversion triple**.

Example 2.2.1. There are 7 coinversion triples below: (0, 2, 4), (0, 2, 7), (3,4,∞), (0,4,7), (4,5,∞), (1,9,∞), and (0,9,∞).



We note that Definition 2.2.2, and that of a triple, depends not merely on the tuple of skew partitions $\boldsymbol{\beta}/\boldsymbol{\gamma}$, but on the individual tuples of partitions $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$. Indeed, if in Example 2.2.1, we made the superficial change in the third skew shape from (1)/(0) to (2,2)/(2,1), then we would introduce another coinversion triple $(0, 9, \infty)$. Likewise if we consider the third shape being instead (1,0)/(0,0), then we introduce the coinversion triples $(0, 8, \infty)$ and $(0, 6, \infty)$. It's easily seen that any extra coinversion triples present are independent of the filling T.

Definition 2.2.3. Let $\boldsymbol{\beta}/\boldsymbol{\gamma}$ be a tuple of skew partitions. The **coinversion LLT polynomial** is the generating function

$$\mathcal{L}_{\boldsymbol{\beta}/\boldsymbol{\gamma}}(x;q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\beta}/\boldsymbol{\gamma})} q^{\mathrm{coinv}(T)} x^T \tag{2.7}$$

where $\mathrm{coinv}(T)$ is the number of coinversion triples of T.

In light of the preceding remarks, we note that if $\boldsymbol{\beta}/\boldsymbol{\gamma}$ and $\boldsymbol{\beta}'/\boldsymbol{\gamma}'$ are two representations of the same skew shapes, then their coinversion LLT polynomials differ by an overall power of q.

Note that in a semistandard filling T on some tuple of skew partitions, a pair of attacking entries forms an inversion if and only if they are in a (unique) inversion triple. Indeed, if $b < a \leq c$, then (a, b) is an attacking inversion, and likewise if $a \leq c < b$, then (b, c) is an attacking inversion. Hence, we have the identity

$$\mathcal{L}_{\boldsymbol{\beta}/\boldsymbol{\gamma}}(X;q) = q^m \mathcal{G}_{\boldsymbol{\beta}/\boldsymbol{\gamma}}(X;q^{-1}) \tag{2.8}$$

where $m = m(\boldsymbol{\beta}/\boldsymbol{\gamma})$ is the total number of triples in $\boldsymbol{\beta}/\boldsymbol{\gamma}$.

Remark 2.2.1. A simplified version of Definition 2.2.3, in which each shape in $\boldsymbol{\beta}/\boldsymbol{\gamma}$ consists of a single row, can be found in [10], in which (2.8) is essentially Proposition 4.5.3. The general case is in [11]. In both, the coinversion LLT polynomials are first defined, via the action of a Hecke algebra, as a polynomial truncation of a certain formal power series. It is then shown that this algebraic definition results in the combinatorial definition above. We will expound on this later in Chapter 5 when we discuss combinatorial formulas for LLT polynomials in general Lie type.

An explicit formula for $m(\boldsymbol{\beta}/\boldsymbol{\gamma})$ can be given when $\boldsymbol{\gamma}$ is empty.

Proposition 2.2.1. *Let $\boldsymbol{\beta}$ be a tuple of partitions. Then,*

$$m(\boldsymbol{\beta}) = \#\{a<b,\ i,j \mid 0 \le \beta_j^{(b)} - j + i < \beta_i^{(a)}\} + \sum_{\substack{a<b\\ i,j}} \max(\min(\beta_i^{(a)} - i, \beta_j^{(b)} - j) + \min(i,j), 0) \tag{2.9}$$

Proof. We count triples by their cell labelled v in (2.5), as this cell is always in the shape $\boldsymbol{\beta}$. Fix a cell $v = (i,\ell) \in \beta^{(a)}$. If there is a triple (u,v,w), then u,w must lie in some (or adjacent to some) $\beta^{(b)}$ for $b>a$. For each row $\beta_j^{(b)}$, let u,w be the unique pair of cells in this row with w on the same content line as v and u directly to the left of w. Then, (u,v,w) form a triple if either (1) u,w are both in $\beta_j^{(b)}$, (2) u is the cell $(j,0)$ just before the beginning of the row, or (3), u is the cell $(j,\beta_j^{(b)})$ at the end of the row. In other words, (u,v,w) is a triple exactly when $\beta_j^{(b)}$ has a cell of content $c(v)$ or $c(v)-1$. As the set of contents in the row $\beta_j^{(b)}$ is precisely the interval $[1-j, \beta_j^{(b)} - j]$, then

$$(u,v,w) \text{ is a triple} \iff 1-j \le c(v) \le \beta_j^{(b)} - j + 1 \iff i-j \le \ell - 1 \le \beta_j^{(b)} + i - j$$

As $\ell - 1$ ranges over the interval $[0, \beta_i^{(a)} - 1]$, after summing over ℓ, i, j and $a<b$, we find that the number of triples (u,v,w) is

$$m(\boldsymbol{\beta}) = \sum_{\substack{a<b\\ i,j}} \#\left([0,\beta_i^{(a)}-1] \cap [i-j, \beta_j^{(b)} + i - j]\right) = \sum_{\substack{a<b\\ i,j}} \#\left([-i,\beta_i^{(a)}-i-1] \cap [-j, \beta_j^{(b)} - j]\right) \tag{2.10}$$

The intersection of the intervals in the summand in (2.10) has size

$$\max(\min(\beta_i^{(a)} - i - 1, \beta_j^{(b)} - j) + \min(i,j) + 1, 0) \tag{2.11}$$

Casing on whether or not $\beta_i^{(a)} - i - 1$ is the minimum, we can rewrite this as

$$\max(\min(\beta_i^{(a)} - i, \beta_j^{(b)} - j) + \min(i,j), 0) + \begin{cases} 1 & : -\min(i,j) \le \beta_j^{(b)} - j \le \beta_i^{(a)} - i - 1 \\ 0 & : \text{else} \end{cases} \tag{2.12}$$

The condition in the piecewise component is equivalent to $i - \min(i,j) \leq \beta_j^{(b)} - j + i < \beta_i^{(a)}$, for which the first inequality is seen to be equivalent to $0 \leq \beta_j^{(b)} - j + i$ in either case $i \leq j$ or $j \leq i$. ☐

When $\boldsymbol{\beta}$ is a tuple of partitions all of which are single rows, then we arrive at the following simpler form for $m(\boldsymbol{\beta})$.

Corollary 2.2.1. *Let μ be a partition and let β be any rearrangement of its parts. Identify β with the tuple of partitions $\boldsymbol{\beta}$ with a single non-negative part in each component. Then,*

$$m(\beta) = n(\mu) + \mathrm{inv}(\beta) \tag{2.13}$$

where $n(\mu) = \sum_i (i-1)\mu_i$ and $\mathrm{inv}(\beta) = \#\{i < j \mid \beta_i > \beta_j\}$.

Proof. As $\boldsymbol{\beta}$ consists of single rows, the only non-zero terms in (2.9) are when $i = j = 1$. Thus,

$$\begin{aligned} m(\beta) &= \#\{a < b \mid 0 \leq \beta^{(b)} < \beta^{(a)}\} + \sum_{a<b} \max(\min(\beta^{(a)} - 1, \beta^{(b)} - 1) + 1, 0) \\ &= \mathrm{inv}(\beta) + \sum_{a<b} \min(\beta^{(a)}, \beta^{(b)}) \end{aligned}$$

The result follows from the identity $n(\mu) = \sum_{a<b} \min(\beta^{(a)}, \beta^{(b)})$. ☐

The quantity $\mathrm{inv}(\beta)$ defined in Corollary 2.2.1 will make key appearances in Chapter 5 when we count statistics for classical type LLT polynomials.

Definition 2.2.1 of the inversion LLT polynomials was first given in [30], however it is not related in an obvious way to the original spin-generating functions defined in [57]. For completeness, we also give the original definition of LLT polynomials. We define the **spin** of a ribbon R to be $\mathrm{ht}(R) - 1$, where $\mathrm{ht}(R)$ is the number of rows in R. The spin of a semistandard ribbon tableau is the sum of the spins of the ribbons in its tiling.

Definition 2.2.4. Let λ/μ be a skew partition and fix an integer k. The spin LLT polynomial is defined as

$$G_{\lambda/\mu}^{(k)}(X;q) = \sum_{T \in \mathrm{SSRT}_k(\lambda/\mu)} q^{\mathrm{spin}(T)} x^T$$

where $\mathrm{SSRT}_k(\lambda/\mu)$ denotes the set of semistandard k-ribbon tableaux of shape λ/μ.

It was shown in [30] that if $T \in \mathrm{SSRT}_k(\lambda/\mu)$ corresponds to $S \in \mathrm{SSYT}(\mathrm{quot}_k(\lambda/\mu))$ under (2.3), then there is some constant e depending only on the shape λ/μ such that $\mathrm{spin}(T) = -2\mathrm{inv}(S) + e$. Hence,

Proposition 2.2.2. *Let λ/μ be a skew partition with $\mathrm{quot}_k(\lambda/\mu) = \boldsymbol{\beta}/\boldsymbol{\gamma}$. Then, there is a constant e depending only on the shape λ/μ such that*

$$G_{\lambda/\mu}^{(k)}(X;q) = q^e \mathcal{G}_{\boldsymbol{\beta}/\boldsymbol{\gamma}}(X;q^{-2}) \tag{2.14}$$

Another inversion statistic was proposed by Schilling, Shimozono, and White in [80], which maps exactly with the (co)spin statistic, without any constant error factor q^e.

2.3 Representation theory of GL_n

We briefly review the basics of the representation theory of GL_n as it relates to the tableau combinatorics discussed in the previous sections. Some familiarity with the material, and representation theory of finite groups, will be assumed, however the reader is welcome to reference a typical textbook on the material, e.g. [24, 25, 39, 79]. We will almost always work over the ground field $k = \mathbb{C}$, although many of these texts work in more generality than we do. We then review the basics of symmetric functions, which can also be reviewed e.g. in [72, 84]. While these results are all well-known, they serve as a template for subsequent chapters, when we give analogous statements for Sp_{2n} which are not as widely disseminated.

Let V be a finite dimensional complex vector space. The **general linear group** $\mathrm{GL}(V)$ is the group of all invertible linear transformations from V to V. After the identification $V \simeq \mathbb{C}^n$, we identify $\mathrm{GL}(V) \simeq \mathrm{GL}_n(\mathbb{C})$ with the group of $n \times n$ invertible matrices. A **representation** of $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$ is a group homomorphism $\rho : \mathrm{GL}_n \to \mathrm{GL}(V)$ for some vector space V over $\mathbb{C}$. In this way V becomes a G-module, and we often interchange ρ and the V when referring to a representation. We say a representation is **irreducible** if it has no nontrivial invariant subspace, and reducible otherwise.

We will primarily be interested in **polynomial representations** of GL_n, which are representations ρ in which the matrix entries $\rho(g)$ are polynomial functions in the entries of g. The reason being that as an algebraic variety, GL_n is the open set in the affine n^2 space of $n \times n$ matrices defined by the non-vanishing of its determinant. Hence, regular functions on GL_n are generated by polynomials in the entries of g and the multiplicative inverse of the determinant $(\det g)^{-1}$. If the entries of $\rho(g)$ are regular functions of the entries of g, we say that ρ is a **rational representation**. The group GL_n is an example of a reductive Lie group, which is to say that all of its finite dimensional rational representations are completely reducible, i.e. they can be written as a direct sum of irreducible representations.

To a representation ρ, we define its **character** $\chi : \mathrm{GL}_n \to \mathbb{C}$ sending g to the trace $\operatorname{tr} \rho(g)$. Characters are examples of class functions, which is to say that they are constant on conjugacy classes. As the subgroup of diagonalizable matrices is dense in GL_n, and χ continuous, a character is determined by what it does to diagonal matrices. In particular, we consider $\chi = \chi(x_1, \ldots, x_n)$ as a function on the entries $x_1, \ldots, x_n$ of a diagonal matrix g, or more generally as a function on the eigenvalues of an arbitrary g. A representation is determined by its character, and we say that a character affords the representation V.

The group of diagonal matrices is a **maximal torus** for GL_n and is acted upon by its **Weyl group**. The Weyl group of GL_n is the symmetric group S_n and it acts on diagonal matrices and hence the characters χ by permuting the entries $x_1, \ldots, x_n$. The irreducible representations for GL_n are indexed by their **highest weight**, a non-increasing list λ of integers of length n. Below we give examples of representations of GL_n along with their associated characters χ_λ.

Example 2.3.1. 1. The standard representation $V \simeq \mathbb{C}^n$ of $\mathrm{GL}_n(\mathbb{C})$ is an irreducible representation whose character is

$$\chi_{(1)}(x_1, \ldots, x_n) = x_1 + \cdots + x_n \tag{2.15}$$

2. The determinant representation is the one-dimensional representation $\det : \mathrm{GL}_n \to \mathbb{C}$ whose character is

$$\chi_{(1^n)}(x_1, \ldots, x_n) = x_1 \cdots x_n \tag{2.16}$$

Every rational representation has the form $\det^{-k} \otimes \rho$ for some polynomial representation ρ and integer k. A representation is polynomial iff its highest weight λ has $\lambda_n \geq 0$, that is λ is a partition.

3. The symmetric power $S^k(V)$ is an irreducible representation whose character is

$$\chi_{(k)}(x_1, \ldots, x_n) = h_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k} \tag{2.17}$$

the kth homogenous symmetric polynomial.

4. The exterior power $\Lambda^k(V)$ is an irreducible representation whose character is

$$\chi_{(1^k)}(x_1, \ldots, x_n) = e_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k} \tag{2.18}$$

the kth elementary symmetric polynomial.

5. We define the element $e_\lambda(x_1, \ldots, x_n) = e_{\lambda_1} \cdots e_{\lambda_n}$ and similarly for $h_\lambda(x_1, \ldots, x_n)$. The former is the character of $\Lambda^{\lambda_1}(V) \otimes \cdots \otimes \Lambda^{\lambda_n}(V)$ and the latter the character of $S^{\lambda_1}(V) \otimes \cdots \otimes S^{\lambda_n}(V)$. These representations are in general not irreducible and will decompose according to **Pieri rules**, stated below.

6. If χ affords the representation V of GL_n, the contragredient representation is the dual V^*, whose character is

$$\chi^*(x_1, \ldots, x_n) = \chi(x_1^{-1}, \ldots, x_n^{-1}) \tag{2.19}$$

If $V = V_\lambda$ is irreducible, then the dual representation V_λ^* is irreducible with highest weight $-w_0(\lambda)$, where w_0 is the longest element of S_n, which reverses all the entries of λ. If λ is a partition, then V_λ^* is not a polynomial representation, but we can factor out powers of the determinant to find a polynomial representation:

$$V_\lambda^* \simeq \det^{-k} \otimes V_{\vec{k} - w_0(\lambda)} \tag{2.20}$$

where $\vec{k}$ denotes the weight $(k, \ldots, k)$ of length n.

The irreducible polynomial representation of GL_n with highest weight λ has character given by the **Schur polynomial** s_λ. One definition is given by the Weyl character formula as a ratio of determinants. We define the alternating element

$$a_\lambda = \sum_{w \in S_n} (-1)^{\ell(w)} x^{w(\lambda)} \tag{2.21}$$

where the factor $(-1)^{\ell(w)}$ is the sign of the permutation w. Letting $\rho = (n-1, \ldots, 0)$ as usual, a classical argument gives

$$a_\rho = \sum_{w \in S_n} (-1)^{\ell(w)} x^\rho = \prod_{i<j} (x_i - x_j) = x^\rho \prod_{i<j} \left(1 - \frac{x_j}{x_i}\right) \tag{2.22}$$

so that

$$s_\lambda(x_1, \ldots, x_n) = \frac{a_{\lambda+\rho}}{a_\rho} = \sum_{w \in S_n} w \left(\frac{x^\lambda}{\prod_{i<j}(1 - x_j/x_i)} \right) \tag{2.23}$$

When λ is a partition, we also have the combinatorial formula

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T \tag{2.24}$$

where x^T denotes the monomial $x^{\mathrm{wt}\, T} = \prod_i x_i^{\#i\text{'s in } T}$. When λ is a dominant weight that is not a partition, then we add enough copies of the determinant as in (2.20) to write $s_\lambda = (x_1 \cdots x_n)^{-k} s_{\lambda+\vec{k}}$, where $\lambda + \vec{k}$ has all positive entries. It is often sometimes convenient to extend (2.23) to when λ is not dominant as follows: if $\lambda + \rho$ is a regular weight, then we set

$$s_\lambda = (-1)^{\ell(w)} s_{w(\lambda+\rho)-\rho} \tag{2.25}$$

where $w \in S_n$ is the unique permutation such that $w(\lambda + \rho) = \lambda_+ + \rho$, for λ_+ dominant. In the case $\lambda + \rho$ is not regular, we set $s_\lambda = 0$.

Equation (2.24) suggests a natural basis of an irreducible representation given by semi-standard Young tableaux. Indeed many such constructions exist, historically attributed to Weyl, see e.g. [24]. A representation always breaks into weight spaces, which are eigenspaces for the subgroup T of diagonal matrices. As T-modules, these weight spaces have characters x^μ; we define the **monomial symmetric polynomial** $m_\mu(x_1, \ldots, x_n)$ to be the sum over distinct monomials in the S_n-orbit of x^μ. We can then write (2.24) as

$$s_\lambda(x) = \sum_{\mu \le \lambda} K_{\lambda,\mu} m_\mu(x) \tag{2.26}$$

where $K_{\lambda,\mu} = |\,\mathrm{SSYT}(\lambda, \mu)|$ and $<$ is the dominance order defined in Section 2.1. The coefficients $K_{\lambda,\mu}$ are **weight multiplicities**, known combinatorially as **Kostka numbers**. They can be computed for example using Kostant's weight multiplicity formula. The dominance order on partitions arises naturally from an order on the root lattice, which we discuss in more detail in Chapter 4.

The Schur polynomials are orthonormal with respect to the inner product

$$\langle f, g\rangle = \frac{1}{n!}[x^0] f\overline{g} \prod_{i\neq j}(1 - x_j/x_i) \tag{2.27}$$

where $\overline{g}$ denotes inverting the variables, and $[x^0]$ denotes taking the constant term. In fact, Schur polynomials are uniquely determined by their orthogonality (2.27) and triangularity (2.26). We will see later a q-analogue and non-symmetric analogue of this inner product for Hall-Littlewood and non-symmetric Hall-Littlewood polynomials.

Dualities and decompositions

We review the classical Cauchy identities and Pieri rules for GL_n, as we will shortly give analogues of each, and their accompanying insertion schemes, for Sp_{2n}. The relationships between these dualities for GL_n and the other classical groups is thoroughly explored in [37].

Theorem 2.3.1 (GL_n - GL_m duality)**.** *Let U, V be finite dimensional complex vector spaces. The symmetric algebra $S(U\otimes V)$ decomposes as a $\mathrm{GL}(U)\times\mathrm{GL}(V)$ module as*

$$S(U\otimes V) = \sum_\lambda \chi_\lambda^U \otimes \chi_\lambda^V \tag{2.28}$$

where $\chi_\lambda^U, \chi_\lambda^V$ denote the irreducible representations of $\mathrm{GL}(U), \mathrm{GL}(V)$, respectively, and λ ranges over all partitions of length $\ell(\lambda) \leq \min(\dim U, \dim V)$.

Taking $U = \mathbb{C}^n$ and $V = \mathbb{C}^m$, we can identify $S(U\otimes V)$ with the algebra of polynomial functions on $\mathrm{Mat}_{n\times m}(\mathbb{C})$. As a character identity, we write (2.28) as

$$\prod_{i=1}^n\prod_{j=1}^m \frac{1}{1-x_iy_j} = \sum_\lambda s_\lambda(x_1,\ldots,x_n)s_\lambda(y_1,\ldots,y_m) \tag{2.29}$$

where we treat the left hand side as a geometric series in x_iy_j. Equation (2.29) is known as the **Cauchy identity**, and taking the coefficient of a monomial x^λ on the left hand side, we find also that

$$\prod_{i=1}^n\prod_{j=1}^m \frac{1}{1-x_iy_j} = \sum_\lambda m_\lambda(x_1,\ldots,x_n)h_\lambda(y_1,\ldots,y_m) \tag{2.30}$$

There also exist the **dual Cauchy identities**

$$\prod_{i,j}^{n,m}(1+x_iy_j) = \sum_\lambda s_\lambda(x_1,\ldots,x_n)s_{\lambda'}(y_1,\ldots,y_m) = \sum_\lambda m_\lambda(x_1,\ldots,x_n)e_\lambda(y_1,\ldots,y_m) \tag{2.31}$$

Theorem 2.3.2 (Schur-Weyl duality). *Let V be the standard representation of $GL(V)$. Under the action of $GL(V) \times S_d$, the space $V^{\otimes d}$ decomposes into irreducibles as*

$$V^{\otimes d} = \sum_{\substack{\lambda \vdash d \\ \ell(\lambda) \leq \dim(V)}} \mathbf{V}^\lambda \otimes \mathbf{S}_\lambda \tag{2.32}$$

where $\mathbf{V}^\lambda, \mathbf{S}_\lambda$ denote the irreducible representations of $GL(V), S_d$, respectively.

As a character identity, Theorem 2.3.2 translates to

$$(x_1 + \cdots + x_n)^d = \sum_{\lambda \vdash d} s_\lambda(x_1, \ldots, x_n) f^\lambda \tag{2.33}$$

where f^λ denotes the number of standard Young tableaux of shape λ.

Both Theorem 2.3.1, 2.3.2 can be proven using the Robinson-Schensted-Knuth (RSK) algorithm, an insertion scheme that provides bijections between certain words or 2-lined arrays and pairs of standard or semistandard Young tableaux.

The following Pieri rules are combinatorial rules governing how the representations in Example 2.3.1(5) decompose into irreducible representations. They are given by

$$e_k(x)s_\mu(x) = \sum_{\substack{\lambda, \\ \lambda/\mu \in V_k}} s_\lambda(x), \qquad h_k(x)s_\mu(x) = \sum_{\substack{\lambda, \\ \lambda/\mu \in H_k}} s_\lambda(x) \tag{2.34}$$

the sums over λ such that λ/μ is a vertical (resp. horizontal) strip of size k. In particular, the decompositions (2.34) are multiplicity free, which is a property special to GL_n. Iterating (2.34) for h_k and viewing a semistandard Young tableau as a chain of horizontal strips gives

$$h_\mu(x) = \sum_{\lambda \geq \mu} K_{\lambda,\mu} s_\lambda(x) \tag{2.35}$$

and hence we can also interpret $K_{\lambda,\mu}$ as the multiplicity of the irreducible representation V^λ inside the tensor product $S^{\mu_1}(V) \otimes \cdots \otimes S^{\mu_\ell}(V)$ of symmetric powers of the standard representation V of GL_n. We note that (2.35) is technically an infinite sum, however only finitely many λ have $K_{\lambda,\mu} \neq 0$.

More generally we have

$$s_\lambda(x)s_\mu(x) = \sum_\nu c^\nu_{\lambda,\mu} s_\nu(x) \tag{2.36}$$

where $c^\nu_{\lambda,\mu}$ is the multiplicity of an irreducible representation V^ν inside the tensor product $V^\lambda \otimes V^\mu$ of irreducible representations for GL_n. The coefficients $c^\nu_{\lambda,\mu}$ are known as **Littlewood-Richardson coefficients** and possess a myriad of combinatorial models, e.g. they are known to count tableaux on the shape ν/λ and weight μ whose reading words are Yamanouchi.

The notion of a semistandard ribbon tableau is not too different from the notion of a semistandard Young tableau. Both are thought of as sequences of certain strips and in fact both are combinatorial manifestations of a certain Pieri rule; the latter appearing in (2.34) and the former appearing in the plethystic Pieri rule

$$h_\mu(x_1^k, x_2^k, \ldots, x_n^k) = \sum_{T \in \mathrm{SSRT}_k(\cdot,\mu)} (-1)^{\mathrm{spin}\, T} s_{\mathrm{sh}\, T}(x_1, x_2, \ldots, x_n) \tag{2.37}$$

This formula can be used to evaluate certain Hall-Littlewood polynomials at roots of unity, which was the original motivation for LLT polynomials [57].

2.4 Symmetric functions

We mention that all the definitions and combinatorial formulas above hold with infinitely many variables. However, so as to avoid confusion between the finite and infinite paradigms, we will refer to the infinite counterparts as *functions* rather than *polynomials*.

We let $\Lambda = \Lambda_R(X)$ be the algebra of symmetric functions in an infinite alphabet of variables $X = x_1, x_2, \ldots$ with coefficients in a ring R. We will often take $R = \mathbb{Q}(q)$, where q is a formal indeterminant. As λ ranges over all partitions, each of $\{e_\lambda\}, \{h_\lambda\}, \{m_\lambda\}, \{s_\lambda\}$ forms a basis for Λ.

The **omega involution** is the unique algebra involution $\omega : \Lambda \to \Lambda$ defined on the elementary basis by $\omega(e_k) = h_k$. On the Schur basis, it maps $\omega(s_\lambda) = s_{\lambda'}$. The **Hall inner product** $\langle -, - \rangle$ on Λ is defined on the bases by

$$\langle s_\lambda, s_\mu \rangle = \langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu} \tag{2.38}$$

and it's shown that bases $\{u_\lambda\}, \{v_\lambda\}$ are dual with respect to the Hall inner product if and only if they satisfy the Cauchy identity $\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_\lambda u_\lambda(X) v_\lambda(Y)$. We caution that the Hall inner product and omega involution are really features of an infinite alphabet. Indeed, ω sends e_k to h_k, but the former is 0 if we take the number of variables to be less than k, whereas the latter is not.

We review the type A Hall-Littlewood polynomials, although all the explicit formulas with finitely many variables can be straightforwardly modified for the other classical Lie types.

The **Hall-Littlewood polynomial** $P_\lambda(x_1, \ldots, x_n; q)$ is defined in $n \geq \ell(\lambda)$ variables by

$$P_\lambda(x_1, \ldots, x_n; q) = \frac{1}{W_\lambda(q)} \sum_{w \in S_n} w \left(\frac{x^\lambda \prod_{i<j}(1 - q x_j / x_i)}{\prod_{i<j}(1 - x_j/x_i)} \right) \tag{2.39}$$

where $W_\lambda(q) = \sum_{w \in \mathrm{Stab}(\lambda)} q^{\ell(w)}$. For GL_n, the polynomials $P_\lambda(x_1, \ldots, x_n; q)$ stabilize in the limit $n \to \infty$, so $P_\lambda(X; q)$ makes sense formally in infinitely many variables. We have the specializations

$$P_\lambda(x_1, \ldots, x_n; 0) = s_\lambda(x_1, \ldots, x_n) \qquad P_\lambda(x_1, \ldots, x_n; 1) = m_\lambda(x_1, \ldots, x_n) \tag{2.40}$$

The **Hall-Littlewood series** $\mathbf{H}_\mu(x_1,\dots,x_n;q)$ is the dual basis to $P_\lambda(x_1,\dots,x_n;q)$ with respect to the inner product (2.27) in which the irreducible characters are orthogonal. It has the explicit formula

$$\mathbf{H}_\mu(x_1,\dots,x_n;q) = \sum_{w\in S_n} w\left(\frac{x^\mu}{\prod_{i<j}(1-x_j/x_i)\prod_{i<j}(1-qx_i/x_j)}\right) \tag{2.41}$$

where again the factors in the denominator are understood as geometric series. We view the function $\mathbf{H}_\mu(x_1,\dots,x_n;q)$ as an infinite formal sum of irreducible GL_n characters with coefficients in $\mathbb{Q}(q)$. From the Weyl character formula (2.23) and (2.25), it follows that the coefficient of $\chi_\lambda(x)$ in $\mathbf{H}_\mu(x_1,\dots,x_n;q)$ is Lusztig's q-analog of Kostant's weight multiplicity formula

$$K_{\lambda,\mu}(q) := \langle\chi_\lambda\rangle\mathbf{H}_\mu(x_1,\dots,x_n;q) = \sum_w (-1)^{\ell(w)}\mathcal{P}_q(w(\lambda+\rho)-(\mu+\rho)) \tag{2.42}$$

where

$$\mathcal{P}_q(\beta) = \langle x^\beta\rangle\prod_{i<j}\frac{1}{1-qx_i/x_j} \tag{2.43}$$

is the q-partition generating function which enumerates multisets S of positive roots that sum to β, with weight $q^{|S|}$. The **transformed Hall-Littlewood polynomial** $H_\mu(x_1,\dots,x_n;q)$ is the polynomial truncation

$$H_\mu(x_1,\dots,x_n;q) = \mathbf{H}_\mu(x_1,\dots,x_n;q)_{\mathrm{pol}} \tag{2.44}$$

where pol refers to truncating to irreducible characters χ_λ with $\lambda_n \geq 0$. From (2.42) and the fact that $\mathbf{H}_\mu(x;q)$ are dual to $P_\lambda(x;q)$, there holds the expansions

$$H_\mu(x;q) = \sum_{\lambda\geq\mu} K_{\lambda,\mu}(q)s_\lambda(x) \qquad s_\lambda(x) = \sum_{\mu\leq\lambda} K_{\lambda,\mu}(q)P_\lambda(x;q) \tag{2.45}$$

The coefficients $K_{\lambda,\mu}(q)$ are known as **Kostka-Foulkes polynomials** and have been well-studied, with several combinatorial, geometric, and representation-theoretic interpretations. Their positivity was a subject of much interest, with geometric proofs given by Hotta and Springer [36, 82] and Lusztig [66]. A combinatorial formula in type A was provided by Lascoux and Schützenberger [58], in which they exhibit $K_{\lambda,\mu}(q)$ as a generating function weighted with their renowned charge statistic. A excellent survey of the charge statistic can be found in the thesis of Butler [15].

Kato [49] and Lusztig [67] showed also that the Kostka-Foulkes polynomials coincide with certain affine Kazhdan-Lusztig polynomials, from which positivity is known by other geometric means that we will briefly touch upon in Section 4.6 when we deal with general type LLT polyomials. The reader can find many more properties and applications of the Kostka-Foulkes polyomials in the surveys [33, 76, 86] and the references therein.

There also exists a q-inner product

$$\langle f, g\rangle_q = \frac{1}{n!}\langle x^0\rangle f\overline{g}\prod_{i\neq j}\frac{1-x_i/x_j}{1-qx_i/x_j} \tag{2.46}$$

This inner product depends on the number of variables n; however, one can normalize so that the dependence stabilizes as $n\to\infty$. In this limit, (2.46) coincides with the q-Hall inner product of Macdonald [72, §III.4].

With respect to (2.46) (or the q-Hall inner product in infinitely many variables), the Hall-Littlewood polynomials $P_\lambda(x;q)$ are uniquely characterized by the conditions

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} C_{\lambda,\mu}(q)m_\mu \tag{2.47}$$

$$\langle P_\lambda, P_\mu\rangle_q = 0, \qquad \mu\neq\lambda \tag{2.48}$$

for some coefficients $C_{\lambda,\mu}(q)$, where $<$ is the dominance order on partitions. The polynomials $H_\mu(x;q)$ are similarly characterized, but with an upper triangularity on the monomial basis.

We will also make use of the **modified Hall-Littlewood polynomials** $\widetilde{H}_\mu(X;q)$ defined via

$$\widetilde{H}_\mu(X;q) = q^{n(\mu)}H_\mu(X;q^{-1}) \tag{2.49}$$

where $n(\mu) = \sum_i (i-1)\mu_i$. Similarly, we define the modified Kostka-Foulkes polynomials $\widetilde{K}_{\lambda,\mu}(q) := q^{n(\mu)}K_{\lambda,\mu}(q^{-1})$. The modification is a superficial change, done so that $\widetilde{H}_\mu(x;q)$ aligns more naturally with the character of a graded S_n-module [26].

The following is due to [57], albeit in a different form than stated below.

