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Grouplike Symmetric Monoidal Categories and Quiver Tensor Products

by

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requirements for the degree of

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Committee in charge:

Professor Constantin Teleman, Chair
Professor David Nadler
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Abstract

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In this thesis we develop a framework for understanding certain functors between the derived categories of toric varieties and stacks. We take the viewpoint that, just as toric varieties can be understood by studying commutative monoids, toric stacks can be understood by studying grouplike symmetric monoidal categories (i.e. those whose objects form an abelian group under the symmetric monoidal structure). We apply this perspective to describe the convolution product on the derived category of quasicoherent sheaves on a smooth toric stack $[\mathbb{A}^{\mathsf{n}}/\mathsf{G}]$ (viewed as a commutative monoid stack via coordinatewise multiplication). By combining this with Halpern-Leistner's theory of windows in GIT and with Hanlon–Hicks–Lazarev's resolutions of the diagonal, we produce descriptions of "quiver tensor products" on certain toric varieties, answering a natural question in homological mirror symmetry. We also construct a novel family of monoidal structures on the derived categories of projective spaces. Finally, we present a sketch of how these ideas can be extended to form a theory of "derived toric geometry" controlling push-pull functors between derived categories of toric varieties and stacks.

To you, dear reader, for trying.

Contents

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Chapter 0

Introduction

The goal of this thesis is to understand the derived categories and derived algebraic geometry of toric stacks (including toric varieties) in terms of subcategories of line bundles.

Our starting point is Beilinson's theorem that the line bundles $\mathscr{O}_{\mathbb{P}^n}, \mathscr{O}_{\mathbb{P}^n}(1), \ldots, \mathscr{O}_{\mathbb{P}^n}(n)$ form a so-called *full strong exceptional collection* (FSEC) of line bundles in the perfect derived (dg-)category of $\mathbb{P}^n$. This induces an equivalence $\mathsf{D}(\mathbb{P}^n) \simeq \mathsf{Fun}(\mathsf{Q}_n, \mathsf{D}(k))$ between the (unbounded) derived category of $\mathbb{P}^n$ and the dg-category of (derived) representations of a certain quiver Q_n depicted in Fig. 0.1 (where the representations are linear over an algebraically closed field k of characteristic zero).

$$q_0 \underset{x_n}{\overset{x_0}{\underset{\cdots}{\rightrightarrows}}} q_1 \underset{x_n}{\overset{x_0}{\underset{\cdots}{\rightrightarrows}}} \cdots \underset{x_n}{\overset{x_0}{\underset{\cdots}{\rightrightarrows}}} q_n$$

Figure 0.1: The Beilinson quiver for $\mathbb{P}^n$ (with relations $x_i x_j = x_j x_i$ for all i, j).

On the category $\mathsf{Fun}(\mathsf{Q}_n, \mathsf{Perf}(k))$, there is a natural "quiver tensor product" $\otimes_{\mathsf{Q}_n}$, given by vertexwise tensor products of quiver representations. One of our goals in this thesis is to answer the following:

Question 0.0.0.1. *Can the quiver tensor product $\otimes_{\mathsf{Q}_n}$ be understood in terms of the geometry of $\mathbb{P}^n$?*

Some basic computations indicate that $\otimes_{\mathsf{Q}_n}$ is closely related to the convolution product on the derived dg-category $\mathsf{D}(\mathsf{T})$ of quasicoherent sheaves on the dense torus $\mathsf{T} \subset \mathbb{P}^n$. However, the quiver tensor product $\otimes_{\mathsf{Q}_n}$ cannot be realized as a genuine convolution product, since $\mathbb{P}^n$ is not a group scheme (or even a monoid scheme).

However, all is not lost: by the theory of *windows* (as developed in [Hal15] and others), we know that there is a close relationship between the derived dg-categories $\mathsf{D}(\mathbb{P}^n)$ and $\mathsf{D}([\mathbb{A}^{n+1}/\mathbb{G}_m])$. Thinking of complexes of quasicoherent sheaves on $[\mathbb{A}^{n+1}/\mathbb{G}_m]$ as graded modules gives a quiver description of $\mathsf{D}([\mathbb{A}^{n+1}/\mathbb{G}_m])$. More precisely, there is an equivalence $\mathsf{D}([\mathbb{A}^{n+1}/\mathbb{G}_m]) \simeq \mathsf{Fun}(\mathsf{Q}_{n,\infty}, \mathsf{D}(k))$, where $\mathsf{Q}_{n,\infty}$ is the infinite quiver depicted in Fig. 0.2.

$$\cdots \underset{x_n}{\overset{x_0}{\underset{\longrightarrow}{\overset{\longrightarrow}{\cdots}}}} \mathsf{q}_{-1} \underset{x_n}{\overset{x_0}{\underset{\longrightarrow}{\overset{\longrightarrow}{\cdots}}}} \mathsf{q}_0 \underset{x_n}{\overset{x_0}{\underset{\longrightarrow}{\overset{\longrightarrow}{\cdots}}}} \mathsf{q}_1 \underset{x_n}{\overset{x_0}{\underset{\longrightarrow}{\overset{\longrightarrow}{\cdots}}}} \cdots$$

Figure 0.2: The infinite Beilinson quiver for $[\mathbb{A}^{n+1}/\mathbb{G}_m]$ (with relations $x_i x_j = x_j x_i$ for all i, j). Vertices are indexed by $\mathbb{X}^\bullet(\mathbb{G}_m) = \mathbb{Z}$.

It is possible (but not entirely trivial) to show that the induced quiver tensor product $\otimes_{\mathsf{Q}_{n,\infty}}$ on $\mathsf{D}([\mathbb{A}^n/\mathbb{G}_m])$ agrees with the convolution product induced by coordinatewise multiplication on the stack $[\mathbb{A}^n/\mathbb{G}_m]$. We thus have an entirely toric / geometric description of $\otimes_{\mathsf{Q}_{n,\infty}}$.

To describe the quiver tensor product $\otimes_{\mathsf{Q}_n}$ on $\mathsf{D}(\mathbb{P}^n)$, we use a trick involving the window functor $W : \mathsf{D}(\mathbb{P}^n) \to \mathsf{D}([\mathbb{A}^{n+1}/\mathbb{G}_m])$ and its right adjoint H to transport the quiver tensor product from $\mathsf{D}([\mathbb{A}^{n+1}/\mathbb{G}_m])$ to $\mathsf{D}(\mathbb{P}^n)$. More precisely, we show that we can write

$$\mathscr{F} \otimes_{\mathsf{Q}_n} \mathscr{G} = \mathsf{H}((W\mathscr{F}) \otimes_{\mathsf{Q}_{n,\infty}} (W\mathscr{G}))$$

for all $\mathscr{F}, \mathscr{G} \in \mathsf{D}(\mathbb{P}^n)$. This general strategy (with a few modifications) allows us to write $\otimes_{\mathsf{Q}_n}$ as a composite of push-pull functors along toric stacks, and we are led to the general question:

Question 0.0.0.2. *Let*

$$\begin{tikzcd} & \mathsf{Y}_1 \arrow[dl, "p_1"'] \arrow[dr, "q_1"] & & \mathsf{Y}_2 \arrow[dl, "p_2"'] \arrow[dr, "q_2"] & \\ \mathsf{X}_1 & & \mathsf{X}_2 & & \mathsf{X}_3 \end{tikzcd}$$

be a diagram of toric stacks. How can one describe the composite functor $q_{2*}p_2^*q_{1*}p_1^* : \mathsf{D}(\mathsf{X}_1) \to \mathsf{D}(\mathsf{X}_3)$*?*

In some sense the answer to this question is obvious: derived algebraic geometry tells us that the composite is given by push-pull along the derived fiber product $\mathsf{Y}_1 \times^{\mathbf{R}}_{\mathsf{X}_2} \mathsf{Y}_2$. However, this runs into the problem that *(derived) fiber products of toric stacks typically fail to be toric.* Thus, even though each push-pull functor can be understood via the combinatorial language of (stacky) fans, the same is not true for the composite.

A good portion of this thesis (specifically Chapter 2, Chapter 3, and Chapter 6) is dedicated to constructing a provisional formalism of $\mathbb{E}_\infty$-*stacky fans* which fixes this problem. We provide many sample computations illustrating how to work with $\mathbb{E}_\infty$-stacky fans, and we discuss questions and prospective applications relevant to this formalism in Chapter 6.

Ultimately, we answer Question 0.0.0.1 without directly using the formalism of $\mathbb{E}_\infty$-fans. A discussion of our answer can be found in Chapter 5. Our methods also generalize to solve an analogous problem for other toric varieties of "Bondal–Ruan type." These results have applications to mirror symmetry for toric varieties and (conjecturally) to the Cox categories of [Bal+24].

Furthermore, we can use these methods to produce a new family of monoidal structures on $\mathsf{Perf}(\mathbb{P}^n)$ related to finite-dimensional algebras. This family enables us to construct "categorical compactifications" of many well-known algebraic groups G, i.e. smooth proper dg-categories with monoidal structures extending the convolution product of G.

0.1 Conventions

Throughout most of this thesis we use the "implicit ∞" convention, i.e. all references to categories and functors are implicitly references to ∞-categories and functors between these. When we want to consider the more classical situation we will typically reference "1-categories" or "discrete categories." In particular, the notation QC will typically refer to the *derived* (dg- / ∞-)category of quasicoherent sheaves. Any deviation from this convention will be explicitly noted. We refer to [HTT] and [HA] for the foundations of ∞-category theory.

We will typically let k be a fixed algebraically closed field of characteristic zero, though many of our results hold in greater generality and we will indicate this when such generality is interesting.

0.2 Outline

Chapter 1 is devoted to background material that is useful for subsequent sections.

In Chapter 2 we introduce a notion of "grouplike symmetric monoidal ∞-category," i.e. a symmetric monoidal ∞-category for which the objects form a group under the symmetric monoidal operation. These can be thought of as a combinatorial analogue of the full subcategory of $\mathsf{D}([\mathsf{Y}/\mathsf{G}])$ spanned by line bundles for Y an affine toric variety and G a reductive algebraic group. We prove a structure theorem (Theorem 2.3.2.1), allowing us to understand grouplike symmetric monoidal ∞-categories in terms of $\mathbb{E}_\infty$-monoids.

Chapter 3 discusses how to use grouplike symmetric monoidal ∞-categories $\mathcal{C}$ to produce derived stacks $\mathfrak{Y}_\mathcal{C}$ using the Tannakian formalism. We show that, under additional hypotheses, the stacks $\mathfrak{Y}_\mathcal{C}$ are quotient stacks of affine derived schemes (Proposition 3.2.3.6), and we compute the derived categories $\mathsf{QC}(\mathfrak{Y}_\mathcal{C})$ (Proposition 3.2.3.8). The stacks $\mathfrak{Y}_\mathcal{C}$ also admit natural $\mathbb{E}_\infty$-monoid structures (internal to the ∞-category of derived stacks).

In Chapter 4 we consider the more general situation of a (nice) $\mathbb{E}_n$-monoid $\mathfrak{M}$ in the ∞-category of derived stacks together with an affine homomorphism $\mathfrak{M} \to \mathsf{BG}$ where BG is a commutative reductive group. We discuss how to understand the derived category $\mathsf{QC}(\mathfrak{M})$

as a "category of modules over an infinite quiver with relations." We prove (Theorem 4.2.3.1) that the convolution product on $\mathsf{QC}(\mathfrak{M})$ induced by the monoid structure on $\mathfrak{M}$ agrees with a "vertexwise tensor product" obtained from this quiver with relations.

Chapter 5 uses the results of Chapter 4 to study some questions about the derived categories of $\mathbb{P}^n$ and more general toric varieties. We establish a novel family of monoidal structures on the derived category of $\mathbb{P}^n$ related to finite dimensional algebras (Proposition 5.3.0.1). We also solve a problem in mirror symmetry for toric varieties, giving a "B-model" construction of the mirror to the tensor product of constructible sheaves on certain stratified real tori (Proposition 5.4.0.2).

In Chapter 6 we discuss how the results of Chapter 2 and Chapter 3 may be used to develop a theory of "derived toric geometry." We indicate some problems and directions for future research.

Chapters 1, 4, and 5 here are based on joint work with Daigo Ito and originally appeared (in modified form) in [IN25].

Chapter 1

Background

This section contains background material which will be useful for the rest of the thesis. We recommend [Gro20], [BFN10], [Sch23] for more extended (but still elementary and conceptual) discussions of many of the ideas mentioned here. Almost everything in this section is well-known to the experts, except for possibly the technical results on transporting monoidal structures along localizations in Section 1.2.5.

1.1 Generalities on ∞-categories

We begin by reviewing some relevant results on ∞-category theory.

Notation 1.1.0.1. In this section and the next, to avoid confusion when introducing new concepts, we will not use the "implicit ∞" convention. We will change notation in Notation 1.3.0.1.

1.1.1 Presentability and compact generation

To make our claims about homotopy coherence rigorous, we use the language of ∞-categories as developed in [HTT] and [HA] (among other references). This subsection is devoted to a rapid review of the theory of *presentable* ∞-categories discussed in [HTT].

Remark 1.1.1.1. There is a (classical) analogous theory of presentable 1-categories, but it is less central to applications. Many of the consequences of the classical theory (e.g. existence of adjoint functors) are easy to check "by hand" as needed. By contrast, proving results "by hand" in the ∞-categorical context is often much more difficult, and we are left with no recourse but to use categorical methods.

The essential idea of ∞-category theory is to replace sets in ordinary category theory by *spaces*, a.k.a. (weak) *homotopy types*. The category of spaces is denoted $\mathcal{S}$. There are many equivalent point-set models one can use to understand and construct $\mathcal{S}$. For example, one may obtain $\mathcal{S}$ from the category of CW-complexes and continuous maps by "inverting homotopy equivalences" in a suitable sense.

Write Cat_∞ for the (large) ∞-category of small ∞-categories and $\widehat{\mathsf{Cat}}_\infty$ for the (very large) ∞-category of large ∞-categories.[1] For any $\mathcal{C}_0 \in \mathsf{Cat}_\infty$, we define the ∞-category of *presheaves on* $\mathcal{C}_0$ as $\mathsf{PSh}(\mathcal{C}_0) = \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. As usual, we have a Yoneda embedding $\mathcal{C}_0 \hookrightarrow \mathsf{PSh}(\mathcal{C}_0)$.

Presheaf categories are key examples of presentable ∞-categories. An ∞-category $\mathcal{C}$ is *presentable* if there exists an ∞-category $\mathcal{C}_0$ and a functor $\mathsf{L} : \mathsf{PSh}(\mathcal{C}_0) \to \mathcal{C}$ which is:

- a *localization*, i.e. L has a fully faithful right adjoint L^{R}, and
- *accessible*, i.e. L^{R} preserves κ-filtered colimits for some regular cardinal κ.

Such a category $\mathcal{C}$ possesses many useful potential features of large ∞-categories – in particular, $\mathcal{C}$ is complete and cocomplete. However, the behavior of $\mathcal{C}$ is still controlled by that of the small ∞-category $\mathcal{C}_0$.

The adjoint functor theorem [HTT, Corollary 5.5.2.9] states that a functor $\mathsf{F} : \mathcal{C} \to \mathcal{C}'$ between presentable ∞-categories has a right adjoint if and only if F preserves colimits. Motivated by this definition, we let Pr^{L} denote the ∞-category of presentable ∞-categories and colimit-preserving functors. We write $\mathsf{Fun}^{\mathsf{L}}(\mathcal{C}, \mathcal{C}')$ for the ∞-categories of functors in Pr^{L} from $\mathcal{C}$ to $\mathcal{C}'$. Letting Pr^{R} be the ∞-category of presentable ∞-categories and functors which preserve limits and κ-filtered colimits (for some regular cardinal κ), the adjoint functor theorem upgrades to an equivalence $(\mathsf{Pr}^{\mathsf{L}})^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Pr}^{\mathsf{R}}$ which is the identity on objects.

The category Pr^{L} has a natural symmetric monoidal structure, the *Lurie tensor product* $\otimes$, defined so that colimit-preserving functors $\mathcal{C}_1 \otimes \mathcal{C}_2 \to \mathcal{C}_3$ are functors $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_3$ which preserve colimits in each variable separately. Our later variants of Pr^{L} will all inherit a corresponding Lurie tensor product.

If $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}$, we say that:

- An object $\mathsf{c}_0 \in \mathcal{C}_0$ is *compact* if the functor $\mathrm{Hom}_{\mathcal{C}}(\mathsf{c}_0, -)$ preserves filtered colimits.
- A small full subcategory $\mathcal{C}_0 \subset \mathcal{C}$ *generates* $\mathcal{C}$ if a morphism $\mathsf{f} : \mathsf{c} \to \mathsf{c}'$ is an isomorphism if and only if the induced map $\mathrm{Hom}_{\mathcal{C}}(\mathsf{c}_0, \mathsf{c}) \to \mathrm{Hom}_{\mathcal{C}}(\mathsf{c}_0, \mathsf{c}')$ is an isomorphism for all $\mathsf{c}_0 \in \mathcal{C}$. Equivalently, every object of $\mathcal{C}$ may be expressed as a colimit of objects of $\mathcal{C}_0$.

If the objects of $\mathcal{C}_0$ are compact and $\mathcal{C}_0$ generates $\mathcal{C}$, we say that $\mathcal{C}$ is *compactly generated* by $\mathcal{C}_0$. In this case, there is a natural equivalence $\mathcal{C} = \mathsf{Ind}(\mathcal{C}_0)$, where the functor $\mathsf{Ind} : \mathsf{Cat}_\infty \to \mathsf{Pr}^{\mathsf{L}}$ "freely adjoins filtered colimits" to its input (see [HTT, §5.3.5] for a precise definition). In particular, every object of $\mathcal{C}$ can be obtained (canonically) as a colimit of objects of $\mathcal{C}_0$. We let $\mathsf{Pr}^{\mathsf{L}}_\omega$ denote the ∞-category of compactly generated ∞-categories and functors which preserve colimits and compact objects.

[1] Standard techniques from the theory of *Grothendieck universes* allow us to deal with most set-theoretic "size issues" when they arise.

Let $(\mathcal{V}, \otimes)$ be a *presentably symmetric monoidal* ∞*-category*, i.e. an object of $\mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}}, \otimes)$. There exists a rich theory of "$\mathcal{V}$-enriched ∞-categories" – see e.g. [MS24, Appendix A] for a highly readable account of the subject. The above results extend (with some mild modifications) to the $\mathcal{V}$-enriched setting. In fact, by [MS24, Theorem A.3.8], a $\mathcal{V}$-enriched presentable ∞-category is the same as a module over $(\mathcal{V}, \otimes)$ in Pr^{L}. In particular, we note that $\mathcal{V}$ is automatically enriched over itself, and that any small $\mathcal{V}$-enriched category $\mathcal{C}$ admits a Yoneda embedding $\mathcal{C} \hookrightarrow \mathsf{Fun}_{\mathcal{V}}(\mathcal{C}^{\mathrm{op}}, \mathcal{V})$. We may use this result to reduce the study of presentable ∞-categories enriched in $(\mathcal{V}, \otimes)$ to the study of unenriched presentable ∞-categories.

Two key examples of the above are as follows:

- When $(\mathcal{V}, \otimes) = (\mathsf{D}(\mathsf{k}), \otimes_{\mathsf{k}})$, where $\mathsf{D}(\mathsf{k})$-enriched ∞-categories are the same as k-*linear* ∞*-categories*. We write $\mathsf{Cat}_{\mathsf{k}}$ for the large category of small k-linear ∞-categories.

- We may also take $(\mathcal{V}, \otimes) = (\mathsf{Sp}, \otimes)$, the ∞-category of *spectra*. Here $\otimes$ is the *smash product*.

1.1.2 Stable ∞-categories

Recall that an ∞-category $\mathcal{C}$ is *stable* if ([HA, Proposition 1.1.3.4]):

- $\mathcal{C}$ admits finite limits and finite colimits, and

- A commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

  is a pullback square if and only if it is a pushout square.

These conditions imply that $\mathcal{C}$ has a zero object 0. If the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ 0 \arrow[r] & C \end{tikzcd}$$

is a pullback square, we say that A is the *fiber* of g (denoted $\mathrm{fib}(g)$) and that C is the *cofiber* of f (denoted $\mathrm{cofib}(g)$). (These notions actually make sense in any ∞-category with a zero object.)

In any stable ∞-category $\mathcal{C}$, the endofunctor $[1] : \mathcal{C} \to \mathcal{C}$ defined by $A[1] = \mathrm{cofib}(A \to 0)$[2] is an autoequivalence, called the *shift autoequivalence*. Its inverse is $[-1] : \mathcal{C} \to \mathcal{C}$ defined by $B[-1] = \mathrm{fib}(0 \to B)$.

[2] When working in the ∞-categorical context, such colimits are typically nontrivial.

The ∞-category of pretriangulated dg-categories over a discrete commutative ring k is equivalent to the ∞-category of stable ∞-categories enriched over $\mathsf{D}(\mathsf{k})$. Via this equivalence, cones in a pretriangulated dg-category correspond to cofibers in the corresponding stable ∞-category. In other words, the *homological algebra* of cones, shifts, etc. in dg-categories translates into the *homotopical algebra* of (co)limits in stable ∞-categories. The stable ∞-categories of primary interest to us are enriched over $\mathsf{D}(\mathsf{k})$, so readers will not lose much by thinking of these stable ∞-categories as pretriangulated dg-categories.

Remark 1.1.2.1. Every stable ∞-category is canonically enriched over $(\mathsf{Sp}, \otimes)$. We can view Sp as the category of modules over the *sphere spectrum* $\mathbb{S}$, so every stable ∞-category is "enriched in some category of modules." Of course, it is typically much harder to perform computations in Sp than within any $\mathsf{D}(\mathsf{k})$, so this remark is best understood as a general philosophy.

Remark 1.1.2.2. The *homotopy category* of a stable ∞-category / pretriangulated dg-category is a triangulated category. We work in the context of stable ∞-categories to avoid various difficulties in the theory of triangulated categories, e.g. non-existence of tensor products of categories, non-functoriality of cones, poor behavior of Fourier-Mukai transforms, etc.

Write $\mathsf{Pr}^{\mathsf{L}}_{\mathrm{st}}$ for the ∞-category of stable, presentable ∞-categories and colimit-preserving functors. For a discrete commutative ring k, $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{k}}$ be the category of $\mathsf{D}(\mathsf{k})$-modules in $\mathsf{Pr}^{\mathsf{L}}_{\mathrm{st}}$. To avoid repeating ourselves too much, we will state the following results for $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{k}}$ (though the analogues for $\mathsf{Pr}^{\mathsf{L}}_{\mathrm{st}}$ also hold).

The theory of compact generation admits some simplifications in the stable setting. For $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}_{\mathsf{k}}$, we have the following (straightforward) results:

- An object $\mathsf{c} \in \mathcal{C}$ is compact if $\mathrm{Hom}(\mathcal{C}, -)$ commutes with infinite direct sums (and hence with all colimits).
- A full subcategory $\mathcal{C}_0 \subset \mathcal{C}$ generates $\mathcal{C}$ if, whenever $\mathsf{c} \in \mathcal{C}$ satisfies $\mathrm{Hom}(\mathsf{c}_0, \mathsf{c}) = 0$ for all $\mathsf{c}_0 \in \mathcal{C}_0$, we must have $\mathsf{c} = 0$.

If $\mathcal{C}$ is compactly generated by $\mathcal{C}_0$, there is a natural equivalence $\mathcal{C} = \mathsf{Ind}(\mathcal{C}_0)$, where we use a k-linear version of Ind. In fact, we have $\mathsf{Ind}(\mathcal{C}_0) = \mathsf{Fun}_{\mathsf{k}}(\mathcal{C}_0, \mathsf{D}(\mathsf{k}))$ in the stable k-linear setting.

Let $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{k},\omega}$ be the ∞-category of stable compactly generated k-linear ∞-categories with morphisms given by k-linear functors that preserve colimits and compact objects. Letting $\mathsf{Pr}^{\mathsf{R}}_{\mathsf{k},\omega}$ be the category with the same objects but with morphisms given by k-linear functors that preserve both limits and colimits, the adjoint functor theorem gives an equivalence $(\mathsf{Pr}^{\mathsf{L}}_{\mathsf{k},\omega})^{\mathrm{op}} = \mathsf{Pr}^{\mathsf{R}}_{\mathsf{k},\omega}$.

The study of $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{k},\omega}$ may be reduced to the study of certain small k-linear ∞-categories as follows. Following [AG14], we say that a k-linear ∞-category $\mathcal{C}_0$ is *perfect* if $\mathcal{C}_0$ is stable, and every idempotent $\mathsf{p} : \mathsf{c}_0 \to \mathsf{c}_0$ in $\mathcal{C}_0$ induces a direct sum decomposition $\mathsf{c}_0 = \mathsf{c}_0' \oplus \mathsf{c}_0''$ with $\mathsf{c}_0', \mathsf{c}_0'' \in \mathcal{C}_0$. Let $\mathsf{Cat}^{\mathrm{perf}}_{\mathsf{k}}$ denote the category of perfect k-linear ∞-categories and exact (i.e. finite (co)limit-preserving) k-linear functors. For $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}_{\mathsf{k},\omega}$, let $\mathcal{C}^{\omega}$ be the full subcategory consisting of all compact objects in $\mathcal{C}$. Then $(-)^{\omega}$ defines an equivalence $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{k},\omega} \xrightarrow{\sim} \mathsf{Cat}^{\mathrm{perf}}_{\mathsf{k}}$, with inverse given by $\mathsf{Ind} : \mathsf{Cat}^{\mathrm{perf}}_{\mathsf{k}} \to \mathsf{Pr}^{\mathsf{L}}_{\mathsf{k},\omega}$.

Remark 1.1.2.3. Using [BGT13, Theorem 1.10], we may obtain the category $\mathsf{Cat}^{\mathrm{perf}}_{\mathsf{k}}$ from $\mathsf{Cat}_{\mathsf{k}}$ by "localizing along the Morita equivalences." That is, the functor

$$\begin{aligned} \mathsf{Cat}_{\mathsf{k}} &\to \mathsf{Cat}^{\mathrm{perf}}_{\mathsf{k}} \\ \mathcal{C}_0 &\mapsto \mathsf{Ind}(\mathcal{C}_0)^{\omega} \end{aligned}$$

is the universal functor out of $\mathsf{Cat}_{\mathsf{k}}$ inverting Morita equivalences of k-linear ∞-categories (including e.g. Morita equivalences of k-algebras). Its right adjoint is the inclusion $\mathsf{Cat}^{\mathrm{perf}}_{\mathsf{k}} \hookrightarrow \mathsf{Cat}_{\mathsf{k}}$, which is necessarily fully faithful.

1.2 $\mathscr{O}$-algebras

In this section we recall some of the basics of $\mathscr{O}$-algebras (for some operad $\mathscr{O}$) and their relationship with adjunctions. In Section 1.2.5 we will use these prove some seemingly novel technical results about transporting monoidal structures along adjunctions which will be useful in our construction of quiver tensor products.

The results here are stated in the "non-enriched" (i.e., enriched in spaces) context in which they originally appeared in the literature. In the rest of the paper, we will largely use versions of these results enriched in spectra or $\mathsf{D}(\mathsf{k})$. One may use the results of [MS24, Appendix A] (especially [MS24, Theorem A.3.8]) to upgrade the results here to their corresponding enriched versions by regarding presentable ∞-categories enriched in a presentably symmetric monoidal ∞-category $\mathcal{V}$ as presentable ∞-categories with $\mathcal{V}$-actions.

1.2.1 $\mathbb{E}_n$-algebras

The ∞-categorical theory of (commutative) algebras in a (symmetric) monoidal category is subsumed by the theory of $\mathbb{E}_n$*-algebras* (for $1 \leqslant n \leqslant \infty$), or more generally *algebras over an* ∞*-operad*, as developed in [HA]. We'll focus on the $\mathbb{E}_n$-case here for simplicity. In fact, in this work, we are only truly interested in the case $n = 1$ or $n = \infty$. We use the terminology of $\mathbb{E}_n$-algebras primarily as an efficient method of covering both commutative and noncommutative cases with the same statement.

An $\mathbb{E}_n$-algebra in a symmetric monoidal ∞-category can be thought of as an algebra object with n compatible associative multiplications. The existence of these multiplications enforces a sort of commutativity on the operations involved. In particular:

- $\mathbb{E}_1$-algebras are the same as associative algebras.
- $\mathbb{E}_\infty$-algebras are the same as commutative algebras.

Note a key difference with the classical case: commutativity is no longer a *property* of the multiplication but an extra *structure* witnessed by the infinitely many compatible multiplications.

Remark 1.2.1.1. When working in a 1-category, $\mathbb{E}_n$-algebras for $n \geqslant 2$ are always commutative (i.e. the same as $\mathbb{E}_\infty$-algebras). The differences between $\mathbb{E}_2$-algebras and $\mathbb{E}_\infty$-algebras appear only when we allow nontrivial 2-morphisms.

Remark 1.2.1.2. When working in Cat_∞ (or similar ∞-categories such as Cat, the 2-category of discrete categories):

- $\mathbb{E}_1$-algebras are the same as monoidal categories.
- $\mathbb{E}_2$-algebras are the same as braided monoidal categories.
- $\mathbb{E}_\infty$-algebras are the same as symmetric monoidal categories.

In Cat_∞, we may take this as a definition of "monoidal / braided monoidal category" (though we need to bootstrap the definitions so that we may consider Cat_∞ as a symmetric monoidal category).

We write $\mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C}) = \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C}, \otimes)$ for the ∞-category of $\mathbb{E}_n$-algebras in a symmetric monoidal ∞-category $(\mathcal{C}, \otimes)$.[3] Lax $\mathbb{E}_n$-monoidal functors induce functors between the corresponding ∞-categories of algebras (this is automatic from the definition, [HA, 2.1.3.1]). There is also an analogous statement for $\mathbb{E}_n$-monoidal adjunctions ([HA, 7.3.2.13], restated as Proposition 1.2.4.4 here and strengthened in Proposition 1.2.4.6).

1.2.2 $\mathscr{O}$-algebras in general

In this subsection we connect our discussion of $\mathbb{E}_n$-algebras with the notation of [HA, §2] and recall some more precise definitions.

Fix an ∞-operad $\mathcal{O}^\otimes$ and write $\mathcal{O}$ for $\mathcal{O}^\otimes_{\langle 1\rangle}$. Suppose that $\mathcal{O}^\otimes$ is single-colored, i.e., $\mathcal{O}$ is contractible (e.g. this is true for $\mathcal{O} = \mathbb{E}_n$ for some $n \in \{1, \ldots, \infty\}$).[4]

Notation 1.2.2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{O}$-monoidal categories, i.e. fix coCartesian fibrations $p_\mathcal{C} : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ and $p_\mathcal{D} : \mathcal{D}^\otimes \to \mathcal{O}^\otimes$ of ∞-operads with $\mathcal{C}^\otimes \times_{\mathcal{O}^\otimes} \mathrm{pt} = \mathcal{C}$ and $\mathcal{D}^\otimes \times_{\mathcal{O}^\otimes} \mathrm{pt} = \mathcal{D}$. Let

$$\mathsf{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D}) = \mathsf{Fun}^{\mathcal{O},\mathrm{lax}}(\mathcal{C}, \mathcal{D})$$

denote the full subcategory of $\mathsf{Fun}_{\mathcal{O}^\otimes}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$ spanned by the maps of ∞-operads. By abuse of notation, a lax $\mathcal{O}$-monoidal functor $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ is a functor $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ together with $\mathsf{F}^\otimes \in \mathsf{Fun}^{\mathrm{lax}}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$ inducing F on the underlying ∞-categories. When $\mathcal{C} = \mathcal{O}$ and $p_\mathcal{C} = \mathrm{id}_\mathcal{O}$, we write $\mathsf{Alg}_{/\mathcal{O}}(\mathcal{D}) := \mathsf{Alg}_{\mathcal{O}/\mathcal{O}}(\mathcal{D})$.

[3] This definition actually makes sense when $(\mathcal{C}, \otimes)$ is only assumed to be $\mathbb{E}_n$-monoidal itself.

[4] This restriction is not logically necessary, but it makes the statements cleaner.

When $\mathcal{O}^\otimes = \mathbb{E}_\infty^\otimes \simeq \mathsf{N}(\mathsf{Fin}_*)$ is the commutative operad, we write $\mathsf{Alg}_{\mathcal{C}}(\mathcal{D}) := \mathsf{Alg}_{\mathcal{C}/\mathbb{E}_\infty}(\mathcal{D})$. We set

$$\mathsf{CAlg}(\mathcal{C}) := \mathsf{Alg}_{\mathbb{E}_\infty/\mathbb{E}_\infty}(\mathcal{C}) = \mathsf{Alg}_{\mathbb{E}_\infty}(\mathcal{C}) = \mathsf{Alg}_{/\mathbb{E}_\infty}(\mathcal{C})$$

and set

$$\mathsf{Fun}^{\otimes,\mathrm{lax}}(\mathcal{C},\mathcal{D}) := \mathsf{Fun}^{\mathcal{O},\mathrm{lax}}(\mathcal{C},\mathcal{D}).$$

By abuse of notation, if $\mathcal{C}$ is a symmetric monoidal ∞-category, we say $\mathsf{c} \in \mathcal{C}$ is an $\mathcal{O}$-algebra object if we may write $\mathsf{c} = \mathsf{F}(*) \in \mathcal{C}$ for some $\mathsf{F} \in \mathsf{Alg}_{\mathcal{O}}(\mathcal{C}) = \mathsf{Fun}^{\otimes,\mathrm{lax}}(\mathcal{O},\mathcal{C})$.

Remark 1.2.2.2. Note that the above is not an extreme abuse of notation in the case of classical operads. Let $\mathcal{C}$ be a symmetric monoidal (ordinary) category, and equip the nerve $\mathsf{N}(\mathcal{C})$ with the symmetric monoidal ∞-category structure of [HA, 2.1.2.21]. Then $\mathsf{CAlg}(\mathsf{N}(\mathcal{C}))$ can be identified with the nerve of the category of commutative algebra objects on $\mathcal{C}$ in the classical sense.

Remark 1.2.2.3. If $(\mathcal{C},\otimes)$ is a symmetric monoidal category and $\mathcal{O}^\otimes$ is any ∞-operad, we may view $\mathcal{C}$ as an $\mathcal{O}$-monoidal category (by pulling back the structure map $\mathcal{C}^\otimes \to \mathsf{N}(\mathsf{Fin}_*)$ along $\mathcal{O}^\otimes \to \mathsf{N}(\mathsf{Fin}_*)$.) By the universal property of fiber products, we see that $\mathsf{Alg}_{/\mathcal{O}}(\mathcal{C}^\otimes \times_{\mathsf{N}(\mathsf{Fin}_*)} \mathcal{O}^\otimes) = \mathsf{Alg}_{\mathcal{O}}(\mathcal{C}^\otimes)$. That is, $\mathcal{O}$-algebras in $\mathcal{C}$ (viewed as an $\mathcal{O}$-monoidal ∞-category) are the same as $\mathcal{O}$-algebras in $\mathcal{C}$ viewed as a symmetric monoidal ∞-category.

1.2.3 Day convolution

If $\mathcal{C}$ is a small $\mathcal{O}$-monoidal ∞-category, then the presheaf ∞-category $\mathsf{PSh}(\mathcal{C}) := \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ can be equipped with a *Day convolution* $\mathcal{O}$-monoidal structure: for an n-ary operation f in $\mathcal{O}$, we define

$$\big(\otimes_{\mathsf{f}} \{\mathscr{F}_{\mathsf{i}}\}_{\mathsf{i}=1}^{\mathsf{n}}\big)(\mathsf{c}) = \operatorname*{colim}_{\otimes_{\mathsf{f}}\{\mathsf{c}_{\mathsf{i}}\}_{\mathsf{i}=1}^{\mathsf{n}} \to \mathsf{c}} \prod_{\mathsf{i}=1}^{\mathsf{n}} \mathscr{F}_{\mathsf{i}}(\mathsf{c}_{\mathsf{i}}). \tag{1.2.3.1}$$

We recall some useful properties of Day convolution as follows – proofs of these (or very similar statements) can be found in [HA, §2.2.6] and [Tor23, §2 and §3].

Proposition 1.2.3.2. *Let $\mathcal{C}$ be a small $\mathcal{O}$-monoidal ∞-category. Then Eq.* (1.2.3.1) *defines the unique colimit-preserving $\mathcal{O}$-monoidal structure on $\mathsf{PSh}(\mathcal{C})$ such that the Yoneda embedding $\mathsf{y}_{\mathcal{C}} : \mathcal{C} \to \mathsf{PSh}(\mathcal{C})$ is $\mathcal{O}$-monoidal. Furthermore:*

1. *If $\mathcal{D}$ is a small $\mathcal{O}$-monoidal ∞-category and $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ is an oplax $\mathcal{O}$-monoidal functor, then the pullback functor $\mathsf{F}^* : \mathsf{PSh}(\mathcal{D}) \to \mathsf{PSh}(\mathcal{C})$ is naturally lax $\mathcal{O}$-monoidal.*

2. *If $\mathcal{E}$ is a cocomplete ∞-category with a colimit-preserving $\mathcal{O}$-monoidal structure, then left Kan extension along the Yoneda embedding $\mathsf{y}_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$ induces an equivalence*

$$\mathsf{Fun}^{\mathcal{O},\mathsf{L}}(\mathsf{PSh}(\mathcal{C}),\mathcal{E}) \xrightarrow{\sim} \to \mathsf{Fun}^{\mathcal{O}}(\mathcal{C},\mathcal{E}).$$

The same is true with $\mathsf{Fun}^{\mathcal{O}}$ replaced by $\mathsf{Fun}^{\mathcal{O},\mathrm{lax}}$ on both sides.

3. *In particular, if* $\mathcal{D}$ *is a small* $\mathcal{O}$*-monoidal* ∞*-category and* $\mathsf{F}:\mathcal{C}\to\mathcal{D}$ *is an* $\mathcal{O}$*-monoidal functor (resp. lax* $\mathcal{O}$*-monoidal functor), then the left Kan extension functor* $\mathsf{F}_!:\mathsf{PSh}(\mathcal{C})\to\mathsf{PSh}(\mathcal{D})$ *is naturally* $\mathcal{O}$*-monoidal (resp. lax* $\mathcal{O}$*-monoidal).*

4. *There is a natural equivalence* $\mathsf{Alg}_{/\mathcal{O}}(\mathsf{PSh}(\mathcal{C}))\simeq\mathsf{Fun}^{\mathcal{O},\mathrm{lax}}(\mathcal{C}^{\mathrm{op}},\mathcal{S})$.

Proof. The fact that Eq. (1.2.3.1) defines an $\mathcal{O}$-monoidal structure on the Yoneda embedding is well-known (e.g. [Tor23, Lemma 2.4]) and the uniqueness is clear as $\mathsf{PSh}(\mathcal{C})$ is generated by $\mathcal{C}$ via colimits.

(1) is [Tor23, Corollary 2.12]. The lax monoidal case of (2) is [Tor23, Proposition 2.7], from which the lax monoidal case of (3) immediately follows (cf. [Tor23, Definition 2.14]). The strong monoidal cases of both statements follow by direct computation (using the fact that Day convolution preserves colimits to reduce to looking at representable functors). Finally, (4) is [HA, 2.2.6.8]. □

In particular, we may use Day convolution to construct $\mathcal{O}$-algebra structures on mapping spaces. The following is essentially contained in the proof of [Nik16, Corollary 6.8]:

Corollary 1.2.3.3. *Let* $\mathcal{C}$ *be a symmetric monoidal* ∞*-category. Let* $\mathsf{c}_1\in\mathsf{Alg}_{\mathcal{O}}(\mathcal{C}^{\mathrm{op}})$ *and let* $\mathsf{c}_2\in\mathsf{CAlg}(\mathcal{C})$*. Then* $\mathrm{Map}_{\mathcal{C}}(\mathsf{c}_1,\mathsf{c}_2)$ *is naturally an* $\mathcal{O}$*-algebra in* $(\mathcal{S},\times)$*,*[5] *and this* $\mathcal{O}$*-algebra structure is functorial in* c_1 *and* c_2*.*

Proof. We may assume $\mathcal{C}$ is small. As $\mathsf{y}_{\mathcal{C}}$ is symmetric monoidal, Proposition 1.2.3.2 (4) gives

$$\mathsf{y}_{\mathcal{C}}(\mathsf{c}_2)\in\mathsf{CAlg}(\mathsf{PSh}(\mathcal{C}))\simeq\mathsf{Fun}^{\otimes,\mathrm{lax}}(\mathcal{C}^{\mathrm{op}},\mathcal{S}).$$

Thus $\mathsf{y}_{\mathcal{C}}(\mathsf{c}_2)$ defines a functor $\mathsf{Alg}_{\mathcal{O}}(\mathcal{C}^{\mathrm{op}})\to\mathsf{Alg}_{\mathcal{O}}(\mathcal{S})$ (by composition). In particular, $\mathsf{y}_{\mathcal{C}}(\mathsf{c}_2)\circ\mathsf{c}_1$ gives an $\mathcal{O}$-algebra structure to $\mathrm{Map}_{\mathcal{C}}(\mathsf{c}_1,\mathsf{c}_2)=\mathsf{y}_{\mathcal{C}}(\mathsf{c}_2)(\mathsf{c}_1)$. The functoriality follows as c_1 also defines a functor $\mathsf{CAlg}(\mathsf{PSh}(\mathcal{C}))\to\mathsf{Alg}_{\mathcal{O}}(\mathcal{S})$ by precomposition. □

1.2.4 $\mathcal{O}$-monoidal adjunctions and $\mathcal{O}$-algebra objects

We will implicitly use the following fundamental theorem on adjunctions between $\mathcal{O}$-monoidal ∞-categories:

Proposition 1.2.4.1 ([Hau+23, Proposition A], [Tor23, Theorem 1.1])**.** *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be* $\mathcal{O}$*-monoidal* ∞*-categories, and let* $\mathsf{L}:\mathcal{C}\rightleftarrows\mathcal{D}:\mathsf{R}$ *be an adjunction of the underlying* ∞*-categories. Then the data of an oplax* $\mathcal{O}$*-monoidal structure on* L *is equivalent to the data of a lax* $\mathcal{O}$*-monoidal structure on* R*.*

With this in mind, we define:

[5] As in Notation 1.2.2.1, this $\mathcal{O}$-algebra structure is encoded formally by a functor $\mathcal{O}^{\otimes}\to\mathcal{S}^{\times}$.

Definition 1.2.4.2. A *lax $\mathcal{O}$-monoidal adjunction* $\mathsf{L} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathsf{R}$ is an adjunction between $\mathcal{O}$-monoidal ∞-categories together with an $\mathcal{O}$-monoidal structure on L. An *$\mathcal{O}$-monoidal adjunction* is a lax $\mathcal{O}$-monoidal adjunction as above in which L is (strong) $\mathcal{O}$-monoidal.

Example 1.2.4.3. If $\mathsf{L} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathsf{R}$ is an $\mathcal{O}$-monoidal adjunction, the functor R need not be $\mathcal{O}$-monoidal. For example, if $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ is any morphism of schemes, then the adjunction $\mathsf{f}^* : (\mathsf{QC}(\mathsf{X}), \otimes_{\mathscr{O}_\mathsf{X}}) \rightleftarrows (\mathsf{QC}(\mathsf{Y}), \otimes_{\mathscr{O}_\mathsf{Y}}) : \mathsf{f}_*$ is symmetric monoidal, but the functor f_* typically is not symmetric monoidal.

Lax $\mathcal{O}$-monoidal functors induce functors on ∞-categories of $\mathcal{O}$-algebra objects. In particular, there is the following standard result:

Proposition 1.2.4.4 ([HA, 7.3.2.13])**.** *Let* $\mathsf{L} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathsf{R}$ *be an $\mathcal{O}$-monoidal adjunction. Then* L *and* R *induce an adjunction*

$$\mathsf{L} : \mathsf{Alg}_{/\mathcal{O}}(\mathcal{C}) \rightleftarrows \mathsf{Alg}_{/\mathcal{O}}(\mathcal{D}) : \mathsf{R}.$$

We may use Proposition 1.2.4.4 to understand the behavior of Corollary 1.2.3.3 with respect to symmetric monoidal functors:

Corollary 1.2.4.5. *Let* $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ *be a symmetric monoidal functor between symmetric monoidal ∞-categories. Let* $\mathsf{c}_1 \in \mathsf{Alg}_\mathcal{O}(\mathcal{C}^{\mathrm{op}})$ *and let* $\mathsf{c}_2 \in \mathsf{CAlg}(\mathcal{C})$*. Then the natural map* $\mathsf{F}_{\mathsf{c}_1,\mathsf{c}_2} : \mathrm{Map}_\mathcal{C}(\mathsf{c}_1, \mathsf{c}_2) \to \mathrm{Map}_\mathcal{D}(\mathsf{F}(\mathsf{c}_1), \mathsf{F}(\mathsf{c}_2))$ *is naturally an $\mathcal{O}$-algebra homomorphism.*

Proof. We may write

$$\begin{aligned} \mathrm{Map}_\mathcal{D}(\mathsf{F}(\mathsf{c}_1), \mathsf{F}(\mathsf{c}_2)) &= \mathsf{y}_\mathcal{D}\big(\mathsf{F}(\mathsf{c}_2)\big)\big(\mathsf{F}(\mathsf{c}_1)\big) \\ &= \mathsf{F}_!\big(\mathsf{y}_\mathcal{C}(\mathsf{c}_2)\big)\big(\mathsf{F}(\mathsf{c}_1)\big) \\ &= \big((\mathsf{F}^*\mathsf{F}_!)(\mathsf{y}_\mathcal{C}(\mathsf{c}_2))\big)(\mathsf{c}_1) \text{ by the definition of } \mathsf{F}^*. \end{aligned}$$

The map $\mathsf{F}_{\mathsf{c}_1,\mathsf{c}_2}$ is equivalent to the map $\mathsf{y}_\mathcal{C}(\mathsf{c}_2)(\mathsf{c}_1) \to \big((\mathsf{F}^*\mathsf{F}_!)(\mathsf{y}_\mathcal{C}(\mathsf{c}_2))\big)(\mathsf{c}_1)$ induced by the unit map

$$\eta_{\mathsf{y}_\mathcal{C}(\mathsf{c}_2)} : \mathsf{y}_\mathcal{C}(\mathsf{c}_2) \to (\mathsf{F}^*\mathsf{F}_!)\big(\mathsf{y}_\mathcal{C}(\mathsf{c}_2)\big).$$

By the functoriality of the construction of Corollary 1.2.3.3, to show that $\mathsf{F}_{\mathsf{c}_1,\mathsf{c}_2}$ is a homomorphism of $\mathcal{O}$-algebras in $\mathcal{S}$, it suffices to show that $\eta_{\mathsf{y}_\mathcal{C}(\mathsf{c}_2)}$ upgrades to a homomorphism of commutative algebras in $\mathsf{PSh}(\mathcal{C})$.

The adjunction

$$\mathsf{F}_! : \mathsf{PSh}(\mathcal{C}) \rightleftarrows \mathsf{PSh}(\mathcal{D}) : \mathsf{F}^*$$

is symmetric monoidal by Proposition 1.2.3.2, so by Proposition 1.2.4.4 we get an adjunction

$$\mathsf{F}_! : \mathsf{CAlg}\big(\mathsf{PSh}(\mathcal{C})\big) \rightleftarrows \mathsf{CAlg}\big(\mathsf{PSh}(\mathcal{D})\big) : \mathsf{F}^*.$$

The unit map of this adjunction (at $\mathsf{y}_\mathcal{C}(\mathsf{c}_2) \in \mathsf{CAlg}\big(\mathsf{PSh}(\mathcal{C})\big)$) gives the desired homomorphism of commutative algebras upgrading $\eta_{\mathsf{y}_\mathcal{C}(\mathsf{c}_2)}$. $\square$

In the body of the paper, we will need the following modest strengthening of Proposition 1.2.4.4.

Proposition 1.2.4.6. *Let* $\mathsf{L} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathsf{R}$ *be a lax* $\mathcal{O}$*-monoidal adjunction, and suppose that* $i : \mathcal{C}' \to \mathcal{C}$ *is the inclusion of a full* $\mathcal{O}$*-monoidal subcategory of* $\mathcal{C}$ *such that the restriction* $\mathsf{L} \circ i$ *is* $\mathcal{O}$*-monoidal. Then, for all* $c' \in \mathsf{Alg}_{/\mathcal{O}}(\mathcal{C}')$ *and* $d \in \mathsf{Alg}_{/\mathcal{O}}(\mathcal{D})$*, there is a natural equivalence*

$$\mathrm{Map}_{\mathsf{Alg}_{/\mathcal{O}}(\mathcal{D})}\big(\mathsf{L}(i(c')), d\big) \simeq \mathrm{Map}_{\mathsf{Alg}_{/\mathcal{O}}(\mathcal{C})}\big(i(c'), \mathsf{R}(d)\big).$$

Proof. By [Tor23, Lemma 2.16] (passing to a larger universe as necessary to avoid set-theoretic issues) there is a lax $\mathcal{O}$-monoidal adjunction

$$\mathsf{L}_! : \mathsf{PSh}(\mathcal{C}) \rightleftarrows \mathsf{PSh}(\mathcal{D}) : \mathsf{R}_!.$$

Composing this with the $\mathcal{O}$-monoidal adjunction $i_! : \mathsf{PSh}(\mathcal{C}') \rightleftarrows \mathsf{PSh}(\mathcal{C}) : i^*$, we obtain an adjunction

$$(\mathsf{L} \circ i)_! : \mathsf{PSh}(\mathcal{C}') \rightleftarrows \mathsf{PSh}(\mathcal{D}) : i^*\mathsf{R}_!$$

which is $\mathcal{O}$-monoidal because $\mathsf{L} \circ i$ is. By Proposition 1.2.4.4, this induces an adjunction on the corresponding categories of $\mathcal{O}$-algebras.

The Yoneda lemma allows us to embed $\mathsf{Alg}_{/\mathcal{O}}(\mathcal{C}') \subset \mathsf{Alg}_{/\mathcal{O}}(\mathsf{PSh}(\mathcal{C}'))$ and $\mathsf{Alg}_{/\mathcal{O}}(\mathcal{D}) \subset \mathsf{Alg}_{/\mathcal{O}}(\mathsf{PSh}(\mathcal{D}))$ by Proposition 1.2.3.2(4). Thus, for $c' \in \mathsf{Alg}_{/\mathcal{O}}(\mathcal{C}')$ and $d \in \mathsf{Alg}_{/\mathcal{O}}(\mathcal{D})$, we have

$$\begin{aligned}
\mathrm{Map}_{\mathsf{Alg}_{/\mathcal{O}}(\mathcal{D})}\big(\mathsf{L}(i(c')), d\big) &= \mathrm{Map}_{\mathsf{Alg}_{/\mathcal{O}}(\mathsf{PSh}(\mathcal{D}))}\big((\mathsf{L} \circ i)_!(c'), d\big) \\
&= \mathrm{Map}_{\mathsf{Alg}_{/\mathcal{O}}(\mathsf{PSh}(\mathcal{C}'))}\big(c', i^*\mathsf{R}_!(d)\big) \\
&= \mathrm{Map}_{\mathsf{Alg}_{/\mathcal{O}}(\mathsf{PSh}(\mathcal{C}))}\big(i_!(c'), \mathsf{R}_!(d)\big) \text{ because } i_! \text{ is fully faithful} \\
&= \mathrm{Map}_{\mathsf{Alg}_{/\mathcal{O}}(\mathcal{C})}\big(i(c'), \mathsf{R}(d)\big).
\end{aligned}$$

□

1.2.5 Right localization of $\mathbb{E}_n$-monoidal structures

We can also ask: given an adjunction of ∞-categories in which one category is $\mathcal{O}$-monoidal, is there a natural $\mathcal{O}$-monoidal structure on the other such that the adjunction upgrades to a (lax) $\mathcal{O}$-monoidal adjunction?

When the left adjoint is a localization in the sense of [HTT, 5.2.7.2] (i.e. when the right adjoint is fully faithful), necessary and sufficient conditions are well-known – see [HA, 2.2.1.9] for the general case as well as [HA, 4.1.7.4] for the (symmetric) monoidal case. In this subsection, we collect some results on the dual problem:

- Let $\mathcal{D}$ be an $\mathcal{O}$-monoidal ∞-category, and let $\mathsf{L} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathsf{R}$ be an adjunction. Assume that L is fully faithful. What other conditions guarantee that there exists a natural $\mathcal{O}$-monoidal structure on $\mathcal{C}$ such that $\mathsf{R} : \mathcal{D} \to \mathcal{C}$ upgrades to a (lax) $\mathcal{O}$-monoidal functor?

We will restrict to the case $\mathcal{O} = \mathbb{E}_n$ (for some $n \in \{1, \ldots, \infty\}$) for notational convenience. We begin by collecting some well-known results on the behavior of such adjunctions:

Proposition 1.2.5.1 ([HA, 4.1.7.4]). [6] *Let* $\mathsf{L} : (\mathcal{C}, \otimes_{\mathcal{C}}) \rightleftarrows (\mathcal{D}, \otimes_{\mathcal{D}}) : \mathsf{R}$ *be a lax* $\mathbb{E}_n$*-monoidal adjunction. Suppose that* L *is fully faithful and* R *is strong* $\mathbb{E}_n$*-monoidal. Then:*

1. $\otimes_{\mathcal{C}}$ *can be computed via the formula*
$$c_1 \otimes_{\mathcal{C}} c_2 = \mathsf{R}(\mathsf{L}(c_1)) \otimes_{\mathcal{C}} \mathsf{R}(\mathsf{L}(c_2)) = \mathsf{R}\big(\mathsf{L}(c_1) \otimes_{\mathcal{D}} \mathsf{L}(c_2)\big).$$
for all $c_1, c_2 \in \mathcal{C}$.

2. *The* $\mathbb{E}_n$*-monoidal functor* $(\mathcal{D}, \otimes_{\mathcal{D}}) \to (\mathcal{C}, \otimes_{\mathcal{C}})$ *is universal among* $\mathbb{E}_n$*-monoidal functors with source* $(\mathcal{D}, \otimes_{\mathcal{D}})$ *whose underlying functors factor through* $\mathcal{C}$*. Symbolically:*
$$\mathsf{Fun}^{\otimes}\big((\mathcal{C}, \otimes_{\mathcal{C}}), (\mathcal{E}, \otimes_{\mathcal{E}})\big) = \mathsf{Fun}^{\otimes}\big((\mathcal{D}, \otimes_{\mathcal{D}}), (\mathcal{E}, \otimes_{\mathcal{E}})\big) \times_{\mathsf{Fun}(\mathcal{D}, \mathcal{E})} \mathsf{Fun}(\mathcal{C}, \mathcal{E}).$$
In particular, $\otimes_{\mathcal{C}}$ *is unique whenever it exists.*

In the stable case, $\mathbb{E}_n$-monoidal adjunctions can be constructed from quotients by (two-sided) thick $\otimes$-ideals. This is well-known to the experts, though we include the details for completeness.

Definition 1.2.5.2. Let $(\mathcal{C}, \otimes_{\mathcal{C}})$ be a stably $\mathbb{E}_n$-monoidal ∞-category. A *two-sided thick* $\otimes$*-ideal of* $\mathcal{C}$ is a stable full subcategory $\mathcal{I} \subset \mathcal{C}$ such that:

- If $c \oplus c' \in \mathcal{I}$, then $c \in \mathcal{I}$ and $c' \in \mathcal{I}$.
- If $i \in \mathcal{I}$ and $c \in \mathcal{C}$, then $i \otimes_{\mathcal{C}} c$ and $c \otimes_{\mathcal{C}} i$ are both in $\mathcal{I}$.

Lemma 1.2.5.3 (Stable, dual version of [HA, 2.2.1.9]). *Let* $(\mathcal{D}, \otimes_{\mathcal{D}})$ *be a stably* $\mathbb{E}_n$*-monoidal* ∞*-category, and let* $\mathcal{C}$ *be a stable* ∞*-category. Let* $\mathsf{L} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathsf{R}$ *be an adjunction. Suppose:*

1. L *is fully faithful, and*

2. *The full subcategory*
$$\ker \mathsf{R} := \{d \in \operatorname{ob} \mathcal{D} \mid \mathsf{R}(d) \simeq 0\}$$
is a two-sided thick $\otimes$*-ideal of* $\mathcal{D}$.

Then there exists a unique $\mathbb{E}_n$*-monoidal structure on* $\mathcal{C}$ *such that* $\mathsf{L} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathsf{R}$ *is* $\mathbb{E}_n$*-monoidal.*

---

[6] The claims in [HA, 4.1.7.4] are made for the (symmetric) monoidal case only, but the arguments work for $\mathbb{E}_n$-monoidal structures for any n.

Proof. By the dual of [HA, 2.2.1.9], it suffices to show that, if f and g are morphisms in $\mathcal{D}$ such that $\mathsf{R}(\mathsf{f})$ and $\mathsf{R}(\mathsf{g})$ are equivalences in $\mathcal{C}$, then $\mathsf{R}(\mathsf{f} \otimes_{\mathcal{D}} \mathsf{g})$ is also an equivalence in $\mathcal{D}$. Factoring $\mathsf{f} \otimes_{\mathcal{D}} \mathsf{g}$ as $(\mathsf{f} \otimes_{\mathcal{D}} \mathrm{id}_{\mathsf{d}_1}) \circ (\mathrm{id}_{\mathsf{d}_2} \otimes_{\mathcal{D}} \mathsf{g})$, we see that it suffices to show that, if f is a morphism in $\mathcal{D}$ such that $\mathsf{R}(\mathsf{f})$ is an equivalence in $\mathcal{C}$, then $\mathsf{R}(\mathsf{f} \otimes_{\mathcal{D}} \mathrm{id}_{\mathsf{d}})$ is an equivalence for all $\mathsf{d} \in \mathcal{D}$.[7]

Since $\mathsf{R}(\mathsf{f})$ is an equivalence, we have $\mathsf{R}(\operatorname{fib} \mathsf{f}) = \operatorname{fib} \mathsf{R}(\mathsf{f}) \simeq 0$, so $\operatorname{fib} \mathsf{f} \in \ker \mathsf{R}$. Because $\ker \mathsf{R}$ is a $\otimes_{\mathcal{D}}$-ideal, we have $\operatorname{fib}(\mathsf{f} \otimes_{\mathcal{D}} \mathrm{id}_{\mathsf{d}}) = \operatorname{fib} \mathsf{f} \otimes_{\mathcal{D}} \mathsf{d} \in \ker \mathsf{R}$. Thus $\operatorname{fib} \mathsf{R}(\mathsf{f} \otimes \mathrm{id}_{\mathsf{d}}) = 0$ and $\mathsf{R}(\mathsf{f} \otimes_{\mathcal{D}} \mathrm{id}_{\mathsf{d}})$ is an equivalence. □

Remark 1.2.5.4. In the context of Lemma 1.2.5.3, the category $\ker \mathsf{R}$ is automatically stable and closed under taking direct summands. Thus it suffices to check that if $\mathsf{i} \in \ker \mathsf{R}$ and $\mathsf{d} \in \mathcal{D}$, then $\mathsf{i} \otimes_{\mathcal{D}} \mathsf{d}$ and $\mathsf{d} \otimes_{\mathcal{D}} \mathsf{i}$ are both in $\mathcal{D}$.