Proposition 2.4.1. *Let μ be a partition, viewed as a tuple of rows, each placed on zero content line. Then,*

$$\mathcal{G}_\mu(X;q) = \widetilde{H}_\mu(X;q). \tag{2.50}$$

where we recall that $\mathcal{G}_\mu(X;t)$ denotes the inversion LLT polynomial.

This proposition is proven in [57] by showing that the inversion statistic for LLT polynomials aligns with a specific geometric intepretation of Kostka-Foulkes polynomials. A proof using the combinatorics of a vertex model was given recently in [21]. Using (2.49), (2.8) and (2.13), in terms of the coinversion LLT polynomials we have

$$\mathcal{L}_\mu(X;q) = q^{\mathrm{inv}(\mu)}H_\mu(X;q) \tag{2.51}$$

where we recall that $\mathrm{inv}(\mu) = \#\{i<j \mid \mu_i>\mu_j\}$.

Lastly, it will be useful to record the following identity for when we apply the omega involution to coinversion LLT polynomials.

Proposition 2.4.2 ([10])**.** *Let $\boldsymbol{\beta}/\boldsymbol{\gamma}$ be a tuple of skew partitions. We say a filling T on $\boldsymbol{\beta}/\boldsymbol{\gamma}$ is a **negative tableau** if the rows are strictly increasing and the columns are weakly increasing, i.e. T is the transpose of a semistandard filling. Then,*

$$\omega \mathcal{L}_{\boldsymbol{\beta}/\boldsymbol{\gamma}}(X;q) = \sum_T q^{\overline{\mathrm{coinv}}(T)} x^T \tag{2.52}$$

*where the sum is over negative tableaux T and $\overline{\mathrm{coinv}}(T)$ is the number of **negative coinversions** of T, where a negative coinversion is a triple of boxes of the form (2.2.2), but with the strict inequalities $a < b < c$.*

Proof. This follows from the quasi-symmetric function expansion in [31, (82)] and the same argument that gives (2.8). □

Chapter 3

Symplectic Combinatorics

Our main interest is in the combinatorics at play for $G = \mathrm{Sp}_{2n}$, and so we introduce those objects here. We also briefly review their connections to the representation theory of Sp_{2n}. Many of these objects and their properties can be found in the excellent exposition by Sundaram [87].

3.1 Representation theory of Sp_{2n}

The **symplectic group** $\mathrm{Sp}(V)$ is the group of linear transformations preserving a non-degenerate skew-symmetric bilinear form on the finite dimensional complex vector space V. Identifying $V \simeq \mathbb{C}^{2n}$, we can identify $\mathrm{Sp}(V) \simeq \mathrm{Sp}_{2n}(\mathbb{C})$ with a group of $2n \times 2n$ matrices M satisfying $M^T JM = J$, where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. It is a simply connected, semisimple Lie group with semisimple Lie algebra $\mathfrak{sp}_{2n}$ the set of traceless $2n \times 2n$ matrices M satisfying $JM + M^T J = 0$.

The Weyl group for Sp_{2n} is the **hyperoctahedral group**, also known as the group of **signed permutations**. A signed permutation π acts on the set $\{\pm 1, \ldots, \pm n\}$, with π sending $i \mapsto j$ iff it sends $-i \mapsto -j$. The group of signed permutations is generated by the same simple reflections $\{s_1, \ldots, s_{n-1}\}$ as for S_n, along with the generator s_n which swaps n with $-n$. An arbitrary element can both permute and negate entries.

We identify the character ring of Sp_{2n} with the algebra of Laurent polynomials in the variables $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$ that are invariant under signed permutations. We review the characters of several representations of Sp_{2n}.

Example 3.1.1. 1. The standard representation $V \simeq \mathbb{C}^{2n}$ of Sp_{2n} is an irreducible representation whose character is

$$\chi_{std} = x_1 + x_1^{-1} + \cdots + x_n + x_n^{-1} \tag{3.1}$$

2. The symmetric power $S^k(V)$ of the standard representation is an irreducible representation whose character is $h_k(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$, where h_k is the complete homogenous symmetric polynomial.

3. The exterior power $\Lambda^k(V)$ of the standard representation has character $e_k(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$, where e_k is the elementary symmetric polynomial. This is not an irreducible character, as the symplectic form lives in $\Lambda^2(V)$ and is definitionally invariant under Sp_{2n}. The kernel of the contraction map $\Lambda^k(V) \to \Lambda^{k-2}(V)$ is the irreducible representation with highest weight the k^{th} fundamental weight, and has character $e_k - e_{k-2}$, see [25].

There is the following decomposition of the d-th tensor power of the standard representation:

Theorem 3.1.1 ([13])**.** *Let V denote the standard representation of Sp_{2n}. Then,*

1. *There is an algebra $B_d(-2n) \subseteq \mathrm{End}(V^{\otimes d})$ for which $B_d(-2n)$ and Sp_{2n} are centralizers of each other in $\mathrm{End}(V^{\otimes d})$.*

2. *The irreducible representations W_λ of $B_d(-2n)$ are indexed by partitions λ with $\ell(\lambda) \leq n$ and have dimension $\dim W_\lambda = \tilde{f}^d_\lambda(n)$, where $\tilde{f}^d_\lambda(n)$ counts the number of n-oscillating tableaux of shape λ and d steps (to be defined in Definition 3.2.2).*

3. *We have the following decomposition as a $\mathrm{Sp}_{2n} \times B_d(-2n)$ module*

$$V^{\otimes d} \simeq \bigoplus_{\lambda} V^\lambda \otimes W_\lambda \tag{3.2}$$

where V^λ denotes the irreducible character of Sp_{2n} with highest weight λ.

3.2 Symplectic and oscillating tableaux

In searching for a symplectic analogue of semistandard tableaux, there are two natural candidates. The first is a combinatorial object that encodes the weight multiplicities for an irreducible representation of Sp_{2n}. These "symplectic tableaux" were proposed independently by Kashiwara/Nakashima [47] and King [51, 52]. Their definitions are quite different, the former more compatible with crystal operations, and the latter more compatible with weight multiplicities and restriction to subgroups. An intricate bijection between the two tableaux was given by Sheats [81]. We opt to use King's tableaux, with a slight modification given by Sundaram [89].

Definition 3.2.1. A **symplectic tableau** T of shape λ is a filling of the Ferrers diagram of λ with the letters $1 < \overline{1} < 2 < \cdots < n < \overline{n}$ such that

1. T is semistandard with respect to the above ordering

2. The entries $\bar{i}$ must be in row $\leq i$.

We let $\mathrm{Symp}(\lambda)$ denote the set of symplectic tableaux of shape λ. The second condition is often referred to as the *symplectic condition*. For convenience, we will denote the entries with their ordering above as the set $[\pm n]$.

The utility of these objects is that the irreducible character χ_λ of Sp_{2n} becomes a generating function for symplectic tableaux of shape λ, just as the irreducible characters of GL_n are generating functions for semistandard Young tableaux.

Proposition 3.2.1. *Let λ be a partition of length $\ell(\lambda) \leq n$. If sp_λ denotes the irreducible character of Sp_{2n} with highest weight λ, then*

$$\mathrm{sp}_\lambda(x_1^{\pm 1}, \dots, x_n^{\pm 1}) = \sum_{T \in \mathrm{Symp}(\lambda)} x^T \tag{3.3}$$

where $x^T = \prod_i x_i^{\# i - \# \bar{i} \text{ in } T}$.

A combinatorial proof of the fact that the irreducible characters of Sp_{2n} are signed-symmetric is detailed in [87] using (3.3) and a symplectic version of the Bender-Knuth involution.

Another analogue of semistandard tableaux comes first from standard tableaux, which encode, via Schur-Weyl duality, multiplicities of irreducible constituents in tensor powers of the standard representation of GL_n, see (2.33). In the Schur-Weyl duality for Sp_{2n} given in Theorem 3.1.1, the symmetric group is replaced by a Brauer algebra, whose irreducible representations have a basis given by the following objects.

Definition 3.2.2. Let λ, μ be partitions. An **n-oscillating tableau** of shape λ/μ is a sequence

$$\mu = \nu^0, \nu^1, \nu^2, \dots, \lambda \tag{3.4}$$

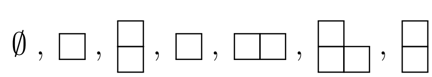
of partitions such that for each i,

(i) ν^i differs from ν^{i-1} by a single box.

(ii) $\ell(\nu^i) \leq n$.

In the literature [89] this is also known as an **n-symplectic up-down tableau**. When the length restriction is implicit or not imposed, we will drop the n and simply refer to this as an oscillating tableau or an up-down tableau.

Example 3.2.1.



is a 2-oscillating tableau of shape (1,1) with 6 steps.

Semistandard tableaux more generally count the multiplicities of irreducible constituents in tensor powers of symmetric powers of the standard representation of GL_n, see (2.35). The following objects take their place in Sp_{2n}.

Definition 3.2.3. Let λ, μ be straight shapes. An n**-horizontal (**n**-vertical) semistandard oscillating tableau** of shape λ/μ is a sequence

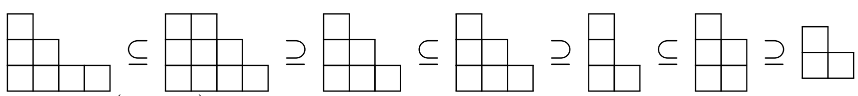
$$\mu = \alpha^0 = \beta^0 \subseteq \alpha^1 \supseteq \beta^1 \subseteq \alpha^2 \supseteq \beta^2 \subseteq \cdots \supseteq \lambda \tag{3.5}$$

of partitions such that for each i,

(i) α^i/β^{i-1} and α^i/β^i is a horizontal (vertical) strip.

(ii) α^i, β^i have all row (column) lengths $\leq n$.

For brevity, we will denote n-hSSOT as the set of n-horizontal semistandard oscillating tableau, and likewise for n-vSSOT. Again, we may often drop the n to avoid clutter or if the condition is not imposed. The **weight** of a horizontal or vertical semistandard oscillating tableau is the composition ν, where $\nu_i = |\alpha^i/\beta^{i-1}| + |\alpha^i/\beta^i|$. We note that a horizontal or vertical semistandard oscillating tableau of weight $(1, \ldots, 1)$ is simply an oscillating tableau, with the same length restrictions.

Example 3.2.2.  has shape $(2,1)/(4,2,1)$ and weight $(5,2,3)$.

The following Pieri rule was shown e.g. in [77, 89].

Proposition 3.2.2. *Let λ, μ be partitions of lengths at most n and let $e_\lambda(x, x^{-1})$ denote the elementary symmetric polynomial in the variables $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$ as in Example 3.1.1.*

$$e_\lambda(x, x^{-1}) \,\mathrm{sp}_\mu(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = \sum_\nu |n\text{-}\,\mathrm{vSSOT}(\nu/\mu, \lambda)| \,\mathrm{sp}_\nu(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) \tag{3.6}$$

A Pieri rule for multiplication by h_λ was also given, although we haven't defined the necessary combinatorial objects here. If we did, they would likely be called "n-horizontal down-up tableaux", from which the reader can infer their definition, see also [54].

Now, it is obvious and yet miraculous that one can view a standard tableau as a special case of a semistandard tableau. Indeed, the former counts the dimension of the (1^n) weight space in an irreducible representation of GL_n, given combinatorially in (2.26), whereas the latter counts the multiplicity of an irreducible representation inside tensor powers of the standard representation, given combinatorially in (2.35). These two coefficients given in (2.26) and (2.35) are not innately related and yet they coincide for GL_n.

For Sp_{2n}, the above definition of a symplectic tableau has no such obvious reformulation to connect to an oscillating tableau (nor does the Kashiwara/Nakashima definition). A priori, an oscillating tableau seems to be a fundamentally different object than a symplectic tableau. One result we present, stated below, is that there is in fact an analogous specialization of symplectic tableaux to oscillating tableaux.

Theorem 3.2.1. *Fix positive integers N, n and let λ be a partition contained in an (N^n) rectangle. There is a bijection*

$$\Phi_{N,n} : \left\{ \begin{array}{c} N\text{-hSSOT of} \\ \text{shape } \lambda \text{ and } n \text{ steps} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{symplectic tableaux of} \\ \text{shape } \lambda^c \text{ and entries in } [\pm n] \end{array} \right\} \tag{3.7}$$

where λ^c denotes the complement of λ in an (N^n) box.

The complement shape is perhaps not surprising when we compare to the situation in GL_n. Indeed, the identity of Schur polynomials

$$(x_1 \cdots x_k)^n s_\lambda(x_1^{-1}, \ldots, x_k^{-1}) = s_{\lambda^c}(x_1, \ldots, x_k) \tag{3.8}$$

where the complement is taken in a (n^k) box, implies a bijection between semistandard Young tableaux of shape λ^c and those of shape λ, with the weight μ mapping to $(n - \mu_k, \ldots, n - \mu_1)$. This bijection is given in [84, Ex. 7.41] and we adapt it to our current case with oscillating tableaux and symplectic tableaux.

Proof of Theorem 3.2.1. We first associate to a horizontal semistandard oscillating tableau a tableau with set-valued entries[1] inside the (N^n) rectangle. More specifically, to each cell in the $N \times n$ rectangle, we will assign a subset of entries in $\{1, \bar{1}, \ldots, n, \bar{n}\}$, viz., if in the i^{th} step of the hSSOT a cell was added or removed, then we add i or $\bar{i}$, respectively, to that cell's label. Figure 3.1 serves as an example.

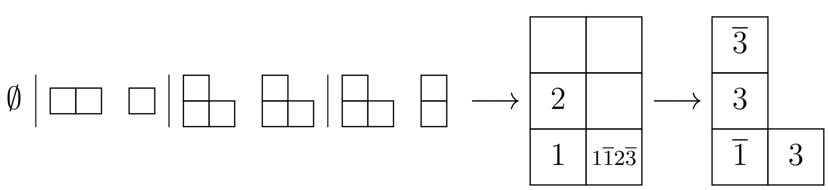


Figure 3.1: An example of the bijection from a hSSOT to a symplectic tableau. The left sequence of partitions is the 2-hSSOT $\emptyset \subseteq (2) \supseteq (1) \subseteq (2,1) \supseteq (2,1) \subseteq (2,1) \supseteq (1,1)$. The middle diagram is the tableaux T with set-valued entries we associate to the 2-hSSOT and the rightmost diagram is the resulting symplectic tableaux $\widetilde{T}$ with entries in $[\pm 3]$.

Denote T the resulting tableau with set-valued entries, consisting of the cells in (N^n) labelled with a (possibly empty) set. Let $\nu^1, \ldots, \nu^N$ be the (possibly zero) columns of T, left

[1] We don't refer to this as a *set-valued tableau* because that term already has two other definitions in the literature that do not seem to apply in this context.

to right. Let $\widetilde{\nu}^i$ be the column whose entries are

$$\{1, \dots, n\} - \{i \mid i \in \nu^i\} \cup \{\bar{i} \mid \bar{i} \in \nu^i\}$$

arranged in increasing order. Let $\widetilde{T}$ be the tableau with columns $\widetilde{\nu}^N, \dots, \widetilde{\nu}^1$, left to right.

The fact that $\widetilde{T}$ is semistandard follows from the same reasoning as in the case of semistandard tableaux, but now with the ordered alphabet including barred entries. We only have left to show that $\widetilde{T}$ satisfies the symplectic condition.

We suppose on the contrary that there is an $\bar{i}$ in some row above row i, say in column $\tilde{c}$. We can take i to be minimal and assume that $\bar{i}$ is in the $(i+1)^{th}$ row, so that the cell below this $\bar{i}$ must be i. The intermediate tableau T with set-valued entries will have an $\bar{i}$ and no i in the complement column, say column c. In other words, the semistandard oscillating tableau removes a cell at the ith step without first adding that cell in the ith step. It must then have added that cell at some step $j < i$. We can pick j maximal so that the semistandard oscillating tableau adds the cell at step j, and does nothing to that cell until it removes it in step i. So, T will have a j but no $\bar{j}$ in column c, and also no k or $\bar{k}$ for $j < k < i$. In other words, in column $\tilde{c}$, $\widetilde{T}$ will have neither j nor $\bar{j}$, but will have k for $j < k < i$. By column strictness, the cell at row j in this column will be strictly less than j, which contradicts the minimality of i if this entry is barred, and otherwise contradicts column strictness. □

We mention that Theorem 3.2.1, or a version thereof, was stated and proven independently in [63] in order to give a crystal structure on King tableaux.

Remark 3.2.1. (a) We note that $\Psi_{n,N}$ is not quite weight preserving. Given an N-hSSOT $\vec{o}$ with n steps, let T be the intermediate tableau with set-valued entries we associate to $\vec{o}$. Recall that the weight of $\vec{o}$ is the composition $\mu = (\mu_1, \dots, \mu_n)$ where

$$\mu_i = |\alpha^i/\beta^{i-1}| + |\alpha^i/\beta^i| = \#i\text{'s in T} + \#\bar{i}\text{'s in T}$$

The weight of the resulting symplectic tableau $\widetilde{T}$ will be $\nu = (\nu_1, \dots, \nu_n)$ where

$$\nu_i = \#i\text{'s in } \widetilde{T} - \#\bar{i}\text{'s in } \widetilde{T} = (N - \#i\text{'s in T}) - (\#\bar{i}\text{'s in T}) = N - \mu_i = (\mu^c)_{n-i}$$

While ν is not always a partition, we can apply the symplectic Bender-Knuth involution to $\widetilde{T}$ to get a symplectic tableau with partition weight.

(b) Secondly, note that given a hSSOT, there is some ambiguity as to what symplectic tableau it bijects to, and vice versa. More specifically, any N-hSSOT is also an $(N+1)$-hSSOT and likewise an hSSOT with n steps is also an hSSOT with $n+1$ steps (just as a symplectic tableau with entries in $[\pm n]$ is also a symplectic tableau with entries in $[\pm(n+1)]$). The corresponding relation between $\Psi_{n,N}$ and $\Psi_{n,N+1}$ is as follows: when increasing N to $N+1$, one adds a full column to the resulting symplectic tableau with entries $1, \dots, n$; the inverse map will add a column of i boxes to all the shapes in the i^{th} step. The corresponding relation between $\Psi_{n,N}$ and $\Psi_{n+1,N}$ is as follows: when increasing n to $n+1$, one adds a full row to the resulting symplectic tableau all with entries $n+1$; the inverse map will add an additional step at the end that is just adding a single horizontal strip of length N.

(c) The set of horizontal semistandard oscillating tableaux of shape λ is an infinite set, as one could just add and remove the same horizontal strip infinitely many times. Similarly, at any given step, one could add and remove a horizontal strip of arbitrary length. Thus, to get a finite set, we impose restrictions on the number of steps and on the sizes of the first part. One might wonder if this is a worry at all, since the set of symplectic tableaux of a specified shape is also an infinite set if we allow unbounded entries. However, in the map from hSSOT to symplectic tableau, the shape of the resulting tableau is a complement shape in a box whose dimensions depend on the number of steps and the maximum part size.

We also have the dual statement

Theorem 3.2.2. *Fix positive integers N, n and let λ be a partition contained in an (n^N) rectangle. There is a bijection*

$$\Psi_{n,N} : \left\{ \begin{array}{c} N\text{-vSSOT of} \\ \text{shape } \lambda \text{ and } n \text{ steps} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{symplectic tableaux of} \\ \text{shape } \mu \text{ and entries in } [\pm n] \end{array} \right\} \tag{3.9}$$

where $\mu = (\lambda')^c$ is the complement transpose of λ in an (n^N) box.

Proof. The bijection $\Psi_{n,N}$ is the bijection in Theorem 3.2.1, precomposed with the map that transposes every intermediate partition in a vertical SSOT. □

Restricting either statement to oscillating tableaux gives

Corollary 3.2.1. *There is a bijection between n-oscillating tableaux from $\emptyset$ to λ in d steps and symplectic tableaux of shape λ^c (or its transpose) and weight $((d-1)^n)$, the complement taken in a (d^n) box.*

We also mention that a notion of skew symplectic tableaux was introduced by Koike and Terada in [53], which follow similar restrictions as for King tableaux. Theorems 3.2.1, 3.2.2 carry over with a slight tweak. We only give the statement for vertical skew symplectic tableaux.

Theorem 3.2.3. *Fix partitions λ, μ and positive integers N, n, k such that $N \geq \ell(\mu), \ell(\lambda)$ and $k \geq \mu_1$. There is a bijection*

$$\Psi_{k,n,N} : \left\{ \begin{array}{c} N\text{-vSSOT of} \\ \text{shape } \lambda/\mu \text{ and } n \text{ steps} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{skew symplectic tableaux of} \\ \text{shape } \tau/\sigma \text{ and entries in} \\ \{\pm(k+1), \ldots, \pm(k+n)\} \end{array} \right\} \tag{3.10}$$

where $\tau = (\lambda')^c$ is the complement transpose of λ in a $((n+k)^N)$ box and $\sigma = (\mu')^c$ is the complement transpose of μ in a (k^N) box.

Proof. We let S be the "standard" N-vSSOT of shape μ in k steps, which adds a column of length $(\mu')_i$ at the i^{th} step (padding μ' with zeros if necessary). Given T an N-vSSOT of shape λ/μ and n steps, we prepend S to get an N-vSSOT of shape λ and $n+k$ steps. Applying $\Psi_{n+k,N}$ maps this to a symplectic tableau of shape τ and entries in $[\pm(n+k)]$, where τ is the complement transpose of λ in an $((n+k)^N)$ box.

We let σ denote the complement transpose of μ in the (k^N) box. The choice of S implies that the tableau $\Psi_{k,N}(S)$ of shape σ in $\Psi_{n+k,N}(S \circ T)$ will be the complement of the superstandard tableau, namely the tableau with a maximal number of k's, then $(k-1)$'s, etc. In particular, the entries in the skew shape τ/σ are the only entries determined by T, and only contain entries in $\{\pm(k+1), \ldots, \pm(k+n)\}$. □

3.3 Cauchy identities

In this section we prove Cauchy identities akin to (2.29) for the symplectic group. Our main tool will be an insertion algorithm due to Berele [7], which serves as an analogue to the RSK insertion algorithm. We briefly review Berele's algorithm, although we again refer the reader to the excellent text [87].

Berele insertion involves inserting words in the alphabet $[\pm n] = \{1 < \overline{1} < \cdots < n < \overline{n}\}$ according to same rules as for RSK, with a slight modification when the symplectic condition is violated. Given a letter $a \in [\pm n]$ and a symplectic tableau T, we denote $T \xleftarrow{B} a$ to be the result of the following algorithm.

Row-insert a into T á la Robinson-Schensted.
If the result is symplectic, then do nothing.
else there is a unique $\overline{i}$ bumped out of row i into row $i+1$ by an i.
  delete both this i and $\overline{i}$ to yield a punctured tableau.
  slide the hole out via jeu de taquin until a normal tableau remains.

We supplement the algorithm with an example.

Example 3.3.1. We insert the word $w = 1\overline{1}\overline{2}11\overline{2}21$ into $\emptyset$.

$$\emptyset, \ \begin{array}{|c|}\hline 1 \\ \hline\end{array}, \ \begin{array}{|c|c|}\hline 1 & \overline{1} \\ \hline\end{array}, \ \begin{array}{|c|c|c|}\hline 1 & \overline{1} & \overline{2} \\ \hline\end{array}, \ \begin{array}{|c|}\hline 1 \\ \hline\end{array} \circ \begin{array}{|c|}\hline \overline{2} \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & \overline{2} \\ \hline\end{array}, \ \begin{array}{|c|c}\hline \overline{2} & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}, \ \begin{array}{|c|cc}\hline \overline{2} & \multicolumn{2}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{\overline{2}} \\ \hline\end{array}, \ \begin{array}{|c|c|c}\hline \overline{2} & \overline{2} & \multicolumn{1}{c}{} \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}, \ \begin{array}{c|c|c}\cline{2-2} \circ & \overline{2} & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{|c|}{1} & 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} \to \begin{array}{|c|cc}\hline \overline{2} & \multicolumn{2}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline\end{array}$$

Noting that the shapes of the intermediate steps give an oscillating tableau, Berele used his insertion algorithm to prove the following combinatorial manifestation of Sp_{2n} Schur-Weyl duality.

Proposition 3.3.1 (Berele [7])**.**

$$(x_1 + x_1^{-1} + \ldots + x_n + x_n^{-1})^m = \sum_{\lambda, \ell(\lambda) \leq n} sp_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) \widetilde{f}_m^\lambda(n) \tag{3.11}$$

where $\widetilde{f}_m^\lambda(n)$ is the number of n-oscillating tableaux of shape λ and m steps.

We can use our bijections between symplectic tableaux and oscillating tableaux to give a more general version of this statement, and of Berele insertion itself, just as Knuth generalized the Robinson-Schensted correspondence. We first recall a variation of RSK known as the (dual) Burge correspondence [14]. We will say a 2-lined array

$$\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

is arranged in **antilexicographic order** if $a_i \geq a_{i+1}$ and $a_i = a_{i+1} \implies b_i < b_{i+1}$. In one guise, the dual Burge correspondence is a bijection between 2-lined arrays in antilexicographic order and pairs of SSYT (P, Q) with conjugate shapes via row bumping $b_r b_{r-1} \cdots b_1$ to form P and placing $a_r a_{r-1} \cdots a_1$ in the newly added cell of the conjugate shape to form Q. We give an analogue for Berele insertion:

Corollary 3.3.1. *Let* $\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$ *be a 2-lined array arranged in antilexicographic order, with the top entries $a_i \in [m]$ and the bottom entries $b_j \in [\pm n]$. The following procedure gives a bijection to pairs of symplectic tableaux $(\widehat{P}, \widehat{Q})$ with conjugate complement shapes:*

- *Row Berele insert $b_r b_{r-1} \cdots b_1$ to form $\widehat{P}$.*
- *Keep track of the intermediate shapes as a n-vSSOT with m steps and weight ν, where $\nu_i = \#\{j \mid a_j = i\}$, and then apply Theorem 3.2.2 to form $\widehat{Q}$.*

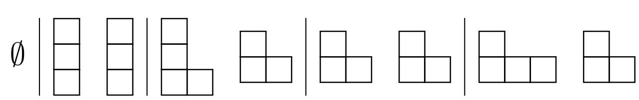
Example 3.3.2. We consider the 2-lined array

$$\begin{pmatrix} 4 & 4 & 2 & 2 & 1 & 1 & 1 \\ 1 & \bar{1} & \bar{1} & 2 & 1 & \bar{2} & 3 \end{pmatrix}$$

Berele inserting the bottom row from right to left gives

$$\emptyset\,,\ \begin{ytableau} 3 \end{ytableau}\,,\ \begin{ytableau} 3 \\ \bar{2} \end{ytableau}\,,\ \begin{ytableau} 3 \\ \bar{2} \\ 1 \end{ytableau}\,,\ \begin{ytableau} 3 \\ \bar{2} \\ 1 & 2 \end{ytableau}\,,\ \begin{ytableau} 3 \\ \none[\circ] \\ 1 & \bar{1} \end{ytableau} \to \begin{ytableau} 3 \\ 1 & \bar{1} \end{ytableau}\,,\ \begin{ytableau} 3 \\ 1 & \bar{1} & \bar{1} \end{ytableau}\,,\ \begin{ytableau} 3 \\ 1 & \none[\circ] & \bar{1} \end{ytableau} \to \begin{ytableau} 3 \\ 1 & \bar{1} \end{ytableau} = \widehat{P}$$

The intermediate shapes form the following vSSOT of weight $(3, 2, 0, 2)$:



Under the map $\Psi_{4,3}$ of Theorem 3.2.2, this bijects to

$$\begin{ytableau} 1\bar{2} & & & \\ 1 & & & \\ 1 & 2 & 4\bar{4} & \end{ytableau} \longrightarrow \begin{ytableau} 4 \\ 3 & 4 \\ \bar{2} & 3 & \bar{4} \\ 2 & 2 & 3 \end{ytableau} = \widehat{Q}$$

We note that $\widehat{P}$ and $\widehat{Q}$ are conjugate complement shapes in the (3^4) box.

As a corollary, we get the following Cauchy-like identity

Corollary 3.3.2.

$$\prod_{i=1}^{n}\prod_{j=1}^{m}(x_i + x_i^{-1} + y_j + y_j^{-1}) = \sum_{\lambda\subseteq(m^n)} sp_\lambda(x_1^{\pm 1}, \dots, x_n^{\pm 1}) sp_{(\lambda')^c}(y_1^{\pm 1}, \dots, y_m^{\pm 1}) \tag{3.12}$$

This is a curious identity, as a similar identity holds for Schur functions, dating back to Littlewood [84, Ex. 7.42]. It appears in [78], where it is referred to as a "Morris Identity", referencing [75, Thm. IV]. A stronger form was shown by Mimachi [74], where he proved the identity for Koornwinder polynomials. A super-version also exists for the orthosymplectic Lie superalgebra [18, §5.3], which reduces to this form. In particular, Corollary 3.3.2 should be a statement about Sp_{2n} - $\mathfrak{sp}_{2m}$ Howe duality, although at present the author has not worked through the details. A crystal interpretation is discussed in [63] and [35].

Nonetheless, in GL_n the representation theoretic statements of GL_m - GL_n duality and Schur-Weyl duality are bundled into one cohesive combinatorial algorithm known as RSK; the previous corollaries exhibit the same such bundling with Berele insertion for the analogous dualities in Sp_{2n}. In particular, one can take the top line in the 2-lined array in Corollary 3.3.1 to be all 1's to recover Berele's original algorithm. Alternatively, taking the coefficient of $y_1^{n-1} \cdots y_m^{n-1}$ in (3.12) and applying Theorem 3.2.2 recovers (3.11).

We also note that taking the coefficient of x^λ on the left hand side of (3.12) gives another Cauchy identity.

Corollary 3.3.3.

$$\prod_{i=1}^{n}\prod_{j=1}^{m}(x_i + x_i^{-1} + y_j + y_j^{-1}) = \sum_{\lambda\subseteq(m^n)} m_\lambda(x_1^{\pm 1}, \dots, x_n^{\pm 1}) e_{\lambda^c}(y_1^{\pm 1}, \dots, y_m^{\pm 1}) \tag{3.13}$$

where the complement is taken in an (m^n) box and m_λ denotes the signed monomial symmetric function, i.e. the W-orbit of x^λ for W the group of signed permutations.

Proof. We recall the generating function identity

$$\sum_k x^k e_k(y_1^{\pm 1}, \dots, y_m^{\pm 1}) = \prod_{j=1}^{m}(1 + xy_j)(1 + xy_j^{-1}) = x^m \prod_{j=1}^{m}(x + x^{-1} + y_j + y_j^{-1}) \tag{3.14}$$

Taking the product over all $x_1, \dots, x_n$ gives

$$\sum_{\lambda_1,\dots,\lambda_n} x^\lambda e_\lambda(y_1^{\pm 1}, \dots, y_m^{\pm 1}) = (x_1 \cdots x_n)^m \prod_{i=1}^{n}\prod_{j=1}^{m}(x_i + x_i^{-1} + y_j + y_j^{-1}) \tag{3.15}$$

from which the result follows after the observation that $e_{\lambda^c} = e_{m-\lambda_1} \cdots e_{m-\lambda_n}$ and that $e_k(y_1^{\pm 1}, \dots, y_m^{\pm 1}) = e_{2m-k}(y_1^{\pm 1}, \dots, y_m^{\pm 1})$. □

Chapter 4

Root Systems, Weyl Groups, Hecke Algebras and All That

The purpose of this chapter is to acquaint the reader with the necessary background on root systems, Weyl groups and Hecke algebras needed to give a definition of an LLT polynomial in general Lie type. Many of these topics have already been widely exposited, see e.g. [9, 25, 39, 40]; however, most material covered in these typical graduate textbooks overlook the affine setting, or treat them differently than they do the finite case.