When $\mathcal{C}$ and $\mathcal{D}$ are compactly generated, we may provide a criterion for the existence of a corresponding $\mathbb{E}_n$-monoidal structure on $\mathcal{C}$ that requires us to understand only the behavior of L (rather than R). We use this result in the body of the paper (see Proposition 5.2.1.5).

Proposition 1.2.5.5. *Fix $n \in \{1, \dots, \infty\}$. Suppose $(\mathcal{D}, \otimes_{\mathcal{D}})$ is a compactly generated, stable, and presentably $\mathbb{E}_n$-monoidal ∞-category.*[8] *Let $\mathsf{G} : \mathcal{D} \otimes \mathcal{D} \to \mathcal{D}$ be the unique colimit-preserving functor such that $- \otimes_{\mathcal{D}} - = \mathsf{G}(- \boxtimes -)$.*[9] *Let $\mathcal{C}$ be a compactly generated stable ∞-category, and let $\mathsf{L} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathsf{R}$ be an adjunction. Suppose:*

1. *L is fully faithful,*
2. *L preserves compact objects,*
3. *G has a left adjoint $\mathsf{F} : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$, and*
4. *The image of $\mathsf{F} \circ \mathsf{L}$ lies in the full subcategory of $\mathcal{D} \otimes \mathcal{D}$ compactly generated by*

$$\left\{ \mathsf{L}(\mathsf{c}_1) \boxtimes \mathsf{L}(\mathsf{c}_2) \,\middle|\, \mathsf{c}_1, \mathsf{c}_2 \in \mathcal{C}^\omega \right\}.$$

Then there exists a unique $\mathbb{E}_n$-monoidal structure $\otimes_{\mathcal{C}}$ on $\mathcal{C}$ such that $\mathsf{R} : (\mathcal{D}, \otimes_{\mathcal{D}}) \to (\mathcal{C}, \otimes_{\mathcal{C}})$ is $\mathbb{E}_n$-monoidal.

Proof. By Lemma 1.2.5.3, it suffices to check that, if $\mathsf{d}_1 \in \ker \mathsf{R}$ and $\mathsf{d}_2 \in \mathcal{D}$, then $\mathsf{d}_1 \otimes_{\mathcal{D}} \mathsf{d}_2 \in \ker \mathsf{R}$.[10] By the Yoneda lemma, it suffices to show that $\operatorname{Hom}_{\mathcal{C}}\big(\mathsf{c}, \mathsf{R}(\mathsf{d}_1 \otimes_{\mathcal{D}} \mathsf{d}_2)\big) = 0$ for all $\mathsf{c} \in \mathcal{C}$. Observe that

$$\operatorname{Hom}_{\mathcal{C}}\big(\mathsf{c}, \mathsf{R}(\mathsf{d}_1 \otimes_{\mathcal{D}} \mathsf{d}_2)\big) = \operatorname{Hom}_{\mathcal{C}}\big(\mathsf{c}, \mathsf{R}(\mathsf{G}(\mathsf{d}_1 \boxtimes \mathsf{d}_2))\big) = \operatorname{Hom}_{\mathcal{D} \otimes \mathcal{D}}\big(\mathsf{F}(\mathsf{L}(\mathsf{c})), \mathsf{d}_1 \boxtimes \mathsf{d}_2\big).$$

[7] Of course, when $n = 1$, we must also show the same result for $\mathsf{R}(\mathrm{id}_{\mathsf{d}} \otimes_{\mathcal{D}} \mathsf{f})$, but the same argument will apply.

[8] We do *not* assume that the monoidal structure $\otimes_{\mathcal{D}}$ preserves compact objects.

[9] G exists by the universal property of the Lurie tensor product, since $- \otimes_{\mathcal{D}} - : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ preserves colimits in each variable separately.

[10] When $n = 1$, we must show $\mathsf{d}_2 \otimes_{\mathcal{D}} \mathsf{d}_1 \in \ker \mathsf{R}$, but the same argument works.

By hypothesis, $\mathsf{F}(\mathsf{L}(\mathsf{c}))$ can be written as a colimit of terms $\mathsf{L}(\mathsf{c}_1) \boxtimes \mathsf{L}(\mathsf{c}_2)$ for $\mathsf{c}_1, \mathsf{c}_2 \in \mathcal{C}^\omega$. Thus it suffices to show that $\mathrm{Hom}_{\mathcal{D}\otimes\mathcal{D}}\left(\mathsf{L}(\mathsf{c}_1) \boxtimes \mathsf{L}(\mathsf{c}_2), \mathsf{d}_1 \boxtimes \mathsf{d}_2\right) = 0$ for all $\mathsf{c}_1, \mathsf{c}_2 \in \mathcal{C}^\omega$. But this is just a direct computation using the Künneth formula:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}\otimes\mathcal{D}}\left(\mathsf{L}(\mathsf{c}_1) \boxtimes \mathsf{L}(\mathsf{c}_2), \mathsf{d}_1 \boxtimes \mathsf{d}_2\right) &= \mathrm{Hom}_{\mathcal{D}}\left(\mathsf{L}(\mathsf{c}_1), \mathsf{d}_1\right) \otimes \mathrm{Hom}_{\mathcal{D}}\left(\mathsf{L}(\mathsf{c}_2), \mathsf{d}_2\right) \\
&= \mathrm{Hom}_{\mathcal{C}}\left(\mathsf{c}_1, \mathsf{R}(\mathsf{d}_1)\right) \otimes \mathrm{Hom}_{\mathcal{C}}\left(\mathsf{L}(\mathsf{c}_2), \mathsf{d}_2\right) \\
&= 0 \otimes \mathrm{Hom}_{\mathcal{C}}\left(\mathsf{L}(\mathsf{c}_2), \mathsf{d}_2\right) = 0.
\end{aligned}$$

□

1.3 Derived algebraic geometry and perfect stacks

Notation 1.3.0.1. From this point on, whenever we reference "categories," we will implicitly mean ∞-categories. We will indicate more classical situations by explicitly referencing 1-categories or discrete categories.

Derived algebraic geometry is the study of (geometric) stacks on a suitable site of *derived affine schemes*. As in classical algebraic geometry, the category dAff_k of derived affine schemes over k is (defined to be) the opposite of the category of *derived commutative* k-*algebras* $\mathsf{dCAlg}_\mathsf{k}$. Write $\mathrm{Spec} : \mathsf{dCAlg}_\mathsf{k}^{\mathrm{op}} \xrightarrow{\sim} \mathsf{dAff}_\mathsf{k}$ for the equivalence.

What one means by "derived commutative rings" varies depending on one's goals, but a few standard definitions include:

- The category of connective commutative dg-algebras.
- The category of simplicial commutative rings.
- The category of connective $\mathbb{E}_\infty$-ring spectra.

When working over a field k of characteristic zero (as we will typically do), all of these approaches are equivalent, so we will not concern ourselves for the moment with the differences between these. Readers will not lose much if they restrict to the more familiar case of commutative dg-algebras.

We shall equip dAff_k with the étale topology (see e.g. [GR19, §1.2.2]). A *derived stack* over k is a presheaf $\mathfrak{X} \in \mathsf{PSh}(\mathsf{dAff}_\mathsf{k})$ which satisfies descent for the étale topology. Write dStk_k for the category of derived stacks.

Remark 1.3.0.2. As usual for the theory of stacks, there are many other topologies one could consider (e.g. smooth, fppf, fpqc, …). However, most stacks that appear in practice (including those we will deal with) satisfy descent for all of the standard choices of topology.

We may define the (derived) category of *quasicoherent sheaves* on a derived stack $\mathfrak{X}$ as[11]

$$\mathsf{QC}(\mathfrak{X}) = \lim_{\mathrm{Spec}\,\mathsf{R} \to \mathfrak{X}} \mathsf{D}(\mathsf{R}).$$

[11]This limit is *a priori* large but can be reduced to a small limit by writing $\mathfrak{X}$ as a small colimit of derived affine schemes as in [BFN10, §3.1].

Within this category there is a full subcategory of *perfect complexes*

$$\mathsf{Perf}(\mathfrak{X}) = \lim_{\operatorname{Spec} \mathsf{R} \to \mathfrak{X}} \mathsf{Perf}(\mathsf{R}),$$

where $\mathsf{Perf}(\mathsf{R}) = \mathsf{D}(\mathsf{R})^{\omega}$. Both $\mathsf{QC}(\mathfrak{X})$ and $\mathsf{Perf}(\mathfrak{X})$ are k-linear stable categories. The category $\mathsf{QC}(\mathfrak{X})$ is presentable, and $\mathsf{Perf}(\mathfrak{X})$ is perfect. From the definition as limits, one sees that both $\mathsf{QC}(\mathfrak{X})$ and $\mathsf{Perf}(\mathfrak{X})$ admit natural tensor products $\otimes_{\mathscr{O}_{\mathfrak{X}}}$, and a morphism $\mathsf{f} : \mathfrak{X} \to \mathfrak{Y}$ induces a symmetric monoidal functor $\mathsf{f}^* : \left(\mathsf{QC}(\mathfrak{Y}), \otimes_{\mathscr{O}_{\mathfrak{Y}}}\right) \to \left(\mathsf{QC}(\mathfrak{X}), \otimes_{\mathscr{O}_{\mathfrak{X}}}\right)$. For notational simplicity, we shall often write $\otimes_{\mathscr{O}}$ for $\otimes_{\mathscr{O}_{\mathfrak{X}}}$ when the stack $\mathfrak{X}$ is clear from context.

Furthermore , the usual t-structures on $\mathsf{D}(\mathsf{R})$ (where $\mathsf{D}^{\leqslant 0}(\mathsf{R})$ consists of connective modules) induce t-structures on the categories $\mathsf{QC}(\mathfrak{X})$ and $\mathsf{Perf}(\mathfrak{X})$. (see e.g. [GR19, §3.1.5]). The functors $\otimes_{\mathscr{O}}$ and f^* are right t-exact for the usual t-structure on $\mathsf{QC}(-)$. Note that the hearts of these t-structures may behave poorly: if R is a derived ring, there is no "abelian category of R-modules" equivalent to $\mathsf{D}(\mathsf{R})^{\heartsuit}$. We shall not make heavy use of these t-structures, and we mention them only so that we may appeal to [BH17, Theorem 1.3] later on.

In general, perfect complexes may not be compact in $\mathsf{QC}(\mathfrak{X})$, and they may fail to generate $\mathsf{QC}(\mathfrak{X})$. Following [BFN10], we say a derived stack $\mathfrak{X}$ is *perfect* if $\mathfrak{X}$ has affine diagonal and $\mathsf{Perf}(\mathfrak{X})$ compactly generates $\mathsf{QC}(\mathfrak{X})$. Write $\mathsf{dStk}_{\mathsf{k}}^{\mathrm{perf}}$ for the full subcategory of $\mathsf{dStk}_{\mathsf{k}}$ consisting of perfect stacks.

Most "small" derived stacks in characteristic zero are perfect – see [BFN10, §3.3] for sufficient criteria. In particular, when $\operatorname{charop} \mathsf{k} = 0$, the classifying stack BG of any linear algebraic group G is perfect. Furthermore, if $\mathfrak{Y}$ is perfect and $\mathsf{f} : \mathfrak{X} \to \mathfrak{Y}$ is affine, then $\mathfrak{X}$ is perfect.

Morphisms of perfect stacks enjoy many of the standard sheaf-theoretic identities of algebraic geometry. If $\mathsf{f} : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of perfect stacks, then the pullback functor $\mathsf{f}^* : \mathsf{QC}(\mathfrak{Y}) \to \mathsf{QC}(\mathfrak{X})$ admits a colimit-preserving right adjoint $\mathsf{f}_* : \mathsf{QC}(\mathfrak{X}) \to \mathsf{QC}(\mathfrak{Y})$. These satisfy base change and the projection formula by [BFN10, Proposition 3.10]. There is also an equivalence $\mathsf{QC}(\mathsf{X} \times \mathsf{Y}) = \mathsf{QC}(\mathfrak{X}) \otimes_{\mathsf{k}} \mathsf{QC}(\mathfrak{Y})$ by [BFN10, Theorem 4.7]. In Section 1.3.4 we will construct a "three-functor formalism" on $\mathsf{dStk}_{\mathsf{k}}$, ensuring that all of the higher homotopy coherence relations for these functors behave "as expected."

Perfect stacks also support a good theory of Fourier-Mukai transforms by [BFN10, Theorem 1.2]. More precisely, if $\mathfrak{X}$ and $\mathfrak{Y}$ are perfect stacks, then there is a pair of mutually inverse equivalences

$$\mathscr{K}_{-} : \mathsf{Fun}_{\mathsf{k}}^{\mathsf{L}}\left(\mathsf{QC}(\mathfrak{X}), \mathsf{QC}(\mathfrak{Y})\right) \xleftrightarrow{\sim} \mathsf{QC}(\mathfrak{X} \times \mathfrak{Y}) : \Phi_{-},$$

where $\Phi_{\mathscr{F}} = \pi_{2*}(\pi_1^*(-) \otimes \mathscr{F})$ is the *Fourier-Mukai transform* associated with $\mathscr{F} \in \mathsf{QC}(\mathfrak{X} \times \mathfrak{Y})$, and $\mathscr{K}_{\mathsf{F}}$ is the *Fourier-Mukai kernel* associated with $\mathsf{F} \in \mathsf{Fun}_{\mathsf{k}}^{\mathsf{L}}\left(\mathsf{QC}(\mathfrak{X}), \mathsf{QC}(\mathfrak{Y})\right)$. The analogous claim for perfect complexes holds when $\mathfrak{X}$ and $\mathfrak{Y}$ are smooth and proper over k. Note the distinction between the above result and the triangulated theory – here every reasonable functor is automatically uniquely / functorially a Fourier-Mukai transform!

1.3.1 Resolutions of the diagonal

Suppose $\mathfrak{X}$ is a perfect stack. The relationship between generators of $\mathfrak{X}$ and resolutions of the diagonal of $\mathfrak{X}$ is well-known and classical (going back to [Bei78]). We shall review this relationship in modern language for future reference.

Let $\Delta_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ be the diagonal morphism of $\mathfrak{X}$. The identity functor $\mathrm{id}_{\mathfrak{X}} : \mathsf{QC}(\mathfrak{X}) \to \mathsf{QC}(\mathfrak{X})$ may be understood as the Fourier-Mukai transform with kernel $\Delta_{\mathfrak{X}*}\mathscr{O}_{\mathfrak{X}} \in \mathsf{QC}(\mathfrak{X} \times \mathfrak{X})$.

Lemma 1.3.1.1. *Let $\mathfrak{X}$ be a perfect stack, and suppose that $\Delta_{\mathfrak{X}*}\mathscr{O}_{\mathfrak{X}} = \operatorname{colim}_{i \in I} \mathscr{A}_i \boxtimes \mathscr{B}_i$ for some families $\{\mathscr{A}_i\}_{i\in I}, \{\mathscr{B}_i\}_{i \in I} \subset \mathsf{QC}(\mathfrak{X})$. Then, for any $\mathscr{F} \in \mathsf{QC}(\mathfrak{X})$, we have*

$$\mathscr{F} = \operatorname*{colim}_{i \in I} \Gamma(\mathfrak{X}, \mathscr{F} \otimes \mathscr{A}_i) \otimes_k \mathscr{B}_i.$$

Proof. For $i = 1, 2$, let $\pi_i : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ be projection onto the ith factor. Note that π_{2*} preserves colimits because π_2 is a morphism of perfect stacks ([BFN10, Proposition 3.10]). Thus we may compute

$$\begin{aligned}
\mathscr{F} &= \pi_{2*}(\pi_1^*\mathscr{F} \otimes \Delta_{\mathfrak{X}*}\mathscr{O}_{\mathfrak{X}}) \\
&= \pi_{2*}\left(\pi_1^*\mathscr{F} \otimes_{\mathscr{O}} \operatorname*{colim}_{i\in I}(\mathscr{A}_i \boxtimes \mathscr{B}_i)\right) \\
&= \operatorname*{colim}_{i\in I} \pi_{2*}\big((\mathscr{F} \otimes_{\mathscr{O}} \mathscr{A}_i) \boxtimes \mathscr{B}_i\big) \text{ because all functors involved commute with colimits} \\
&= \operatorname*{colim}_{i\in I} \pi_{2*}\big(\pi_1^*(\mathscr{F} \otimes_{\mathscr{O}} \mathscr{A}_i) \otimes_{\mathscr{O}} \pi_2^*\mathscr{B}_i\big) \\
&= \operatorname*{colim}_{i\in I} \pi_{2*}\big(\pi_1^*(\mathscr{F} \otimes_{\mathscr{O}} \mathscr{A}_i)\big) \otimes_k \mathscr{B}_i \text{ by the projection formula} \\
&= \operatorname*{colim}_{i\in I} \Gamma(\mathfrak{X}, \mathscr{F} \otimes_{\mathscr{O}} \mathscr{A}_i) \otimes_k \mathscr{B}_i \text{ by base change.}
\end{aligned}$$

□

In the situation of Lemma 1.3.1.1, we see that the family $\{\mathscr{B}_i\}_{i\in I}$ generates $\mathsf{QC}(\mathfrak{X})$. Conversely, if $\{\mathscr{B}_i\}_{i\in I}$ generates $\mathsf{QC}(\mathfrak{X})$, then $\{\mathscr{B}_i \boxtimes \mathscr{B}_j\}_{i,j\in I}$ generates $\mathsf{QC}(\mathfrak{X} \times \mathfrak{X}) = \mathsf{QC}(\mathfrak{X}) \boxtimes \mathsf{QC}(\mathfrak{X})$, so we may write $\Delta_{\mathfrak{X}*}\mathscr{O}_{\mathfrak{X}}$ as a colimit over these sheaves.

We may use this perspective to rewrite any colimit-preserving functor $\mathsf{F} : \mathsf{QC}(\mathfrak{X}) \to \mathsf{QC}(\mathfrak{Y})$ (where $\mathfrak{X}$ and $\mathfrak{Y}$ are perfect stacks) in terms of a given resolution of the diagonal of $\mathfrak{X}$. More precisely:

Proposition 1.3.1.2. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be perfect stacks, and write $\Delta_{\mathfrak{X}*}\mathscr{O}_{\mathfrak{X}} = \operatorname{colim}_{i\in I} \mathscr{A}_i \boxtimes \mathscr{B}_i$. If $\mathsf{F} : \mathsf{QC}(\mathfrak{X}) \to \mathsf{QC}(\mathfrak{Y})$ is a colimit-preserving functor, then $\mathscr{K}_{\mathsf{F}} = \operatorname{colim}_{i\in I} \mathscr{A}_i \boxtimes \mathsf{F}(\mathscr{B}_i)$.*

Proof. This is a direct computation using Lemma 1.3.1.1:

$$\begin{aligned}
\mathsf{F}(\mathscr{F}) &= \mathsf{F}\left(\operatorname*{colim}_{i \in I} \Gamma(\mathfrak{Y}, \mathscr{F} \otimes \mathscr{A}_i) \otimes_k \mathscr{B}_i \right) \\
&= \operatorname*{colim}_{i \in I} \Gamma(\mathfrak{Y}, \mathscr{F} \otimes \mathscr{A}_i) \otimes_k \mathsf{F}(\mathscr{B}_i) \\
&= \operatorname*{colim}_{i \in I} \pi_{2*}\big((\mathscr{F} \otimes \mathscr{A}_i) \boxtimes \mathsf{F}(\mathscr{B}_i) \big) \\
&= \pi_{2*}\left(\pi_1^* \mathscr{F} \otimes \operatorname*{colim}_{i \in I} \big(\mathscr{A}_i \boxtimes \mathsf{F}(\mathscr{B}_i) \big) \right).
\end{aligned}$$

□

1.3.2 Three-functor formalisms in general

To rigorously construct monoidal convolution products on categories of sheaves, it is useful to have access to a "three-functor formalism" for said categories. One can define this loosely as follows. Fix some category of "geometric spaces," e.g. dAff_k or dStk_k. A three-functor formalism should assign to each object X a category $\mathsf{D}(\mathsf{X})$ of "sheaves on X" together with:

- For every X, a symmetric monoidal tensor product $\otimes$ on $\mathsf{D}(\mathsf{X})$,
- For every $f : \mathsf{X} \to \mathsf{Y}$, a $*$-pullback functor $f^* : \mathsf{D}(\mathsf{Y}) \to \mathsf{D}(\mathsf{X})$, and
- For "nice" morphisms $f : \mathsf{X} \to \mathsf{Y}$, a !-pushforward functor $f_! : \mathsf{D}(\mathsf{X}) \to \mathsf{D}(\mathsf{Y})$,

satisfying base change and the projection formula.

Three-functor formalisms, as well as the stronger notion of six-functor formalisms, are made mathematically precise (in the language of $(\infty, 1)$-categories) in [Man22, Appendix A.5], building on previous works [LZ12; GR19]. We recall (a slightly simplified version of) Mann's definition.

Let $(\mathcal{C}, \mathsf{E})$ be a pair where $\mathcal{C}$ is a category with finite limits and E is a collection of morphisms in $\mathcal{C}$ which contains all isomorphisms and is stable under homotopy, composition, and pullback. One can define a category $\mathsf{Corr}(\mathcal{C}, \mathsf{E})$ of *correspondences in* $\mathcal{C}$ *with right leg in* E. The objects of $\mathsf{Corr}(\mathcal{C}, \mathsf{E})$ are the objects of $\mathcal{C}$. Morphisms $\mathsf{X}_1 \to \mathsf{X}_2$ in $\mathsf{Corr}(\mathcal{C}, \mathsf{E})$ are given by correspondences (also called "spans") $\mathsf{X}_1 \xleftarrow{f} \mathsf{Y} \xrightarrow{g} \mathsf{X}_2$ where f is a morphism in $\mathcal{C}$

and g is a morphism in E.[12] Composition of correspondences is given by fiber products[13]

$$
\begin{tikzcd}
 & & \mathsf{Y}_1 \times_{\mathsf{X}_2} \mathsf{Y}_2 \arrow[dl] \arrow[dr] & & \\
 & \mathsf{Y}_1 \arrow[dl] \arrow[dr] & & \mathsf{Y}_2 \arrow[dl] \arrow[dr] & \\
\mathsf{X}_1 & & \mathsf{X}_2 & & \mathsf{X}_3.
\end{tikzcd}
$$

The category $\mathsf{Corr}(\mathcal{C}, \mathsf{E})$ has a symmetric monoidal structure $\times$ defined using the Cartesian monoidal structure on $\mathcal{C}$.

Definition 1.3.2.1 ([Man22, Definition A.5.6])**.** A *three-functor formalism* on $(\mathcal{C}, \mathsf{E})$ is a lax symmetric monoidal functor $\mathsf{Sh} : \big(\mathsf{Corr}(\mathcal{C}, \mathsf{E}), \times\big) \to \big(\widehat{\mathsf{Cat}}, \times\big)$, where $\widehat{\mathsf{Cat}}$ is the category of large categories (in some universe). This induces the following "three functors:"

- The inclusion map $\mathcal{C}^{\mathrm{op}} \hookrightarrow \mathsf{Corr}(\mathcal{C}, \mathsf{E})$, given on morphisms by $(\mathsf{X} \xrightarrow{\mathsf{f}} \mathsf{Y}) \mapsto (\mathsf{Y} \xleftarrow{\mathsf{f}} \mathsf{X} \xrightarrow{\mathrm{id}_{\mathsf{X}}} \mathsf{X})$, is symmetric monoidal when viewed as a functor $(\mathcal{C}^{\mathrm{op}}, \times) \to (\mathsf{Corr}(\mathcal{C}, \mathsf{E}), \times)$. Every object of $\mathcal{C}^{\mathrm{op}}$ is uniquely a commutative algebra for $\times$ (the coproduct in $\mathcal{C}^{\mathrm{op}}$) by [HA, 2.4.3.10]. Thus every category $\mathsf{Sh}(\mathsf{X})$ for $\mathsf{X} \in \mathcal{C}$ has a natural symmetric monoidal structure $\otimes_{\mathsf{Sh}(\mathsf{X})}$.

- Similarly, for a morphism $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ in $\mathcal{C}$, we obtain a symmetric monoidal functor $\mathsf{f}^* : (\mathsf{Sh}(\mathsf{Y}), \otimes_{\mathsf{Sh}(\mathsf{Y})}) \to (\mathsf{Sh}(\mathsf{X}), \otimes_{\mathsf{Sh}(\mathsf{X})})$.

- Let $\mathcal{C}_{\mathsf{E}}$ be the subcategory of $\mathcal{C}$ containing all objects of $\mathcal{C}$ but only containing the morphisms in E. Consider the inclusion map $\mathcal{C}_{\mathsf{E}} \hookrightarrow \mathsf{Corr}(\mathcal{C}, \mathsf{E})$ given on morphisms by $(\mathsf{X} \xrightarrow{\mathsf{f}} \mathsf{Y}) \mapsto (\mathsf{X} \xleftarrow{\mathrm{id}_{\mathsf{X}}} \mathsf{X} \xrightarrow{\mathsf{f}} \mathsf{Y})$. For a morphism $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ in E, we obtain a functor $\mathsf{f}_! : \mathsf{Sh}(\mathsf{X}) \to \mathsf{Sh}(\mathsf{Y})$.

If all functors $\otimes$, f^*, and $\mathsf{f}_!$ admit right adjoints, then Sh is called a *six-functor formalism.*

Remark 1.3.2.2. We note that, at the level of objects, the lax symmetric monoidal structure on Sh corresponds to an "external tensor product" $\boxtimes : \mathsf{Sh}(\mathsf{X}) \times \mathsf{Sh}(\mathsf{Y}) \to \mathsf{Sh}(\mathsf{X} \times \mathsf{Y})$. The fact that Sh is a lax symmetric monoidal *functor* implies that $\boxtimes$ satisfies the "expected" compatibilities with the pullback / pushforward functors, e.g. $\mathscr{F} \otimes \mathscr{G} = \Delta_{\mathsf{X}}^*(\mathscr{F} \boxtimes \mathscr{G})$ for $\mathsf{X} \in \mathcal{C}$, $\mathscr{F}, \mathscr{G} \in \mathsf{Sh}(\mathsf{X})$, and $\Delta_{\mathsf{X}} : \mathsf{X} \to \mathsf{X} \times \mathsf{X}$ the diagonal. (We will explicitly mention these compatibilities when needed.)

[12] In particular, the morphism spaces in $\mathsf{Corr}(\mathcal{C}, \mathsf{E})$ are typically no longer small. Any set-theoretic difficulties this presents can be circumvented by standard universe-based arguments, and we will ignore such difficulties for simplicity of exposition.

[13] Because fiber products are only unique up to natural equivalence, composition is only well-defined up to coherent homotopy. In particular, the rigorous definition of $\mathsf{Corr}(\mathcal{C}, \mathsf{E})$ is somewhat technical (as usual for the subject).

Six-functor formalisms behave best in the presentable case:

Definition 1.3.2.3. A *presentable six-functor formalism* is a six-functor formalism Sh such that every category $\mathsf{Sh}(\mathsf{X})$ (for $\mathsf{X} \in \mathcal{C}$) is presentable.

In other words, a presentable six-functor formalism is a lax symmetric monoidal functor $\mathsf{Sh} : \big(\mathsf{Corr}(\mathcal{C}, \mathsf{E}), \times\big) \to \big(\mathsf{Pr}^{\mathsf{L}}, \times\big)$. As noted in [Man22, proof of Lemma A.5.11], any presentable six-functor formalism defines a lax symmetric monoidal functor[14]

$$\mathsf{Sh} : \big(\mathsf{Corr}(\mathcal{C}, \mathsf{E}), \times\big) \to \big(\mathsf{Pr}^{\mathsf{L}}, \otimes\big)$$

by postcomposition with the lax symmetric monoidal identity functor $\big(\mathsf{Pr}^{\mathsf{L}}, \times\big) \to \big(\mathsf{Pr}^{\mathsf{L}}, \otimes\big)$.

In fact, there is even more structure on the categories $\mathsf{Sh}(\mathsf{X})$ when Sh is presentable. If Sh is a presentable six-functor formalism, then all categories $\mathsf{Sh}(\mathsf{X})$ and all functors $\otimes$, f^*, and $\mathsf{f}_!$ are naturally enriched and tensored over $\mathsf{Sh}(\mathrm{pt})$, where pt is the terminal object of $\mathcal{C}$ (see [MS24, Theorem A.3.8]). This allows us to view Sh as a lax symmetric monoidal functor

$$\mathsf{Sh} : \big(\mathsf{Corr}(\mathcal{C}, \mathsf{E}), \times\big) \to \big(\mathsf{Pr}^{\mathsf{L}}_{\mathsf{Sh}(\mathrm{pt})}, \otimes^{\mathsf{L}}_{\mathsf{Sh}(\mathrm{pt})}\big), \tag{1.3.2.4}$$

where $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{Sh}(\mathrm{pt})}$ is the symmetric monoidal category of presentably $\mathsf{Sh}(\mathrm{pt})$-enriched categories and $\mathsf{Sh}(\mathrm{pt})$-enriched left adjoint functors. We will often abuse notation and refer to the lax symmetric monoidal functor of (1.3.2.4) as a "presentable six-functor formalism."

1.3.3 Three-functor formalisms and convolution

In this subsection we shall discuss the *convolution products* on categories of sheaves obtained from a three-functor formalism. We begin with a cautionary remark.

Remark 1.3.3.1. Let $(\mathcal{C}, \mathsf{E})$ be as in Section 1.3.2. The category $\mathcal{C}_{\mathsf{E}}$ may not have products (and even if it does, these products need not agree with the products in $\mathcal{C}$). Indeed, for $\mathsf{X}_1, \mathsf{X}_2 \in \mathcal{C}$, there is no reason that the projection morphisms $\pi_i : \mathsf{X}_1 \times \mathsf{X}_2 \to \mathsf{X}_i$ must be in E. Thus it does not make sense *a priori* to claim that the functor $(\mathcal{C}_{\mathsf{E}}, \times) \to (\mathsf{Corr}(\mathcal{C}, \mathsf{E}), \times)$ is symmetric monoidal.

Under the given hypotheses on E, we expect that the Cartesian product of $\mathcal{C}$ defines a non-Cartesian symmetric monoidal structure (also denoted $\times$) on $\mathcal{C}_{\mathsf{E}}$. Furthermore, the inclusion functor $(\mathcal{C}_{\mathsf{E}}, \times) \to (\mathsf{Corr}(\mathcal{C}, \mathsf{E}), \times)$ should be symmetric monoidal. A proof of these claims under an additional assumption (which does not hold in our case of interest) is given in [HM24, Proposition 2.3.7]. However, proving these claims in our case of interest appears to involve more tinkering with the "internal machinery" of categories of correspondences than we care to do here.

Thus, *in this section, we will assume for simplicity that* E *consists of all morphisms in* $\mathcal{C}$. (In practice, we may often restrict to this case by replacing $(\mathcal{C}, \mathsf{E})$ by $(\mathcal{C}', \operatorname{Mor}\mathcal{C}')$ where $\mathcal{C}'$ is a full subcategory of "nice objects" of $\mathcal{C}$.)

[14] Here $\otimes$ denotes the Lurie tensor product on Pr^{L}.

Notation 1.3.3.2. When E is the class of all morphisms in $\mathcal{C}$, we will write $\mathsf{Corr}(\mathcal{C}) = \mathsf{Corr}(\mathcal{C}, \mathsf{E})$ and call $\mathsf{Corr}(\mathcal{C})$ the *category of correspondences in* $\mathcal{C}$.

The inclusion $(\mathcal{C}, \times) \to (\mathsf{Corr}(\mathcal{C}), \times)$ is symmetric monoidal. Indeed, because $\mathsf{Corr}(\mathcal{C})$ is equivalent to its opposite category, it suffices to show that $(\mathcal{C}^{\mathrm{op}}, \times) \to (\mathsf{Corr}(\mathcal{C}), \times)$ is symmetric monoidal. But this was noted above.

Let $\mathsf{Sh} : \mathsf{Corr}(\mathcal{C}) \to \widehat{\mathsf{Cat}}$ be a three-functor formalism. Then the composite functor

$$\mathsf{Sh}_! : (\mathsf{Corr}(\mathcal{C}), \times) \to \left(\widehat{\mathsf{Cat}}, \times\right)$$

is lax symmetric monoidal. In particular, for an $\mathbb{E}_n$-algebra object M of $\mathcal{C}$, applying $\mathsf{Sh}_!$ to M yields an $\mathbb{E}_n$-monoidal *convolution product* $\star_M$ on $\mathsf{Sh}(M)$. As a binary operation, $\star_M$ is given by $\mathscr{F} \star_M \mathscr{G} = \mu_!(\mathscr{F} \boxtimes \mathscr{G})$ for $\mathscr{F}, \mathscr{G} \in \mathsf{Sh}(M)$, where $\mu : M \times M \to M$ is the binary multiplication. The construction of $\star_M$ is functorial: if $f : M \to N$ is a homomorphism of $\mathbb{E}_n$-monoids in $\mathcal{C}_{\mathsf{E}}$, then $f_! : (\mathsf{Sh}(M), \star_M) \to (\mathsf{Sh}(N), \star_N)$ is $\mathbb{E}_n$-monoidal.

Under some additional hypotheses, the convolution product combined with the diagonal map $\Delta_M : M \to M \times M$ equips $\mathsf{Sh}(M)$ with the structure of an $(\mathbb{E}_n, \mathbb{E}_\infty)$-bialgebra. Before proving this, we recall the ∞-categorical definition of bialgebras:

Definition 1.3.3.3. Let $(\mathcal{D}, \otimes_{\mathcal{D}})$ be a symmetric monoidal category. For any $n, m \in \mathbb{N} \cup \{\infty\}$, the category of $(\mathbb{E}_n, \mathbb{E}_m)$-*bialgebras in* $(\mathcal{D}, \otimes_{\mathcal{D}})$ is

$$\mathsf{BiAlg}_{\mathbb{E}_n, \mathbb{E}_m}(\mathcal{D}, \otimes_{\mathcal{D}}) := \mathsf{Alg}_{\mathbb{E}_m}\left((\mathsf{Alg}_{\mathbb{E}_n}(\mathcal{D}, \otimes_{\mathcal{D}}))^{\mathrm{op}}, \otimes_{\mathcal{D}}\right)^{\mathrm{op}}.$$

We will frequently leave the tensor product $\otimes_{\mathcal{D}}$ implicit, writing $\mathsf{BiAlg}_{\mathbb{E}_n, \mathbb{E}_m}(\mathcal{D}) := \mathsf{BiAlg}_{\mathbb{E}_n, \mathbb{E}_m}(\mathcal{D}, \otimes_{\mathcal{D}})$.

Informally, an $(\mathbb{E}_n, \mathbb{E}_m)$-bialgebra in a symmetric monoidal category $\mathcal{D}$ is an object d of $\mathcal{D}$ together with:

- an $\mathbb{E}_n$-algebra structure on d,
- an $\mathbb{E}_m$-coalgebra structure on d, and
- extra data controlling the (homotopy coherent) compatibility of the two structures.

Because (strong) symmetric monoidal functors preserve $\mathbb{E}_n$-algebras and $\mathbb{E}_m$-coalgebras, a symmetric monoidal functor $\mathcal{D}_1 \to \mathcal{D}_2$ induces a functor $\mathsf{BiAlg}_{\mathbb{E}_n, \mathbb{E}_m}(\mathcal{D}_1) \to \mathsf{BiAlg}_{\mathbb{E}_n, \mathbb{E}_m}(\mathcal{D}_2)$.

Proposition 1.3.3.4. *Let* $\mathsf{Sh} : (\mathsf{Corr}(\mathcal{C}), \times) \to (\mathsf{Pr}^{\mathsf{L}}, \otimes)$ *be a presentable six-functor formalism. Assume that the induced functor* $\mathsf{Sh} : (\mathcal{C}, \times) \to (\mathsf{Pr}^{\mathsf{L}}_{\mathsf{Sh}(\mathrm{pt})}, \otimes_{\mathsf{Sh}(\mathrm{pt})})$ *is (strong) symmetric monoidal. If* M *is an* $\mathbb{E}_n$*-monoid in* $\mathcal{C}$*, then:*

1. $*$*-pullback along the structure maps and the diagonal of* M *make* $\mathsf{Sh}(M)$ *into an* $(\mathbb{E}_\infty, \mathbb{E}_n)$*-bialgebra in* $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{Sh}(\mathrm{pt})}$.

2. *!-pushforward along the structure maps and the diagonal of* M *make* $\mathsf{Sh}(\mathsf{M})$ *into an* $(\mathbb{E}_n, \mathbb{E}_\infty)$*-bialgebra in* $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{Sh}(\mathrm{pt})}$.

Furthermore, these constructions are functorial in M.

Proof. (1) The category $\mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C})$ is Cartesian monoidal, so by [HA, 2.4.3.10] there is an equivalence $\mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C}) \xrightarrow{\sim} \mathsf{BiAlg}_{\mathbb{E}_n,\mathbb{E}_\infty}(\mathcal{C})$. Thus we may view M as an $(\mathbb{E}_n, \mathbb{E}_\infty)$-bialgebra in $\mathcal{C}$. Reversing the direction of arrows makes M into an $(\mathbb{E}_\infty, \mathbb{E}_n)$-bialgebra in $\mathcal{C}^{\mathrm{op}}$. The composite functor

$$(\mathcal{C}^{\mathrm{op}}, \times) \to (\mathsf{Corr}(\mathcal{C}), \times) \to (\mathsf{Pr}^{\mathsf{L}}_{\mathsf{Sh}(\mathrm{pt})}, \otimes_{\mathsf{Sh}(\mathrm{pt})})$$

is symmetric monoidal, hence must send $(\mathbb{E}_\infty, \mathbb{E}_n)$-bialgebras to $(\mathbb{E}_\infty, \mathbb{E}_n)$-bialgebras.

(2) The same argument works here. □

Remark 1.3.3.5. Part (1) of Proposition 1.3.3.4 does not require that E is the class of all morphisms in $\mathcal{C}$, though part (2) does.

We state Proposition 1.3.3.4 in terms of the induced functor $(\mathcal{C}, \times) \to (\mathsf{Pr}^{\mathsf{L}}_{\mathsf{Sh}(\mathrm{pt})}, \otimes_{\mathsf{Sh}(\mathrm{pt})})$ rather than $(\mathcal{C}, \times) \to (\mathsf{Pr}^{\mathsf{L}}_{\mathsf{Sh}(\mathrm{pt})}, \otimes_{\mathsf{Sh}(\mathrm{pt})})$ because the former is rarely symmetric monoidal while the latter often is so. This will be the case for the six-functor formalism $\mathsf{QC} : (\mathsf{Corr}(\mathsf{dStk}^{\mathrm{perf}}_{\mathsf{k}}), \times) \to (\mathsf{Pr}^{\mathsf{L}}_{\mathsf{k}}, \otimes_{\mathsf{k}})$ discussed below (see Proposition 1.3.4.3(3)).

1.3.4 A three-functor formalism for quasicoherent sheaves

For our applications, we would like to use a three-functor formalism for quasicoherent sheaves on stacks in which the "!-pushforward functors" are the right adjoints to the usual $*$-pullback functors (in reasonable cases, these are the usual $*$-pushforward functors). The analogous three-functor formalism for qcqs schemes is discussed in [Sch23, §8.3]. We will need to extend the formalism to general derived stacks – this can be done using the results of [BFN10].

We begin by clarifying the notion of "nice" morphisms in this context.

Definition 1.3.4.1. A morphism of stacks $\mathsf{f} : \mathfrak{X} \to \mathfrak{Y}$ is *very perfect* if f has affine diagonal and, for every morphism of stacks $\mathsf{g} : \mathfrak{Z} \to \mathfrak{Y}$ with $\mathfrak{Z}$ perfect, the fiber product $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z}$ is perfect. Let VP be the class of very perfect morphisms in $\mathsf{dStk}_{\mathsf{k}}$.

Lemma 1.3.4.2. *The class* VP *has the following properties:*

1. VP *is stable under composition.*[15]

2. VP *is stable under base change.*

3. VP *contains all affine morphisms.*

4. VP *satisfies a cancellation law: if* $\mathsf{f} : \mathfrak{X} \to \mathfrak{Y}$ *and* $\mathsf{g} : \mathfrak{Y} \to \mathfrak{Z}$ *are such that* $\mathsf{g} \in \mathrm{VP}$ *and* $\mathsf{g} \circ \mathsf{f} \in \mathrm{VP}$*, then* $\mathsf{f} \in \mathrm{VP}$.

[15] It is not clear *a priori* whether this holds for the *perfect morphisms* of [BFN10, Definition 3.2].

5. *Any morphism between perfect stacks is in* VP.

Proof. This is standard diagram chasing and references to [BFN10]. We first note that the class of morphisms with affine diagonal is stable under composition and base change, e.g. by (the argument of) [Vak24, Theorem 11.1.2].

(1). Let $\mathsf{f} : \mathfrak{X} \to \mathfrak{Y}$ and $\mathsf{g} : \mathfrak{Y} \to \mathfrak{Z}$ be very perfect. To see that $\mathsf{g} \circ \mathsf{f}$ is very perfect, let $\mathcal{W} \to \mathfrak{Z}$ be a morphism with $\mathcal{W}$ perfect. Then $\mathfrak{X} \times_{\mathfrak{Z}} \mathcal{W} = (\mathfrak{Y} \times_{\mathfrak{Z}} \mathcal{W}) \times_{\mathfrak{Y}} \mathfrak{X}$. Because g is very perfect, the stack $\mathfrak{Y} \times_{\mathfrak{Z}} \mathcal{W}$ is perfect. Then, because f is very perfect, the stack $\mathfrak{X} \times_{\mathfrak{Z}} \mathcal{W}$ must also be perfect. It follows that $\mathsf{g} \circ \mathsf{f}$ is very perfect.

(2). Let $\mathfrak{X} \to \mathfrak{Y}$ be very perfect and let $\mathfrak{Z} \to \mathfrak{Y}$ be any morphism. If $W \to \mathfrak{Z}$ is a morphism with W perfect, then $(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z}) \times_{\mathfrak{Z}} W = \mathfrak{X} \times_{\mathfrak{Y}} W$ is perfect. Thus the base change $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z} \to \mathfrak{Z}$ is very perfect.

(3). Any affine morphism has affine diagonal. Furthermore, if $\mathfrak{X} \to \mathfrak{Y}$ is affine and $\mathfrak{Z} \to \mathfrak{Y}$ is any morphism with $\mathfrak{Z}$ perfect, then $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z} \to \mathfrak{Z}$ is affine. Then $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z}$ is perfect by [BFN10, Proposition 3.21], so $\mathfrak{X} \to \mathfrak{Y}$ must be very perfect.

(4). This is a consequence of (the argument of) [Vak24, Theorem 11.1.1], where we use (3) above to see that the diagonal of a very perfect morphism is very perfect.

(5). By (4) it suffices to show that, if $\mathfrak{X}$ is a perfect stack, then the natural map $\pi : \mathfrak{X} \to \operatorname{Spec} \mathsf{k}$ is very perfect. The map π has affine diagonal because the same is true for $\mathfrak{X}$. Furthermore, if $\mathfrak{Y}$ is a perfect stack, then $\mathfrak{X} \times \mathfrak{Y}$ is perfect by [BFN10, Proposition 3.24], giving the claim. □

By parts (1) and (2) of Lemma 1.3.4.2, the pair $(\mathsf{dStk}_{\mathsf{k}}, \mathrm{VP})$ satisfies the conditions discussed in Section 1.3.2, so the category $\mathsf{Corr}(\mathsf{dStk}_{\mathsf{k}}, \mathrm{VP})$ is well-defined. We may now define our three-functor formalisms of interest:

Proposition 1.3.4.3. 1. *There is a three-functor formalism*

$$\mathsf{QC} : (\mathsf{Corr}(\mathsf{dStk}_{\mathsf{k}}, \mathrm{VP}), \times) \to (\widehat{\mathsf{Cat}}_{\infty}, \times)$$

sending a correspondence $\mathfrak{X} \xleftarrow{\mathsf{f}} \mathfrak{Z} \xrightarrow{\mathsf{g}} \mathfrak{Y}$ *with* g *very perfect to*

$$\mathsf{QC}(\mathfrak{X}) \xrightarrow{\mathsf{f}^*} \mathsf{QC}(\mathfrak{Z}) \xrightarrow{\mathsf{g}_*} \mathsf{QC}(\mathfrak{Y}).$$

2. *The same formula defines a presentable six-functor formalism*

$$\mathsf{QC} : (\mathsf{Corr}(\mathsf{dStk}_{\mathsf{k}}^{\mathrm{perf}}), \times) \to (\mathsf{Pr}^{\mathsf{L}}, \otimes)$$

3. *The induced functor*

$$\mathsf{QC} : (\mathsf{Corr}(\mathsf{dStk}_{\mathsf{k}}^{\mathrm{perf}}), \times) \to (\mathsf{Pr}^{\mathsf{L}}_{\mathsf{k}}, \otimes_{\mathsf{k}})$$

is (strong) symmetric monoidal.

Proof. (1). We have already constructed QC as a functor $\mathsf{dStk}_k^{\mathrm{op}} \to \mathsf{CAlg}(\widehat{\mathsf{Cat}})$ (sending $f : \mathfrak{X} \to \mathfrak{Y}$ to $f^* : \mathsf{QC}(\mathfrak{Y}) \to \mathsf{QC}(\mathfrak{X})$. By [Man22, Proposition A.5.10],[16] to extend QC to a three-functor formalism, it suffices to show that:

- VP satisfies the cancellation law of Lemma 1.3.4.2 (4).
- For $f : \mathfrak{X} \to \mathfrak{Y}$ in VP and any $g : \mathfrak{Z} \to \mathfrak{Y}$, let $f' : \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z} \to \mathfrak{Z}$ and $g' : \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z} \to \mathfrak{X}$ be the induced maps. Then the base change formula $g^* f_* = (f')_* (g')^*$ holds.
- The projection formula holds for $f \in$ VP.

The former condition holds by Lemma 1.3.4.2 (4), while the latter two conditions hold by [BFN10, Proposition 3.10].

(2). Note that the categories $\mathsf{QC}(\mathfrak{X})$ are automatically presentable (even if $\mathfrak{X}$ is not perfect). Thus it suffices to show that, if $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of perfect stacks, then the functors $\otimes_{\mathscr{O}_{\mathfrak{X}}}$, f^*, and f_* all admit right adjoints. For $\otimes_{\mathscr{O}_{\mathfrak{X}}}$ and f^* this is standard (and does not use perfectness of the stacks involved). For f_*, we know by [BFN10, Proposition 3.10] that f_* preserves all colimits, so the existence of a right adjoint to f_* follows from the adjoint functor theorem [HTT, 5.5.2.9].

(3). This follows from [BFN10, Theorem 4.7] and the above discussion. □

Corollary 1.3.4.4. *If $\mathfrak{M}$ is an $\mathbb{E}_n$-monoid object in $\mathsf{dStk}_k^{\mathrm{perf}}$, then:*

1. *$*$-pullback along the structure maps and the diagonal of $\mathfrak{M}$ make $\mathsf{QC}(\mathfrak{M})$ into an $(\mathbb{E}_\infty, \mathbb{E}_n)$-bialgebra in $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{Sh}(\mathrm{pt})}$.*
2. *$*$-pushforward along the structure maps and the diagonal of $\mathfrak{M}$ make $\mathsf{QC}(\mathfrak{M})$ into an $(\mathbb{E}_n, \mathbb{E}_\infty)$-bialgebra in $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{Sh}(\mathrm{pt})}$.*

Proof. This follows by applying Proposition 1.3.3.4 to the presentable six-functor formalism of parts (2) and (3) of Proposition 1.3.4.3. □

Remark 1.3.4.5. Although part (2) of Proposition 1.3.4.3 defines a presentable six-functor formalism, the right adjoint to f_* behaves poorly in general, even for proper morphisms. For example, f_* typically does not preserve compact objects, so its right adjoint does not preserve colimits. Dealing with these issues is one of the main reasons for the existence of the formalism of ind-coherent sheaves (as developed in [GR19] among others). We will not need to make direct use of the right adjoint of f_*, so we content ourselves with using QC.

[16] More precisely: [Man22, Proposition A.5.10] applies to a "suitable decomposition" of VP into classes of morphisms I and P satisfying certain conditions. Here we simplify things by taking $\mathsf{P} =$ VP and taking I to be the class of isomorphisms in dStk_k.

Chapter 2

Grouplike symmetric monoidal categories

Our goal in this chapter is to study the following particularly nice class of symmetric monoidal categories:

Definition 2.0.0.1. A symmetric monoidal category $\mathcal{C}$ is *grouplike* if its $\mathbb{E}_\infty$-monoid of objects $\iota_0\mathcal{C}$ is an $\mathbb{E}_\infty$-group.

Remark 2.0.0.2. One should think of a grouplike SMC $\mathcal{C}$ as somewhat analogous to the full subcategory of $\mathsf{QC}^{\leqslant 0}(\mathfrak{Y})$ consisting of line bundles (for some derived stack $\mathfrak{Y}$). Of course, there is one key difference here: grouplike SMCs are enriched in $\mathcal{S}$, while we will typically think of $\mathsf{QC}^{\leqslant 0}(\mathfrak{Y})$ as enriched over some $\mathsf{D}^{\leqslant 0}(\mathsf{k})$. By k-linearizing $\mathcal{C}$, we can obtain something that behaves much more similarly to a full subcategory of some $\mathsf{QC}^{\leqslant 0}(\mathfrak{Y})$. However, we shall see that the structure theory of grouplike symmetric monoidal categories is quite transparent (as opposed to that of their k-linear counterparts).

Our key results (Proposition 2.3.1.1 and Theorem 2.3.2.1) will allow us to understand presentations of grouplike SMCs in terms of $\mathbb{E}_\infty$-monoids, thus “decreasing our category level” and offering a more concrete approach to studying grouplike SMCs. These presentations will be written using the language of “skew quotients” introduced in Section 2.2.

Skew quotient presentations of grouplike SMCs will be used in Chapter 3 to make the analogy with categories of line bundles on derived stacks much more precise. In this chapter, we will also use to establish some technical results on localizations of grouplike SMCs which are necessary for the proofs of key results in derived toric geometry (see Chapter 6).

2.1 $\mathbb{E}_\infty$-monoids

We will begin with a review of the theory of $\mathbb{E}_\infty$-monoids (i.e. commutative algebra objects in the symmetric monoidal category $(\mathcal{S}, \times)$). Parts of this section (especially the material on $\mathbb{E}_\infty$-groups) are well-known to the experts, and we have tried to give references where possible.

2.1.1 Some basic definitions and examples

We begin by recalling some definitions we will use throughout.

Definition 2.1.1.1. The *category of* $\mathbb{E}_\infty$*-monoids* is $\mathsf{CMon} := \mathsf{CAlg}(\mathcal{S}, \times)$. If M is an $\mathbb{E}_\infty$-monoid, we will write M^0 for the connected component of the identity in M and $\pi_0(M)$ for the discrete commutative monoid of connected components of M.

Notation 2.1.1.2. We will typically use additive notation for $\mathbb{E}_\infty$-monoids, i.e. if M is an $\mathbb{E}_\infty$-monoid then $0 := 0_M$ is the unit and $+ := +_M$ is the binary operation.

We would be remiss if we did not include some examples:

Example 2.1.1.3. Any discrete commutative monoid is an $\mathbb{E}_\infty$-monoid via the symmetric monoidal inclusion $(\mathsf{Set}, \times) \hookrightarrow (\mathcal{S}, \times)$.

Example 2.1.1.4. The free $\mathbb{E}_\infty$-monoid on one generator, which we will call $\mathbb{F}$, has underlying space given by $\sqcup_{n\in\mathbb{N}} B\Sigma_n$, where Σ_n is the symmetric group on n generators. Under the equivalence between spaces and groupoids, $\mathbb{F}$ corresponds to the symmetric monoidal 1-groupoid of finite sets and equivalences (with symmetric monoidal structure given by disjoint union).

Example 2.1.1.5. Any topological commutative monoid determines an $\mathbb{E}_\infty$-monoid by taking its weak homotopy type. In particular, we may view $S^1 = \mathrm{U}(1)$ as an $\mathbb{E}_\infty$-monoid.

For any $\mathbb{E}_\infty$-monoid M, we may construct a symmetric monoidal category BM with one object $*$ and with $\mathrm{Map}_{BM}(*, *) = M$. This defines a fully faithful, colimit-preserving functor $B : \mathsf{CMon} \to \mathsf{CAlg}(\mathsf{Cat})$. See [GH15, Corollary 6.3.11] for the ∞-categorical details.

Given an $\mathbb{E}_\infty$-monoid M, we may define a category of *spaces with (right)* M*-action* as

$$\mathcal{S}^M := \mathsf{Fun}(BM^{\mathrm{op}}, \mathcal{S}) = \mathsf{PSh}_{\mathcal{S}}(BM).$$

The category $\mathcal{S}^M$ is a topos, and the forgetful map $\mathcal{S}^M \to \mathcal{S}$ (i.e. pullback along the inclusion $\mathrm{pt} \to BM$) preserves both limits and colimits. This additional structure means that it is often easier to manipulate limits and colimits in CMon by working at the level of underlying spaces (or underlying M-spaces for some M). We shall do so throughout.

One of the easiest classes of $\mathbb{E}_\infty$-monoids to work with (and the class that is most frequently relevant to applications) is the following:

Definition 2.1.1.6. An $\mathbb{E}_\infty$*-group* (sometimes called a *grouplike* $\mathbb{E}_\infty$-monoid) is an $\mathbb{E}_\infty$-monoid A such that $\pi_0(A)$ is a group. We let CGrp denote the full subcategory of CMon spanned by the $\mathbb{E}_\infty$-groups.

The inclusion $\mathsf{CGrp} \hookrightarrow \mathsf{CMon}$ admits a right adjoint $(-)^{\pm}$ (the *group of units* functor) and a left adjoint $(-)^{\mathrm{gp}}$ (the *group completion* functor). These may both be described more concretely. For an $\mathbb{E}_\infty$-monoid M, the $\mathbb{E}_\infty$-group $M^{\pm}$ is the union of all connected components of M which are invertible in $\pi_0(M)$. By the classical *group completion theorem* (see [Leh24, Proposition 2.8] for an ∞-categorical discussion), the $\mathbb{E}_\infty$-group M^{gp} is the loop space $\Omega_0(\mathrm{pt}/M)$.

The category CGrp enjoys a useful collection of properties that do not hold in general in CMon: namely, CGrp is *prestable* in the sense of [Lur18, Definition C.1.2.1]. Recall that this means:

- The category CGrp has a zero object and admits all finite colimits.
- The functor $\mathsf{CGrp} \to \mathsf{CGrp}$, $A \mapsto \mathsf{B}A = 0 \coprod_A 0$, is fully faithful.
- If $\phi : A \to \mathsf{B}A'$ is a homomorphism, then $\mathrm{fib}(\phi)$ exists in CGrp, and the square

$$\begin{tikzcd} \mathrm{fib}(\phi) \arrow[r] \arrow[d] & A \arrow[d, "\phi"] \\ 0 \arrow[r] & \mathsf{B}A' \end{tikzcd}$$

is both a pullback square and a pushout square.

Example 2.1.1.7. Let $\mathbb{F}$ be the free $\mathbb{E}_\infty$-monoid on one generator. Then $\mathbb{F}^{\mathrm{gp}}$, the free $\mathbb{E}_\infty$-group on one generator, fits into the pushout square

$$\begin{tikzcd} \mathbb{F} \arrow[r] \arrow[d, "\Delta_{\mathbb{F}}"'] & 0 \arrow[d] \\ \mathbb{F} \times \mathbb{F} \arrow[r] & \mathbb{F}^{\mathrm{gp}}. \end{tikzcd}$$

The quotient $\mathsf{B}\mathbb{F}^{\mathrm{gp}} = \mathrm{pt}/\mathbb{F}^{\mathrm{gp}}$ corepresents the functor $\Omega_0 M$, i.e. $\mathsf{B}\mathbb{F}^{\mathrm{gp}}$ is the free connected $\mathbb{E}_\infty$-monoid with a nontrivial generator of its loop space. This quotient also fits into a pushout square

$$\begin{tikzcd} \mathbb{F}^{\mathrm{gp}} \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \mathsf{B}\mathbb{F}^{\mathrm{gp}}. \end{tikzcd}$$

Because $\mathbb{F}$ is compact in CMon, the finite colimit presentations of $\mathbb{F}^{\mathrm{gp}}$ and $\mathsf{B}\mathbb{F}^{\mathrm{gp}}$ imply that $\mathbb{F}^{\mathrm{gp}}$ and $\mathsf{B}\mathbb{F}^{\mathrm{gp}}$ are compact in CMon. Even though $\mathbb{F}$ is a 1-groupoid, $\mathbb{F}^{\mathrm{gp}}$ has homotopy groups in infinitely many degrees: see Example 2.1.2.2.

Remark 2.1.1.8. Because $\mathbb{F}$ compactly generates CMon, we see that

$$(-)^{\mathrm{gp}} : \mathsf{CMon} \to \mathsf{CGrp} \hookrightarrow \mathsf{CMon}$$

and

$$\Omega_0 : \mathsf{CMon} \to \mathsf{CGrp} \hookrightarrow \mathsf{CMon}$$

preserve compact objects.

2.1.2 Presentations of $\mathbb{E}_\infty$-groups

A version of May's recognition principle for infinite loop spaces (see [HA, 5.2.6.26]) gives an equivalence

$$\mathsf{H} : \mathsf{CGrp} \xrightarrow{\sim} \mathsf{Sp}^{\leqslant 0}$$

sending an $\mathbb{E}_\infty$-group A to the connective spectrum $\mathsf{H}A$ described by the sequence

$$\{\dots, \Omega_0^2 A, \Omega_0 A, A, \mathsf{B}A, \mathsf{B}^2 A, \dots\}.$$

In particular, we have $\pi^{-i}(\mathsf{H}A) = \pi_i(A)$. The equivalence H allows us to use stable homotopy theory to classify $\mathbb{E}_\infty$-groups. This classification will not be strictly necessary for our theoretical discussions later on, but it can be quite useful for constructing and understanding examples.

Example 2.1.2.1. If A is a discrete abelian group, then $\mathsf{H}A$ is the spectrum representing ordinary cohomology with coefficients in A. Thus our usage of H for the equivalence $\mathsf{CGrp} \xrightarrow{\sim} \mathsf{Sp}^{\leqslant 0}$ is compatible with classical notation.

Example 2.1.2.2. Let $\mathbb{F}^{\mathrm{gp}}$ be the free $\mathbb{E}_\infty$-group on one generator. Then $\mathsf{H}\mathbb{F}^{\mathrm{gp}} = \mathbb{S}$, the sphere spectrum.

If A is an $\mathbb{E}_\infty$-group, then we have $\pi_0(A) = \mathrm{cofib}(A^0 \to A)$. Applying H gives a cofiber sequence in Sp:

$$\mathsf{H}(A^0) \longrightarrow \mathsf{H}A \longrightarrow \mathsf{H}(\pi_0(A)) \longrightarrow .$$

In particular, we may write $\mathsf{H}A = \mathrm{fib}\big(\mathsf{H}(\pi_0(A)) \to \mathsf{H}(A^0)[1]\big)$ (where the fiber is taken in Sp, not $\mathsf{Sp}^{\leqslant 0}$). Note that $\mathsf{H}(A^0)[1] = \mathsf{H}(\Omega_0 A)[2]$ and that

$$\pi_0 \mathrm{Map}_{\mathsf{Sp}}\big(\mathsf{H}(\pi_0(A)), \mathsf{H}(\Omega_0 A)[2]\big) = \mathsf{H}(\Omega_0 A)^2\big(\mathsf{H}(\pi_0(A))\big).$$

Thus we obtain:

Proposition 2.1.2.3. *Any $\mathbb{E}_\infty$-group A is classified by an element $\eta_A \in \mathsf{H}(\Omega_0 A)^2\big(\mathsf{H}(\pi_0(A))\big)$.*

This "classification" is nothing but a precise way of saying that the data of A is determined by:

- the discrete abelian group $\pi_0(A)$,
- the $\mathbb{E}_\infty$-group $\Omega_0 A$, and

- the interaction between the two, which is controlled by an element of $\mathsf{H}(\Omega_0 A)^2\big(\mathsf{H}(\pi_0(A))\big)$.

Because $\Omega_0 A$ is itself an $\mathbb{E}_\infty$-group, this perspective does not generally reduce the classification of $\mathbb{E}_\infty$-groups into a simpler problem. However, it can be quite useful in classifying highly truncated $\mathbb{E}_\infty$-groups.

Example 2.1.2.4. [JO12, Corollary 2.8] gives the following explicit classification of all $\mathbb{E}_\infty$-groups with nontrivial homotopy groups only in degrees 0 and 1. For any discrete abelian groups π_0 and π_1, the elements of $(\mathsf{H}\pi_1)^2(\mathsf{H}\pi_0)$ may be represented by symmetric bilinear maps $\sigma_{-,-} : \pi_0 \times \pi_0 \to \pi_1$. Given such a σ, we may construct a symmetric monoidal 1-groupoid $A(\pi_0, \pi_1, \sigma)$ with the following data:

- $\mathrm{ob}\, A = \pi_0$.
- Morphism spaces:
  - $\mathrm{Map}_A(x, x) = \pi_1$ for $x \in \pi_0$
  - $\mathrm{Map}_A(x, y) = \varnothing$ for $x \neq y$.
- Composition of morphisms is multiplication in π_0.
- Tensor product:
  - $\otimes : \mathrm{ob}\, A \times \mathrm{ob}\, A \to \mathrm{ob}\, A$ is addition in π_0.
  - $\otimes$ is defined on morphisms by addition in π_1.
- Associators $\alpha_{xyz} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ are given by the identity morphisms.
- Braiding maps $x \otimes y \to y \otimes x$ are given by $\sigma_{x,y} \in \pi_1 = \mathrm{Map}_{\mathcal{C}}(x + y, y + x)$.

In other words, $A = \pi_0 \times \mathsf{B}\pi_1$ as $\mathbb{E}_1$-monoids, but the symmetric monoidal structure is twisted by σ. Every 1-truncated $\mathbb{E}_\infty$-group arises as $A(\pi_0, \pi_1, \sigma)$ for some π_0, π_1, and σ.

Example 2.1.2.5. In the situation of Example 2.1.2.4, we have $\pi_0 \times \mathsf{B}\pi_1 = A(\pi_0, \pi_1, 0)$ as symmetric monoidal 1-groupoids, where 0 refers to the zero map.

Example 2.1.2.6. In the situation of Example 2.1.2.4, take $\pi_0 = \mathbb{Z}/2 = \{0, 1\}$ and $\pi_1 = \mu_2 = \{\pm 1\}$. We may define $\sigma : \mathbb{Z}/2 \times \mathbb{Z}/2 \to \mu_2$ by $\sigma_{i,j} = (-1)^{ij}$. Then $A = A(\mathbb{Z}/2, \mu_2, \sigma)$ corresponds to $\mathbb{Z}/2 \times \mathsf{B}\mu_2$ as $\mathbb{E}_1$-groups, but the symmetric monoidal structure is twisted by the Koszul sign rule.

The class in $(\mathsf{H}\mu_2)^2\big(\mathsf{H}(\mathbb{Z}/2)\big) \simeq \mathsf{H}(\mathbb{Z}/2)^2\big(\mathsf{H}(\mathbb{Z}/2)\big)$ classifying σ (and hence A) is nontrivial.[1] Indeed, we know that the nontrivial class in $(\mathsf{H}\mu_2)^2\big(\mathsf{H}(\mathbb{Z}/2)\big)$ can be represented by some symmetric bilinear map $\mathbb{Z}/2 \times \mathbb{Z}/2 \to \mu_2$, and σ is the only nonzero option. Thus A is not isomorphic to $\mathbb{Z}_2 \times \mathsf{B}\mu_2$ as $\mathbb{E}_\infty$-groups.

[1] In terms of cohomology theories, σ corresponds to the Steenrod square $\mathrm{Sq}^2 : \mathsf{H}^\bullet(-; \mathbb{Z}/2) \to \mathsf{H}^{\bullet+2}(-; \mathbb{Z}/2)$.

Example 2.1.2.7. In the situation of Example 2.1.2.4, let R be a discrete commutative ring, and let $\mu : R \otimes R \to R$ be the multiplication map. Then $A(R, R, \mu)$ is a symmetric monoidal 1-groupoid. This generalizes Example 2.1.2.6 (once we have identified $\mu_2 \simeq \mathbb{Z}/2$). It would be interesting to understand what conditions on $A(R, R, \mu)$ correspond to standard ring-theoretic conditions on R (e.g. integrality).

We conclude this discussion with some remarks about how to classify more general $\mathbb{E}_\infty$-groups. (This is standard and we mostly include it for readers' convenience.)

By [HA, 1.4.3.6(3)], the standard t-structure on Sp is left complete. In other words, for any spectrum E, we may write $E = \lim_{n\to\infty} \tau^{\geqslant -n} E$. Let's now suppose $E = HA$ for some $\mathbb{E}_\infty$ group A.[2] Then $\tau^{\geqslant -n} HA = H(\tau_{\leqslant n} A)$ for any n, and in particular $\tau^{\geqslant n} HA = 0$ for any $n > 0$. The corresponding sequence of spectra

$$\dots \longrightarrow H(\tau_{\leqslant 2} A) \longrightarrow H(\tau_{\leqslant 1} A) \longrightarrow H(\tau_{\leqslant 0} A) \longrightarrow 0$$

is called the *Postnikov tower* of HA. The Postnikov tower lets us inductively construct A from discrete data.

More precisely, for all $n \geqslant 0$, note that

$$\mathrm{fib}(H(\tau_{\leqslant n+1} A) \to H(\tau_{\leqslant n} A)) = \pi^n(HA)[n] = \pi_n(A)[n].$$

Thus, because Sp is stable, we also have $H(\tau_{\leqslant n+1} A) = \mathrm{cofib}(H(\tau_{\leqslant n} A)[-1] \to \pi_n(A)[n])$. It follows that $H(\tau_{\leqslant n+1} A)$ (and hence $\tau_{\leqslant n+1} A$) is classified by an element

$$k_n \in \pi_0 \mathrm{Map}_{\mathsf{Sp}}(H(\tau_{\leqslant n} A)[-1], \pi_n(A)[n]).$$

Now

$$\begin{aligned} \pi_0 \mathrm{Map}_{\mathsf{Sp}}(H(\tau_{\leqslant n} A)[-1], \pi_n(A)[n]) &= \pi_0 \mathrm{Map}_{\mathsf{Sp}}(H(\tau_{\leqslant n} A), \pi_n(A)[n+1]) \\ &= \pi_{n+1} \mathrm{Map}_{\mathsf{Sp}}(H(\tau_{\leqslant n} A), \pi_n(A)) \\ &= \big(H\pi_n(A)\big)^{n+1}\big(H(\tau_{\leqslant n} A)\big) \end{aligned}$$

so we may as well view k_n as an element of the cohomology group $\big(H\pi_n(A)\big)^{n+1}\big(H(\tau_{\leqslant n} A)\big)$. These k_n are called the *(Postnikov)* k-*invariants* of A.

Thus an $\mathbb{E}_\infty$-group A is classified by:

- Discrete abelian groups $\pi_n(A)$ for all n, and
- k-invariants $k_n \in \big(H\pi_n(A)\big)^n\big(H(\tau_{\leqslant n} A)\big)$ for all $n \geqslant 0$.

Computing the groups $\big(H\pi_n(A)\big)^{n+1}\big(H(\tau_{\leqslant n} A)\big)$ is difficult to do in general but can give useful information about the corresponding $\mathbb{E}_\infty$-groups.

[2] In fact, the same argument works whenever E is "eventually connective," though this case will not interest us.

2.1.3 Sharp $\mathbb{E}_\infty$-monoids

So far we have studied only $\mathbb{E}_\infty$-monoids which behave quite similarly to groups. We will also find it useful to consider the other extreme:

Definition 2.1.3.1. An $\mathbb{E}_\infty$-monoid M is *sharp* if $\mathsf{M}^{\pm} = 0$. We write $\mathsf{CMon}^{\sharp}$ for the full subcategory of CMon spanned by the sharp $\mathbb{E}_\infty$-monoids.

Remark 2.1.3.2. This is an ∞-categorical version of the more well-known notion of "sharp discrete commutative monoid" (see e.g. [Ogu18, §1.2]).

Proposition 2.1.3.3. *For an $\mathbb{E}_\infty$-monoid M, the following are equivalent:*

1. *M is sharp.*
2. *For all $\mathbb{E}_\infty$-groups A, we have $\mathrm{Hom}_{\mathsf{CMon}}(\mathsf{A}, \mathsf{M}) = 0$.*
3. *$\mathsf{M}^0 = 0$ and $\pi_0(\mathsf{M})^{\pm} = 0$.*
4. *$\Omega_e \mathsf{M} = 0$ and $\pi_0(\mathsf{M})^{\pm} = 0$.*

Proof. The equivalence of (1) and (2) follows from the definition of $\mathsf{M}^{\pm}$ as the right adjoint to the inclusion $\mathsf{CGrp} \hookrightarrow \mathsf{CMon}$. The equivalence of (3) and (4) follows from the equivalence $\mathsf{M}^0 = 0/\Omega_0 \mathsf{M}$.

(2 $\Rightarrow$ 3). For the first claim, as M^0 is an $\mathbb{E}_\infty$-group, we have $\mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}^0, \mathsf{M}^0) = \mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}^0, \mathsf{M}) = 0$. This implies that $\mathrm{id}_{\mathsf{M}_0}$ is contractible, i.e. $\mathsf{M}^0 = 0$. For the second claim, if $\mathbb{F}^{\mathrm{gp}}$ is the free $\mathbb{E}_\infty$-group on one generator, then

$$\pi_0(\mathsf{M})^{\pm} = \pi_0(\mathsf{M}^{\pm}) = \pi_0(\mathrm{Hom}_{\mathsf{CMon}}(\mathbb{F}^{\mathrm{gp}}, \mathsf{M})) = \pi_0(\mathrm{pt}) = 0.$$

(3 $\Rightarrow$ 1). Since $\pi_0(\mathsf{M})^{\pm} = \pi_0(\mathsf{M}^{\pm}) = 0$, we must have $\mathsf{M}^{\pm} = \mathsf{M}^0 = 0$. □

Here are some key examples of sharp $\mathbb{E}_\infty$-monoids:

Example 2.1.3.4. Let $\mathbb{F}$ be the free $\mathbb{E}_\infty$-monoid on one generator. Then $\mathbb{F}$ is sharp. Indeed, we may write $\mathbb{F} = \sqcup_{n \in \mathbb{N}} (\mathrm{pt}/\Sigma_n)$ where Σ_n is the symmetric group on n elements. Then $\pi_0(\mathbb{F}) = (\mathbb{N}, +)$ and $\mathbb{F}^0 = \mathrm{pt}$.

Example 2.1.3.5. A discrete commutative monoid is sharp (as an $\mathbb{E}_\infty$-monoid) if and only if it is sharp in the classical sense.

Example 2.1.3.6. Let N and M be $\mathbb{E}_\infty$-monoids, and suppose M is sharp. Then $\mathrm{Hom}_{\mathsf{CMon}}(\mathsf{N}, \mathsf{M})$ is sharp (when considered as an $\mathbb{E}_\infty$-monoid with pointwise addition). Indeed, because a homomorphism $\phi : \mathsf{N} \to \mathsf{M}$ has an additive inverse if and only if all values $\phi(n)$ have additive inverses, we have

$$\mathrm{Hom}_{\mathsf{CMon}}(\mathsf{N}, \mathsf{M})^{\pm} = \mathrm{Hom}_{\mathsf{CMon}}(\mathsf{N}, \mathsf{M}^{\pm}) = \mathrm{Hom}_{\mathsf{CMon}}(\mathsf{N}, 0) = 0.$$

Any $\mathbb{E}_\infty$-monoid may be understood as an extension of a sharp $\mathbb{E}_\infty$-monoid by an $\mathbb{E}_\infty$-group. In fact:

Proposition 2.1.3.7. *The inclusion* $\mathsf{CMon}^\sharp \hookrightarrow \mathsf{CMon}$ *admits a left adjoint given by* $\mathsf{M} \mapsto \mathsf{M}^\sharp := \mathsf{M}/\mathsf{M}^\pm$*.*

Proof. Let M be an $\mathbb{E}_\infty$-monoid. If N is any sharp $\mathbb{E}_\infty$-monoid, then

$$\begin{aligned}\mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}/\mathsf{M}^\pm, \mathsf{N}) &= \mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}, \mathsf{N}) \times_{\mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}^\pm, \mathsf{N})} 0 \\ &= \mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}, \mathsf{N}) \times_0 0 \text{ because } \mathsf{N} \text{ is sharp} \\ &= \mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}, \mathsf{N}).\end{aligned}$$

Thus it suffices to show that $\mathsf{M}/\mathsf{M}^\pm$ is sharp.

We check that the conditions of Proposition 2.1.3.3(3) hold. First observe that the discrete monoid

$$\pi_0(\mathsf{M}/\mathsf{M}^\pm) = \pi_0(\mathsf{M})/\pi_0(\mathsf{M}^\pm) = \pi_0(\mathsf{M})/\pi_0(\mathsf{M})^\pm$$

is sharp. Furthermore, the components of M which map to the identity component of $\mathsf{M}^\pm$ are precisely the components of $\mathsf{M}^\pm$. Thus $(\mathsf{M}/\mathsf{M}^\pm)^0 = \mathsf{M}^\pm/\mathsf{M}^\pm = 0$. □

It follows from Proposition 2.1.3.7 that the limit of a diagram of sharp $\mathbb{E}_\infty$-monoids is sharp. The analogous statement for colimits fails (since $\mathbb{F}^{\mathrm{gp}}$ may be viewed as a colimit of sharp $\mathbb{E}_\infty$-monoids as in Example 2.1.1.7). However, we do have the following:

Proposition 2.1.3.8. *A filtered colimit of sharp* $\mathbb{E}_\infty$*-monoids is sharp. In particular, the inclusion* $\mathsf{CMon}^\sharp \hookrightarrow \mathsf{CMon}$ *preserves filtered colimits.*

Proof. Suppose $\{\mathsf{M}_i\}_{i\in I}$ is a filtered system of sharp $\mathbb{E}_\infty$-monoids. The functor $(-)^\pm$: $\mathsf{CMon} \to \mathsf{CMon}$ is corepresented by $\mathbb{F}^{\mathrm{gp}}$, which is compact by Example 2.1.1.7. Thus

$$(\operatorname*{colim}_i \mathsf{M}_i)^\pm = \operatorname*{colim}_i \mathsf{M}_i^\pm = \operatorname*{colim}_i 0 = 0.$$ □

Corollary 2.1.3.9. *The functor* $(-)^\sharp : \mathsf{CMon} \to \mathsf{CMon}^\sharp$ *exhibits* $\mathsf{CMon}^\sharp$ *as an* ω*-accessible localization of* CMon*. In particular,* $\mathsf{CMon}^\sharp$ *is compactly generated.*

2.1.4 Faces, localizations, and quotients of $\mathbb{E}_\infty$-monoids

One way to obtain sharp $\mathbb{E}_\infty$-monoids is by taking quotients by faces of other $\mathbb{E}_\infty$-monoids. Faces also give a natural context for discussing localizations of $\mathbb{E}_\infty$-monoids. Our definition of a *face* closely parallels the classical definition:

Definition 2.1.4.1. Let M be an $\mathbb{E}_\infty$-monoid. A *face* of M is a submonoid $\mathsf{S} \subset \mathsf{M}$ such that, if $m_1, m_2 \in \mathsf{M}$ and $m_1 + m_2 \in \mathsf{S}$, then $m_1, m_2 \in \mathsf{M}$. Let $\mathrm{Face}(\mathsf{M})$ denote the poset of faces of M ordered by inclusion.

Examples of faces abound:

Example 2.1.4.2. If M is an $\mathbb{E}_\infty$-group, then M is the only face of M, and $M/M = \mathrm{pt}$ is sharp.

Example 2.1.4.3. If M is any $\mathbb{E}_\infty$-monoid, then $M^\pm$ is a face of M. The quotient $M/M^\pm$ is sharp by Proposition 2.1.3.7.

Example 2.1.4.4. Let $M = \mathbb{N}^n$, and let e_i for the ith standard basis vector of $\mathbb{N}^n$. The faces of M are exactly the submonoids of $\mathbb{N}^n$ of the form $\mathbb{N}^I = \oplus_{i\in I}\mathbb{N}e_i$ for some $I \subset \{1, \ldots, n\}$. For any such I, the corresponding quotient $\mathbb{N}^n / \oplus_{i\in I} \mathbb{N}e_i$ may be identified with $\mathbb{N}^{\{1,\ldots,n\}\setminus I}$. This quotient is sharp.

Note that $\mathrm{Face}(M) = \mathrm{Face}(\pi_0(M))$, so the classification of faces of $\mathbb{E}_\infty$-monoids may be understood by studying only discrete monoids. In particular, the classification in Example 2.1.4.4 also classifies faces of any $\mathbb{E}_\infty$-monoid M with $\pi_0(M) = \mathbb{N}$ (e.g. $M = \mathbb{F}$).

The above examples (and many more) can all be obtained from the following construction.

Proposition 2.1.4.5. *If* $\phi : M \to N$ *is any homomorphism of* $\mathbb{E}_\infty$*-monoids and* N *is sharp, then* $\mathrm{fib}(\phi)$ *is a face of* M*.*

Proof. By Proposition 2.1.3.3, we have $N^0 = 0$. That is, the natural morphism $0 \to N$ is an inclusion. Because inclusions pull back, the natural morphism $\mathrm{fib}(\phi) \to M$ is an inclusion, and we may identify $\mathrm{fib}(\phi)$ with the submonoid of M consisting of components which are sent to zero by ϕ. If $m_1, m_2 \in M$ satisfy $\phi(m_1 + m_2) = 0$, then $\phi(m_1)$ and $\phi(m_2)$ are both units in N, hence are zero because N is sharp. Thus $m_1, m_2 \in \mathrm{fib}(\phi)$. Since m_1 and m_2 were arbitrary, $\mathrm{fib}(\phi)$ is a face of N. ☐

Let M be an $\mathbb{E}_\infty$-monoid, and let $S \subset M$ be a face. Then the *localization of* M *at* S, written $M[-S]$, is the $\mathbb{E}_\infty$-monoid satisfying the universal property

$$\mathrm{Hom}_{\mathsf{CMon}}(M[-S], N) = \{\phi : M \to N \mid \phi(s) \text{ is a unit for all } s \in S\}.$$

In the case of $\mathbb{E}_\infty$-monoids (as opposed to rings), localizing and quotienting by faces turn out to be closely related. For example, it is easy to see that $M[-S]$ can be explicitly constructed as the cofiber of the diagonal map $S \to M \times S$.

Remark 2.1.4.6. We do not need to restrict to faces at all to define localization: if $\phi : N \to M$ is any homomorphism, we may define the localization $M[-N]$ as the universal $\mathbb{E}_\infty$-monoid with a homomorphism $M \to M[-N]$ inverting all components meeting the image of ϕ. As above, we may compute that

$$M[-N] = \mathrm{cofib}((\phi, \mathrm{id}_N) : N \to M \times N).$$

If S is the smallest face of M containing the image of ϕ. then $M[-N] = M[-S]$, so no generality is lost by considering only localizations at faces.

Moreover, we have:

Proposition 2.1.4.7. *Let* M *be an* $\mathbb{E}_\infty$*-monoid, and let* $\mathsf{S} \subset \mathsf{M}$ *be a face. Then* $\mathsf{M}[-\mathsf{S}]^{\pm} = \mathsf{S}^{\mathrm{gp}}$*, and* $\mathsf{M}[-\mathsf{S}]/\mathsf{S}^{\mathrm{gp}} = \mathsf{M}/\mathsf{S}$*.*

Proof. We may use [Leh24, Theorem 8.4] to identify $\mathsf{M}[-\mathsf{S}]$ with the groupoid completion $(\mathsf{M}\vec{/}\mathsf{S})^{\mathrm{gpd}}$of a certain category $\mathsf{M}\vec{/}\mathsf{S}$,[3] i.e. the groupoid obtained by inverting all morphisms in $\mathsf{M}\vec{/}\mathsf{S}$. Likewise, $\mathsf{S}^{\mathrm{gp}} = \mathsf{S}[-\mathsf{S}]$ may be identified with the groupoid completion of a full subcategory $\mathsf{S}\vec{/}\mathsf{S} \subset \mathsf{M}\vec{/}\mathsf{S}$. By the construction of $\mathsf{M}\vec{/}\mathsf{S}$ and the fact that $\mathsf{S} \subset \mathsf{M}$ is a face, any morphism with domain or codomain in $\mathsf{S}\vec{/}\mathsf{S}$ must in fact lie entirely in $\mathsf{S}\vec{/}\mathsf{S}$. Thus there is a natural inclusion

$$\mathsf{S}^{\mathrm{gp}} = (\mathsf{S}\vec{/}\mathsf{S})^{\mathrm{gpd}} \hookrightarrow (\mathsf{M}\vec{/}\mathsf{S})^{\mathrm{gpd}} = \mathsf{M}[-\mathsf{S}].$$

To see that $\mathsf{S}^{\mathrm{gp}} = (\mathsf{M}[-\mathsf{S}])^{\pm}$, it now suffices to show that $\pi_0(\mathsf{S}^{\mathrm{gp}}) = \pi_0(\mathsf{M}[-\mathsf{S}])^{\pm}$. Any unit of $\mathsf{M}[-\mathsf{S}]$ is a unit in M up to adding an element of S^{gp}, and any unit in M lies in S (because S is a face). Therefore $\pi_0(\mathsf{S}^{\mathrm{gp}}) \subset \pi_0(\mathsf{M}[-\mathsf{S}])^{\pm}$. The reverse inclusion is obvious, so we get $\mathsf{S}^{\mathrm{gp}} = (\mathsf{M}[-\mathsf{S}])^{\pm}$.

To see that $\mathsf{M}[-\mathsf{S}]/\mathsf{S}^{\mathrm{gp}} = \mathsf{M}/\mathsf{S}$, we use the universal property: for any $\mathbb{E}_\infty$-monoid N, we have

$$\begin{aligned}\mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}[-\mathsf{S}]/\mathsf{S}^{\mathrm{gp}}, \mathsf{N}) &= \mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}[-\mathsf{S}], \mathsf{N}) \times_{\mathrm{Hom}(\mathsf{S}^{\mathrm{gp}},\mathsf{N})} 0 \\ &= \mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}, \mathsf{N}) \times_{\mathrm{Hom}(\mathsf{S},\mathsf{N})} 0 \text{ because the zero morphism inverts } \mathsf{S} \\ &= \mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}/\mathsf{S}, \mathsf{N}). \qquad \square\end{aligned}$$

Corollary 2.1.4.8. *Let* M *be an* $\mathbb{E}_\infty$*-monoid, and let* $\mathsf{S} \subset \mathsf{M}$ *be a face. Then* M/S *is sharp.*

Proof. Write $\mathsf{M}/\mathsf{S} = \mathsf{M}[-\mathsf{S}]/(\mathsf{M}[-\mathsf{S}]^{\pm})$ as in Proposition 2.1.4.7, and apply Proposition 2.1.3.7 to see that $\mathsf{M}[-\mathsf{S}]/(\mathsf{M}[-\mathsf{S}]^{\pm})$ is sharp. □

Remark 2.1.4.9. We have shown that any homomorphism of $\mathbb{E}_\infty$-monoids $\phi : \mathsf{M} \to \mathsf{N}$ with N sharp induces a face $\mathrm{fib}(\phi) \subset \mathsf{M}$, and the quotient $\mathsf{M}/\mathrm{fib}(\phi)$ is sharp. However, we typically do not have $\mathsf{N} = \mathsf{M}/\mathrm{fib}(\phi)$. Indeed, this fails for the addition map $+ : \mathbb{N}^2 \to \mathbb{N}$, since $\mathrm{fib}(+) = 0$ but $\mathbb{N}^2/0 \not\simeq \mathbb{N}$.

The above subtlety does not appear when considering a quotient map $\mathsf{M} \to \mathsf{M}/\mathsf{S}$ where S is a face of M. Indeed, quotients by faces behave more like quotients in CGrp than like arbitrary surjective morphisms in CMon, as the following results show.

Proposition 2.1.4.10. *Let* M *be an* $\mathbb{E}_\infty$*-monoid, and let* S *be a face of* M*. Then* $\mathrm{fib}(\mathsf{M} \to \mathsf{M}/\mathsf{S}) = \mathsf{S}$*.*

Proof. By Corollary 2.1.4.8, M/S is sharp. Thus Proposition 2.1.4.5 tells us that $\mathrm{fib}(\mathsf{M} \to \mathsf{M}/\mathsf{S})$ is a face of M. To show that $\mathrm{fib}(\mathsf{M} \to \mathsf{M}/\mathsf{S}) = \mathsf{S}$, it now suffices to show that both sides have the same connected components, which is straightforward. □

[3] We will discuss this construction in much more detail in Section 2.2.

2.1.5 M-spaces and quotients

We would like to better understand the relationship between fiber products and cofibers of homomorphisms of $\mathbb{E}_\infty$-monoids.

Proposition 2.1.5.1. *Let $\phi : \mathsf{N} \to \mathsf{M}$ be a homomorphism of $\mathbb{E}_\infty$-monoids. Then the quotient functor $(-)/\mathsf{N} : \mathcal{S}^{\mathsf{M}} \to \mathcal{S}^{\mathsf{M}/\mathsf{N}}$ preserves fiber products.*

Remark 2.1.5.2. The quotient functor $(-)/\mathsf{N} : \mathcal{S}^{\mathsf{M}} \to \mathcal{S}^{\mathsf{M}/\mathsf{N}}$ does not preserve all finite limits, simply because it fails to preserve final objects. This is easy to fix: we can use Proposition 2.1.5.1 to show that the functor $(-)/\mathsf{N} : \mathcal{S}^{\mathsf{M}} \to (\mathcal{S}^{\mathsf{M}/\mathsf{N}})_{/(\mathrm{pt}/\mathsf{N})}$ preserves finite limits.

Our proof of Proposition 2.1.5.1 will be delayed until after Lemma 2.1.5.6, since we'll first need to discuss a more explicit way to work with quotients of M-spaces.

These quotients can be computed by working in the category

$$\mathsf{s}\mathcal{S} := \mathsf{PSh}_{\mathcal{S}}(\Delta) = \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$$

of *simplicial spaces*. Viewing sets as discrete spaces gives an inclusion $\mathsf{sSet} \hookrightarrow \mathsf{s}\mathcal{S}$, so all of our favorite simplicial sets (simplices Δ^n, horns Λ^n_k, *et cetera*) may be viewed as simplicial spaces.

For any M-space X, the quotient X/M is nothing but the geometric realization of the *action simplicial space*

$$\mathrm{actss}_{\mathsf{M}}(\mathsf{X}) := \mathsf{X} \times \mathsf{M}^\bullet.$$

The action simplicial space construction $\mathrm{actss}_{\mathsf{M}}$ defines a functor $\mathcal{S}^{\mathsf{M}} \to \mathsf{s}\mathcal{S}$. In fact, we can do a little better: since $\mathcal{S}^{\mathsf{M}} = (\mathcal{S}^{\mathsf{M}})_{/\mathrm{pt}}$, $\mathrm{actss}_{\mathsf{M}}$ is well-defined as a functor

$$\mathrm{actss}_{\mathsf{M}} : \mathcal{S}^{\mathsf{M}} = (\mathcal{S}^{\mathsf{M}})_{/\mathrm{pt}} \to (\mathsf{s}\mathcal{S})_{\mathrm{pt}\times\mathsf{M}^\bullet} = \mathsf{s}\mathcal{S}_{/\mathsf{M}^\bullet}. \tag{2.1.5.3}$$

Lemma 2.1.5.4. *The functor $\mathrm{actss}_{\mathsf{M}} : \mathcal{S}^{\mathsf{M}} \to \mathsf{s}\mathcal{S}_{/\mathsf{M}^\bullet}$ admits a left adjoint $\mathsf{L} : \mathsf{s}\mathcal{S}_{/\mathsf{M}^\bullet} \to \mathcal{S}^{\mathsf{M}}$. For any simplex Δ^n (considered together with an arbitrary structure map $\mathsf{f} : \Delta^n \to \mathsf{M}^\bullet$), there is a natural equivalence $\mathsf{L}(\Delta^n) = \mathsf{M}$.*

Proof. We shall show that L exists by writing the functor $\mathrm{actss}_{\mathsf{M}}$ as a composite of right adjoints. The embedding

$$\mathcal{S}^{\mathsf{M}} = \mathsf{Fun}(\mathsf{BM}, \mathcal{S}) \hookrightarrow \mathsf{Fun}(\mathsf{BM}, \mathsf{Cat})$$

admits a left adjoint (given by applying the groupoid completion functor $(-)^{\mathrm{gpd}}$ pointwise). The Grothendieck construction gives a natural equivalence

$$\mathsf{Fun}(\mathsf{BM}, \mathsf{Cat}) = \mathsf{Cat}^{\mathrm{coCart}}_{/\mathsf{BM}},$$

and the inclusion

$$\mathsf{Cat}^{\mathrm{coCart}}_{/\mathsf{BM}} \to \mathsf{Cat}_{/\mathsf{BM}}$$

admits a left adjoint by the dual of [GHN17, Theorem 4.5]. By viewing ∞-categories as complete Segal spaces (as in [Lur09, §1.2]), we get a fully faithful embedding $\mathsf{Cat} \hookrightarrow \mathsf{s}\mathcal{S}$ which admits a left adjoint. Because $\mathsf{Cat} \hookrightarrow \mathsf{s}\mathcal{S}$ sends BM to $\mathsf{M}^\bullet$, we actually obtain a functor

$$\mathsf{Cat}_{/\mathsf{BM}} \hookrightarrow \mathsf{s}\mathcal{S}_{/\mathsf{M}^\bullet}$$

which admits a left adjoint. Chasing the definitions shows that the composite of all of these functors is $\mathrm{actss}_\mathsf{M} : \mathcal{S}^\mathsf{M} \to \mathsf{s}\mathcal{S}_{/\mathsf{M}^\bullet}$, so $\mathrm{actss}_\mathsf{M}$ has a left adjoint L.

Now let $\mathsf{f} : \Delta^n \to \mathsf{M}^\bullet$ be any map (corresponding to $(m_1, \dots, m_n) \in \mathsf{M}^n$), and let $\mathsf{X} \in \mathcal{S}^\mathsf{M}$ be arbitrary. Then we have

$$\begin{aligned}
\mathrm{Map}_{\mathcal{S}^\mathsf{M}}(\mathsf{L}(\Delta^n), \mathsf{X}) &= \mathrm{Map}_{\mathsf{s}\mathcal{S}_{/\mathsf{M}^\bullet}}(\Delta^n, \mathsf{X} \times \mathsf{M}^\bullet) \\
&= \mathrm{Map}_{\mathsf{s}\mathcal{S}}(\Delta^n, \mathsf{X} \times \mathsf{M}^\bullet) \times_{\pi_2 \circ -, \mathrm{Map}_{\mathsf{s}\mathcal{S}}(\Delta^n, \mathsf{M}^\bullet), \mathsf{f}} \mathrm{pt} \\
&= (\mathsf{X} \times \mathsf{M}^n) \times_{\pi_2 \circ -, \mathsf{M}^n, (m_1, \dots, m_n)} \mathrm{pt} \\
&= \mathsf{X} \\
&= \mathrm{Map}_{\mathcal{S}^\mathsf{M}}(\mathsf{M}, \mathsf{X}),
\end{aligned}$$

so $\mathsf{L}(\Delta^n) = \mathsf{M}$ by the Yoneda lemma. □

Remark 2.1.5.5. Lemma 2.1.5.4 fails when we view $\mathrm{actss}_\mathsf{M}$ as a functor $\mathcal{S}^\mathsf{M} \to \mathsf{s}\mathcal{S}$. Indeed, because $\mathrm{actss}_\mathsf{M}(\mathrm{pt}) = \mathsf{M}^\bullet$ is not final in $\mathsf{s}\mathcal{S}$, the functor $\mathrm{actss}_\mathsf{M} : \mathcal{S}^\mathsf{M} \to \mathsf{s}\mathcal{S}$ does not preserve final objects, hence cannot be a right adjoint.

As shown in [Maz25], $\mathsf{s}\mathcal{S}$ admits a "Kan–Quillen" model structure (behaving much like the Kan–Quillen model structure on sSet). We will not need to use the full strength of Mazel-Gee's ∞-categorical theory of model categories here, but we will need the following notion from [Maz25, Definition 4.1]: a map $\mathsf{f} : \mathsf{X}_\bullet \to \mathsf{Y}_\bullet$ in $\mathsf{s}\mathcal{S}$ is a *Kan fibration* if, for all horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ and all commutative squares

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d] & \mathsf{X}_\bullet \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathsf{Y}_\bullet,
\end{tikzcd}$$

there exists a (typically non-unique) lift $\Delta^n \to \mathsf{X}_\bullet$ making the diagram commute.

Lemma 2.1.5.6. *Let* M *be an* $\mathbb{E}_\infty$*-monoid, and let* $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ *be a morphism in* $\mathcal{S}^\mathsf{M}$. *Then the induced map* $\mathrm{actss}_\mathsf{M}(\mathsf{f}) : \mathrm{actss}_\mathsf{M}(\mathsf{X}) \to \mathrm{actss}_\mathsf{M}(\mathsf{Y})$ *is a Kan fibration in* $\mathsf{s}\mathcal{S}$.

Proof. Suppose we are given a commutative diagram

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d] & \mathrm{actss}_\mathsf{M}(\mathsf{X}) \arrow[d] \\
\Delta^n \arrow[r] & \mathrm{actss}_\mathsf{M}(\mathsf{Y})
\end{tikzcd}$$

in $\mathsf{s}\mathcal{S}$. We need to construct a lift $\Delta^n \to \mathrm{actss}_M(X)$.

Because $\mathrm{actss}_M(X) \to \mathrm{actss}_M(Y)$ is a morphism in $\mathsf{s}\mathcal{S}_{/M^\bullet}$, we may view this commutative diagram as a diagram in $\mathsf{s}\mathcal{S}_{/M^\bullet}$. This allows us to use Lemma 2.1.5.4 to transform this commutative diagram into a diagram

$$\begin{tikzcd} L(\Lambda^n_k) \arrow[r] \arrow[d] & X \arrow[d] \\ L(\Delta^n) \arrow[r] & Y \end{tikzcd}$$

in $\mathcal{S}^M$ (where L is as in Lemma 2.1.5.4). We know that $L(\Delta^n) = M$. Furthermore, Λ^n_k can be written as an iterated pushout of simplices $\Delta^{n'}$, so $L(\Lambda^n_k)$ can be written as an iterated pushout of copies of $L(\Delta^{n'}) = M$. All transition maps in this presentation of $L(\Lambda^n_k)$ are equivalences, so we see that $L(\Lambda^n_k) = M$, and the natural map $L(\Lambda^n_k) \to L(\Delta^n)$ agrees with id_M.

Because the map $L(\Lambda^n_k) \to L(\Delta^n)$ is invertible, we get a lift $L(\Delta^n) \to X$ defined as the composite $L(\Delta^n) \xrightarrow{\sim} L(\Lambda^n_k) \to X$. Using the adjunction of Lemma 2.1.5.4, we obtain the needed lift $\Delta^n \to \mathrm{actss}_M(X)$. □

Proof of Proposition 2.1.5.1. Suppose we are given maps $p : X \to Z$ and $q : Y \to Z$ in $\mathcal{S}^M$. We need to show that the natural map

$$(X \times_Z Y)/N \to X/N \times_{Z/N} Y/N.$$

is an equivalence. It suffices to check this at the level of underlying spaces, where we can write $X/N = |\mathrm{actss}_N(X)|$ (and likewise for the other spaces in question). Thus we are studying the natural map

$$|\mathrm{actss}_N(X \times_Z Y)| \to |\mathrm{actss}_N(X)| \times_{|\mathrm{actss}_N(Z)|} |\mathrm{actss}_N(Y)|.$$

This is an equivalence by [Maz25, Corollary 6.7], which applies because $\mathrm{actss}_N(X) \to \mathrm{actss}_N(Y)$ is a Kan fibration by Lemma 2.1.5.6. □

We now discuss some key applications of Proposition 2.1.5.1 to the interplay of fiber products, quotients, and localizations in CMon.

Corollary 2.1.5.7. *Let $\phi : M_1 \to M_3$ and $\psi : M_2 \to M_3$ be homomorphisms of $\mathbb{E}_\infty$-monoids.*

1. *If $\theta : N \to M_1 \times_{M_3} M_2$ is any homomorphism of $\mathbb{E}_\infty$-monoids, then*

$$(M_1 \times_{M_3} M_2)/N = M_1/N \times_{M_3/N} M_2/N$$

as $\mathbb{E}_\infty$-monoids.

2. *If* $S \subset M_1 \times_{M_3} M_2$ *is any face, then*

$$(M_1 \times_{M_3} M_2)[-S] = M_1[-S] \times_{M_3[-S]} M_2[-S]$$

as $\mathbb{E}_\infty$*-monoids.*

Proof. 1. As usual, there is a natural homomorphism of $\mathbb{E}_\infty$-monoids $(M_1 \times_{M_3} M_2)/N \to M_1/N \times_{M_3/N} M_2/N$, and to check that this is an equivalence, it suffices to work at the level of underlying spaces. The natural homomorphisms $M_1 \times_{M_3} M_2 \to M_i$ (for $i = 1, 2, 3$) allow us to view each M_i as an object of $\mathcal{S}^{M_1 \times_{M_3} M_2}$. The result then follows by applying Proposition 2.1.5.1 to the quotient map $(-)/N : \mathcal{S}^{M_1 \times_{M_3} M_2} \to \mathcal{S}^{(M_1 \times_{M_3} M_2)/N}$.

2. We may write

$$\begin{aligned}
(M_1 \times_{M_3} M_2)[-S] &= \big((M_1 \times_{M_3} M_2) \times S\big)/S \\
&= \big((M_1 \times S) \times_{(M_3 \times S)} (M_2 \times S)\big)/S \\
&= (M_1 \times S)/S \times_{(M_3 \times S)/S} (M_2 \times S)/S \text{ by part (1)} \\
&= M_1[-S] \times_{M_3[-S]} M_2[-S].
\end{aligned}$$

□

Corollary 2.1.5.8 (Quotient maps pull back)**.** *Let* M*,* M'*, and* N *be* $\mathbb{E}_\infty$*-monoids, and let* $\phi : N \to M$ *and* $\psi : M' \to M/_\phi N$ *be homomorphisms of* $\mathbb{E}_\infty$*-monoids. Then there is a natural equivalence*

$$(M \times_{M/N} M')/N := \mathrm{cofib}\big((\phi, 0) : N \to (M \times_{M/N} M')\big) \xrightarrow{\sim} M'.$$

Proof. The action of N on M/N and M' is trivial, so we have $(M/N)/N = (M/N) \times \mathrm{pt}/N$ and $M'/N = M' \times \mathrm{pt}/N$. Thus

$$\begin{aligned}
(M \times_{M/N} M')/N &= M/N \times_{M/N \times \mathrm{pt}/N} (M' \times \mathrm{pt}/N) \text{ by Corollary 2.1.5.7(1).} \\
&= M/N \times_{M/N} M' \\
&= M'.
\end{aligned}$$

□

2.1.6 Integral $\mathbb{E}_\infty$-monoids

The following class of $\mathbb{E}_\infty$-monoids does not have the same theoretical importance as sharp $\mathbb{E}_\infty$-monoids, but it is often still useful to consider (as it contains many key examples).

Definition 2.1.6.1. An $\mathbb{E}_\infty$-monoid M is *integral* if the natural map $M \to M^{\mathrm{gp}}$ is an inclusion. Write $\mathsf{CMon}^{\mathrm{int}}$ for the full subcategory of CMon spanned by the integral $\mathbb{E}_\infty$-monoids.

Proposition 2.1.6.2. *For an* $\mathbb{E}_\infty$*-monoid* M*, the following are equivalent:*

1. M *is integral.*

2. *M is a union of connected components of some $\mathbb{E}_\infty$-group A.*

3. *Every connected component of M is an M^0-torsor and $\pi_0(M)$ satisfies the cancellation law: if $a, b, c \in \pi_0(M)$ and $a + b = a + c$, then $b = c$.*

Proof. ($1 \Rightarrow 2$). Obvious (take $A = M^{\mathrm{gp}}$).

($2 \Rightarrow 3$). For the first claim, we may embed M into A and reduce to the case where M is a group. The claim then follows from the group completion theorem: in a loop space ΩX, every component is a torsor for $(\Omega X)^0$.

For the second claim, we note that $\pi_0(M) \subset \pi_0(A)$, i.e. $\pi_0(M)$ is a submonoid of a discrete abelian group. The cancellation law then follows from classical algebra.

($3 \Rightarrow 1$). Because $\pi_0(M)$ satisfies the cancellation law, the map $\pi_0(M) \to \pi_0(M)^{\mathrm{gp}} = \pi_0(M^{\mathrm{gp}})$ is injective. Every component of M^{gp} is an $(M^{\mathrm{gp}})^0$-torsor, so to show that $M \to M^{\mathrm{gp}}$ is an inclusion, it suffices to show that the natural map $M^0 \to (M^{\mathrm{gp}})^0$ is an equivalence. Because M^0 and $(M^{\mathrm{gp}})^0$ are connected, it in fact suffices to show that the natural map $\Omega_0 M \to \Omega_0(M^{\mathrm{gp}})$ is an equivalence.

By [Leh24, Theorem 1.2], we may write

$$\Omega_0(M^{\mathrm{gp}}) = \operatorname*{colim}_{m \in M\vec{/}M} \Omega_m M.$$

where $M\vec{/}M$ is the *action category* of M (cf. [Leh24, Definition 3.1]).[4] By [Leh24, Lemma 3.2], for $m_1, m_2 \in M\vec{/}M$, we have a Cartesian square

$$\begin{tikzcd} \mathrm{Map}_{M\vec{/}M}(m_1, m_2) \arrow[r] \arrow[d] & M \arrow[d, "\cdot m_1"] \\ \mathrm{pt} \arrow[r, "m_2"] & M \end{tikzcd}$$

For $i \in \{1, 2\}$, let M_i be the connected component of M containing M_i. By the cancellation law, there is at most one connected component $M_3 \subset M$ such that $(-) + m_1$ maps M_3 to M_2. If such an M_3 exists, then $(-) + m_1 : M_3 \to M_2$ is a morphism of M^0-torsors, and therefore an equivalence. Thus

$$\mathrm{Map}_{M\vec{/}M}(m_1, m_2) = \begin{cases} \mathrm{pt} & \text{there exists } m_3 \text{ such that } m_2 = m_3 + m_1 \\ \varnothing & \text{otherwise.} \end{cases}$$

That is, $M\vec{/}M$ is a poset. Furthermore, the induced maps $\Omega_{m_1} M \to \Omega_{m_2} M$ are all equivalences (as if $m_2 = m_3 + m_1$, then $m_3 + (-) : M_1 \to M_2$ is a morphism of M^0-torsors, hence an equivalence). It follows that

$$\Omega_0 M = \operatorname*{colim}_{m \in M\vec{/}M} \Omega_m M = \Omega_1 M^{\mathrm{gp}}.$$

□

[4] The reason for the notation $M\vec{/}M$ will become clear in Section 2.2.

Here are a few relevant examples:

Example 2.1.6.3. A discrete commutative monoid is integral (as an $\mathbb{E}_\infty$-monoid) if and only if it is integral in the classical sense (see [Ogu18, §1.2]). In particular, $\mathbb{N}$ is integral.

Example 2.1.6.4. Any $\mathbb{E}_\infty$-group is integral.

Remark 2.1.6.5. Every sharp integral monoid M is discrete. Indeed, every connected component of M is a torsor for $\mathsf{M}^0 = \mathrm{pt}$.

Proposition 2.1.6.6. *The inclusion* $\mathsf{CMon}^{\mathrm{int}} \hookrightarrow \mathsf{CMon}$ *admits a left adjoint* $\mathsf{M} \mapsto \mathsf{M}^{\mathrm{int}}$, *where* $\mathsf{M}^{\mathrm{int}}$ *is the union of the connected components of* M^{gp} *meeting the image of* M *in* M^{gp}.

Proof. It is clear from Proposition 2.1.6.2(2) that $\mathsf{M}^{\mathrm{int}}$ is integral. Furthermore, if N is integral, then the inclusion $\mathsf{N} \subset \mathsf{N}^{\mathrm{gp}}$ gives

$$\mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}, \mathsf{N}) \subset \mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}, \mathsf{N}^{\mathrm{gp}}) = \mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}^{\mathrm{gp}}, \mathsf{N}^{\mathrm{gp}}),$$

where $\mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}, \mathsf{N})$ is identified with the submonoid of $\mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}^{\mathrm{gp}}, \mathsf{N}^{\mathrm{gp}})$ consisting of maps which send $\mathsf{M}^{\mathrm{int}} \subset \mathsf{M}^{\mathrm{gp}}$ to $\mathsf{N} \subset \mathsf{N}^{\mathrm{gp}}$. But this submonoid is exactly $\mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}^{\mathrm{int}}, \mathsf{N})$, so we have $\mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}, \mathsf{N}) = \mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}^{\mathrm{int}}, \mathsf{N})$ as needed. $\square$

It follows from Proposition 2.1.6.6 that the limit of a diagram of integral $\mathbb{E}_\infty$-monoids is integral. The analogous statement for colimits fails.

2.2 Skew quotients

The following construction gives a key method of producing symmetric monoidal categories from $\mathbb{E}_\infty$-monoids.

Definition 2.2.0.1. Let $\phi : \mathsf{N} \to \mathsf{M}$ be a homomorphism of $\mathbb{E}_\infty$-monoids. The *skew quotient* $\mathsf{M}\vec{/}_\phi\mathsf{N}$ is the small SMC characterized by the following universal property: for every small SMC $\mathcal{C}$, there is a natural equivalence

$$\mathsf{Fun}^{\oplus}(\mathsf{M}\vec{/}_\phi\mathsf{N}, \mathcal{C}) = \mathsf{Fun}^{\oplus}(\mathsf{N}, \mathcal{C}_{0/}) \times_{\mathsf{Fun}^{\oplus}(\mathsf{N},\mathcal{C})} \mathsf{Fun}^{\oplus}(\mathsf{M}, \mathcal{C}).$$

We typically omit ϕ from the notation, writing $\mathsf{M}\vec{/}\mathsf{N} = \mathsf{M}\vec{/}_\phi\mathsf{N}$.

Example 2.2.0.2. For any $\mathbb{E}_\infty$-monoid N, the skew quotient $0\vec{/}\mathsf{N}$ is equivalent to the symmetric monoidal category BN. Indeed, for any symmetric monoidal category $(\mathcal{C}, \oplus)$, we have

$$\begin{aligned}\mathsf{Fun}^{\oplus}(0\vec{/}\mathsf{N}, \mathcal{C}) &= \mathsf{Fun}^{\oplus}(\mathsf{N}, \mathcal{C}_{0/}) \times_{\mathsf{Fun}^{\oplus}(\mathsf{N},\mathcal{C})} \mathsf{Fun}^{\oplus}(0, \mathcal{C}) \\ &= \mathsf{Fun}^{\oplus}(\mathsf{N}, \mathcal{C}_{0/}) \times_{\mathsf{Fun}^{\oplus}(\mathsf{N},\mathcal{C})} 0 \text{ because } 0 \text{ is initial in } \mathsf{CAlg}(\mathsf{Cat}) \\ &= \mathsf{Fun}^{\oplus}(\mathsf{N}, \mathcal{C}_{0/} \times_{\mathcal{C}} 0) \\ &= \mathrm{Hom}_{\mathsf{CMon}}(\mathsf{N}, {}_{0}(\mathcal{C})) \\ &= \mathsf{Fun}^{\oplus}(\mathsf{BN}, \mathcal{C}),\end{aligned}$$

so the result follows from the Yoneda lemma.

From an $(\infty, 2)$-categorical perspective, $\mathsf{M}\vec{/}\mathsf{N}$ fits into a universal lax pushout square in $\mathsf{CAlg}(\mathsf{Cat})$:

$$\begin{tikzcd} \mathsf{N} \arrow[r] \arrow[d] & \mathsf{M} \arrow[d] \\ 0 \arrow[r] \arrow[ur, Rightarrow] & \mathsf{M}\vec{/}\mathsf{N} \end{tikzcd}$$

That is, for any diagram of $\mathbb{E}_\infty$-categories

$$\begin{tikzcd} \mathsf{N} \arrow[r] \arrow[d] & \mathsf{M} \arrow[d] \\ 0 \arrow[r] \arrow[ur, Rightarrow] & \mathcal{C} \end{tikzcd} \tag{2.2.0.3}$$

there is an induced map $\mathsf{M}\vec{/}\mathsf{N} \to \mathcal{C}$ such that the outer square of the diagram

$$\begin{tikzcd} \mathsf{N} \arrow[r] \arrow[d] & \mathsf{M} \arrow[d] \arrow[dr] & \\ 0 \arrow[r] \arrow[ur, Rightarrow] & \mathsf{M}\vec{/}\mathsf{N} \arrow[r] & \mathcal{C} \end{tikzcd}$$

agrees with (2.2.0.3) above.

2.2.1 Skew quotients via the Grothendieck construction

Using cocompleteness of $\mathsf{CAlg}(\mathsf{Cat})$, it is possible to show skew quotients $\mathsf{M}\vec{/}\mathsf{N}$ always exist. However, it is difficult to to understand the behavior of $\mathsf{M}\vec{/}\mathsf{N}$ using only the universal property and this abstract existence result. Instead, we offer a more explicit construction of skew quotients using the Grothendieck construction (following the approach of [Leh24, §8]). This will give us more information about the objects and morphisms of $\mathsf{M}\vec{/}\mathsf{N}$.

By [Leh24, Proposition 3.11], for an $\mathbb{E}_\infty$-monoid N, there is an equivalence

$$\mathsf{CMon}_{\mathsf{N}/} \simeq \mathsf{CAlg}(\mathsf{Fun}(\mathsf{BN}, \mathcal{S})).$$

This equivalence sends $\phi : \mathsf{N} \to \mathsf{M}$ to the map $\mathsf{F}_\phi : \mathsf{BN} \to \mathcal{S}$ given as follows:

- Let $*$ be the unique object of BN. Then $\mathsf{F}_\phi(*) = \mathsf{M}$.
- For $n \in \mathsf{N} = \mathrm{Map}_{\mathsf{BN}}(*,*)$, $\mathsf{F}_\phi(n) : \mathsf{M} \to \mathsf{M}$ is multiplication by $\phi(n)$.

The inclusion $(\mathcal{S}, \times) \hookrightarrow (\mathsf{Cat}, \times)$ induces a symmetric monoidal functor $(\mathsf{Fun}(\mathsf{BN}, \mathcal{S}), \star) \hookrightarrow (\mathsf{Fun}(\mathsf{BN}, \mathsf{Cat}), \star)$. The Grothendieck construction (also called "unstraightening") gives a symmetric monoidal equivalence

$$(\mathsf{Fun}(\mathsf{BN}, \mathsf{Cat}), \star) \simeq (\mathsf{Cat}^{\mathrm{coCart}}_{/\mathsf{BN}}, \times_{\mathsf{BN}})$$

where $\mathsf{Cat}^{\mathrm{coCart}}_{/\mathsf{BN}}$ is the category of coCartesian fibrations over BN and functors which preserve coCartesian morphisms. (See [HTT, §3.2] for the standard reference, [Maz15] for a model-independent discussion, and [Ram22, Theorem A] for discussion of the symmetric monoidal structure on the equivalence.) The forgetful functor $(\mathsf{Cat}^{\mathrm{coCart}}_{/\mathsf{BN}}, \times_{\mathsf{BN}}) \to (\mathsf{Cat}, \times)$ (given by $(\mathcal{E} \to \mathsf{BN}) \mapsto \mathcal{E}$) is lax symmetric monoidal, where the lax $\mathbb{E}_\infty$-structure on the functor corresponds to the maps

$$\mathcal{E}_1 \times_{\mathsf{BN}} \mathcal{E}_2 \to \mathcal{E}_1 \times \mathcal{E}_2$$

induced by the map $\mathsf{BN} \to \mathrm{pt}$.Combining all of these together gives a lax symmetric monoidal functor

$$(\mathsf{Fun}(\mathsf{BN}, \mathcal{S}), \times) \to (\mathsf{Cat}^{\mathrm{coCart}}_{/\mathsf{BN}}, \times \mathsf{BN}) \to (\mathsf{Cat}, \times)$$

and thus a functor

$$(-)\vec{/}'\mathsf{N} : \mathsf{CMon}_{\mathsf{N}/} \simeq \mathsf{CAlg}(\mathsf{Fun}(\mathsf{BN}, \mathcal{S})) \to \mathsf{CAlg}(\mathsf{Cat}) \tag{2.2.1.1}$$

The following is a refinement of [Leh24, Theorem 8.4]:

Proposition 2.2.1.2. *The functor* $(-)\vec{/}'\mathsf{N}$ *of* (2.2.1.1) *is equivalent to* $(-)\vec{/}\mathsf{N}$ *as functors* $\mathsf{CMon}_{\mathsf{N}/} \to \mathsf{CAlg}(\mathsf{Cat})$.

Proof. Both are computed by the same weighted colimit (by definition for $(-)\vec{/}\mathsf{N}$, and by [GHN17, Theorem 1.1] for $(-)\vec{/}'\mathsf{N}$). □

As a consequence, we may use the results of [Leh24, §8] to compute $\mathsf{M}\vec{/}\mathsf{N}$ as a *limit* of familiar $\mathbb{E}_\infty$-categories.

Corollary 2.2.1.3. *Let* $\phi : \mathsf{N} \to \mathsf{M}$ *be a homomorphism of* $\mathbb{E}_\infty$*-monoids. Then*

$$\mathsf{M}\vec{/}\mathsf{N} = (\mathsf{BM})_{0/} \times_{\mathsf{BM}} \mathsf{BN}.$$

Proof. The equivalence $\mathsf{M}\vec{/}'\mathsf{N} \simeq (\mathsf{BM})_{0/} \times_{\mathsf{BM}} \mathsf{BN}$ follows from [Leh24, Definition 8.1] and the discussion immediately following it. Now use $\mathsf{M}\vec{/}\mathsf{N} = \mathsf{M}\vec{/}'\mathsf{N}$. □

Example 2.2.1.4. If $\phi : \mathsf{M} \to \mathsf{M}$ is the identity, then Corollary 2.2.1.3 implies $\mathsf{M}\vec{/}\mathsf{M} = (\mathsf{BM})_{0/}$.

Combining Corollary 2.2.1.3 and [Leh24, Lemma 8.2], we obtain the following useful result.

Corollary 2.2.1.5 ([Leh24, Lemma 8.2])**.** *Let $\phi : N \to M$ be a homomorphism of $\mathbb{E}_\infty$-monoids. For $m_1, m_2 \in M$ (with corresponding objects $[m_1], [m_2] \in M\vec{/}N$), there is a Cartesian square*

$$\begin{tikzcd} \mathrm{Map}_{M\vec{/}N}([m_1],[m_2]) \arrow[r] \arrow[d] & N \arrow[d, "\phi(-)+m_1"] \\ \mathrm{pt} \arrow[r, "m_2"] & M. \end{tikzcd}$$

In particular, for all $m \in M$, we have $\mathrm{Map}_{M\vec{/}N}(0_{M\vec{/}N}, m) = \phi^{-1}(m)$.

It is not hard to see that the natural map $M \to M\vec{/}N$ is essentially surjective.[5] Thus Corollary 2.2.1.5 allows us to describe *all* morphisms in $M\vec{/}N$. This can be quite useful, as we shall see presently.

If $\phi : N \to M$ is a homomorphism of $\mathbb{E}_\infty$-monoids and $m_1, m_2 \in M$, the long exact sequence for a fibration gives a long exact sequence:

$$\ldots \longrightarrow \pi_i(\mathrm{Map}_{M\vec{/}N}([m_1],[m_2])) \longrightarrow \pi_i(N) \longrightarrow \pi_i(M) \longrightarrow \ldots \longrightarrow \pi_0(M). \tag{2.2.1.6}$$

Proposition 2.2.1.7. *Let $\phi : N \to M$ be a homomorphism of $\mathbb{E}_\infty$-monoids. If N is discrete and $\pi_i(M)$ is concentrated in degrees 0 and 1, then $M\vec{/}N$ is a discrete category.*

Proof. Every object of $M\vec{/}N$ is of the form $[m]$ for some $m \in M$, so it suffices to show that all mapping spaces $\mathrm{Map}_{M\vec{/}N}([m_1],[m_2])$ are discrete. Fix $m_1, m_2 \in M$, and consider the sequence

$$\ldots \longrightarrow \pi_{i+1}(M) \longrightarrow \pi_i(\mathrm{Map}_{M\vec{/}N}([m_1],[m_2])) \longrightarrow \pi_i(N) \longrightarrow \pi_i(M) \longrightarrow \ldots$$

contained within the long exact sequence Eq. (2.2.1.6). For all $i \geqslant 1$, we have $\pi_i(N) = 0$ and $\pi_{i+1}(M) = 0$, so $\pi_i(\mathrm{Map}_{M\vec{/}N}([m_1],[m_2])) = 0$. Since this holds for all $i \geqslant 1$, $\mathrm{Map}_{M\vec{/}N}([m_1],[m_2])$ is discrete. □

2.2.2 Path monoids

The following construction will allow us to understand skew quotients entirely in terms of an $(\infty,1)$-categorical universal property.

[5] For an explicit proof of a more general claim, see Proposition 2.2.4.1(1).

Definition 2.2.2.1. Let $\mathcal{C}$ be a small SMC. The *(absolute) path monoid* of $\mathcal{C}$, written $\mathrm{P}_0\mathcal{C}$, is the fiber product appearing in the following Cartesian square of $\mathbb{E}_\infty$-monoids:

$$\begin{tikzcd} \mathrm{P}_0\mathcal{C} \arrow[r] \arrow[d] & \iota_0 \mathsf{Arr}(\mathcal{C}) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ \iota_0\mathcal{C} \arrow[r, "\mathsf{c}\mapsto(0,\mathsf{c})"] & \iota_0\mathcal{C}\times\iota_0\mathcal{C}. \end{tikzcd}$$

where the $\mathbb{E}_\infty$-monoid structure on $\iota_0\,\mathsf{Arr}(\mathcal{C})$ is defined pointwise, and ev_0 (resp. ev_1) sends a morphism to its domain (resp. codomain). More succinctly, we can write $\mathrm{P}_0\mathcal{C} = \iota_0(\mathcal{C}_{0/})$.