Much of the theory on affine Lie algebras (or symmetrizable Kac-Moody algebras in general) will not come into play, although they do make an key appearance in [28], where the definition of general type LLT polynomials first appeared. Only affine Weyl groups and their associated affine Hecke algebras will be of importance to us, although in order to maintain a semblance of completeness, we choose to include a review of affine root systems, much of which can also be found e.g. in [17, 45, 71]. We will assume some previous knowledge of Lie theory.

Our presentation follows a very similar outline to that of [32], and like that work, we will often state things without proof, as our goal is to ultimately arrive at a parseable definition of LLT polynomials in general Lie type. We do however strive to supplement much of the material with examples.

4.1 Root systems

In anticipation of working with affine root systems and Weyl groups, we build out from generalized Cartan matrices.

Definition 4.1.1. A **generalized Cartan matrix** is an $n \times n$ matrix $A = (a_{ij})$ such that

(i) $a_{ii} = 2$ for $i = 1, \dots, n$

(ii) a_{ij} are non-positive integers for $i \neq j$

(iii) $a_{ij} = 0$ implies $a_{ij} = 0$

The **Dynkin diagram** of A is the graph with nodes $i = 1, \dots, n$ and an edge $\{i, j\}$ for each $a_{ij} \neq 0$, along with some weight or marking to indicate the values a_{ij}, a_{ji}. We say that A is **indecomposable** if its Dynkin diagram is connected.

Definition 4.1.2. A **root system** X is a collection $(P, \Delta, \Delta^\vee)$, where

(i) P is a finite-rank free abelian group with dual lattice $P^\vee := \mathrm{Hom}(P, \mathbb{Z})$,

(ii) $\Delta = \{\alpha_1, \dots, \alpha_n\} \subseteq P$ consists of *simple roots*,

(iii) $\Delta^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subseteq P^\vee$ consists *simple coroots*,

(iv) the matrix A with entries $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$ is a generalized Cartan matrix.

It is well known that generalized Cartan matrices fall into the following trichotomy:

Theorem 4.1.1. *Let A be an indecomposable generalized Cartan matrix. Then, A is exactly one of finite type, affine type, or indefinite type, where*

(i) A has finite type iff all its principal minors have positive determinant.

(ii) A has affine type iff $\det A = 0$ *and all proper principal minors have positive determinant.*

(iii) A has indefinite type iff A satisfies neither of the above.

Let $X = (P, \Delta, \Delta^\vee)$ be a root system with generalized Cartan matrix A. We say X is **finite** (resp. **affine**) if A is finite (resp. affine). The **dual** of X is the root system $X^\vee := (P^\vee, \Delta^\vee, \Delta)$, with associated generalized Cartan matrix A^t. The dual $X^\vee$ is finite (resp. affine) if and only if X is finite (resp. affine).

Cartan matrices of finite type were classified by Dynkin, and share the same classification as that of finite-dimensional semisimple Lie algebras over an algebraically closed field. By incorporating a weight lattice into our definition of a root system, one has that finite root systems classify reductive algebraic groups G over an algebraically closed field. Under this classification, the weight lattice P is the character group of a maximal torus in G. For a complex reductive Lie group G, we often refer to $(P, \Delta, \Delta^\vee)$ as the **Cartan data** specifying G.

Cartan matrices of affine type are classified in Kac and Macdonald, albeit with different labellings. A good compendium is given in Carter [17], in which he refers to the differing nomenclatures as the "Dynkin name" and the "Kac name". Following Carter, the Dynkin names (resp. Kac names) of the Cartan matrices of affine type are: the **untwisted types** $\widetilde{Z_n}$ (resp. $Z_n^{(1)}$), where $Z_n = A_n, B_n, C_n, D_n, E_{6,7,8}, F_4, G_2$ is a Cartan matrix of finite type; the **dual untwisted types** $\widetilde{B_n}^\vee, \widetilde{C_n}^\vee, \widetilde{F_4}^\vee$ and $\widetilde{G_2}^\vee$ (resp. $A_{2n-1}^{(2)}, D_{n+1}^{(2)}, E_6^{(2)}$, and $D_4^{(3)}$); and the

mixed type $\widetilde{BC_n}$ (resp. $A_{2n}^{(2)}$). The untwisted types whose duals not listed are isomorphic to their dual.

We mention that there also exist Cartan matrices of affine type coming from a *non-reduced* root system, which we choose not to consider. Similarly, we also will not consider generalized Cartan matrices of indefinite type in this treatise.

Each simple root $\alpha_i \in \Delta$ gives rise to a linear automorphism s_{α_i} of P via

$$s_{\alpha_i}(\beta) = \beta - \langle \beta, \alpha_i^\vee \rangle \alpha_i \tag{4.1}$$

The map s_{α_i} is a reflection, as it fixes the hyperplane $\langle \beta, \alpha_i^\vee \rangle = 0$ and sends α_i to $-\alpha_i$. Likewise, we have an analogous reflection $s_{\alpha_i^\vee}$ on $P^\vee$, which we will often identify with s_{α_i} via the non-degenerate pairing $\langle -, - \rangle$. We say s_{α_i} is a **simple reflection** and denote it by s_i.

The **Weyl group** W is the group of automorphisms of P (and of $P^\vee$) generated by the simple reflections s_i. The set of **roots** R and **coroots** $R^\vee$ are

$$R = \bigcup_i W(\alpha_i), \qquad R^\vee = \bigcup_i W(\alpha_i^\vee)$$

The **root lattice** Q and **coroot lattice** $Q^\vee$ are

$$Q = \mathbb{Z}\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{Z}P, \qquad Q^\vee = \mathbb{Z}\{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subseteq \mathbb{Z}P^\vee$$

The **positive roots (coroots)** are $R_+ = R \cap Q_+$ and $R_+^\vee = R^\vee \cap Q_+^\vee$ where $Q_+ = \mathbb{N}\{\alpha_1, \ldots, \alpha_n\}$ is the cone generated by the simple roots in Q, and $Q_+^\vee$ the respective cone in $Q^\vee$. We set $R_- = -R_+$ and $R_-^\vee = -R_+^\vee$ to be the **negative roots (coroots)**, respectively. The roots will decompose as $R = R_+ \cup R_-$, so that every root is either positive or negative. The **dominant weights** are elements of

$$P_+ = \{\lambda \in P \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for all } i\} \tag{4.2}$$

and the set of **strictly dominant weights** P_{++} consisting of those $\lambda \in P_+$ with a strict inequality in (4.2) for all i. The dominant weights are ordered by the **dominance ordering**, where we set $\lambda < \mu$ if $\mu - \lambda$ is in the positive root lattice Q_+. We write $P_\mathbb{R}$ for the extension by scalars $P \otimes \mathbb{R}$.

In the case that X is finite, then A is invertible, and R and W are finite sets. In the case X is affine, then corank $A = 1$, and R and W are infinite. If X is of untwisted type, then the affine roots can be decomposed as $R = R_0 + \mathbb{Z}\delta$, where R_0 are the roots of some finite root system, and δ is known is the **nullroot**. The positive roots are of the form $(R_0 + \mathbb{Z}_{>0}\delta) \cup (R_0)_+$. A more explicit description of W will be given in Section 4.2.

We say that A is **symmetrizable** if there is a diagonal matrix D such that DA is symmetric. In other words, there exist non-zero integers d_i (which can be assumed positive) such that $\langle \alpha_j, d_i\alpha_i^\vee \rangle = \langle \alpha_i, d_j\alpha_j^\vee \rangle$. One shows that if X is finite or affine, then A is symmetrizable (and vice versa). If A is indecomposable, then the integers d_i are unique up to an overall common factor, and we call d_i the **length** of the root α_i. If there are only two root lengths, we call them **long** and **short**; if there is only one root length, every root is both long and short.

Example 4.1.1. Let $P = \mathbb{Z}^n$ with the standard inner product so that the unit vectors e_i are orthogonal. We identify P with its dual $P^\vee$.

(a) The root system of GL_n has simple roots and coroots $\alpha_i = \alpha_i^\vee = e_i - e_{i+1}$ for $i = 1, \ldots, n-1$. The positive roots are of the form $e_i - e_j$ for $i < j$ and the dominant weights are non-increasing integer sequences $(\lambda_1 \geq \cdots \geq \lambda_n)$.

(b) Consider the constant vector $\vec{1} = e_1 + \cdots + e_n$, which satisfies $\langle \vec{1}, \alpha_i^\vee \rangle = 0$ for all i. Replacing P with $P/(\mathbb{Z}\vec{1})$ and keeping the same simple roots and coroots, considered now in the quotient, gives the root system of SL_n. The dominant weights are now an equivalence class of non-increasing integer sequences as before, up to translation by the vector $\vec{1}$; we identify these with partitions $(\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ of length at most n.

(c) Replacing P with the root lattice Q of GL_n gives the root system of the adjoint group PGL_n. It is dual to the root system of SL_n. All three of $\mathrm{GL}_n, \mathrm{SL}_n$, and PGL_n have the Cartan matrix of type A_{n-1}.

(d) Keeping $\alpha_i, \alpha_i^\vee$ the same as above for $i = 1, \ldots, n-1$, and further letting $\alpha_n = 2e_n$ and $\alpha_n^\vee = e_n$ gives the root system of Sp_{2n}. The positive roots are $\{e_i \pm e_j \mid i < j\} \cup \{2e_i\}$ and the dominant weights are non-increasing integer sequences $(\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$. The Cartan matrix is of type C_n.

(e) The root system of SO_{2n+1} is dual to the root system of Sp_{2n}. The simple roots and coroots for $i = 1, \ldots, n-1$ are the same as before, but now $\alpha_n = e_n$ and $\alpha_n^\vee = 2e_n$. The positive roots are $\{e_i \pm e_j \mid i < j\} \cup \{e_i\}$ and the dominant weights are non-increasing integer sequences $(\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$. Extending the weight lattice to $P = \mathbb{Z}^n \oplus \mathbb{Z}(\frac{1}{2}\vec{1})$ gives the root system of the simply connected form $\mathrm{Spin}(2n+1)$. In this case, the dominant weights are allowed to be sequences with half-integers, however with the condition $\lambda_i - \lambda_j \in \mathbb{Z}$. For both SO_{2n+1} and $\mathrm{Spin}(2n+1)$, the Cartan matrix is of type B_n.

(f) For the root system of SO_{2n}, the simple roots and coroots are the same as above for $i = 1, \ldots, n-1$, and we set $\alpha_n = \alpha_n^\vee = e_{n-1} + e_n$. The positive roots are $e_i \pm e_j$ for $i < j$ and the dominant weights are sequences of integers $(\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n|)$. The Cartan matrix is of type D_n.

4.2 Weyl groups

We recall some basic facts about Weyl groups. To start, the Weyl group W together with its generating set S of simple reflections s_i is a Coxeter group, with relations

$$(s_i s_j)^{m_{ij}} = 1 \tag{4.3}$$
$$s_i^2 = 1 \tag{4.4}$$

where m_{ij} is either 0, 2, 3, 4, or 6, depending on the product $a_{ij}a_{ij}$ of entries in the associated generalized Cartan matrix A. We will refer to (4.4) as the quadratic relation and (4.3) as the braid relations.

The **length** of a permutation $w \in W$ is the minimal ℓ such that $w = s_{i_1} \dots s_{i_\ell}$. Such an expression is called a reduced factorization and is in general not unique. Geometrically, if we view the Weyl group as acting on the set of roots R, the length of w is precisely the number of positive roots that are carried into negative roots by w. We define $\mathrm{Inv}(w) := \{\alpha \in R_+ \mid w(\alpha) \in R_-\}$ so that

$$\ell(w) = |\operatorname{Inv}(w)| = |R_+ \cap w^{-1}(R_-)| \tag{4.5}$$

In particular, α_i is the only positive root α such that $s_i(\alpha) \in R_-$, so that $\ell(ws_i) < \ell(w)$ if and only if $w(\alpha_i) \in R_-$. Likewise, $\ell(s_i w) < \ell(w)$ iff $w^{-1}(\alpha_i) \in R_-$. Equivalently, $\ell(ws_i) < \ell(w)$ (resp. $\ell(s_i w) < \ell(w)$) if and only if there is some reduced factorization of w that ends in (resp. begins with) s_i. We may also at times let $\mathrm{Inv}(w)$ denote the positive coroots that are sent to negative coroots by w.

The **Bruhat order** is the partial order on W that is the transitive closure of relation $u < v$ if $u = vs$ and $\ell(u) = \ell(v) + 1$ for some $s \in S$. More explicitly, $u \leq v$ if there is some (equivalently every) reduced word for v contains as a subword a reduced word for u. If $u \leq v$, then $\ell(u) \leq \ell(v)$, however the converse is not always true. Many other equivalent characterizations can be found e.g. in [9].

The complement of the hyperplanes $\langle \alpha^\vee, \cdot \rangle = 0$ over all $\alpha \in R$ is disconnected and each connected component is called a **chamber**. The **dominant chamber** is the chamber consisting of dominant weights; the faces of this chamber are given by the hyperplanes $\langle \Lambda_i, \cdot \rangle = 0$, where Λ_i are the **fundamental weights**, defined by $\Lambda_i(\alpha_j^\vee) = \delta_{ij}$.

The Weyl group W acts transitively on the set of chambers with the dominant chamber a fundamental domain for this action. In other words, for every $\lambda \in P$, there is a unique dominant weight $\lambda_+ \in P_+$ in the orbit of λ. We say λ is **regular** if it is in the interior of a chamber. In this case, λ_+ is strictly dominant and the permutation $w \in W$ with $w(\lambda_+) = \lambda$ is unique.

Given $J \subseteq S$, the **parabolic subgroup** $W_J \subseteq W$ is the subgroup generated by J. For $W = S_n$, the parabolic subgroups are isomorphic to the Young subgroups $S_{r_1} \times \cdots \times S_{r_j}$. We write W^J for minimal length coset representatives of W/W_J and likewise ${}^J W$ for those of W_J/W. These are precisely the subgroup of permutations with no reduced word ending (resp. beginning) with $s \in J$. If W is the Weyl group for a Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, then a choice of J is a choice of simple roots, from which one can construct a parabolic subalgebra $\mathfrak{p}$. The parabolic subalgebra $\mathfrak{p}$ has a Levi decomposition, say with a Levi factor $\mathfrak{l}$, which gives rise to a **Levi subgroup** L of G. For GL_n, the Levi subgroups are of the form $\mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k}$, where $r_1 + \cdots + r_k = n$. For Sp_{2n}, the Levi subgroups are of the form $\mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_{k-1}} \times \mathrm{Sp}_{2r_k}$.

Example 4.2.1. (a) The Weyl group for GL_n and SL_n is the symmetric group S_n, acting on the weight lattice $\mathbb{Z}^n$ and generated by the simple transpositions $s_i = (i, i+1)$ for $i = 1, \ldots, n-1$.

(b) The Weyl group for Sp_{2n} and SO_{2n+1} is the group of signed permutations as discussed in the beginning of Section 3.1. We recall that it is generated by the simple reflections $s_1, \ldots, s_{n-1}$ as above, along with the generator s_n which swaps n with $-n$.

(c) The Weyl group for SO_{2n} again acts on the set $[\pm n]$, and is generated by $s_1, \ldots, s_{n-1}$, along with the generator s_n which swaps $n-1$ with n and negates both. An arbitrary element can permute entries and negate an even number of entries in $[\pm n]$.

Affine Weyl groups

There are two notions of an "affine Weyl group": the first being the Weyl group of an affine root system, and the second being the affinization of a finite Weyl group. Although related, we take the time to review both, and for clarity we refer only to the latter as an affine Weyl group.

If $X = (P, \Delta, \Delta^\vee)$ is a finite root system with root lattice Q, then we define its **affine Weyl group** W_a to be the semidirect product $W \ltimes Q$. Concretely, if we let $\tau(\alpha) \in W_a$ denote the element corresponding to $\alpha \in Q$, then W_a is generated by the subgroups W and Q, with the additional relation

$$s_i \tau(\alpha) s_i = \tau(s_i \alpha) \tag{4.6}$$

As it will make an essential appearance later, we also define the **extended affine Weyl group** $\widetilde{W}$ to be the semidirect product $W \ltimes P$. We can extend the action of W on P to $\widetilde{W}$ by setting

$$\tau(\mu)(\lambda) = \lambda + \mu \tag{4.7}$$

so that the element $\tau(\mu)$ corresponds to translation by μ. The reason we call this group W_a an affine Weyl group is because it is generated by *affine reflections*, namely the reflections $s_{\alpha,m}$, where

$$s_{\alpha,m}(\lambda) = \lambda - (\langle \lambda, \alpha^\vee \rangle + m)\alpha \tag{4.8}$$

for $\alpha \in R$ and $m \in \mathbb{Z}$. The map $s_{\alpha,m}$ is precisely a reflection about the affine hyperplane $\langle \alpha^\vee, \cdot \rangle = -m$. In fact, W_a is isomorphic to the group generated by the $s_{\alpha,m}$, via

$$s_{\alpha,m} \mapsto s_\alpha \tau(m\alpha) \tag{4.9}$$

Indeed,

$$s_{\alpha,m}(\lambda) = \lambda - (\langle \lambda, \alpha^\vee \rangle + m)\alpha = s_\alpha(\lambda) - m\alpha = s_\alpha(\lambda + m\alpha) = (s_\alpha \tau(m\alpha))(\lambda)$$

If θ is the dominant short root (so that $\theta^\vee$ is the highest coroot), then the orbit $W(\theta)$ consists of all short roots and hence spans Q. We define the affine reflection s_0 as

$$s_0 := \tau(\theta)s_\theta : \lambda \mapsto s_\theta(\lambda) + \theta \tag{4.10}$$

Then, W_a is generated by $S = \{s_0, s_1, \ldots, s_n\}$, and in fact (W_a, S) forms a Coxeter system, i.e. the generators satisfy the usual braid relations. We note however that $\widetilde{W}$ is not in general a Coxeter group.

Example 4.2.2. For GL_n, the dominant root (and coroot) is $e_1 - e_n$ and so

$$s_0(\lambda) = \lambda - (\lambda_1 - \lambda_n - 1)(e_1 - e_n) = (\lambda_n + 1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_1 - 1)$$

For Sp_{2n}, the dominant short root is $e_1 + e_2$ and $s_0(\lambda) = (-\lambda_2, -\lambda_1, \lambda_3, \ldots, \lambda_n)$. For SO_{2n+1}, the dominant short root is e_1 and $s_0(\lambda) = (-\lambda_1, \lambda_2, \ldots, \lambda_n)$.

We now come to the essential correspondence for affine Weyl groups.

Proposition 4.2.1. *Given the constructions W_a and $\widetilde{W}$ associated to a finite root system X, there exists an affine root system Y whose Weyl group W is (1) isomorphic to W_a and (2) has a semidirect extension $\Pi \ltimes W$ isomorphic to $\widetilde{W}$. Conversely, given any affine root system Y, its Weyl group W is isomorphic to the affine Weyl group $W_a = W_0 \ltimes Q_0$ of some finite root system X.*

We give a cursory explanation of how this correspondence comes about, as the details will come into play when we discuss a Lie-theoretic interpretation of a k-core.

Suppose first that we have $X = (P, \Delta, \Delta^\vee)$ a finite root system, with Weyl group W and affine Weyl groups $W_a = W \ltimes Q$ and $\widetilde{W} = W \ltimes P$ constructed prior. We construct an affine root system Y as follows. Set $P' = P^\vee \oplus \mathbb{Z}$, fix a non-zero $\delta \in \mathbb{Z}$, and extend the pairing $\langle P^\vee, P \rangle \to \mathbb{Z}$ by declaring $\langle \delta, P \rangle = 0$.

Let $\theta^\vee \in P^\vee$ be a dominant coroot, associated to the root $\theta \in P$. Define the elements

$$\alpha_0^\vee = \delta - \theta^\vee, \qquad \alpha_0 = -\theta$$

and set $\Delta' = \{\alpha_0^\vee\} \cup \Delta^\vee$ and $(\Delta')^\vee = \{\alpha_0\} \cup \Delta$. If X is type Z_n, then the root system $Y := (P', \Delta', (\Delta')^\vee)$ is an affine root system of the following type: (1) if $\theta^\vee$ is the highest coroot, then Y is of untwisted type $\widetilde{Z_n}$; (2) if $\theta^\vee$ is a dominant short coroot (so θ is the highest root), then Y is of dual untwisted type $\widetilde{Z_n}^\vee$; (3) if one takes $\theta^\vee$ to be one half a long coroot or twice a short coroot in a non-reduced finite root system containing X of type B_n or C_n, then Y is the mixed type $\widetilde{BC_n}$.

In the first case, when $\theta^\vee$ is the highest coroot, then the Weyl group $W(Y)$ is isomorphic to $W_a = W \ltimes Q$. Indeed, $W(Y)$ is generated by elements $s_0, s_1, \dots, s_n$, where s_i fixes δ and acts via its original action on $P^\vee$ for $i \neq 0$, and

$$s_0(\lambda^\vee) = \lambda^\vee - \langle \lambda^\vee, \alpha_0 \rangle \alpha_0^\vee = \lambda^\vee - \langle \lambda^\vee, -\theta \rangle (\delta - \theta^\vee) = s_\theta(\lambda^\vee) + \langle \lambda^\vee, \theta \rangle \delta \tag{4.11}$$

Having the subgroup $Q \subseteq W_a$ act on P' via

$$\tau(\mu)(\lambda^\vee) = \lambda^\vee - \langle \lambda^\vee, \mu \rangle \delta \tag{4.12}$$

identifies s_0 with the generator $\tau(\theta)s_\theta \in W_a$ defined in (4.10).

Now let $Y = (P, \Delta, \Delta^\vee)$ be an affine root system, with generalized Cartan matrix A and Weyl group $W = W(Y)$. We label the simple roots and coroots as

$$\Delta = \{\alpha_0, \dots, \alpha_n\} \qquad \Delta^\vee = \{\alpha_0^\vee, \dots, \alpha_n^\vee\} \tag{4.13}$$

As Y is affine, A is singular and so we can find a with $Aa = 0$. Moreover, letting $\Delta_0 = \{\alpha_1, \dots, \alpha_n\}$ and $\Delta_0^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$, the root system $Y_0 = (P, \Delta_0, \Delta_0^\vee)$ is finite. Without loss we suppose $a_0 = 1$ and define the elements

$$\delta = \alpha_0 + a_1\alpha_1 + \cdots + a_n\alpha_n, \qquad \theta = \delta - \alpha_0 \tag{4.14}$$

We note that

$$\langle \delta, \alpha_i^\vee \rangle = (Aa)_{ij} = 0 \qquad (i = 0, \dots, n) \tag{4.15}$$

$$\langle \theta, \alpha_i^\vee \rangle = -\langle \alpha_0, \alpha_i^\vee \rangle \geq 0, \qquad (i = 1, \dots, n) \tag{4.16}$$

from which we conclude that W fixes δ and θ is a dominant root of P.

We consider the finite root system that is the dual of Y_0, namely $X = (P^\vee, \Delta_0^\vee, \Delta_0)$. If Y is of untwisted type, then from above we conclude that $W \simeq W_a$, where W_a is the affine Weyl group of X. In the case that Y is of dual untwisted type, then $W \simeq W(Y^\vee) \simeq W_a'$, where W_a' is the affine Weyl group of $X^\vee = Y_0$.

To summarize, an affine Weyl group is isomorphic to the Weyl group of an affine root system that is of dual type to the original finite root system. In light of this, one may often see $\widetilde{W}$ defined as $W \ltimes P^\vee$. There is similarly a left handed presentation as $P \rtimes W$; this will only make a difference when dealing for example with double affine Hecke algebas, which we will not concern ourselves with here.

For the extended affine Weyl group, it will be convenient to visualize its action, and that of W_a, with the following alcove picture.

We fix a finite root system $X = (P, \Delta, \Delta^\vee)$ and untwisted affine root system $Y = (P', \Delta', (\Delta')^\vee)$ with $W(Y) \simeq W_a(X)$. Following the constructions prior, we write

$$P' = P^\vee \oplus \mathbb{Z}\delta \qquad \Delta = \{\alpha_1, \dots, \alpha_n\}, \qquad \Delta^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \tag{4.17}$$

$$\Delta' = \{\alpha_0^\vee\} \cup \Delta^\vee, \qquad (\Delta')^\vee = \{\alpha_0\} \cup \Delta$$

We keep the elements δ, θ as defined in (4.14), so that $\theta^\vee \in P^\vee$ is the highest coroot. For any $k \in \mathbb{R}$, we define the **level k plane** H_k to be

$$H_k = \{x \in (P')^\vee_{\mathbb{R}} \mid \langle \lambda, \delta \rangle = k\} \tag{4.18}$$

If we define the "0th fundamental weight" $\Lambda_0 \in (P')^\vee$ by

$$\langle \Lambda_0, \alpha_j^\vee \rangle = \delta_{j0} \tag{4.19}$$

then $\langle \Lambda_0, \delta \rangle = 1$, the elements $\{\delta, \alpha_1, \ldots, \alpha_n, \Lambda_0\}$ are linearly independent, and in the case when they form a basis of $(P')^\vee$, then H_k consists of weights λ whose coefficient of Λ_0 is k.

The Weyl groups $W(Y) \simeq W_a(X)$ and $\widetilde{W} = W(X) \ltimes P$ fix δ, hence preserve each H_k. Orthogonally projecting onto $P_{\mathbb{R}}$ induces an affine action of $\widetilde{W}$, known as the **level k action**. The permutations $w \in W(X)$ act as usual, and the translations $P \subseteq \widetilde{W}$ act by

$$\tau(\mu)(\lambda) = \lambda - k\mu, \qquad \mu \in P \tag{4.20}$$

The connected components bounded by all the affine root hyperplanes tessellate the level k plane into **alcoves**. The fact that θ is the highest short root makes the following **level k fundamental alcove** a fundamental domain for the level k action:

$$\mathcal{A}_k = \{x \in P_{\mathbb{R}} \mid \langle \theta^\vee, x \rangle < k \text{ and } \langle \alpha_i^\vee, x \rangle \geq 0 \text{ for } i = 1, \ldots, n\} \tag{4.21}$$

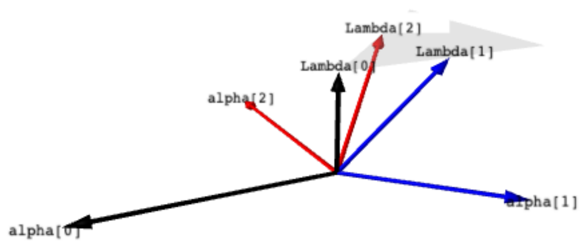
In particular, the level 1 fundamental alcove is bounded by the hyperplanes H_{α_i} for the simple roots α_i. In Figure 4.1, we see the projection of the weight lattice of type $\widetilde{C_2}$ to $\mathbb{R}\{\alpha_1, \alpha_2, \Lambda_0\}$ (including the δ direction would give a 4-dimensional picture). Shaded on the $\Lambda_0 = 1$ plane is the level 1 fundamental chamber, and further projecting to the weight lattice of C_2 gives the level 1 alcove picture. The level 2 fundamental chamber, for example, would be on the $\Lambda_0 = 2$ plane, and projecting down would result in a fundamental alcove that is twice as big as pictured.

Heuristically, for a fixed level k, we can visualize the roots of the affine root system Y as lying on planes stacked on top of each other, in the δ direction, that is, each stack is the plane $\delta = a$. The Weyl group $W(Y)$ acts by affine reflections, perhaps also moving a weight to a different stack. Projecting down collapses the stacks into the alcove setting pictured in Figure 4.1b.

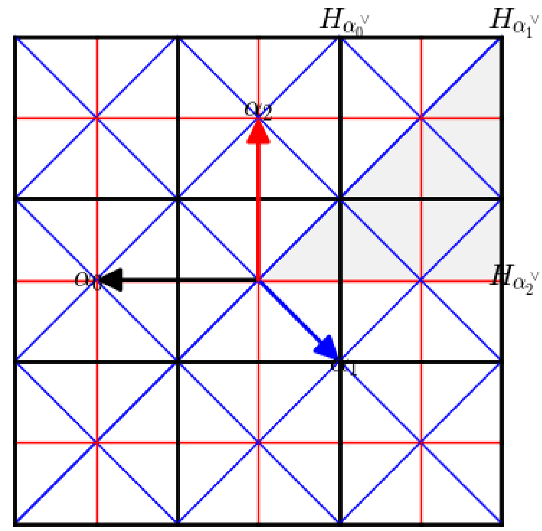
Example 4.2.3. Let $G = \mathrm{GL}_3$, with alcoves pictured in Figure 4.2. Each alcove $w(\mathcal{A})$ corresponds to an affine reflection $w \in W_a$. The level 1 fundamental alcove is labelled 1. Let $\omega = e_1$ and consider the element $\pi = s_2 s_0 \tau(\omega)$. Note that π stabilizes the fundamental alcove and permutes its walls, in particular cycles the hyperplanes $H_{\alpha_0} \to H_{\alpha_1} \to H_{\alpha_2} \to H_{\alpha_0}$.

If we define $\Pi \subseteq \widetilde{W}$ to be the stabilizer of $\mathcal{A} := \mathcal{A}_1$, we note that Π permutes the walls of $\mathcal{A}$. We can identify the walls of $\mathcal{A}$ with the simple reflections $S = \{s_0, \ldots, s_n\}$, so that

$$\pi(H_{\alpha_i}) = H_{\alpha_j} \iff \pi s_i = s_j \pi \tag{4.22}$$



(a) The orthogonal projection of the weight lattice of $\widetilde{C_2}$ to the weight lattice of C_2 and $\mathbb{R}\Lambda_0$. Pictured are the simple roots $\alpha_0, \alpha_1, \alpha_2$ and their associated fundamental weights $\Lambda_0, \Lambda_1, \Lambda_2$. Shaded is the fundamental chamber on the level 1 plane.



(b) The further projection of the weight lattice on the left to that of C_2. The fundamental chamber is shaded and the level 1 fundamental alcove is the shaded alcove nearest the origin.

Figure 4.1: The alcove picture for C_2, drawn using SageMath.

Hence Π normalizes $S \subseteq W_a$, and we arrive at

$$\widetilde{W} \simeq \Pi \ltimes W_a \tag{4.23}$$

Given the presentations $W_a \simeq W \ltimes Q$ and $\widetilde{W} \simeq W \ltimes P$, together with (4.23), it follows that

$$\Pi \simeq P/Q \tag{4.24}$$

and so Π is in bijection with the set of **minuscule weights**, i.e. dominant weights ω such that $\langle \omega, \theta^\vee \rangle \leq 1$. To be precise, every $\pi \in \Pi$ can be written $\pi = \tau(\omega)v$, where $v \in W$ and ω is a minuscule weight.

4.3 Extended affine Weyl groups

To recap, given a finite Weyl group W and associated affine Weyl group W_a, we have the following two descriptions of the extended affine Weyl group $\widetilde{W}$:

$$\widetilde{W} \simeq W \ltimes P \simeq \Pi \ltimes W_a \tag{4.25}$$

where Π is the stabilizer of the fundamental alcove, defined in the previous section. We remarked earlier that $\widetilde{W}$ is not Coxeter group; however, it can still be given a Bruhat order

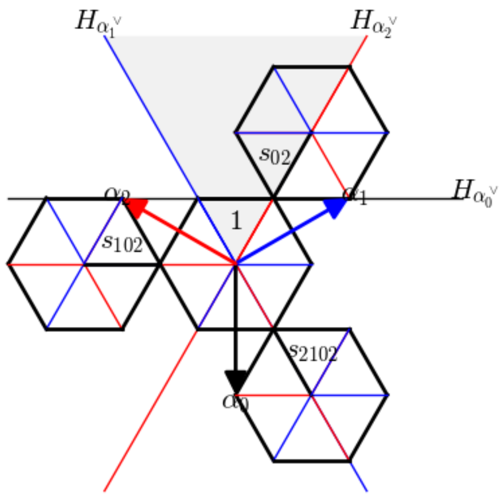


Figure 4.2: Collection of alcoves in level 1 alcove picture for GL_3, drawn by SageMath.

and length function. In particular, given $\pi w \in \widetilde{W}$ with $w \in W_a$, we define $\ell(\pi w) = \ell_{W_a}(w)$ and $\pi w \leq \pi' v$ if $w \leq v$ in W_a. Pictorially, $\ell(\pi w)$ is still the number of hyperplanes separating the fundamental alcove $\mathcal{A}$ from $\pi w(\mathcal{A})$. With this description, we note that Π is precisely the elements of length zero. Explicitly, for $w \in W$ and $\lambda \in P$, we have

$$\ell(w\tau(\lambda)) = \sum_{\alpha^\vee \in \mathrm{Inv}(w)} |\langle \lambda, \alpha^\vee \rangle + 1| + \sum_{\alpha^\vee \in R_+^\vee \setminus \mathrm{Inv}(w)} |\langle \lambda, \alpha^\vee \rangle| \tag{4.26}$$

When $\lambda \in P_+$, then $\ell(\tau(\lambda)) = \langle 2\rho, \lambda \rangle$, where ρ is the usual Weyl vector. From (4.26), when $w \in W, \lambda \in P_+$, then

$$\ell(w\tau(\lambda)) = \ell(w) + \ell(\tau(\lambda)) \tag{4.27}$$

so that $w\tau(\lambda)$ is a reduced expression. Similarly, $\tau(\lambda)w$ is a reduced expression for $w \in W$, $\lambda \in P_-$.

In light of (4.25), we can identify P with the the set of minimal coset representatives in $\widetilde{W}/W$. The Bruhat order on $\widetilde{W}$ then induces a partial order $<$ on P, which we also refer to as the Bruhat order. In particular, for $\lambda \in P$, we let $\tilde{\lambda} \in \widetilde{W} \cdot \lambda$ be the unique element in $P \cap \mathcal{A}$, and let v_λ be minimal such that $v_\lambda(\tilde{\lambda}) = \lambda$. For $\lambda, \mu \in P$, we write

$$\lambda < \mu \text{ if and only if } \tilde{\lambda} = \tilde{\mu} \text{ and } v_\lambda < v_\mu \tag{4.28}$$

To be more explicit, for a weight λ and root $\alpha \in R$, consider the root string $\lambda + \mathbb{Z}\alpha$. The Bruhat order on this root string is the total order $\lambda < \mu$ if $|\langle \lambda, \alpha^\vee \rangle| < |\langle \mu, \alpha^\vee \rangle|$ or if $\langle \lambda, \alpha^\vee \rangle = -\langle \mu, \alpha^\vee \rangle > 0$, i.e. λ is on the positive side of the α_i-hyperplane, and μ is on the negative side. The Bruhat order is the transitive closure of these relations over all positive roots.

Alternatively, for weights λ, μ in the same W-orbit the Bruhat order is the transitive closure of the relation $s_i\lambda > \lambda$ if $\langle \lambda, \alpha_i^\vee \rangle > 0$. For arbitrary weights $\lambda, \mu \in P$, we have

$$\lambda < \mu \text{ if and only if } \lambda_+ < \mu_+ \text{ or } \lambda_+ = \mu_+ \text{ and } \lambda \leq \mu \tag{4.29}$$

where $\lambda_+ < \mu_+$ refers to the dominance order defined in Section 4.2.

In any of the formulations, we see that the minimal elements with respect to the Bruhat order are the minuscule weights, and the minimal and maximal elements in an orbit $W \cdot \lambda$ are λ_+ and λ_-, respectively.

Example 4.3.1. Consider the level 1 alcove picture of GL_3 pictured in Figure 4.2. The alcoves pictured are alcoves in the double coset $W\tau(\lambda)W$ where $\lambda = 2e_1$ (along with alcoves parameterized by the finite Weyl group). The minimal elements of each coset $w\tau(\lambda)W$ are labelled.

Relationship to cores and quotients

We take a brief detour in this section to discuss how the level k action of an extended affine Weyl group relates to the combinatorics of k-cores and k-quotients. Much of this theory can be found in [28, §6]. We briefly set $G = \mathrm{GL}_n$.

Recall that if a skew shape μ/ν is a k-ribbon, then $\mu+\delta$ is some permutation of $\nu+\delta+ke_i$ for a unit vector e_i, where $\delta = \delta_n = (-1, \ldots, -n)$. More generally, if μ/ν can be tiled by k-ribbons, then $\mu + \delta$ is some permutation of $\nu + \delta + k\lambda$ for some $\lambda \in P$. In other words,

$$\mu/\nu \text{ can be tiled by k-ribbons} \iff \mu + \delta \in \widetilde{W} \cdot (\nu + \delta) \tag{4.30}$$

where $\widetilde{W}$ acts via the level k action given in (4.20). If $\eta = \mathrm{core}_k(\mu)$, then $\eta + \delta$ is a minimal strict partition in its orbit $\widetilde{W} \cdot (\eta + \delta)$. More specifically, η is of the form

$$k > \eta_1 = \cdots = \eta_{r_1} > \eta_{r_1+1} = \cdots = \eta_{r_1+r_2} > \cdots > \eta_{n-r_\ell+1} = \cdots = \eta_n \geq 0 \tag{4.31}$$

Given k, every weight μ is in the orbit $\widetilde{W} \cdot \eta$ for some η, as one can translate μ to reduce it mod k and then rearrange to form η in (4.31). This choice of k and η is equivalent to choosing a weight in the level k fundamental alcove, defined in (4.21). The fact that every μ is in some $\widetilde{W} \cdot \eta$ is merely the fact that the fundamental alcove is a fundamental domain for the level k action.

To see how the k-quotient appears in this story, we discuss another object indexed by dominant weights in $\mathcal{A}_k$. Let G be arbitrary now and say we choose $k \in \mathbb{N}$ and $\eta \in P_+$ on the level k fundamental alcove as in (4.31), that is we have $\eta \in P_+$ and $\langle \eta, \theta^\vee \rangle < k$, where $\theta^\vee$ is the highest coroot. The walls on which η lie determine a parabolic subgroup, namely we set $J = \{j \mid \langle \eta, \alpha_j^\vee \rangle = 0\}$. For $G = \mathrm{GL}_n$, the parabolic subgroup determined by the choice of η in (4.31) is the Young subgroup $W_J = S_{r_1} \times \cdots \times S_{r_\ell}$.

It follows that $\mathrm{Stab}^{\widetilde{W}}(\eta) = W_J$, where again $\widetilde{W}$ acts via the level k action, and so we can identify the cosets $\widetilde{W}/W_J$ with the orbit $\widetilde{W} \cdot \eta$. We then identify the double cosets $W \backslash \widetilde{W}/W_J$

with W-orbits in $\widetilde{W}\cdot\eta$, each of which contains a unique dominant weight $\mu \in P_+$. We arrive at a bijection

$$P_+ \cap \widetilde{W}\cdot\eta \xrightarrow{\sim} W\backslash\widetilde{W}/W_J \quad \mu = w\tau(\beta)\cdot\eta \mapsto W\tau(\beta)W_J \tag{4.32}$$

There is a second indexing of double cosets, owing to the presentation $\widetilde{W} \simeq W \ltimes P$. One canonically identifies $\widetilde{W}/W$ with the weight lattice P, and consequently $W_J\backslash\widetilde{W}/W$ with W_J orbits in P. Each W_J orbit of P has a unique L-dominant weight $\beta \in P_+(L)$, and so we arrive at a bijection

$$P_+(L) \simeq W_J\backslash\widetilde{W}/W, \quad \beta \mapsto W_J\tau(\beta)W \tag{4.33}$$

Composing (4.32) with the canonical bijection

$$W\backslash\widetilde{W}/W_J \simeq W_J\backslash\widetilde{W}/W \qquad WwW_J \mapsto W_Jw^{-1}W \tag{4.34}$$

we have a correspondence

$$P_+(L) \simeq P_+ \cap \widetilde{W}, \qquad \beta \mapsto \mu := w(\eta + k\beta) \tag{4.35}$$

where w is such that $w(\eta + k\beta)$ is dominant. We note that restricting (4.35) to regular weights gives the same bijection between $P_{++}(L)$ and P_{++}. We come to the following

Proposition 4.3.1. *Set $G = \mathrm{GL}_n$ and $\delta = \delta_n = (-1, \dots, -n)$.*

1. *There is a bijective correspondence*

$$k\text{-cores } \nu \text{ with } \ell(\nu) \leq n \leftrightarrow \text{dominant weights } \eta \text{ in the } k\text{th fundamental alcove} \tag{4.36}$$

   More precisely, to a k-core ν we find $\eta \in \mathcal{A}_k$ such that $\nu + \delta \in \widetilde{W}\cdot\eta$. This is accomplished by reducing $\nu + \delta$ mod k and then rearranging into nonincreasing order. In the other direction, given $\eta \in \mathcal{A}_k$, let L be the Levi determined by the walls on which η lies and let δ_L be the concatenation of the weights δ_{r_i} for each of the Levi factors. Set $\nu + \delta$ to be the strictly dominant weight in the orbit of $\eta + k\delta_L$.

2. *If μ has $\mathrm{core}_k(\mu) = \nu$, where ν corresponds to $\eta \in \mathcal{A}_k$, then we can write $\mu + \delta = w(\eta + k(\beta + \delta_L))$, where $\beta \in P_+(L)$.*

3. *Set $\mu + \delta = w(\eta + k(\beta + \delta_L))$ as in (ii) and write*

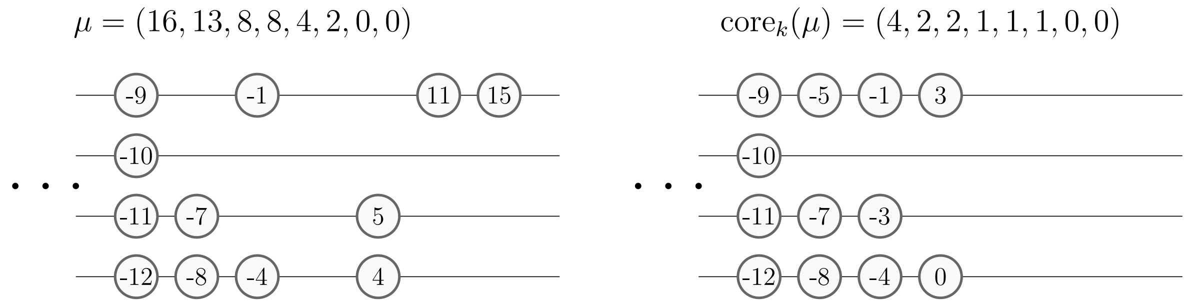
$$\beta = (\beta^{(1)}, \dots, \beta^{(k)}) \qquad \text{where} \quad \beta^{(i)} = (\beta^{(i)}_1 \geq \cdots \geq \beta^{(i)}_{r_i}) \tag{4.37}$$

   If $\mathrm{quot}_k(\mu)$ has shapes whose origins are placed on the content lines $q_1, \dots, q_k$, then $\beta^{(k-i+1)}/(q_i^{r_i})$ is precisely the i^{th} shape in $\mathrm{quot}_k(\mu)$.

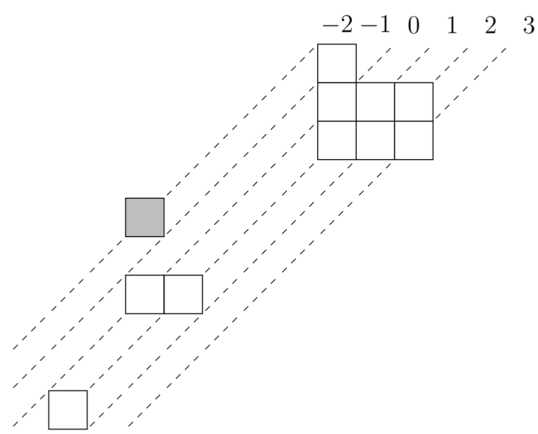
Remark 4.3.1. We make note that the bijection (4.35) depends on the choice of k and η. Combinatorially, we may view this for GL_n as the fact that β is a quotient of infinitely many partitions μ, which is determined by choosing k and a k-core.

All of these correspondences are best illustrated with an example, as we do below in Example 4.3.2. More details can also be found in [28, Prop. 6.18].

Example 4.3.2. Let $G = \mathrm{GL}_8$ and $k = 4$. We draw the abacus of a partition μ and its k-core below.



The quotient $\mathrm{quot}_k(\mu)$ is the tuple of skew shapes



where the shaded box denotes the empty partition placed on the -2 content line. We place the shapes with their origins on the content lines $1, 0, -2, 1$ because the four ribbons that can be added to $\mathrm{core}_k(\mu)$ have contents $c_i = q_i k + (i-1)$, where $q_1 = 1, q_2 = 0, q_3 = -2, q_3 = 1$. One can read the values q_i on the abacus by noticing that the four beads that can be moved right, one on each rung read bottom to top, will be moved to occupy spots in columns $1, 0, -2, 1$, where column 0 denotes the column with the 0 bead.

Under the correspondence (4.36), the number of beads on the rung corresponding to residue r in the abacus of the k-core is the multiplicity of r in η. In our case, we have $\eta = (3, 3, 3, 1, 1, 0, 0, 0)$ (ignoring the infinitely many negative beads after -8), which determines the Levi $L = \mathrm{GL}_3 \times \mathrm{GL}_0 \times \mathrm{GL}_2 \times \mathrm{GL}_3$. We write

$$\mu + \delta = w(\eta + k(\beta + \delta_L)), \qquad \beta = (4, 4, 2, 2, 0, 2, 1, 1) \tag{4.38}$$

Writing β as in (4.37) gives $\beta^{(1)} = (4, 4, 2), \beta^{(2)} = \emptyset, \beta^{(3)} = (2, 0), \beta^{(4)} = (2, 1, 1)$. We note that for each $1 \le i \le k$, $\beta^{(k-i+1)}/(q_i^{r_i})$ is precisely the i^{th} skew shape in the quotient $\mathrm{quot}_k(\mu)$.

We note that in GL_n, the weight η in (4.31) is what is known as a $(k-1)$-bounded partition. The correspondence (4.36) is then a bijection between k-cores and $(k-1)$-bounded partitions, for which there is already a known bijection given by Lapointe and Morse [56] in their study of k-atoms. However, the two bijections are indeed different: to read off the $(k-1)$-bounded partition λ in the spirit of [56], one counts the number of gaps from one bead in the abacus of a k-core to the spot directly left of it. Continuing with Example 4.3.2, we have $\lambda = (2,1,1,1,1,1,0,0)$, since for example there are two missing beads between the bead 3 and the bead -1 in $\mathrm{core}_k(\mu)$, namely the spot at 2 and the spot at 1.

There is also a bijection between k-cores and minimal length coset reps of $\widetilde{S_k}/S_k$, which is discussed in [8] and related to the Lapointe-Morse bijection; at the present we don't know of any direct connections to the construction of η here.

4.4 Hecke algebras

In this section we introduce the Hecke algebra, keeping in mind that our ultimate goal is toward a definition of LLT polynomials in general Lie type. We choose to review finite Hecke algebras first, although we will be mainly interested in Hecke algebras arising from an extended affine Weyl group. We attempt to keep the material self-contained, although a previous acquaintance may be of use. Any prerequisites concerning Hecke algebras can be found e.g. in [40] or [9].

Definition 4.4.1. Let $W = (W,S)$ be a Coxeter system. The **Hecke algebra** $\mathcal{H} = \mathcal{H}(W,q)$ is the $\mathbb{Z}[q^{\pm 1}]$-algebra with linear basis $\{T_w \mid w \in W\}$ and relations

$$T_u T_w = T_{uw} \qquad \text{if } \ell(uw) = \ell(u) + \ell(w) \tag{4.39}$$

$$(T_s - q)(T_s + 1) = 0 \qquad s \in S \tag{4.40}$$

The presence of q^{-1} means that the basis elements are invertible, with

$$T_s^{-1} = q^{-1} T_s + (q^{-1} - 1) \tag{4.41}$$

The relations (4.39)-(4.40) are often combined as

$$T_s T_w = \begin{cases} T_{sw} & : \ell(sw) > \ell(w) \\ (q-1)T_w + qT_{sw} & : \ell(sw) < \ell(w) \end{cases} \tag{4.42}$$

We let $\mathcal{H}_{\mathbb{C}}$ denote the free $\mathbb{C}[q^{\pm 1}]$-module $\mathcal{H} \otimes_{\mathbb{Z}} \mathbb{C}$. We note that the parameter q is treated here as a formal invertible variable, and as such $\mathcal{H}$ is often referred to as a *generic Hecke algebra*. Specializing q and W in various ways leads to previously well studied algebras.

For example, (4.40) specializes at $q = 1$ to the quadratic relation in W, and hence $\mathcal{H}_{\mathbb{C}}$ is a q-deformation of the group algebra $\mathbb{C}W$. When W is finite, Tits' deformation argument [12, Ch.IV, §2] shows that in fact $\mathcal{H}_{\mathbb{C}}$ is generically semisimple and isomorphic to $\mathbb{C}W$; however, in order to write down an explicit bijection, one needs to enlarge the ground ring to include $q^{1/2}$. As such, often the Hecke algebra has as its ground ring $\mathbb{Z}[q^{1/2}, q^{-1/2}]$, and the second relation may be written as

$$(T'_s - q^{1/2})(T'_s + q^{-1/2}) = 0 \tag{4.43}$$

which is seen to be equivalent to the first definition by the substitution $T'_s = q^{-1/2}T_s$. It is an annoying fact of the literature that this definition seems to be non-standardized. We will choose to include half integer powers of q, although we keep the relation (4.40).

When W is a finite Weyl group and q a prime power, $\mathcal{H}$ coincides with the *Iwahori-Hecke* algebra of B-bi-invariant functions on G, where G is a Chevalley group over $\mathbb{F}_q$ with Weyl group W and B a Borel subgroup [43]. When W is an affine Weyl group and q prime, the same conclusion holds [44], now with G a split q-adic Chevalley group and I an Iwahori subgroup of G. In both cases, the multiplication in $\mathcal{H}$ is interpreted as convolution of functions in the latter algebras.

The case when W is an affine Weyl group can be further extended to generic q by replacing functions supported on $I \setminus G/I$ by perverse sheaves on a certain affine flag variety. This leads to many beautiful and deep statements in geometric representation theory, which we unfortunately will not touch here, although we may at points make mention of. The interested reader is encouraged to reference [20].

Define the **bar involution**, denoted $\bar{\cdot}$, to be the unique $\mathbb{Z}$-linear involution on $\mathcal{H}$ which sends $\overline{T_w} = T_{w^{-1}}^{-1}$ and $\overline{q} = q^{-1}$. The **Kazhdan-Lusztig basis** $\{C_w\}_{w\in W}$ is uniquely determined by the properties

$$\overline{C_w} = C_w \tag{4.44}$$

$$C_w = (-1)^{\ell(w)} q^{\ell(w)/2} \sum_{z\leq w} (-q)^{-\ell(z)} P_{z,w}(q^{-1}) T_z \tag{4.45}$$

where $P_{z,w}(q) \in \mathbb{Z}[q]$ is a polynomial in q of degree $\leq \frac{1}{2}(\ell(w) - \ell(z) - 1)$ for $z < w$ and $P_{w,w} = 1$. The polynomials $P_{z,w}$ are **Kazhdan-Lusztig polynomials** and both these and the basis elements C_w were introduced by Kazhdan and Lusztig [50] in their exploration of singularities of Schubert varieties.

Remark 4.4.1. Kazhdan and Lusztig also defined another basis C'_w that is invariant under $\bar{\cdot}$, and given instead by

$$C'_w = q^{-\ell(w)/2} \sum_{z\leq w} P_{z,w}(q) T_z \tag{4.46}$$

In [50], the basis elements C_w are used to construct representations of $\mathcal{H}$; the 1-dimensional span of C_{w_0} affords the sign representation, whereas the span of C'_{w_0} affords the trivial representation.

The Kazhdan-Lusztig polynomials have many geometric interpretations and the question of their positivity as been a subject of much study in the recent decades. While a modified proof of their positivity will be the reason why general type LLT polynomials are an $\mathbb{N}[q]$-linear combination of irreducible characters, the reader should rest assured that no knowledge of the underlying geometry will be needed in the definition of general type LLT polynomials.

4.5 Extended affine Hecke algebras

We fix the Cartan data $X = (P, \Delta, \Delta^\vee)$ specifying a complex reductive Lie group G. Let $W, W_a, \widetilde{W}$ be the associated Weyl group, affine Weyl group, and extended affine Weyl group, respectively, constructed in the previous sections.

For the general definition of LLT polynomials that will follow, we will be interested in the **extended affine Hecke algebra** $\widetilde{\mathcal{H}} = \mathcal{H}(\widetilde{W}, q)$. To be precise, $\widetilde{W}$ is not a Coxeter group, however it still has a Bruhat order and length function, outlined in Section 4.3, and so we can still define $\widetilde{\mathcal{H}}$ as in Definition 4.4.1.

In line with the two presentations $\Pi \ltimes W_a$ and $W \ltimes P$ for $\widetilde{W}$, the algebra $\widetilde{\mathcal{H}}$ has two convenient presentations. For the former, the subgroup Π acts on $\mathcal{H}(W_a)$ with the same action as in (4.22), namely if $\pi s_i = s_j \pi$, then $\pi T_i = T_j \pi$ for $\pi \in \Pi$. We have that

$$\widetilde{\mathcal{H}} \simeq \Pi \ltimes \mathcal{H}(W_a) \tag{4.47}$$

and hence $\widetilde{\mathcal{H}}$ is generated by Π and $\{T_w \mid w \in W_a\}$, subject to the relation $\pi f = \pi(f)\pi$ for $f \in \mathcal{H}(W_a), \pi \in \Pi$, along with the Hecke relations (4.39), (4.40). We will refer to this as the *Coxeter presentation* of $\widetilde{\mathcal{H}}$. The second presentation was given by Bernstein [65] and is detailed below.

For $\lambda \in P$, write $\lambda = \mu - \nu$ where $\mu, \nu \in P_+$. Define

$$Y^\lambda = T_{\tau(\mu)}(T_{\tau(\nu)})^{-1} \tag{4.48}$$

The Y^λ are well-defined and satisfy

$$Y^\lambda Y^\mu = Y^{\lambda+\mu} = Y^\mu Y^\lambda \tag{4.49}$$

for all $\lambda, \mu \in P$. Indeed, (4.49) is satisfied for $\lambda, \mu \in P_+$, since $T_{\tau(\lambda)} T_{\tau(\mu)} = T_{\tau(\lambda+\mu)}$ in that case, and the general case follows immediately thereafter.

Proposition 4.5.1. *The sets $\{T_w Y^\lambda \mid w \in W, \lambda \in P\}$ and $\{Y^\lambda T_w \mid w \in W, \lambda \in P\}$ are both bases for $\widetilde{\mathcal{H}}$, subject to the usual multiplication laws* (4.39), (4.40), *the commutativity* (4.49), *and the additional relation*

$$T_{s_i} Y^\lambda - Y^{s_i(\lambda)} T_{s_i} = (q-1)\frac{Y^\lambda - Y^{s_i(\lambda)}}{1 - Y^{-\alpha_i}} \tag{4.50}$$

for all simple roots α_i, $i \neq 0$.

Remark 4.5.1. (1) One can also define the extended affine Hecke algebra as a quotient of an *extended affine Braid group*. This Braid group is generated by $\{T_w \mid w \in W\}$ and $\{y^\lambda \mid \lambda \in P\}$, with the relation (4.39) for the operators T_w and the additional relation

$$T_i y^\lambda = y^\lambda T_i \qquad \text{if } s_i\lambda = \lambda \tag{4.51}$$

$$T_i y^\lambda T_i = y^{s_i\lambda} \qquad \text{if } \langle \lambda, \alpha_i^\vee \rangle = 1 \tag{4.52}$$

Further imposing the quadratic relation (4.40) then implies (4.50).

(2) From the Bernstein presentation, we see immediately that there are two subalgebras sitting inside $\widetilde{\mathcal{H}}$: the subalgebra of $\widetilde{\mathcal{H}}$ generated by the T_w for $w \in W$ is isomorphic to the finite Hecke algebra $\mathcal{H}(W)$, and the subalgebra generated by the Y^λ is isomorphic to the weight lattice P.

The elements Y^λ are q-deformations of the translation elements $\tau(\lambda)$, although care has to be taken when defining Y^λ for λ not a dominant weight. Like the subgroup of translations in $\widetilde{W}$, the subalgebra $\mathcal{Y} := \mathbb{Z}[u^{\pm 1}]Y^P$ is commutative; however if one had simply set $Y^\lambda = T_{\tau(\lambda)}$ for any $\lambda \in P$, then the Y^λ would no longer commute in general.

Example 4.5.1. Let $\widetilde{W}$ be the extended affine Weyl group of $\mathfrak{g} = \mathfrak{sl}_3$, pictured below.

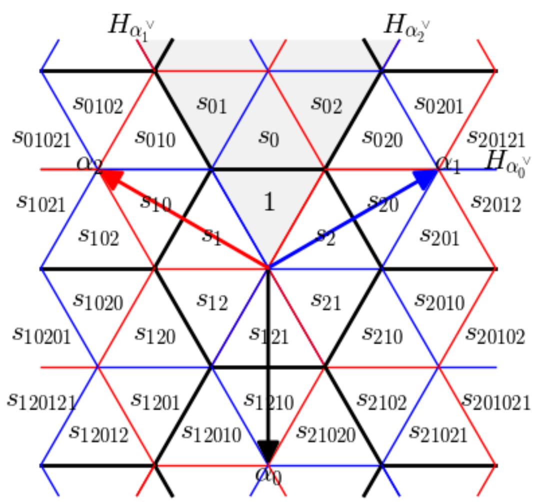


Figure 4.3: Alcove picture for $\mathfrak{sl}_3$, drawn by SageMath.

(a) Set $\lambda = (1, 0, -1)$. Then, λ is a dominant weight that is also in the root lattice, and so $\tau(\lambda)$ can be written as a product of affine simple reflections s_i for $i \in \{0, 1, 2\}$. In particular, $\tau(\lambda) = s_0 s_1 s_2 s_1$ and hence

$$Y^{(1,0,-1)} = T_{\tau(\lambda)} = T_0 T_1 T_2 T_1$$

(b) Set $\lambda = (1, -1, 0)$. This is not a dominant weight, so write $\lambda = (2, 0, 0) - (1, 1, 0)$. We have

$$\tau(2, 0, 0) = s_0 s_2 s_1 s_0 \pi, \qquad \tau(1, 1, 0) = s_0 s_1 \pi$$

where π is defined in Example 4.2.3 to be the element that rotates the fundamental alcove counter-clockwise. So,

$$Y^{(1,-1,0)} = Y^{(2,0,0)}(Y^{(1,1,0)})^{-1} = T_0 T_2 T_1 T_0 \pi \pi^{-1} T_1^{-1} T_0^{-1} = T_0 T_2 T_0^{-1} T_1$$

Note that $\tau(1, -1, 0) = s_0 s_2 s_0 s_1$, but $Y^{(1,-1,0)} \neq T_0 T_2 T_0 T_1$.

Example 4.5.2. We do the case $G = \mathrm{GL}_n$. The Hecke algebra $\widetilde{\mathcal{H}}$ is generated by elements $\{T_0, T_1, \ldots, T_{n-1}, \pi\}$, subject to the relations (4.39),(4.40) for $T_0, \ldots, T_{n-1}$ and the additional relation

$$\pi T_i = T_{i+1}\pi \tag{4.53}$$

where the indices are taken modulo n. Alternatively, $\widetilde{\mathcal{H}}$ is generated by the elements $\{T_1, \ldots, T_{n-1}, Y_1, \ldots, Y_n\}$ subject to the relations

$$Y_iY_j = Y_jY_i \tag{4.54}$$
$$T_iY_j = Y_jT_i, \quad j \neq i, i+1 \tag{4.55}$$
$$T_iY_iT_i = Y_{i+1} \tag{4.56}$$

where again indices are taken modulo n. The element Y_i in this presentation is the element Y^{e_i} in Bernstein's definition (4.48). The relation (4.50) implies the relations (4.55), (4.56). The presentations are related by

$$T_0 = Y_1Y_n^{-1}T_1^{-1} \cdots T_{n-2}^{-1}T_{n-1}^{-1}T_{n-2}^{-1} \cdots T_1^{-1} \tag{4.57}$$
$$\pi = Y_1T_1^{-1}T_2^{-1} \cdots T_{n-1}^{-1} \tag{4.58}$$

In general, the Coxeter and Bernstein presentations are related as follows. If θ denotes the highest short root, then

$$T_0 = Y^{\theta}T_{s_\theta}^{-1} \tag{4.59}$$

owing to the fact that $\tau(\theta) = s_0s_\theta$ is reduced. If $\pi \in \Pi$, following (4.24) we can write $\pi = \tau(\omega)v$ where $v \in W$ and ω is a minuscule weight. Then

$$\pi = Y^{\omega}\overline{T_v} \tag{4.60}$$

In light of Remark 4.5.1(2), from any representation φ of the finite Hecke algebra $\mathcal{H}$, we can construct the induced representation $\mathrm{Ind}_{\mathcal{H}}^{\widetilde{\mathcal{H}}}(\varphi)$ of the full extended affine Hecke algebra. In particular, taking the trivial representation $\mathbb{1}$ in which each T_i acts by the scalar q gives rise to the **polynomial representation** of $\widetilde{\mathcal{H}}$, with an explicit description we now review.

We define the elements e^+ and e^- in $\widetilde{\mathcal{H}}$ by

$$e^+ = C'_{w_0} = q^{-\ell(w_0)/2}\sum_{w\in W} T_w, \qquad e^- = C_{w_0} = q^{-\ell(w_0)/2}\sum_{w\in W}(-q)^{\ell(w_0)-\ell(w)}T_w \tag{4.61}$$

where C_{w_0} is the Kazhdan-Lusztig basis element defined in (4.44) and C'_{w_0} is described in Remark 4.4.1. The elements e^+, e^- are idempotents and satisfy

$$(T_{s_i} - q)e^+ = 0, \qquad (T_{s_i} + 1)e^- = 0 \tag{4.62}$$

for all simple reflections s_i. The spaces $\mathbb{Z}[q^{\pm 1}]e^+, \mathbb{Z}[q^{\pm 1}]e^-$ are the one-dimensional trivial and sign representations, respectively, of the finite Hecke algebra $\mathcal{H}$. The left $\widetilde{\mathcal{H}}$-modules

$\widetilde{\mathcal{H}}e^+$, $\widetilde{\mathcal{H}}e^-$ are then identified with the induced trivial and sign representations from $\mathcal{H}$ up to the full extended Hecke algebra. From the presentation of $\widetilde{\mathcal{H}}$ in Proposition 4.5.1, these modules are isomorphic to the subalgebras $\mathcal{Y}e^+, \mathcal{Y}e^-$, which we can further identify with the group algebra of the weight lattice, namely

$$\widetilde{\mathcal{H}}e^+ \simeq \mathcal{Y}e^+ \simeq \mathbb{Z}[q^{\pm 1}]P \tag{4.63}$$

in which we map $Y^\lambda e^+ \mapsto x^\lambda$. Under this identification, the relation (4.50) translates to the generator T_i acting by the Demazure-Lusztig $q-$divided difference operators:

$$T_i = qs_i + (q-1)\frac{1}{1-x^{-\alpha_i}}(1-s_i) \qquad (i \neq 0) \tag{4.64}$$

The elements Y^λ act by multiplication on a monomial x^μ.

After one identifies $\mathbb{Z}[q^{\pm 1}]P$ with a Laurent polynomial ring, the action of the Hecke algebra $\widetilde{\mathcal{H}}$ via the divided difference operators (4.64) was due originally to Bernstein and Zelevinsky and detailed by Lusztig in [68]. The monomials x^λ for $\lambda \in P$ form an obvious basis of the space; however, a more convenient basis for us will be the basis of non-symmetric Hall-Littlewood polynomials.