At the level of underlying spaces, we have

$$\mathrm{P}_0\mathcal{C} = \operatorname*{colim}_{\mathsf{c}\in\iota_0\mathcal{C}} \mathrm{Map}_{\mathcal{C}}(0,\mathsf{c}) \tag{2.2.2.2}$$

where the automorphism groups $\mathrm{Aut}_{\mathcal{C}}(\mathsf{c})$ act on $\mathrm{Map}_{\mathcal{C}}(0,\mathsf{c})$ by postcomposition.

By construction, the skew quotient $\mathsf{M}\vec{/}_\phi\mathsf{N}$ satisfies

$$\iota_0\,\mathsf{Fun}^{\oplus}(\mathsf{M}\vec{/}_\phi\mathsf{N},\mathcal{C}) = \mathrm{Hom}_{\mathsf{CMon}}(\mathsf{N},\mathrm{P}_0\mathcal{C}) \times_{\mathrm{Hom}_{\mathsf{CMon}}(\mathsf{N},\iota_0\mathcal{C})} \mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M},\iota_0\mathcal{C}). \tag{2.2.2.3}$$

for any SMC $\mathcal{C}$. In other words, the functor $(\mathsf{N}\to\mathsf{M})\mapsto\mathsf{M}\vec{/}\mathsf{N}$ is the left adjoint to the functor

$$\begin{aligned} \mathsf{CAlg}(\mathsf{Cat}) &\to \mathsf{Arr}(\mathsf{CMon}) \\ \mathcal{C} &\mapsto (\mathrm{P}_0\mathcal{C}\to\iota_0\mathcal{C}). \end{aligned}$$

We will eventually show (in Theorem 2.3.2.1) that this adjunction induces an equivalence between certain subcategories of $\mathsf{CAlg}(\mathsf{Cat})$ and $\mathsf{Arr}(\mathsf{CMon})$. To accomplish this, we should first understand what values $\mathrm{P}_0\mathcal{C}$ can potentially take.

Proposition 2.2.2.4. *Let $\mathcal{C}$ be a SMC. Then $\mathrm{P}_0\mathcal{C}$ is sharp.*

Proof. The identity morphism id_0 is the initial object of $\mathcal{C}_{0/}$. Thus, given $\mathsf{f}\in(\mathrm{P}_0\mathcal{C})^{\pm}$, we have $\mathrm{Hom}_{\mathcal{C}_{0/}}(\mathrm{id}_0,\mathsf{f})\simeq\mathrm{pt}$ and

$$\mathrm{Hom}_{\mathcal{C}_{0/}}(\mathsf{f},\mathrm{id}_0)\simeq\mathrm{Hom}_{\mathcal{C}_{0/}}\big(\mathrm{id}_0,-\mathsf{f}\big)\simeq\mathrm{pt}.$$

It follows that $\mathsf{f}\simeq\mathrm{id}_0$ in $\mathcal{C}_{0/}$, and this equivalence is unique up to contractible choices. Therefore $(\mathrm{P}_0\mathcal{C})^{\pm}\simeq\mathrm{pt}$. □

In fact, this is the only obstruction: *every* sharp $\mathbb{E}_\infty$-monoid arises as a path monoid, as we see from the following example.

Example 2.2.2.5. Suppose M is an $\mathbb{E}_\infty$-monoid. Then $\mathrm{P}_0\mathsf{BM} = \mathsf{M}/\mathsf{M}^{\pm}$. Indeed, since BM has only one object 0, $\mathrm{P}_0\mathsf{BM}$ is the quotient of $\mathrm{Map}_{\mathsf{BM}}(0,0)$ by $\mathrm{Aut}_{\mathsf{BM}}(0)$ (acting by postcomposition). The former space is M, while the latter space is $\mathsf{M}^{\pm}$, so $\mathrm{P}_0\mathsf{BM} = \mathsf{M}/\mathsf{M}^{\pm}$. In particular, if M is sharp, we obtain $\mathrm{P}_0\mathsf{BM} = \mathsf{M}$.

Here are some other computations we can do with path monoids:

Example 2.2.2.6. Suppose M is a $\mathbb{E}_\infty$-monoid, viewed as a $\mathbb{E}_\infty$-groupoid (possibly with many objects). Then $\mathrm{P}_0\mathsf{M}$ is just the space of paths at M based at the origin, so $\mathrm{P}_0\mathsf{M} = \mathrm{pt}$ by contractibility of path spaces.

Example 2.2.2.7. Suppose $\phi : \mathsf{N} \to \mathsf{M}$ is a homomorphism of $\mathbb{E}_\infty$-monoids and N is an $\mathbb{E}_\infty$-group. If $\mathcal{C}$ is any SMC, we have $\mathrm{Hom}_{\mathsf{CMon}}(\mathsf{N}, \mathrm{P}_0\mathcal{C}) \simeq \mathrm{pt}$ by Proposition 2.2.2.4. Thus

$$\begin{aligned} \iota_0 \,\mathsf{Fun}^{\oplus}(\mathsf{M}\vec{/}\mathsf{N}, \mathcal{C}) &= \mathrm{pt} \times_{\mathrm{Hom}_{\mathsf{CMon}}(\mathsf{N},\iota_0\mathcal{C})} \mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}, \iota_0\mathcal{C}) \\ &= \mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}/\mathsf{N}, \iota_0\mathcal{C}) \\ &= \iota_0 \,\mathsf{Fun}^{\oplus}(\mathsf{M}/\mathsf{N}, \mathcal{C}), \end{aligned}$$

i.e. $\mathsf{M}\vec{/}\mathsf{N}$ is just the usual quotient M/N in CMon.

2.2.3 Relative path monoids

The following generalization of path monoids will be useful in constructing some presentations of SMCs as skew quotients.

Definition 2.2.3.1. Let $\mathcal{C}$ be a small SMC, let N be an $\mathbb{E}_\infty$-monoid, and let $\psi : \mathsf{N} \to \iota_0\mathcal{C}$ be a homomorphism. The *path monoid of* $\mathcal{C}$ *relative to* ψ, written $\mathrm{P}_{0,\mathsf{N}}\,\mathcal{C}$, is the fiber product appearing in the following Cartesian square:

$$\begin{tikzcd} \mathrm{P}_{0,\mathsf{N}}\,\mathcal{C} \arrow[r] \arrow[d] & \iota_0\,\mathsf{Arr}(\mathcal{C}) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ 0 \times \mathsf{N} \arrow[r, "0\times\psi"] & \iota_0\mathcal{C} \times \iota_0\mathcal{C}. \end{tikzcd}$$

Remark 2.2.3.2. Note that $\mathrm{P}_{0,\iota_0\mathcal{C}}\,\mathcal{C} = \mathrm{P}_0\mathcal{C}$, so this generalizes our original notion of path monoid.

Heuristically, $\mathrm{P}_{0,\mathsf{N}}\,\mathcal{C}$ is the $\mathbb{E}_\infty$-monoid of pairs $(n \in \mathsf{N}, \mathsf{f} : 0 \to \psi(n))$, though some care needs to be taken in defining the morphisms correctly. At the level of underlying spaces, we have

$$\mathrm{P}_{0,\mathsf{N}}\,\mathcal{C} = \operatorname*{colim}_{n\in\mathsf{N}} \mathrm{Map}_{\mathcal{C}}(0, \psi(n)), \tag{2.2.3.3}$$

generalizing Eq. (2.2.2.2).

Relative path monoids typically are not sharp. In fact, *any* $\mathbb{E}_\infty$-monoid can arise as a relative path monoid:

Example 2.2.3.4. Let $\mathcal{C}$ be a small SMC. Then $\mathrm{P}_{0,0}\,\mathcal{C} = \mathrm{End}_{\mathcal{C}}(0)$. In particular, if $\mathcal{C} = \mathsf{BM}$ for some $\mathsf{M} \in \mathsf{CMon}$, then $\mathrm{P}_{0,0}\,\mathcal{C} = \mathsf{M}$.

2.2.4 Properties of skew quotients

Let's list some important skew quotient computations for future reference.

Proposition 2.2.4.1. *Let $\phi : \mathsf{N} \to \mathsf{M}$ be a homomorphism of $\mathbb{E}_\infty$-monoids. Then:*

1. $\iota_0(\mathsf{M}\vec{/}\mathsf{N}) = \mathsf{M}/\mathsf{N}^{\pm}$.
2. ${}_0(\mathsf{M}\vec{/}\mathsf{N}) = \mathrm{pt} \times_{\mathsf{M}} \mathsf{N}$.
3. $(\mathsf{M}\vec{/}\mathsf{N})^{\mathrm{gpd}} = \mathsf{M}/\mathsf{N}$.
4. $\mathrm{cofib}(\mathsf{M} \to \mathsf{M}\vec{/}\mathsf{N}) = \mathsf{BN}$.
5. *If $\phi' : \mathsf{N}' \to \mathsf{M}$ is any homomorphism of $\mathbb{E}_\infty$-monoids, then $\mathrm{P}_{0,\mathsf{N}'}(\mathsf{M}\vec{/}\mathsf{N}) = \mathsf{N}' \times_{\mathsf{M}} \mathsf{N}$. In particular, $\mathrm{P}_{0,\mathsf{M}}\,(\mathsf{M}\vec{/}\mathsf{N}) = \mathsf{N}$.*

Proof. (1). Using Corollary 2.2.1.3, we compute

$$\begin{aligned}
\iota_0(\mathsf{M}\vec{/}\mathsf{N}) &= \iota_0(\mathsf{BM}_{0/}) \times_{\iota_0\mathsf{BM}} \iota_0\mathsf{BN} \text{ because } \iota_0 \text{ preserves limits.}\\
&= \iota_0(\mathsf{BM}_{0/}) \times_{\iota_0\mathsf{BM}} \mathsf{BN}^{\pm}\\
&= \iota_0(\mathsf{M}\vec{/}\mathsf{N}^{\pm})\\
&= \iota_0(\mathsf{M}/\mathsf{N}^{\pm}) \text{ by Example 2.2.2.7.}
\end{aligned}$$

(2). This is the special case of Corollary 2.2.1.5 with $m_1 = m_2 = 0_{\mathsf{M}}$.

(3). This is [Leh24, Theorem 8.4]. One may also prove this directly from the skew quotient definition of $\mathsf{M}\vec{/}\mathsf{N}$: the functor $\mathsf{Arr}(\mathsf{CMon}) \to \mathsf{CMon}$ defined by

$$(\mathsf{M} \to \mathsf{N}) \mapsto (\mathsf{M}\vec{/}\mathsf{N})^{\mathrm{gpd}}$$

is the left adjoint to the functor $\mathsf{CMon} \to \mathsf{Arr}(\mathsf{CMon})$ defined by

$$\mathsf{Q} \mapsto (\mathrm{P}_0\mathsf{Q} \to \mathsf{Q}) = (\mathrm{pt} \to \mathsf{Q}).$$

The left adjoint to this latter functor is the usual quotient $(\mathsf{M} \to \mathsf{N}) \mapsto \mathsf{M}/\mathsf{N}$, so $\mathsf{M}\vec{/}\mathsf{N} = \mathsf{M}/\mathsf{N}$ by uniqueness of left adjoints.

(4). For an SMC $\mathcal{C}$, we have

$$\begin{aligned}
\mathsf{Fun}^{\oplus}(\mathrm{cofib}(\mathsf{M} \to \mathsf{M}\vec{/}\mathsf{N}), \mathcal{C}) &= \mathsf{Fun}^{\oplus}(\mathsf{M}\vec{/}\mathsf{N}, \mathcal{C}) \times_{\mathsf{Fun}^{\oplus}(\mathsf{M},\mathcal{C})} 0\\
&= \mathsf{Fun}^{\oplus}(\mathsf{N}, \mathrm{P}_0\mathcal{C}) \times_{\mathsf{Fun}^{\oplus}(\mathsf{N},\mathcal{C})} \mathsf{Fun}^{\oplus}(\mathsf{M}, \mathcal{C}) \times_{\mathsf{Fun}^{\oplus}(\mathsf{M},\mathcal{C})} 0\\
&= \mathsf{Fun}^{\oplus}(\mathsf{N}, \mathrm{P}_0\mathcal{C}) \times_{\mathsf{Fun}^{\oplus}(\mathsf{N},\mathcal{C})} 0\\
&= \mathsf{Fun}^{\oplus}(\mathrm{pt}\vec{/}\mathsf{N}, \mathcal{C})\\
&= \mathsf{Fun}^{\oplus}(\mathsf{BN}, \mathcal{C}).
\end{aligned}$$

The result follows by Yoneda.

(5). We first construct a homomorphism of $\mathbb{E}_\infty$-monoids $\mathrm{P}_{0,\mathsf{N}'}(\mathsf{M}\vec{/}\mathsf{N}) \to \mathsf{N}' \times_{\mathsf{M}} \mathsf{N}$. There is a natural homomorphism $\mathrm{P}_{0,\mathsf{N}'}(\mathsf{M}\vec{/}\mathsf{N}) \to \mathsf{N}'$ coming from the definition of relative path monoids. We may also obtain a homomorphism $\mathrm{P}_{0,\mathsf{N}'}(\mathsf{M}\vec{/}\mathsf{N}) \to \mathsf{N}$ as a composite

$$\mathrm{P}_{0,\mathsf{N}'}(\mathsf{M}\vec{/}\mathsf{N}) \to \mathrm{P}_{0,\mathsf{N}'}(0\vec{/}\mathsf{N}) \to \mathrm{P}_{0,0}\,\mathsf{BN} = \mathsf{N}.$$

The induced homomorphisms $\mathrm{P}_{0,\mathsf{N}'}(\mathsf{M}\vec{/}\mathsf{N}) \to \mathsf{M}$ both agree (they both send a pair $(n', f : 0 \to \phi'(n'))$ to $\phi(n')$), so we obtain the desired homomorphism $\mathrm{P}_{0,\mathsf{N}'}(\mathsf{M}\vec{/}\mathsf{N}) \to \mathsf{N}' \times_{\mathsf{M}} \mathsf{N}$.

At the level of underlying spaces, we may write

$$\begin{aligned}
\mathrm{P}_{0,\mathsf{N}'}(\mathsf{M}\vec{/}\mathsf{N}) &= \operatorname*{colim}_{n'\in\mathsf{N}'} \mathrm{Map}_{\mathsf{M}\vec{/}\mathsf{N}}(0, n') \text{ by (2.2.2.2)} \\
&= \operatorname*{colim}_{n'\in\mathsf{N}'} \mathrm{pt} \times_{n',\mathsf{M},\phi} \mathsf{N} \text{ by Corollary 2.2.1.5} \\
&= (\operatorname*{colim}_{n'\in\mathsf{N}'} \mathrm{pt}) \times_{\mathsf{M}} \mathsf{N} \\
&= \mathsf{N}' \times_{\mathsf{M}} \mathsf{N}.
\end{aligned}$$

This equivalence of underlying spaces agrees with the homomorphism constructed above. Since homomorphisms which induce equivalences of underlying spaces are themselves equivalences, we see that $\mathrm{P}_{0,\mathsf{N}'}(\mathsf{M}\vec{/}\mathsf{N}) = \mathsf{N}' \times_{\mathsf{M}} \mathsf{N}$ as $\mathbb{E}_\infty$-monoids. □

2.3 Grouplike symmetric monoidal categories

We are now ready to study grouplike SMCs (Definition 2.0.0.1). We should also introduce some notation for the category of grouplike SMCs:

Definition 2.3.0.1. Write $\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})$ for the full subcategory of $\mathsf{CAlg}(\mathsf{Cat})$ spanned by the grouplike SMCs.

Note that the inclusion $\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat}) \hookrightarrow \mathsf{CAlg}(\mathsf{Cat})$ admits a right adjoint, given by sending a small SMC $\mathcal{C}$ to the full subcategory $\mathcal{C}^{\mathrm{gplike}} \subset \mathcal{C}$ spanned by the units in $\iota_0\mathcal{C}$.

2.3.1 Presenting grouplike SMCs as skew quotients

If $\phi : \mathsf{M} \to \mathsf{N}$ is a homomorphism of $\mathbb{E}_\infty$-monoids and N is an $\mathbb{E}_\infty$-group, then $\mathsf{M}\vec{/}\mathsf{N}$ is a grouplike SMC by Proposition 2.2.4.1(1). Our goal is to show that every grouplike SMC arises in this way. In fact, the following result controls skew quotient presentations of grouplike SMCs.

Proposition 2.3.1.1. *Let $\mathcal{C}$ be a grouplike SMC, let M be an $\mathbb{E}_\infty$-group, and let $\phi : \mathsf{M} \to \iota_0\mathcal{C}$ be a homomorphism. If ϕ is essentially surjective, then there is a natural equivalence $\mathsf{F}_\phi : \mathsf{M}\vec{/}\,\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C} \xrightarrow{\sim} \mathcal{C}$.*

Proof. Using Eq. (2.2.2.3), we obtain an equivalence

$$\iota_0 \operatorname{Fun}^{\oplus}(M\vec{/}\mathrm{P}_{0,M}\,\mathcal{C}, \mathcal{C}) = \mathrm{Hom}_{\mathsf{CMon}}(\mathrm{P}_{0,M}\,\mathcal{C}, \mathrm{P}_0\mathcal{C}) \times_{\mathrm{Hom}_{\mathsf{CMon}}(\mathrm{P}_{0,M}\,\mathcal{C}, \iota_0\mathcal{C})} \mathrm{Hom}_{\mathsf{CMon}}(M, \iota_0\mathcal{C}).$$

The desired functor $\mathsf{F}_\phi : M\vec{/}\mathrm{P}_{0,M}\,\mathcal{C} \to \mathcal{C}$ is induced by ϕ and the forgetful homomorphism $\mathrm{P}_{0,M}\,\mathcal{C} \to \mathrm{P}_0\mathcal{C}$. The functor F_ϕ is obviously essentially surjective. For fully faithfulness, we use Lemma 2.3.1.2 below. □

Lemma 2.3.1.2. *Let $\mathcal{C}$ be a small SMC, let M be an $\mathbb{E}_\infty$-group, and let $\phi : M \to \iota_0\mathcal{C}$ be a homomorphism. Then there is a natural fully faithful functor $\mathsf{F}_\phi : M\vec{/}\mathrm{P}_{0,M}\,\mathcal{C} \hookrightarrow \mathcal{C}$.*

Proof. The functor F_ϕ is constructed exactly as in the proof of Proposition 2.3.1.1 above. Because M is grouplike, to show that F_ϕ is fully faithful, it suffices to show that the induced map

$$\mathrm{Map}_{M\vec{/}\mathrm{P}_{0,M}\,\mathcal{C}}(0, [m]) \to \mathrm{Map}_{\mathcal{C}}(0, \phi(m))$$

is an equivalence for all $m \in M$. By Corollary 2.2.1.5 and the Cartesian square defining $\mathrm{P}_{0,m}$, there is a commutative diagram of Cartesian squares

$$\begin{tikzcd}
\mathrm{Map}_{M\vec{/}\mathrm{P}_{0,M}\,\mathcal{C}}(0, [m]) \arrow[r] \arrow[d] & \mathrm{P}_{0,M}\,\mathcal{C} \arrow[r] \arrow[d] & \iota_0 \mathsf{Arr}(\mathcal{C}) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\
\mathrm{pt} \arrow[r, "m"] & M \arrow[r, "0\times\phi"] & \iota_0\mathcal{C} \times \iota_0\mathcal{C}.
\end{tikzcd}$$

Thus the outer square is Cartesian. But the fiber product of the outer square is just $\mathrm{Map}_{\mathcal{C}}(0, \phi(m))$, so we get $\mathrm{Map}_{M\vec{/}\mathrm{P}_{0,M}\,\mathcal{C}}(0, [m]) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{C}}(0, \phi(m))$ and F_ϕ is fully faithful. □

2.3.2 A structure theorem for grouplike SMCs

By restricting Proposition 2.3.1.1 to the case of absolute path monoids, we can prove a useful structure theorem for grouplike SMCs.

Theorem 2.3.2.1 (Structure of grouplike SMCs)**.** *The adjunction*

$$(-)\vec{/}(-) : \mathsf{Arr}(\mathsf{CMon}) \rightleftarrows \mathsf{CAlg}(\mathsf{Cat}) : \big(\mathrm{P}_0(-) \to \iota_0(-)\big)$$

restricts to an equivalence of categories $\mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon}) = \mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})$.

Proof. It is clear that the adjunction in question restricts to an adjunction

$$\mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon}) \rightleftarrows \mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat}).$$

We need to show that this is an equivalence. One direction follows from Proposition 2.3.1.1: if $\mathcal{C}$ is a grouplike SMC, then $\mathcal{C} = \iota_0\mathcal{C}\vec{/}\mathrm{P}_{0,\iota_0\mathcal{C}}\,\mathcal{C} = \iota_0\mathcal{C}\vec{/}\mathrm{P}_0\mathcal{C}$.

For the other direction of the equivalence, suppose we are given $(\phi : \mathsf{N} \to \mathsf{M}) \in \mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon})$. We have $\iota_0(\mathsf{M}\vec{/}\mathsf{N}) = \mathsf{M}/\mathsf{N}^{\pm} = \mathsf{M}$ by Proposition 2.2.4.1(1) and the hypothesis that N is sharp. By Proposition 2.2.4.1(5), we have

$$\mathrm{P}_0(\mathsf{M}\vec{/}\mathsf{N}) = \iota_0(\mathsf{M}\vec{/}\mathsf{N}) \times_{\mathsf{M}} \mathsf{N} = \mathsf{M} \times_{\mathsf{M}} \mathsf{N} = \mathsf{N}.$$

It is clear that the induced morphism $\mathrm{P}_0(\mathsf{M}\vec{/}\mathsf{N}) \to \iota_0(\mathsf{M}\vec{/}\mathsf{N})$ (viewed as an object of $\mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{M}\vec{/}\mathsf{N})$) is naturally equivalent to the original morphism $(\phi : \mathsf{N} \to \mathsf{M})$. □

We can derive many useful results about the category $\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})$ from Theorem 2.3.2.1. Here are just a few:

Corollary 2.3.2.2. *The category* $\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})$ *is presentable and compactly generated.*

Proof. There is a Cartesian square of categories

$$\begin{tikzcd}
\mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon}) \arrow[r] \arrow[d] & \mathsf{Arr}(\mathsf{CMon}) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
\mathsf{CMon}^{\sharp} \times \mathsf{CMon}^{\mathrm{gp}} \arrow[r] & \mathsf{CMon} \times \mathsf{CMon}.
\end{tikzcd}$$

The categories and arrows not involving $\mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon})$ all lie in $\mathsf{Pr}^{\mathsf{R}}_{\omega}$. Thus, by [HTT, 5.5.7.6], $\mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon})$ must also be presentable and compactly generated. Now use the equivalence $\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat}) = \mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon})$ of Theorem 2.3.2.1. □

Corollary 2.3.2.3. *Finite products agree with finite coproducts in* $\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})$*.*

Proof. We may use the equivalence $\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat}) \simeq \mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon})$ and the fact that limits and colimits in diagram categories are computed pointwise to reduce this to the claim that finite products agree with finite coproducts in $\mathsf{CMon}^{\sharp}$ and CGrp. The claim then follows from the fact that finite products agree with finite coproducts in CMon and that finite (co)products of sharp $\mathbb{E}_\infty$-monoids (resp. $\mathbb{E}_\infty$-groups) are sharp $\mathbb{E}_\infty$-monoids (resp. $\mathbb{E}_\infty$-groups). □

Corollary 2.3.2.4. *The functors* $\mathrm{P}_0 : \mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat}) \to \mathsf{CMon}$ *and* $\iota_0 : \mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat}) \to \mathsf{CMon}$ *preserve filtered colimits.*

Proof. Under the equivalence of Theorem 2.3.2.1, we may identify the functors in question with the evaluation maps $\mathrm{ev}_0, \mathrm{ev}_1 : \mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon}) \to \mathsf{CMon}$. The functors $\mathrm{ev}_0 : \mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon}) \to \mathsf{CMon}^{\sharp}$ and $\mathrm{ev}_1 : \mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon}) \to \mathsf{CGrp}$ preserve filtered colimits because colimits in diagram categories are computed pointwise. Thus it remains to show that the inclusions $\mathsf{CMon}^{\sharp} \hookrightarrow \mathsf{CMon}$ and $\mathsf{CGrp} \hookrightarrow \mathsf{CMon}$ preserve filtered colimits. The former claim is Proposition 2.1.3.8, while the latter claim holds because the inclusion $\mathsf{CGrp} \hookrightarrow \mathsf{CMon}$ is a left adjoint. □

Theorem 2.3.2.1 gives a simple procedure for computing limits of grouplike SMCs. Given a diagram $\mathcal{C}_{(-)} : \mathsf{I} \to \mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})$, we replace each $\mathcal{C}_i$ by $(\mathrm{P}_0\mathcal{C}_i \to \iota_0\mathcal{C}_i) \in \mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon})$. Limits of sharp $\mathbb{E}_\infty$-monoids are sharp, and limits of $\mathbb{E}_\infty$-groups are $\mathbb{E}_\infty$-groups, so the inclusion $\mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon}) \hookrightarrow \mathsf{Arr}(\mathsf{CMon})$ preserves limits. Thus we may compute the limit in $\mathsf{Arr}(\mathsf{CMon})$, and we obtain

$$\lim_i \mathcal{C}_i \simeq \left(\lim_i \iota_0\mathcal{C}_i\right) \vec{\Big/} \left(\lim_i \mathrm{P}_0\mathcal{C}_i\right). \tag{2.3.2.5}$$

Example 2.3.2.6. Let $\phi, \psi : \mathbb{N} \to \mathbb{Z}$ be any homomorphisms. We compute

$$\mathsf{B}\mathbb{N} \times_{\phi,\mathsf{B}\mathbb{Z},\psi} \mathsf{B}\mathbb{N}.$$

Under the equivalence of Theorem 2.3.2.1, $\mathsf{B}\mathbb{N}$ corresponds to $(\mathbb{N} \to \mathrm{pt}) \in \mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon})$, while $\mathsf{B}\mathbb{Z}$ corresponds to $(\mathrm{pt} \to \mathsf{B}\mathbb{Z}) \in \mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon})$. Thus $\mathsf{B}\mathbb{N} \times_{\phi,\mathsf{B}\mathbb{Z},\psi} \mathsf{B}\mathbb{N}$ corresponds to

$$(\mathbb{N} \times_{\mathrm{pt}} \mathbb{N} \to \mathrm{pt} \times_{\mathsf{B}\mathbb{Z}} \mathrm{pt}) \simeq \left(\mathbb{N}^2 \xrightarrow{\phi+\psi} \mathbb{Z}\right) \in \mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon}).$$

We conclude that

$$\mathsf{B}\mathbb{N} \times_{\phi,\mathsf{B}\mathbb{Z},\psi} \mathsf{B}\mathbb{N} \simeq \mathbb{Z} \vec{/}_{\phi+\psi} \mathbb{N}^2.$$

Remark 2.3.2.7. There is also a version of (2.3.2.5) involving relative path monoids. Namely, suppose $\mathcal{C}_- : \mathrm{I} \to \mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})$ is any diagram, and let $\phi : \mathsf{M} \to \lim_i \iota_0\mathcal{C}_i$ be a surjective homomorphism of $\mathbb{E}_\infty$-groups. Then

$$\lim_i \mathcal{C}_i = \big(\lim_i \iota_0\mathcal{C}_i\big) \vec{/} \big(\lim_i \mathrm{P}_{0,\mathsf{M}}\, \mathcal{C}_i\big)$$

Indeed, this follows from Proposition 2.3.1.1 and the fact that both ι_0 and $\mathrm{P}_{0,\mathsf{M}}$ commute with limits.

We may use (2.3.2.5) to construct a general pullback / pushout square describing grouplike SMCs.

Proposition 2.3.2.8. *Let $\mathcal{C}$ be a grouplike symmetric monoidal category. Then the square*

$$\begin{tikzcd} \iota_0\mathcal{C} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ 0 \arrow[r] & \mathsf{B}\,\mathrm{P}_0\mathcal{C}. \end{tikzcd}$$

is both a pullback square and a pushout square.

Proof. Using (2.3.2.5), we see that

$$0 \times_{\mathsf{B}\,\mathrm{P}_0\mathcal{C}} \mathcal{C} = (0 \times_0 \iota_0\mathcal{C}) \vec{/} (0 \times_{\mathrm{P}_0\mathcal{C}} \mathrm{P}_0\mathcal{C}) = \iota_0\mathsf{C}\mathsf{c} \vec{/} 0 = \iota_0\mathcal{C}.$$

Thus the square in question is a pullback square.

Because skew quotients commute with colimits, we have

$$\mathcal{C}/\iota_0\mathcal{C} = (\iota_0\mathcal{C}/\iota_0\mathcal{C}) \vec{/} (\mathrm{P}_0\mathcal{C}/0) = 0 \vec{/} \mathrm{P}_0\mathcal{C} = \mathsf{B}\,\mathrm{P}_0\mathcal{C}.$$

This is the same as saying that the square in question is a pullback square. □

2.3.3 Grouplike SMCs as fiber products

Theorem 2.3.2.1 allows us to recover a grouplike SMC $\mathcal{C}$ from its $\mathbb{E}_\infty$-group of objects $\iota_0\mathcal{C}$, its path monoid $\mathrm{P}_0\mathcal{C}$, and the natural map $\mathrm{P}_0\mathcal{C} \to \iota_0\mathcal{C}$. However, $\iota_0\mathcal{C}$ is not the only $\mathbb{E}_\infty$-group one can obtain from $\mathcal{C}$.

Indeed, the groupoid completion $\mathcal{C}^{\mathrm{gpd}} = \iota_0\mathcal{C}/\mathrm{P}_0\mathcal{C}$ is also an $\mathbb{E}_\infty$-group. It is thus natural to ask whether $\mathcal{C}$ can be recovered from $\mathcal{C}^{\mathrm{gpd}}$ together with $\mathrm{P}_0\mathcal{C}$ and some data controlling the interaction between these $\mathbb{E}_\infty$-monoids. To obtain such a description, we first give a description of $\mathcal{C}^{\mathrm{gpd}}$ completely in terms of $\mathbb{E}_\infty$-groups:

Proposition 2.3.3.1. *Let $\phi : \mathsf{N} \to \mathsf{M}$ be a homomorphism of $\mathbb{E}_\infty$-monoids, and suppose M is an $\mathbb{E}_\infty$-group. Then $(\mathsf{M}\vec{/}\mathsf{N})^{\mathrm{gpd}} = \mathsf{M}/\mathsf{N}^{\mathrm{gp}}$.*

Proof. This is a direct computation:

$$\begin{aligned}(\mathsf{M}\vec{/}\mathsf{N})^{\mathrm{gpd}} &= \mathsf{M}/\mathsf{N} \text{ by Proposition 2.2.4.1(3)}\\ &= (\mathsf{M}/\mathsf{N})^{\mathrm{gp}} \text{ because } \mathsf{M}/\mathsf{N} \text{ is already an } \mathbb{E}_\infty\text{-group}\\ &= \mathsf{M}^{\mathrm{gp}}/\mathsf{N}^{\mathrm{gp}} \text{ because } (-)^{\mathrm{gp}} \text{ preserves colimits}\\ &= \mathsf{M}/\mathsf{N}^{\mathrm{gp}} \text{ because } \mathsf{M} \text{ is already an } \mathbb{E}_\infty\text{-group.}\end{aligned}$$

□

Corollary 2.3.3.2. *If $\mathcal{C}$ is a grouplike SMC, then $\mathcal{C}^{\mathrm{gpd}} = \iota_0\mathcal{C}/(\mathrm{P}_0\mathcal{C})^{\mathrm{gp}}$.*

Proof. By Theorem 2.3.2.1 we may write $\mathcal{C} = \iota_0\mathcal{C}\vec{/}\mathrm{P}_0\mathcal{C}$. The result follows from Proposition 2.3.3.1. □

In particular, we see that if $\mathcal{C}$ is a grouplike SMC, then there is a natural map

$$\mathcal{C}^{\mathrm{gpd}} \to \mathsf{B}(\mathrm{P}_0\mathcal{C})^{\mathrm{gpd}} = \mathsf{B}((\mathrm{P}_0\mathcal{C})^{\mathrm{gp}}).$$

We may use this map to characterize $\mathcal{C}$ as a pullback:

Proposition 2.3.3.3. *Let $\mathcal{C}$ be a grouplike SMC. Then $\mathcal{C} = \mathcal{C}^{\mathrm{gpd}} \times_{(\mathsf{B}\,\mathrm{P}_0\mathcal{C})^{\mathrm{gpd}}} \mathsf{B}\,\mathrm{P}_0\mathcal{C}$.*

Proof. We compute

$$\begin{aligned}\mathcal{C}^{\mathrm{gpd}} \times_{(\mathsf{B}\,\mathrm{P}_0\mathcal{C})^{\mathrm{gpd}}} \mathsf{B}\,\mathrm{P}_0\mathcal{C} &= (\mathcal{C}^{\mathrm{gpd}}\vec{/}0) \times_{\mathsf{B}(\mathrm{P}_0\mathcal{C})^{\mathrm{gp}}\vec{/}0} (0\vec{/}\,\mathrm{P}_0\mathcal{C}) \text{ by Theorem 2.3.2.1}\\ &= \left(\mathcal{C}^{\mathrm{gpd}} \times_{\mathsf{B}(\mathrm{P}_0\mathcal{C})^{\mathrm{gp}}} \mathrm{pt}\right)\vec{/}(0 \times_0 \mathrm{P}_0\mathcal{C}) \text{ by (2.3.2.5)}\\ &= \left(\iota_0\mathcal{C}/(\mathrm{P}_0\mathcal{C})^{\mathrm{gp}} \times_{\mathsf{B}(\mathrm{P}_0\mathcal{C})^{\mathrm{gp}}} 0\right)\vec{/}\,\mathrm{P}_0\mathcal{C}\\ &= \mathrm{fib}\left(\iota_0\mathcal{C}/(\mathrm{P}_0\mathcal{C})^{\mathrm{gp}} \to \mathsf{B}(\mathrm{P}_0\mathcal{C})^{\mathrm{gp}}\right)\vec{/}\,\mathrm{P}_0\mathcal{C}\\ &= \iota_0\mathcal{C}\vec{/}\,\mathrm{P}_0\mathcal{C} \text{ because } \mathsf{CGrp} \text{ is prestable}\\ &= \mathcal{C} \text{ by Theorem 2.3.2.1.}\end{aligned}$$

□

Corollary 2.3.3.4. *Let $\mathcal{C}$ be a grouplike SMC. Then $\iota_0\mathcal{C} = \mathrm{fib}(\mathcal{C}^{\mathrm{gpd}} \to (\mathsf{B}\,\mathrm{P}_0\mathcal{C})^{\mathrm{gpd}})$.*

Proof. Apply ι_0 to the Cartesian square of Proposition 2.3.3.3 and note that $\iota_0 \mathsf{B}\,\mathrm{P}_0\mathcal{C} = \mathsf{B}(\mathrm{P}_0\mathcal{C})^{\pm} = \mathsf{B}(\mathrm{pt})$. $\square$

The limit description of Proposition 2.3.3.3 may be upgraded to an equivalence of categories:

Proposition 2.3.3.5. *The commutative square of categories*

$$\begin{tikzcd}
\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat}) \arrow[r, "(-)^{\mathrm{gpd}} \to (\mathsf{B}\,\mathrm{P}_0(-))^{\mathrm{gpd}}"] \arrow[d, "\mathrm{P}_0"] & \mathsf{Arr}(\mathsf{CMon}^{\mathrm{gp}}) \arrow[d, "\mathrm{ev}_1"] \\
\mathsf{CMon}^{\sharp} \arrow[r, "\mathsf{B}(-)^{\mathrm{gpd}}"] & \mathsf{CMon}^{\mathrm{gp}}
\end{tikzcd}$$

is Cartesian.

Proof. The square clearly commutes, so we obtain a natural map

$$\begin{aligned}
\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat}) &\to \mathsf{CMon}^{\sharp} \times_{\mathsf{CMon}^{\mathrm{gp}}} \mathsf{Arr}(\mathsf{CMon}^{\mathrm{gp}}) \\
\mathcal{C} &\mapsto (\mathrm{P}_0\mathcal{C}, \mathcal{C}^{\mathrm{gpd}} \to (\mathsf{B}\,\mathrm{P}_0\mathcal{C})^{\mathrm{gpd}}).
\end{aligned}$$

Taking pullbacks as in Proposition 2.3.3.3 defines a one-sided inverse $\mathsf{CMon}^{\sharp} \times_{\mathsf{CMon}^{\mathrm{gp}}} \mathsf{Arr}(\mathsf{CMon}^{\mathrm{gp}}) \to \mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})$ to the above map. We need to show that this is in fact a two-sided inverse. That is, given a pair $(\mathsf{N}, \mathsf{M} \to (\mathsf{BN})^{\mathrm{gpd}}) \in \mathsf{CMon}^{\sharp} \times_{\mathsf{CMon}^{\mathrm{gp}}} \mathsf{Arr}(\mathsf{CMon}^{\mathrm{gp}})$, we need to show that:

- $\mathrm{P}_0(\mathsf{M} \times_{(\mathsf{BN})^{\mathrm{gpd}}} \mathsf{BN}) = \mathsf{N}$ and
- $(\mathsf{M} \times_{(\mathsf{BN})^{\mathrm{gpd}}} \mathsf{BN})^{\mathrm{gpd}} = \mathsf{M}$.

For the first claim, note that

$$\mathsf{M} \times_{(\mathsf{BN})^{\mathrm{gpd}}} \mathsf{BN} = (\mathsf{M}\vec{/}0) \times_{\mathsf{BN}^{\mathrm{gp}}\vec{/}0} 0\vec{/}\mathsf{N} = (\mathsf{M} \times_{\mathsf{BN}^{\mathrm{gp}}} 0)\vec{/}\mathsf{N}$$

By Proposition 2.2.4.1(5), we have $\mathrm{P}_0(\mathsf{M} \times_{(\mathsf{BN})^{\mathrm{gpd}}} \mathsf{BN}) = \mathsf{N}$, giving the claim.

For the second claim, we compute

$$\begin{aligned}
\big((\mathsf{M} \times_{(\mathsf{BN})^{\mathrm{gpd}}} 0)\vec{/}\mathsf{N}\big)^{\mathrm{gpd}} &= (\mathsf{M} \times_{\mathsf{BN}^{\mathrm{gp}}} 0)^{\mathrm{gp}}\vec{/}\mathsf{N}^{\mathrm{gp}} \text{ by (2.3.2.5)} \\
&= (\mathsf{M} \times_{\mathsf{BN}^{\mathrm{gp}}} 0)/\mathsf{N}^{\mathrm{gp}} \text{ because } \mathsf{M} \times_{\mathsf{BN}^{\mathrm{gp}}} 0 \text{ is already an } \mathbb{E}_\infty\text{-group} \\
&= \mathsf{M} \text{ by Corollary 2.1.5.8.}
\end{aligned}$$

$\square$

Informally, Proposition 2.3.3.5 says that $\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})$ is equivalent to the category of diagrams

$$\begin{tikzcd}
 & \mathsf{M} \arrow[d] \\
\mathsf{BN} \arrow[r] & \mathsf{BN}^{\mathrm{gp}}
\end{tikzcd}$$

with N a sharp $\mathbb{E}_\infty$-monoid and M an $\mathbb{E}_\infty$-group.

2.3.4 Localizations and faces

To understand the geometry associated with grouplike SMCs, we need to introduce a suitable notion of "locality."

Definition 2.3.4.1. Let $\mathcal{C}$ be a grouplike SMC. For a set of morphisms S in $\mathcal{C}$, the *localization of* $\mathcal{C}$ *with respect to* S is the grouplike SMC $\mathcal{C}$ satisfying the following universal property: for any grouplike SMC $\mathcal{C}'$, we have

$$\mathsf{Fun}^{\oplus}(\mathcal{C}, \mathcal{C}') = \{\mathsf{F} \in \mathsf{Fun}^{\oplus}(\mathcal{C}, \mathcal{C}') \,|\, \mathsf{F}(\mathsf{f}) \text{ is invertible for all } \mathsf{f} \in \mathsf{S}\}.$$

We say that a localization is *elementary* if S may be taken to be a single morphism. The collection of localizations of $\mathcal{C}$ forms a full subcategory $(\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat}))^{\mathrm{loc}}_{\mathcal{C}/} \subset (\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat}))_{\mathcal{C}/}$.

Remark 2.3.4.2. The assumption that $\mathcal{C}'$ is grouplike here is not significant: if $\mathcal{C}'$ is arbitrary, an analogous universal property holds (by combining the above universal property with that of $(\mathcal{C}')^{\mathrm{gplike}} \subset \mathcal{C}'$). We restrict to the case of grouplike $\mathcal{C}'$ for technical convenience, as $\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})$ is easier to work with than $\mathsf{CAlg}(\mathsf{Cat})$.

We seek to show that localizations of a grouplike SMC $\mathcal{C}$ always exist. In fact, Theorem 2.3.2.1 will allow us to give a skew quotient presentation of any localization of $\mathcal{C}$.

Proposition 2.3.4.3. *Let $\mathcal{C}$ be a grouplike SMC. Then:*

1. *Any localization of $\mathcal{C}$ may be written as the localization of $\mathcal{C}$ with respect to a face of $\mathrm{P}_0\mathcal{C}$.*

2. *Given a face $\mathsf{S} \subset \mathrm{P}_0\mathcal{C}$, the localization $\mathcal{C}[\mathsf{S}^{-1}]$ exists, and there is a natural equivalence*

$$\mathcal{C}[\mathsf{S}^{-1}] = \big(\iota_0\mathcal{C}/\mathsf{S}^{\mathrm{gp}}\big)\vec{/}\big((\mathrm{P}_0\mathcal{C})/\mathsf{S}\big).$$

3. *Given a face $\mathsf{S} \subset \mathrm{P}_0\mathcal{C}$ and a surjective homomorphism of $\mathbb{E}_\infty$-groups $\phi : \mathsf{M} \to \iota_0\mathcal{C}$, there is a natural equivalence*

$$\mathcal{C}[\mathsf{S}^{-1}] = \mathsf{M}\vec{/}\big((\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C})[-\mathsf{S}]\big).$$

   In particular, $\mathcal{C}[\mathsf{S}^{-1}] = \iota_0\mathcal{C}\vec{/}\big(\,\mathrm{P}_0\mathcal{C}[-\mathsf{S}]\big)$.

4. *The assignment $\mathsf{S} \mapsto (\mathcal{C} \to \mathcal{C}[\mathsf{S}^{-1}])$ defines an equivalence $\mathrm{Face}(\mathrm{P}_0\mathcal{C}) \xrightarrow{\sim} (\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat}))^{\mathrm{loc}}_{\mathcal{C}/}$.*

Proof. (1). If $\mathsf{f} : \mathsf{c}_1 \to \mathsf{c}_2$ is any morphism in $\mathcal{C}$, then because $\mathcal{C}$ is grouplike, there exists an object $-\mathsf{c}_1 \in \mathcal{C}$ such that $\mathsf{c}_1 \oplus -\mathsf{c}_1 \simeq 0_{\mathcal{C}}$. The morphism f is then equivalent (as an object of $\mathsf{Arr}(\mathcal{C})$) to $(\mathsf{f} \oplus \mathrm{id}_{-\mathsf{c}_1}) \oplus \mathrm{id}_{\mathsf{c}_1}$. Localizing with respect to f is equivalent to localizing with respect to $(\mathsf{f} \oplus \mathrm{id}_{-\mathsf{c}_1})$, which is an object of $\mathrm{P}_0\mathcal{C}$. It follows that any localization of $\mathcal{C}$ may be described as a localization with respect to a collection of morphisms in $\mathrm{P}_0\mathcal{C}$. We may always assume that such a collection is closed under equivalence, i.e. is a subspace of $\mathrm{P}_0\mathcal{C}$.

Now suppose that S is a subspace of $\mathrm{P}_0\mathcal{C}$. We claim that localizing with respect to S is equivalent to localizing with respect to a face of $\mathrm{P}_0\mathcal{C}$. If $\mathsf{f}_1, \mathsf{f}_2 \in \mathsf{S}$, then localizing with respect to S automatically inverts $\mathsf{f}_1 + \mathsf{f}_2$, so we may assume S is a submonoid of $\mathrm{P}_0\mathcal{C}$. If $\mathsf{g}_1, \mathsf{g}_2 \in \mathrm{P}_0\mathcal{C}$ and $\mathsf{g}_1 + \mathsf{g}_2 \in \mathsf{S}$, then localizing with respect to S must invert g_1 (because a suitable translate of $\mathsf{g}_2 - (\mathsf{g}_1 + \mathsf{g}_2)$ is an inverse for g_1). Likewise, any such localization must also invert g_2. It follows that we may take S to be a face of $\mathrm{P}_0\mathcal{C}$.

(2). Let $\mathcal{C}'$ be any grouplike SMC. Under the equivalence $\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat}) = \mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon})$ of Theorem 2.3.2.1, any symmetric monoidal functor $\mathsf{F} : \mathcal{C} \to \mathcal{C}'$ which inverts S corresponds to a commutative square

$$\begin{tikzcd} \mathrm{P}_0\mathcal{C} \arrow[r] \arrow[d] & \iota_0\mathcal{C} \arrow[d] \\ \mathrm{P}_0\mathcal{C}' \arrow[r] & \iota_0\mathcal{C}' \end{tikzcd}$$

such that the leftmost vertical arrow sends S to 0 (because $\mathrm{P}_0\mathcal{C}'$ is sharp), and vice versa. Thus the grouplike SMC $\mathcal{C}[\mathsf{S}^{-1}]$ corresponds to the morphism $(\mathsf{N} \to \mathsf{M}) \in \mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathcal{C})$ equipped with a universal commutative square

$$\begin{tikzcd} \mathrm{P}_0\mathcal{C} \arrow[r] \arrow[d] & \iota_0\mathcal{C} \arrow[d] \\ \mathsf{N} \arrow[r] & \mathsf{M} \end{tikzcd}$$

such that the leftmost vertical arrow sends S to 0. By Corollary 2.1.4.8, the quotient $\mathrm{P}_0\mathcal{C}/\mathsf{S}$ is sharp, so the morphism $\mathrm{P}_0\mathcal{C}/\mathsf{S} \to (\iota_0\mathcal{C})/\mathsf{S}^{\mathrm{gp}}$ is an object of $\mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon})$. The commutative square

$$\begin{tikzcd} \mathrm{P}_0\mathcal{C} \arrow[r] \arrow[d] & \iota_0\mathcal{C} \arrow[d] \\ \mathrm{P}_0\mathcal{C}/\mathsf{S} \arrow[r] & \iota_0\mathcal{C}/\mathsf{S}^{\mathrm{gp}} \end{tikzcd}$$

(viewed as a morphism in $\mathsf{Arr}^{\sharp,\mathrm{gp}}(\mathsf{CMon})$) clearly satisfies the desired universal property, so we have $\mathcal{C}[\mathsf{S}^{-1}] = (\iota_0\mathcal{C}/\mathsf{S}^{\mathrm{gp}})\vec{/}(\mathrm{P}_0\mathcal{C}/\mathsf{S})$.

(3). Let $\mathcal{C}'$ be a grouplike SMC. Then $\iota_0 \,\mathsf{Fun}^{\oplus}(\mathsf{M}\vec{/}\big((\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C})[-\mathsf{S}]\big), \mathcal{C}')$ is, by definition, equivalent to the fiber product

$$\mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}, \iota_0\mathcal{C}') \times_{\mathrm{Hom}_{\mathsf{CMon}}((\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C})[-\mathsf{S}], \iota_0\mathcal{C}')} \mathrm{Hom}_{\mathsf{CMon}}((\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C})[-\mathsf{S}], \mathrm{P}_0\mathcal{C}').$$

Because $\iota_0\mathcal{C}'$ is an $\mathbb{E}_\infty$-group, we have

$$\mathrm{Hom}_{\mathsf{CMon}}((\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C})[-\mathsf{S}], \iota_0\mathcal{C}') = \mathrm{Hom}_{\mathsf{CMon}}((\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C}[-\mathsf{S}])^{\mathrm{gp}}, \iota_0\mathcal{C}') = \mathrm{Hom}_{\mathsf{CMon}}((\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C})^{\mathrm{gp}}, \iota_0\mathcal{C}').$$

There is a natural inclusion $\mathrm{Hom}_{\mathsf{CMon}}((\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C})[-\mathsf{S}], \mathrm{P}_0\mathcal{C}') \subset \mathrm{Hom}_{\mathsf{CMon}}(\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C}, \mathrm{P}_0\mathcal{C}')$. Because inclusions pull back, we obtain an inclusion

$$\begin{aligned}\iota_0\,\mathsf{Fun}^{\oplus}(\mathsf{M}\vec{/}\big((\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C})[-\mathsf{S}]\big), \mathcal{C}') &\subset \mathrm{Hom}_{\mathsf{CMon}}(\mathsf{M}, \iota_0\mathcal{C}') \times_{\mathrm{Hom}_{\mathsf{CMon}}((\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C})^{\mathrm{gp}}, \iota_0\mathcal{C}')} \mathrm{Hom}_{\mathsf{CMon}}(\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C}, \mathrm{P}_0\mathcal{C}') \\ &= \iota_0\,\mathsf{Fun}^{\oplus}(\mathsf{M}\vec{/}\,\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C}, \mathcal{C}') \\ &= \iota_0\,\mathsf{Fun}^{\oplus}(\mathcal{C}, \mathcal{C}')\end{aligned}$$

identifying $\iota_0\,\mathsf{Fun}^{\oplus}(\mathsf{M}\vec{/}\big((\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C})[-\mathsf{S}]\big), \mathcal{C}')$ with the subspace of $\iota_0\,\mathsf{Fun}^{\oplus}(\mathcal{C}, \mathcal{C}')$ consisting of symmetric monoidal functors which invert S. But this is exactly saying that

$$\iota_0\,\mathsf{Fun}^{\oplus}\left(\mathsf{M}\vec{/}\big((\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C})[-\mathsf{S}]\big), \mathcal{C}'\right) = \iota_0\,\mathsf{Fun}^{\oplus}(\mathcal{C}[\mathsf{S}^{-1}], \mathcal{C}'),$$

so $\mathsf{M}\vec{/}\big((\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C})[-\mathsf{S}]\big) = \mathcal{C}[\mathsf{S}^{-1}]$ by Yoneda.

(4). By the preceding results, this functor is well-defined and essentially surjective. To see that this assignment is fully faithful, we note that both $\mathrm{Face}(\mathrm{P}_0\mathcal{C})$ and $(\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat}))^{\mathrm{loc}}_{\mathcal{C}/}$ are (the categories associated with) posets, and for faces $\mathsf{S}_1, \mathsf{S}_2 \subset \mathrm{P}_0\mathcal{C}$, we have $\mathsf{S}_1 \subset \mathsf{S}_2$ if and only if $\mathcal{C}[\mathsf{S}_1^{-1}]$ maps to $\mathcal{C}[\mathsf{S}_2^{-1}]$ over $\mathcal{C}$. □

We may use Proposition 2.3.4.3 to give a more intrinsic characterization of localizations.

Proposition 2.3.4.4. *Let $\mathsf{F} : \mathcal{C} \to \mathcal{C}'$ be a symmetric monoidal functor between grouplike SMCs. Then F is a localization if and only if:*

- *The induced morphism $\mathrm{P}_0\mathcal{C}/\,\mathrm{fib}(\mathrm{P}_0\mathsf{F}) \to \mathrm{P}_0\mathcal{C}'$ is an equivalence,*
- *F induces a surjection $\pi_0(\iota_0\mathcal{C}) \to \pi_0(\iota_0\mathcal{C}')$, and*
- *The induced morphism $\mathrm{fib}(\mathrm{P}_0\mathsf{F})^{\mathrm{gp}} \to \mathrm{fib}(\iota_0\mathsf{F})$ is an equivalence.*

Furthermore, in this case, $\mathrm{fib}(\mathrm{P}_0\mathsf{F})$ is a face of $\mathrm{P}_0\mathcal{C}$, and F is the localization of $\mathcal{C}$ with respect to $\mathrm{fib}(\mathrm{P}_0\mathsf{F})$.

Proof. ($\Rightarrow$). Suppose $\mathsf{F} : \mathcal{C} \to \mathcal{C}'$ is a localization. By Proposition 2.3.4.3, we may assume F is the localization with respect to a face $\mathsf{S} \subset \mathrm{P}_0\mathcal{C}$, and we have $\mathrm{P}_0\mathcal{C}' = \mathrm{P}_0\mathcal{C}/\mathsf{S}$ and $\iota_0\mathcal{C}' = \iota_0\mathcal{C}/\mathsf{S}^{\mathrm{gp}}$. By Proposition 2.1.4.10, we have $\mathsf{S} = \mathrm{fib}(\mathrm{P}_0\mathsf{F})$, so the "furthermore" statement holds and $\mathrm{P}_0\mathcal{C}' = \mathrm{P}_0\mathcal{C}/\mathsf{S} = \mathrm{P}_0\mathcal{C}/\,\mathrm{fib}(\mathrm{P}_0\mathsf{F})$.

Because $\iota_0\mathsf{F}$ is a quotient map, $\iota_0\mathsf{F}$ must be surjective on π_0. It remains to show that $\mathrm{fib}(\iota_0\mathsf{F}) = \mathrm{fib}(\mathrm{P}_0\mathsf{F})^{\mathrm{gp}} = \mathsf{S}^{\mathrm{gp}}$. This is straightforward because CGrp is prestable:

$$\mathrm{fib}(\iota_0\mathsf{F}) = \mathrm{fib}(\mathsf{M} \to \mathsf{M}/\mathsf{S}^{\mathrm{gp}}) = \mathrm{fib}(\mathsf{M} \to \mathrm{cofib}(\mathsf{S}^{\mathrm{gp}} \to \mathsf{M})) = \mathsf{S}^{\mathrm{gp}}.$$

($\Leftarrow$). Suppose F satisfies the stated conditions. First observe that $\iota_0\mathsf{F} : \iota_0\mathcal{C} \to \iota_0\mathcal{C}'$ is equivalent to the quotient map $\iota_0\mathcal{C} \to \iota_0\mathcal{C}/\,\mathrm{fib}(\iota_0\mathsf{F})^{\mathrm{gp}}$. Indeed, this follows from the prestability of CGrp: because $\iota_0\mathsf{F}$ is surjective on π_0, there is a natural equivalence

$$\iota_0\mathcal{C}' = \mathsf{M}/\,\mathrm{fib}(\iota_0\mathsf{F}) = \mathsf{M}/\,\mathrm{fib}(\mathrm{P}_0\mathsf{F})^{\mathrm{gp}}.$$

We may therefore use Theorem 2.3.2.1 to write $\mathcal{C}'$ as

$$\iota_0\mathcal{C}' \vec{/}\, \mathrm{P}_0\mathcal{C}' = \big(\iota_0\mathcal{C}/\operatorname{fib}(\mathrm{P}_0\mathsf{F})^{\mathrm{gp}}\big)\vec{/}\big(\mathrm{P}_0\mathcal{C}/\operatorname{fib}(\mathrm{P}_0\mathsf{F})\big)$$

Because $\mathrm{P}_0\mathcal{C}'$ is sharp, $\operatorname{fib}(\mathrm{P}_0\mathsf{F})$ must be a face of $\mathrm{P}_0\mathcal{C}$ by Proposition 2.1.4.5. Combining this with Proposition 2.3.4.3(2), we get $\mathcal{C}' = \mathcal{C}[\operatorname{fib}(\mathrm{P}_0\mathsf{F})^{-1}]$, as desired. □

The faces of $\mathbb{N}^n$ are classified in Example 2.1.4.4. Using Proposition 2.3.4.4, we may extend this to a classification of the localizations of grouplike SMCs $\mathcal{C}$ with $\mathrm{P}_0\mathcal{C} \simeq \mathbb{N}^n$.This result underpins the Cox construction discussed in Chapter 6.

Example 2.3.4.5. Let $\mathcal{C}$ be a grouplike SMC, and suppose $\mathrm{P}_0\mathcal{C} \simeq \mathbb{N}^n$ for some n. Then localizations of $\mathcal{C}$ correspond to commutative squares

$$\begin{tikzcd} \mathbb{N}^n \arrow[r, "\phi"] \arrow[d, "\pi_J"] & \iota_0\mathcal{C} \arrow[d, "\psi"] \\ \mathbb{N}^J \arrow[r, "\phi'"] & M \end{tikzcd}$$

such that:

- π_J is projection onto coordinates indexed by $J \subset \{1, \dots, n\}$,
- ψ induces a surjection $\pi_0(\iota_0\mathcal{C}) \twoheadrightarrow \pi_0(M)$, and
- The induced map $\mathbb{Z}^{n-|J|} = \operatorname{fib}(\pi_J)^{\mathrm{gp}} \to \operatorname{fib}(\psi)$ is an equivalence.

Indeed, any quotient of $\mathbb{N}^n$ by a face agrees with π_J for some J by Example 2.1.4.4, so the equivalence follows from Proposition 2.3.4.4 applied to the skew quotient map $\mathcal{C} \to M\vec{/}\mathbb{N}^J$.

We note that localizations of grouplike SMCs interact well with colimits:

Proposition 2.3.4.6 (Localizations push out)**.** *If $\mathcal{C}$ and $\mathcal{C}'$ are grouplike SMCs, $\mathcal{C} \to \mathcal{C}[S^{-1}]$ is a localization, and $F : \mathcal{C} \to \mathcal{C}'$ is a symmetric monoidal functor, then there is a pushout square*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{C}' \arrow[d] \\ \mathcal{C}[S^{-1}] \arrow[r] & \mathcal{C}'[F(S)^{-1}]. \end{tikzcd}$$

Proof. Immediate from the universal properties. □

2.3.5 Localizations and limits

In this subsection we collect some useful results on the interaction between localizations of grouplike SMCs and limits. These algebraic results control the locality properties of $\mathbb{E}_\infty$-fans sketched in Chapter 6.

We begin by showing that localizations of grouplike SMCs pull back. Note that localizations of $\mathbb{E}_\infty$-monoids *do not* pull back, so this result may be somewhat surprising: the discrepancy is essentially due to the fact that limits of grouplike SMCs retain more information than the corresponding limits of $\mathbb{E}_\infty$-monoids.

Proposition 2.3.5.1 (Localizations of grouplike SMCs pull back)**.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be grouplike SMCs, let $\mathsf{S} \subset \mathrm{P}_0\mathcal{C}$ be a face with corresponding localization $\mathsf{G} : \mathcal{C} \to \mathcal{C}[\mathsf{S}^{-1}]$, and let $\mathsf{F} : \mathcal{C}' \to \mathcal{C}[\mathsf{S}^{-1}]$ be a symmetric monoidal functor. Let*

$$\begin{tikzcd} \mathcal{C}'' \arrow[r, "\mathsf{G}'"] \arrow[d] & \mathcal{C} \arrow[d, "\mathsf{G}"] \\ \mathcal{C}' \arrow[r, "\mathsf{F}"] & \mathcal{C}[\mathsf{S}^{-1}] \end{tikzcd}$$

be a pullback square. Then the natural functor $\mathsf{G} : \mathcal{C}'' \to \mathcal{C}'$ is a localization.

Proof. Note that $\mathsf{S} = \mathrm{fib}(\mathrm{P}_0\mathsf{G})$ by Proposition 2.3.4.4. We check that the criteria of Proposition 2.3.4.4 hold for G'.