Definition 4.5.1. Given $\gamma \in P$, the **non-symmetric Hall-Littlewood polynomial** $E_\gamma(x;q)$ is

$$E_\gamma(x;q) := q^{-\ell(w)}T_w(x^{\gamma_+}) \tag{4.65}$$

where $\gamma_+ \in P_+$ is the unique dominant weight in the orbit of γ, and $w \in W$ with $w(\gamma_+) = \gamma$.

If γ is not a regular weight, then the w above is not unique; however, since $q^{-1}T_{s_i}$ fixes x^μ if $s_i\mu = \mu$, the formula for $E_\gamma(x;q)$ is independent of the choice of w and normalized so that it has the monic form

$$E_\gamma(x;q) = x^\gamma + \sum_{\beta<\gamma} c_\beta x^\beta \tag{4.66}$$

where $<$ is the Bruhat ordering defined in Section 4.3. In fact, with respect to the Bruhat order, the operators T_i have the triangular form

$$T_i x^\lambda = \begin{cases} qx^{s_i\lambda} + (q-1)x^\lambda + \text{lower order terms} & : \langle \lambda, \alpha_i^\vee \rangle > 0 \\ x^{s_i\lambda} + \text{lower order terms} & : \langle \lambda, \alpha_i^\vee \rangle < 0 \\ qx^\lambda & : \langle \lambda, \alpha_i^\vee \rangle = 0 \end{cases} \tag{4.67}$$

from which the monic form (4.66) then follows.

Example 4.5.3. Set $G = \mathrm{GL}_3$ and make the identification $\mathbb{Z}[q^{\pm 1}]P \simeq \mathbb{Z}[q^{\pm 1}][x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$. Then,

$$E_{(1,0,1)}(x;q) = q^{-1}T_2 x^{(1,1,0)} = q^{-1}\left(qx_1x_3 + (q-1)\frac{x_1x_2 - x_1x_3}{1-x_3/x_2}\right) = x_1x_3 + (1-q^{-1})x_1x_2 \tag{4.68}$$

These polynomials will be discussed further in Chapter 5. The pertinent property for us now is that $\{E_\gamma\}_{\gamma\in P}$ form a basis for the space $\widetilde{\mathcal{H}}e^+$.

More generally, we fix a parabolic subset J with corresponding standard parabolic $W_J \subseteq W$ and Levi subgroup $L \subseteq G$. We define the element e_J^- which satisfies $(T_j+1)e_J^- = 0$ for all $j \in J$, and has the explicit formula

$$e_J^- = C_{w_0^J} = q^{-\ell(w_0^J)/2} \sum_{w\in W_J} (-q)^{\ell(w_0^J)-\ell(w)} T_w \tag{4.69}$$

where w_0^J is the longest permutation of W_J. Just as above, the space $\widetilde{\mathcal{H}}e_J^- \simeq e_J^-\widetilde{\mathcal{H}} \simeq \mathrm{Ind}_{\mathcal{H}_J}^{\widetilde{\mathcal{H}}}(\varepsilon)$ is an induced sign representation, where $\mathcal{H}_J$ is the subalgebra of the finite Hecke algebra $\mathcal{H}$ spanned by $\{T_w \mid w \in W_J\}$.

Proposition 4.5.2. *The set $\{e_J^- E_\gamma \mid \gamma \in P_{++}(L)\}$ forms a basis for the space $e_J^-\widetilde{\mathcal{H}}e^+$.*

The proof will be postponed to when we discuss positivity in the next section.

4.6 General type LLT polynomials

We finally come to the definition of a general type LLT polynomial, which we give now so as to help orient the reader. The remainder of the section is devoted to discussing two prominent properties of LLT polynomials, the first being that they expand as an $\mathbb{N}[q]$-linear combination of irreducible characters, and the second being that they are Weyl-invariant polynomials.

Fix the objects $G, P, J, L, \widetilde{W}, \widetilde{\mathcal{H}}$ as above.

Definition 4.6.1. Let $\beta, \gamma \in P_{++}(L)$. The **LLT series** associated to G is the formal series

$$\mathcal{L}^G_{L,\beta,\gamma}(x;q) = \sum_{\lambda\in P_+} Q^\lambda_{\beta,\gamma}(q^{-1})\chi_\lambda(x) \tag{4.70}$$

where the coefficients $Q^\lambda_{\beta,\gamma}(q)$ are defined by

$$\chi_\lambda e_J^- E_\gamma(x;q) = \sum_\beta Q^\lambda_{\beta,\gamma}(q) e_J^- E_\beta(x;q) \tag{4.71}$$

i.e. they are the matrix coefficients of multiplying an irreducible character χ_λ by a basis element $e_J^- E_\gamma$ and expanding into basis elements $e_J^- E_\beta$.

We make several observations about the LLT series.

Remark 4.6.1. (a) The presence of q^{-1} in (4.70) is simply because the non-symmetric Hall-Littlewood polynomials are technically polynomials in q^{-1}.

(b) The LLT series is a formal series, since the sum is over all dominant weights. Regardless, we may be careless and refer to these objects as LLT polynomials, with the understanding that they contain an infinite number of terms. In the case $G = \mathrm{GL}_n$, we will see shortly how a certain truncation of this series will coincide with the LLT polynomials (which are truly polynomials) defined in Section 2.2.

(c) When $q = 1$, the basis element $e_J^- E_\gamma$ specializes to

$$e_J^- E_\gamma = e_J^- x^\gamma = \sum_{w \in W_J} (-1)^{\ell(w)} x^{w(\gamma)} \tag{4.72}$$

This is an antisymmetric W_J-invariant element, and hence divisible by the partial Vandermonde determinant

$$a_{\rho_L} := x^{\rho_L} \cdot \prod_{\alpha \in R_+(L)} (1 - x^{-\alpha}) \tag{4.73}$$

where $\rho_L \in P$ satisfies $\langle \rho_L, \alpha_j^\vee \rangle = 1$ for all $j \in J$. The weight ρ_L is not uniquely determined, however is such that $P_{++}(L) = P_+(L) + \rho_L$. From the Weyl character formula, the quotient of (4.72) by a_{ρ_L} is precisely the irreducible character $\chi_{\gamma - \rho_L}(L)$ of L. Hence, the expression (4.71) for the coefficient of χ_λ in $\mathcal{L}^G_{L,\beta,\gamma}$ is also the multiplicity of the irreducible representation $\chi_{\beta - \rho_L}(L)$ in $\chi_\lambda|_L \otimes \chi_{\gamma - \rho_L}(L)$, whence the formal identity of characters

$$\mathcal{L}^G_{L,\beta,\gamma}(x; 1) = \mathrm{Ind}_L^G(\chi_{\beta - \rho_L}(L) \otimes \chi^*_{\gamma - \rho_L}(L)) \tag{4.74}$$

When $G = \mathrm{GL}_n$, $L = \mathrm{GL}_{r_1} \times \cdots \mathrm{GL}_{r_k}$, $\beta - \rho_L = (\beta^{(1)}, \ldots, \beta^{(k)})$, and $\gamma - \rho_L = 0$, this is merely the statement that the LLT polynomial at $q = 1$ is the product of Schur polynomials $s_{\beta^{(1)}}(x) \cdots s_{\beta^{(k)}}(x)$.

The reader is welcome to skip the following two subsections without any worry of discontinuity.

Positivity

Before we tackle the notion of positivity of the coefficients, we will need to modify Definition 4.6.1 to a more suitable form. To that end, we delve a bit more into the space $e_J^- \widetilde{\mathcal{H}} e^+$. We start by presenting an alternate basis, which we will see presently is nothing more than a relabelling of the basis $e_J^- E_\beta$ given in Proposition 4.5.2. We first note that an element $e_J^- T_x e^+$ only depends, up to a power of q, on the double coset $W_J x W$. Indeed, if we suppose x is minimal in its double coset and write $y = wxv$, where $w \in W_J$ and $v \in W$, then

$$e_J^- T_y e^+ = e_J^- T_w T_x T_v e^+ = (-1)^{\ell(w)} q^{\ell(v)} e_J^- T_x e^+ \tag{4.75}$$

We say a double coset $W_J x W$ is **regular** if it is a regular orbit for the left action of W_J on $\widetilde{W}/W$, or equivalently, the right action of W on $W_J \backslash W$.

Proposition 4.6.1 ([28]). *As w ranges over any choice of coset representatives for the regular double cosets $W_J w W$, the elements $e_J^- T_w e^+$ form a basis for $e_J^- \widetilde{\mathcal{H}} e^+$.*

Without the regularity condition, it follows from (4.75) that the set in question spans. We restrict to regular double cosets simply because if x is a representative for $W_J x W$ that is not regular, then one shows that $e_J^- T_x e^+ = 0$, due to the alternating factor e_J^-.

Extending the bijection (4.33) to regular double cosets gives

$$P_{++}(L) \simeq (W_J \backslash \widetilde{W}/W)_{reg}, \quad \beta \mapsto W_J \tau(\beta) W \tag{4.76}$$

A convenient choice of coset representatives are the *minimal representatives.* In other words, as β ranges over $P_{++}(L)$, with v the minimal element of $W_J \tau(\beta) W$, the elements

$$e_J^- T_v e^+ \sim \sum_{w \in W_J \tau(\beta) W} T_w \tag{4.77}$$

form a linear basis for $e_J^- \widetilde{\mathcal{H}} e^+$, where we write $\sim$ to mean "up to a power of q". This choice is referred to as the **standard basis** of the module $e_J^- \widetilde{\mathcal{H}} e^+$. The following is Proposition 6.3(i) in [28].

Proposition 4.6.2. *Let $\beta \in P_{++}(L)$. Let $w \in W^J$ be such that $w(\beta) \in P_+$. If v is the minimal element of $W_J \tau(\beta) W$, then $v w_0 = w^{-1} \tau(w(\beta))$, with both sides reduced. In particular, $e_J^- T_v e^+ = q^{-\ell(w w_0)} e_J^- E_\beta$.*

Proof. We only give a few details. Setting $\lambda = w(\beta)$, one shows that $w^{-1}\tau(\lambda)$ is minimal in $W_J w^{-1} \tau(\lambda)$ and also maximal in $w^{-1}\tau(\lambda) W$, from which one concludes the identity of reduced factorizations. For the last part, we use that $\lambda \in P_+$ and the identification $x^\lambda = Y^\lambda e^+ = T_{\tau(\lambda)} e^+$ to see that

$$e_J^- T_v e^+ = e_J^- T_{v w_0} T_{w_0}^{-1} e^+ = q^{-\ell(w_0)} e_J^- T_{w^{-1}} T_{\tau(\lambda)} e^+ = q^{-\ell(w_0)} e_J^- T_{w^{-1}} x^\lambda = q^{-\ell(w w_0)} e_J^- E_\beta \tag{4.78}$$

$\square$

We let $P_{v,w}^\lambda(q)$ denote the matrix coefficients defined by

$$\chi_\lambda e_J^- T_w e^+ = \sum_v P_{v,w}^\lambda(q) e_J^- T_v e^+ \tag{4.79}$$

where $e_J^- T_w e^+$ and $e_J^- T_v e^+$ are standard basis elements. If v, w are the minimal coset representatives for $W_J \tau(\beta) W$ and $W_J \tau(\gamma) W$, respectively, then it follows from Proposition 4.6.2 that

$$P_{v,w}^\lambda(q) = q^d Q_{\beta,\gamma}^\lambda(q) \tag{4.80}$$

for some integer d. Putting everything together, we arrive at the following alternate definition of LLT polynomials.

Definition 4.6.2. Let $\beta, \gamma \in P_{++}(L)$. Let v, w be the minimal elements of $W_J\tau(\beta)W$ and $W_J\tau(\gamma)W$, respectively. Then, there is some power d such that

$$\mathcal{L}^G_{L,\beta,\gamma}(x;q) = q^d \sum_\lambda P^\lambda_{v,w}(q)\chi_\lambda(x) \tag{4.81}$$

where $P^\lambda_{v,w}(q)$ is defined in (4.79).

The power d can be given a more explicit description, as is done in [28, Remark 5.10]. This may seem like a trivial relabelling of basis elements, however it leads to a more natural interpretation of positivity.

Theorem 4.6.1 ([28])**.** *The coefficients of χ_λ in the LLT series $\mathcal{L}^G_{L,\beta,\gamma}(x;q)$ are polynomials in q with non-negative coefficients.*

The proof is beyond the scope of this thesis, although we speak a bit about how the result transpires. Long-standing conjectures in Kazhdan-Lusztig theory have revolved around the positivity of the coefficients of various Kazhdan-Lusztig polynomials. When $J = \varnothing$, the positivity of the original polynomials $P^\lambda_{v,w}(q)$ was first proved by Dyer and Lehrer [23], who showed that the operator of multiplication by a Kazhdan-Lusztig basis element C_v has positive matrix coefficients on the standard basis $\{T_w\}$. Preceding this proof was a theorem of Springer and Lusztig [69, 83] that the same operator has positive matrix coefficients on the basis $\{C_w\}$. Kashiwara and Tanisaki [48] extended these results to the positivity of Deodhar's parabolic Kazhdan-Lusztig polynomials.

The standard proof of positivity is shown by identifying the Hecke algebra with a convolution algebra of constructible sheaves on a flag variety, with the classical stratification by Schubert cells. The Kazhdan-Lusztig basis element C_w is identified with an intersection cohomology sheaf on the Schubert cell corresponding to w. The standard basis T_w is identified with a pushfoward (with proper support) of the constant sheaf on a Schubert cell. The bar involution is identified with a natural duality functor. With this perspective, the matrix coefficients are interpreted as graded decomposition factors of irreducible constructible sheaves in the convolution of two sheaves, and hence innately positive.

In [28], the authors extend this argument to a new hybrid basis $\{CT_w\}$ they define, which depends on a parabolic subgroup W_J and interpolates between C_w (when $W_J = W$) and T_w (when $W_J = 1$). They show that the operator C_v on the basis $\{CT_w\}$ has positive matrix coefficients and that these coefficients can be naturally identified with our polynomials $P^\lambda_{v,w}(q)$ (up to a power of q), whence Theorem 4.6.1. While we know the coefficient of an irreducible character in an LLT series has coefficients in $\mathbb{N}[q^{\pm 1/2}]$, it still remains to compute these coefficients combinatorially and manifestly exhibit their positivity. This is achieved in certain cases in the next chapter, and we again reassure the reader that no knowledge of this geometric background will be needed in the combinatorics that follow.

Symmetry

The question of Weyl group invariance is straightforward from Definition 4.6.1, but not at all obvious from any of the combinatorial definitions 2.2.1, 2.2.3, or 2.2.4 in type A. Several proofs of the fact that LLT polynomials are symmetric already exist [21, 30, 57], and we spend some time relating the original algebraic proof to the current setting.

To briefly summarize the argument given in [57], the reason that LLT polynomials are symmetric is because the coefficient of x^λ is essentially the same as the coefficient $\langle \beta \mid V_\lambda \mid \gamma \rangle$, where $V_\lambda = V_{\lambda_1} \cdots V_{\lambda_\ell}$ is an operator acting on a basis element $|\gamma\rangle$ of some module. One has that $[V_i, V_j] = 0$, hence the coefficient of x^λ is the same as the coefficient of $x^{w(\lambda)}$ for any permutation w.

In the language of physicists and of [57], the module in question is the (fermionic) Fock space for $\mathcal{U}_q(\widehat{\mathfrak{sl}}_n)$, which is a certain subspace of infinite wedge products. The Fock space is also endowed with an action of the center of an infinite affine Hecke algebra, which is where the operators V_i lie, and the reason why they commute. This interpretation leads more naturally to the spin definition of LLT polynomials.

The action of this Hecke algebra on the Fock space is the same action as by the Demazure-Lusztig q-divided difference operators defined in (4.64). We won't go over the exact details of how to relate this construction back to the Fock space construction, as it requires some finnicky details with choosing a certain level and multicharge of the Fock space. The interested reader is encouraged to refer to [46] for a classical series of lectures on affine Lie algebras and Fock spaces, and to [59] for how the Fock space relates to LLT polynomials.

Chapter 5

Combinatorial Formulas for Classical Type LLT Polynomials

This chapter is devoted to our main results, which give combinatorial formulas for the LLT series defined in the previous chapter for Sp_{2n}, albeit with certain conditions. We offer an outline of this chapter so as to help clarify the various conditions and specializations.

We start in Section 5.1 by reviewing the theory of non-symmetric Hall-Littlewood polynomials, as they are essential objects for computing LLT series, and also defining the inversion statistic that will ultimately be q-counted by the LLT polynomials. In Section 5.2, we introduce a twisted analogue and prove various identities of LLT series in arbitrary Lie type when the Levi L is specialized to a maximal torus T. In Section 5.3, we specialize G to the general linear and symplectic groups. For the general linear group, we recall the proof that a polynomial truncation of the LLT series coincides with the combinatorial LLT polynomials, given as a $q-$generating function over tuples of semistandard tableaux. Most of the material up until this point can be found in [10] in the case when $G = \mathrm{GL}_n$.

We then adapt the proof for GL_n to the symplectic case, where we define polynomial truncation and a new combinatorial object, called an out-in tableau (Definition 5.3.1), which is nothing more than an extension of a vertical semistandard oscillating tableau to compositions. We show that the symplectic LLT polynomials are generating functions for these out-in-tableau in the case when all the parts are sufficiently far from zero. Most of what is proven for the symplectic case also carries over into the orthogonal groups, which we briefly discuss in Section 5.4.

In the final section, we give another combinatorial formula for symplectic LLT polynomials, this time for an arbitrary Levi L, however at the specialization $q = 1$. In this case, we use our bijection between semistandard oscillating tableaux and symplectic tableaux given in Chapter 3 to write the LLT polynomials as generating functions over symplectic tableaux.

5.1 Non-symmetric Hall-Littlewood polynomials

We recall the notion of a non-symmetric Hall-Littlewood polynomial, first introduced in Section 4.4,

$$E_\gamma(x;q) = q^{-\ell(w)} T_w x^{\gamma_+} \tag{5.1}$$

where $\gamma_+ \in P_+$ is the unique dominant weight in the Weyl orbit of γ and $w(\gamma_+) = \gamma$.

We will also make use of their twisted variants and duals

Definition 5.1.1. Given $\sigma \in W$, the **twisted non-symmetric Hall-Littlewood polynomials** $E^\sigma_\lambda(x;q)$ and their **duals** $F^\sigma_\gamma(x;q)$ are defined as

$$E^\sigma_\gamma(x;q) = q^{d_\sigma(\gamma)} T^{-1}_{\sigma^{-1}} E_{\sigma^{-1}(\gamma)}(x;q) \tag{5.2}$$

$$F^\sigma_\gamma(x;q) = \overline{E^{\sigma w_0}_{-\gamma}(x;q)} = E^{\sigma w_0}_{-\gamma}(x^{-1};q^{-1}) \tag{5.3}$$

where $d_\sigma(\lambda) = |\{\alpha^\vee \in \mathrm{Inv}(\sigma^{-1}) \mid \langle \gamma, \alpha^\vee \rangle \geq 0\}|$ and w_0 is the longest element of W.

Recall here that the inversions of $\sigma \in W$, denoted $\mathrm{Inv}(\sigma)$, are the positive roots (or coroots) sent to negative roots (or coroots). To aid in keeping track of powers of q, we extend this notion to weights.

Definition 5.1.2. Let $\gamma \in P$. An **inversion** of γ is a positive coroot $\alpha^\vee \in R^\vee_+$ with $\langle \gamma, \alpha^\vee \rangle > 0$. The set of inversions of γ is denoted $\mathrm{Inv}(\gamma)$ and the number of inversions denoted $\mathrm{inv}(\gamma)$.[1]

In the case of GL_n, the weight lattice can be identified with $\mathbb{Z}^n$, in which case an inversion of $\gamma \in \mathbb{Z}^n$ is a pair $i < j$ with $\gamma_i > \gamma_j$. In the case of Sp_{2n}, an inversion of $\gamma \in \mathbb{Z}^n$ is either (1) a pair $i < j$ with $\gamma_i > \gamma_j$, (2) a pair $i < j$ with $\gamma_i > -\gamma_j$, or (3) an index i with $\gamma_i > 0$.

In general, if γ_- is the unique antidominant weight in the Weyl orbit of γ, and say $w(\gamma_-) = \gamma$ with w minimal, then $|\mathrm{Inv}(\gamma)| = \ell(w)$. In particular, an antidominant weight will have no inversions, and conversely a strictly dominant weight will have $|R_+|$ inversions.

The power $d_\sigma(\gamma)$ in Definition 5.1.1 is defined precisely to make E^σ_γ monic, with the same form as (4.66). More specifically, there holds the recurrence

$$E^\sigma_\gamma = \begin{cases} q^{-\delta_{s_i\gamma \leq \gamma}} T_i E^{s_i\sigma}_{s_i\gamma}, & s_i\sigma > \sigma \\ q^{\delta_{s_i\gamma \geq \gamma}} T_i^{-1} E^{s_i\sigma}_{s_i\gamma}, & s_i\sigma < \sigma \end{cases} \tag{5.4}$$

where δ_P is 1 if the condition P is true and 0 otherwise. The twisted variants are determined by (5.4), along with the initial condition $E^\sigma_\gamma(x;q) = x^\gamma$ if $\gamma \in P_+$ is dominant. If $\sigma = e$, then the twisted variants $E^\sigma_\gamma, F^\sigma_\gamma$ reduce to the usual E_γ, F_γ. In particular, we arrive at a

[1] It may be wiser to call these coinversions, as we will see that they align more naturally with the coinversion statistic of type A LLT polynomials (not to mention that the prefix matches that of coroot).

useful alternate formula for $F_\gamma(x;q)$: if $\gamma_- \in -P_+$ and v *minimal* with $v(\gamma_-) = \gamma$, then (5.4) implies

$$F_\gamma(x;q) = \overline{E^{w_0}_{-\gamma}(x;q)} = \overline{T^{-1}_{v^{-1}} E^{v^{-1}w_0}_{-v^{-1}\gamma}(x;q)} = \overline{T^{-1}_{v^{-1}} x^{-(\gamma_-)}} = T_v x^{\gamma_-} \tag{5.5}$$

There are unfortunately several different definitions of a non-symmetric Hall-Littlewood polynomial in the literature. Our E_γ, F_γ coincide with specializations of non-symmetric Macdonald polynomials considered by Ion in [42] and Haiman, Haglund, Loehr in [29], as $q \to 0$ for E_γ and $q \to \infty$ for F_γ, and t replaced with q^{-1}. The twisted E^σ_γ is a specialization of the permuted basement non-symmetric Macdonald polynomials for GL_n studied by Alexandersson in [3].

The polynomials E^σ_γ and F^σ_γ are dual in the following sense, which can be viewed as a non-symmetric version of the inner product (2.46).

Proposition 5.1.1. *For $\sigma \in W$, the polynomials $E^\sigma_\gamma(x;q)$ and $\overline{F^\sigma_\gamma}(x;q)$ are dual bases of $\mathbb{Q}(q)P$ with respect to the inner product defined by*

$$\langle f, g\rangle_q = \langle x^0\rangle fg \prod_{\alpha \in R_+} \frac{1 - x^\alpha}{1 - q^{-1}x^\alpha} \tag{5.6}$$

In other words, $\langle E^\sigma_\gamma, \overline{F^\sigma_\beta}\rangle = \delta_{\gamma,\beta}$ for all $\lambda, \mu \in P$ and $w \in W$.

This is Proposition 4.3.2 in [10], and the proof can be found therein. We only remark that the proof relies on the fact that the operators T_i are self-adjoint with respect to $\langle -, -\rangle_q$. Part of Macdonald's inner product formula [19, 70] shows that the W-symmetrization of (5.6) coincides with the inner product (2.46) for Hall-Littlewood polynomials $P_\lambda(x;q^{-1})$. In particular, one equivalently defines $P_\lambda(x;q^{-1})$ by

$$P_\lambda(x;q^{-1}) = \frac{1}{W_\lambda(q)} \sum_{w \in W} q^{\ell(w_0)-\ell(w)} E_{w\lambda}(x;q) = \sum_{\beta \in W\cdot\lambda} q^{\operatorname{inv}\beta} E_\beta(x;q) \tag{5.7}$$

where $W_\lambda(q) = \sum_{w \in \mathrm{Stab}(\lambda)} q^{\ell(w)}$.

The non-symmetric Hall-Littlewood polynomials have the property that if $\lambda \in P_+$, then

$$E_{w_0\lambda}(x;\infty) = \chi_\lambda(x) \tag{5.8}$$

where χ_λ is the irreducible character of G with highest weight λ. In fact, a more general result holds. At $q^{-1} = 0$, the action of $q^{-1}T_i$ in (4.64) specializes to

$$(q^{-1}T_i)\big|_{q^{-1}=0} = s_i + \frac{1 - s_i}{1 - x^{-\alpha_i}} = \frac{1 - x^{-\alpha_i}s_i}{1 - x^{-\alpha_i}} =: \partial_i \tag{5.9}$$

where ∂_i is a **Demazure operator**, defined in [22]. The Demazure operators satisfy the braid relations and so one unambiguously defines $\partial_w = \partial_{s_1} \cdots \partial_{s_r}$ for any reduced factorization $w = s_1 \cdots s_r$, from which we get the more general identity

$$E_{w\lambda}(x;\infty) = \partial_w x^\lambda \tag{5.10}$$

for any dominant weight $\lambda \in P_+$ and $w \in W$. It was moreover shown that $\partial_{w_0} x^\lambda = \chi_\lambda$, from which (5.8) then follows.

The polynomials $K_\gamma := \partial_w x^{\gamma_+}$ are known as **key polynomials** or **Demazure characters** in the literature. The dual basis elements F_γ specialize to what are known as **Demazure atoms**, given as

$$F_{w\lambda}(x;0) = \Theta_w(x^\lambda) \tag{5.11}$$

where $\Theta_i := \frac{1-s_i}{x^{\alpha_i}-1}$ also satisfy the braid relations. The reader is warned that Demazure atoms and characters are often conflated and our terminology here may be in contrast to other sources.

5.2 LLT polynomials for the torus

We recall that the LLT series of G, specialized to when the Levi $L = T$ is a maximal torus, is given by

$$\langle \chi_\lambda \rangle \mathcal{L}_{\beta,\gamma}(x;q^{-1}) = \langle E_\beta(x;q) \rangle \chi_\lambda E_\gamma(x;q) \tag{5.12}$$

where $\langle f \rangle$ denotes the coefficient of f. We also define their twisted analogs in this case.

Definition 5.2.1. Let $\sigma \in W$. The **twisted LLT series** $\mathcal{L}^\sigma_{\beta,\gamma}$ are defined by

$$\langle \chi_\lambda \rangle \mathcal{L}^{\sigma^{-1}}_{\beta,\gamma}(x;q^{-1}) = \langle E^\sigma_\beta(x;q) \rangle \chi_\lambda E^\sigma_\gamma(x;q) \tag{5.13}$$

While the statement below is not given directly in [10], it is essentially proven in the proof that E^σ_λ and F^σ_λ are dual bases. We provide a proof for completeness.

Proposition 5.2.1. *Let $\sigma \in W$ and $\beta, \gamma \in P$. Then,*

$$\mathcal{L}^\sigma_{\beta,\gamma}(x;q) = q^d \mathcal{L}_{\sigma(\beta),\sigma(\gamma)}(x;q) \tag{5.14}$$

where $d = |\operatorname{Inv}(\sigma) \cap (\operatorname{Inv}(\beta+\rho) \setminus \operatorname{Inv}(\gamma+\rho))| - |\operatorname{Inv}(\sigma) \cap (\operatorname{Inv}(\gamma+\rho) \setminus \operatorname{Inv}(\beta+\rho))|$

Proof. For sake of notation, we prove (5.14) at q^{-1}. In that case, the coefficient of χ_λ on the left hand side is given by

$$\langle \chi_\lambda \rangle \mathcal{L}^\sigma_{\beta,\gamma}(x;q^{-1}) = \langle \overline{F^{\sigma^{-1}}_\beta(x;q)}, \chi_\lambda E^{\sigma^{-1}}_\gamma(x;q) \rangle_q = \langle E^{\sigma^{-1}w_0}_{-\beta}, \chi_\lambda E^{\sigma^{-1}}_\gamma \rangle_q \tag{5.15}$$

Since $\ell(w_0) = \ell(\sigma^{-1}w_0) + \ell(\sigma^{-1})$, it follows from repeated applications of (5.4) that

$$E^{\sigma^{-1}w_0}_{-\beta}(x;q) = q^h T_{\sigma^{-1}} E^{w_0}_{-\sigma(\beta)}(x;q) \tag{5.16}$$

where

$$\begin{aligned} h &= d_{\sigma^{-1}w_0}(-\beta) - d_{w_0}(-\sigma(\beta)) \\ &= |\{\alpha^\vee \in \operatorname{Inv}(w_0\sigma) \mid \langle -\beta, \alpha^\vee \rangle \geq 0\}| - |\{\alpha^\vee \in R^\vee_+ \mid \langle -\sigma(\beta), \alpha^\vee \rangle \geq 0\}| \\ &= |\{\alpha^\vee \notin \operatorname{Inv}(\sigma) \mid \langle \beta, \alpha^\vee \rangle \leq 0\}| - |\{\alpha^\vee \in R^\vee_+ \mid \langle \beta, \sigma^{-1}(\alpha^\vee) \rangle \leq 0\} \end{aligned} \tag{5.17}$$

where in the last equality we've used the W-invariance of $\langle \cdot, \cdot \rangle$ and the fact that $\alpha^\vee \in \mathrm{Inv}(w_0\sigma)$ is equivalent to $\alpha^\vee \notin \mathrm{Inv}(\sigma)$. We can split the second term, casing on whether a coroot is an inversion of σ^{-1} or not. In the case $\alpha^\vee \notin \mathrm{Inv}(\sigma^{-1})$, we find

$$|\{\alpha^\vee \notin \mathrm{Inv}(\sigma^{-1}) \mid \langle \beta, \sigma^{-1}(\alpha^\vee)\rangle \leq 0\}| = |\{\alpha^\vee \notin \mathrm{Inv}(\sigma) \mid \langle \beta, \alpha^\vee \rangle \leq 0\}| \tag{5.18}$$

after reindexing $\alpha^\vee \mapsto \sigma^{-1}(\alpha^\vee)$, which cancels with the first term in (5.17). Similarly, in the case $\alpha^\vee \in \mathrm{Inv}(\sigma^{-1})$, we find

$$|\{\alpha^\vee \in \mathrm{Inv}(\sigma^{-1}) \mid \langle \beta, \sigma^{-1}(\alpha^\vee)\rangle \leq 0\}| = |\{\alpha^\vee \in \mathrm{Inv}(\sigma) \mid \langle \beta, \alpha^\vee \rangle \geq 0\}| = d_{\sigma^{-1}}(\beta) \tag{5.19}$$

where we reindex $\alpha^\vee \mapsto -\sigma^{-1}(\alpha^\vee)$, and hence $h = -d_{\sigma^{-1}}(\beta)$. Now, as the operators T_i are self-adjoint with respect to $\langle -, - \rangle_q$, (5.15) together with (5.16) simplifies to

$$\begin{aligned}
\langle \chi_\lambda \rangle \mathcal{L}^\sigma_{\beta,\gamma}(x; q^{-1}) &= q^h \langle T_{\sigma^{-1}} E^{w_0}_{-\sigma(\beta)}(x;q), \chi_\lambda E^{\sigma^{-1}}_\gamma(x;q)\rangle_q \\
&= q^{d_{\sigma^{-1}}(\gamma) - d_{\sigma^{-1}}(\beta)} \langle E^{w_0}_{-\sigma(\beta)}(x;q), \chi_\lambda E_{\sigma(\gamma)}(x;q)\rangle_q \\
&= q^{d_{\sigma^{-1}}(\gamma) - d_{\sigma^{-1}}(\beta)} \langle \overline{F_{\sigma(\beta)}(x;q)}, \chi_\lambda E_{\sigma(\gamma)}(x;q)\rangle_q \\
&= q^{d_{\sigma^{-1}}(\gamma) - d_{\sigma^{-1}}(\beta)} \langle \chi_\lambda \rangle \mathcal{L}_{\sigma(\beta),\sigma(\gamma)}(x; q^{-1})
\end{aligned} \tag{5.20}$$

where

$$\begin{aligned}
d_{\sigma^{-1}}(\gamma) - d_{\sigma^{-1}}(\beta) &= |\{\alpha^\vee \in \mathrm{Inv}(\sigma) \mid \ \langle \gamma, \alpha^\vee \rangle \geq 0\}| - |\{\alpha^\vee \in \mathrm{Inv}(\sigma) \mid \langle \beta, \alpha^\vee \rangle \geq 0\}| \\
&= |\{\alpha^\vee \in \mathrm{Inv}(\sigma) \mid \langle \gamma, \alpha^\vee \rangle \geq 0, \langle \beta, \alpha^\vee \rangle < 0\}| \\
&\quad - |\{\alpha^\vee \in \mathrm{Inv}(\sigma) \mid \langle \gamma, \alpha^\vee \rangle < 0, \langle \beta, \alpha^\vee \rangle \geq 0\}|
\end{aligned} \tag{5.21}$$

which one see matches the desired identity. □

5.3 General linear and symplectic cases

We review how the LLT polynomials defined in Definition 4.6.1 relate to the combinatorial LLT polynomials in type A, given in Definition 2.2.3. We set $G = \mathrm{GL}_n$ and make the identifications $P \simeq \mathbb{Z}^n$ and $\mathbb{Z}[q^{\pm 1}]P \simeq \mathbb{Z}[q^{\pm 1}][x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The exact statement is as follows

Proposition 5.3.1. *Let $\beta, \gamma \in \mathbb{Z}^n$ and suppose $\gamma_i \leq \beta_i$ for all i. Let $\boldsymbol{\beta}/\boldsymbol{\gamma} = (\beta_1/\gamma_1, \dots, \beta_n/\gamma_n)$ denote the tuple of skew shapes with contents, with a single row in each skew shape. Then,*

$$\mathcal{L}^{\mathrm{GL}_n}_{T,\beta,\gamma}(x_1, \dots, x_n; q)_{\mathrm{pol}} = \mathcal{L}_{\boldsymbol{\beta}/\boldsymbol{\gamma}}(x_1, \dots, x_n; q) \tag{5.22}$$

where pol *denotes the "polynomial truncation" to* GL_n *characters $s_\lambda(x)$ with $\lambda_n \geq 0$.*

This is Proposition 4.5.3 in [10] and Corollary 6.19 in [28] when $L = T$. As the proof techniques therein will be essential to us, we provide the details below. The essential step will be to find the coefficient of $E_\beta(x;q)$ in the product $e_k(x)E_\gamma(x;q)$, where $e_k(x) = e_k(x_1, \dots, x_n)$ denotes the elementary symmetric polynomial in n variables. To that end, we show

Lemma 5.3.1 ([10]). *Let $G = \mathrm{GL}_n$, $\gamma \in \mathbb{Z}^n$ and $k \in \mathbb{N}$. Then,*

$$e_k(x)E_\gamma(x;q) = \sum_{|I|=k} q^{-h_I} E_{\gamma+\varepsilon_I}(x;q) \tag{5.23}$$

where $I \subseteq \{1,\dots,n\}$ has k elements, $\varepsilon_I = \sum_{i\in I} \varepsilon_I$ is the indicator vector of I, and

$$h_I = |\operatorname{Inv}(\gamma+\varepsilon_I) \setminus \operatorname{Inv}(\gamma)| \tag{5.24}$$

Equivalently, h_I is the number of pairs $i < j$ such that $i \in I, j \notin I$ and $\gamma_i = \gamma_j$.

Proof. We note that because $e_k(x)$ is symmetric, it commutes with any T_w, so that

$$e_k(x)E_\gamma = q^{-\ell(w)} T_w e_k(x) x^{\gamma_+} = q^{-\ell(w)} \sum_{|I|=k} T_w x^{\gamma_+ + \varepsilon_I} \tag{5.25}$$

Setting $\lambda := \gamma_+ + \varepsilon_I$, we note that λ can fail to be dominant at worst by having entries $\lambda_i = \lambda_j + 1$ for some $i > j$ with $(\gamma_+)_i = (\gamma_+)_j$. We pick v_I minimal so that $v_I(\lambda) = \lambda_+$, that is v_I moves indices $i \in I$ as above to the left within each constant block in γ_+. We note that $T_i x_{i+1} = x_i$, and more generally

$$T_i(x_i^a x_{i+1}^{a+1}) = x_i^{a+1} x_{i+1}^a \tag{5.26}$$

from which it follows that $T_{v_I}(\lambda) = \lambda_+$. Hence,

$$x^{\gamma_+ + \varepsilon_I} = T_{v_I}^{-1} x^{(\gamma_+ + \varepsilon_I)_+} \tag{5.27}$$

Now, we can pick w maximal in its coset $w\,\mathrm{Stab}(\gamma_+)$, and since v_I only permutes entries in constant blocks of γ_+, it follows that $\ell(wv_I^{-1}) = \ell(w) - \ell(v_I)$, so that $T_w T_{v_I}^{-1} = T_{wv_I^{-1}}$. Putting everything together, we see that (5.25) becomes

$$e_k(x)E_\gamma = q^{-\ell(w)} \sum_{|I|=k} T_{wv_I^{-1}} x^{(\gamma_+ + \varepsilon_I)_+} = \sum_{|I|=k} q^{-\ell(v_I)} E_{\gamma + w(\varepsilon_I)} \tag{5.28}$$

It remains to compute $\ell(v_I)$. We let $\beta = \gamma + w(\varepsilon_I) = w(\lambda)$. The length $\ell(v_I)$ is precisely the number of pairs $i' > j'$ with $\lambda_{i'} = \lambda_{j'} + 1$ and $(\gamma_+)_{i'} = (\gamma_+)_{j'}$. As w is maximal, these are also the pairs $i < j$, where $i = w(i'), j = w(j')$, with $\beta_i = \beta_j + 1$ and $\gamma_i = \gamma_j$. In other words, $e_i^\vee - e_j^\vee \in \operatorname{Inv}(\beta) \setminus \operatorname{Inv}(\gamma)$. As I ranges over subsets of size k, so too does $w(I)$, giving (5.23). $\square$

We remark that the exact same proof carries over for the multiplication $e_k(x^{-1})E_\gamma(x;q)$, yielding the expansion

$$e_k(x^{-1})E_\gamma(x;q) = \sum_{|J|=k} q^{-h_I} E_{\gamma - \varepsilon_J}(x;q) \tag{5.29}$$

where $J \subseteq \{1,\dots,n\}$ has k elements, and $h_J = |\operatorname{Inv}(\gamma - \varepsilon_J) \setminus \operatorname{Inv}(\gamma)|$.

Proof of Proposition 5.3.1. If we consider γ drawn in French notation, then the weight $\gamma+\varepsilon_I$ as in (5.23) differs from γ by the vertical strip ε_I, where by vertical strip we mean that $(\gamma+\varepsilon_I)/\gamma$ has at most one box in each row. It follows then from Lemma 5.3.1 that the coefficient of E_β in the product $e_\lambda(x)E_\gamma(x;q)$ is nonzero if and only if β/γ can be decomposed as a sequence of vertical strips $\varepsilon_{I_1},\dots,\varepsilon_{I_\ell}$ whose sizes are $\lambda_1,\dots,\lambda_\ell$. We record this data by placing an i in every cell added in the i^{th} vertical strip, as in Figure 5.1. What results is a negative tableau of shape β/γ, as defined in Proposition 2.4.2. On the negative tableau, we

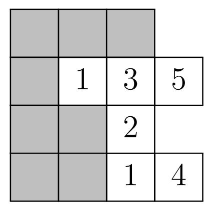


Figure 5.1: Starting at $\gamma=(2,2,1,3)$, we add the indicator vectors, in order, of the subsets $I_1=\{1,3\}, I_2=\{2\}, I_3=\{3\}, I_4=\{1\}, I_5=\{3\}$. The respective inversions added at each step are $1,1,0,3,1$, so that if $\beta=(4,3,4,3)$, then the coefficient of E_β in the product $e_5(x)E_\gamma(x;q)$ has a term $q^6=q^{1+1+0+3+1}$.

can recast an inversion as follows: After adding ε_{I_i}, the statistic h_{I_i} counts the number of pairs of cells u,v in which (1) v has label i, (2) u is in the column directly left on a strictly higher row than v, and (3) either u is in γ or has label strictly less than i. Adding all the statistics h_{I_i} gives

$$\langle E_\beta(x;q)\rangle e_\lambda(x)E_\gamma(x;q)=\sum_{T\in\mathrm{NegTab}(\beta/\gamma,\lambda)} q^{-h(T)} \tag{5.30}$$

where $h(T)$ is the number of triples of boxes of the form

$$\begin{array}{|c|c|} \hline a & c \\ \hline \end{array} \atop \begin{array}{c} \\ \end{array}\begin{array}{|c|} \hline b \\ \hline \end{array} \tag{5.31}$$

where $a<b<c$, and we set $a=0$ if a is a cell in γ and $c=\infty$ if it is not present. In other words, $h(T)$ is precisely the number of negative coinversions of the negative tableau T. Extending to infinitely many variables, we find

$$\omega\mathcal{L}_{\beta/\gamma}(X;q)=\sum_{T\in\mathrm{NegTab}(\beta/\gamma)} q^{\overline{\mathrm{coinv}}(T)}x^T=\sum_\lambda m_\lambda(X)\langle E_\beta(y)\rangle e_\lambda(y)E_\gamma(y;q) \tag{5.32}$$

where we have introduced another set of variables $y_1,\dots,y_n$ and ω is the involution on the ring of symmetric functions defined in Section 2.4. Finally, using the dual Cauchy identity

(2.31) and applying ω in the X variables, we write (5.32) as

$$\mathcal{L}_{\beta/\gamma}(X;q) = \sum_{\lambda} s_\lambda(X)\langle E_\beta(y)\rangle s_\lambda(y) E_\gamma(y;q) \tag{5.33}$$

the right hand side being precisely the image of $\mathcal{L}^{\mathrm{GL}_n}_{T,\beta,\gamma}(x;q)_{\mathrm{pol}}$ in infinitely many variables. $\square$

We wish now to derive an analogue of Proposition 5.3.1 when $G = \mathrm{Sp}_{2n}$, that is, we want a symplectic combinatorial definition that could replace the right hand side of (5.22).

For the remainder of this section, we set $G = \mathrm{Sp}_{2n}$, so that $W = \{s_1, \ldots, s_n\}$ is the group of signed permutations as in Chapter 3. We again make the identifications $P \simeq \mathbb{Z}^n$ and $\mathbb{Z}[q^{\pm 1}]P \simeq \mathbb{Z}[q^{\pm 1}][x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

There are two key components to the proof of Proposition 5.3.1: (1) a combinatorial formula for multiplying an $e_k(x)$ by an $E_\gamma(x;q)$ as in (5.23) and (2) a Cauchy identity for which $e_k(x)$ is dual to (the involution ω of) a monomial symmetric polynomial. For the latter, (3.13) gives an appropriate analogue, involving the elementary symmetric polynomial $e_k(x, x^{-1})$ in the variables x and their inverses. We are then left with finding a combinatorial formula for the coefficients $c^{(k)}_{\beta,\gamma}(q)$ in

$$e_k(x, x^{-1}) E_\gamma(x;q) = \sum_{\beta} c^{(k)}_{\beta,\gamma}(q) E_\beta(x;q) \tag{5.34}$$

If we naively try the same argument as in the proof of Lemma 5.3.1, we come to the following expression

$$e_k(x, x^{-1}) E_\gamma = \sum_{|I|+|J|=k} q^{-\ell(w)} T_w x^{\gamma_+ + \varepsilon_I - \varepsilon_J} \tag{5.35}$$

at which point we hit three immediate barriers, owing to the fact that $\lambda = \gamma_+ + \varepsilon_I - \varepsilon_J$ can fail to be dominant in more ways than in the GL_n case.

To start, $\lambda = \gamma_+ + \varepsilon_I - \varepsilon_J$ could fail to be dominant at worst by having some entries $\lambda_j = \lambda_i + 2$ for some $i < j$ with $(\gamma_+)_i = (\gamma_+)_j$. This happens in the case $i \in J \setminus I$ and $j \in I \setminus J$. We can use the identity

$$x_{i+1}^2 = T_i^{-1} x_i^2 + (q^{-1} - 1) x_i x_{i+1} \tag{5.36}$$

to straighten x^λ, which results not in a single monomial, but rather a polynomial with coefficients $q^{-a}(q^{-1}-1)^b$. This can quickly become unwieldy for arbitrary $\varepsilon_I, \varepsilon_J$.

The second impediment is when λ fails to be dominant by having entries $\lambda_j = \lambda_i + 1$ for some $i < j$ with $(\gamma_+)_i = (\gamma_+)_j + 1$, again in the case $i \in J \setminus I$ and $j \in I \setminus J$. In this case, we can straighten $x^\lambda = T_v^{-1} x^{\lambda_+}$ with only a single monomial, however it might no longer be the case that $T_w T_v^{-1} = T_{wv^{-1}}$. Indeed, this relied on the fact that we chose w maximal and v only permuted along constant blocks of γ_+. As v no longer stabilizes γ_+, it could happen that $\ell(wv^{-1}) \neq \ell(w) - \ell(v^{-1})$ and so we need to use the Hecke relation $T_w T_s^{-1} = q^{-1}T_{ws} + (q^{-1}-1)T_w$, hence arriving at the same problem above of straightening x^λ into a polynomial with coefficients $q^{-a}(q^{-1}-1)^b$.

A third obstacle occurs when λ fails to be dominant by having an entry $\lambda_j = -1$ for some j with $(\gamma_+)_j = 0$. If there are many such occurrences, then straightening λ involves again using the identity (5.36), leading to the same unwieldy expression.

The problem common to all three impediments is that even with a meticulous tracking of all the coefficients, we arrive at an expression for the coefficient $c^{(k)}_{\beta,\gamma}$ in (5.34) as a polynomial in q^{-1} and $(q^{-1}-1)$. It still remains to exhibit $c^{(k)}_{\beta,\gamma}$ as a polynomial in q^{-1} with *positive* coefficients. After all, by Theorem 4.6.1, since $e_k - e_{k-2}$ is an irreducible character, we know that the coefficients $c^{(k)}_{\beta,\gamma}$ are in $\mathbb{N}[q^{-1}]$. This positivity must follow then from an intricate and rather mysterious cancelling of terms. Before continuing with this line, we prove two cases in which the obstacles listed above can be overcome.

Lemma 5.3.2. *Let $\gamma \in (\mathbb{Z}_{>0})^n$ and $k \in \mathbb{N}$. Then,*

$$e_k(x, x^{-1})E_\gamma(x;q) = \sum_{\substack{I,J \\ |I|+|J|=k}} q^{-h_{I,J}} E_{\gamma+\varepsilon_I-\varepsilon_J}(x;q) \tag{5.37}$$

where $I, J \subseteq \{1,\dots,n\}$, and

$$h_{I,J} = |\operatorname{Inv}(\gamma+\varepsilon_I) \setminus \operatorname{Inv}(\gamma)| + |\operatorname{Inv}(\gamma+\varepsilon_I-\varepsilon_J) \setminus \operatorname{Inv}(\gamma+\varepsilon_I)| \tag{5.38}$$

Proof. Under the assumption $\gamma \in (\mathbb{Z}_{>0})^n$, we can write $E_\gamma(x;q) = q^{-\ell(w)}T_w x^{\gamma_+}$, where $w(\gamma_+) = \gamma$, then w is in fact a permutation in S_n. Then, using the identity $e_k(x,x^{-1}) = \sum_{a+b=k} e_a(x)e_b(x^{-1})$, we have

$$e_k(x, x^{-1})E_\gamma(x;q) = \sum_{a+b=k} e_b(x^{-1}) \left(e_a(x) q^{-\ell(w)} T_w x^{\gamma_+}\right) \tag{5.39}$$

where the term in parentheses can now be viewed as the product of $e_a(x)$ with the non-symmetric Hall-Littlewood polynomial $E_\gamma(x;q)$ for GL_n. Following (5.23), the product will decompose into a linear combination of terms $E_{\gamma+\varepsilon_I}$, which again can be viewed as an object for GL_n, since $\gamma+\varepsilon_I$ also has all positive terms, whose subsequent product with $e_b(x^{-1})$ can then be decomposed following (5.29). Putting everything together, we have

$$\begin{aligned} e_k(x,x^{-1})E_\gamma(x;q) &= \sum_{a+b=k} \sum_{|I|=a} q^{-h_I} e_b(x^{-1}) E_{\gamma_+ + \varepsilon_I} &(5.40)\\ &= \sum_{\substack{I,J \\ |I|+|J|=k}} q^{-(h_I+h_J)} E_{\gamma_+ + \varepsilon_I - \varepsilon_J} &(5.41)\end{aligned}$$

where $h_I + h_J$ is exactly the desired expression (5.38), albeit technically with inversions restricted to coroots $\alpha^\vee$ that are also coroots in GL_n; however, with the assumption $\gamma \in (\mathbb{Z}_{>0})^n$, there are in fact no inversions of the form $e_i^\vee + e_j^\vee$ that are in the set difference of either term in (5.38). $\square$

Corollary 5.3.1. *Let $\gamma \in \mathbb{Z}^n$ with no zero entries and let $\sigma \in W$ be the involution which negates all negative entries of γ. Then,*

$$e_k(x, x^{-1}) E_\gamma^\sigma(x; q) = \sum_{\substack{I,J \\ |I|+|J|=k}} q^{-h_{I,J}^\sigma} E_{\gamma+\varepsilon_I-\varepsilon_J}^\sigma(x; q) \tag{5.42}$$

where $I, J \subseteq \{1, \ldots, n\}$, and

$$\begin{aligned} h_{I,J}^\sigma = &\,|\operatorname{Inv}(\sigma) \cap (\operatorname{Inv}(\beta+\rho) \setminus \operatorname{Inv}(\nu+\rho))| + |\operatorname{Inv}(\sigma) \cap (\operatorname{Inv}(\nu+\rho) \setminus \operatorname{Inv}(\gamma+\rho))| \\ &+ |(\operatorname{Inv}(\beta) \setminus \operatorname{Inv}(\nu)) \setminus \operatorname{Inv}(\sigma)| + |(\operatorname{Inv}(\nu) \setminus \operatorname{Inv}(\gamma)) \setminus \operatorname{Inv}(\sigma)| \end{aligned} \tag{5.43}$$

where $\nu = \gamma + \varepsilon_I$ and $\beta = \gamma + \varepsilon_I - \varepsilon_J$.

Proof. For fixed I, J, we set $\nu = \gamma + \varepsilon_I$ and $\beta = \gamma + \varepsilon_I - \varepsilon_J$. From Proposition 5.2.1 relating twisted LLT series with the untwisted ones, we have

$$\langle E_\beta^\sigma(x; q) \rangle e_k(x, x^{-1}) E_\gamma^\sigma(x; q) = q^{-d} \langle E_{\sigma(\beta)}(x; q) \rangle e_k(x, x^{-1}) E_{\sigma(\gamma)}(x; q) \tag{5.44}$$

where

$$d = |\{\alpha^\vee \in \operatorname{Inv}(\sigma) \mid \langle \beta, \alpha^\vee \rangle \geq 0, \langle \gamma, \alpha^\vee \rangle < 0\}| - |\{\alpha^\vee \in \operatorname{Inv}(\sigma) \mid \langle \beta, \alpha^\vee \rangle < 0, \langle \gamma, \alpha^\vee \rangle \geq 0\}| \tag{5.45}$$

As $\sigma(\gamma)$ has all positive entries, from (5.38) the coefficient on the right hand side of (5.44) is q^{-h}, where

$$h = |\operatorname{Inv}(\sigma(\nu)) \setminus \operatorname{Inv}(\sigma(\gamma))| + |\operatorname{Inv}(\sigma(\beta)) \setminus \operatorname{Inv}(\sigma(\nu))| \tag{5.46}$$

Following the proof of Proposition 5.2.1, specifically the identities (5.19), (5.18), we rewrite

$$\begin{aligned} |\operatorname{Inv}(\sigma(\nu)) \setminus \operatorname{Inv}(\sigma(\gamma))| &= |\{\alpha^\vee \in \operatorname{Inv}(\sigma) \mid \langle \nu, \alpha^\vee \rangle < 0, \langle \gamma, \alpha^\vee \rangle \geq 0\}| \\ &\quad + |\{\alpha^\vee \notin \operatorname{Inv}(\sigma) \mid \langle \nu, \alpha^\vee \rangle > 0, \langle \gamma, \alpha^\vee \rangle \leq 0\}| \end{aligned} \tag{5.47}$$

$$\begin{aligned} |\operatorname{Inv}(\sigma(\beta)) \setminus \operatorname{Inv}(\sigma(\nu))| &= |\{\alpha^\vee \in \operatorname{Inv}(\sigma) \mid \langle \beta, \alpha^\vee \rangle < 0, \langle \nu, \alpha^\vee \rangle \geq 0\}| \\ &\quad + |\{\alpha^\vee \notin \operatorname{Inv}(\sigma) \mid \langle \beta, \alpha^\vee \rangle > 0, \langle \nu, \alpha^\vee \rangle \leq 0\}| \end{aligned} \tag{5.48}$$

The terms in (5.47), (5.48) counting inversions in $\operatorname{Inv}(\sigma)$ combine with (5.45) to give

$$|\{\alpha^\vee \in \operatorname{Inv}(\sigma) \mid \langle \beta, \alpha^\vee \rangle \geq 0, \langle \nu, \alpha^\vee \rangle < 0\}| + |\{\alpha^\vee \in \operatorname{Inv}(\sigma) \mid \langle \nu, \alpha^\vee \rangle \geq 0, \langle \gamma, \alpha^\vee \rangle < 0\}| \tag{5.49}$$

Combining with the terms in (5.47), (5.48) counting inversions not in $\operatorname{Inv}(\sigma)$ gives the desired result. ∎

We also have a dual statement, which we note is not a direct result of Corollary 5.3.1, as that situation only applies to choices σ, γ with $\sigma(\gamma)$ having all positive entries.

Lemma 5.3.3. *Let $\gamma \in (\mathbb{Z}_{<0})^n$ and $k \in \mathbb{N}$. Then,*

$$e_k(x, x^{-1}) F_\gamma(x; q) = \sum_{\substack{I,J \\ |I|+|J|=k}} q^{h_{I,J}} F_{\gamma+\varepsilon_I-\varepsilon_J}(x; q) \tag{5.50}$$

where $I, J \subseteq \{1, \dots, n\}$, and

$$h_{I,J} = |\operatorname{Inv}(\gamma - \varepsilon_J) \setminus \operatorname{Inv}(\gamma)| + |\operatorname{Inv}(\gamma - \varepsilon_J + \varepsilon_I) \setminus \operatorname{Inv}(\gamma - \varepsilon_J)| \tag{5.51}$$

Proof. The proof is almost identical to that of Lemma 5.3.2, using the formula for F_γ given in (5.5). □

Lemmas 5.3.2, 5.3.3 can be combined using the following combinatorial gadget, which is merely an extension of semistandard oscillating tableaux to when the indexing shapes are compositions with possibly negative parts.

Definition 5.3.1. Let $\beta, \gamma \in \mathbb{Z}^n$. An **out-in tableau** of shape β/γ is a sequence

$$\gamma = \upsilon^0 = \delta^0 \subseteq \upsilon^1 \supseteq \delta^1 \subseteq \upsilon^2 \supseteq \delta^2 \subseteq \cdots \supseteq \beta \tag{5.52}$$

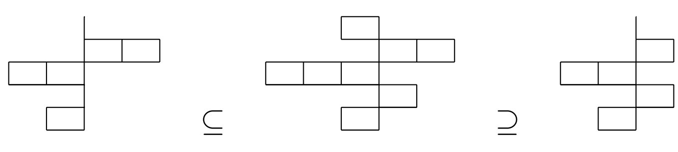
where we write $\upsilon \subseteq \delta$ to mean $|\upsilon| \subseteq |\delta|$, such that

(i) the skew shapes $|\upsilon^i|/|\delta^{i-1}|$ and $|\upsilon^i|/|\delta^i|$ are vertical strips, and

(ii) each $\upsilon^i, \delta^i \in \mathbb{Z}^n$.

Note that if all intermediate shapes are partitions, then an out-in tableau of shape β/γ is simply a vertical semistandard oscillating tableau. We have chosen to use a different terminology for when β, γ are not dominant weights mostly for clarity.

Example 5.3.1. Let $\gamma = (-1, 0, -2, 2, 0)$. The following is an out-in tableau starting from γ with one "out" step and one "in" step.



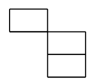
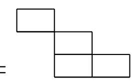
We let $\operatorname{OutIn}(\beta/\gamma), \operatorname{OutIn}(\beta/\gamma, \nu)$ denote the set of out-in tableaux of shape β/γ and weight ν, where the weight is the same as for semistandard oscillating tableaux, namely ν_i is the total number cells added or removed at the i^{th} "out" and "in" steps.

Remark 5.3.1. Note that for parts $\gamma_i = 0$, there are two options for the out step υ_i, namely we could have $\upsilon_i = \pm 1$. There will be times when we only want to consider one of these options, and so we define $\operatorname{OutIn}^+(\beta/\gamma)$ to be the subset of $\operatorname{OutIn}(\beta/\gamma)$ for which the oscillation $0 \mapsto -1$ is not allowed, and likewise we define $\operatorname{OutIn}^-(\beta/\gamma)$ to be the subset for which the oscillation $0 \mapsto 1$ is not allowed.

Definition 5.3.2. Let T be an out-in tableau of shape β/γ as in (5.52). Define $\operatorname{inv} T$ to be the quantity

$$\operatorname{inv} T = \sum_i |\operatorname{Inv}(\upsilon^{i+1}) \setminus \operatorname{Inv}(\delta^i)| + |\operatorname{Inv}(\delta^{i+1}) \setminus \operatorname{Inv}(\upsilon^{i+1})| \tag{5.53}$$

Example 5.3.2. Let $G = \mathrm{Sp}_6$. Let $T = \gamma \subseteq \beta \supseteq \beta$ be as follows

$\gamma = (1, 1, -1) =$  $\beta = (2, 1, -1) =$ 

Then, $e_1^\vee - e_2^\vee$ and $e_1^\vee + e_3^\vee$ are inversions of β but not of γ, hence $\operatorname{inv} T = 2$.

It follows immediately that we can rewrite (5.37), and similarly (5.50), as

$$e_k(x, x^{-1})E_\gamma(x; q) = \sum_T q^{-\operatorname{inv} T} E_{\operatorname{end} T}(x; q) \tag{5.54}$$

the sum over out-in tableaux starting at γ and ending at $\operatorname{end} T$, with weight (k).

We now tackle the more general condition of finding an expression for $c^{(k)}_{\beta,\gamma}$ in (5.34) when $\gamma \in \mathbb{Z}^n$. Our method of proof will follow the same format as the proof of Lemma 5.3.1. In that vein, it will be helpful to have a more concrete description of what a permutation w does to a dominant weight γ_+, where $w(\gamma_+) = \gamma$ and w is maximal in its coset $w \operatorname{Stab}(\gamma_+)$. The following order on indices will be useful.

Definition 5.3.3. Let $\gamma \in \mathbb{Z}^n$. We order the indices of γ so that all nonpositive parts of γ come first in their original order, followed by the positive parts in reverse order. More precisely, we say $i \prec j$ iff one of the following conditions holds

(a) $\gamma_i, \gamma_j \leq 0$ and $i < j$,

(b) $\gamma_i \leq 0$ and $\gamma_j > 0$, or

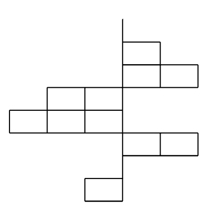
(c) $\gamma_i, \gamma_j > 0$ and $i > j$.

When w is maximal with $w(\gamma_+) = \gamma$, we make the following useful observation:

$$\text{If } |\gamma_i| = |\gamma_j|, \text{ then } i \prec j \iff w^{-1}(i) < w^{-1}(j) \tag{5.55}$$

where we view w^{-1} as acting on the indices, forgetting any sign changes.

Example 5.3.3. Let $\gamma = (-1, 0, 2, -3, -2, 2, 1, 0)$, drawn in French notation below.



The order on the indices is $1 \prec 2 \prec 4 \prec 5 \prec 8 \prec 7 \prec 6 \prec 3$. We have $\gamma_+ = (3,2,2,2,1,1,0,0)$. If w is maximal with $w(\gamma_+) = \gamma$, then as a permutation on the indices, w is given in 2-line notation as

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 3 & 1 & 7 & 2 & 8 \end{pmatrix}$$

We see for example that $|\gamma_5| = |\gamma_6| = |\gamma_3|$, $5 \prec 6 \prec 3$, and $w^{-1}(5) < w^{-1}(6) < w^{-1}(3)$.

We now establish the following lemma, in which we show that for fixed I, the terms in (5.35) ranging over all J, when grouped together, can themselves be written as a $\mathbb{N}[q^{-1}]$-linear combination of E_β's.

Lemma 5.3.4. *Let $\gamma \in \mathbb{Z}^n$ with $\gamma_i \neq 0$ for all i, and $k \in \mathbb{N}$. Fix $I \subseteq n$ and w maximal such that $w(\gamma_+) = \gamma$. Then,*

$$q^{-\ell(w)} T_w(x^{\gamma_+ + \varepsilon_I} e_b(x^{-1})) = \sum_T q^{-\operatorname{inv} T} E_{\operatorname{end} T}(x; q) \tag{5.56}$$

where the sum is over out-in tableaux $T = (\gamma \subseteq \nu \supseteq \beta)$, with $\nu = \gamma + w(\varepsilon_I)$ and $\beta = \nu - \varepsilon_J$ for some $J \subseteq [n], |J| = b$.

Proof. We proceed by induction on $|I|$. When $I = \emptyset$, then the proof of Lemma 5.3.1 carries over. More specifically, each monomial in $x^{\gamma_+} e_b(x^{-1})$ is of the form $x^{\gamma_+ - \varepsilon_J}$ for some subset $J \subseteq [n]$ of size b. Setting $\lambda = \gamma_+ - \varepsilon_J$, then as γ_+ has no trailing zeroes, λ fails to be dominant at worst by having entries $\lambda_i = \lambda_j - 1$ for some $i < j$ with $(\gamma_+)_i = (\gamma_+)_j$. We can pick v_J minimal such that $v_J(\lambda) = \lambda_+$, that is v_J moves indices $i \in J$ as above to the right within each constant block in γ_+. Using the identity $T_i(x_i^{a-1} x_{i+1}^a) = x_i^a x_{i+1}^{a-1}$, we have that

$$x^{\gamma_+ - \varepsilon_J} = T_{v_J}^{-1} x^{(\gamma_+ - \varepsilon_J)_+} \tag{5.57}$$

Now, as w is maximal, it follows that $T_w T_{v_J}^{-1} = T_{wv_J^{-1}}$, since $\ell(wv_J^{-1}) + \ell(v_J) = \ell(w)$. Hence, each monomial on the left hand side of (5.56) is of the form

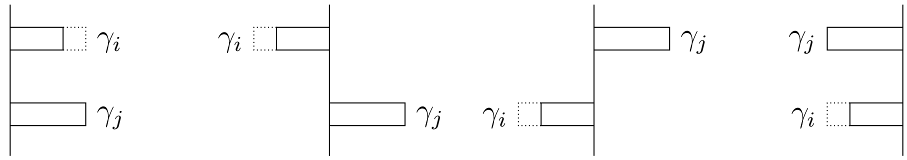
$$q^{-\ell(w)} T_{wv_J^{-1}} x^{(\gamma_+ - \varepsilon_J)_+} = q^{-\ell(w)} q^{\ell(wv_J^{-1})} E_{wv_J^{-1}(\gamma_+ - \varepsilon_J)_+} = q^{-\ell(v_J)} E_{\gamma - w(\varepsilon_J)} \tag{5.58}$$

where we use the fact that $v_J^{-1}(\gamma_+ - \varepsilon_J)_+ = \gamma_+ - \varepsilon_J$.