- We have

$$\begin{aligned} \mathrm{P}_0\mathcal{C}''/\,\mathrm{fib}(\mathrm{P}_0\mathsf{G}) &= (\mathrm{P}_0\mathcal{C} \times_{\mathrm{P}_0(\mathcal{C}[\mathrm{fib}(\mathrm{P}_0\mathsf{G})^{-1}])} \mathrm{P}_0\mathcal{C}')/\,\mathrm{fib}(\mathrm{P}_0\mathsf{G}) \\ &= (\mathrm{P}_0\mathcal{C} \times_{\mathrm{P}_0\mathcal{C}/\,\mathrm{fib}(\mathrm{P}_0\mathsf{G})} \mathrm{P}_0\mathcal{C}')/\,\mathrm{fib}(\mathrm{P}_0\mathsf{G}) \text{ by Proposition 2.3.4.3(2)} \\ &= \mathrm{P}_0\mathcal{C}' \text{ by Corollary 2.1.5.8.} \end{aligned}$$

- As $\iota_0\mathsf{G}'$ is a pullback of $\iota_0\mathsf{G}$, we see that $\pi_0(\iota_0\mathsf{G}')$ is surjective on π_0 by [HTT, Proposition 6.2.3.7] (applied in $\mathcal{S}$) and the corresponding claim for $\mathcal{S}$.

- We have

$$\mathrm{fib}(\mathrm{P}_0\mathsf{G}')^{\mathrm{gp}} = \mathrm{fib}(\mathrm{P}_0\mathsf{G})^{\mathrm{gp}} = \mathrm{fib}(\iota_0\mathsf{G}) = \mathrm{fib}(\iota_0\mathsf{G}').$$

Thus G' is the localization of $\mathcal{C}''$ along the face $\mathrm{fib}(\mathrm{P}_0\mathsf{G}) \subset \mathrm{P}_0\mathcal{C}''$. □

To extend Proposition 2.3.5.1 to a more general class of limits, we will need the following technical (but certainly well-known) bit of abstract nonsense.

Lemma 2.3.5.2. *Let $(\mathrm{I}, \leqslant)$ be a nonempty finite poset admitting finite nonempty joins.*[6] *Let $\mathsf{C}_- : \mathrm{I} \to \mathcal{C}$ be any diagram. Then $\lim_{\mathsf{i}\in\mathrm{I}} \mathsf{C}_\mathsf{i}$ can be computed as an iterated fiber product of the C_i.*

[6] Actually, as we shall see, all we will need are binary joins of minimal elements of I.

Proof. Let $i_1, \dots, i_n$ be the minimal elements of I. Let I' be the subposet of I consisting of all binary joins $\{i_a \vee i_b\}_{a,b \in \{1,\dots,n\}}$. Note that each $i_a = i_a \vee i_a$ is an element of I'. We first claim that I' is coinitial in I.

By the dual of [HTT, Theorem 4.1.3.1], I' is coinitial in I if and only if, for any $i \in I$, we have $(I' \times_I I_{/i})^{\mathrm{gpd}} = \mathrm{pt}$. By our choice of the i_a, there exists some $i_a \leqslant i$, so $I' \times_I I_{/i}$ is nonempty. Furthermore, if $i_a \vee i_b \leqslant i$ and $i_{a'} \vee i_{b'} \leqslant i$, then there exists a zigzag

$$\begin{tikzcd}
& i_a \arrow[dl] \arrow[dr] & & i_{b'} \arrow[dl] \arrow[dr] & \\
i_a \vee i_b & & i_a \vee i_{b'} & & i_{a'} \vee i_{b'}.
\end{tikzcd}$$

It follows that $(I' \times_I I_{/i})^{\mathrm{gpd}}$ is connected. Since $I' \times_I I_{/i}$ is itself a poset, the groupoid completion $(I' \times_I I_{/i})^{\mathrm{gpd}}$ is a discrete set. The only discrete connected set is pt, so $(I' \times_I I_{/i})^{\mathrm{gpd}} = \mathrm{pt}$, and I' is coinitial in I.

We may therefore write $\lim_{i \in I} C_i = \lim_{i' \in I'} C_{i'}$. The limit of this latter diagram can be computed as an iterated fiber product as follows. First, for all a, b, compute $C_{i_a} \times_{C_{i_a \vee i_b}} C_{i_b}$. Then, for all a, b, and c, compute the limit of all terms involving a, b, and c as

$$\big((C_{i_a} \times_{C_{i_a \vee i_b}} C_{i_b}) \times_{C_{i_b}} (C_{i_b} \times_{C_{i_b \vee i_c}} C_{i_c})\big) \times_{C_{i_a \vee i_c}} (C_{i_a} \times_{C_{i_a \vee i_c}} C_{i_c})$$

Repeat this process for all a, b, c, d, *et cetera*, until one has obtained the limit of all of I'.[7] □

Corollary 2.3.5.3. *Let* I *be a nonempty finite poset admitting finite nonempty joins. Let* $\mathcal{C}_- : I \to \mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})$ *be a finite diagram such that all transition maps* $\mathcal{C}_i \to \mathcal{C}_j$ *are localizations. Then all structure maps* $\lim_{i \in I} \mathcal{C}_i \to \mathcal{C}_j$ *are localizations.*

Proof. By Lemma 2.3.5.2, the limit in question may be constructed as an iterated fiber product of the $\mathcal{C}_i$. All structure maps in question may therefore be written as compositions of structure maps of fiber products and transition maps $\mathcal{C}_i \to \mathcal{C}_j$. The result then follows from Proposition 2.3.5.1 and the fact that compositions of localizations are localizations. □

Proposition 2.3.5.4. *Let* $\mathsf{F} : \mathcal{C}_1 \to \mathcal{C}_3$ *and* $\mathsf{G} : \mathcal{C}_2 \to \mathcal{C}_3$ *be morphisms in* $\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})$*. Let* $\mathsf{S} \subset \mathrm{P}_0(\mathcal{C}_1 \times_{\mathcal{C}_3} \mathcal{C}_2)$ *be a face. Then*

$$(\mathcal{C}_1 \times_{\mathcal{C}_3} \mathcal{C}_2)[\mathsf{S}^{-1}] = \mathcal{C}_1[\mathsf{S}^{-1}] \times_{\mathcal{C}_3[\mathsf{S}^{-1}]} \mathcal{C}_2[\mathsf{S}^{-1}].$$

Proof. Let $\mathsf{A} = \iota_0\mathcal{C}_1 \times_{\iota_0\mathcal{C}_3} \iota_0\mathcal{C}_2$. Replacing each $\mathcal{C}_i$ by the full subcategory $\mathcal{C}'_i$ spanned by the objects in the image of the natural map $\mathsf{A} \to \iota_0\mathcal{C}_i$ does not change the fiber product: we have $\mathcal{C}_1 \times_{\mathcal{C}_3} \mathcal{C}_2 = \mathcal{C}'_1 \times_{\mathcal{C}'_3} \mathcal{C}'_2$. Furthermore, the image of S in each $\mathcal{C}_i$ lies in $\mathcal{C}'_i$, so $\mathcal{C}'_i[\mathsf{S}^{-1}]$ is the full subcategory of $\mathcal{C}_i[\mathsf{S}^{-1}]$ spanned by the objects lying in the image of $\mathsf{A} \to \iota_0\mathcal{C}_i[\mathsf{S}^{-1}]$.

[7] One can of course make this process more rigorous using an inductive argument at the cost of a significant reduction in clarity.

Thus $\mathcal{C}_1[S^{-1}] \times_{\mathcal{C}_3[S^{-1}]} \mathcal{C}_2[S^{-1}] = \mathcal{C}'_1[S^{-1}] \times_{\mathcal{C}'_3[S^{-1}]} \mathcal{C}'_2[S^{-1}]$. It follows that we may assume without loss of generality that $\mathcal{C}_i = \mathcal{C}'_i$ for all i, i.e. we may assume that the natural map $A \to \mathcal{C}_i$ is surjective on π_0.

Now we can write

$$\begin{aligned}
(\mathcal{C}_1 \times_{\mathcal{C}_3} \mathcal{C}_2)[S^{-1}] &= (A \vec{/} \mathrm{P}_{0,A}(\mathcal{C}_1 \times_{\mathcal{C}_3} \mathcal{C}_2))[S^{-1}] \text{ by Proposition 2.3.1.1} \\
&= A \vec{/} \big(\mathrm{P}_{0,A}(\mathcal{C}_1 \times_{\mathcal{C}_3} \mathcal{C}_2)[-S]\big) \text{ by Proposition 2.3.4.3(3)} \\
&= (A \times_A A) \vec{/} \big(\mathrm{P}_{0,A}\, \mathcal{C}_1 \times_{\mathrm{P}_{0,A}\, \mathcal{C}_3} \mathrm{P}_{0,A}\, \mathcal{C}_2)[-S]\big) \text{ because } \mathrm{P}_{0,A} \text{ preserves limits} \\
&= (A \times_A A) \vec{/} \big(\mathrm{P}_{0,A}\, \mathcal{C}_1[-S] \times_{\mathrm{P}_{0,A}\, \mathcal{C}_3[-S]} \mathrm{P}_{0,A}\, \mathcal{C}_2[-S])\big) \text{ by Corollary 2.1.5.7} \\
&= (A \vec{/} \mathrm{P}_{0,A}\, \mathcal{C}_1[-S]) \times_{A \vec{/} \mathrm{P}_{0,A}\, \mathcal{C}_3[-S]} (A \vec{/} \mathrm{P}_{0,A}\, \mathcal{C}_2[-S]) \text{ by Remark 2.3.2.7} \\
&= \mathcal{C}_1[S^{-1}] \times_{\mathcal{C}_3[S^{-1}]} \mathcal{C}_2[S^{-1}] \text{ by Proposition 2.3.4.3(3)}
\end{aligned}$$

where the final step uses the assumption that $A \to \mathcal{C}_i$ is surjective for all i. □

Corollary 2.3.5.5. *Let* I *be a nonempty finite poset admitting finite nonempty joins. Let* $\mathcal{C}_- : I \to \mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})$ *be a finite diagram, and let* $S \subset \lim_i \mathrm{P}_0\mathcal{C}_i$ *be a face. Then* $(\lim_i \mathcal{C}_i)[S^{-1}] = \lim_i(\mathcal{C}_i[S^{-1}])$.

Proof. By Lemma 2.3.5.2, we may write $\lim_i \mathcal{C}_i$ as an iterated fiber product of the $\mathcal{C}_i$. The result then follows from Proposition 2.3.5.4. □

2.3.6 Integral grouplike SMCs

Just as with integral $\mathbb{E}_\infty$-monoids, we can define a notion of integral grouplike SMC. This notion is not especially important for the general theory, but many grouplike SMCs of interest are integral, and it is worth noting some nice properties of integral grouplike SMCs.

Definition 2.3.6.1. A grouplike SMC $\mathcal{C}$ is *integral* if the natural functor $\mathcal{C} \to \mathcal{C}^{\mathrm{gpd}}$ is faithful, i.e. all maps

$$\mathrm{Map}_{\mathcal{C}}(c_1, c_2) \to \mathrm{Map}_{\mathcal{C}^{\mathrm{gpd}}}(c_1, c_2)$$

(for $c_1, c_2 \in \mathcal{C}$) are inclusions.

Proposition 2.3.6.2. *Let* $\mathcal{C}$ *be a grouplike SMC. The following are equivalent:*

1. $\mathcal{C}$ *is integral.*
2. *For every skew quotient presentation* $\mathcal{C} = M \vec{/}_\phi N$, N *is integral.*
3. $\mathrm{P}_0\mathcal{C}$ *is integral (hence discrete by Remark 2.1.6.5).*
4. *There exists a skew quotient presentation* $\mathcal{C} = M \vec{/}_\phi N$ *with* N *integral.*

Proof. First observe that, if we write $\mathcal{C} = \mathsf{M}\vec{/}\mathsf{N}$ for some $\phi : \mathsf{M} \to \mathsf{N}$, then for all $\mathfrak{m} \in \mathsf{M}$, the bottom and outer squares of

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(0,[\mathfrak{m}]) \arrow[r] \arrow[d] & \mathsf{N} \arrow[d] \\
\mathrm{Map}_{\mathcal{C}^{\mathrm{gpd}}}(0,[\mathfrak{m}]) \arrow[r] \arrow[d] & \mathsf{N}^{\mathrm{gp}} \arrow[d, "\phi^{\mathrm{gp}}"] \\
\mathrm{pt} \arrow[r, "\mathfrak{m}"] & \mathsf{M}
\end{tikzcd} \tag{2.3.6.3}$$

are Cartesian by Corollary 2.2.1.5. By the two pullbacks lemma, the top square of (2.3.6.3) must also be Cartesian.

(1) $\Rightarrow$ (2). Given a presentation $\mathcal{C} = \mathsf{M}\vec{/}_{\phi}\mathsf{N}$, we need to show that the map $\mathsf{N} \to \mathsf{N}^{\mathrm{gp}}$ is an inclusion. Because $\mathcal{C}$ is integral, all of the maps $\mathrm{Map}_{\mathcal{C}}(0,[\mathfrak{m}]) \to \mathrm{Map}_{\mathcal{C}^{\mathrm{gpd}}}(0,[\mathfrak{m}])$ are inclusions. From the diagram (2.3.6.3), we see that this means the map $\mathsf{N} \to \mathsf{N}^{\mathrm{gp}}$ becomes an inclusion whenever we base change to some $\mathfrak{m} \in \mathsf{M}$. A map of spaces is an inclusion if and only if all induced maps of fibers are inclusions, so $\mathsf{N} \to \mathsf{N}^{\mathrm{gp}}$ is an inclusion.

(2) $\Rightarrow$ (3). This follows from Theorem 2.3.2.1 by writing $\mathcal{C} = \iota_0\mathcal{C}\vec{/}\mathrm{P}_0\mathcal{C}$.

(3) $\Rightarrow$ (4). Trivial.

(4) $\Rightarrow$ (1). Suppose we have a presentation $\mathcal{C} = \mathsf{M}\vec{/}_{\phi}\mathsf{N}$ with N integral. We want to show that, for all $\mathsf{c}_1, \mathsf{c}_2 \in \mathcal{C}$, the map $\mathrm{Map}_{\mathcal{C}}(\mathsf{c}_1, \mathsf{c}_2) \to \mathrm{Map}_{\mathcal{C}^{\mathrm{gpd}}}(\mathsf{c}_1, \mathsf{c}_2)$ is an inclusion. Because $\mathcal{C}$ is grouplike, it suffices to consider the case where $\mathsf{c}_1 = 0$. We may also write $\mathsf{c}_2 = [\mathfrak{m}]$ for some $\mathfrak{m} \in \mathsf{M}$. The top square of (2.3.6.3) is Cartesian, and the map $\mathsf{N} \to \mathsf{N}^{\mathrm{gp}}$ is an inclusion because N is integral. Thus the natural map $\mathrm{Map}_{\mathcal{C}}(0,[\mathfrak{m}]) \to \mathrm{Map}_{\mathcal{C}^{\mathrm{gpd}}}(0,[\mathfrak{m}])$ must be an inclusion, precisely as needed. $\square$

Corollary 2.3.6.4. *A limit of integral grouplike SMCs is integral.*

Chapter 3

Geometrization of grouplike SMCs

In this chapter we discuss how to construct derived stacks from grouplike SMCs. The essential idea is relatively simple: given a grouplike SMC $\mathcal{C}$, we use the modern perspective on the Tannakian formalism (as in e.g. [Ste23]) to define a stack $\mathfrak{Y}_{\mathcal{C}}$ from the symmetric monoidal category $(\mathsf{PSh}_{\mathsf{k}}(\mathcal{C}), \star)$. In favorable cases, we can show that the stack $\mathfrak{Y}_{\mathcal{C}}$ admits an explicit presentation as a quotient stack (Proposition 3.2.3.6) and that there actually is an equivalence $(\mathsf{QC}(\mathfrak{Y}_{\mathcal{C}}), \otimes_{\mathscr{O}}) \simeq (\mathsf{PSh}_{\mathsf{k}}(\mathcal{C}), \star)$.

3.1 Presheaves on grouplike SMCs

We will begin by discussing the general behavior of presheaf categories $\mathsf{PSh}_{\mathsf{k}}(\mathcal{C})$ when $\mathcal{C}$ is grouplike. The main result of this section, Proposition 3.1.2.3, allows us to think of $\mathsf{PSh}_{\mathsf{k}}(\mathcal{C})$ as a category of "graded modules" (for a certain ∞-categorical interpretation of "grading").

3.1.1 The projection formula for presheaves

We will first need to establish that presheaves on grouplike SMCs satisfy a version of the projection formula. This is a straightforward but lengthy calculation, and actually holds in greater generality:

Proposition 3.1.1.1. *Let $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor between small SMCs, and suppose that every object of $\mathcal{C}$ is dualizable.*[1] *For $\mathscr{F} \in \mathsf{PSh}_{\mathcal{V}}(\mathcal{C})$ and $\mathscr{G} \in \mathsf{PSh}_{\mathcal{V}}(\mathcal{D})$, we have*

$$\mathsf{F}^*(\mathsf{F}_!\mathscr{F} \star \mathscr{G}) = \mathscr{F} \star \mathsf{F}^*\mathscr{G}.$$

Proof. The functors F^*, $\mathsf{F}_!$, and $\star$ all commute with colimits and $\mathcal{V}$-tensors, so it suffices to consider the case where $\mathscr{F} = \mathfrak{y}_{\mathcal{C}}(\mathsf{c})$ and $\mathscr{G} = \mathfrak{y}_{\mathcal{D}}(\mathsf{d})$ for some $\mathsf{c} \in \mathcal{C}$ and $\mathsf{d} \in \mathcal{D}$. Then, for

[1] This holds if $\mathcal{C}$ is grouplike, and this is the only situation we will need to consider in the rest of the paper.

$c' \in \mathcal{C}$, we have

$$\begin{aligned}
\big(F^*(F_! y_{\mathcal{C}}(c) \star y_{\mathcal{D}}(d))\big)(c') &= \big(F_! y_{\mathcal{C}}(c) \star y_{\mathcal{D}}(d)\big)(F(c')) \\
&= \big(y_{\mathcal{D}}(F(c)) \star y_{\mathcal{D}}(d)\big)(F(c')) \\
&= \mathrm{Hom}_{\mathsf{PSh}_{\mathcal{V}}(\mathcal{D})}\big(y_{\mathcal{D}}(F(c')), y_{\mathcal{D}}(F(c)) \star y_{\mathcal{D}}(d)\big) \\
&= \mathrm{Hom}_{\mathsf{PSh}_{\mathcal{V}}(\mathcal{D})}\big(y_{\mathcal{D}}(F(c')) \star y_{\mathcal{D}}(F(c))^{\vee}, y_{\mathcal{D}}(d)\big) \\
&= \mathrm{Hom}_{\mathsf{PSh}_{\mathcal{V}}(\mathcal{D})}\big(y_{\mathcal{D}}(F(c')) \star y_{\mathcal{D}}(F(c^{\vee})), y_{\mathcal{D}}(d)\big) \\
&= \mathrm{Hom}_{\mathsf{PSh}_{\mathcal{V}}(\mathcal{D})}\big(y_{\mathcal{D}}(F(c' \otimes_{\mathcal{C}} c^{\vee})), y_{\mathcal{D}}(d)\big) \\
&= \mathrm{Hom}_{\mathsf{PSh}_{\mathcal{V}}(\mathcal{D})}\big(F_!(y_{\mathcal{C}}(c') \star y_{\mathcal{C}}(c^{\vee})), y_{\mathcal{D}}(d)\big) \\
&= \mathrm{Hom}_{\mathsf{PSh}_{\mathcal{V}}(\mathcal{C})}\big(y_{\mathcal{C}}(c') \star y_{\mathcal{C}}(c^{\vee}), F^* y_{\mathcal{D}}(d)\big) \\
&= \mathrm{Hom}_{\mathsf{PSh}_{\mathcal{V}}(\mathcal{C})}\big(y_{\mathcal{C}}(c'), y_{\mathcal{C}}(c) \star F^* y_{\mathcal{D}}(d)\big) \\
&= \big(y_{\mathcal{C}}(c) \star F^* y_{\mathcal{D}}(d)\big)(c').
\end{aligned}$$

Since $c' \in \mathcal{C}$ was arbitrary, we get $F^*(F_! y_{\mathcal{C}}(c) \star y_{\mathcal{D}}(d)) = y_{\mathcal{C}}(c) \star F^* y_{\mathcal{D}}(d)$ as desired. □

3.1.2 $\mathsf{PSh}_k(\mathcal{C})$ as a category of graded modules

For an $\mathbb{E}_\infty$-monoid M, let $\mathsf{Loc}_k(M) = \mathsf{Fun}(M, \mathsf{Mod}_k)$ be the category of $\mathcal{V}$-valued *local systems* on M. Because M is a groupoid, there is a natural equivalence $M \simeq M^{\mathrm{op}}$, and in particular we have $\mathsf{Loc}_k(M) \simeq \mathsf{PSh}_k(M)$. We will view $\mathsf{Loc}_k(M)$ as symmetric monoidal under the Day convolution product $\star_M$.

The category $\mathsf{Loc}_k(M)$ serves as an ∞-categorical version of the category of "M-graded k-modules," as the following example makes clear.

Example 3.1.2.1. Let M be a discrete commutative monoid. Then objects of $\mathsf{Loc}_k(M)$ are given by families $V = (V_m)_{m \in M}$ in Mod_k, and morphisms $V \to V'$ are just families of morphisms $V_m \to V'_m$ for all $m \in M$ (with no coherence data involved). The symmetric monoidal structure on $\mathsf{Loc}_k(M)$ is given by

$$(V \star_M V')_m = \bigsqcup_{m_1 + m_2 = m} V_{m_1} \otimes_k V_{m_2}.$$

Remark 3.1.2.2. There is a natural t-structure on the category $\mathsf{Loc}_k(M)$ given by $\mathsf{Loc}_k^{\leqslant 0}(M) = \mathsf{Fun}(M, \mathsf{Mod}_k^{\leqslant 0})$. More generally, we may define a natural t-structure on any $\mathsf{PSh}_k(\mathcal{C})$ by $\mathsf{PSh}_k^{\leqslant 0}(\mathcal{C}) = \mathsf{Fun}(\mathcal{C}, \mathsf{Mod}_k^{\leqslant 0})$.

We would like to give a description of $\mathsf{PSh}_k(\mathcal{C})$ (for $\mathcal{C}$ a grouplike SMC) as a category of "graded modules over a graded algebra."

Proposition 3.1.2.3. *Let $\mathcal{C}$ be a grouplike SMC, and let $\phi : M \to \iota_0\mathcal{C}$ be a homomorphism of $\mathbb{E}_\infty$-groups which is surjective on π_0. Write ψ for the natural homomorphism* $\mathrm{P}_{0,M}\,\mathcal{C} \to M$

and $\pi_{\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C}}$ *for the natural symmetric monoidal functor* $\mathcal{C} \to \mathsf{B}\,\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C}$*. Then there is a* t*-exact symmetric monoidal equivalence*

$$\big(\mathsf{PSh}_{\mathsf{k}}(\mathcal{C}), \star_{\mathcal{C}}\big) = \big(\mathsf{Mod}_{\psi_! \pi^*_{\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C}}\mathsf{k}}(\mathsf{Loc}_{\mathcal{V}}(\mathsf{M})), \otimes_{\phi_! \pi^*_{\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C}}\mathsf{k}}\big)$$

Proof. Consider the adjunction

$$\phi_! : \mathsf{Loc}_{\mathsf{k}}(\mathsf{M}) \rightleftarrows \mathsf{PSh}_{\mathsf{k}}(\mathcal{C}) : \phi^*.$$

Note that:

- The pair $(\phi_!, \phi^*)$ satisfies the projection formula by Proposition 3.1.1.1.
- The functor ϕ^* preserves arbitrary colimits (since colimits in $\mathsf{PSh}_{\mathcal{V}}(\mathsf{M}\vec{/}\mathsf{N})$ are computed pointwise).
- The functor ϕ^* is conservative (since a morphism in $\mathsf{PSh}_{\mathsf{k}}(\mathcal{C})$ is an isomorphism if and only if it is an isomorphism pointwise).

Thus this adjunction satisfies the hypotheses of the symmetric monoidal Barr–Beck theorem [MNN17, Proposition 5.29] and we obtain a symmetric monoidal equivalence

$$\big(\mathsf{PSh}_{\mathsf{k}}(\mathcal{C}), \star_{\mathcal{C}}\big) = \big(\mathsf{Mod}_{\phi^*\phi_!\mathbb{1}_{\mathsf{Loc}_{\mathsf{k}}(\mathsf{M})}}(\mathsf{Loc}_{\mathsf{k}}(\mathsf{M})), \otimes_{\phi^*\phi_!\mathbb{1}_{\mathsf{Loc}_{\mathsf{k}}(\mathsf{M})}}\big).$$

Thus it suffices to show that

$$\phi^*\phi_!\mathbb{1}_{\mathsf{Loc}_{\mathcal{V}}(\mathsf{M})} = \psi_!\pi^*_{\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C}}\mathsf{k}.$$

In fact, because $\phi_! : (\mathsf{Loc}_{\mathsf{k}}(\mathsf{M}), \star_{\mathsf{M}}) \to (\mathsf{PSh}_{\mathsf{k}}(\mathcal{C}), \star_{\mathcal{C}})$ is symmetric monoidal, we have

$$\phi_!\mathbb{1}_{\mathsf{Loc}_{\mathcal{V}}(\mathsf{M})} = \mathbb{1}_{\mathsf{PSh}_{\mathsf{k}}(\mathcal{C})} = \eta_{\mathcal{C}!}\mathsf{k}$$

where $\eta_{\mathcal{C}} : 0 \to \mathcal{C}$ is the unit morphism. We may therefore reduce to showing that

$$\phi^*\eta_{\mathcal{C}!}\mathsf{k} = \psi_!\pi^*_{\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C}}\mathsf{k},$$

and this last claim follows from Lemma 3.1.2.4 below. □

Lemma 3.1.2.4. *Let* $\mathcal{C}$ *be a grouplike SMC, and let* $\phi : \mathsf{M} \to \iota_0\mathcal{C}$ *be a homomorphism of* $\mathbb{E}_\infty$*-groups which is surjective on* π_0*. Write:*

- ψ *for the natural homomorphism* $\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C} \to \mathsf{M}$
- $\pi_{\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C}}$ *for the natural symmetric monoidal functor* $\mathcal{C} \to \mathsf{B}\,\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C}$*, and*
- $\eta_{\mathcal{C}}$ *for the unit map* $0 \to \mathcal{C}$*.*

Then, for all $\mathfrak{m} \in \mathsf{M}$ *and* $\mathsf{V} \in \mathcal{V}$*, there is a natural equivalence*

$$\big(\phi^*\eta_{\mathcal{C}!}(\mathsf{V})\big)(\mathfrak{m}) = \operatorname*{colim}_{\psi^{-1}(\mathfrak{m})} \mathsf{V} = \big(\psi_!\pi^*_{\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C}}\mathsf{V}\big)(\mathfrak{m}).$$

Thus there is a natural equivalence $\phi^*\eta_{\mathcal{C}!} = \psi_!\pi^*_{\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C}}$.

Proof. Both equivalences follow from the pointwise formula for left Kan extensions. More precisely, for the first equivalence, note that

$$\big(\phi^*\eta_{\mathcal{C}!}(\mathsf{V})\big)(\mathfrak{m}) = \big(\eta_{\mathcal{C}!}(\mathsf{V})\big)(\phi(\mathfrak{m})) = \operatorname*{colim}_{\eta_{\mathcal{C}}(0)\to\phi(\mathfrak{m})} \mathsf{V} = \operatorname*{colim}_{\mathrm{Map}_{\mathcal{C}}(0_{\mathcal{C}},[\mathfrak{m}])} \mathsf{V},$$

and Corollary 2.2.1.5 gives $\mathrm{Map}_{\mathcal{C}}(0_{\mathcal{C}}, [\mathfrak{m}]) = \psi^{-1}(\mathfrak{m})$. For the second equivalence, note that

$$\big(\psi_!\pi^*_{\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C}}\mathsf{V}\big)(\mathfrak{m}) = \operatorname*{colim}_{\psi(\mathfrak{n})\to\mathfrak{m}} \pi^*_{\mathrm{P}_{0,\mathsf{M}}\,\mathcal{C}}(\mathsf{V})(\mathfrak{n}) = \operatorname*{colim}_{\psi(\mathfrak{n})\to\mathfrak{m}} \mathsf{V},$$

whih agrees with $\operatorname{colim}_{\psi^{-1}(\mathsf{M})} \mathsf{V}$ because M is a groupoid. □

3.2 Geometrization

We are now prepared to discuss the construction of stacks $\mathfrak{Y}_{\mathcal{C}}$ from grouplike SMCs $\mathcal{C}$. More precisely:

Definition 3.2.0.1. We define a *geometrization* functor $\mathfrak{Y}_{-} : \mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})^{\mathrm{op}} \to \mathsf{dStk}$ by the formula

$$\mathfrak{Y}_{\mathcal{C}}(\mathsf{R}) = \iota_0\,\mathsf{Fun}^{\otimes}(\mathcal{C}, \mathsf{Mod}_{\mathsf{R}}^{\leqslant 0})$$

for all $\mathsf{R} \in \mathsf{dCAlg}$.

To see that $\mathfrak{Y}_{\mathcal{C}}$ is indeed a derived stack, note that if $\operatorname{Spec}\mathsf{R}^\bullet \to \operatorname{Spec}\mathsf{R}$ is a Čech cover corresponding to an fppf covering sieve of $\operatorname{Spec}\mathsf{R}$, then

$$\begin{aligned}\lim \mathfrak{Y}_{\mathcal{C}}(\mathsf{R}^\bullet) &= \lim \iota_0\,\mathsf{Fun}^{\otimes}(\mathcal{C}, \mathsf{Mod}_{\mathsf{R}^\bullet}^{\leqslant 0})\\ &= \iota_0\,\mathsf{Fun}^{\otimes}(\mathcal{C}, \lim \mathsf{Mod}_{\mathsf{R}^\bullet}^{\leqslant 0})\\ &= \iota_0\,\mathsf{Fun}^{\otimes}(\mathcal{C}, \mathsf{Mod}_{\mathsf{R}}^{\leqslant 0}) \text{ by faithfully flat descent}\\ &= \mathfrak{Y}_{\mathcal{C}}(\mathsf{R})\end{aligned}$$

as needed.

Of course, we could have made this definition much earlier: the reason we have postponed it until now is that we are now able to prove algebraicity results for $\mathfrak{Y}_{\mathcal{C}}$ (when certain additional properties hold).

We will begin by discussing some useful basic properties and examples of geometrization Section 3.2.1. In Section 3.2.2, we will discuss how applying the geometrization functor to $\mathbb{E}_\infty$-groups produces $\mathbb{E}_\infty$-group objects in the category of derived stacks. Finally, in Section 3.2.3, we will use the preceding discussion and the characterizations of presheaf categories in Section 3.1 to prove algebraicity results for geometrizations of grouplike SMCs with “trivializable symmetry” (see Definition 3.2.3.1).

3.2.1 Basics of geometrization

Here are a few key examples to keep in mind when discussing geometrization:

Example 3.2.1.1. For any $\mathbb{E}_\infty$-monoid M, we have $\mathfrak{Y}_{\mathsf{BM}} = \operatorname{Spec} \mathsf{k}\langle \mathsf{M}\rangle$. Indeed, for all $\mathsf{R} \in \mathsf{dCAlg}_{\mathsf{k}}$, any symmetric monoidal functor $\mathsf{BM} \to \mathsf{Mod}_{\mathsf{R}}^{\leqslant 0}$ must send the unique object of BM to $\mathsf{R} \in \mathsf{Mod}_{\mathsf{R}}$, so

$$\mathfrak{Y}_{\mathsf{BM}}(\mathsf{R}) = \mathrm{Map}_{\mathsf{dCAlg}_{\mathsf{k}}}(\mathsf{k}\langle \mathsf{M}\rangle, \mathsf{R}) = (\operatorname{Spec} \mathsf{k}\langle \mathsf{M}\rangle)(\mathsf{R}).$$

Example 3.2.1.2. If $\mathbb{F}^{\mathrm{gp}}$ is the free $\mathbb{E}_\infty$-group on one generator, then $\mathfrak{Y}_{\mathbb{F}^{\mathrm{gp}}} = \mathrm{BGL}_1$. Indeed, for any $\mathsf{R} \in \mathsf{dCAlg}_{\mathsf{k}}$, we have

$$\mathfrak{Y}_{\mathbb{F}^{\mathrm{gp}}}(\mathsf{R}) = \iota_0 \mathsf{Fun}^{\otimes}(\mathbb{F}^{\mathrm{gp}}, \mathsf{Mod}_{\mathsf{R}}^{\leqslant 0}) = \mathrm{Pic}(\mathsf{R}) = \mathrm{BGL}_1(\mathsf{R}).$$

Note that the derived stacks $\mathfrak{Y}_{\mathsf{BM}}$ and $\mathfrak{Y}_{\mathbb{F}^{\mathrm{gp}}}$ admit natural $\mathbb{E}_\infty$-monoid structures in the category of derived stacks. This in fact holds much more generally:

Proposition 3.2.1.3. *Let $\mathcal{C}$ be a grouplike SMC. Then the derived stack $\mathfrak{Y}_{\mathcal{C}}$ has a natural $\mathbb{E}_\infty$-monoid structure, i.e. $\mathfrak{Y}_{\mathcal{C}} \in \mathsf{CAlg}(\mathsf{dStk})$. If $\mathcal{C} = \mathsf{A}$ is an $\mathbb{E}_\infty$-group, then $\mathfrak{Y}_{\mathsf{A}}$ is an $\mathbb{E}_\infty$-group object in dStk.*

Proof. Considering $(\mathsf{Cat}, \times)$ as a Cartesian monoidal category, [HA, Corollary 2.4.3.10] implies that every object of Cat admits a natural unique $\mathbb{E}_\infty$-coalgebra structure in Cat (and this extends to an equivalence of categories). Passing to commutative algebra objects in Cat, we see that the forgetful map $\mathsf{BiAlg}_{\mathbb{E}_\infty,\mathbb{E}_\infty}(\mathsf{Cat}) \to \mathsf{CAlg}(\mathsf{Cat})$ is an equivalence. Corollary 1.2.3.3 then implies that, for any $\mathsf{R} \in \mathsf{dCAlg}$, there is a natural $\mathbb{E}_\infty$-monoid structure on the mapping space $\iota_0 \mathsf{Fun}^{\otimes}(\mathcal{C}, \mathsf{Mod}_{\mathsf{R}}^{\leqslant 0}) = \mathfrak{Y}_{\mathcal{C}}(\mathsf{R})$. Combining these algebra structures for all R gives the $\mathbb{E}_\infty$-monoid structure on $\mathfrak{Y}_{\mathcal{C}}$.

If $\mathcal{C} = \mathsf{A}$ is an $\mathbb{E}_\infty$-group, then we in fact have

$$\mathfrak{Y}_{\mathsf{A}}(\mathsf{R}) = \iota_0 \mathsf{Fun}^{\otimes}(\mathsf{A}, \mathrm{Pic}(\mathsf{R})) = \mathrm{Hom}_{\mathsf{CGrp}}(\mathsf{A}, \mathrm{Pic}(\mathsf{R}))$$

where $\mathrm{Pic}(\mathsf{R})$ here denotes the $\mathbb{E}_\infty$-group of line bundles over R. Because homomorphisms between $\mathbb{E}_\infty$-groups form an $\mathbb{E}_\infty$-group, we see that $\mathfrak{Y}_{\mathsf{A}}(\mathsf{R})$ is itself an $\mathbb{E}_\infty$-group. Combining this for all R shows that $\mathfrak{Y}_{\mathsf{A}}$ is in fact an $\mathbb{E}_\infty$-group object in dStk. $\square$

We conclude with a remark which will not be directly useful here, but which will motivate some of the discussion in Chapter 6:

Remark 3.2.1.4. The geometrization functor $\mathfrak{Y} : \mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})^{\mathrm{op}} \to \mathsf{dStk}$ preserves all limits. Indeed, we have

$$\mathfrak{Y}_{\mathrm{colim}_i \mathcal{C}_i}(\mathsf{R}) = \iota_0 \mathsf{Fun}^{\otimes}\left(\operatorname*{colim}_i \mathcal{C}_i, \mathsf{Mod}_{\mathsf{R}}^{\leqslant 0}\right) = \lim_i \iota_0 \mathsf{Fun}^{\otimes}\left(\mathcal{C}_i, \mathsf{Mod}_{\mathsf{R}}^{\leqslant 0}\right) = \lim_i \mathfrak{Y}_{\mathcal{C}_i}(\mathsf{R})$$

for any diagram $\mathcal{C}_- : \mathrm{I} \to \mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})^{\mathrm{op}}$ and any $\mathsf{R} \in \mathsf{dCAlg}$.

3.2.2 Geometrizing $\mathbb{E}_\infty$-groups

Our goal in this subsection is to discuss the behavior of the geometrization functor on certain $\mathbb{E}_\infty$-groups.

Notation 3.2.2.1. Fix an $\mathbb{E}_\infty$-group A, and let $G_A = \operatorname{Spec} k\langle A\rangle$.

Here are some key examples to consider:

Example 3.2.2.2. If $A = \mathbb{Z}$, then $G_\mathbb{Z} = \operatorname{Spec} k[t, t^{-1}] = \mathbb{G}_m$. More generally, if A is free of rank n, then G_A is an algebraic torus of rank n.

Example 3.2.2.3. If $A = \mathbb{Z}/n$, then $G_{\mathbb{Z}/n} = \operatorname{Spec} k[t]/(t^n - 1) = \mu_n$.

By Proposition 3.2.1.3, G_A is an $\mathbb{E}_\infty$-group stack. Thus it makes sense to talk about the classifying stack BG_A. Using the comonadic version of the Barr–Beck theorem, one can prove the following result,

Proposition 3.2.2.4 ([Mou21, Theorem 4.1 and Proposition 8.2])**.** [2] *Suppose A is discrete. Then there is a t-exact symmetric monoidal equivalence*

$$(\mathsf{Loc}_k(A), \star_A) \xrightarrow{\sim} (\mathsf{QC}(BG_A), \otimes_\mathscr{O})$$

This equivalence sends $\mathsf{Loc}_k^{\leqslant 0}(A)$ to $\mathsf{QC}^{\leqslant 0}(BG_A)$.

Corollary 3.2.2.5. *If A is a discrete abelian group, then $\mathfrak{Y}_A = BG_A$, and there is a natural symmetric monoidal equivalence*

$$(\mathsf{Loc}_k(A), \star_A) \xrightarrow{\sim} (\mathsf{QC}(\mathfrak{Y}_A), \otimes_\mathscr{O}).$$

Proof. For any $R \in \mathsf{dCAlg}_k$, we may write

$$\begin{aligned}
\mathfrak{Y}_A(R) &= \iota_0 \operatorname{Fun}^\otimes(A, \mathsf{Mod}_R^{\leqslant 0}) \\
&= \iota_0 \operatorname{Fun}_k^{\otimes, L}\left((\mathsf{Loc}_k^{\leqslant 0}(A), \star_A), (\mathsf{Mod}_R^{\leqslant 0}, \otimes_R)\right) \\
&= \iota_0 \operatorname{Fun}_k^{\otimes, L}\left((\mathsf{QC}^{\leqslant 0}(BG_A), \otimes_\mathscr{O}), (\mathsf{Mod}_R^{\leqslant 0}, \otimes_R)\right) \\
&= BG_A(R) \text{ by [Ste23, Theorem 1.0.3]},
\end{aligned}$$

giving the first claim. The second claim is then just a restatement of Proposition 3.2.2.4. $\square$

Remark 3.2.2.6. Some sort of discreteness hypothesis is necessary for Corollary 3.2.2.5 to hold. Indeed, both claims in Corollary 3.2.2.5 may fail when we allow A to be non-discrete:

[2] Moulinos states the claim only for $A = \mathbb{Z}$, but the argument works for any discrete A.

- If we take $A = \mathbb{F}^{\mathrm{gp}}$, then by Example 3.2.1.2, we have $\mathfrak{Y}_{\mathbb{F}^{\mathrm{gp}}} = \mathrm{BGL}_1$. Because we are working over a field k of characteristic zero, we have $\mathrm{GL}_1 = \mathbb{G}_m$, so $\mathfrak{Y}_{\mathbb{F}^{\mathrm{gp}}} = \mathfrak{Y}_{\mathbb{Z}}$. However, we have

$$\mathsf{QC}(\mathfrak{Y}_{\mathbb{F}^{\mathrm{gp}}}) = \mathsf{QC}(\mathfrak{Y}_{\mathbb{Z}}) = \mathsf{Loc}_k(\mathbb{Z}) \neq \mathsf{Loc}_k(\mathbb{F}^{\mathrm{gp}}),$$

  for example because the unit object of $\mathsf{Loc}_k(\mathbb{F}^{\mathrm{gp}})$ has

$$\mathsf{H}^0 \operatorname{End}_k(\mathbb{1}_{\mathsf{Loc}_k(\mathbb{F}^{\mathrm{gp}})})) = \mathsf{H}^0(k\langle \Omega_0 \mathbb{F}^{\mathrm{gp}} \rangle) = k\langle \pi_1(\mathbb{F}^{\mathrm{gp}}, 0)\rangle = k\langle \mathbb{Z}/2\rangle \neq k.$$

- If we take $A = \mathsf{B}\mathbb{Z}$, then by Example 3.2.1.1, we have $\mathfrak{Y}_{\mathsf{B}\mathbb{Z}} = \mathbb{G}_m$, so $\mathfrak{Y}_{\mathsf{B}\mathbb{Z}}$ is not a classifying stack. However, we do have $\mathsf{Loc}_k(\mathsf{B}\mathbb{Z}) = \mathsf{Mod}_{k\langle \mathbb{Z}\rangle} = \mathsf{QC}(\mathfrak{Y}_A)$. The only problem here is that the unit map $\mathrm{pt} \to \mathfrak{Y}_{\mathsf{B}\mathbb{Z}}$ is not a cover in any reasonable topology on dAff_k.

3.2.3 $\mathsf{PSh}_k(\mathcal{C})$ versus $\mathsf{QC}(\mathfrak{Y}_{\mathcal{C}})$

We would like to extend Corollary 3.2.2.5 to the stacks $\mathfrak{Y}_{\mathcal{C}}$ for more general grouplike SMCs $\mathcal{C}$. To avoid the first counterexample of Remark 3.2.2.6, we will need to impose some extra conditions on the grouplike SMCs we study:

Definition 3.2.3.1. An $\mathbb{E}_\infty$-group A has *trivializable symmetry* if the natural map $A \to \pi_0(A)$ splits in CGrp, i.e. there is a (typically non-canonical) equivalence of $\mathbb{E}_\infty$-groups $A = \pi_0(A) \times A^0$. A grouplike SMC $\mathcal{C}$ has *trivializable symmetry* if the $\mathbb{E}_\infty$-group of objects $\iota_0 A$ does.

Example 3.2.3.2. A grouplike symmetric monoidal 1-category $\mathcal{C}$ has trivializable symmetry if and only if, up to natural symmetric monoidal equivalence, we may assume that the grading σ of $\mathcal{C}$ satisfies $\sigma_{C,C} = \mathrm{id}_{C \oplus C}$ for all $C \in \mathcal{C}$.

By Proposition 2.1.2.3, an $\mathbb{E}_\infty$-group A has trivializable symmetry if and only if the extension class $\eta_A \in \mathsf{H}(\Omega_0 A)^2\big(\mathsf{H}(\pi_0(A))\big)$ is zero. Thus:

Example 3.2.3.3. Any discrete $\mathbb{E}_\infty$-group has trivializable symmetry.

Example 3.2.3.4. Any connected $\mathbb{E}_\infty$-group has trivializable symmetry.

Example 3.2.3.5. The $\mathbb{E}_\infty$-group $\mathbb{F}^{\mathrm{gp}}$ does not have trivializable symmetry. Indeed, by the discussion in [JO12, §3], $\eta_{\mathbb{F}^{\mathrm{gp}}} \in \mathsf{H}(\Omega_0 \mathbb{F}^{\mathrm{gp}})^2(\mathsf{H}\mathbb{Z})$ maps to the nonzero element of $\mathsf{H}(\mathbb{Z}/2)^2\mathsf{H}(\mathbb{Z})$, hence is itself nonzero.

We may now show that, for every grouplike SMC $\mathcal{C}$, the stack $\mathfrak{Y}_{\mathcal{C}}$ admits a global quotient presentation. This presentation is not canonical. The lack of canonicity is closely related to issues with the Cox construction for toric varieties in the presence of toric factors, which we plan to discuss in future work.

Proposition 3.2.3.6. *Let $\mathcal{C}$ be a grouplike SMC with trivializable symmetry. Then there is a (typically non-canonical) equivalence $\mathfrak{Y}_{\mathcal{C}} \simeq [(\operatorname{Spec} \mathsf{k}\langle \mathrm{P}_{0,\pi_0(\iota_0\mathcal{C})}\, \mathcal{C}\rangle)/\mathsf{G}_{\mathsf{A}}]$.*

Proof. Choose a splitting of the forgetful map $\iota_0\mathcal{C} \to \pi_0(\iota_0\mathcal{C})$. This gives an essentially surjective symmetric monoidal functor $\pi_0(\iota_0\mathcal{C}) \to \mathcal{C}$. By Proposition 2.3.1.1, we obtain a skew quotient presentation $\mathcal{C} = \pi_0(\iota_0\mathcal{C})\vec{/}\,\mathrm{P}_{0,\pi_0(\iota_0\mathcal{C})}\,\mathcal{C}$. Proposition 3.1.2.3 then gives a symmetric monoidal equivalence

$$\big(\mathsf{PSh}_{\mathsf{k}}(\mathcal{C}), \star_{\mathcal{C}}\big) \simeq \Big(\mathsf{Mod}_{\mathsf{A}}\,\big(\mathsf{Loc}_{\mathsf{k}}(\pi_0(\iota_0\mathcal{C})\big), \otimes_{\mathsf{A}}\big)$$

where $\mathsf{A} := \mathsf{k}\langle \mathrm{P}_{0,\pi_0(\iota_0\mathcal{C})}\,\mathcal{C}\rangle$ is viewed as a local system over $\pi_0(\iota_0\mathcal{C})$ with fiber over $\mathsf{p} \in \pi_0(\iota_0\mathcal{C})$ given by $\mathsf{k}\langle(\mathrm{P}_{0,\pi_0(\iota_0\mathcal{C})}\,\mathcal{C}) \times_{\pi_0(\iota_0\mathcal{C})} \{\mathsf{p}\}\rangle$. By Corollary 3.2.2.5 this gives a t-exact symmetric monoidal equivalence

$$\big(\mathsf{PSh}_{\mathsf{k}}(\mathcal{C}), \star_{\mathcal{C}}\big) \simeq \Big(\mathsf{Mod}_{\mathsf{A}}\,\big(\mathsf{QC}(\mathsf{BG}_{\pi_0(\iota_0\mathcal{C})})\big), \otimes_{\mathsf{A}}\Big)$$

or equivalently

$$\big(\mathsf{PSh}_{\mathsf{k}}(\mathcal{C}), \star_{\mathcal{C}}\big) \simeq \Big(\mathsf{QC}\,\big([(\operatorname{Spec}\mathsf{A})/\mathsf{G}_{\pi_0(\iota_0\mathcal{C})}]\big), \otimes_{\mathscr{O}}\Big). \tag{3.2.3.7}$$

By Tannaka duality ([Ste23, Theorem 1.0.3] again), for all $\mathsf{R} \in \mathsf{dCAlg}$, we get

$$\begin{aligned}
[(\operatorname{Spec}\mathsf{A})/\mathsf{G}_{\pi_0(\iota_0\mathcal{C})}](\mathsf{R}) &= \mathsf{Fun}^{\otimes,\mathsf{L}}(\mathsf{QC}^{\leqslant 0}([(\operatorname{Spec}\mathsf{A})/\mathsf{G}_{\pi_0(\iota_0\mathcal{C})}]), \mathsf{Mod}_{\mathsf{R}}^{\leqslant 0}) \\
&\simeq \mathsf{Fun}^{\otimes,\mathsf{L}}(\mathsf{PSh}_{\mathsf{k}}(\mathcal{C})^{\leqslant 0}, \mathsf{Mod}_{\mathsf{R}}^{\leqslant 0}) \\
&= \mathsf{Fun}^{\otimes,\mathsf{L}}(\mathcal{C}, \mathsf{Mod}_{\mathsf{R}}^{\leqslant 0}) \\
&= \mathfrak{Y}_{\mathcal{C}}(\mathsf{R}).
\end{aligned}$$

This equivalence is natural in R (but not in $\mathcal{C}$), so we get $\mathfrak{Y}_{\mathcal{C}} \simeq [(\operatorname{Spec}\mathsf{A})/\mathsf{G}_{\pi_0(\iota_0\mathcal{C})}](\mathsf{R})$. $\square$

Even though the equivalence of Proposition 3.2.3.6 is not canonical, we can obtain a canonical description of the category $\mathsf{QC}(\mathfrak{Y}_{\mathcal{C}})$:

Proposition 3.2.3.8. *Let $\mathcal{C}$ be a grouplike SMC with trivializable symmetry. Then there is a natural t-exact symmetric monoidal equivalence*

$$\big(\mathsf{PSh}_{\mathsf{k}}(\mathcal{C}), \star_{\mathcal{C}}\big) \tilde{\to} \big(\mathsf{QC}(\mathfrak{Y}_{\mathcal{C}}), \otimes_{\mathscr{O}}\big).$$

Proof. To construct our alleged equivalence $\bar{\mathsf{F}}$, we first construct a natural symmetric monoidal functor $\mathsf{F} : \mathcal{C} \to \big(\mathsf{QC}(\mathfrak{Y}_{\mathcal{C}}), \otimes_{\mathscr{O}}\big)$. By definition we have

$$\mathsf{QC}(\mathfrak{Y}_{\mathcal{C}}) = \lim_{\operatorname{Spec}\mathsf{R}\to\mathfrak{Y}_{\mathcal{C}}} \mathsf{Mod}_{\mathsf{R}} = \lim_{\mathcal{C}\to\mathsf{Mod}_{\mathsf{R}}^{\leqslant 0}} \mathsf{Mod}_{\mathsf{R}}$$

where the last limit is taken over all symmetric monoidal functors $\mathcal{C} \to \mathsf{Mod}_{\mathsf{R}}^{\leqslant 0}$ with R a derived commutative ring. There is a tautological symmetric monoidal functor $\mathcal{C} \to$

$\lim_{\mathcal{C}\to\mathsf{Mod}_{\mathsf{R}}^{\leqslant 0}} \mathsf{Mod}_{\mathsf{R}}^{\leqslant 0}$, and composing this with the inclusion $\lim_{\mathcal{C}\to\mathsf{Mod}_{\mathsf{R}}^{\leqslant 0}} \mathsf{Mod}_{\mathsf{R}}^{\leqslant 0} \subset \mathsf{QC}(\mathfrak{Y}_{\mathcal{C}})$ gives the desired F. The universal property of Day convolution then gives a symmetric monoidal functor $\bar{\mathsf{F}} : \big(\mathsf{PSh}_{\mathsf{k}}(\mathcal{C}), \star_{\mathcal{C}}\big) \to \big(\mathsf{QC}(\mathfrak{Y}_{\mathcal{C}}), \otimes_{\mathscr{O}}\big)$, and it is clear from the definition that this is t-exact.

To show that $\bar{\mathsf{F}}$ is an equivalence, we use the non-canonical equivalence of Proposition 3.2.3.6. Indeed, post-composing $\bar{\mathsf{F}}$ with the induced equivalence

$$\mathsf{QC}(\mathfrak{Y}_{\mathcal{C}}) \simeq \mathsf{QC}([(\operatorname{Spec}\mathsf{A})/\mathsf{G}_{\pi_0(\iota_0\mathcal{C})}])$$

(where $\mathsf{A} = \mathsf{k}\langle \mathrm{P}_{0,\pi_0(\iota_0\mathcal{C})}\,\mathcal{C}\rangle$), we obtain the equivalence Eq. (3.2.3.7). By the two-out-of-three property for equivalences, we see that $\bar{\mathsf{F}}$ is itself an equivalence. $\square$

Chapter 4

Convolution and Tensor Products

4.1 Quivers and derived categories of quotient stacks

Notation 4.1.0.1. Throughout this section, fix a commutative reductive group G (i.e. a product of a torus and a finite abelian group). Let $\phi : \mathfrak{Y} \to \mathsf{BG}$ be a morphism of derived stacks which is affine and almost of finite type. Equivalently, there exists a unique affine derived scheme Y almost of finite type over $\operatorname{Spec} \mathsf{k}$ and a G-action $\mathsf{G} \curvearrowright \mathsf{Y}$ such that $\mathfrak{Y} = [\mathsf{Y}/\mathsf{G}]$.

The derived category $\mathsf{QC}(\mathfrak{Y}) = \mathsf{QC}([\mathsf{Y}/\mathsf{G}])$ may be understood as the derived category of G-equivariant quasicoherent sheaves on Y. Because G is reductive, for $\mathscr{F}, \mathscr{G} \in \mathsf{QC}(\mathfrak{Y})$, the Hom-complexes $\operatorname{Hom}_{\mathfrak{Y}}(\mathscr{F}, \mathscr{G})$ may be computed by taking the G-invariants of the Hom-complexes of the corresponding quasicoherent sheaves on Y. We write $\mathscr{O}_{\mathfrak{Y}}(\chi)$ for the pullback of the line bundle (a.k.a. 1-dimensional representation) $\mathscr{O}_{\mathsf{BG}}(\chi) \in \mathsf{QC}(\mathsf{BG})$ to $\mathfrak{Y}$.

We are interested in describing the symmetric monoidal category $(\mathsf{QC}(\mathfrak{Y}), \otimes_{\mathscr{O}})$ as a functor category out of some small k-linear category.

Definition 4.1.0.2. The *total weight quiver* $\mathsf{Q}(\phi)$ is the essentially small full symmetric monoidal subcategory of $(\mathsf{Perf}(\mathfrak{Y}), \otimes_{\mathscr{O}})$ with objects given by $\{\mathscr{O}_{\mathfrak{Y}}(\chi)\}_{\chi \in \mathbb{X}^\bullet(\mathsf{G})}$. More generally, for any subset $\mathsf{S} \subset \mathbb{X}^\bullet(\mathsf{G})$, the *partial weight quiver* $\mathsf{Q}_\mathsf{S}(\phi)$ corresponding to S is the full (not necessarily monoidal) subcategory of $\mathsf{Perf}(\mathfrak{Y})$ with objects $\{\mathscr{O}_{\mathfrak{Y}}(\chi)\}_{\chi \in \mathsf{S}}$.

Remark 4.1.0.3. Suppose $\mathsf{Y} = \operatorname{Spec} \mathsf{R}$ for a derived ring R. Then specifying the G-action on Y is the same as specifying a $\mathbb{X}^\bullet(\mathsf{G})$-grading on the derived ring R, say $\mathsf{R} = \oplus_{\chi \in \mathbb{X}^\bullet(\mathsf{G})} \mathsf{R}_\chi$. From this perspective, quasicoherent sheaves on $\mathfrak{Y}$ correspond to $\mathbb{X}^\bullet(\mathsf{G})$-graded R-modules. In this case, we have the following explicit (but less homotopy-coherent) description of the partial weight quivers $\mathsf{Q}_\mathsf{S}(\phi)$:

- $\operatorname{ob} \mathsf{Q}_\mathsf{S}(\phi) = \mathsf{S}$.
- For $\chi_1, \chi_2 \in \mathsf{S}$, we have $\operatorname{Hom}_{\mathsf{Q}_\mathsf{S}(\phi)}(\chi_1, \chi_2) = \mathsf{R}_{\chi_2 - \chi_1}$.

- Composition of morphisms in $\mathsf{Q}_{\mathsf{S}}(\phi)$ is multiplication in R.

The symmetric monoidal structure on the total weight quiver $\mathsf{Q}(\phi)$ is given on objects by addition in $\mathbb{X}^\bullet(\mathsf{G})$ and on morphisms[1] by multiplication in R.

Example 4.1.0.4. If $\mathsf{G} = \mathbb{G}_m$ and $\mathfrak{Y} = [\mathbb{A}^{n+1}/\mathbb{G}_m]$, the total weight quiver $\mathsf{Q}(\phi)$ is the k-linearization of the infinite Beilinson quiver appearing in Fig. 0.2. The partial weight quiver for $\{0, \ldots, n\} \subset \mathbb{Z} = \mathbb{X}^\bullet(\mathbb{G}_m)$ is the k-linearization of the Beilinson quiver appearing in Fig. 0.1.

Due to the above interpretations, we will often implicitly identify the objects of the category $\mathsf{Q}_{\mathsf{S}}(\phi)$ with $\mathsf{S} \subset \mathbb{X}^\bullet(\mathsf{G})$. Following Example 4.1.0.4 (and Definition 5.1.4.1 below), we will generally think of $\mathsf{Q}_{\mathsf{S}}(\phi)$ as a "k-linear quiver with relations" and think of k-linear functors from $\mathsf{Q}_{\mathsf{S}}(\phi)$ (or $\mathsf{Q}_{\mathsf{S}}(\phi)^{\mathrm{op}}$) to $\mathsf{D}(\mathsf{k})$ as derived quiver representations.

By Proposition 1.2.3.2, there exists a unique k-linear colimit-preserving symmetric monoidal structure $\odot_{\mathsf{Q}(\phi)}$ on $\mathsf{D}(\mathsf{Q}(\phi)^{\mathrm{op}})$ extending the natural symmetric monoidal structure on $\mathsf{Q}(\phi)$ (viewed as a full subcategory of $\mathsf{D}(\mathsf{Q}(\phi)^{\mathrm{op}})$ via the Yoneda embedding). More concretely, for $\mathsf{V}_1, \mathsf{V}_2 \in \mathsf{D}(\mathsf{Q}(\phi)^{\mathrm{op}}) = \mathsf{Fun}_{\mathsf{k}}(\mathsf{Q}(\phi)^{\mathrm{op}}, \mathsf{D}(\mathsf{k}))$ and $\chi \in \mathbb{X}^\bullet(\mathsf{G})$, we have

$$(\mathsf{V}_1 \odot_{\mathsf{Q}(\phi)} \mathsf{V}_2)(\chi) = \bigoplus_{\chi_1+\chi_2=\chi} \mathsf{V}_1(\chi_1) \otimes_{\mathsf{k}} \mathsf{V}_2(\chi_2).$$

With the above definitions, it is not hard to give a description of $(\mathsf{QC}(\mathfrak{Y}), \otimes_{\mathscr{O}})$ in terms of $\mathsf{Q}(\phi)$. Special cases and variants of this statement are well-known in the literature, especially in the context of equivariant mirror symmetry for affine toric varieties (see e.g. [BO19, Proposition 2.1], as well as [Mou21, Theorem 1.1] and [BH25, Proposition 3.3.1] for statements over the sphere spectrum).

Proposition 4.1.0.5. *The inclusion* $\mathsf{Q}(\phi) \hookrightarrow \mathsf{QC}(\mathfrak{Y})$ *induces a symmetric monoidal equivalence*

$$\Psi : (\mathsf{D}(\mathsf{Q}(\phi)^{\mathrm{op}}), \odot_{\mathsf{Q}(\phi)}) \xrightarrow{\sim} (\mathsf{QC}(\mathfrak{Y}), \otimes_{\mathscr{O}}).$$

Proof. The functor Ψ is symmetric monoidal by Proposition 1.2.3.2(1). It suffices to show that $\mathsf{QC}(\mathfrak{Y})$ is compactly generated by the line bundles $\{\mathscr{O}_{\mathfrak{Y}}(\chi)\}_{\chi \in \mathbb{X}^\bullet(\mathsf{G})}$ (as $\otimes_{\mathscr{O}}$ must then be the unique colimit-preserving k-linear extension of the symmetric monoidal structure on $\mathsf{Q}(\phi)$). This is standard, though we provide an "intrinsically derived" proof.

Because G is a reductive group, $\mathsf{QC}(\mathsf{BG})$ is compactly generated by the line bundles $\{\mathscr{O}_{\mathsf{BG}}(\chi)\}_{\chi \in \mathbb{X}^\bullet(\mathsf{G})}$ (see e.g. [BFN10, Corollary 3.22]). The natural map $\mathfrak{Y} \to \mathsf{BG}$ is affine, so we may write

$$\mathsf{QC}(\mathfrak{Y}) = \mathsf{Mod}_{\mathscr{A}}(\mathsf{QC}(\mathsf{BG}))$$

[1] The behavior on morphisms is forced on us by the fact that the underlying category of $\mathsf{Q}(\phi)$ is grouplike.

for some $\mathscr{A} \in \mathsf{CAlg}(\mathsf{QC}^{\leqslant 0}(\mathsf{BG}))$. Observe that $\mathsf{Mod}_{\mathscr{A}}(\mathsf{QC}(\mathsf{BG}))$ is compactly generated by $\{\mathscr{A}(\chi)\}_{\chi \in \mathbb{X}^\bullet(\mathsf{G})}$. In fact, if $\mathscr{F} \in \mathsf{Mod}_{\mathscr{A}}(\mathsf{QC}(\mathsf{BG}))$ satisfies $\mathrm{Hom}_{\mathsf{Mod}_{\mathscr{A}}(\mathsf{QC}(\mathsf{BG}))}(\mathscr{A}(\chi), \mathscr{F}) = 0$ for all $\chi \in \mathbb{X}^\bullet(\mathsf{G})$, then by adjunction

$$\mathrm{Hom}_{\mathsf{QC}(\mathsf{BG})}(\mathscr{O}_{\mathsf{BG}}(\chi), \mathscr{F}) = \mathrm{Hom}_{\mathsf{Mod}_{\mathscr{A}}(\mathsf{QC}(\mathsf{BG}))}(\mathscr{A}(\chi), \mathscr{F}) = 0$$

for all $\chi \in \mathbb{X}^\bullet(\mathsf{G})$, so the underlying object of the $\mathscr{A}$-module $\mathscr{F}$ is zero and we must have $\mathscr{F} = 0$. It follows that $\mathsf{QC}(\mathfrak{Y})$ is compactly generated by $\{\mathscr{O}_{\mathfrak{Y}}(\chi)\}_{\chi \in \mathbb{X}^\bullet(\mathsf{G})}$. □

Combining Proposition 4.1.0.5 with Tannakian reconstruction theory gives a useful moduli interpretation of $\mathfrak{Y}$.

Corollary 4.1.0.6. *For all* $\mathsf{R} \in \mathsf{dCAlg}_{\mathsf{k}}$*, we have*

$$\mathfrak{Y}(\mathsf{R}) = \mathsf{Fun}^{\otimes}(\mathsf{Q}(\phi), \mathsf{Perf}^{\leqslant 0}(\mathsf{R}))^{\simeq}.$$

Proof. By [BH17, Theorem 1.3], we may identify $\mathfrak{Y}(\mathsf{R})$ with the subspace of $\mathsf{Fun}^{\mathsf{L},\otimes}_{\mathsf{k}}(\mathsf{QC}(\mathfrak{Y}), \mathsf{D}(\mathsf{R}))^{\simeq}$ consisting of functors which preserve connective objects. By Proposition 4.1.0.5 and Proposition 1.2.3.2(2), we may write

$$\mathsf{Fun}^{\mathsf{L},\otimes}_{\mathsf{k}}(\mathsf{QC}(\mathfrak{Y}), \mathsf{D}(\mathsf{R}))^{\simeq} = \mathsf{Fun}^{\otimes}(\mathsf{Q}(\phi), \mathsf{D}(\mathsf{R}))^{\simeq},$$

and the subspace of $\mathsf{Fun}^{\mathsf{L},\otimes}(\mathsf{QC}(\mathfrak{Y}), \mathsf{D}(\mathsf{R}))^{\simeq}$ consisting of functors which preserve connective objects is identified with $\mathsf{Fun}^{\otimes}(\mathsf{Q}(\phi), \mathsf{D}^{\leqslant 0}(\mathsf{R}))^{\simeq}$. Because every object of $\mathsf{Q}(\phi)$ is invertible with respect to $\otimes_{\mathsf{Q}}$, the image of every symmetric monoidal functor $\mathsf{Q}(\phi) \to \mathsf{D}^{\leqslant 0}(\mathsf{R})$ must consist of line bundles, hence must lie in $\mathsf{Perf}^{\leqslant 0}(\mathsf{R})$. Thus $\mathsf{Fun}^{\otimes}(\mathsf{Q}(\phi), \mathsf{D}^{\leqslant 0}(\mathsf{R}))^{\simeq} = \mathsf{Fun}^{\otimes}(\mathsf{Q}(\phi), \mathsf{Perf}^{\leqslant 0}(\mathsf{R}))^{\simeq}$. □

4.2 Convolution and quiver tensor products

Notation 4.2.0.1. Let G be a commutative reductive group. Let $\phi : \mathfrak{M} \to \mathsf{BG}$ be a homomorphism of $\mathbb{E}_n$-monoid derived stacks which is (as a morphism of derived stacks) affine and almost of finite type, so $\mathfrak{M}$ is a perfect stack. In particular, taking $\mathsf{M} = \mathfrak{M} \times_{\mathsf{BG}} \operatorname{Spec} \mathsf{k}$, the affine derived scheme M is an $\mathbb{E}_n$-monoid derived scheme, and the natural morphism $\mathsf{M} \to \mathfrak{M}$ is a morphism of $\mathbb{E}_n$-monoids. Write $\mu : \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$ for the binary multiplication map.

Remark 4.2.0.2. In general, we expect that such an $\mathbb{E}_n$-homomorphism $\mathfrak{M} \to \mathsf{BG}$ is obtained from a "normal" $\mathbb{E}_n$-homomorphism $\mathsf{G} \to \mathsf{M}$ (defined analogously to the inclusion of a normal subgroup). However, we are not aware of a workable characterization of normal $\mathbb{E}_n$-homomorphisms in the ∞-categorical context. In our classical applications, the $\mathbb{E}_n$-structure on $\mathfrak{M} \to \mathsf{BG}$ can be checked "by hand," so we leave development of a theory of normal $\mathbb{E}_n$-homomorphisms to future work.

By Proposition 4.1.0.5, we know that $\mathsf{QC}(\mathfrak{M}) \simeq \mathsf{D}(\mathsf{Q}(\phi)^{\mathrm{op}})$ and that the usual tensor product $\otimes_{\mathscr{O}}$ on $\mathsf{Q}(\phi)$ corresponds to the Day convolution product on $\mathsf{D}(\mathsf{Q}(\phi)^{\mathrm{op}})$. Under the above hypotheses, $\mathsf{QC}(\mathfrak{M})$ also admits an $\mathbb{E}_n$-monoidal convolution product $\star_{\mathfrak{M}}$ by the results of Section 1.3.3. Our goal in this section is to show (Theorem 4.2.3.1) that $\star_{\mathfrak{M}}$ corresponds to a "quiver tensor product" $\otimes_{\mathsf{Q}}$ on $\mathsf{D}(\mathsf{Q}(\phi)^{\mathrm{op}})$. Compared with Proposition 4.1.0.5, establishing Theorem 4.2.3.1 as a homotopy-coherent $\mathbb{E}_n$-monoidal equivalence is nontrivial, and we are not aware of versions of this result in the literature (even at the level of homotopy categories).

4.2.1 The quiver tensor product

To rigorously construct the quiver tensor product on $\mathsf{D}(\mathsf{Q}(\phi)^{\mathrm{op}})$, first recall that we may view $\mathsf{QC}(\mathfrak{M})$ as an $(\mathbb{E}_n, \mathbb{E}_\infty)$-bialgebra in $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{k}}$ by Corollary 1.3.4.4. The multiplication on $\mathsf{QC}(\mathfrak{M})$ is the usual tensor product of quasicoherent sheaves, while the (binary) comultiplication is the pullback functor

$$\mu^* : \mathsf{QC}(\mathfrak{M}) \to \mathsf{QC}(\mathfrak{M} \times \mathfrak{M}) = \mathsf{QC}(\mathfrak{M}) \otimes_{\mathsf{k}} \mathsf{QC}(\mathfrak{M}),$$

where $\mu : \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$ is the binary multiplication associated with the monoid structure on $\mathfrak{M}$. In particular, for $\chi \in \mathbb{X}^\bullet(\mathsf{G})$, we have

$$\mu^* \mathscr{O}_{\mathfrak{M}}(\chi) = \mathscr{O}_{\mathfrak{M}}(\chi) \boxtimes \mathscr{O}_{\mathfrak{M}}(\chi), \tag{4.2.1.1}$$

so the $\mathbb{E}_n$-coalgebra structure on $\mathsf{QC}(\mathfrak{M}) \simeq \mathsf{D}(\mathsf{Q}(\phi)^{\mathrm{op}})$ restricts to an $\mathbb{E}_n$-coalgebra structure on $\mathsf{Q}(\phi)$. Because $\mathsf{Q}(\phi)$ is closed under the usual tensor product on $\mathsf{QC}(\mathfrak{M})$, we actually get a stronger statement: the $(\mathbb{E}_\infty, \mathbb{E}_n)$-bialgebra structure on $\mathsf{QC}(\mathfrak{M})$ (defined by pullbacks) restricts to an $(\mathbb{E}_\infty, \mathbb{E}_n)$-bialgebra structure on $\mathsf{Q}(\phi)$.

Example 4.2.1.2. Consider the setting of Definition 5.1.4.1. Because the category Cat of small discrete categories is Cartesian monoidal, every small category $\mathcal{C}$ is (uniquely) an $\mathbb{E}_\infty$-coalgebra in Cat by [HA, 2.4.3.10].[2] The comultiplication on $\mathcal{C}$ is given by the diagonal $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$. In particular, the category $\mathsf{Q}^{\mathrm{pre}}(\phi)$ obtains an $\mathbb{E}_\infty$-coalgebra structure in this way, and the above $\mathbb{E}_\infty$-coalgebra structure on $\mathsf{Q}(\phi)$ is the k-linearization of that on $\mathsf{Q}^{\mathrm{pre}}(\phi)$.

By Corollary 1.2.3.3, the $\mathbb{E}_n$-coalgebra structure on $\mathsf{Q}(\phi)$ combines with the symmetric monoidal structure $\otimes_{\mathsf{k}}$ on $\mathsf{D}(\mathsf{k})$ to produce a *quiver tensor product* $\otimes_{\mathsf{Q}} := \otimes_{\mathsf{Q}(\phi)}$ on $\mathsf{D}(\mathsf{Q}(\phi)^{\mathrm{op}}) = \mathsf{Fun}_{\mathsf{k}}(\mathsf{Q}(\phi)^{\mathrm{op}}, \mathsf{D}(\mathsf{k}))$. More concretely, this is defined for $\mathsf{V}_1, \mathsf{V}_2 : \mathsf{Q}(\phi)^{\mathrm{op}} \to \mathsf{D}(\mathsf{k})$ and $\chi \in \mathbb{X}^\bullet(\mathsf{G})$ by

$$(\mathsf{V}_1 \otimes_{\mathsf{Q}} \mathsf{V}_2)(\chi) = \mathsf{V}_1(\chi) \otimes_{\mathsf{k}} \mathsf{V}_2(\chi).$$

The $\mathbb{E}_n$-coalgebra structure on $\mathsf{Q}(\phi)$ is used to define the behavior of $\otimes_{\mathsf{Q}}$ on morphisms.

[2] This does not require discreteness – the only reason we mention "discreteness" here is that the categories $\mathsf{Q}^{\mathrm{pre}}(\phi)$ of Definition 5.1.4.1 are discrete.

4.2.2 Moduli of representations and comparison functors

To establish Theorem 4.2.3.1, we will use the moduli stack of representations of a small k-linear ∞-category, as developed in [TV07] and subsequent papers (see in particular [AG14, §5] for a discussion in the language of ∞-categories). We review the theory of these moduli stacks here.

Definition 4.2.2.1 ([TV07, Definition 3.2])**.** Let $\mathfrak{Rep} : \mathsf{Cat}_{\mathsf{k}}^{\mathrm{op}} \to \mathsf{dStk}_{\mathsf{k}}$ be the functor defined by

$$\mathfrak{Rep}(\mathcal{C})(\mathsf{R}) = \mathsf{Fun}_{\mathsf{k}}(\mathcal{C}, \mathsf{Perf}(\mathsf{R}))^{\simeq}$$

for $\mathcal{C} \in \mathsf{Cat}_{\mathsf{k}}$ and $\mathsf{R} \in \mathsf{dCAlg}_{\mathsf{k}}$. (This defines a derived stack for the étale topology by [TV07, Lemma 3.1] – see also [AG14, Lemma 5.4] for an ∞-categorical treatment.) We call $\mathfrak{Rep}(\mathcal{C})$ the *moduli stack of representations of* $\mathcal{C}$.

Remark 4.2.2.2. We use conventions opposite to those of [TV07], which refers to what we call $\mathfrak{Rep}(\mathcal{C})$ as the *moduli stack of (pseudo-perfect)* $\mathcal{C}^{\mathrm{op}}$*-modules.*

Remark 4.2.2.3. The moduli stacks $\mathfrak{Rep}(\mathcal{C})$ are typically very "large" and poorly behaved. For example, $\mathfrak{Rep}(\mathsf{k})$ is the moduli stack of all perfect complexes. This will not present difficulties for us: we are not interested in the stacks $\mathfrak{Rep}(\mathcal{C})$ themselves. Instead, we want to study maps $\mathfrak{X} \to \mathfrak{Rep}(\mathcal{C})$ where $\mathfrak{X}$ is "small" and well-behaved (e.g. perfect).

Recall that:

- $\mathsf{Ind}(-) = \mathsf{Fun}((-)^{\mathrm{op}}, \mathsf{D}(\mathsf{k}))$ sends a small k-linear ∞-category $\mathcal{C}$ to the ∞-category of k-linear presheaves on $\mathcal{C}$.
- $(-)^{\omega}$ sends a presentable k-linear ∞-category to its (small) full subcategory of compact objects.

The left adjoint to the inclusion $\mathsf{Cat}_{\mathsf{k}}^{\mathrm{perf}} \hookrightarrow \mathsf{Cat}_{\mathsf{k}}$ is given by $\mathcal{C} \mapsto (\mathsf{Ind}\,\mathcal{C})^{\omega}$. Thus $\mathfrak{Rep}(\mathcal{C}) \simeq \mathfrak{Rep}(\mathsf{Ind}(\mathcal{C})^{\omega})$ for all $\mathcal{C} \in \mathsf{Cat}_{\mathsf{k}}$. We will implicitly identify these moduli stacks. As a consequence, we lose no generality by restricting the domain of $\mathfrak{Rep}$ to $(\mathsf{Cat}_{\mathsf{k}}^{\mathrm{perf}})^{\mathrm{op}}$.

Viewing $\mathfrak{Rep}$ as a functor $(\mathsf{Cat}_{\mathsf{k}}^{\mathrm{perf}})^{\mathrm{op}} \to \mathsf{dStk}_{\mathsf{k}}$ allows us to find an explicit left adjoint to $\mathfrak{Rep}$. In fact, we can do even better: $\mathfrak{Rep}$ fits into a lax symmetric monoidal adjunction (cf. Definition 1.2.4.2). Here subscripts $(-)_*$, resp. superscripts $(-)^*$, are used to indicate that a functor sends morphisms to the corresponding pushforward functors, resp. pullback functors.

Proposition 4.2.2.4. *There are lax symmetric monoidal adjunctions*

$$(\mathsf{Perf}^*)^{\mathrm{op}} : (\mathsf{dStk}_{\mathsf{k}}, \times) \rightleftarrows \left(\left(\mathsf{Cat}_{\mathsf{k}}^{\mathrm{perf}}\right)^{\mathrm{op}}, \otimes_{\mathsf{k}}\right) : \mathfrak{Rep}$$

and

$$\mathsf{Ind}\,\mathsf{Perf}_* : (\mathsf{dStk}_{\mathsf{k}}, \times) \rightleftarrows \left(\mathsf{Pr}^{\mathsf{R}}_{\mathsf{k},\omega}, \otimes\right) : \mathfrak{Rep}((-)^{\omega}).$$

Proof. By using the symmetric monoidal equivalence

$$\mathsf{Ind} : \left(\left(\mathsf{Cat}_k^{\mathrm{perf}}\right)^{\mathrm{op}}, \otimes_k\right) \xleftrightarrow{\sim} \left(\mathsf{Pr}^{\mathsf{R}}_{k,\omega}, \otimes_k\right) : (-)^{\omega}$$

we see that it suffices to construct the former adjunction. Existence of this adjunction is standard (cf. [Toë07, Proposition 3.4]), though we recall the argument in ∞-categorical language for completeness.

Let $\mathcal{C} \in \mathsf{Cat}_k^{\mathrm{perf}}$. Any derived stack $\mathfrak{X} : \mathsf{dAff}_k^{\mathrm{op}} \to \mathcal{S}$ may be written as a colimit of representables, i.e. $\mathfrak{X} = \mathrm{colim}_i \operatorname{Spec} \mathsf{R}_i$ for some R_i. We may then compute

$$\begin{aligned}
\mathrm{Map}_{\mathsf{dStk}_k}\left(\mathfrak{X}, \mathfrak{Rep}(\mathcal{C})\right) &= \mathrm{Map}_{\mathsf{dStk}_k}\left(\operatorname*{colim}_i \operatorname{Spec} \mathsf{R}_i, \mathfrak{Rep}(\mathcal{C})\right)\\
&= \lim_i \mathrm{Map}_{\mathsf{dStk}_k}\left(\operatorname{Spec} \mathsf{R}_i, \mathfrak{Rep}(\mathcal{C})\right)\\
&= \lim_i \mathsf{Fun}_k\left(\mathcal{C}, \mathsf{Perf}(\mathsf{R}_i)\right) \text{ by definition of } \mathfrak{Rep}(\mathcal{C})\\
&= \mathsf{Fun}_k\left(\mathcal{C}, \lim_i \mathsf{Perf}(\mathsf{R}_i)\right)\\
&= \mathsf{Fun}_k\left(\mathcal{C}, \mathsf{Perf}(\mathfrak{X})\right) \text{ by definition of } \mathsf{Perf}(\mathfrak{X}).
\end{aligned}$$

This is the needed adjunction.

To get the lax symmetric monoidal structure on the adjunction, we just need to find a natural oplax symmetric monoidal structure on the left adjoint $(\mathsf{Perf}^*)^{\mathrm{op}}$. The lax symmetric monoidal structure on $\mathsf{QC}^* : \mathsf{dStk}_k^{\mathrm{op}} \to \mathsf{Pr}^{\mathsf{L}}_k$ restricts to a lax symmetric monoidal structure on $\mathsf{Perf}^* : \mathsf{dStk}_k^{\mathrm{op}} \to \mathsf{Cat}_k^{\mathrm{perf}}$. Reversing the direction of arrows, we obtain an oplax symmetric monoidal structure on $(\mathsf{Perf}^*)^{\mathrm{op}} : \mathsf{dStk}_k \to \left(\mathsf{Cat}_k^{\mathrm{perf}}\right)^{\mathrm{op}}$ as needed. □

In particular, for a derived stack $\mathfrak{X} \in \mathsf{dStk}_k$ and a small k-linear ∞-category $\mathcal{C} \in \mathsf{Cat}_k$, the following data are equivalent:

- A morphism of derived stacks $f_{\mathscr{E}_\bullet} : \mathfrak{X} \to \mathfrak{Rep}(\mathcal{C}) \simeq \mathfrak{Rep}(\mathsf{Ind}(\mathcal{C})^{\omega})$.
- A k-linear functor $\mathscr{E}_\bullet : \mathcal{C} \to \mathsf{Perf}(\mathfrak{X})$.
- A k-linear functor of the form $\mathrm{Hom}(\mathscr{E}_\bullet, -) : \mathsf{Ind}\,\mathsf{Perf}(\mathfrak{X}) \to \mathsf{Ind}(\mathcal{C})$.

Here $\mathrm{Hom}(\mathscr{E}_\bullet, -)$ refers to the functor satisfying

$$\mathrm{Hom}(\mathscr{E}_\bullet, \mathscr{F})(c) = \mathrm{Hom}_{\mathsf{Ind}\,\mathsf{Perf}(\mathfrak{X})}(\mathscr{E}_c, \mathscr{F})$$

for $\mathscr{F} \in \mathsf{Ind}\,\mathsf{Perf}(\mathfrak{X})$ and $c \in \mathcal{C}$. The functors of this form are precisely the functors from $\mathsf{Ind}\,\mathsf{Perf}(\mathfrak{X})$ to $\mathsf{Ind}(\mathcal{C})$ in $\mathsf{Pr}^{\mathsf{R}}_{k,\omega}$ by a version of Yoneda's lemma.

Suppose that $\mathcal{C} \in \mathsf{Alg}_{\mathbb{E}_n}(\mathsf{Cat}_k^{\mathrm{op}})$, i.e. $\mathcal{C}$ is an $\mathbb{E}_n$-coalgebra in small k-linear ∞-categories. Then $\mathsf{Ind}(\mathcal{C})$ is an $\mathbb{E}_n$-coalgebra in $\mathsf{Pr}^{\mathsf{L}}_{k,\omega}$. Applying the equivalence $(\mathsf{Pr}^{\mathsf{L}}_{k,\omega})^{\mathrm{op}} \simeq \mathsf{Pr}^{\mathsf{R}}_{k,\omega}$, we see that $\mathsf{Ind}(\mathcal{C})$ is an $\mathbb{E}_n$-algebra in $\mathsf{Pr}^{\mathsf{R}}_{k,\omega}$, i.e. an $\mathbb{E}_n$-monoidal category such that the monoidal structure preserves colimits and limits (formed in the powers $\mathsf{Ind}(\mathcal{C})^{\boxtimes k}$).

Furthermore, for $\mathsf{R} \in \mathsf{dCAlg}_k$, Corollary 1.2.3.3 lets us combine the symmetric monoidal structure $\otimes_{\mathscr{O}}$ on $\mathsf{Perf}(\mathsf{R})$ with the $\mathbb{E}_n$-coalgebra structure on $\mathcal{C}$ to produce an $\mathbb{E}_n$-monoid structure on the space $\mathfrak{Rep}(\mathcal{C})(\mathsf{R}) = \mathsf{Fun}_k(\mathcal{C}, \mathsf{Perf}(\mathsf{R}))^{\simeq}$. Thus $\mathfrak{Rep}(\mathcal{C})$ is an $\mathbb{E}_n$-monoid object of dStk_k. This $\mathbb{E}_n$-monoid structure gives a useful source of symmetric monoidal functors into $(\mathsf{Ind}(\mathcal{C}), \otimes_{\mathcal{C}})$:

Proposition 4.2.2.5. *Let $\mathfrak{X} \in \mathsf{Alg}_{\mathbb{E}_n}(\mathsf{dStk}_k^{\mathrm{perf}})$ be a perfect $\mathbb{E}_n$-monoid derived stack, and let $\mathcal{C} \in \mathsf{Alg}_{\mathbb{E}_n}(\mathsf{Cat}_k^{\mathrm{op}})$. There is a natural equivalence*

$$\mathrm{Map}_{\mathsf{Alg}_{\mathbb{E}_n}(\mathsf{dStk}_k)}(\mathfrak{X}, \mathfrak{Rep}(\mathcal{C})) \simeq \mathsf{Fun}_k^{\mathbb{E}_n,\mathsf{R}}\left((\mathsf{QC}(\mathfrak{X}), \star_{\mathfrak{X}}), (\mathsf{Ind}(\mathcal{C}), \otimes_{\mathcal{C}})\right)^{\simeq}$$

enhancing the adjunction equivalence of Proposition 4.2.2.4. In particular, if $\mathsf{f}_{\mathscr{E}_\bullet} : \mathfrak{X} \to \mathfrak{Rep}(\mathcal{C})$ is a homomorphism of commutative monoid derived stacks, then the functor

$$\mathrm{Hom}(\mathscr{E}_\bullet, -) : (\mathsf{QC}(\mathfrak{X}), \star_{\mathfrak{X}}) \to (\mathsf{Ind}(\mathcal{C}), \otimes_{\mathcal{C}})$$

is $\mathbb{E}_n$-monoidal.

Proof. The adjunction of Proposition 4.2.2.4 is only lax symmetric monoidal in general. However, on the full subcategory $\mathsf{dStk}_k^{\mathrm{perf}} \subset \mathsf{dStk}_k$, the functor $\mathsf{Ind}\,\mathsf{Perf}_*$ is symmetric monoidal and agrees with QC_*. The result follows from Proposition 1.2.4.6 (keeping in mind Remark 1.2.2.3), using the fact that $\mathsf{Fun}_k^{\mathbb{E}_n,\mathsf{R}}(-,-)^{\simeq}$ is the space of morphisms in $\mathsf{Alg}_{\mathbb{E}_n}\left(\mathsf{Pr}_k^{\mathsf{R}}\right)$ by definition. $\square$

4.2.3 Proof of the $\mathbb{E}_n$-monoidal equivalence

We have now assembled all of the ingredients necessary to upgrade the underlying equivalence Proposition 4.1.0.5 to a symmetric monoidal equivalence

Theorem 4.2.3.1. *The underlying equivalence of categories of Proposition 4.1.0.5 upgrades to an $\mathbb{E}_n$-monoidal equivalence*

$$\left(\mathsf{QC}(\mathfrak{M}), \star_{\mathfrak{M}}\right) \simeq \left(\mathsf{D}(\mathsf{Q}(\phi)^{\mathrm{op}}), \otimes_{\mathsf{Q}(\phi)^{\mathrm{op}}}\right).$$

Proof. Let $\mathsf{R} \in \mathsf{dCAlg}_k$. There is a natural $\mathbb{E}_n$-monoid structure on $\mathsf{Fun}^{\otimes}(\mathsf{Q}(\phi), \mathsf{Perf}^{\leqslant 0}(\mathsf{R}))^{\simeq}$ constructed as follows. Note that $\mathsf{Q}(\phi)$ is an $(\mathbb{E}_\infty, \mathbb{E}_n)$-bialgebra in $(\mathsf{Cat}_k^{\mathrm{perf}}, \otimes_k)$, or equivalently an $\mathbb{E}_n$-coalgebra in $\left(\mathsf{CAlg}(\mathsf{Cat}_k^{\mathrm{perf}}), \otimes_k\right)$. Furthermore, $\mathsf{Perf}^{\leqslant 0}(\mathsf{R})$ is a symmetric monoidal ∞-category, i.e. an object of $\mathsf{CAlg}(\mathsf{Cat}_k^{\mathrm{perf}}) = \mathsf{CAlg}(\mathsf{CAlg}(\mathsf{Cat}_k^{\mathrm{perf}}))$ (by [HA, 3.2.4.5]). Thus the mapping space

$$\mathsf{Fun}^{\otimes}(\mathsf{Q}(\phi), \mathsf{Perf}^{\leqslant 0}(\mathsf{R}))^{\simeq} = \mathrm{Map}_{\mathsf{CAlg}(\mathsf{Cat}_k^{\mathrm{perf}})}(\mathsf{Q}(\phi), \mathsf{Perf}^{\leqslant 0}(\mathsf{R}))$$

obtains an $\mathbb{E}_n$-algebra structure from Corollary 1.2.3.3.

By Corollary 4.1.0.6, we may identify $\mathfrak{M}(\mathsf{R}) \simeq \mathsf{Fun}^{\otimes}(\mathsf{Q}(\phi), \mathsf{Perf}^{\leqslant 0}(\mathsf{R}))^{\simeq}$ as $\mathbb{E}_{n}$-monoids. Forgetting the symmetric monoidal structure on functors and applying the inclusion $\mathsf{Perf}^{\leqslant 0}(\mathsf{R}) \hookrightarrow \mathsf{Perf}(\mathsf{R})$, we obtain a map

$$\mathsf{f}_{\mathscr{O}(-),\mathsf{R}} : \mathfrak{M}(\mathsf{R}) \simeq \mathsf{Fun}^{\otimes}(\mathsf{Q}(\phi), \mathsf{Perf}^{\leqslant 0}(\mathsf{R}))^{\simeq} \to \mathsf{Fun}(\mathsf{Q}(\phi), \mathsf{Perf}(\mathsf{R}))^{\simeq} =: \mathfrak{Rep}(\mathsf{Q}(\phi))(\mathsf{R}).$$

Each map $\mathsf{f}_{\mathscr{O}(-),\mathsf{R}}$ is a homomorphism of $\mathbb{E}_{n}$-monoid spaces by Corollary 1.2.3.3 and Corollary 1.2.4.5. The maps $\mathsf{f}_{\mathscr{O}(-),\mathsf{R}}$ are natural in R and thus assemble into a map of $\mathbb{E}_{n}$-monoid derived stacks $\mathsf{f}_{\mathscr{O}(-)} : \mathfrak{M} \to \mathfrak{Rep}(\mathsf{Q}(\phi))$.

By Proposition 4.2.2.5, $\mathsf{f}_{\mathscr{O}(-)}$ gives an $\mathbb{E}_{n}$-monoidal functor

$$\begin{aligned} \mathrm{Hom}(\mathscr{O}(-), -) : \big(\mathsf{QC}(\mathfrak{M}), \star_{\mathfrak{M}}\big) &\to \big(\mathsf{D}(\mathsf{Q}(\phi)^{\mathrm{op}}), \otimes_{\mathsf{Q}}\big) \\ \mathscr{F} &\mapsto \big(\chi \mapsto \mathrm{Hom}(\mathscr{O}_{\mathfrak{M}}(\chi), \mathscr{F})\big) \end{aligned}$$

The underlying functor of this $\mathbb{E}_{n}$-monoidal functor is the inverse of the underlying equivalence of Proposition 4.1.0.5. Thus $\mathrm{Hom}(\mathscr{O}(-), -)$ gives the desired $\mathbb{E}_{n}$-monoidal equivalence. □

Remark 4.2.3.2. The proof of Theorem 4.2.3.1 works just as well over the sphere spectrum $\mathbb{S}$ provided that we know $\mathsf{QC}(\mathsf{BG})$ is compactly generated by invertible objects. In particular, taking $\mathsf{G} = \mathbb{G}_{m}$ (the flat multiplicative group over $\mathbb{S}$) and $\mathsf{M} = \mathbb{A}^{1}$ (the flat affine line over $\mathbb{S}$), we extend [Mou21, Theorem 1.1] to an equivalence

$$\big(\mathsf{QC}([\mathbb{A}^{1}/\mathbb{G}_{m}]), \star\big) \simeq \big(\mathsf{Fun}((\mathbb{Z}, \leqslant)^{\mathrm{op}}, \mathsf{Sp}), \otimes_{\mathbb{S}}\big).$$

It would be interesting to understand the topological implications of this statement.

Chapter 5

Extended convolution and applications

So far we have been largely concerned with the behavior of “basic” toric stacks (and generalizations of the same), i.e. quotients $[\operatorname{Spec} \mathsf{R}/\mathsf{G}]$ with G a commutative reductive group. In particular, Theorem 4.2.3.1 identifies convolution products on such stacks with “quiver tensor products” for certain generalized quivers $\mathsf{Q}(\phi)$. Now we shall use these results to prove analogous results for more general toric stacks (and variations thereof – most crucially $\mathbb{P}^n$. Our key results include:

- Proposition 5.3.0.1, which constructs a family of hitherto-unknown monoidal structures on $\mathsf{Perf}\,\mathbb{P}^n$, and
- Proposition 5.4.0.2, which describes the mirror to a certain natural tensor product on constructible sheaves arising in toric mirror symmetry.

To construct these tensor products, we will use a version of the theory of *windows* (as discussed in [Hal15]) to relate the derived category of a GIT quotient to that of the corresponding “basic” quotient stack. In Section 5.1, we discuss some of the aspects of window theory that we will need. Afterwards, in Section 5.2, we discuss how to use windows to transport monoidal structures from quotient stacks to GIT quotients. We finish by giving explicit applications of this theory in Section 5.3 and Section 5.4.

5.1 Quiver presentations of derived categories

We begin by recalling the relationship between full strong exceptional collections of line bundles, quiver presentations of derived categories, and the window theory of [Hal15].

Notation 5.1.0.1. In this section we continue with the notation of Notation 4.1.0.1. Let $\mathfrak{X}$ be an open substack of $\mathfrak{Y}$, and write $\mathsf{j}_{\mathfrak{X}} : \mathfrak{X} \hookrightarrow \mathfrak{Y}$ for the corresponding open immersion. Write $\mathsf{X} = \mathfrak{X} \times_{\mathfrak{Y}} \mathsf{Y}$, so $\mathfrak{X} = [\mathsf{X}/\mathsf{G}]$.

5.1.1 Windows and quivers

There is a deep relationship between the derived category of $\mathfrak{Y}$ and that of an open substack $\mathfrak{X} \subset \mathfrak{Y}$ (see e.g. [Hal15]). In particular, in many cases, there is a nice embedding $\mathsf{QC}(\mathfrak{X}) \hookrightarrow \mathsf{QC}(\mathfrak{Y})$. We can formalize one notion of "nice embedding" as follows:

Definition 5.1.1.1. A *window* is a functor $W : \mathsf{QC}(\mathfrak{X}) \hookrightarrow \mathsf{QC}(\mathfrak{Y})$ such that:

- $j_{\mathfrak{X}}^* W = \mathrm{id}_{\mathsf{QC}(\mathfrak{X})}$,
- W preserves colimits, and
- W preserves compact objects (i.e. perfect complexes).

The adjoint functor theorem [HTT, 5.5.2.9] implies that W admits a right adjoint $\mathsf{H} : \mathsf{QC}(\mathfrak{Y}) \to \mathsf{QC}(\mathfrak{X})$, which we call the *Hitchcock functor* corresponding to W.

Remark 5.1.1.2. Our hypotheses on W guarantee that W can be recovered from the restriction $W|_{\mathsf{Perf}(\mathfrak{X})} : \mathsf{Perf}(\mathfrak{X}) \to \mathsf{Perf}(\mathfrak{Y})$. In the literature, the term "window" more frequently refers to this restriction (or to closely related notions, e.g. with Perf replaced by Coh). The windows that interest us necessarily send perfect complexes to perfect complexes, so we do not gain anything by replacing QC by $\mathsf{Ind}\,\mathsf{Coh}$.

We are particularly interested in windows coming from certain collections of line bundles on $\mathfrak{X}$. For such a collection to determine a window, the line bundles should be induced by weights of G satisfying the following conditions.

Definition 5.1.1.3. A collection of weights $\mathsf{S} \subset \mathbb{X}^\bullet(\mathsf{G})$ is *transparent* for $\mathfrak{X} \subset \mathfrak{Y}$ if:

- $\{\mathscr{O}_{\mathfrak{X}}(\chi)\}_{\chi \in \mathsf{S}}$ compactly generates $\mathsf{QC}(\mathfrak{X})$, and
- $\mathrm{Hom}_{\mathfrak{X}}\left(\mathscr{O}_{\mathfrak{X}}(\chi_1), \mathscr{O}_{\mathfrak{X}}(\chi_2)\right) = \mathrm{Hom}_{\mathfrak{Y}}\left(\mathscr{O}_{\mathfrak{Y}}(\chi_1), \mathscr{O}_{\mathfrak{Y}}(\chi_2)\right)$ for all $\chi_1, \chi_2 \in \mathsf{S}$.

Because Y is affine and G is reductive, for all $\chi_1, \chi_2 \in \mathbb{X}^\bullet(\mathsf{G})$, we have

$$\mathrm{Hom}^i_{\mathfrak{Y}}\left(\mathscr{O}_{\mathfrak{Y}}(\chi_1), \mathscr{O}_{\mathfrak{Y}}(\chi_2)\right) = 0 \text{ for all } i > 0.$$

In particular, if S is transparent, then for all $\chi_1, \chi_2 \in \mathsf{S}$, we have

$$\mathrm{Hom}_{\mathfrak{X}}\left(\mathscr{O}_{\mathfrak{X}}(\chi_1), \mathscr{O}_{\mathfrak{X}}(\chi_2)\right) = \mathrm{Hom}_{\mathfrak{Y}}\left(\mathscr{O}_{\mathfrak{Y}}(\chi_1), \mathscr{O}_{\mathfrak{Y}}(\chi_2)\right) = \mathsf{R}_{\chi_2 - \chi_1} \tag{5.1.1.4}$$

using the notation of Remark 4.1.0.3. The following is one of the primary examples we should keep in mind.

The notions of "transparent collection of weights" and "full strong exceptional collection of line bundles" are related but distinct. The key distinction for our purposes is that transparent collections are defined relative to an embedding $\mathfrak{X} \subset \mathfrak{Y}$, while full strong exceptional collections are defined solely in terms of $\mathfrak{X}$. Transparent collections also exist more frequently than full strong exceptional collections when $\mathfrak{X}$ is not proper. (For example, the entire collection $\mathbb{X}^\bullet(\mathsf{G})$ is always transparent for $\mathfrak{Y} \subset \mathfrak{Y}$.) Nevertheless, in many cases, the two notions coincide:

Proposition 5.1.1.5. *Assume that* Y *is an (affine) underived algebraic variety and* $\mathrm{Hom}_{\mathfrak{Y}}(\mathscr{O}_{\mathfrak{Y}}, \mathscr{O}_{\mathfrak{Y}}) = \mathsf{k}[0]$*, i.e.* $\mathscr{O}_{\mathfrak{Y}}$ *is an exceptional object of* $\mathsf{QC}(\mathfrak{Y})$*.*

1. *If* $\mathsf{S} \subset \mathbb{X}^\bullet(\mathsf{G})$ *is transparent for* $\mathfrak{X} \subset \mathfrak{Y}$*, then the line bundles* $\{\mathscr{O}_{\mathfrak{X}}(\chi)\}_{\chi \in \mathsf{S}}$ *form a full strong exceptional collection in* $\mathsf{Perf}(\mathfrak{X})$*.*

2. *Suppose* Y *is normal and* $\mathrm{codim}_{\mathsf{Y}}(\mathsf{Y}\backslash\mathsf{X}) \leqslant 2$*. If* $\mathsf{S} \subset \mathbb{X}^\bullet(\mathsf{G})$ *is such that the line bundles* $\{\mathscr{O}_{\mathfrak{X}}(\chi)\}_{\chi \in \mathsf{S}}$ *form a full strong exceptional collection in* $\mathsf{Perf}(\mathfrak{X})$*, then* S *is transparent for* $\mathfrak{X} \subset \mathfrak{Y}$*.*

3. *Suppose* $\mathsf{Y} = \mathrm{Spec}\, \mathsf{R}$ *where* R *is a unique factorization domain, and suppose* $\mathrm{Pic}(\mathfrak{X}) = \mathbb{X}^\bullet(\mathsf{G})$*. Then transparent collections of weights for* $\mathfrak{X} \subset \mathfrak{Y}$ *correspond precisely to full strong exceptional collections in* $\mathsf{Perf}(\mathfrak{X})$ *via the correspondence* $\mathsf{S} \mapsto \{\mathscr{O}_{\mathfrak{X}}(\chi)\}_{\chi \in \mathsf{S}}$*.*

Proof. Throughout we use the equivalence between line bundles on a stack quotient $[\mathsf{Z}/\mathsf{G}]$ and G-equivariant line bundles on Z.

(1). By supposition, $\mathscr{O}_{\mathfrak{X}}(\chi)$ is exceptional in $\mathsf{Perf}(\mathfrak{X})$ for every $\chi \in \mathbb{X}^\bullet(\mathsf{G})$. Also, for $\chi \in \mathbb{X}^\bullet(\mathsf{G})$, note that only one of $\mathsf{H}^0(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}(\chi))$ and $\mathsf{H}^0(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}(-\chi))$ can be non-zero. This gives a partial order on S. Thus, if $\mathsf{S} \subset \mathbb{X}^\bullet(\mathsf{G})$ is transparent, then $\{\mathscr{O}_{\mathfrak{X}}(\chi)\}_{\chi \in \mathsf{S}}$ can be ordered to be a full strong exceptional collection.

(2). It suffices to show that $\mathrm{Hom}^0_{\mathfrak{X}}\left(\mathscr{O}_{\mathfrak{X}}(\chi_1), \mathscr{O}_{\mathfrak{X}}(\chi_2)\right) = \mathrm{Hom}^0_{\mathfrak{Y}}\left(\mathscr{O}_{\mathfrak{Y}}(\chi_1), \mathscr{O}_{\mathfrak{Y}}(\chi_2)\right)$ for all $\chi_1, \chi_2 \in \mathsf{S}$. In fact, this holds for all $\chi_1, \chi_2 \in \mathbb{X}^\bullet(\mathsf{G})$, as we now show.

We may identify $\mathrm{Hom}^0_{\mathfrak{X}}\left(\mathscr{O}_{\mathfrak{X}}(\chi_1), \mathscr{O}_{\mathfrak{X}}(\chi_2)\right)$ with the subspace of $\mathsf{H}^0(\mathsf{X}, \mathscr{O}_{\mathsf{X}})$ on which G acts with weight $\chi_2 - \chi_1$. Any section of $\mathsf{H}^0(\mathsf{X}, \mathscr{O}_{\mathsf{X}})$ extends uniquely to a section of $\mathsf{H}^0(\mathsf{Y}, \mathscr{O}_{\mathsf{Y}})$ by the algebraic Hartogs' lemma (see e.g. [Stacks, Tag 031T]), and the uniqueness implies that G acts on this new section with the same weight. Thus we may identify $\mathrm{Hom}^0_{\mathfrak{X}}\left(\mathscr{O}_{\mathfrak{X}}(\chi_1), \mathscr{O}_{\mathfrak{X}}(\chi_2)\right)$ with the subspace of $\mathsf{H}^0(\mathsf{Y}, \mathscr{O}_{\mathsf{Y}})$ on which G acts with weight $\chi_2 - \chi_1$. This last subspace may itself be identified with $\mathrm{Hom}^0_{\mathfrak{Y}}\left(\mathscr{O}_{\mathfrak{Y}}(\chi_1), \mathscr{O}_{\mathfrak{Y}}(\chi_2)\right)$.

(3). By hypothesis, every line bundle on Y is (non-equivariantly) trivial. It follows that a G-equivariant line bundle on Y is determined by the character by which G acts on a (non-equivariant) non-vanishing section. In other words, there is a natural surjection $\mathbb{X}^\bullet(\mathsf{G}) \twoheadrightarrow \mathrm{Pic}(\mathfrak{Y})$. Applying the restriction map $\mathrm{Pic}(\mathfrak{Y}) \to \mathrm{Pic}(\mathfrak{X}) = \mathbb{X}^\bullet(\mathsf{G})$, we see that the identity map $\mathbb{X}^\bullet(\mathsf{G}) \to \mathbb{X}^\bullet(\mathsf{G})$ factors through the surjection $\mathbb{X}^\bullet(\mathsf{G}) \twoheadrightarrow \mathrm{Pic}(\mathfrak{Y})$, and thus $\mathrm{Pic}(\mathfrak{Y}) = \mathbb{X}^\bullet(\mathsf{G})$. In particular, we see that the restriction $\mathrm{Pic}(\mathfrak{Y}) \to \mathrm{Pic}(\mathfrak{X})$ is an isomorphism, i.e. every G-equivariant line bundle on X extends to a unique G-equivariant line bundle on Y.

We claim that this implies $\mathrm{codim}_{\mathsf{Y}}(\mathsf{Y} \setminus \mathsf{X}) \geqslant 2$. Indeed, if this is not true, we can choose a nonempty divisor $\mathsf{D} \subset \mathsf{Y}$ which is entirely contained in $\mathsf{Y} \setminus \mathsf{X}$. Replacing D by $\mathsf{G} \cdot \mathsf{D}$, we may assume without loss of generality that D is G-invariant. Thus $\mathscr{O}_{\mathsf{Y}}(\mathsf{D})$ is naturally G-equivariant. Note that $\mathscr{O}_{\mathsf{Y}}(\mathsf{D})$ restricts to the trivial G-equivariant line bundle on X, so $\mathscr{O}_{\mathsf{Y}}(\mathsf{D})$ must be G-equivariantly trivial on Y. That is, there exists $\mathsf{f} \in \mathsf{H}^0(\mathsf{Y}, \mathscr{O}_{\mathsf{Y}})^{\mathsf{G}}$ such that the effective divisor D is linearly equivalent to $\mathsf{V}(\mathsf{f})$. But the hypothesis $\mathsf{H}^0(\mathsf{Y}, \mathscr{O}_{\mathsf{Y}})^{\mathsf{G}} =$

$\mathrm{Hom}_{\mathfrak{Y}}(\mathscr{O}_{\mathfrak{Y}}, \mathscr{O}_{\mathfrak{Y}}) = \mathsf{k}$ then implies D must be empty, a contradiction. Thus $\mathrm{codim}_{\mathsf{Y}}(\mathsf{Y}\backslash\mathsf{X}) \geqslant 2$ and we are in the situation of (2). ☐

We now introduce the windows arising from transparent collections of weights. The following is known to experts,[1] and we state it partially to establish precise notation for later.

Proposition 5.1.1.6. *Suppose* S *is transparent for* $\mathfrak{X} \subset \mathfrak{Y}$. *Then:*

1. *There is a natural equivalence* $\mathsf{QC}(\mathfrak{X}) = \mathsf{D}(\mathsf{Q}_{\mathsf{S}}(\phi)^{\mathrm{op}})$.
2. *There exists a window* $W_{\mathsf{S}} : \mathsf{QC}(\mathfrak{X}) \hookrightarrow \mathsf{QC}(\mathfrak{Y})$ *such that* $W_{\mathsf{S}}(\mathscr{O}_{\mathfrak{X}}(\chi)) = \mathscr{O}_{\mathfrak{Y}}(\chi)$ *for all* $\chi \in \mathsf{S}$.

Proof. (1). Let $\chi_1, \chi_2 \in \mathsf{S}$. By Eq. (5.1.1.4), we have $\mathrm{Hom}_{\mathfrak{X}}(\mathscr{O}_{\mathfrak{X}}(\chi_1), \mathscr{O}_{\mathfrak{X}}(\chi_2)) = \mathrm{Hom}_{\mathsf{Q}_{\mathsf{S}}(\phi)}(\chi_1, \chi_2)$. Thus the full subcategory of $\mathsf{QC}(\mathfrak{X})$ with objects $\{\mathscr{O}_{\mathfrak{X}}(\chi)\}_{\chi\in\mathsf{S}}$ is equivalent to $\mathsf{Q}_{\mathsf{S}}(\phi)$. This full subcategory compactly generates $\mathsf{QC}(\mathfrak{X})$, yielding the claim.

(2). The inclusion $\mathfrak{i}_{\mathsf{S}} : \mathsf{Q}_{\mathsf{S}}(\phi) \hookrightarrow \mathsf{Q}(\phi)$ left Kan extends to a functor $(\mathfrak{i}_{\mathsf{S}})_! : \mathsf{D}(\mathsf{Q}_{\mathsf{S}}(\phi)^{\mathrm{op}}) \to \mathsf{D}(\mathsf{Q}(\phi)^{\mathrm{op}})$. Note that $(\mathfrak{i}_{\mathsf{S}})_!$ preserves colimits and compact objects by definition. Applying part (1) to $\mathbb{X}^\bullet(\mathsf{G})$ (which is transparent for $\mathfrak{Y} \subset \mathfrak{Y}$), we get an equivalence $\mathsf{QC}(\mathfrak{Y}) = \mathsf{D}(\mathsf{Q}(\phi)^{\mathrm{op}})$. Let $W_{\mathsf{S}} : \mathsf{QC}(\mathfrak{X}) \to \mathsf{QC}(\mathfrak{Y})$ be the functor corresponding to $(\mathfrak{i}_{\mathsf{S}})_!$ via the above equivalences. Chasing the definitions shows that $W_{\mathsf{S}}(\mathscr{O}_{\mathfrak{X}}(\chi)) = \mathscr{O}_{\mathfrak{Y}}(\chi)$ for all $\chi \in \mathsf{S}$. In particular, since $\mathfrak{j}_{\mathfrak{X}}^* W_{\mathsf{S}}(\mathscr{O}_{\mathfrak{X}}(\chi)) = \mathscr{O}_{\mathfrak{Y}}(\chi)$ and $\{\mathscr{O}_{\mathfrak{X}}(\chi)\}_{\chi\in\mathsf{S}}$ generates $\mathsf{QC}(\mathfrak{X})$, we see that $\mathfrak{j}_{\mathfrak{X}}^* W_{\mathsf{S}} = \mathrm{id}_{\mathsf{QC}(\mathfrak{X})}$, so W_{S} is a window. ☐

Remark 5.1.1.7. If $\mathsf{S} \subset \mathbb{X}^\bullet(\mathsf{G})$ is any collection of weights, note that the compact objects of $\mathsf{D}(\mathsf{Q}_{\mathsf{S}}(\phi)^{\mathrm{op}})$ are those which can be written as a finite colimit (possibly with shifts) of the generators $\left\{\mathscr{O}_{\mathfrak{Y}}(\chi)\right\}_{\chi\in\mathsf{S}}$. In general, we cannot expect $\mathsf{D}(\mathsf{Q}_{\mathsf{S}}(\phi)^{\mathrm{op}})^\omega = \mathsf{Fun}_{\mathsf{k}}(\mathsf{Q}_{\mathsf{S}}(\phi)^{\mathrm{op}}, \mathsf{Perf}(\mathsf{k}))$, though this is true if S arises from a finite full strong exceptional collection as in Proposition 5.1.1.5.

Notation 5.1.1.8. If S is a transparent collection of weights for $\mathfrak{X} \subset \mathfrak{Y}$, we fix the following notation:

- W_{S} is the window of Proposition 5.1.1.6.
- H_{S} is the corresponding Hitchcock functor (i.e. right adjoint of W_{S}).
- $\mathscr{K}_{\mathsf{S}}$ is the Fourier-Mukai kernel of W_{S}.

By the construction in Proposition 5.1.1.6, we see that the window W_{S} may be identified with the left Kan extension functor $\mathfrak{i}_{\mathsf{S}!} : \mathsf{D}(\mathsf{Q}_{\mathsf{S}}(\phi)^{\mathrm{op}}) \to \mathsf{D}(\mathsf{Q}(\phi)^{\mathrm{op}})$. The right adjoint H_{S} may therefore be identified with the "restriction of quiver representations" functor $\mathfrak{i}_{\mathsf{S}}^* : \mathsf{D}(\mathsf{Q}(\phi)^{\mathrm{op}}) \to \mathsf{D}(\mathsf{Q}_{\mathsf{S}}(\phi)^{\mathrm{op}})$.

[1]We thank Daniel Halpern-Leistner for bringing a version of this statement to our attention.

Example 5.1.1.9. Let $\mathfrak{X} = \mathbb{P}^n \subset \mathfrak{Y} = [\mathbb{A}^{n+1}/\mathbb{G}_m]$, and let $\mathsf{S} = \{0, \ldots, n\} \subset \mathbb{Z} = \mathbb{X}^\bullet(\mathbb{G}_m)$ (so S is transparent for $\mathfrak{X} \subset \mathfrak{Y}$). Direct computations using the "restriction of quiver representations" description of the Hitchcock functor H_S show that

$$\mathsf{H}_\mathsf{S}\big(\mathscr{O}_{[\mathbb{A}^{n+1}/\mathbb{G}_m]}(\ell)\big) = \begin{cases} \mathscr{O}_{[\mathbb{A}^{n+1}/\mathbb{G}_m]}(\ell) & \ell \geqslant 0 \\ 0 & \ell < 0 \end{cases}$$

for all $\ell \in \mathbb{Z}$. In particular, H_S is distinct from the geometric pullback functor j^* as functors from $\mathsf{D}(\mathsf{Q}(\phi)^{\mathrm{op}})$ to $\mathsf{D}(\mathsf{Q}_\mathsf{S}(\phi)^{\mathrm{op}})$, although the two functors agree on the image of $\mathcal{W}_\mathsf{S}$.

In general, it is difficult to give a geometric description of the windows $\mathcal{W}_\mathsf{S}$ or the Hitchcock functors H_S in terms of Fourier-Mukai transforms. However, in certain favorable cases, the windows $\mathcal{W}_\mathsf{S}$ arise via push-pull along compactifications of the diagonal of $\mathfrak{X}$ as in [BDF17] (see Remark 5.1.4.12). We show this claim for a class of toric varieties in Proposition 5.1.4.10. Proposition 5.1.3.2 allows us to leverage this to provide geometric descriptions of the Hitchcock functors for the opposite collections of weights. We use this to understand certain windows and Hitchcock functors for the inclusion $\mathbb{P}^n \subset [\mathbb{A}^{n+1}/\mathbb{G}_m]$ in Example 5.1.2.4 and Example 5.1.3.4.

5.1.2 Windows and resolutions of the diagonal

A standard method (introduced in [Bei78]) for showing an exceptional collection in $\mathsf{Perf}(\mathfrak{X})$ is full is to show that the collection can be used to produce a resolution of the diagonal of $\mathfrak{X}$. When the exceptional collection in question corresponds to a collection of weights $\mathsf{S} \subset \mathbb{X}^\bullet(\mathsf{G})$ which is transparent for $\mathfrak{X} \subset \mathfrak{Y}$, this resolution of the diagonal may be used to concretely understand the window $\mathcal{W}_\mathsf{S}$.

Proposition 5.1.2.1. *Suppose* $\mathsf{S} \subset \mathbb{X}^\bullet(\mathsf{G})$ *is such that*

$$\mathrm{Hom}_\mathfrak{X}(\mathscr{O}_\mathfrak{X}(\chi_1), \mathscr{O}_\mathfrak{X}(\chi_2)) = \mathrm{Hom}_\mathfrak{Y}(\mathscr{O}_\mathfrak{Y}(\chi_1), \mathscr{O}_\mathfrak{Y}(\chi_2))$$

for all $\chi_1, \chi_2 \in \mathsf{S}$. *Then* S *is transparent for* $\mathfrak{X} \subset \mathfrak{Y}$ *if and only if there is an expression*

$$\Delta_{\mathfrak{X}*}\mathscr{O}_\mathfrak{X} = \operatorname*{colim}_{i \in I} \mathscr{A}_i \boxtimes \mathscr{O}_\mathfrak{X}(\chi_i)$$

with all $\chi_i \in \mathsf{S}$ *and all* $\mathscr{A}_i \in \mathsf{QC}(\mathfrak{X})$. *In this case,* $\mathscr{K}_\mathsf{S} = \mathrm{colim}_{i \in I}\, \mathscr{A}_i \boxtimes \mathscr{O}_\mathfrak{Y}(\chi_i) \in \mathsf{QC}(\mathfrak{X} \times \mathfrak{Y})$.

Proof. With the given hypotheses, transparency of S is equivalent to the claim that $\{\mathscr{O}_\mathfrak{X}(\chi)\}_{\chi \in \mathsf{S}}$ generates $\mathsf{QC}(\mathfrak{X})$. (Compactness of the sheaves $\{\mathscr{O}_\mathfrak{X}(\chi)\}_{\chi \in \mathsf{S}}$ is obvious because $\mathfrak{X}$ is perfect.) The equivalence in the statement of the Proposition now follows from Lemma 1.3.1.1. The computation of $\mathscr{K}_\mathsf{S}$ is Proposition 1.3.1.2. □

Example 5.1.2.2. Let $\mathfrak{X} = \mathbb{P}^n \subset [\mathbb{A}^{n+1}/\mathbb{G}_m]$. The Beilinson complex

$$\Omega^n_{\mathbb{P}^n}(n) \boxtimes \mathscr{O}_{\mathbb{P}^n}(-n) \longrightarrow \Omega^{n-1}_{\mathbb{P}^n}(n-1) \boxtimes \mathscr{O}_{\mathbb{P}^n}(-n+1) \longrightarrow \dots \longrightarrow \mathscr{O}_{\mathbb{P}^n} \boxtimes \mathscr{O}_{\mathbb{P}^n}$$

may be understood as a colimit over all of its terms (building the complex up one term at a time by a sequence of iterated mapping cones / cofibers). This complex is quasi-isomorphic to $\Delta_{\mathbb{P}^n *}\mathscr{O}_{\mathbb{P}^n}$, so by Proposition 5.1.2.1, we see that $S = \{-n, \dots, 0\}$ is transparent for $\mathbb{P}^n \subset [\mathbb{A}^{n+1}/\mathbb{G}_m]$. The complex $\mathscr{K}_S$ is given by

$$\Omega^n_{\mathbb{P}^n}(n) \boxtimes \mathscr{O}_{[\mathbb{A}^{n+1}/\mathbb{G}_m]}(-n) \longrightarrow \dots \longrightarrow \mathscr{O}_{\mathbb{P}^n} \boxtimes \mathscr{O}_{[\mathbb{A}^{n+1}/\mathbb{G}_m]}.$$

When the resolution of the diagonal of $\mathfrak{X}$ extends to a resolution of the pushforward of the structure sheaf of a perfect stack over $\mathfrak{X} \times \mathfrak{Y}$, we may obtain a more geometric description of W_S:

Proposition 5.1.2.3. *Suppose that there exist a collection of weights* $S \subset \mathbb{X}^\bullet(G)$*, a perfect stack* $\mathcal{W}_S$*, and a morphism* $q = (q_1, q_2) : \mathcal{W}_S \to \mathfrak{X} \times \mathfrak{Y}$ *such that:*

- $\mathrm{Hom}_{\mathfrak{X}}(\mathscr{O}_{\mathfrak{X}}(\chi_1), \mathscr{O}_{\mathfrak{X}}(\chi_2)) = \mathrm{Hom}_{\mathfrak{Y}}(\mathscr{O}_{\mathfrak{Y}}(\chi_1), \mathscr{O}_{\mathfrak{Y}}(\chi_2))$ *for all* $\chi_1, \chi_2 \in S$.
- *There is an expression* $q_*\mathscr{O}_{\mathcal{W}_S} = \mathrm{colim}_{i \in I}\, \mathscr{A}_i \boxtimes \mathscr{O}_{\mathfrak{Y}}(\chi_i)$ *with all* $\chi_i \in S$.
- *There is a Cartesian square*

$$\begin{tikzcd} \mathfrak{X} \arrow[r] \arrow[d, "\Delta_{\mathfrak{X}}"] & \mathcal{W}_S \arrow[d, "q"] \\ \mathfrak{X} \times \mathfrak{X} \arrow[r, "(\mathrm{id}_{\mathfrak{X}}, j)"] & \mathfrak{X} \times \mathfrak{Y}. \end{tikzcd}$$

Then S *is transparent for* $\mathfrak{X} \subset \mathfrak{Y}$*, and* $W_S = q_{2*}q_1^*$.