It remains to compute $\ell(v_J)$, for which we use the same argument as in the proof of Lemma 5.3.1, with a minor tweak. Suppose $T = (\gamma \subseteq \nu \supseteq \beta)$ is an out-in tableau with $\gamma = \nu$ and $\beta = \gamma - w(\varepsilon_J)$. The length $\ell(v_J)$ is precisely the number of pairs $i' < j'$ with $\lambda_{i'} = \lambda_{j'} - 1$ and $(\gamma_+)_{i'} = (\gamma_+)_{j'}$. Being maximal, w carries these to pairs $i = w(i'), j = w(j')$ such that $i \prec j$ with $|\gamma_i| = |\gamma_j|$ and $|\beta_i| = |\beta_j| - 1$.

We claim that these pairs are exactly the positive coroots $\alpha^\vee = e_i^\vee \pm e_j^\vee$ that are inversions of β and not γ, so that $\ell(v_J) = \operatorname{inv} T$. We case on the signs of γ_i, γ_j, pictured below in Figure 5.2. We let S denote the set $\operatorname{Inv}(\beta) \setminus \operatorname{Inv}(\gamma)$.

If $\gamma_i, \gamma_j > 0$, then $i > j$, $\gamma_i = \gamma_j$, and $\beta_i = \beta_j - 1 \geq 0$. So, $\alpha^\vee = e_j^\vee - e_i^\vee \in S$. If $\gamma_i < 0$ and $\gamma_j > 0$, then $\gamma_i = -\gamma_j$ and $\beta_i = -(\beta_j - 1)$. So $\alpha^\vee = e_i^\vee + e_j^\vee \in S$. If $\gamma_i, \gamma_j < 0$, then $i < j$, $\gamma_i = \gamma_j$, and $\beta_i = -(-\beta_j - 1) = \beta_j + 1$, so $\alpha^\vee = e_i^\vee - e_j^\vee \in S$. The case $\gamma_i > 0$ and $\gamma_j < 0$ is not possible, as $i \prec j$.



(a) $\gamma_i, \gamma_j > 0$ (b) $\gamma_i < 0, \gamma_j > 0$ (c) $\gamma_i, \gamma_j < 0$

Figure 5.2: Cases in which $i \prec j$ and there is an inversion $\alpha^\vee = \varepsilon_i^\vee \pm \varepsilon_j^\vee$ in $\mathrm{Inv}(\beta) \setminus \mathrm{Inv}(\gamma)$.

Now suppose $|I| \geq 1$ and write $I = \{i_1\} \cup I'$, where i_1 is the minimum element of I. Let v_1 be minimal such that $v_1(\gamma_+ + \varepsilon_{i_1}) = (\gamma_+ + \varepsilon_{i_1})_+$, that is v_1 moves the index i_1 to the left in its constant block of γ_+. Let $\xi = \gamma + w(\varepsilon_{i_1})$, so $\xi_+ = (\gamma_+ + \varepsilon_{i_1})_+$ and $wv_1^{-1}(\xi_+) = \xi$.

With our choice of i_1, it follows that v_1 fixes I', so that

$$v_1(\gamma_+ + \varepsilon_I) = v_1(\gamma_+ + \varepsilon_{i_1} + \varepsilon_{I'}) = \xi_+ + \varepsilon_{I'} \tag{5.59}$$

and $x^{\gamma_+ + \varepsilon_I} = T_{v_1}^{-1} x^{\xi_+ + \varepsilon_{I'}}$. Moreover, our assumption that γ_+ has no zero entries means that v_1 is in fact a permutation in S_n (a priori it is an element of the Weyl group of signed permutations). The left hand side of (5.56) now becomes

$$q^{-\ell(w)} T_{wv_1^{-1}}(x^{\xi_+ + \varepsilon_{I'}} e_b(x^{-1})) \tag{5.60}$$

where we've appealed to the facts that the S_n-invariant polynomial $e_b(x^{-1})$ commutes with T_u for any $u \in S_n$, and again that w is maximal so $T_w T_{v_1}^{-1} = T_{wv_1^{-1}}$.

Now, wv_1^{-1} may not be maximal with $wv_1^{-1}(\xi_+) = \xi$. However, if $(\gamma_+)_{i_1} = a$, then ξ_+ and γ_+ differ only at the index $v_1(i_1)$, where γ_+ takes the value a and ξ_+ takes the value $a + 1$. Moreover, by our choice of i_1, $(\gamma_+)_j$ and $(\xi_+)_j$ are both $\leq a$ for all $j \in I'$. Thus, if we let u be maximal in its coset $u\,\mathrm{Stab}(\xi_+)$, then since w is maximal, it follows that u and wv_1^{-1} only differ by some permutation of the indices j with $(\xi_+)_j = a + 1$. In other words, we can write $u = wv_1^{-1} y$, where $y \in \mathrm{Stab}(\xi_+)$ fixes $\varepsilon_{I'}$. By maximality of u, we have that $\ell(uy^{-1}) = \ell(u) - \ell(y)$ and so $\ell(w) = \ell(u) - \ell(y) + \ell(v_1)$ and $T_{wv_1^{-1}} = T_{uy^{-1}} = T_u T_y^{-1}$. Then, (5.60) reduces to

$$q^{-\ell(u)+\ell(y)-\ell(v_1)} T_u T_y^{-1}(x^{\xi_+ + \varepsilon_{I'}} e_b(x^{-1})) = q^{-\ell(v_1)} q^{-\ell(u)} T_u(x^{\xi_+ + \varepsilon_{I'}} e_b(x^{-1})) \tag{5.61}$$

for which we can apply our inductive hypothesis to arrive at

$$q^{-\ell(w)} T_w(x^{\gamma_+ + \varepsilon_I} e_b(x^{-1})) = q^{-\ell(v_1)} \sum_{T'} q^{-\operatorname{inv} T'} E_{\operatorname{end} T}(x; q) \tag{5.62}$$

the sum over out-in tableaux $T' = (\xi \subseteq \nu \supseteq \beta)$ in which $\nu = \xi + u(\varepsilon_{I'})$ and $\beta = \nu - \varepsilon_J$ for some $J \subseteq [n], |J| = b$. First note that $\nu = \gamma + w(\varepsilon_I)$, and so we can view T' as an out-in tableau T starting instead at γ. The desired identity (5.56) is then established, provided we verify that

$$\ell(v_1) = \operatorname{inv} T - \operatorname{inv} T' = |\operatorname{Inv}(\nu) \setminus \operatorname{Inv}(\gamma)| - |\operatorname{Inv}(\nu) \setminus \operatorname{Inv}(\xi)| \tag{5.63}$$

The length of v_1 is the number of indices $p < i_1$ such that $(\gamma_+)_p = (\gamma_+)_{i_1}$. As usual, setting $p' = w(p), i' = w(i_1)$, by the maximality of w, this is the number of indices $p' \prec i'$ with $|\gamma_{p'}| = |\gamma_{i'}|$. In other words, $\ell(v_1) = |\operatorname{Inv}(\xi) \setminus \operatorname{Inv}(\gamma)|$ and we aim to prove

$$|\operatorname{Inv}(\nu) \setminus \operatorname{Inv}(\gamma)| = |\operatorname{Inv}(\nu) \setminus \operatorname{Inv}(\xi)| + |\operatorname{Inv}(\xi) \setminus \operatorname{Inv}(\gamma)| \tag{5.64}$$

To that end, note that any inversion $\alpha^\vee \in \operatorname{Inv}(\nu) \setminus \operatorname{Inv}(\gamma)$ is either in $\operatorname{Inv}(\nu) \setminus \operatorname{Inv}(\xi)$ or $\operatorname{Inv}(\xi) \setminus \operatorname{Inv}(\gamma)$, depending on whether $\langle \xi, \alpha^\vee \rangle$ is non-positive or positive, respectively. Conversely, the sets on the right hand side of (5.64) are clearly disjoint, and so we show that any inversion in either term is also in $\operatorname{Inv}(\nu) \setminus \operatorname{Inv}(\gamma)$. Below we set $i' = w(i_1)$. It will be helpful to note that $|\gamma_{i'}|$ is maximal among $|\gamma_{w(j)}|$ for $j \in I$, and is the minimum such index with respect to the order $\prec$ on indices.

We suppose first that $\alpha^\vee \in \operatorname{Inv}(\nu) \setminus \operatorname{Inv}(\xi)$. If on the contrary $\langle \gamma, \alpha^\vee \rangle > 0$, then $\alpha^\vee \in \operatorname{Inv}(\gamma) \setminus \operatorname{Inv}(\xi)$, which can only happen if there is some $p' \succ i'$ with $|\gamma_{p'}| = |\gamma_{i'}| + 1$ and $|\xi_{p'}| = |\xi_{i'}|$. However, by assumption we also have $|\nu_{p'}| = |\nu_{i'}| + 1$, and so

$$|\nu_{p'}| = |\nu_{i'}| + 1 = |\xi_{i'}| + 1 = |\xi_{p'}| + 1 \tag{5.65}$$

which implies that $p := w^{-1}(p') \in I'$. However, this is impossible as $|\gamma_{p'}| > |\gamma_{i'}|$ contradicts our choice of i_1.

We suppose next that $\alpha^\vee \in \operatorname{Inv}(\xi) \setminus \operatorname{Inv}(\gamma)$. If on the contrary $\langle \nu, \alpha^\vee \rangle \leq 0$, then $\alpha^\vee \in \operatorname{Inv}(\xi) \setminus \operatorname{Inv}(\nu)$, which can only happen if there is some $p' \prec i'$ with $|\xi_{p'}| = |\xi_{i'}| - 1$ and $|\nu_{p'}| = |\nu_{i'}|$. So,

$$|\nu_{p'}| = |\nu_{i'}| = |\xi_{i'}| = |\xi_{p'}| + 1 \tag{5.66}$$

which implies that $p := w^{-1}(p') \in I'$. However, by assumption we also have $|\gamma_{p'}| = |\gamma_{i'}|$, which together with $p' \prec i'$ contradicts our choice of i_1. This establishes (5.64) and completes the proof of the lemma. □

We now come to one of our key results,

Proposition 5.3.2. *Let $\gamma \in \mathbb{Z}^n$ with $\gamma_i \neq 0$ for all i, and $k \in \mathbb{N}$. Then,*

$$e_k(x, x^{-1}) E_\gamma(x; q) = \sum_T q^{-\operatorname{inv} T} E_{\operatorname{end} T}(x; q) \tag{5.67}$$

where T is an out-in tableau starting at γ, with weight (k).

Proof. As $e_k(x,x^{-1})$ is W-symmetric, it commutes with any T_w, and hence

$$e_k(x,x^{-1})E_\gamma(x;q) = q^{-\ell(w)}T_w e_k(x,x^{-1})x^{\gamma_+} \tag{5.68}$$

$$= \sum_{a+b=k}\sum_{|I|=a} q^{-\ell(w)}T_w(x^{\gamma_+ + \varepsilon_I}e_b(x^{-1})) \tag{5.69}$$

Noting that as I ranges over subsets of size a, so too does $w(I)$, (5.67) follows from (5.69) after applying Lemma 5.3.4. □

A straightforward repeated use of Proposition 5.3.2 gives

Corollary 5.3.2. *Let μ be a partition and $k \in \mathbb{N}$. For $\gamma \in \mathbb{Z}^n$, write*

$$e_\mu(x,x^{-1})E_\gamma(x;q) = \sum_\beta c^\mu_{\beta,\gamma}(q)E_\beta(x;q) \tag{5.70}$$

If for all i, either $|\gamma_i| + |\beta_i| >> 0$ (in particular greater than $\ell(\mu)$), then

$$c^\mu_{\beta,\gamma}(q) = \sum_{T\in \mathrm{OutIn}(\beta/\gamma,\mu)} q^{-\operatorname{inv} T} \tag{5.71}$$

We recover the following special case of the Pieri rule as a further corollary to Proposition 5.3.2.

Corollary 5.3.3. *Let μ, λ be partitions with all parts sufficiently far from 0. Let $\chi_\lambda(x)$ denote the irreducible character of Sp_{2n} with highest weight λ. Then,*

$$e_\mu(x,x^{-1})\chi_\lambda(x) = \sum_\nu |\operatorname{vSSOT}(\nu/\lambda,\mu)|\chi_\nu \tag{5.72}$$

Proof. We let $\lambda \in P_+$ be a dominant weight and consider $\gamma = w_0\lambda$. From (5.8), specializing at $q = \infty$ yields

$$e_\mu(x,x^{-1})\chi_\lambda = \sum_\beta c^\mu_{\beta,\lambda}E_\beta(x;\infty) \tag{5.73}$$

where $c^\mu_{\beta,\lambda} = \#\{T \in \mathrm{OutIn}(\beta/\gamma,\mu) \mid \operatorname{inv} T = 0\}$. Now, if T is an out-in tableau of shape β/γ with $\operatorname{inv} T = 0$, then γ being antidominant means that any intermediate shape in T must also be antidominant. Applying w_0 to all intermediate shapes in T then gives a vertical semistandard oscillating tableau from λ to $\nu := w_0(\beta)$, hence arriving at (5.72). □

We note the Pieri rule above is already known [77, 89], even without the condition that the parts be far from 0. However, we also find the following new Demazure Pieri rule in the case when no part of γ is zero.

Corollary 5.3.4. *Given $\gamma = w(\lambda)$ for $\gamma \in P, w \in W, \lambda \in P_+$, we let $\mathcal{K}_\gamma = \partial_w x^\lambda$ denote the Demazure character as defined in (5.10). If γ has no zero parts, then*

$$e_k(x, x^{-1})\mathcal{K}_\gamma(x) = \sum_\nu \#\{T \in \mathrm{OutIn}(\beta/\gamma, k) \mid \mathrm{inv}\, T = 0\}\mathcal{K}_\beta(x) \tag{5.74}$$

We are now in a position to give a partial combinatorial formula for symplectic LLT polynomials. To start, we give a suitable symplectic analogue of polynomial truncation.

Definition 5.3.4. Let $\beta, \gamma \in \mathbb{Z}^n$. We define the **polynomial truncation** of $\mathcal{L}_{\beta,\gamma}(x; q)$ to be

$$\mathcal{L}_{\beta,\gamma}(x; q)\big|_{\mathrm{pol}} := \sum_{\lambda \subseteq (n^n)} Q^{(\lambda')^c}_{\beta,\gamma}(q^{-1})\chi_\lambda(x) \tag{5.75}$$

where $(\lambda')^c$ denotes the complement transpose of λ and $Q^\lambda_{\beta,\gamma}(q^{-1}) = \langle E_\beta(x; q)\rangle\chi_\lambda E_\gamma(x; q)$ is the usual coefficient of χ_λ in $\mathcal{L}_{\beta,\gamma}$. In other words, $\mathcal{L}_{\beta,\gamma}\big|_{\mathrm{pol}}$ is the truncation of the formal series $\mathcal{L}_{\beta,\gamma}$ to weights contained in an $n \times n$ rectangle, and then swapping all the coefficients of χ_λ with those of $\chi_{(\lambda')^c}$.

With this definition in hand, we prove

Theorem 5.3.1. *Let $\gamma \in \mathbb{Z}^n$ with $|\gamma_i| + |\beta_i| > n$. Then,*

$$\mathcal{L}_{\beta,\gamma}(x; q)\big|_{\mathrm{pol}} = \sum_T q^{\mathrm{inv}\, T} m_{(\mathrm{wt}\, T)^c}(x, x^{-1}) \tag{5.76}$$

the sum over out-in tableaux T of shape β/γ and partition weight $\mathrm{wt}\, T \subseteq (n^n)$.

Proof. We introduce another set of variables $y_1^{\pm 1}, \ldots, y_n^{\pm 1}$. Then, using the Cauchy identity (3.13), we have

$$\mathcal{L}_{\beta,\gamma}(x; q)\big|_{\mathrm{pol}} = \sum_{\lambda \subseteq (n^n)} \langle E_\beta(y; q^{-1})\rangle\chi_\lambda(y)E_\gamma(y; q^{-1})\chi_{(\lambda')^c}(x) \tag{5.77}$$

$$= \sum_{\lambda \subseteq (n^n)} \langle E_\beta(y; q^{-1})\rangle e_\lambda(y, y^{-1})E_\gamma(y; q^{-1})m_{\lambda^c}(x, x^{-1}) \tag{5.78}$$

$$= \sum_{\lambda \subseteq (n^n)} \sum_{T \in \mathrm{OutIn}(\beta/\gamma, \lambda)} q^{\mathrm{inv}\, T} m_{\lambda^c}(x, x^{-1}) \tag{5.79}$$

where the last equality is from Corollary 5.3.2. □

While this definition of truncation might seem strange, it becomes more natural under the bijection Ψ between semistandard oscillating tableaux and symplectic tableaux in Theorem 3.2.2. More specifically, given an out-in tableau of shape β/γ, we can view each part as a 1-vSSOT of shape β_i/γ_i in n steps. We can biject each of these to a symplectic tableau that is a single column of length $n - (|\beta_i| - |\gamma_i|)$. We then recall from Remark 3.2.1(a) that Ψ is not weight-preserving, but rather maps a semistandard oscillating tableau of weight μ to a symplectic tableau of weight μ^c. Hence, through Ψ, the sum in (5.76) transforms to

$$\mathcal{L}_{\beta,\gamma}(x;q)|_{\mathrm{pol}} = \sum_T q^{\mathrm{inv}\,T} x^T \tag{5.80}$$

where the sum is over tuples of symplectic tableaux on the columns $n - (|\beta_i| - |\gamma_i|)$, and $\mathrm{inv}\,T$ is some undetermined statistic on tuples of symplectic tableaux. The upside of this reformulation is that at $q = 1$, we witness $\mathcal{L}_{\beta,\gamma}(x;q)|_{\mathrm{pol}}$ as a product of irreducible characters as is the case with LLT polynomials for GL_n at $q = 1$, and as we noted in Remark 4.6.1(c). More details on the case $q = 1$ will be given in Section 5.5.

5.4 Other classical Lie types

Most of what has been proven up until now for Sp_{2n} carries over to analogous statements for the orthogonal Lie types, where one replaces $e_k(x, x^{-1})$ with the character of the k^{th} exterior power of the standard representation. More specifically, we have

Proposition 5.4.1. *Let $G = \mathrm{SO}_{2n+1}$ or $\mathrm{Spin}(2n+1)$ and $\gamma \in P$. If either*

1. *$G = \mathrm{SO}_{2n+1}$ and $\gamma_i \neq 0$ for all i, or*
2. *$G = \mathrm{Spin}(2n+1)$ and $\gamma_i \neq 0, \pm 1/2$ for all i,*

then

$$e_k(x, x^{-1}, 1)E_\gamma(x;q) = \sum_T q^{-\mathrm{inv}\,T} E_{\mathrm{end}\,T}(x;q) \tag{5.81}$$

where T is an out-in tableau starting at γ, with weight k or $k-1$.

Proof. The proof boils down to proving a counterpart to Lemma 5.3.4, which we note relies on the following facts

(1) The weight $\lambda = \gamma_+ - \varepsilon_J$ fails to be dominant at worst by having $\lambda_i = \lambda_j - 1$ for some $i < j$ with $(\gamma_+)_i = (\gamma_+)_j$, so that one can sort

$$x^{\gamma_+ - \varepsilon_J} = T_{v_J}^{-1} x^{(\gamma_+ - \varepsilon_J)_+} \tag{5.82}$$

where $v_J \in S_n$.

(2) The length of v_J is the number of inversions in $\mathrm{Inv}(\beta) \setminus \mathrm{Inv}(\gamma)$, where $\beta = \gamma - w(\varepsilon_J)$ and w is maximal with $w(\gamma_+) = \gamma$.

Both of these facts hold for G having Cartan type B_n, as the Weyl group is the same as for type C_n, and the conditions on γ still ensure that $v_J \in S_n$. The sum changes to a sum over out-in tableau with weight k or $k-1$ simply because the terms in $e_k(x, x^{-1}, 1)$ are of the form $x^{\varepsilon_I - \varepsilon_J}$, where $|I| + |J| = k$ or $k-1$. □

Conjecture 5.4.1. *Let $G = \mathrm{SO}_{2n}$ or $\mathrm{Spin}(2n)$ and $\gamma \in P$. If either*

1. *$G = \mathrm{SO}_{2n}$ and $\gamma_i \neq 0$ for all i, or*
2. *$G = \mathrm{Spin}(2n)$ and $\gamma_i \neq 0, \pm 1/2$ for all i,*

then

$$e_k(x, x^{-1}) E_\gamma(x; q) = \sum_T q^{-\operatorname{inv} T} E_{\operatorname{end} T}(x; q) \tag{5.83}$$

where T is an out-in tableau starting at γ, with weight k.

We would like to follow the same outline as for the odd orthogonal groups, namely we consider the ways in which the weight $\lambda = \gamma_+ - \varepsilon_J$ can fail to be dominant. There is a new case now, when we have $\lambda_{n-1} = |\lambda_n| - 1$ and $\gamma_{n-1} = -\gamma_n$. We can still pick v_J minimal such that $v_J(\lambda) = \lambda_+$, although now a reduced decomposition of v_J will include s_n. There holds the identity $T_n(x_{n-1}^a x_n^{-a-1}) = x_{n-1}^{a+1} x_n^{-a}$ for $a \in \mathbb{Z}$, however the identity (5.82) no longer necessarily holds, for example in the case when $\gamma = (a, a, \ldots, a, -a)$ and $\varepsilon_J = (0, \ldots, 0, -1, -1)$.

For G an orthogonal or spin group, we would like to use Proposition 5.4.1 to give a formula for some polynomial truncation of its LLT series akin to Theorem 5.3.1; alas we don't presently know of any useful Cauchy identity in this case. We discuss progress towards this a bit in Chapter 6. Nonetheless, the Pieri rule for Demazure characters (Corollary 5.3.4) still holds in the orthogonal types.

5.5 Symplectic LLT polynomials at $q = 1$

In this section we let G be arbitrary and work over a general Levi L. We again make the usual identification of $\mathbb{Z}[q^{\pm 1}]P$ with a Laurent polynomial ring. We aim to explicitly compute the polynomials $Q^\lambda_{\beta,\gamma}(q)$ as defined in (4.71). To that end, we recapitulate the constructions of $\eta \in P_+$ and $k \in \mathbb{N}$ given in Section 4.3.

Given a parabolic subgroup W_J with corresponding Levi subgroup L, we pick a dominant weight η in the level k fundamental alcove, where the only walls on which η lies are the walls given by J. With η and k in hand, we have the correspondence (4.35) between $\beta \in P_{++}(L)$ and $\mu \in P_{++} \cap \widetilde{W} \cdot \eta$ with the relationship $\mu = w(\eta + k\beta)$. Following [28], we relabel the elements $e_J^- E_\beta(x; q)$ as $|\mu\rangle$. The upshot of this relabelling is that a W-invariant polynomial $f = \sum_\lambda a_\lambda x^\lambda$ acts on these basis elements by

$$f \cdot |\mu\rangle = \sum_\lambda a_\lambda |\mu - k\lambda\rangle \tag{5.84}$$

With this new indexing of basis elements, the matrix coefficients $Q^\lambda_{\beta,\gamma}(q)$ are equal to polynomials $Q^\lambda_{\mu,\nu}(q)$, defined by

$$\chi_\lambda |\nu\rangle = \sum_\mu Q^\lambda_{\mu,\nu}(q) |\mu\rangle \tag{5.85}$$

where $|\nu\rangle = e_J^- E_\gamma$ and $|\mu\rangle = e_J^- E_\beta$. We note that there is an implicit dependence on k and η everywhere - changing k and η will change the labelling $|\mu\rangle$ for a basis element $e_J^- E_\beta$, and also how an irreducible character χ_λ acts on $|\mu\rangle$. However, the polynomials $Q^\lambda_{\beta,\gamma}(q)$ are independent of any choices.

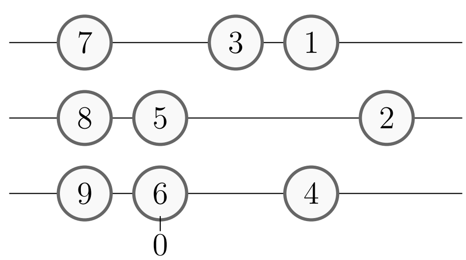
To compute the coefficients $Q^\lambda_{\mu,\nu}(q)$, one must be able to write the elements $\mu - k\lambda$ in (5.84), which are in general not strictly dominant weights, as a linear combination of basis elements. This is accomplished by applying the following **straightening rules**, which can be found in [28, Prop. 6.3(ii)] for arbitrary Lie type and in [60] for GL_n, albeit with slightly different notation.

Proposition 5.5.1. *Let $\mu \in P$ and assume $\langle \mu, \alpha_i^\vee \rangle \leq 0$ for some simple coroot $\alpha_i^\vee$. Write $-\langle \mu, \alpha_i^\vee \rangle = pk + r$, where $0 \leq r < k$. Then,*

$$|\mu\rangle = \begin{cases} 0 & : r = p = 0 \\ -|s_i\mu\rangle & : r = 0, p > 0 \\ q^{-1}|s_i\mu\rangle & : r \neq 0, p = 0 \\ q^{-1}|s_i\mu\rangle + q^{-1}|\mu + r\alpha_i\rangle - |s_i\mu - r\alpha_i\rangle & : r \neq 0, p > 0 \end{cases} \tag{5.86}$$

The action (5.84) and straightening rules can all be visualized with the aid of an abacus. Following Section 4.3, we view $|\mu\rangle$ as an **ordered abacus** on k rungs, where we place a bead with label i at positions μ_i. We read the beads from largest position to smallest, so that if μ is a strictly dominant weight, then the labels are in decreasing order. If μ is not a regular weight, then $|\mu\rangle = 0$.

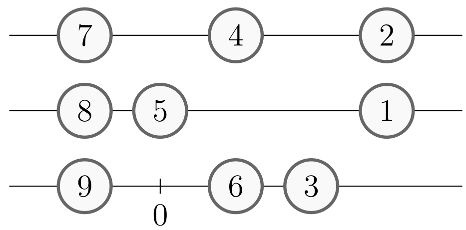
Example 5.5.1. If $k = 3$ and $\mu = (8, 10, 5, 6, 1, 0, -1, -2, -3)$ in GL_9, then we draw $|\mu\rangle$ as the ordered abacus



We caution that labels of beads are not the positions of the beads, as we originally had when we defined abaci in Section 2.1. For example, the bead labelled 2 is in position 10.

With the abacus perspective, a generator x_i in the weight ring acts according to (5.84) by moving the bead with label i on the abacus one unit to the right. Similarly, x_i^{-1} moves the bead with label i one unit to the left. If the labellings of the beads on the abacus are not in decreasing order, then one uses (5.86) to reorder the beads.

Example 5.5.2. We let $k = 3$ and $\mu = (10, 8, 6, 5, 1, 0, -1, -2, -3)$ be the strictly dominant weight in the orbit of μ in Example 5.5.1. The abacus of μ is pictured as before, but with the beads labelled in decreasing order from right to left, top to bottom. The action of x_2x_6 on $|\mu\rangle$ results in the element $|\nu\rangle$, where $\nu = (10, 11, 6, 5, 1, 3, -1, -2, -3)$, and is drawn as



We straighten $|\nu\rangle$ by using the third case in (5.86) to swap beads 5,6 and beads 1,2, yielding

$$x_2^{-1}x_6^{-1} \cdot |\mu\rangle = q^{-2}|\nu_+\rangle \tag{5.87}$$

With Example 5.5.2 as a guide, we come to the following

Proposition 5.5.2. *We let $G = \mathrm{GL}_n$ and $\mu \in P_+$, so that $\mu + \rho \in P_{++}$. Then,*

(i) The only straightening relations that can occur in $e_s(x) \cdot |\mu + \rho\rangle$ are the first and third conditions in (5.86).

(ii) The element $x_i^{-1} \cdot |\mu + \rho\rangle$ is either zero or equal to $q^{-\operatorname{spin} R}|\nu + \rho\rangle$, where $\nu \in P_+$ is gotten from μ by adding a k-ribbon R. In general, if ε_I is the indicator vector for a subset $I \subseteq n$ of size s, then $x^{-\varepsilon_I} \cdot |\mu + \rho\rangle = q^{-\operatorname{spin} T}|\nu + \rho\rangle$, where μ/ν is a horizontal k-ribbon strip T consisting of s ribbons.

This is essentially the argument used in the proof of Proposition 6.8 in [28], which shows that the polynomial truncation of LLT series coincides with combinatorial LLT polynomials, this time with the spin formulation.

For the remainder of this section, we set $G = \mathrm{Sp}_{2n}$.

To compute the coefficients $Q^{\lambda}_{\mu\nu}(q)$ in (5.85), we follow the strategy outlined in Proposition 5.3.1, namely we first analyze the action of an elementary symmetric polynomial $e_s(x, x^{-1})$. Matters become slightly more complicated in this case, as the polynomials $e_s(x, x^{-1})$ can move beads of an abacus $|\mu\rangle$ either left or right (and some beads simultaneously left and right). As a consequence, any of the relations (5.86) can occur. For example, Figure 5.3 depicts a sequence of actions when $e_{(2,1)}(x_1^{\pm 1}, x_2^{\pm 1})$ acts on an abacus with two beads.

The main takeaway from Figure 5.3 is that in contrast to the GL_n case, more than just the first and third conditions in (5.86) can occur when straightening an element $x^{\varepsilon_I - \varepsilon_J} \cdot |\mu\rangle$, where $\mu \in P_{++}$. Drawn in Figure 5.4 are all the ways in which $x^{\varepsilon_I - \varepsilon_J} \cdot |\mu\rangle$ can be out of order and how one straightens in each case. To elaborate on Figure 5.4, case (5.4e) occurs for example when there is an index i with $0 < \mu_i - \mu_{i+1} < k$ and we act by $x_i^{-1}x_{i+1}$. Letting ν denote the result, we have that

$$-2k < \langle \nu, \alpha_i^\vee \rangle = \nu_i - \nu_{i+1} = (\mu_i - k) - (\mu_{i+1} + k) < -k \tag{5.88}$$

Hence, to straighten ν, we are in the case $p = 1, r \neq 0$ of (5.86). The three terms in this case are the three terms pictured as abaci in case (5.4e).

We make special note of case (5.4c). This case occurs when the last index μ_n is less than k. Then, to straighten $|\nu\rangle := x_n|\mu\rangle$ we use the third relation of (5.86) to get $q^{-1}|s_n\nu\rangle$. This moves the bead with label n to the **conjugate runner**, namely if the last bead in μ was on rung r, then it is moved to rung $k - r$, with a power of q^{-1}.

For the remainder of this section, we further set $q = 1$.

As we see from Figure 5.4, each of the four cases in the straightening relations can occur (although with the value p at most 1). We note however that upon specialization to $q = 1$, the cases (5.4e), (5.4f) have terms that vanish, resulting in a more simplified straightening algorithm: to straighten in all cases except the move to the conjugate runner, we only swap beads with a power of either q^{-1} or -1.

The case (5.4c) when a bead moves to the conjugate runner can be avoided with a process we will refer to as **unfolding**, which we will see presently is none other than the map (4.35) between strictly dominant weights for a Levi and strictly dominant weights in the $\widetilde{W}$ orbit of some η. Combinatorially, unfolding is a procedure which takes an abacus with k rungs and outputs an abacus with $\lceil (k+1)/2 \rceil$ rungs, given pictorially in Figure 5.5. More precisely, to a strictly dominant weight ν and $k \in \mathbb{N}$, unfolding proceeds as follows:

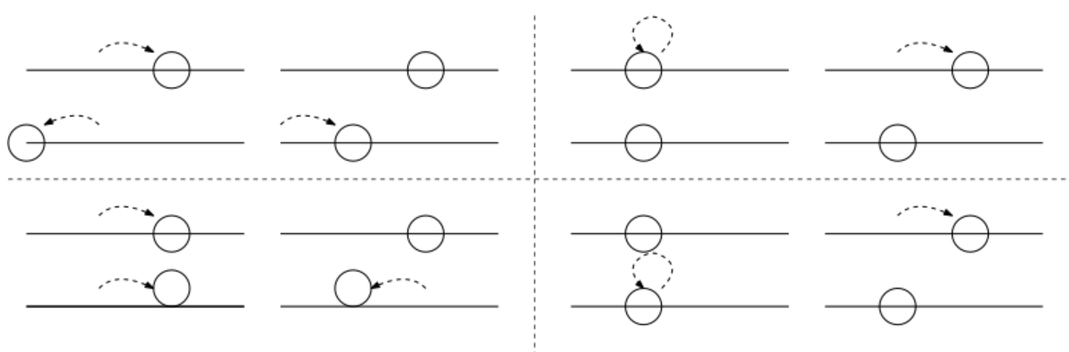
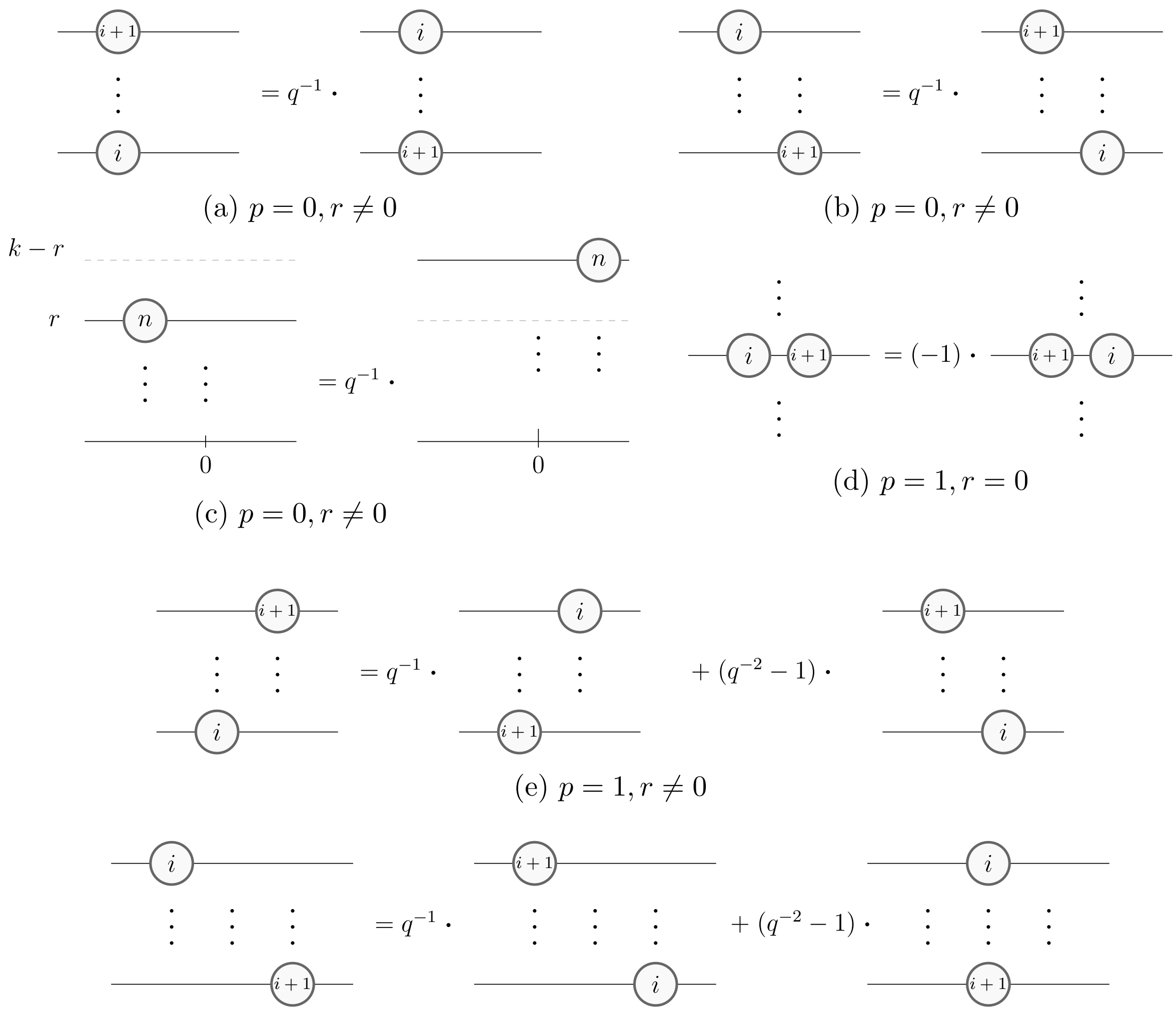


Figure 5.3: We let $|\mu\rangle$ stand for the initial abacus with 2 beads in the same column on different rungs. The top left diagram shows the sequence of actions $x_1^{-1}x_2$ followed by x_2^{-1}. The top right diagram shows the sequence of actions $x_1x_1^{-1}$ followed by x_1^{-1}. The bottom left diagram shows the sequence of actions $x_1^{-1}x_2^{-1}$ followed by x_2. The bottom right diagram shows the sequence of actions $x_2x_2^{-1}$ followed by x_1^{-1}.



(a) $p = 0, r \neq 0$ (b) $p = 0, r \neq 0$ (c) $p = 0, r \neq 0$ (d) $p = 1, r = 0$ (e) $p = 1, r \neq 0$ (f) $p = 1, r \neq 0$

Figure 5.4: All possible ways to straighten $x^{\varepsilon_I - \varepsilon_J} \cdot |\mu\rangle$ for $\mu \in P_{++}$, labelled by the cases in (5.86).

1. Draw the k-quotient of ν, i.e. draw the k-abacus with beads at positions ν_i.
2. For $1 \leq r \leq \lceil k/2 \rceil$, prepend the $(k-r)^{th}$ rung to the r^{th} rung so that if a bead is at the i^{th} position of rung $k - r$, then it is now at the $(-i-1)^{th}$ position of rung r.

The unfolded abacus of ν has the following relation with the choices of k and η in the kth fundamental alcove. If $\eta \in \mathcal{A}_k$, then $\langle \eta, \theta^\vee \rangle < k$, where $\theta^\vee = e_1^\vee + e_2^\vee$ is the highest coroot of Sp_{2n}. We can choose k and η to have the form

$$\eta_1 = \cdots = \eta_{r_1} > \eta_{r_1+1} = \cdots = \eta_{r_1+r_2} > \cdots > \eta_{n-r_\ell+1} = \cdots = \eta_n \geq 0, \qquad k \geq 2\eta_1 + 1 \tag{5.89}$$

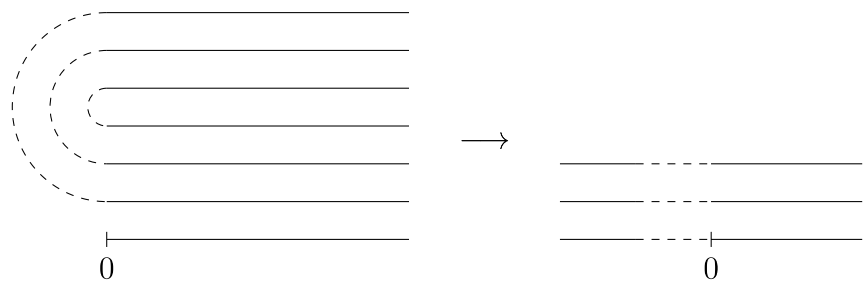


Figure 5.5: An abacus with 7 rungs being unfolded to an abacus with 3 rungs.

Every weight is in the $\widetilde{W}$-orbit of some η, as one can reduce all the entries of ν to be in the interval $(-k, k)$, and then permute. The choice of η determines the Levi

$$L = \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_{\ell-1}} \times \mathrm{Sp}_{2r_\ell} \tag{5.90}$$

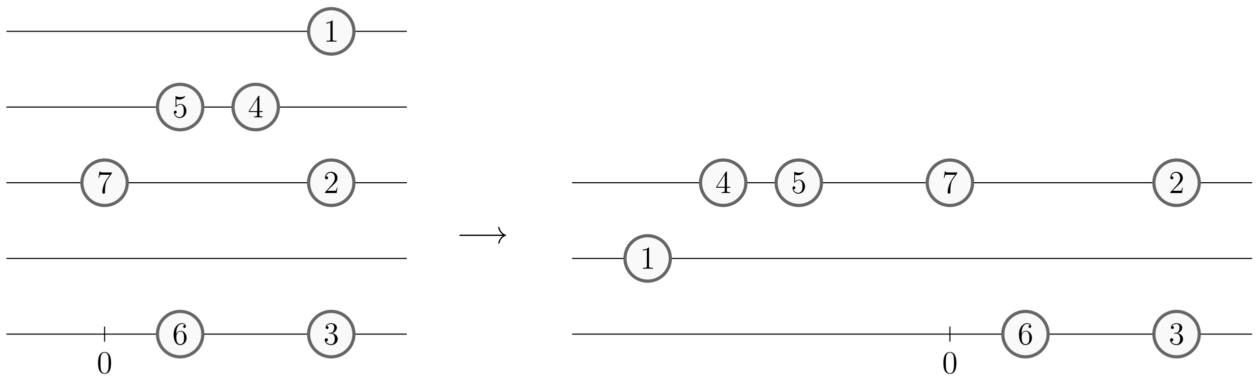
and if we write $\nu = w(\eta + k\gamma)$, then we can pick $w \in W^J$ so that $\gamma \in P_{++}(L)$, which we write as $\gamma = (\gamma^{(1)}, \ldots, \gamma^{(\ell)})$. Also note that $\lceil (k+1)/2 \rceil \geq \eta_1 + 1 \geq \ell$ and so we can moreover choose k so that $\lceil (k+1)/2 \rceil$ is exactly the number of Levi factors.

The unfolded abacus of ν has ℓ rungs whose beads are prescribed by γ in the following way: on the i^{th} rung of the folded abacus of ν (read from top to bottom), there are r_i beads in columns $\gamma_1^{(i)}, \ldots, \gamma_{r_i}^{(i)}$. To reverse unfolding, i.e. to **fold**, one writes $\nu = w(\eta + k\gamma)$, and then

1. for each $\gamma_j^{(r)} \geq 0$, place a bead on the r^{th} rung from the top in column $\gamma_j^{(r)}$
2. for each $\gamma_j^{(r)} < 0$, place a bead on the $(k-r)^{th}$ rung from the top in column $|\gamma_j^{(r)}| - 1$.

More succinctly, the beads on the i^{th} rung of the unfolded abacus of $|\nu\rangle$, read from top to bottom, are in columns $\gamma_j^{(i)}$ for $1 \leq j \leq r_i$. Given this correspondence, we will use the notation $|\gamma\rangle$ interchangeably with the unfolding of the abacus $|\nu\rangle$.

Example 5.5.3. Set $\nu = (19, 17, 15, 13, 8, 5, 2) \in P_{++}$ and $k = 5$. Its ordered abacus and unfolded abacus are drawn below.



Reducing all the entries of ν to be between $-k$ and k, exclusive, we have

$$\begin{aligned} \nu &= (-1, 2, 0, -2, -2, 0, 2) + 5 \cdot (4, 3, 3, 3, 2, 1, 0) \\ &= w((2, 2, 2, 2 \mid 1 \mid 0, 0) + 5 \cdot (3, 0, -2, -3 \mid -4 \mid 3, 1)) \end{aligned} \tag{5.91}$$

so that ν is in the $\widetilde{W}$-orbit of $\eta = (2, 2, 2, 2 \mid 1 \mid 0, 0) \in \mathcal{A}_k$. The weight η determines the Levi $L = \mathrm{GL}_4 \times \mathrm{GL}_1 \times \mathrm{Sp}_2$, and we have $\gamma = (3, 0, -2, -3 \mid -4 \mid 3, 1) \in P_{++}(L)$. We note that the beads on the r^{th} rung of the unfolded abacus of ν (read from top to bottom) are in columns $\gamma_j^{(r)}$.

With unfolding, we come to

Lemma 5.5.1. *Fix $k \in \mathbb{N}$. Let $\nu \in P_{++}$ and write $\nu = w(\eta + k\gamma)$, where η is of the form (5.89) and $\gamma \in P_{++}(L)$ for a Levi L of the form (5.90), which we can assume has number of factors $\ell = \lceil (k+1)/2 \rceil$. We let $|\gamma\rangle$ denote the unfolding of the abacus $|\nu\rangle$.*

1. *The action of $f \in \mathbb{Z}P^W$ on $|\nu\rangle$ descends to an action on $|\gamma\rangle$, with the elementary symmetric polynomials acting by*

$$e_s(x, x^{-1}) \cdot |\gamma\rangle = \sum_{|I|+|J|=s} |\gamma + \varepsilon_I - \varepsilon_J\rangle \tag{5.92}$$

2. *Let $\beta = \gamma + \varepsilon_I - \varepsilon_J$ for some $I, J \subseteq [n]$. If β is not a regular weight for L, then $|\beta\rangle = 0$. Otherwise, if $w \in W_J$ is such that $w(\beta_+) = \beta$, where $\beta_+ \in P_{++}(L)$, then $|\beta\rangle = (-1)^{\ell(w)} |\beta_+\rangle$.*

3. *If $\mu \in P_{++}$ with decomposition $v(\eta + k\beta)$ for $\beta \in P_{++}(L)$, then the coefficient of $|\mu\rangle$ in $e_s(x, x^{-1}) \cdot |\nu\rangle$ is the coefficient of $|\beta\rangle$ in $e_s(x, x^{-1}) \cdot |\gamma\rangle$.*

Proof. For the first part, we recall that a monomial x_i acts on $|\nu\rangle$ by moving the bead with label i one unit to the left, which still holds after unfolding the abacus. It may happen though that $x_n \cdot |\nu\rangle$ needs to be straightened by moving the bead with label n to its conjugate runner. On the unfolded $|\gamma\rangle$, this is exactly moving the bead with label n one unit to the left. The second part follows from any instances of Figure 5.4d when we straighten. The third part follows from the fact that $\beta \in P_{++}(L)$ uniquely determines μ, given fixed η, k. □

Using Lemma 5.5.1, our goal of computing the coefficients $Q^{\lambda}_{\mu,\nu}(1)$ in (5.85) translates to computing the coefficient of $|\beta\rangle$ in $e_s(x, x^{-1}) \cdot |\gamma\rangle$. We interpret this action as a sequence of bead moves on each rung, where for each bead, we perform exactly one of the following 4 weighted moves:

1. move the bead one unit to the right, with weight 1.
2. move the bead one unit to the left, with weight 1.
3. don't move the bead, with weight 2 (i.e. act by $x_i x_i^{-1}$).
4. do nothing, with weight 0.

Consequently, the coefficient of $|\beta\rangle$ in $e_\lambda(x, x^{-1}) \cdot |\gamma\rangle$ is the number of ways to go from a given initial configuration of beads $|\gamma\rangle$ to a given final configuration $|\beta\rangle$ with weight λ, such that

(i) No beads collide.

(ii) If 2 beads on a rung swap places, we count this with a negative sign.

As each rung is independent, we can isolate this problem to a single rung, where we count ways to move from one configuration of beads to another. We visualize this as a time evolution, in which each sequence of moves becomes a tuple of non-intersecting paths as in Figure 5.6. For GL_n, it is well known that a tuple of non-intersecting paths is in bijection

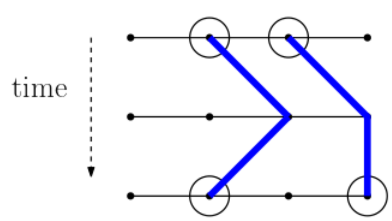


Figure 5.6: The sequence of actions $x_i x_j$ followed by x_i^{-1} on a single rung of $|\gamma\rangle$, viewed as a tuple of non-intersecting paths.

with semistandard Young tableaux, as is used in a combinatorial proof of the Jacobi-Trudi identity using the Lindström-Gessel-Viennot lemma. This gives another way to see that LLT polynomials for GL_n are a q-deformation of a product of Schur polynomials. The combinatorics of non-intersecting paths and their relation to oscillating tableaux are investigated

[54], in which the author proves similar determinantal formulas for the generating functions of semistandard oscillating tableaux.

Proposition 5.5.3. *Let $\gamma+\rho_L, \beta+\rho_L \in P_{++}(L)$ for L of the form (5.90) and fix components $\gamma^{(i)}, \beta^{(i)}$. As before, each determines a configuration of beads on a fixed rung by placing beads in column $\gamma^{(i)}+\rho_{r_i}$ and $\beta^{(i)}+\rho_{r_i}$. We let NIP$(\beta^{(i)}/\gamma^{(i)}, m)$ denote the set of tuples of non-intersecting paths from the bead configuration prescribed by $\gamma^{(i)}$ to the one given by $\beta^{(i)}$ in m steps. Then, there is a sign-reversing involution on NIP$(\beta^{(i)}/\gamma^{(i)}, m)$ for which the fixed points are in bijection with vertical semistandard oscillating tableaux of shape $\widetilde{\beta}^{(i)}/\widetilde{\gamma}^{(i)}$ in m steps, where*

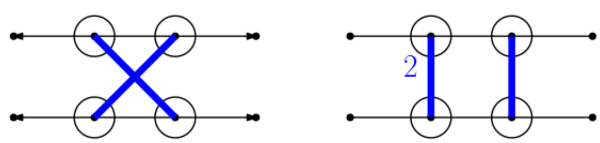
$$\widetilde{\gamma}^{(i)} = \gamma^{(i)} + (R^{r_i}), \qquad \widetilde{\beta}^{(i)} = \beta^{(i)} + (R^{r_i}) \tag{5.93}$$

for some sufficiently large $R \in \mathbb{N}$.

Proof. We aid the proof with an example, in Figure 5.7. We first map a tuple of non-intersecting paths to an out-in tableau T starting at $\gamma^{(i)}$, with the first out step υ and first in step δ constructed as follows.

We fix a bead B in the initial configuration given by $\gamma^{(i)}$, say at position $b = (\gamma^{(i)} + \rho_{r_i})_j$. If B moves in a direction which increases the absolute value of b, then we set υ_j and δ_j to this new position. Likewise, if B moves in a direction which decreases the absolute value of b, then we keep $\upsilon_j = b$ and set δ_j to the new position. If B does not move, with weight 0, then we set $\upsilon_j = \delta_j = b$ and if B does not move, but with weight 2, then we keep $\delta_j = b$ and set $\upsilon_j = b+1$ if b is non-negative and otherwise $b-1$. Continuing in this way for each step of the non-intersecting paths yields an out-in tableau.

We note that the resulting out-in tableau would in fact be a vertical semistandard tableau, provided that all the parts of all intermediate shapes are non-negative, and neither of the following two cases occurs



(5.94)

where we subscript a path with the weight 2 to denote the action by a monomial $x_i x_i^{-1}$. To ensure the non-negativity, we simply add a large enough integer R to each part of $\gamma^{(i)}$ and $\beta^{(i)}$. Setting $R = \max(|\gamma^{(i)}_{r_i}|, |\beta^{(i)}_{r_i}|) + m + r_i$ will suffice.

From Lemma 5.5.1(2), a tuple of non-intersecting paths has weight $(-1)^h$, where h is the number of "crossings", depicted on the left hand side of (5.94). We will uncross this by exchanging it with the tuple of non-intersecting paths on the right hand side of (5.94). Specifically, the involution in question will find the first occurrence of either tuple of paths in (5.94) (if one exists, and in any predetermined order), and exchange it with the other option. The fixed points are those tuples of paths with no occurrences of (5.94). $\square$

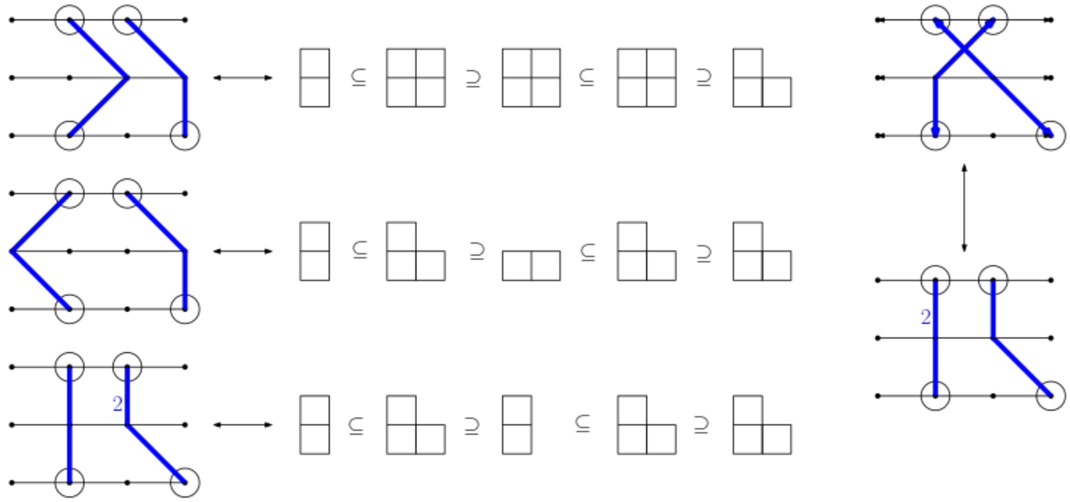


Figure 5.7: Example of bijection between tuples of non-intersecting paths and vertical semi-standard oscillating tableaux. We subscript a path with the weight 2 when we act by a monomial $x_i x_i^{-1}$. The fixed points of the involution are shown on the left.

Combining Proposition 5.5.3 and Theorem 3.2.2 gives a combinatorial tool to count the coefficient $\langle\beta|e_\lambda|\gamma\rangle$, given schematically by

$$\langle\beta \mid e_\lambda|\gamma\rangle \leftrightarrow \#\{\text{non-intersecting paths}\} \leftrightarrow \#\{\text{vertical SSOT}\} \leftrightarrow \#\{\text{symplectic tableaux}\} \tag{5.95}$$

Choosing everything appropriately, we arrive at

Theorem 5.5.1. *Let $G = \mathrm{Sp}_{2n}$ and fix a Levi $L = \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_{\ell-1}} \times \mathrm{Sp}_{2r_\ell}$, and weights $\beta = (\beta^{(1)}, \ldots, \beta^{(\ell)}), \gamma = (\gamma^{(1)}, \ldots, \gamma^{(\ell)}) \in P_+(L)$. For $1 \le j \le \ell$, choose R_j so that $\widetilde{\gamma}^{(j)} := \gamma^{(j)} + (R_j^{r_j})$ and $\widetilde{\beta}^{(j)} := \beta^{(j)} + (R_j^{r_j})$ have all part sizes at least n. For $k >> 0$ sufficiently large, set*

$$\boldsymbol{\tau} := (\widetilde{\beta}')^c, \text{ each complement taken in a } ((k+n)^{r_j}) \text{ box} \tag{5.96}$$

$$\boldsymbol{\sigma} := (\widetilde{\gamma}')^c, \text{ each complement taken in a } (k^{r_j}) \text{ box} \tag{5.97}$$

Then,

$$\mathcal{L}^G_{L,\beta+\rho_L,\gamma+\rho_L}(x_{k+1}, \ldots, x_{k+n}; 1)\big|_{\mathrm{pol}} = \sum_{T \in \mathrm{Symp}(\boldsymbol{\tau}/\boldsymbol{\sigma})} x^T \tag{5.98}$$

the sum over all skew symplectic tableaux of shape $\boldsymbol{\tau}/\boldsymbol{\sigma}$, and where $|_{\mathrm{pol}}$ denotes symplectic polynomial truncation as defined in Definition 5.3.4.

Chapter 6

Conclusions and Further Work

At the outset, we gave several definitions of LLT polynomials, all as q-generating functions for some combinatorial object. In the previous chapter, we reviewed how these combinatorial definitions coincide with polynomial truncations of an LLT series associated to the Lie group $G = \mathrm{GL}_n$, and began our foray into defining an analogous combinatorial LLT polynomial for the other classical Lie types, in particular for Sp_{2n}, that also coincides with some truncation of the associated LLT series. Towards this end, Theorems 5.3.1 and 5.5.1 give partial results for Sp_{2n} and Proposition 5.4.1 and Conjecture 5.4.1 likewise for the orthogonal Lie types. We profess however that these are by no means complete.

To start, Theorem 5.3.1 only holds when the indexing tuple γ is sufficiently far from 0, the reason being that it hinges on Lemma 5.3.4 which gives a formula for the expansion of a Hecke operator T_w applied to a monomial of the form $x^{\gamma_+ + \varepsilon_I - \varepsilon_J}$ when γ has no zeroes. The reason this condition is needed is because when we attempt to straighten the monomial $x^{\gamma_+ + \varepsilon_I - \varepsilon_J}$, we only need to use T_v for $v \in S_n$, and not a priori some signed permutation. Relaxing this condition not only breaks the proof, but invalidates the statement. Accordingly, we are led to modifying the definition of the inversion statistic of an out-in tableau that starts at a weight with possibly zero entries. On the one extreme, when γ is identically 0, we conjecture

Conjecture 6.0.1. *For $k \in \mathbb{N}$,*

$$e_k(x, x^{-1}) = \sum_{\beta} \binom{n - |\beta|}{\frac{k-|\beta|}{2}}_{q^{-2}} q^{-\operatorname{inv} \beta} E_\beta(x; q) \tag{6.1}$$

the sum over $\beta \in W \cdot (1^{k-2r})$ for some r, where $\binom{n}{k}_q$ denotes the q-binomial coefficient and $|\beta| = \#\{i \mid \beta_i = \pm 1\}$.

One can reformat (6.1) as a sum over a subset of out-in tableau with a modified inversion statistic as follows. Recall that the set $\mathrm{OutIn}^-(\beta/\gamma)$, defined in Remark 5.3.1, consists of out-in tableaux of shape β/γ in which the oscillation $0 \to 1 \to 0$ is not allowed. If $T = (\gamma \subseteq \nu \supseteq \beta) \in \mathrm{OutIn}^-(\beta/\gamma)$, we define a **descent** of T to be an index i with $(\gamma_i, \nu_i, \beta_i) = (0, -1, 0)$.

We define the **arm** of an index u to be

$$\operatorname{arm}(u) = \#\{i < u \mid (\gamma_i, \nu_i, \beta_i) = (0,0,0)\} \tag{6.2}$$

With the modified definition

$$\operatorname{inv}' T = \operatorname{inv} T + \sum_{u \in \operatorname{Desc}(T)} \operatorname{arm}(u) \tag{6.3}$$

then (6.1) becomes

$$e_k(x, x^{-1}) = \sum_{\beta} \sum_{T \in \operatorname{OutIn}^-(\beta,k)} q^{-\operatorname{inv}' T} E_\beta(x;q) \tag{6.4}$$

There are other possible modified inversion statistics one could feasibly use, however interpolating between any of these choices and Proposition 5.3.2 in which γ has no zero entries remains unsolved.

Secondly, it is remarked in [28, Remark 6.20] that for GL_n, there is a strengthening of Proposition 5.3.1 in which one can remove the polynomial truncation. More precisely, there holds an identity of LLT series

$$\mathcal{L}^{\mathrm{GL}_n}_{L,\beta,\gamma}(x;q) = (x_1 \cdots x_n)^{-s} \mathcal{L}^{\mathrm{GL}_n}_{L,\beta+(s^n),\gamma}(x;q) \tag{6.5}$$

for every integer s, so that can write the full LLT series for GL_n as an inverse limit of combinatorial LLT polynomials. It would be desirable to have such a restatement for Sp_{2n} so as to remove the somewhat arbitrary polynomial truncation in this case.

Thirdly, we mentioned in the paragraph following Theorem 5.3.1 that one could use the bijection Ψ between semistandard oscillating tableau and symplectic tableaux to rewrite the combinatorial definition of $\mathcal{L}_{\beta,\gamma}(x;q)|_{\mathrm{pol}}$ in Sp_{2n} instead as a sum over symplectic tableaux, which more closely resembles the inversion definition for type A LLT polynomials. It still remains however to determine how $\operatorname{inv} T$ transforms under Ψ.

Alternatively, one could extend the proposed tableaux definition of LLT polynomials given in Theorem 5.5.1 to arbitrary q. This method uses a different approach than considering the product $e_k E_\gamma$; instead one considers the action of e_k on a basis element $|\mu\rangle$, where $\mu \in P_{++}$, which we view as an abacus. The straightening relations involved in this case become more complicated when $q \neq 1$ (see Figure 5.4) and it still remains to overcome this.

Another natural progression would be to provide a combinatorial definition of general LLT polynomials for other Lie types. We mentioned when providing a combinatorial definition of LLT polynomials for Sp_{2n} that two essential ingredients were needed, the first being a formula for $e_k E_\gamma$ and the second being a Cauchy-like identity that relates the elementary polynomial to the irreducible character. In Section 5.4, we give suitable analogues of the former for the orthogonal Lie types. What we still lack is a suitable Cauchy identity as in Corollary 3.3.2.

The combinatorial objects at play for SO_{2n+1} appear in [88]. In place of symplectic tableaux, one has *orthogonal tableaux*, which are symplectic tableaux on the alphabet $1 <$

$\overline{1} < \cdots < n < \overline{n} < \infty$ such that all entries equal to ∞ form a vertical border strip. In place of vertical semistandard oscillating tableaux in the Pieri rule for the product $e_k\chi_\lambda$, one has the set of sequences of shapes

$$\mu = \alpha^0 = \beta^0 \subseteq \alpha^1 \supseteq \beta^1 \subseteq \alpha^2 \supseteq \beta^2 \subseteq \cdots \supseteq \lambda \tag{6.6}$$

such that for each i,

(i) α^i/β^{i-1} and α^i/β^i are vertical strips with $|\alpha^i/\beta^{i-1}| + |\alpha^i/\beta^i| \leq k$.

(ii) α^i, β^i have all lengths $\leq n$.

(iii) If $\ell(\alpha^i) < n$, then $|\alpha^i/\beta^{i-1}| + |\alpha^i/\beta^i| = k$.

The difference between these objects and semistandard oscillating tableaux is in the third condition, when a partition is allowed to "do nothing" if it has maximal length. In [88], the author uses a modified Berele insertion to prove an analogue of Proposition 3.3.1 in SO_{2n+1}. Akin to our work in Chapter 3, one would like some bijection between orthogonal tableaux and these analogues of semistandard oscillating tableaux which in conjunction with the modified Berele insertion would yield a Cauchy identity. The specifics have yet to be quantified and this remains an open problem.

Lastly, recalling that LLT polynomials were used in [31] to give a monomial expansion for Macdonald polynomials, it follows that a combinatorial formula for Sp_{2n} LLT polynomials could illuminate a similar expansion for type C Macdonald polynomials. As it stands, Macdonald polynomials are defined for any root system, but with only a combinatorial (and geometric) understanding in type A. What's more, the general type LLT polynomials coincide with the Hall-Littlewood polynomials $H_\mu(x;q)$ in arbitrary Lie type, when the indexing Levi L is the torus T. Consequently, a combinatorial formula for LLT polynomials could lead towards a formula for general type Kostka-Foulkes polynomials akin to Lascoux and Schutzenberger's celebrated and mysterious charge formula in type A. A charge statistic for other Lie types has been proposed in [61] for Kashiwara-Nakashima tableaux, and in fact the current problem was suggested to the author after first trying to extend the charge formula to type C for King tableaux.

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