Proof. By base change, we have

$$\Delta_{\mathfrak{X}*}\mathscr{O}_{\mathfrak{X}} = (\mathrm{id}_{\mathfrak{X}}, j)^* q_* \mathscr{O}_{\mathcal{W}_S} = \operatorname*{colim}_{i \in I} \mathscr{A}_i \boxtimes \mathscr{O}_{\mathfrak{X}}(\chi_i) \in \mathsf{QC}(\mathfrak{X} \times \mathfrak{X}).$$

Thus the hypotheses of Proposition 5.1.2.1 are satisfied, S is transparent for $\mathfrak{X} \subset \mathfrak{Y}$, and W_S is given by the Fourier-Mukai transform with kernel $q_*\mathscr{O}_{\mathcal{W}_S}$. This Fourier-Mukai transform is exactly $q_{2*}q_1^*$. □

Example 5.1.2.4. In the situation of Example 5.1.2.2, the complex $\mathscr{K}_S$ has cohomology sheaves $\mathscr{H}^i(\mathscr{K}_S)$ concentrated in degree 0. Furthermore, the map $\mathscr{O}_{\mathbb{P}^n} \boxtimes \mathscr{O}_{[\mathbb{A}^{n+1}/\mathbb{G}_m]} \to \mathscr{K}_S$ (induced from the brutal truncation) is a surjection on $\mathscr{H}^0$. That is, $\mathscr{K}_S$ is a quotient of the structure sheaf in the abelian category of quasicoherent sheaves. Thus $\mathscr{K}_S$ is naturally a commutative algebra. Let $\mathcal{W}_S = \underline{\mathrm{Spec}}_{\mathbb{P}^n \times [\mathbb{A}^{n+1}/\mathbb{G}_m]} \mathscr{K}_S$, and let $q = (q_1, q_2) : \mathcal{W} \to \mathbb{P}^n \times [\mathbb{A}^{n+1}/\mathbb{G}_m]$ be the natural map. The conditions of Proposition 5.1.2.3 hold automatically, so we must have $W_S = q_{2*}q_1^*$.

5.1.3 Hitchcock functors via opposite collections of weights

We may use the results of Section 5.1.2 to obtain geometric descriptions of the windows W_S in some nice cases. For our applications, we will also want to understand the Hitchcock functors H_S and their relationship to the aforementioned windows. It turns out that the Fourier-Mukai kernel of H_S agrees with the Fourier-Mukai kernel of a window corresponding to a *different* transparent collection of weights!

Let $S \subset \mathbb{X}^\bullet(G)$ be an arbitrary collection of weights. The *opposite collection* to S is $-S = \{-\chi \,|\, \chi \in S\}$. By taking duals of line bundles, we see that $Q_S(\phi)^{\mathrm{op}} \simeq Q_{-S}(\phi)$.

Proposition 5.1.3.1. *Let S be transparent for $\mathfrak{X} \subset \mathfrak{Y}$. Then $-S$ is also transparent for $\mathfrak{X} \subset \mathfrak{Y}$. Furthermore, for all $\mathscr{F} \in \mathsf{Perf}(\mathfrak{X})$, there is a natural equivalence $W_{-S}(\mathscr{F}) = W_S(\mathscr{F}^\vee)^\vee$.*

Proof. Let $\mathscr{F} \in \mathsf{Perf}(\mathfrak{X})$. By hypothesis, $\mathscr{F}^\vee$ may be obtained from the objects $\mathscr{O}_{\mathfrak{X}}(\chi)$ for $\chi \in S$ via a finite sequence of taking shifts, (co)fibers, and direct summands. Dualizing, we see that $\mathscr{F}$ can be obtained from the objects $\mathscr{O}_{\mathfrak{X}}(-\chi)$ for $\chi \in S$ via a finite sequence of taking shifts, (co)fibers, and direct summands. Because $\mathsf{Perf}(\mathfrak{X})$ compactly generates $\mathsf{QC}(\mathfrak{X})$, the collection $\{\mathscr{O}_{\mathfrak{X}}(-\chi)\}_{\chi \in S}$ also compactly generates $\mathsf{QC}(\mathfrak{X})$.

For all $-\chi_1, -\chi_2 \in S$ and all $i > 0$, we have

$$\mathrm{Hom}^i_{\mathfrak{X}}\left(\mathscr{O}_{\mathfrak{X}}(-\chi_1), \mathscr{O}_{\mathfrak{X}}(-\chi_2)\right) = \mathrm{Hom}^i_{\mathfrak{X}}\left(\mathscr{O}_{\mathfrak{X}}(\chi_2), \mathscr{O}_{\mathfrak{X}}(\chi_1)\right) = 0$$

by transparency of S. Thus $-S$ is transparent.

To see the claim about $W_{-S}(\mathscr{F})$, by writing $\mathscr{F}$ as a canonical colimit of objects $\mathscr{O}_{\mathfrak{X}}(-\chi)$, we reduce to showing that $W_{-S}\big(\mathscr{O}_{\mathfrak{X}}(-\chi)\big) = W_S(\mathscr{O}_{\mathfrak{X}}(-\chi)^\vee)^\vee$ for all $\chi \in S$. But this is obvious, as both $W_{-S}\big(\mathscr{O}_{\mathfrak{X}}(-\chi)\big)$ and $W_S(\mathscr{O}_{\mathfrak{X}}(-\chi)^\vee)^\vee$ are naturally equivalent to $\mathscr{O}_{\mathfrak{Y}}(-\chi)$ by definition. □

We may use knowledge of W_{-S} to compute the Hitchcock functor H_S:

Proposition 5.1.3.2. *Let S be transparent for $\mathfrak{X} \subset \mathfrak{Y}$. Then $\mathscr{K}_{-S}$ is the Fourier-Mukai kernel of the Hitchcock functor $H_S : \mathsf{QC}(\mathfrak{Y}) \to \mathsf{QC}(\mathfrak{X})$.*

Proof. Let $\pi_{\mathfrak{Y}} : \mathfrak{Y} \times \mathfrak{X} \to \mathfrak{Y}$ and $\pi_{\mathfrak{X}} : \mathfrak{Y} \times \mathfrak{X} \to \mathfrak{X}$ be the natural projections. We need to show that

$$\mathrm{Hom}_{\mathfrak{X}}(\mathscr{F}, H_S\mathscr{G}) = \mathrm{Hom}_{\mathfrak{X}}\left(\mathscr{F}, \pi_{\mathfrak{X}*}(\pi_{\mathfrak{Y}}^*\mathscr{G} \otimes \mathscr{K}_{-S})\right)$$

for all $\mathscr{F} \in \mathsf{QC}(\mathfrak{X})$ and all $\mathscr{G} \in \mathsf{QC}(\mathfrak{Y})$. Because $\mathsf{Perf}(\mathfrak{X})$ compactly generates $\mathsf{QC}(\mathfrak{X})$, $\mathsf{Perf}(\mathfrak{Y})$ compactly generates $\mathsf{QC}(\mathfrak{Y})$, and H_S preserves colimits, it suffices to take $\mathscr{F} \in \mathsf{Perf}(\mathfrak{X})$ and

$\mathscr{G} \in \mathsf{Perf}(\mathfrak{Y})$. Then we may compute

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{X}}(\mathscr{F}, \mathsf{H}_{\mathsf{S}}\mathscr{G}) &= \mathrm{Hom}_{\mathfrak{Y}}(\mathsf{W}_{\mathsf{S}}\mathscr{F}, \mathscr{G}) \\
&= \mathrm{Hom}_{\mathfrak{Y}}\left((\mathsf{W}_{-\mathsf{S}}(\mathscr{F}^{\vee}))^{\vee}, \mathscr{G}\right) \\
&= \mathrm{Hom}_{\mathfrak{Y}}\left(\mathscr{G}^{\vee}, \mathsf{W}_{-\mathsf{S}}\mathscr{F}^{\vee}\right) \\
&= \mathrm{Hom}_{\mathfrak{Y}}\left(\mathscr{G}^{\vee}, \pi_{\mathfrak{Y}*}(\pi_{\mathfrak{X}}^{*}\mathscr{F}^{\vee} \otimes \mathscr{K}_{-\mathsf{S}})\right) \\
&= \mathrm{Hom}_{\mathfrak{X}\times\mathfrak{Y}}\left(\pi_{\mathfrak{Y}}^{*}\mathscr{G}^{\vee}, \pi_{\mathfrak{X}}^{*}\mathscr{F}^{\vee} \otimes \mathscr{K}_{-\mathsf{S}}\right) \\
&= \mathrm{Hom}_{\mathfrak{X}\times\mathfrak{Y}}\left(\pi_{\mathfrak{X}}^{*}\mathscr{F}, \pi_{\mathfrak{Y}}^{*}\mathscr{G} \otimes \mathscr{K}_{-\mathsf{S}}\right) \\
&= \mathrm{Hom}_{\mathfrak{X}}\left(\mathscr{F}, \pi_{\mathfrak{X}*}(\pi_{\mathfrak{Y}}^{*}\mathscr{G} \otimes \mathscr{K}_{-\mathsf{S}})\right).
\end{aligned}$$

□

Corollary 5.1.3.3. *Let* S *be transparent for* $\mathfrak{X} \subset \mathfrak{Y}$*, and suppose there exists a perfect stack* $\mathcal{W}_{-\mathsf{S}}$ *and a morphism* $\mathsf{q} = (\mathsf{q}_1, \mathsf{q}_2) : \mathcal{W}_{-\mathsf{S}} \to \mathfrak{X} \times \mathfrak{Y}$ *such that* $\mathsf{W}_{-\mathsf{S}} = \mathsf{q}_{2*}\mathsf{q}_1^{*}$*. Then* $\mathsf{H}_{\mathsf{S}} = \mathsf{q}_{1*}\mathsf{q}_2^{*}$.

Proof. Take $\mathscr{K}_{-\mathsf{S}} = \mathsf{q}_{*}\mathscr{O}_{\mathcal{W}_{-\mathsf{S}}}$ in Proposition 5.1.3.2. □

Example 5.1.3.4. Let $\mathfrak{X} = \mathbb{P}^{n} \subset [\mathbb{A}^{n}/\mathbb{G}_{m}]$, and let $\mathsf{q} : \mathcal{W} \to \mathbb{P}^{n} \times [\mathbb{A}^{n}/\mathbb{G}_{m}]$ be as in Example 5.1.2.4. For $\mathsf{S} = \{0, \ldots, n\}$, we see that $\mathsf{H}_{\mathsf{S}} = \mathsf{q}_{1*}\mathsf{q}_2^{*}$.

5.1.4 The toric case

Our key examples of transparent collections of line bundles will arise from smooth toric stacks. For our purposes, a *smooth toric stack* is an open substack $\mathfrak{X} = [\mathsf{X}/\mathsf{G}]$ of a stack quotient $[\mathbb{A}^{n}/\mathsf{G}]$, where the action $\mathsf{G} \curvearrowright \mathbb{A}^{n}$ is induced by a homomorphism $\mathsf{G} \to \mathbb{G}_{m}^{n}$ and the diagonal action $\mathbb{G}_{m}^{n} \curvearrowright \mathbb{A}^{n}$. As above, we write $\phi : [\mathbb{A}^{n}/\mathsf{G}] \to \mathsf{BG}$ for the structure map.

We begin by introducing a combinatorial description of the total weight quiver $\mathsf{Q}(\phi)$.

Definition 5.1.4.1. Let $\phi : \mathsf{G} \to \mathbb{G}_{m}^{n}$ be a homomorphism. We may define a (non-enriched) symmetric monoidal discrete category $\mathsf{Q}^{\mathrm{pre}}(\phi)$, the *prelinear total weight quiver*, where:

- $\mathrm{ob}\,\mathsf{Q}^{\mathrm{pre}}(\phi) = \mathbb{X}^{\bullet}(\mathsf{G})$.
- $\mathrm{Map}_{\mathsf{Q}^{\mathrm{pre}}(\phi)}(\chi_1, \chi_2)$ consists of monomials of degree $\chi_2 - \chi_1$ in the variables $x_1, \ldots, x_n$.
- Composition is given by multiplication of monomials.
- Multiplication of objects is the group operation in $\mathbb{X}^{\bullet}(\mathsf{G})$.
- Multiplication of morphisms is multiplication of monomials.

For $\mathsf{S} \subset \mathbb{X}^{\bullet}(\mathsf{G})$, we define the *prelinear partial weight quiver* $\mathsf{Q}_{\mathsf{S}}^{\mathrm{pre}}(\phi)$ as the full subcategory of $\mathsf{Q}^{\mathrm{pre}}(\phi)$ with objects S.

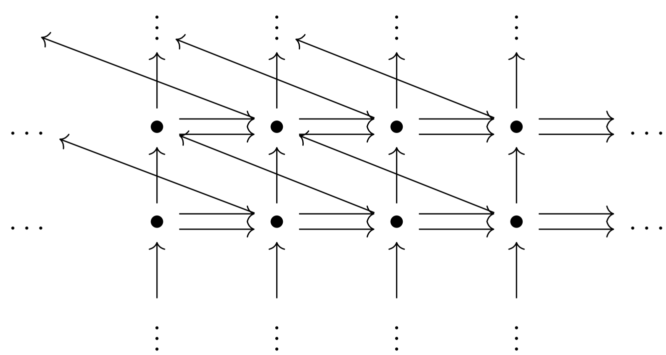
For $\mathsf{S} \subset \mathbb{X}^{\bullet}(\mathsf{G})$, the weight quiver $\mathsf{Q}_{\mathsf{S}}(\phi)$ agrees with the k-linearization of the prelinear weight quiver $\mathsf{Q}_{\mathsf{S}}^{\mathrm{pre}}(\phi)$.

Example 5.1.4.2. For $\mathfrak{X} = \mathbb{P}^n \subset [\mathbb{A}^{n+1}/\mathbb{G}_m]$, the category $\mathsf{Q}^{\mathrm{pre}}(\phi)$ is the infinite Beilinson quiver $\mathsf{Q}_{n,\infty}$ of the Introduction (depicted in Fig. 0.2). The collection $\mathsf{S} = \{0, \dots, n\}$ is transparent for $\mathbb{P}^n \subset [\mathbb{A}^{n+1}/\mathbb{G}_m]$, and the category $\mathsf{Q}^{\mathrm{pre}}_{\mathsf{S}}(\phi)$ is the Beilinson quiver Q_n depicted in Fig. 0.1.

Example 5.1.4.3. Let $\mathfrak{X} = \mathsf{F}_2$, the Hirzebruch surface of type 2. We may realize $\mathfrak{X}$ as an open subset of the quotient $[\mathbb{A}^4/\mathbb{G}_m^2]$, where $\mathbb{G}_m^2$ acts on $\mathbb{A}^4$ by

$$(g_1, g_2) \cdot (y_1, y_2, y_3, y_4) = (g_1 y_1, g_1^{-2} g_2 y_2, g_1 y_3, g_2 y_4).$$

(See e.g. [CLS11, Examples 14.2.17 and 14.2.20].) The category $\mathsf{Q}^{\mathrm{pre}}(\phi)$ is the infinite quiver



where:

- the horizontal arrows are given by the monomials y_1 and y_3,
- the vertical arrows are given by the monomials y_4,
- the slanted diagonal arrows are given by the monomials y_3, and
- there are (implicit) relations $y_i y_j = y_j y_i$.

Identify $\mathbb{X}^\bullet(\mathbb{G}_m^2) = \mathbb{Z}^2$ via the isomorphism corresponding to the x and y axes in our depiction of $\mathsf{Q}^{\mathrm{pre}}(\phi)$. The collection $\mathsf{S} = \{(0,0), (1,0), (-2,1), (-1,1)$ is transparent for $\mathfrak{X} \subset [\mathbb{A}^4/\mathbb{G}_m^2]$ by the results of [Kin97, §6].

Let us say that the open immersion $j_{\mathfrak{X}} : \mathfrak{X} \subset [\mathbb{A}^n/\mathsf{G}]$ is *decent* if $\mathbb{X}^\bullet(\mathsf{G}) = \mathrm{Pic}(\mathfrak{X})$. If $\mathfrak{X}$ is a smooth toric stack and $\mathfrak{X} \hookrightarrow [\mathbb{A}^n/\mathsf{G}]$ is a decent open immersion, then any full strong exceptional collection of line bundles on $\mathfrak{X}$ corresponds to a transparent collection of weights S for $\mathfrak{X} \subset [\mathbb{A}^n/\mathsf{G}]$ by Proposition 5.1.1.5. Thus we obtain a combinatorial description of $\mathsf{QC}(\mathfrak{X})$:

$$\mathsf{QC}(\mathfrak{X}) = \mathsf{Fun}\left(\mathsf{Q}^{\mathrm{pre}}_{\mathsf{S}}(\phi), \mathsf{D}(k)\right).$$

This description applies to many key examples.

Example 5.1.4.4. If $\mathfrak{X}$ is a smooth toric variety without torus factors, the Cox construction (see [CLS11, Chapter 5]) gives a decent open immersion $\mathfrak{X} \hookrightarrow [\mathbb{A}^n/G]$, where n is the number of rays in the fan of $\mathfrak{X}$.

Example 5.1.4.5. Let $d_0, \dots, d_n$ be positive integers, and let $\mathbb{P}(d_0, \dots, d_n)$ be the weighted projective stack with weights $d_0, \dots, d_n$. Assume $n \geqslant 1$.[2] Then the standard open immersion $\mathbb{P}(d_0, \dots, d_n) \hookrightarrow [\mathbb{A}^{n+1}/\mathbb{G}_m]$ is decent.

In the situation of Example 5.1.4.4, the window corresponding to the *Bondal-Thomsen collection* of weights has a particularly nice description whenever it exists. We first recall some relevant definitions.

Definition 5.1.4.6 ([Bon06])**.** Let $\mathfrak{X}$ be a smooth complete toric variety, and let $j : \mathfrak{X} \hookrightarrow \mathfrak{Y} = [\mathbb{A}^n/G]$ be the decent open immersion of Example 5.1.4.4. Let $\phi^* : \mathbb{Z}^n \to \mathbb{X}^\bullet(G)$ be the natural map induced by the action $G \curvearrowright \mathbb{A}^n$. Let $M = \ker \phi^*$, and let $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. Define a (discontinuous) *anti-Bondal-Ruan map* $F_\phi : M_\mathbb{R}/M \to \mathbb{X}^\bullet(G)$ (where $M_\mathbb{R}/M$ is viewed as a real torus) by

$$F_\phi\Big(\sum_i a_i e_i + M\Big) = \phi^*\Big(\sum_i \lfloor a_i \rfloor e_i\Big)$$

The *anti-Bondal-Thomsen collection* (or *anti-BT collection*) is $-\Theta_\mathfrak{X} = \operatorname{im} F_\phi \subset \mathbb{X}^\bullet(G)$. The *Bondal-Thomsen collection* is $\Theta_\mathfrak{X} = -(-\Theta_\mathfrak{X})$. Following [FH25, Definition 5.3], we say that $\mathfrak{X}$ is of *Bondal-Ruan type* if $\Theta_\mathfrak{X}$ (equivalently, $-\Theta_\mathfrak{X}$) is transparent for $\mathfrak{X} \subset [\mathbb{A}^n/G]$.

Remark 5.1.4.7. We use the term "anti" above to stress that we are prioritizing different conventions from those of [Bon06] and subsequent works. This is motivated by Proposition 5.1.3.2: the window corresponding to the Bondal-Thomsen collection has a nice geometric description, so the Hitchcock functor corresponding to the anti-BT collection has a corresponding description. Because of the relevance of Hitchcock functors to the results of Section 5.2, we prefer to use the anti-BT collection.

Remark 5.1.4.8. The generalization of Definition 5.1.4.6 to toric Deligne-Mumford stacks is straightforward and profitable; see e.g. [HHL24, §2.3]. We avoid doing so here due to the difficulties sketched in Remark 5.1.4.12 below.

Intersection-theoretic sufficient conditions for a smooth complete toric variety to be of Bondal-Ruan type are given in [Bon06]. (See also [FH25, Proposition 5.18] for a more detailed proof.) These conditions hold for many examples of interest, e.g. all but two smooth toric Fano threefolds.

Example 5.1.4.9. For $\mathfrak{X} = \mathbb{P}^n$, the Bondal-Thomsen collection of weights is $\{-n, \dots, 0\} \subset \mathbb{Z} = \mathbb{X}^\bullet(\mathbb{G}_m)$. This is transparent for $\mathbb{P}^n \subset [\mathbb{A}^{n+1}/\mathbb{G}_m]$, so $\mathbb{P}^n$ is of Bondal-Ruan type.

[2] When $n = 0$, we have $\mathbb{P}(d_0) = B(\mathbb{Z}/d_0)$, and one can check the existence of a transparent collection of line bundles "by hand."

One should also note that many smooth toric varieties (e.g. Hirzebruch surfaces) are not of Bondal-Ruan type but still admit transparent collections of weights. In such cases, we can construct corresponding windows using Proposition 5.1.1.6, but a geometric interpretation of such windows remains elusive. In the Bondal-Ruan case, the situation is much nicer:

Proposition 5.1.4.10. *Let $\mathfrak{X}$ be a smooth complete toric variety of Bondal-Ruan type, and fix notation as in Definition 5.1.4.6. Let $\mathcal{W}$ be the scheme-theoretic closure (taken in the category of relative schemes over $\mathfrak{X} \times \mathfrak{Y}$) of the diagonal of $\mathfrak{X}$ in $\mathfrak{X} \times \mathfrak{Y}$.*[3] *Let $q = (q_1, q_2) : \mathcal{W} \to \mathfrak{X} \times \mathfrak{Y}$ be the natural closed immersion. Then $W_{\Theta_{\mathfrak{X}}} = q_{2*}q_1^*$.*

Proof. We need only check that q satisfies the hypotheses of Proposition 5.1.2.3 for $S = \Theta_{\mathfrak{X}}$. The first condition follows by the definition of Bondal-Ruan type. The second condition follows from [HHL24, Theorem A], as the structure sheaf $q_*\mathscr{O}_{\mathcal{W}}$ admits a resolution by direct sums of line bundles $\mathscr{O}_{\mathfrak{X}}(\chi_1) \boxtimes \mathscr{O}_{\mathfrak{Y}}(\chi_2)$ with $\chi_1, \chi_2 \in \Theta_{\mathfrak{X}}$. The third condition holds by the definition of $\mathcal{W}$. □

In the case where $\mathfrak{X}$ is $\mathbb{P}^n$ (or more generally a weighted projective stack), we may understand the stack $\mathcal{W}$ and the push-pull explicitly using blowups. This seems to be known to experts (see e.g. [BDF17]), but we include details for completeness.

Example 5.1.4.11. Let $\mathfrak{X} = \mathbb{P}(d_0, \dots, d_n)$ be a weighted projective stack. Consider the weighted stacky blow-up $\mathsf{sBl}_0^w \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ of $\mathbb{A}^{n+1}$ at the origin with weight $w = (d_0, \dots, d_n)$ in the sense of [QR21]. Unpacking the definitions, the stack $\mathsf{sBl}_0^w \mathbb{A}^{n+1}$ is equivalent to the (stacky) relative affine spectrum $\underline{\mathrm{Spec}}_{\mathfrak{X}} \mathrm{Sym}^\bullet_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{\mathfrak{X}}(1)$ (i.e., the (stacky) total space of $\mathscr{O}_{\mathfrak{X}}(-1)$) since they are both equivalent to the quotient stack

$$[(\mathrm{Spec}\, k[x_0, \dots, x_n][u] \setminus V(x_0, \dots, x_n))/\mathbb{G}_m]$$

where the $\mathbb{G}_m$-action on $\mathrm{Spec}\, k[x_0, \dots, x_n][u]$ corresponds to the $\mathbb{Z}$-grading on $k[x_0, \dots, x_n][u]$ given by $\deg x_i = d_i$ and $\deg u = -1$. Now, consider another $\mathbb{G}_m$-action on $\mathrm{Spec}\, k[x_0, \dots, x_n][u]$ corresponding to the $\mathbb{Z}$-grading given by $\deg x_i = 0$ and $\deg u = 1$. Combining these two $\mathbb{G}_m$-actions, we have a $\mathbb{G}_m^2$-action on $\mathrm{Spec}\, k[x_0, \dots, x_n][u] \setminus V(x_0, \dots, x_n)$ and we set

$$\mathfrak{Z} = [(\mathrm{Spec}\, k[x_0, \dots, x_n][u] \setminus V(x_0, \dots, x_n))/\mathbb{G}_m^2].$$

Geometrically, the second $\mathbb{G}_m$-action corresponds to the $\mathbb{G}_m$-action on $\underline{\mathrm{Spec}}_{\mathfrak{X}} \mathrm{Sym}^\bullet_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{\mathfrak{X}}(1) \simeq \mathsf{sBl}_0^w \mathbb{A}^{n+1}$ given by scaling fibers and we have

$$\mathfrak{Z} \simeq [\underline{\mathrm{Spec}}_{\mathfrak{X}} \mathrm{Sym}^\bullet_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{\mathfrak{X}}(1)/\mathbb{G}_m] \simeq [\mathsf{s}\underset{0}{\overset{w}{\mathsf{Bl}}}\, \mathbb{A}^{n+1}/\mathbb{G}_m]$$

[3]Alternatively, note that the map $(\mathrm{id}_{\mathfrak{X}}, j) : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{Y}$ is a locally closed immersion, so its (relative) affinization $\mathcal{W} = \underline{\mathrm{Spec}}_{\mathfrak{X}\times\mathfrak{Y}} \tau^{\leqslant 0}(\mathrm{id}_{\mathfrak{X}}, j)_*\mathscr{O}_{\mathfrak{X}}$ is a closed substack of $\mathfrak{X} \times \mathfrak{Y}$.

in the sense of [Rom05, Remark 2.4]. Now, note that if we consider the diagonal $\mathbb{G}_m$-action of weight w on $\mathbb{A}^{n+1}$, then the stacky weighted blow-up $\mathsf{sBl}_0^w \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ is $\mathbb{G}_m$-equivariant. Thus, we obtain a morphism

$$\mathsf{b} : \mathfrak{Z} \simeq [\mathsf{s}\underset{0}{\overset{w}{\mathsf{Bl}}} \mathbb{A}^{n+1}/\mathbb{G}_m] \to [\mathbb{A}^{n+1}/\mathbb{G}_m] = \mathfrak{Y}.$$

Algebraically, this reduces to saying that the ring homomorphism

$$\begin{aligned} k[x_0, \ldots, x_n] &\to k[x_0, \ldots, x_n][u] \\ x_i &\mapsto x_i u^{d_i} \end{aligned}$$

respects $\mathbb{Z}^2$-gradings given by $\deg x_i = (d_i, 0)$ on $k[x_0, \ldots, x_n]$ and by $\deg x_i = (0, d_i)$ and $\deg u = (1, -1)$ on $k[x_0, \ldots, x_n][u]$, respectively (cf. [QR21, Section 5.2] and [BDF17] for details).

Similarly, the projection $\underline{\mathrm{Spec}}_{\mathfrak{X}} \mathrm{Sym}^\bullet_{\mathscr{O}_\mathfrak{X}} \mathscr{O}_\mathfrak{X}(1) \to \mathfrak{X}$ is $\mathbb{G}_m$-equivariant with respect to the trivial $\mathbb{G}_m$-action on $\mathfrak{X}$, so we obtain a morphism

$$\pi : \mathfrak{Z} \simeq [\underline{\mathrm{Spec}}_{\mathfrak{X}} \mathrm{Sym}^\bullet_{\mathscr{O}_\mathfrak{X}} \mathscr{O}_\mathfrak{X}(1)/\mathbb{G}_m] \to \mathfrak{X} \times \mathrm{B}\mathbb{G}_m \to \mathfrak{X}.$$

In this case, the Bondal-Thomsen collection $\Theta_\mathfrak{X} = \left\{1 - \sum_i d_i, \ldots, 0\right\}$ forms a full strong exceptional collection in $\mathbb{P}(d_0, \ldots, d_n)$. By comparing with the proof of [BDF17, Proposition 4.1.5] (or by direct verification), we can see $W_{\Theta_\mathfrak{X}} \simeq \mathsf{b}_* \pi^*$.

Remark 5.1.4.12. It is not clear how one would obtain the description of $W_{\Theta_\mathfrak{X}}$ in Example 5.1.4.11 from the results of [HHL24] (unless e.g. $(d_0, \ldots, d_n) = (1, \ldots, 1)$), as the diagonal morphism of $\mathbb{P}(d_0, \ldots, d_n)$ is finite but not a closed immersion in general. One would hope that a refined approach can be used to extend the virtual resolutions of [HHL24] to virtual resolutions of diagonals of separated smooth toric Deligne-Mumford stacks. In this case, our results for smooth complete toric varieties of Bondal-Ruan type may be extended to separated smooth toric Deligne-Mumford stacks of Bondal-Ruan type.

5.1.5 Quiver-theoretic examples

We may use the approach of [BP08] to view a full strong exceptional collection of line bundles on a general smooth projective toric variety as transparent for a certain open immersion. This open immersion is constructed in terms of moduli spaces of quiver representations.

Let Q be a k-linear category such that:

- Q has finitely many objects.
- All Hom objects in Q are finite-dimensional vector spaces concentrated in degree zero.
- There is a total order on $\mathrm{ob}\,\mathsf{Q}$ such that $\mathrm{Hom}_\mathsf{Q}(q_1, q_2) = 0$ only if $q_1 \leqslant q_2$

- $\mathrm{Hom}_{\mathsf{Q}}(q, q) = k$ for all $q \in \mathsf{Q}$.

Let Y_{Q} be the parameter space of representations of Q with dimension vector $(1, \ldots, 1)$, i.e.

$$\mathsf{Y} \subset \prod_{q_1, q_2 \in \mathrm{ob}\, \mathsf{Q}} \mathbb{A}\Big(\mathrm{Hom}_k \left(\mathrm{Hom}_{\mathsf{Q}}(q_1, q_2), \mathrm{Hom}_k(k, k) \right) \Big)$$

is the subscheme defined by the composition relations in Q. Let

$$\mathsf{G}_{\mathsf{Q}} = \Big(\prod_{q \in \mathrm{ob}\, \mathsf{Q}} \mathbb{G}_m \Big) \Big/ \mathbb{G}_m^{\pi_0(\mathsf{Q})}$$

where $\pi_0(\mathsf{Q})$ is the set of connected components of Q. Then G_{Q} acts on Y by conjugation, and $\mathfrak{Y}_{\mathsf{Q}} = [\mathsf{Y}_{\mathsf{Q}}/\mathsf{G}_{\mathsf{Q}}]$ is the *rigidified moduli stack of vertexwise one-dimensional representations* of Q.

If $\mathfrak{X}$ is a smooth projective variety and $\{\mathscr{L}_1, \ldots, \mathscr{L}_n\}$ is a full strong exceptional collection of line bundles on $\mathfrak{X}$, we may define a small k-linear category $\mathsf{Q}_{\mathscr{L}}$ as the opposite of the full subcategory of $\mathsf{QC}(\mathfrak{X})$ consisting of the objects $\{\mathscr{L}_1, \ldots, \mathscr{L}_n\}$. Let us identify the objects of $\mathsf{Q}_{\mathscr{L}}$ with $\{1, \ldots, n\}$. There is a tautological map $j : \mathfrak{X} \to \mathfrak{Y}_{\mathsf{Q}_{\mathscr{L}}}$ defined by sending a point $u \in \mathfrak{X}$ to the representation $i \mapsto \mathrm{Hom}_{\mathfrak{X}}(\mathscr{L}_i, \mathscr{O})$. By [BP08, proof of Theorem 2.4], the map j is an open immersion. The vector bundles $\mathscr{L}_i$ are all pullbacks $j^* \mathscr{O}_{\mathfrak{Y}_{\mathsf{Q}_{\mathscr{L}}}}(\chi_i)$ for some $\chi_i \in \mathbb{X}^\bullet(G)$, and the collection $S = \{\chi_1, \ldots, \chi_n\} \subset \mathbb{X}^\bullet(G)$ is transparent for $\mathfrak{X} \subset \mathfrak{Y}$.

Example 5.1.5.1. By [HP11, Theorem 5.14], if $\mathfrak{X}$ is a del Pezzo surface with $\mathrm{rk}\, \mathrm{Pic}\, \mathfrak{X} \leqslant 7$, then $\mathfrak{X}$ admits a full strong exceptional collection of line bundles. We do not know when the corresponding k-linear categories $\mathsf{Q}_{\mathscr{L}}$ admit "quiver tensor products."

The quiver-theoretic presentations of smooth projective varieties are generally larger and harder to control than those arising in our toric examples. In particular, the varieties Y_{Q} are typically singular. We do not know when it is possible to describe the corresponding windows $\mathcal{W}_S$ in terms of push-pull operations as in Proposition 5.1.4.10.

Example 5.1.5.2. For $\mathfrak{X} = \mathbb{P}^n$, if Q is the quiver corresponding to (the opposite of) the Beilinson collection $\{\mathscr{O}_{\mathbb{P}^n}(-n+1), \ldots, \mathscr{O}_{\mathbb{P}^n}$, then Y_{Q} is a closed subscheme of $\mathbb{A}^{n(n+1)}$, and $\mathsf{G}_{\mathsf{Q}} \cong \mathbb{G}_m^n$. By contrast, the toric quotient presentation allows us to view $\mathfrak{X}$ as an open substack of $[\mathbb{A}^n/\mathbb{G}_m]$.

5.2 Extended convolution products

Suppose $\mathfrak{M}$ is a perfect $\mathbb{E}_n$-monoid stack with multiplication $\mu : \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$. If $\mathfrak{X}$ is an open substack of $\mathfrak{M}$, then the multiplication on $\mathfrak{M}$ typically does not induce a multiplication on $\mathfrak{X}$: there is no reason for $\mathfrak{X}$ to be closed under μ.

However, the situation is somewhat different if we categorify. Sometimes the convolution product $\star_{\mathfrak{M}}$ on $\mathsf{QC}(\mathfrak{M})$ (together with a choice of window W_S) induces an "extended convolution" product $\star'_{\mathfrak{M},S}$ on $\mathsf{QC}(\mathfrak{X})$ *even when* $\mathfrak{X}$ *is not closed under* μ. Our goal in this section is to demonstrate this claim and understand the behavior of $\star'_{\mathfrak{M},S}$ in the case where $\mathfrak{M}$ is a (nice) global quotient $[M/G]$.

5.2.1 Definitions and examples

We begin by formalizing the data we will use to construct extended convolution products.

Definition 5.2.1.1. For $1 \leqslant n \leqslant \infty$, an $\mathbb{E}_n$-*extended convolution setup* (or $\mathbb{E}_n$-*EC setup*) is a triple

$$(\phi : \mathfrak{M} \to BG, j : \mathfrak{X} \hookrightarrow \mathfrak{M}, S)$$

where:

- G is a commutative reductive group over k.
- $\phi : \mathfrak{M} \to BG$ is a homomorphism of $\mathbb{E}_n$-monoid derived stacks which is (as a morphism of derived stacks) affine and almost of finite type.
- $j : \mathfrak{X} \hookrightarrow \mathfrak{M}$ is an open immersion.
- $S \subset \mathbb{X}^\bullet(G)$ is a collection of weights which is transparent for $\mathfrak{X} \subset \mathfrak{M}$.

A *geometric* $\mathbb{E}_n$-EC setup is an $\mathbb{E}_n$-EC setup as above together with:

- a morphism of perfect stacks $q = (q_1, q_2) : \mathcal{W}_{-S} \to \mathfrak{X} \times \mathfrak{M}$ such that $W_{-S} = q_{2*}q_1^*$.

Given an $\mathbb{E}_n$-EC setup as above, we write $\mu : \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$ for the binary multiplication.

Before developing the theory of extended convolution, we introduce a few key examples of $\mathbb{E}_n$-EC setups.

Example 5.2.1.2. Let A be a finite dimensional k-algebra. The natural map $k^\times \subset k \to A$ corresponds to a multiplicative homomorphism $\mathbb{G}_m \to \mathbb{A}(A)$, and the quotient stack $[\mathbb{A}(A)/\mathbb{G}_m]$ inherits a monoid structure. Let $\mathfrak{X} = \mathbb{P}(A)$, and identify $\mathbb{X}^\bullet(\mathbb{G}_m) \simeq \mathbb{Z}$ in such a way that $\mathcal{O}_{\mathbb{P}(A)}(1) \in \operatorname{Pic}\mathbb{P}(A) \simeq \mathbb{X}^\bullet(\mathbb{G}_m)$ corresponds to $1 \in \mathbb{Z}$. Write $S = \{0, \dots, \dim_k A\} \subset \mathbb{Z}$. Then

$$(\phi : [\mathbb{A}(A)/\mathbb{G}_m] \to B\mathbb{G}_m, \mathbb{P}(A) \hookrightarrow [\mathbb{A}(A)/\mathbb{G}_m], S)$$

is a geometric $\mathbb{E}_1$-EC setup, where $q : \mathcal{W}_{-S} \to \mathbb{P}(A) \times [\mathbb{A}(A)/\mathbb{G}_m]$ is the morphism of Proposition 5.1.4.10. If A is commutative, the above $\mathbb{E}_1$-EC setup upgrades to an $\mathbb{E}_\infty$-EC setup.

Example 5.2.1.3. Let $\mathfrak{X}$ be a smooth toric stack with a decent open immersion $j : \mathfrak{X} \hookrightarrow [\mathbb{A}^n/G]$. We may view $[\mathbb{A}^n/G]$ as an $\mathbb{E}_\infty$-monoid stack, where $\mathbb{A}^n$ is equipped with its coordinatewise multiplication. Let S be any collection of weights corresponding to a full strong exceptional collection of line bundles in $\mathsf{Perf}(\mathfrak{X})$. Then

$$(\phi : [\mathbb{A}^n/G] \to BG, j : \mathfrak{X} \to [\mathbb{A}^n/G], S)$$

is an $\mathbb{E}_\infty$-EC setup (though there is no reason *a priori* for it to be geometric).

We may do even better in the Bondal-Ruan case:

Example 5.2.1.4. Suppose that $\mathfrak{X}$ is a smooth complete toric variety of Bondal-Ruan type, and let $j : \mathfrak{X} \hookrightarrow [\mathbb{A}^n/G]$ be the open immersion arising from the Cox construction. Then

$$(\phi : [\mathbb{A}^n/G] \to BG, j : \mathfrak{X} \to [\mathbb{A}^n/G], -\Theta_{\mathfrak{X}})$$

is a geometric $\mathbb{E}_\infty$-EC setup by Proposition 5.1.4.10.

Given an $\mathbb{E}_n$-EC setup (ϕ, j, S) with notation as in Definition 5.2.1.1, the category $\mathsf{QC}(\mathfrak{X})$ inherits a convolution product from that of $\mathsf{QC}(\mathfrak{M})$.

Proposition 5.2.1.5. *Let (ϕ, j, S) be an $\mathbb{E}_n$-EC setup with notation as in Definition 5.2.1.1. There exists a unique $\mathbb{E}_n$-monoidal structure $\star'_{\mathfrak{M},S}$ on $\mathsf{QC}(\mathfrak{X})$ such that the Hitchcock functor*

$$H_S : (\mathsf{QC}(\mathfrak{M}), \star_{\mathfrak{M}}) \to (\mathsf{QC}(\mathfrak{X}), \star'_{\mu,S})$$

is $\mathbb{E}_n$-monoidal.

Proof. We check that the adjunction

$$W_S : \mathsf{QC}(\mathfrak{X}) \rightleftarrows \mathsf{QC}(\mathfrak{M}) : H_S$$

satisfies the conditions of Proposition 1.2.5.5. Conditions (1) and (2) are satisfied by the definition of W_S. Condition (3) is satisfied by the definition of the convolution product: $- \star_{\mathfrak{M}} - = \mu_*(- \boxtimes -)$ and μ_* has left adjoint μ^*. To check condition (4), note that $\mathsf{QC}(\mathfrak{X})$ is compactly generated by $\{\mathscr{O}_{\mathfrak{X}}(\chi)\}_{\chi \in S}$ and $W_S(\mathscr{O}_{\mathfrak{X}}(\chi)) = \mathscr{O}_{\mathfrak{M}}(\chi)$ for $\chi \in S$. Since

$$\mu^* \mathscr{O}_{\mathfrak{M}}(\chi) \simeq \mathscr{O}_{\mathfrak{M}}(\chi) \boxtimes \mathscr{O}_{\mathfrak{M}}(\chi)$$

by G-equivariance of the multiplication on $\mathfrak{M}$, condition (4) holds. □

We call $\star'_{\mathfrak{M},S}$ the *extended convolution product* (or *EC product*) associated with the $\mathbb{E}_n$-EC setup (ϕ, j, S). The name is justified by Corollary 5.2.2.6, which implies in particular that $\star'_{\mathfrak{M},S}$ extends the convolution product on any open submonoid stack of $\mathfrak{M}$ contained in $\mathfrak{X}$.

By Proposition 1.2.5.1, the $\mathbb{E}_n$-monoidal functor H_S satisfies the following universal property: if $F : (\mathsf{QC}(\mathfrak{M}), \star_{\mathfrak{M}}) \to (\mathcal{C}, \otimes_{\mathcal{C}})$ is an $\mathbb{E}_n$-monoidal functor and the underlying functor of F satisfies $F = F' \circ H_S$, then there is a unique $\mathbb{E}_n$-monoidal structure on F' such that $F = F' \circ H_S$ as $\mathbb{E}_n$-monoidal functors.

5.2.2 Functoriality of extended convolution

To understand the behavior of EC products geometrically, it is useful (though perhaps not strictly necessary for later developments) to define morphisms of $\mathbb{E}_n$-EC setups. We do so here and show (Proposition 5.2.2.5) that morphisms of $\mathbb{E}_n$-EC setups induce $\mathbb{E}_n$-monoidal functors.

Definition 5.2.2.1. Let (ϕ_1, j_1, S_1) and (ϕ_2, j_2, S_2) be $\mathbb{E}_n$-EC setups with notation as in Definition 5.2.1.1. For notational simplicity, write $j_i = j_{\mathfrak{X}_i}$, $W_i = W_{S_i}$, $H_i = H_{S_i}$, and $\star'_i = \star'_{\mathfrak{M}_i, S_i}$ for $i = 1, 2$. A *morphism of* $\mathbb{E}_n$*-EC setups* $(\alpha, \beta) : (\phi_1, j_1, S_1) \to (\phi_2, j_2, S_2)$ consists of:

- an $\mathbb{E}_n$-homomorphism $\alpha : \mathfrak{M}_1 \to \mathfrak{M}_2$ and
- a group homomorphism $\beta : G_1 \to G_2$,

such that the diagram

$$\begin{tikzcd} \mathfrak{M}_1 \arrow[r, "\alpha"] \arrow[d, "\phi_1"] & \mathfrak{M}_2 \arrow[d, "\phi_2"] \\ BG_1 \arrow[r, "B\beta"] & BG_2 \end{tikzcd}$$

commutes and $\operatorname{im} \alpha^* W_2 \subset \operatorname{im} W_1$.

Example 5.2.2.2. If $G_1 = G_2 = G$ and $S_1 = \mathbb{X}^\bullet(G)$ (so $\mathfrak{X}_1 = \mathfrak{M}_1$), then any morphism $\alpha : \mathfrak{M}_1 \to \mathfrak{M}_2$ over BG gives a morphism of $\mathbb{E}_n$-EC setups $(\alpha, \mathrm{id}_G) : (\phi_1, \mathrm{id}_{\mathfrak{X}_1}, \mathbb{X}^\bullet(G)) \to (\phi_2, j_2, S_2)$.

Example 5.2.2.3. Let $f : A \to B$ be a homomorphism of finite-dimensional k-algebras such that $\dim A \geqslant \dim B$. Then the natural map $\alpha_f : [\mathbb{A}(A)/\mathbb{G}_m] \to [\mathbb{A}(B)/\mathbb{G}_m]$ gives a morphism of $\mathbb{E}_1$-EC setups

$$\begin{aligned}(\alpha_f, \mathrm{id}_{\mathbb{G}_m}) : &\big(\phi_A : [\mathbb{A}(A)/\mathbb{G}_m] \to B\mathbb{G}_m, \mathbb{P}(A) \hookrightarrow [\mathbb{A}(A)/\mathbb{G}_m], \{0, \dots, \dim_k A\}\big) \\ &\to \big(\phi_B : [\mathbb{A}(B)/\mathbb{G}_m] \to B\mathbb{G}_m, \mathbb{P}(B) \hookrightarrow [\mathbb{A}(B)/\mathbb{G}_m], \{0, \dots, \dim_k B\}\big).\end{aligned}$$

If A and B are both commutative, then α_f is in fact a morphism of $\mathbb{E}_\infty$-EC setups.

Morphisms of $\mathbb{E}_n$-EC setups induce functors that preserve the corresponding EC products. To prove this, we first need a lemma allowing us to simplify certain composite functors involving windows and Hitchcock functors.

Lemma 5.2.2.4. *For* $i \in \{1, 2\}$*, fix the following data:*

- *A commutative reductive group* G_i*,*
- *A morphism of derived stacks* $\phi_i : \mathfrak{Y}_i \to BG_i$ *which is affine and almost of finite type,*

- *An open immersion* $j_i : \mathfrak{X}_i \hookrightarrow \mathfrak{Y}_i$*, and*
- *A transparent collection of weights* $S_i \subset \mathbb{X}^\bullet(G_i)$ *for* $\mathfrak{X}_i \subset \mathfrak{Y}_i$*.*

Write $W_i : \mathsf{QC}(\mathfrak{X}_i) \to \mathsf{QC}(\mathfrak{Y}_i)$ *and* $H_i : \mathsf{QC}(\mathfrak{Y}_i) \to \mathsf{QC}(\mathfrak{X}_i)$ *for the corresponding window and Hitchcock functor (respectively). Let* $f : \mathfrak{Y}_1 \to \mathfrak{Y}_2$ *be a morphism such that* $f^*(\operatorname{im} W_2) \subset \operatorname{im} W_1$*.*[4] *Then:*

1. $f^* W_2 = W_1 j_1^* f^* W_2$*.*
2. $H_2 f_* = H_2 f_* j_{1*} H_1$*.*
3. $H_2 f_* W_1 = H_2 f_* j_{1*}$*.*

Proof. (1). Suppose $\mathscr{F} \in \mathsf{QC}(\mathfrak{X}_2)$. Then $f^* W_2 \mathscr{F} = W_1 \mathscr{G}$ for some $\mathscr{G} \in \mathsf{QC}(\mathfrak{X}_1)$. Applying j_1^* gives

$$j_1^* f^* W_2 \mathscr{F} = j_1^* W_1 \mathscr{G} = \mathscr{G},$$

so

$$f^* W_2 \mathscr{F} = W_1 \mathscr{G} = W_1 j_1^* f^* W_2 \mathscr{F}.$$

Naturality of this isomorphism is clear because W_1 is fully faithful.

(2). Take right adjoints of all functors involved in (1).

(3). We compute

$$\begin{aligned} H_2 f_* W_1 &= H_2 f_* j_{1*} H_1 W_1 \text{ by (2)} \\ &= H_2 f_* j_{1*} \operatorname{id}_{\mathfrak{X}_1} \text{ because } W_1 \text{ is fully faithful} \\ &= H_2 f_* j_{1*}. \end{aligned}$$

□

Proposition 5.2.2.5. *Let* $(\alpha, \beta) : (\phi_1, j_1, S_1) \to (\phi_2, j_2, S_2)$ *be a morphism of* $\mathbb{E}_n$*-EC setups with notation as above. Then the functor* $H_2 \alpha_* j_{1*} : \left(\mathsf{QC}(\mathfrak{X}_1), \star_1'\right) \to \left(\mathsf{QC}(\mathfrak{X}_2), \star_2'\right)$ *is symmetric monoidal.*

Proof. By Lemma 5.2.2.4(2), we have $H_2 \alpha_* = H_2 \alpha_* j_{1*} H_1$. Using the universal property of H_1 mentioned in Proposition 5.2.1.5, it suffices to show that $H_2 \alpha_*$ is $\mathbb{E}_n$-monoidal. But this is clear as both H_2 and α_* are $\mathbb{E}_n$-monoidal. □

Corollary 5.2.2.6. *Let* (ϕ, j, S) *be an* $\mathbb{E}_n$*-EC setup with notation as in Definition 5.2.1.1. Let* $\alpha : \mathfrak{N} \to \mathfrak{M}$ *be a homomorphism of* $\mathbb{E}_n$*-monoid derived stacks over* BG*, and assume* α *factors through the inclusion* $j : \mathfrak{X} \hookrightarrow \mathfrak{Y}$*, say* $\alpha = j \circ a$*. Then the pushforward functor* $a_* : \left(\mathsf{QC}(\mathfrak{N}), \star_{\mathfrak{N}}\right) \to \left(\mathsf{QC}(\mathfrak{X}), \star'_{\mathfrak{M},S}\right)$ *is* $\mathbb{E}_n$*-monoidal.*

Proof. In Proposition 5.2.2.5, take $(\phi_1, j_1, S_1) = \left(\phi, \operatorname{id}_{\mathfrak{X}}, \mathbb{X}^\bullet(G)\right)$, $(\phi_2, j_2, S_2) = (\phi, j, S)$, $\alpha = j \circ a$, and $\beta = \operatorname{id}_G$. Then $H_2 \alpha_* j_{1*} H_1 = H_S j_* a_* = a_*$. □

[4] The functor W_1 is fully faithful, so it suffices to check this inclusion on objects.

5.2.3 Fourier-Mukai kernels and geometric descriptions of extended convolution

We would like to understand the operations $\star'_{\mathfrak{M},S}$ using the geometry of $\mathfrak{X}$. This is difficult in general for the simple reason that the window associated with a transparent collection is hard to understand geometrically. However, we shall show that giving a geometric description of $\star'_{\mathfrak{M},S}$ is *no more difficult* than giving a geometric description of the Fourier-Mukai kernel $\mathscr{K}_{-S}$ of W_{-S}. In particular, for geometric $\mathbb{E}_n$-EC setups, we obtain a simple geometric description of $\star'_{\mathfrak{M},S}$.

For simplicity, our claims here will be made for the binary product $\star'_{\mathfrak{M},S} : \mathsf{QC}(\mathfrak{X}) \boxtimes \mathsf{QC}(\mathfrak{X}) \to \mathsf{QC}(\mathfrak{X})$. The analogous claims for n-ary products can be established using the same arguments.

Proposition 5.2.3.1. *Let (ϕ, j, S) be an $\mathbb{E}_n$-EC setup with notation as in Definition 5.2.1.1. View $\mathscr{K}_{-S}$ as an object of $\mathsf{QC}(\mathfrak{M} \times \mathfrak{X})$. Then $\star'_{\mathfrak{M},S}$ is given by the Fourier-Mukai transform with kernel $(j \times j \times \mathrm{id}_{\mathfrak{X}})^*(\mu \times \mathrm{id}_{\mathfrak{X}})^*\mathscr{K}_{-S} \in \mathsf{QC}(\mathfrak{X}^3)$.*

Proof. Let $\mathscr{F}, \mathscr{G} \in \mathsf{QC}(\mathfrak{X})$. By Lemma 5.2.2.4(3), we have

$$\mathscr{F} \star'_{\mathfrak{M},S} \mathscr{G} = H_S(j_*\mathscr{F} \star_{\mathfrak{M}} j_*\mathscr{G}) = H_S \mu_*(j \times j)_*(\mathscr{F} \boxtimes \mathscr{G}).$$

Writing $H_S = \Phi_{\mathscr{K}_{-S}}$ and applying the general formula $\Phi_{\mathscr{K}} \circ f_* = \Phi_{(f \times \mathrm{id})^* \mathscr{K}}$, we see the claim. $\square$

If we have an explicit resolution of the diagonal, we may obtain an explicit algebraic description of the corresponding EC product.

Corollary 5.2.3.2. *Suppose (ϕ, j, S) is an $\mathbb{E}_n$-EC setup with notation as in Definition 5.2.1.1. Write $\Delta_{\mathfrak{X}*}\mathscr{O}_{\mathfrak{X}}$ as a complex*

$$\cdots \longrightarrow \bigoplus_{\chi \in S} \mathscr{O}_{\mathfrak{X}}(-\chi) \boxtimes \mathscr{A}_{i,\chi} \xrightarrow{d_i} \bigoplus_{\chi \in S} \mathscr{O}_{\mathfrak{X}}(-\chi) \boxtimes \mathscr{A}_{i+1,\chi} \longrightarrow \cdots$$

Then the Fourier-Mukai kernel of $\star'_{\mathfrak{M},S}$ is

$$\cdots \longrightarrow \bigoplus_{\chi \in S} \mathscr{O}_{\mathfrak{X}}(-\chi) \boxtimes \mathscr{O}_{\mathfrak{X}}(-\chi) \boxtimes \mathscr{A}_{i,\chi} \xrightarrow{(j \times j \times \mathrm{id}_{\mathfrak{X}})^*(\mu \times \mathrm{id}_{\mathfrak{X}})^* d_i} \bigoplus_{\chi \in S} \mathscr{O}_{\mathfrak{X}}(-\chi) \boxtimes \mathscr{O}_{\mathfrak{X}}(-\chi) \boxtimes \mathscr{A}_{i+1,\chi} \longrightarrow \cdots$$

Proof. By Proposition 5.1.2.1, the kernel $\mathscr{K}_{-S} \in \mathsf{QC}(\mathfrak{M} \times \mathfrak{X})$ is given by

$$\cdots \longrightarrow \bigoplus_{\chi \in S} \mathscr{O}_{\mathfrak{M}}(-\chi) \boxtimes \mathscr{A}_{i,\chi} \xrightarrow{d_i} \bigoplus_{\chi \in S} \mathscr{O}_{\mathfrak{M}}(-\chi) \boxtimes \mathscr{A}_{i+1,\chi} \longrightarrow \cdots$$

Applying Proposition 5.2.3.1 gives the result. $\square$

Proposition 5.2.3.1 also yields a geometric description of the EC products associated with geometric $\mathbb{E}_n$-EC setups.

Proposition 5.2.3.3. *Suppose* (ϕ, j, S, q) *is a geometric* $\mathbb{E}_n$*-EC setup with notation as in Definition 5.2.1.1. Let* $\mathfrak{Z}_{\mu,S}$ *be defined by the Cartesian square (in* dStk_k*)*

$$\begin{tikzcd} \mathfrak{Z}_{\mu,S} \arrow[rr, "p_3"] \arrow[d, "p_1\times p_2"] & & \mathcal{W}_{-S} \arrow[d, "q_2"] \\ \mathfrak{X}\times\mathfrak{X} \arrow[r, "j\times j"] & \mathfrak{M}\times\mathfrak{M} \arrow[r, "\mu"] & \mathfrak{M} \end{tikzcd}$$

Then, for $\mathscr{F}, \mathscr{G} \in \mathsf{QC}(\mathfrak{X})$*, there is a natural isomorphism* $\mathscr{F} \star'_{\mathfrak{M},S} \mathscr{G} = q_{1*}p_{3*}(p_1^*\mathscr{F} \otimes p_2^*\mathscr{G})$.

Proof. By hypothesis, $\mathscr{K}_{-S} = q_*\mathscr{O}_{\mathcal{W}_{-S}}$, so the Fourier-Mukai kernel of $\star'_{\mathfrak{M},S}$ is

$$(j \times j \times \mathrm{id}_{\mathfrak{X}})^*(\mu \times \mathrm{id}_{\mathfrak{X}})^* q_*\mathscr{O}_{\mathcal{W}_{-S}}$$

by Proposition 5.2.3.1. Applying base change for the commutative square

$$\begin{tikzcd} \mathfrak{Z}_{\mu,S} \arrow[rr, "p_3"] \arrow[d, "p_1\times p_2\times(q_1\circ p_3)"'] & & \mathcal{W}_{-S} \arrow[d, "q"] \\ \mathfrak{X}^3 \arrow[r, "j\times j\times \mathrm{id}_{\mathfrak{X}}"] & \mathfrak{M}\times\mathfrak{M}\times\mathfrak{X} \arrow[r, "\mu\times\mathrm{id}_{\mathfrak{X}}"] & \mathfrak{M}\times\mathfrak{X} \end{tikzcd}$$

shows that

$$(j \times j \times \mathrm{id}_{\mathfrak{X}})^*(\mu \times \mathrm{id}_{\mathfrak{X}})^* q_*\mathscr{O}_{\mathcal{W}_{-S}} = (p_{1,2} \times (q_1 \circ p_3))_*\mathscr{O}_{\mathfrak{Z}_{\mu,S}}.$$

For $\mathscr{F}, \mathscr{G} \in \mathsf{QC}(\mathfrak{X})$, we may now compute (letting $\pi_i : \mathfrak{X}^3 \to \mathfrak{X}$ be projection onto the ith coordinate):

$$\begin{aligned} \mathscr{F} \star'_{\mathfrak{M},S} \mathscr{G} &= \pi_{3*}\big(\pi_1^*\mathscr{F} \otimes \pi_2^*\mathscr{G} \otimes (p_1 \times p_2 \times (q_1 \circ p_3))_*\mathscr{O}_{\mathfrak{Z}_{\mu,S}}\big) \\ &= \pi_{3*}\big(p_1 \times p_2 \times (q_1 \circ p_3)\big)_*\big(p_1^*\mathscr{F} \otimes p_2^*\mathscr{G}\big) \text{ by the projection formula} \\ &= q_{1*}p_{3*}(p_1^*\mathscr{F} \otimes p_2^*\mathscr{G}). \end{aligned}$$

□

Example 5.2.3.4. Suppose $\mathfrak{X}$ is a separated scheme and $\mathcal{W}_{-S} \subset \mathfrak{X} \times \mathfrak{M}$ is the restriction of the diagonal closed substack of $\mathfrak{M}$ to $\mathfrak{X} \times \mathfrak{M}$ (this is true for Example 5.2.1.2 and Example 5.2.1.4). Then $\mathfrak{Z}_{\mu,S} \subset \mathfrak{X}^3$ is (the restriction to $\mathfrak{X}^3$ of) the closure of the graph of $\mu \circ (j \times j) : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{M}$. The legs of the correspondence

$$\begin{tikzcd} & \mathfrak{Z}_{\mu,S} \arrow[dl, "p_1\times p_2"'] \arrow[dr, "q_1\circ p_3"] & \\ \mathfrak{X}\times\mathfrak{X} & & \mathfrak{X} \end{tikzcd}$$

are just the projections to the factors when $\mathfrak{Z}_{\mu,S}$ is viewed as a closed substack of $\mathfrak{X}^3$.

Remark 5.2.3.5. If $\alpha : (\phi_1, j_1, S_1) \to (\phi_2, j_2, S_2)$ is a morphism of $\mathbb{E}_n$-EC setups, we may use similar methods to the above to describe the functor $H_2\alpha_*j_{1*}$.

5.2.4 Quiver tensor products via extended convolution

Let (ϕ, j, S) be an $\mathbb{E}_n$-EC setup with notation as in Definition 5.2.1.1. Because $\mu^*\mathscr{O}_{\mathfrak{M}}(\chi) = \mathscr{O}(\chi) \boxtimes \mathscr{O}(\chi)$ for all $\chi \in \mathbb{X}^\bullet(G)$, the $\mathbb{E}_n$-coalgebra structure on $Q(\phi)$ restricts to an $\mathbb{E}_n$-coalgebra structure on $Q_S(\phi)$. As in Section 4.2.1, this produces an $\mathbb{E}_n$-monoidal *quiver tensor product* $\otimes_{Q_S(\phi)}$ on $D(Q_S(\phi)^{\mathrm{op}})$.

Theorem 5.2.4.1. *Let* (ϕ, j, S) *be an* $\mathbb{E}_n$*-EC setup with notation as in Definition 5.2.1.1. Then there is an* $\mathbb{E}_n$*-monoidal equivalence*

$$\begin{aligned} \big(\mathsf{QC}(\mathfrak{X}), \star'_{\mathfrak{M},S}\big) &\xrightarrow{\sim} \big(D(Q_S(\phi)^{\mathrm{op}}), \otimes_{Q_S(\phi)}\big) \\ \mathscr{F} &\mapsto \big(\chi \mapsto \mathrm{Hom}_{\mathfrak{X}}(\mathscr{O}_{\mathfrak{X}}(\chi), \mathscr{F})\big). \end{aligned}$$

Proof. The functor in question is an equivalence by Proposition 5.1.1.6, so it suffices to show that said functor is $\mathbb{E}_n$-monoidal. For this, let $i_S : Q_S(\phi) \to Q(\phi)$ be the inclusion, and recall that the diagram

$$\begin{tikzcd} \mathsf{QC}(\mathfrak{M}) \arrow[r, "H_S"] \arrow[d, "\sim"] & \mathsf{QC}(\mathfrak{X}) \arrow[d, "\sim"] \\ D(Q(\phi)^{\mathrm{op}}) \arrow[r, "i_S^*"] & D(Q_S(\phi)^{\mathrm{op}}) \end{tikzcd}$$

commutes by the definition of H_S. Thus, by the universal property of Proposition 5.2.1.5, it suffices to show that the composite $\mathsf{QC}(\mathfrak{M}) \to D(Q_S(\phi)^{\mathrm{op}})$ is $\mathbb{E}_n$-monoidal. But the equivalence $\big(\mathsf{QC}(\mathfrak{M}), \star_{\mathfrak{M}}\big) \xrightarrow{\sim} \big(D(Q(\phi)^{\mathrm{op}}), \otimes_Q\big)$ is $\mathbb{E}_n$-monoidal by Theorem 4.2.3.1, and functoriality of Corollary 1.2.3.3 implies i_S^* is $\mathbb{E}_n$-monoidal, so the same must be true of the composite. □

Remark 5.2.4.2. As a consequence of Theorem 5.2.4.1 and Remark 5.1.1.7, we see that, if S arises from a finite full strong exceptional collection of line bundles on $\mathfrak{X}$, then the EC product $\star'_{\mathfrak{M},S}$ preserves perfect complexes. (One may also deduce this from the definition of $\star'_{\mathfrak{M},S}$.) However, even in this case, we are not aware of a way to define $\star'_{\mathfrak{M},S}$ geometrically without using the full categories QC. Indeed, convolution on $\mathfrak{M}$ uses the functor μ_*, which typically does not preserve perfect complexes.

5.3 Application: (symmetric) monoidal structures on $\mathsf{Perf}\,\mathbb{P}^d$

Using the methods of Section 5.2, we can construct new (symmetric) monoidal structures on $\mathsf{Perf}\,\mathbb{P}^d$ from finite-dimensional algebras. Recall that we use the term "$\mathbb{E}_n$-monoidal structures" for the sole purpose of stating results about usual monoidal structures ($n = 1$) and symmetric monoidal structures ($n = \infty$) concisely, and no other value of n will be considered in this section (cf. Remark 1.2.1.1 and Remark 1.2.1.2).

Throughout the section, let A be a finite-dimensional k-algebra with $\dim A = d + 1$, and let $\mathbb{A}(A)$ be the corresponding affine monoid scheme. By standard results on algebraic monoids (see e.g. [Vin95, page 1]), the group of units $\mathbb{A}(A)^\times$ is open and Zariski dense in $\mathbb{A}(A)$. We may use this to obtain the following:

Proposition 5.3.0.1. *Let A be a nonzero*[5] *finite-dimensional k-algebra. Write $\mathfrak{Z}_A$ for the closure in $\mathbb{P}(A)^3$ of the graph of the binary multiplication on $[\mathbb{A}(A)^\times/\mathbb{G}_m]$. Then:*

1. *Push-pull along the correspondence*

$$\begin{tikzcd} & \mathfrak{Z}_A \arrow[dl] \arrow[dr] & \\ \mathbb{P}(A) \times \mathbb{P}(A) & & \mathbb{P}(A) \end{tikzcd}$$

   defines a monoidal structure $\star'_A := \star'_{[\mathbb{A}(A)/\mathbb{G}_m], -\Theta_{\mathbb{P}(A)}}$ on $\mathsf{Perf}\,\mathbb{P}(A)$.

2. *If $j' : [\mathbb{A}(A)^\times/\mathbb{G}_m] \hookrightarrow \mathbb{P}(A)$ is the inclusion, then the pushforward functor*

$$j'_* : \big(\mathsf{QC}([\mathbb{A}(A)^\times/\mathbb{G}_m]), \star_{[\mathbb{A}(A)^\times/\mathbb{G}_m]}\big) \to \big(\mathsf{QC}(\mathbb{P}(A)), \star'_A\big)$$

   is monoidal.

3. *The construction of $\star'_A$ is functorial in surjections of finite-dimensional k-algebras.*

When A is commutative, "monoidal" may be upgraded to "symmetric monoidal" throughout.

Proof. We use the geometric $\mathbb{E}_1$-EC setup $(\phi, j, \{0, \dots, \dim_k A\}, q)$ of Example 5.2.1.2. When A is commutative, we upgrade this to an $\mathbb{E}_\infty$-EC setup.

(1). The scheme $\mathfrak{Z}_A$ agrees with the restriction to $\mathbb{P}(A)^3$ of the graph of the multiplication morphism

$$\mathbb{P}(A) \times \mathbb{P}(A) \to [\mathbb{A}(A)/\mathbb{G}_m].$$

Thus Proposition 5.2.3.3 and Example 5.2.3.4 imply the claim.

(2). This is a direct consequence of Corollary 5.2.2.6.

(3). This follows from Example 5.2.2.3 and Proposition 5.2.2.5. □

5.3.1 Computations

One can compute EC products $\mathscr{F} \star'_A \mathscr{G}$ using the equivalence of Theorem 5.2.4.1. More precisely, using Remark 4.1.0.3 with $R = \operatorname{Sym} A^\vee$, the category $\mathsf{Q}_S(\phi)^{\mathrm{op}}$ is given by

$$q_0 \xleftarrow{A^\vee} q_1 \xleftarrow{A^\vee} \dots \xleftarrow{A^\vee} q_n$$

[5]When $A = 0$, we have $\mathbb{P}(A) = \varnothing$, so the corresponding EC-setup exists but the description of the geometric structure fails.

where we have relations $\alpha_1\alpha_2 = \alpha_2\alpha_1 \in \mathrm{Sym}\, A^\vee$ for $\alpha_i \in A^\vee$. Via the equivalence of Theorem 5.2.4.1, $\mathscr{F} \in \mathsf{QC}(\mathsf{Perf}\,\mathbb{P}(A))$ corresponds to the derived $Q_S(\phi)^{\mathrm{op}}$-representation[6]

$$\Gamma(\mathbb{P}(A), \mathscr{F}) \longleftarrow \Gamma(\mathbb{P}(A), \mathscr{F}(-1)) \longleftarrow \dots \longleftarrow \Gamma(\mathbb{P}(A), \mathscr{F}(-n))$$

and similarly for $\mathscr{G}$. The EC product $\mathscr{F} \star'_A \mathscr{G}$ then corresponds to the quiver tensor product

$$\Gamma(\mathbb{P}(A), \mathscr{F}) \otimes_k \Gamma(\mathbb{P}(A), \mathscr{G}) \longleftarrow \dots \longleftarrow \Gamma(\mathbb{P}(A), \mathscr{F}(-n)) \otimes_k \Gamma(\mathbb{P}(A), \mathscr{G}(-n)),$$

where the coalgebra structure on $A^\vee$ is used to construct the tensor product of morphisms.

This is easiest to understand for specific classes of sheaves:

Example 5.3.1.1. For $i \in \{0, \dots, \dim A - 1\}$, the sheaf $\Omega^i_{\mathbb{P}(A)}(i)[i]$ corresponds to the simple $Q_S(\phi)^{\mathrm{op}}$-representation

$$0 \longleftarrow \dots \longleftarrow 0 \longleftarrow k \longleftarrow 0 \longleftarrow \dots \longleftarrow 0$$

sending q_i to k and q_j to 0 for $j \neq i$ by Bott's formula. In particular, we see that

$$\Omega^i_{\mathbb{P}(A)}(i)[i] \star'_A \Omega^j_{\mathbb{P}(A)}(j)[j] = \begin{cases} 0 & i \neq j \\ \Omega^i_{\mathbb{P}(A)}(i)[i] & i = j \end{cases}$$

Example 5.3.1.2. For $[a] \in \mathbb{P}(A)$, the skyscraper sheaf $k([a])$ corresponds to the quiver representation

$$k \xleftarrow{a} k \xleftarrow{a} \dots \xleftarrow{a} k$$

where we view a as an element of $A^{\vee\vee} = \mathrm{Hom}_k(A^\vee, \mathrm{Hom}_k(k, k))$.

Proposition 5.3.1.3. *Let* $a_1, a_2 \in A \setminus \{0\}$*, so* $k([a_1]), k([a_2]) \in \mathsf{Perf}\,\mathbb{P}(A)$. *Then*

$$k([a_1]) \star'_A k([a_2]) = \begin{cases} k\big([\mu_A(a_1, a_2)]\big) & \mu_A(a_1, a_2) \neq 0. \\ \oplus_{i=0}^{\dim A - 1} \Omega^i_{\mathbb{P}(A)}(i)[i] & \mu_A(a_1, a_2) = 0. \end{cases}$$

Proof. Using Example 5.3.1.2 and the fact that the coalgebra structure on $A^\vee$ is the dual of the algebra structure on A, we see that $k([a_1]) \star'_A k([a_2])$ corresponds to the quiver representation

$$k \xleftarrow{\mu_A(a_1,a_2)} k \xleftarrow{\mu_A(a_1,a_2)} \dots \xleftarrow{\mu_A(a_1,a_2)} k$$

If $\mu_A(a_1, a_2) \neq 0$, this representation corresponds to $k\big([\mu_A(a_1, a_2)]\big)$, establishing the first case of the claim.

Otherwise $\mu_A(a_1, a_2) = 0$, so $k([a_1]) \star'_A k([a_2])$ corresponds to the quiver representation

$$k \xleftarrow{0} k \xleftarrow{0} \dots \xleftarrow{0} k$$

[6]Recall that our convention is that all the functors are implicitly derived.

This decomposes as a direct sum of the simple representations of $\mathsf{Q}_S(\phi)^{\mathrm{op}}$. Example 5.3.1.1 lets us convert these quiver representations back into perfect complexes, giving

$$k([a_1]) \star'_A k([a_2]) = \bigoplus_{i=0}^{\dim A-1} \Omega^i_{\mathbb{P}(A)}(i)[i]. \qquad \square$$

5.3.2 Categorical compactifications

The monoidal ∞-categories $\left(\mathsf{Perf}\,\mathbb{P}(A), \star'_A\right)$ give categorified "compactifications" of many well-known algebras and groups. We formalize this notion as follows.

Definition 5.3.2.1. Let $\mathfrak{N}$ be a perfect $\mathbb{E}_n$-monoid stack over k. A *categorical compactification* of $\mathfrak{N}$ is a fully faithful, $\mathbb{E}_n$-monoidal functor

$$\iota : (\mathsf{QC}(\mathfrak{N}), \star_{\mathfrak{N}}) \hookrightarrow (\mathsf{Ind}\,\mathcal{C}, \otimes_{\mathcal{C}})$$

where $(\mathcal{C}, \otimes_{\mathcal{C}}) \in \mathsf{CAlg}\left(\mathsf{Cat}_k^{\mathrm{perf}}\right)$. We will often abuse notation and refer to $(\mathcal{C}, \otimes_{\mathcal{C}})$ as a categorical compactification of $\mathfrak{N}$.

By Corollary 5.2.2.6, if $\mathfrak{N}$ is an open submonoid of $[\mathbb{A}(A)/G]$ such that $\mathfrak{N} \subset \mathbb{P}(A)$, then pushforward along the inclusion $j' : \mathfrak{N} \subset \mathfrak{X}$ exhibits $(\mathsf{Perf}\,\mathbb{P}(A), \star'_{\mathfrak{M},S})$ as a categorical compactification of $\mathfrak{N}$. We obtain the following examples in this way.

Example 5.3.2.2. Any finite-dimensional k-algebra A' admits a categorical compactification. In fact, if we let $A = A' \times k$, then the inclusion $\mathbb{A}(A') \hookrightarrow \mathbb{P}(A)$ exhibits $(\mathsf{Perf}\,\mathbb{P}(A), \star'_{\mathfrak{M},S})$ as a categorical compactification of $\mathbb{A}(A)$.

Example 5.3.2.3. We may also construct categorical compactifications of many linear algebraic groups. We list some such groups G together with algebras A such that $(\mathsf{Perf}\,\mathbb{P}(A), \star'_{\mathfrak{M},S})$ is a categorical compactification of G:

- For $G = \mathbb{G}_m^n$, we can take $A = k^{n+1}$.
- For $G = \mathrm{GL}_n$, we can take $A = \mathrm{End}_k(k^n) \times k$.
- For $G = \mathrm{PGL}_n$, we can take $A = \mathrm{End}_k(k^n)$.
- For $G = \mathbb{G}_a$, we can take $A = k[\epsilon]/\epsilon^2$.
- For $G = \mathrm{B}_n$ the group of invertible upper triangular matrices, let $\overline{\mathrm{B}}_n$ be the k-algebra of all upper triangular matrices. Then we can take $A = \overline{\mathrm{B}}_n \times k$.
- For $G = \mathrm{B}_n/\mathbb{G}_m$ (a Borel subgroup of PGL_n), we can take $A = \overline{\mathrm{B}}_n$.

Given Example 5.3.2.2, it is natural to ask:

Question 5.3.2.4. *Which linear algebraic groups admit categorical compactifications?*

Remark 5.3.2.5. Categorical compactifications are typically far from unique, even if we impose "connectivity" hypotheses on $\mathcal{C}$.[7] It would be interesting to understand whether one can construct a *canonical* categorical compactification of a nice (e.g. semisimple) algebraic group.

Although the EC products in this subsection can be defined geometrically, they may exhibit surprising behavior at infinity.

Example 5.3.2.6. Consider $\mathbb{P}^1 = \mathbb{P}(\mathsf{k}^2)$, so that $\mathsf{Perf}\,\mathbb{P}^1$ is a categorical compactification of $\mathbb{G}_m$. By Proposition 5.3.1.3, we see that $\mathsf{k}([1:0]) \star'_{\mathsf{k}^2} \mathsf{k}([0:1])$ is the (globally supported) complex $\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-1)[1]$. Loosely: we have extended the multiplication of $\mathbb{G}_m$ to $\mathbb{P}^1$, but multiplying zero by infinity produces a perfect complex rather than a number!

Let us mention some additional subtleties that may arise when attempting to apply these methods over non-algebraically closed fields.

Remark 5.3.2.7. Suppose for the context of this remark that k is not algebraically closed, and let $\mathsf{k} \subset \mathsf{F}$ be a finite extension of fields. In this case, we still obtain a symmetric monoidal EC product on $\mathsf{Perf}\,\mathbb{P}(\mathsf{F})$. One might naïvely expect that this EC product agrees with a genuine convolution product, with the multiplication on $\mathbb{P}(\mathsf{F})$ constructed by restricting the multiplication on $\mathbb{A}(\mathsf{F})$ to $\mathbb{A}(\mathsf{F}) \setminus \{0\}$.

However, such a construction is typically not possible: even though the set of k-points $\big(\mathbb{A}(\mathsf{F}) \setminus \{0\}\big)(\mathsf{k}) = \mathsf{F}^\times$ is closed under multiplication, this is no longer true for the set of F-points $\big(\mathbb{A}(\mathsf{F}) \setminus \{0\}\big)(\mathsf{F})$. In particular, the construction of the EC product on $\mathsf{Perf}\,\mathbb{P}(\mathsf{F})$ does not contradict the fact that all complete connected algebraic groups are abelian varieties.

5.3.3 Recovering A from $(\mathsf{Perf}\,\mathbb{P}(A), \star'_A)$

We have the following result allowing us to reconstruct a finite-dimensional algebra A from the corresponding EC product on $\mathsf{Perf}\,\mathbb{P}(\mathsf{A})$:

Proposition 5.3.3.1. *Let A and A' be finite-dimensional k-algebras. Then A and A' are isomorphic as k-algebras if and only if there is a monoidal equivalence $(\mathsf{Perf}(\mathbb{P}(\mathsf{A})), \star'_{\mathsf{A}}) \simeq (\mathsf{Perf}(\mathbb{P}(\mathsf{A}')), \star'_{\mathsf{A}'})$.*

To prove Proposition 5.3.3.1, we will need a linear-algebraic lemma.

Notation 5.3.3.2. Let V and V' be finite-dimensional vector spaces. Given an element $\nu \in \mathsf{V}$ (possibly zero), let $[\nu]$ denote the corresponding class in $\mathsf{V}/\mathsf{k}^\times$. Given a linear map $\phi : \mathsf{V} \to \mathsf{V}'$, let $[\phi]$ denote the corresponding class in $\mathsf{V}/\mathsf{k}^\times \to \mathsf{V}/\mathsf{k}^\times$.

Lemma 5.3.3.3. *Let A and A' be finite-dimensional k-algebras. Suppose $\phi : \mathsf{A} \to \mathsf{A}'$ is a linear map such that $[\phi] : \mathsf{A}/\mathsf{k}^\times \to \mathsf{A}'/\mathsf{k}^\times$ is a homomorphism of monoids (in Set). Then there exists $\mathsf{c} \in \mathsf{k}^\times$ such that $\mathsf{c}\phi : \mathsf{A} \to \mathsf{A}'$ is a monoid homomorphism.*

[7]For example, any full strong exceptional collection on a smooth complete toric variety gives a categorical compactification of the dense torus by Example 5.2.1.4.

Proof. Invertible elements of A are dense in $\mathbb{A}(A)$ (see e.g. [Vin95, page 1]), so we may fix a basis $\{e_0, \dots, e_n\}$ of A such that e_i is invertible in A for all i. Because $[\phi]$ is a monoid homomorphism, the images $[\phi]([e_i])$ are invertible in $A'/k^\times$. Thus the images $\phi(e_i)$ must also be invertible in A'. In particular, for fixed i, the sets $\{\mu_{A'}(\phi(e_i), \phi(e_j))\}_{j=0}^n$ are linearly independent in A'.

For $i, j, \ell \in \{0, \dots, n\}$, we may write

$$\phi\big(\mu_A(e_i, e_j)\big) = c_{i,j}\mu_{A'}\big(\phi(e_i), \phi(e_j)\big)$$

and

$$\phi\big(\mu_A(e_i, e_j + e_\ell)\big) = c_{i,j\ell}\mu_{A'}\big(\phi(e_i), \phi(e_j + e_\ell)\big)$$

for some $c_{i,j}, c_{i,j\ell} \in k^\times$. For any $i, j, \ell \in \{0, \dots, n\}$, we get

$$\begin{aligned} c_{i,j}\mu_{A'}\big(\phi(e_i), \phi(e_j)\big) + c_{i,\ell}\mu_{A'}\big(\phi(e_i), \phi(e_\ell)\big) &= \phi\big(\mu_A(e_i, e_j + e_\ell)\big) \\ &= c_{i,j\ell}\mu_{A'}\big(\phi(e_i), \phi(e_j + e_\ell)\big) \\ &= c_{i,j\ell}\mu_{A'}\big(\phi(e_i), \phi(e_j)\big) + c_{i,j\ell}\mu_{A'}\big(\phi(e_i), \phi(e_\ell)\big) \end{aligned}$$

so that $c_{i,j} = c_{i,j\ell} = c_{i,\ell}$ by the aforementioned linear independence. In particular, for all i, j, we have $c_{i,j} = c_{i,0}$. Repeating the argument with the order of inputs to $\mu_{A'}$ reversed shows that $c_{i,j} = c_{0,j}$. Thus $c_{i,j} = c_{0,0}$ for all i, j, i.e.

$$\phi\big(\mu_A(e_i, e_j)\big) = c_{0,0}\mu_{A'}\big(\phi(e_i), \phi(e_j)\big).$$

Multiplying both sides of this equation by $c_{0,0}$ shows that $(c_{0,0}\phi) \circ \mu_A = \mu_{A'} \circ (c_{0,0}\phi \otimes c_{0,0}\phi)$, i.e. $c_{0,0}\phi$ is a monoid homomorphism. $\square$

Proof of Proposition 5.3.3.1. The "only if" direction follows by the functoriality of the construction of $\star'$ on surjective maps (Proposition 5.2.2.5).

For the "if" direction, we may assume A and A' have the same underlying vector space V. Every monoidal equivalence $(\mathsf{Perf}(\mathbb{P}(A)), \star'_A) \simeq (\mathsf{Perf}(\mathbb{P}(A')), \star'_{A'})$ is induced by an autoequivalence τ of $\mathsf{Perf}(\mathbb{P}(V))$. By [BO01, Theorem 3.1], we can write $\tau = [\phi]_*(- \otimes \mathscr{L}[n])$ for some $\phi \in \mathrm{GL}(V)$ (so $[\phi] \in \mathrm{PGL}(V)$), $\mathscr{L} \in \mathrm{Pic}(\mathbb{P}(V))$, and $n \in \mathbb{Z}$. As $\tau(k([1_A])) = k([1_{A'}])$, we must have $n = 0$, i.e. $\tau = [\phi]_*(- \otimes \mathscr{L})$.[8]

By Lemma 5.3.3.3, it suffices to show that $[\phi] : A/k^\times \to A'/k^\times$ is a monoid homomorphism. To this end, let $x, y \in A$. We may assume $\dim A \geqslant 2$, allowing us to argue by cases:

- If $x = 0$ or $y = 0$, we must have

$$[\phi](\mu_A(x, y)) = [0] = \big[\mu_{A'}\big(\phi(x), \phi(y)\big)\big]. \tag{5.3.3.4}$$

[8] One can also show that in this situation we must have $\mathscr{L} \simeq \mathscr{O}$, though this does not simplify matters for us.

- If x and y are both nonzero and $\mu_A(x,y) = 0$, then $\mu_{A'}(\phi(x),\phi(y)) = 0$, so Eq. (5.3.3.4) still holds. In fact, if we had $\mu_{A'}(\phi(x),\phi(y)) \neq 0$, then
$$k([\phi(x)]) \star'_{A'} k([\phi(y)]) = k\big([\mu_{A'}(x,y)]\big)$$
is indecomposable while Proposition 5.3.1.3 implies $k([x]) \star'_A k([y])$ is decomposable. But then
$$\tau\big(k([x]) \star'_A k([y])\big) = k([\phi(x)]) \star'_{A'} k([\phi(y)])$$
contradicts the assumption that τ is an equivalence.

- Otherwise, $\mu_A(x,y) \neq 0$, so
$$k\big([\phi(\mu_A(x,y))]\big) \simeq \tau\big(k([\mu_A(x,y)])\big) \simeq \tau\big(k([x]) \star'_A k([y])\big) \simeq k([\phi(x)]) \star'_{A'} k([\phi(y)]).$$
Hence, by Proposition 5.3.1.3, we have $[\phi(\mu_A(x,y))] = [\mu_{A'}(\phi(x),\phi(y))]$.

Thus $[\phi] : A/k^\times \to A'/k^\times$ is a monoid homomorphism. □

5.3.4 Invariants

Fix a finite-dimensional algebra A and consider the monoidal category $\big(\mathsf{Perf}\,\mathbb{P}(A), \star'_A\big)$. In this section we compute:

- The Balmer spectrum $\mathrm{Spc}_{\star'_A}\,\mathsf{Perf}(\mathbb{P}(A))$
- The Grothendieck ring $\mathsf{K}^0\big(\mathsf{Perf}\,\mathbb{P}(A), \star'_A\big)$.
- The Picard group $\mathrm{Pic}(\mathsf{Perf}\,\mathbb{P}(A), \star'_A)$.

We observe that the Balmer spectrum and the Grothendieck ring depend only on $\dim A$.

We first compute $\mathrm{Spc}_{\star'_A}\,\mathsf{Perf}(\mathbb{P}(A))$.

Remark 5.3.4.1. For $A = k^{n+1}$ (so $\star'_A$ is the quiver tensor product for the Beilinson quiver Q_n), the Balmer spectrum $\mathrm{Spc}_{\star'_A}\,\mathbb{P}(A) = \mathrm{Spc}_{\otimes}\,\mathsf{D}(Q_n)$ is computed in [LS13, §2]: it is nothing but a disjoint union of $n+1$ points. A general discussion of related phenomena (in the language of semiorthogonal decompositions) can be found in [BIK13, Appendix A]. We will see the same computations hold even if we start with different finite dimensional algebras A, even when A is not commutative. In particular, the Balmer spectrum is a quite coarse invariant: it cannot tell different algebras apart.

Here we follow [NVY22] for a definition of the Balmer spectrum of a stably monoidal ∞-category which need not be symmetric monoidal:

Definition 5.3.4.2. Let $(\mathcal{C}, \otimes_{\mathcal{C}})$ be a small stably monoidal ∞-category. A two-sided thick $\otimes$-ideal $\mathcal{P}$ in $\mathcal{C}$ is *prime* if $\mathcal{P} \neq \mathcal{C}$ and, whenever we have $\mathcal{I} \otimes_{\mathcal{C}} \mathcal{J} \subset \mathcal{P}$ for two-sided thick $\otimes$-ideals $\mathcal{I}, \mathcal{J}$ of $\mathcal{C}$, then either $\mathcal{I} \subset \mathcal{P}$ or $\mathcal{J} \subset \mathcal{P}$. The *Balmer spectrum* of $(\mathcal{C}, \otimes_{\mathcal{C}})$ is

$$\operatorname{Spc}_{\otimes_{\mathcal{C}}} \mathcal{C} = \{\mathcal{P} \,|\, \mathcal{P} \text{ is a prime two-sided thick } \otimes\text{-ideal of } \mathcal{C}\}$$

with a topology defined similarly to the (Zariski) topology of the usual Balmer spectrum (see [NVY22, Section 1.2] for details).

Lemma 5.3.4.3. *Let $(\mathcal{C}, \otimes_{\mathcal{C}})$ be a small stably monoidal ∞-category. Let $S_0, \dots, S_n$ be a collection of exceptional objects of $\mathcal{C}$ with $\langle S_i \rangle \neq \langle S_j \rangle$ for $i \neq j$, and for each i, let $\mathcal{P}_i = \langle S_j \,|\, j \neq i \rangle$. Suppose that:*

1. $\mathcal{C} = \langle S_0, \dots, S_n \rangle$
2. $\mathcal{P}_i = \ker(- \otimes S_i) = \ker(S_i \otimes -)$ *for all* i.
3. $\langle S_i \rangle$ *is a two-sided thick $\otimes$-ideal for all* i.

Then

$$\operatorname{Spc}_{\otimes_{\mathcal{C}}} \mathcal{C} = \bigsqcup_{i=0}^{n} \mathcal{P}_i.$$

Proof. First, $\mathcal{P}_i$ is clearly a two-sided thick $\otimes$-ideal since $\mathcal{P}_i = \ker(- \otimes S_i) = \ker(S_i \otimes -)$. Because $\mathcal{C}/\mathcal{P}_i = \langle S_i \rangle = \mathsf{Perf}(k)$, we see that $\mathcal{P}_i$ is maximal among thick subcategories of $\mathcal{C}$. Thus $\mathcal{P}_i$ is a prime two-sided thick $\otimes$-ideal of $\mathcal{C}$ by [NVY22, Theorem 3.2.3].

Conversely, let $\mathcal{P}$ be a prime thick $\otimes$-ideal of $\mathcal{C}$. Because $\mathcal{P} \neq \mathcal{C}$, there exists i such that $S_i \notin \mathcal{P}$. Then, since $\langle S_i \rangle \otimes \langle S_j \rangle = \langle 0 \rangle \subset \mathcal{P}$ for each $j \neq i$ and $\langle S_i \rangle \not\subset \mathcal{P}$, we have $S_j \in \langle S_j \rangle \subset \mathcal{P}$ for every $j \neq i$. Thus, $\mathcal{P} = \mathcal{P}_i$ as $\mathcal{P}_i$ is maximal. $\square$

Remark 5.3.4.4. When $\otimes_{\mathcal{C}}$ is symmetric monoidal, the result above is also a direct consequence of [BIK13, Theorem A.5].

Proposition 5.3.4.5. *Let A be a finite dimensional algebra A over k. Then*

$$\operatorname{Spc}_{\star'_A} \mathbb{P}(A) = \bigsqcup_{i=0}^{\dim A - 1} \langle \Omega^j(j)[j] \,|\, j \neq i \rangle.$$

In particular, $\operatorname{Spc}_{\star'_A} \mathbb{P}(A)$ *depends only on the dimension of* A.

Proof. This follows from Lemma 5.3.4.3 by taking $S_i = \Omega^i(i)[i]$ for $i = 0, \dots, \dim A - 1$ and using the computation of Example 5.3.1.1. $\square$

Remark 5.3.4.6. Recent work on higher Zariski geometry ([Aok+25]) proves that the ∞-category of 2-rings (i.e., rigid stably symmetric monoidal ∞-categories) embeds into the ∞-category of 2-ringed spaces via an enhanced version of the Balmer spectrum construction. If A is commutative, the above computations can be used to show that the 2-ringed space associated with $(\mathsf{Perf}(\mathbb{P}(A)), \star'_A)$ depends only on $\dim A$, i.e. this 2-ringed space does not distinguish between different commutative algebras of the same dimension. However, it is easy to show directly that $\star'_A$ is not rigid, so we do not contradict the results of [Aok+25].

Example 5.3.1.1 also allows us to compute the Grothendieck ring of $(\mathsf{Perf}\,\mathbb{P}(A), \star'_A)$:

Proposition 5.3.4.7. *Let A be a finite dimensional k-algebra. Then there is an isomorphism*

$$\mathsf{K}^0\big(\mathsf{Perf}\,\mathbb{P}(A), \star'_A\big) \simeq \mathbb{Z}^{\dim A}.$$

Proof. Consider the the basis of $\mathsf{K}^0(\mathsf{Perf}\,\mathbb{P}(A), \star'_A)$ given by $\nu_i := \big[\Omega^i(i)[i]\big]$ for $i = 0, \ldots, \dim A - 1$. By Example 5.3.1.1, regardless of the algebra structure of A, we have $\nu_i^2 = \nu_i$ for all i and $\nu_i \cdot \nu_j = 0$ for all $i \neq j$. Thus the basis $\{\nu_i\}_{i=0}^{\dim A-1}$ gives rise to the desired isomorphism. $\square$

Unlike the Balmer spectrum and the Grothendieck ring, the Picard group of $\mathrm{Pic}\,\big(\mathsf{Perf}\,\mathbb{P}(A), \star'_A\big)$ can depend on the choice of A.

Proposition 5.3.4.8. *Let A be a finite dimensional k-algebra. Then*

$$\mathrm{Pic}\,\big(\mathsf{Perf}\,\mathbb{P}(A), \star'_A\big) = (A^\times)^{\dim A-1}/k^\times \times \mathbb{Z}.$$

Proof. Suppose $\mathscr{F}$ is invertible in $\mathrm{Pic}\,\big(\mathsf{Perf}\,\mathbb{P}(A), \star'_A\big)$. Then the derived quiver representation corresponding to $\mathscr{F}$ must take invertible values at each vertex, i.e. $\mathscr{F}$ corresponds to a derived quiver representation of the form

$$k[i_0] \longleftarrow k[i_1] \longleftarrow \ldots \longleftarrow k[i_{\dim A-1}].$$

We first claim that $i_0 = i_1 = \cdots = i_{\dim A-1}$. Indeed, if this were not the case, then we would necessarily have $i_p \neq i_{p-1}$ for some p. This forces $H^0\big(\mathrm{Hom}_k(k[i_p], k[i_{p-1}])\big) = H^{i_{p-1}-i_p}(\mathrm{Hom}_k(k, k)) = 0$. Thus, for any $\mathscr{G} \in \mathsf{Perf}\,\mathbb{P}(A)$, the derived quiver representation corresponding to $\mathscr{F} \star'_A \mathscr{G}$ would necessarily have 0 as one of its morphisms. In particular, because $\mathscr{F}$ is invertible, one of the morphisms in the quiver representation corresponding to the unit of $\star'_A$ is 0. But this is impossible as the unit of $\star'_A$ is $k(1_A)$, corresponding to the quiver representation

$$k \xleftarrow{1_A} k \xleftarrow{1_A} \ldots \xleftarrow{1_A} k.$$

Thus every invertible object of $\mathrm{Pic}\,\big(\mathsf{Perf}\,\mathbb{P}(A), \star'_A\big)$ corresponds to a derived quiver representation of the form

$$k[i] \longleftarrow k[i] \longleftarrow \ldots \longleftarrow k[i]$$

for some $i \in \mathbb{Z}$. We may define a homomorphism $\alpha : \mathrm{Pic}\left(\mathsf{Perf}\,\mathbb{P}(A), \star_A\right) \to \mathbb{Z}$ sending a quiver representation of the above form to i. The homomorphism

$$\begin{aligned} \mathbb{Z} &\to \mathrm{Pic}\left(\mathsf{Perf}\,\mathbb{P}(A), \star'_A\right) \\ i &\mapsto k(1_A)[i] \end{aligned}$$

defines a section of α.

It remains to show that $\ker \alpha = (A^\times)^{\dim A - 1}/k^\times$. We may define a map $\beta : (A^\times)^{\dim A-1} \to \ker \alpha$ sending $(a_1, \ldots, a_{\dim A - 1})$ to the object of $\ker\alpha$ corresponding to the quiver representation

$$k \xleftarrow{a_1} k \xleftarrow{a_2} \ldots \xleftarrow{a_{\dim A-1}} k$$

It is clear that every object of $\ker\alpha$ arises in this way. Furthermore, any isomorphism $\beta(a_1, \ldots, a_{\dim A-1}) \simeq \beta(a'_1, \ldots, a'_{\dim A-1})$ is witnessed by a commutative diagram

$$\begin{tikzcd} k \arrow[d, "\sim"] & k \arrow[l, "a_1"'] \arrow[d, "\sim"] & \ldots \arrow[l, "a_2"'] & k \arrow[l, "a_{\dim A-1}"'] \arrow[d, "\sim"] \\ k & k \arrow[l, "a'_1"] & \ldots \arrow[l, "a'_2"] & k, \arrow[l, "a'_{\dim A-1}"] \end{tikzcd}$$

where invertibility of each a_p and each a'_p ensures that the vertical arrows are all determined by the rightmost vertical arrow. Thus $\ker\beta = k^\times$ and $\ker\alpha = (A^\times)^{\dim A-1}/\ker\beta = (A^\times)^{\dim A-1}/k^\times$. □

Remark 5.3.4.9. The invariants we have discussed in this section reflect the differences in behavior between the EC products on $\mathsf{Perf}\,\mathbb{P}^d$ and the usual tensor product on $\mathsf{Perf}\,\mathbb{P}^d$. Indeed, for the usual tensor product $\otimes_{\mathscr{O}}$, we have:

- $\mathrm{Spc}_{\otimes_{\mathscr{O}}} \mathsf{Perf}\,\mathbb{P}^d = \mathbb{P}^d$.
- $K^0\left(\mathsf{Perf}\,\mathbb{P}^d, \otimes_{\mathscr{O}}\right) \simeq \mathbb{Z}[\eta]/\eta^{n+1}$ where $\eta = [\mathscr{O}_{\mathbb{P}^d}(1)] - 1$.
- $\mathrm{Pic}(\mathsf{Perf}\,\mathbb{P}(A), \otimes_{\mathscr{O}}) \simeq \mathbb{Z}\cdot\mathscr{O}_{\mathbb{P}^d}(1) \times \mathbb{Z}$ (where the second factor arises from the shift functor [1]).

Example 5.3.4.10. We may use the above results to construct two distinct symmetric monoidal structures on $\mathsf{Perf}\,\mathbb{P}^1$ with the same Balmer spectrum and the same ring structure on K^0. Namely, take $A = k^2$ and $A' = k[\epsilon]/\epsilon^2$. The Balmer spectra and Grothendieck rings of $\left(\mathsf{Perf}\,\mathbb{P}(A), \star'_A\right)$ and $\left(\mathsf{Perf}\,\mathbb{P}(A'), \star'_{A'}\right)$ agree because $\dim A = \dim A'$. However, $\mathrm{Pic}\left(\mathsf{Perf}\,\mathbb{P}(A), \star'_A\right) = k^\times \times k^\times \times \mathbb{Z}$, while $\mathrm{Pic}\left(\mathsf{Perf}\,\mathbb{P}(A'), \star'_{A'}\right) = k \times k^\times \times \mathbb{Z}$, so the symmetric monoidal structures are distinct.

Note that the Picard group is not a complete invariant.

Example 5.3.4.11. Let A be any finite dimensional k-algebra. In general, A is not isomorphic to its opposite algebra A^{op}. However, we have

$$\mathrm{Pic}(\mathsf{Perf}\,\mathbb{P}(A), \star_A) = (A^\times)^{\dim A - 1}/k^\times \times \mathbb{Z} \cong ((A^{\mathrm{op}})^\times)^{\dim A - 1}/k^\times \times \mathbb{Z} = \mathrm{Pic}(\mathsf{Perf}\,\mathbb{P}(A^{\mathrm{op}}), \star_{A^{\mathrm{op}}})$$

where we use the isomorphism $A^\times \cong (A^{\mathrm{op}})^\times = (A^\times)^{\mathrm{op}}$ defined by $a \mapsto a^{-1}$. Note that this isomorphism does not extend to a linear map $A \to A^{\mathrm{op}}$, so the argument of Proposition 5.3.3.1 does not apply.

Example 5.3.4.12. For an example where all algebras involved are commutative, let $A = k[\epsilon]/\epsilon^3$ and $A' = k[x, y]/\langle x^2, xy, y^2\rangle$, then $A^\times \cong k^\times \times k^2 \cong A'^\times$, where the first isomorphism is given by

$$\begin{aligned} A^\times = k^\times \times (1 + \mathfrak{m}_A) &\to k^\times \times k^2 \\ (c, 1 + a\epsilon + b\epsilon^2) &\mapsto (c, a, b - a^2/2). \end{aligned}$$

As before, this isomorphism is not linear, so this does not contradict the argument of Proposition 5.3.3.1.

5.4 Application: Tensor products in toric mirror symmetry

Suppose $\mathfrak{X} = \mathfrak{X}_\Sigma$ is the smooth complete toric variety associated with a fan $\Sigma \subset N_\mathbb{R}$ (where N is a lattice, M is its dual lattice, and $(-)_\mathbb{R} = (-) \otimes_\mathbb{Z} \mathbb{R}$). Homological mirror symmetry for toric varieties (in this context also called the “coherent-constructible correspondence”) gives a symmetric monoidal equivalence

$$\big(\mathsf{QC}(\mathfrak{X}_\Sigma), \otimes_{\mathscr{O}}\big) \simeq \big(\mathsf{Sh}_{\Lambda_\Sigma}(M_\mathbb{R}/M), \star_{M_\mathbb{R}/M}\big) \tag{5.4.0.1}$$

where:

- $M_\mathbb{R}/M$ is a real torus,
- $\Lambda_\Sigma \subset T^*(M_\mathbb{R}/M)$ is a certain Lagrangian defined from the combinatorics of Σ,
- $\mathsf{Sh}_{\Lambda_\Sigma}(M_\mathbb{R}/M)$ is the category of constructible [9] sheaves on $M_\mathbb{R}/M$ with coefficients in k and singular support in Λ_Σ, and
- $\star_{M_\mathbb{R}/M}$ is the convolution product on $T^*(M_\mathbb{R}/M)$.

[9] Here we use constructible in a weak sense, i.e. we do not require any finiteness conditions on stalks. See [Kuw17] for a more comprehensive discussion of finiteness conditions in this context.

There is a vast literature on this subject, and we will content ourselves by noting that a fairly complete (and significantly more general) discussion of Eq. (5.4.0.1) may be found in [Kuw17], which builds on [Bon06; Fan+11] and many other sources.

In this section we will discuss the appearance of quiver tensor products in homological mirror symmetry, with a particular focus on smooth complete toric varieties of Bondal-Ruan type (cf. Definition 5.1.4.6). We begin by noting the following explicit description of quiver tensor products in the Bondal-Ruan case.

Proposition 5.4.0.2. *Let $\mathfrak{X}$ be a smooth complete toric variety of Bondal-Ruan type, and let $\mathfrak{X} \subset \mathfrak{M}$ be the Cox presentation of $\mathfrak{X}$. Write $\mathfrak{Z}_{\mu,-\Theta_{\mathfrak{X}}}$ for the closure in $\mathfrak{X}^3$ of the graph of the binary multiplication on the dense torus in $\mathfrak{X}$. Then push-pull along the correspondence*

$$\begin{tikzcd} & \mathfrak{Z}_{\mu,-\Theta_{\mathfrak{X}}} \arrow[dl] \arrow[dr] & \\ \mathfrak{X} \times \mathfrak{X} & & \mathfrak{X} \end{tikzcd}$$

defines a symmetric monoidal structure $\star'_{\mathfrak{M},-\Theta_{\mathfrak{X}}}$ on $\mathsf{Perf}(\mathfrak{X})$. Furthermore, there is a symmetric monoidal equivalence

$$\Big(\mathsf{Perf}(\mathfrak{X}), \star'_{\mathfrak{M},-\Theta_{\mathfrak{X}}}\Big) \simeq \Big(\mathsf{Fun}\left(\mathsf{Q}^{\mathrm{pre}}_{\Theta_{\mathfrak{X}}}(\phi), \mathsf{Perf}(k)\right), \otimes_{\mathsf{Q}}\Big).$$

Proof. We use the geometric EC setup $(\phi, j, -\Theta_{\mathfrak{X}}, q)$ of Example 5.2.1.4. The first claim follows by combining Proposition 5.2.3.3 and Example 5.2.3.4. The second claim is Theorem 5.2.4.1, noting that $\mathsf{Q}^{\mathrm{pre}}_{-\Theta_{\mathfrak{X}}}(\phi)^{\mathrm{op}} \simeq \mathsf{Q}^{\mathrm{pre}}_{\Theta_{\mathfrak{X}}}(\phi)$ by taking duals of line bundles. ☐

5.4.1 The mirror of the constructible tensor product

Given the underlying equivalence of categories of Eq. (5.4.0.1), it is natural to ask when and whether one can describe the tensor product $\otimes_k$ of constructible sheaves on $M_{\mathbb{R}}/M$ in terms of the algebraic geometry of $\mathfrak{X}_\Sigma$. This question does not always make sense – for $\mathscr{F}, \mathscr{G} \in \mathsf{Sh}_{\Lambda_\Sigma}(M_{\mathbb{R}}/M)$, the singular support of the tensor product $\mathscr{F} \otimes_k \mathscr{G}$ need not lie in Λ_Σ. However, when the Lagrangian Λ_Σ arises from a stratification $Z = \{Z_i\}_{i \in I}$ of $M_{\mathbb{R}}/M$, the category $\mathsf{Sh}_{\Lambda_\Sigma}(M_{\mathbb{R}}/M)$ is closed under $\otimes_k$, and the question does make sense. In this case, we shall write $\mathsf{Sh}_Z(M_{\mathbb{R}}/M) := \mathsf{Sh}_{\Lambda_\Sigma}(M_{\mathbb{R}}/M)$.

By [Bon06] (see also [FH25, §5] for a more detailed presentation and generalization), we know that Λ_Σ arises from a stratification when $\mathfrak{X}$ is of Bondal-Ruan type. More precisely, let $F_\phi : M_{\mathbb{R}}/M \to \mathbb{X}^\bullet(G)$ be the anti-Bondal-Ruan map of Definition 5.1.4.6. We may decompose $M_{\mathbb{R}}/M$ as a disjoint union of the level sets $\{Z_\chi := F_\phi^{-1}(\chi)\}_{\chi \in -\Theta_{\mathfrak{X}}}$.[10]

There is a natural order on $-\Theta_{\mathfrak{X}}$ given by the transitive closure of the following rule: $\chi_1 \leqslant \chi_2$ if $Z_{\chi_1} \subset \overline{Z_{\chi_2}}$ (cf. [FH25, Corollary 5.7 and Definitions 4.8, 4.13, and 4.28]. The order

[10] In [FH25], Z_χ is denoted by S_χ for $\chi \in \widehat{G} = \mathbb{X}^\bullet(G)$. We use the letter Z only to avoid conflict with our notation for transparent collections.

on $-\Theta_{\mathfrak{X}}$ may also be understood algebraically: by the discussion at the beginning of [FH25, §5.2], we have $\chi_1 \leqslant \chi_2$ if and only if $\mathsf{H}^0(\mathfrak{X}, \mathscr{O}(\chi_2 - \chi_1)) \neq 0$. In particular, $\mathsf{Z} = \{\mathsf{Z}_\chi\}_{\chi \in -\Theta_{\mathfrak{X}}}$ is a (non-conical) stratification, which we call the *anti-Bondal-Ruan stratification*, of $\mathsf{M}_{\mathbb{R}}/\mathsf{M}$ by the poset $-\Theta_{\mathfrak{X}}$ (cf. [FH25, Definition 4.2]).

Proposition 5.4.1.1. *Let $\mathfrak{X}$ be a smooth complete toric variety of Bondal-Ruan type, and let Z be the corresponding anti-Bondal-Ruan stratification of $\mathsf{M}_{\mathbb{R}}/\mathsf{M}$. There is a symmetric monoidal equivalence*

$$\big(\mathsf{QC}(\mathfrak{X}), \star'_{[\mathbb{A}^n/\mathsf{G}], -\Theta_{\mathfrak{X}}}\big) \simeq \big(\mathsf{Sh}_{\mathsf{Z}}(\mathsf{M}_{\mathbb{R}}/\mathsf{M}), \otimes_{\mathsf{k}}\big)$$

Proof. Let $\mathsf{Ex}_{\mathsf{Z}}(\mathsf{M}_{\mathbb{R}}/\mathsf{M})$ be the ∞-category of *exit paths* of the stratification Z. We refer to [HPT24, Proposition 2.2.10] for a precise definition of $\mathsf{Ex}_{\mathsf{Z}}(\mathsf{M}_{\mathbb{R}}/\mathsf{M})$, but we note as a heuristic (cf. [HPT24, Observation 5.1.6]) that:

- Objects of $\mathsf{Ex}_{\mathsf{Z}}(\mathsf{M}_{\mathbb{R}}/\mathsf{M})$ are points of Z.
- Morphisms are *exit paths* in $\mathsf{M}_{\mathbb{R}}/\mathsf{M}$, i.e. paths $\gamma : [0,1] \to \mathsf{M}_{\mathbb{R}}/\mathsf{M}$ (from the domain to the codomain) such that if $t_1 \leqslant t_2$ and $\gamma(t_i) \in \mathsf{Z}_{\chi_i}$ for $i = 1, 2$, then $\chi_1 \leqslant \chi_2$.
- Composition is concatenation of paths.

By [HPT24, Theorem 0.4.2 and Example 5.3.10], there is a symmetric monoidal equivalence

$$\big(\mathsf{Sh}_{\mathsf{Z}}(\mathsf{M}_{\mathbb{R}}/\mathsf{M}), \otimes_{\mathsf{k}}\big) = \big(\mathsf{Fun}(\mathsf{Ex}_{\mathsf{Z}}(\mathsf{M}_{\mathbb{R}}/\mathsf{M}), \mathsf{D}(\mathsf{k})), \otimes_{\mathsf{k}}\big)$$

given by sending a constructible sheaf to the collection of its stalks and specialization maps.[11] By [FH25, Proposition 5.5 and Proposition 5.6], there is a natural equivalence

$$\mathsf{Q}^{\mathrm{pre}}_{-\Theta_{\mathfrak{X}}}(\phi)^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Ex}_{\mathsf{Z}}(\mathsf{M}_{\mathbb{R}}/\mathsf{M})$$

given by sending an object $\mathscr{O}_{\mathfrak{X}}(\chi) \in \mathsf{Q}^{\mathrm{pre}}_{-\Theta_{\mathfrak{X}}}(\phi)^{\mathrm{op}}$ to the corresponding stratum Z_χ. Thus we obtain a chain of equivalences

$$\begin{aligned}\big(\mathsf{Sh}_{\mathsf{Z}}(\mathsf{M}_{\mathbb{R}}/\mathsf{M}), \otimes_{\mathsf{k}}\big) &= \big(\mathsf{Fun}(\mathsf{Ex}_{\mathsf{Z}}(\mathsf{M}_{\mathbb{R}}/\mathsf{M}), \mathsf{D}(\mathsf{k})), \otimes_{\mathsf{k}}\big)\\ &= \big(\mathsf{Fun}(\mathsf{Q}^{\mathrm{pre}}_{-\Theta_{\mathfrak{X}}}(\phi)^{\mathrm{op}}, \mathsf{D}(\mathsf{k})), \otimes_{\mathsf{k}}\big)\\ &= \big(\mathsf{D}(\mathsf{Q}_{-\Theta_{\mathfrak{X}}}(\phi)^{\mathrm{op}}), \otimes_{\mathsf{Q}}\big)\\ &= \big(\mathsf{QC}(\mathfrak{X}), \star'_{[\mathbb{A}^n/\mathsf{G}], \Theta_{\mathfrak{X}}}\big).\end{aligned}$$

□

[11] The symmetric monoidal structure on $\mathsf{Fun}(\mathsf{Ex}_{\mathsf{Z}}(\mathsf{M}_{\mathbb{R}}/\mathsf{M}), \mathsf{D}(\mathsf{k}))$ is just the objectwise tensor product, so saying that this equivalence is symmetric monoidal is just saying that stalks and specialization maps commute with tensor products.

Remark 5.4.1.2. Proposition 5.4.1.1 may also be stated for the usual Bondal-Ruan stratification, though in this case the geometric interpretation of the EC product $\star'_{[\mathbb{A}^n/G],-\Theta_{\mathfrak{X}}}$ on $\mathsf{QC}(\mathfrak{X})$ is less clear. Going between the Bondal-Ruan and anti-Bondal-Ruan stratifications corresponds to taking the negative of the corresponding Lagrangians. This does not affect the existence of the homological mirror symmetry equivalence, as both $\mathsf{QC}(\mathfrak{X})$ and $\mathsf{Sh}_{\Lambda_\Sigma}(M_\mathbb{R}/M)$ are self-dual.

When $\mathfrak{X}$ is not of Bondal-Ruan type, the situation can be more subtle:

Example 5.4.1.3. Let $\mathfrak{X} = \mathfrak{X}_\Sigma = \mathsf{F}_n$ be a Hirzebruch surface of type $n \geqslant 2$. By [Kin97, Proposition 6.1], $\mathfrak{X}$ admits a full strong exceptional collection of line bundles. This corresponds (by Example 5.1.4.4) to a collection of weights S which is transparent for an embedding $\mathfrak{X} \subset [\mathbb{A}^4/\mathbb{G}_m^2]$. Let $\mathsf{Q}_\mathsf{S}^{\mathrm{pre}}(\phi)$ be the discrete category of Definition 5.1.4.1. By Example 4.2.1.2, we obtain a symmetric monoidal structure $\otimes_{\mathsf{Q}_\mathsf{S}(\phi)}$ on $\mathsf{D}(\mathsf{Q}_\mathsf{S}(\phi)^{\mathrm{op}}) \simeq \mathsf{Sh}_{\Lambda_\Sigma}(M_\mathbb{R}/M)$.

However, this symmetric monoidal structure *does not agree* with the tensor product of constructible sheaves on $M_\mathbb{R}/M$. In fact, the category $\mathsf{Sh}_{\Lambda_\Sigma}(M_\mathbb{R}/M)$ is not closed under $\otimes_k$! (This relates to the fact that Λ_Σ is not the Lagrangian of conormals to a stratification of $M_\mathbb{R}/M$.) The discrepancy arises because, even though $\mathfrak{X}$ has a full strong exceptional collection of line bundles, the equivalence of Eq. (5.4.0.1) is not induced by said collection.

5.4.2 A conjecture about Cox categories

The recent paper [Bal+24] introduces a "Cox category" $\mathsf{QC}_{\mathrm{Cox}}(\mathfrak{X})$ (for an arbitrary semiprojective toric variety $\mathfrak{X}$) in which the Bondal-Thomsen collection $\Theta_\mathfrak{X}$ behaves as if it were "transparent" in a suitable sense. In general, the $\mathsf{QC}_{\mathrm{Cox}}(\mathfrak{X})$ is not the derived category of any genuine stack $\mathfrak{X}$, though it has a geometric interpretation in terms of "gluing birational models of $\mathfrak{X}$." Via homological mirror symmetry, $\mathsf{QC}_{\mathrm{Cox}}(\mathfrak{X})$ is expected to correspond to a category $\mathsf{Sh}_{\Lambda_{\mathrm{Cox}}}(M_\mathbb{R}/M)$ where the singular support Lagrangian Λ_{Cox} is obtained by taking the unions of the Lagrangians of these birational models.

Conjecture 5.4.2.1. *Let $\mathfrak{X}$ be a smooth semiprojective toric variety. The category $\mathsf{Sh}_{\Lambda_{\mathrm{Cox}}}(M_\mathbb{R}/M)$ is closed under $\otimes_k$, and there is a symmetric monoidal equivalence*

$$\big(\mathsf{QC}_{\mathrm{Cox}}(\mathfrak{X}), \star'\big) \simeq \big(\mathsf{Sh}_{\Lambda_{\mathrm{Cox}}}(M_\mathbb{R}/M), \otimes_k\big)$$

where $\star'$ is a "birationally glued EC product" extending the convolution on the maximal torus of $\mathfrak{X}$.

Chapter 6

A sketch of derived toric geometry

Here we provide a brief discussion of how to use our results on grouplike SMCs to establish a theory of *derived toric stacks*, i.e. a generalization of the notion of toric stacks which is closed under taking fiber products. A more complete account of these ideas (including proofs of the claims) will appear in the near future.

We should first indicate some reasons why we desire such a theory:

1. Many natural functors between derived categories of toric varieties or stacks arise as push-pull functors along spans

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

   of toric stacks and morphisms. To compose such push-pull functors, we need to take derived fiber products of the toric stacks involved. Because everything is toric, we should expect that the derived fiber products can be understood via the combinatorics of (some variants of) cones and fans. Derived toric geometry is an attempt to do so.

2. Toric geometry is a useful testing ground for conjectures and ideas in algebraic geometry, largely because it replaces the hard commutative algebra of arbitrary commutative rings with more tractable combinatorics. We can expect derived toric geometry to play a similar role with respect to general derived algebraic geometry.

3. There has been a recent proliferation of different flavors of "derived geometry" (e.g. derived algebraic geometry, spectral algebraic geometry, rigid analytic geometry, *et cetera*). All of these theories should have some analogue of "toric varieties," and it is natural to ask to what extent classic results from toric geometry (e.g. the Cox construction) reappear in these new contexts.

4. Derived algebraic geometry is often pitched as a more refined approach to intersection theory (essentially transforming Serre's intersection formula into a *definition*). We hope

that a derived approach to toric geometry can illuminate and categorify questions about (toric) intersection theory on toric varieties, though we will not pursue this direction here.

With that motivation in mind, let's proceed to discussing the basics of derived toric geometry.

6.1 $\mathbb{E}_\infty$-cones

Our "local building blocks" in derived toric geometry are $\mathbb{E}_\infty$-cones, which are patterned on grouplike SMCs in much the same way that affine schemes are patterned on commutative rings.

Remark 6.1.0.1. Our choice to work with grouplike SMCs rather than $\mathbb{E}_\infty$-monoids reflects the fact that we are interested in the general theory of toric *stacks*. Since toric stacks are not Zariski-locally determined by affine toric varieties, we cannot just use the standard theory of fans to describe toric stacks. Instead, we are left with two options:

- Take the Cox construction as a *definition* and replace the notion of "fan" by "stacky fan" (as in [BCS05] or more generally in [GS15a]). This is intrinsically global: the combinatorics of stacky fans control the action of a commutative reductive group G on a toric variety X, and the toric stack associated with a stacky fan is the global quotient stack X/G. Note however that [GS15b, Theorem 6.1] gives a mostly local[1] algebro-geometric characterization of toric stacks as defined in [GS15a].

- Replace the local models (i.e. cones) appearing in classical toric geometry by something which accounts for the stacky points of toric stacks. By the results of Chapter 3 and Section 4.1, we can understand certain quotient stacks by commutative reductive groups in terms of grouplike SMCs. Thus these local models should encode the data of grouplike SMCs. (We do not know of a reference that treats this perspective explicitly.)

We shall take the latter approach here, since we would prefer to be able to work fully locally while still manipulating objects which are semi-combinatorial in nature.

Given a grouplike SMC $\mathcal{C}$, we will define the corresponding $\mathbb{E}_\infty$-cone $\mathrm{Cone}(\mathcal{C})$ as a topological space together with a structure sheaf encapsulating the data of $\mathcal{C}$. In nice enough cases, the topological space underlying $\mathrm{Cone}(\mathcal{C})$ will be the (Alexandrov) topological space associated with a finite poset, justifying our claim that $\mathbb{E}_\infty$-cones are semi-combinatorial in nature.

[1] The only non-local part of the characterization in [GS15b] is the rather mild assumption that the toric stacks they work with have affine diagonal. We will make a similar assumption when defining stacky $\mathbb{E}_\infty$-fans.

6.1.1 Spectra of $\mathbb{E}_\infty$-monoids

We begin by recalling some standard definitions (see [Ogu18, §1.3]) that will allow us to define the topological space underlying Cone($\mathcal{C}$).

Definition 6.1.1.1. Let M be an $\mathbb{E}_\infty$-monoid.

- A *(naïve) ideal* of M is an M-equivariant subspace $I \subset M$ (i.e. a subspace such that, for all $i \in I$ and $m \in M$, we have $i + m \in I$).
- A *radical ideal* of M is an ideal $I \subset M$ such that, if $m \in M$ satisfies $n \cdot m \in I$ for some $n \in \mathbb{N}$, then $m \in M$. The *radical* $\sqrt{I}$ of an ideal I is the smallest radical ideal containing I.
- A *prime ideal* of M is an ideal $\mathfrak{p} \subset M$ such that $M \setminus \mathfrak{p}$ is a submonoid (equivalently, a face) of M.

The *spectrum* of M, denoted $\operatorname{Spec} M$, is the set of prime ideals of M. We consider $\operatorname{Spec} M$ as a topological space with the *Zariski topology*, generated by the basic open sets

$$D(m) = \{\mathfrak{p} \in \operatorname{Spec} M \mid m \notin \mathfrak{p}\}$$

for $m \in M$. We will write $\mathsf{Open}^{\text{basic}}(\operatorname{Spec} M)$ for the poset of open subsets of $\operatorname{Spec} M$ (ordered by inclusion).

Remark 6.1.1.2. We have $\operatorname{Spec} M = \operatorname{Spec} \pi_0(M)$, and the following discussion could be performed entirely in the language of discrete commutative monoids without losing any generality.

Example 6.1.1.3. Consider the case of $M = \mathbb{N}^n$. By Example 2.1.4.4, the faces of $\mathbb{N}^n$ are exactly the submonoids $\mathbb{N}^I$ for some $I \subset \{1, \dots, n\}$. The prime ideals are then given by

$$\mathfrak{p}_J = \left\{ \sum_{i=1}^{n} c_i e_i \;\middle|\; c_j > 0 \text{ for some } j \in J \right\}.$$

for $J \subset \{1, \dots, n\}$. Thus we may identify $\operatorname{Spec} \mathbb{N}^n$ with the powerset $\mathcal{P}(\{1, \dots, n\})$ via $\mathfrak{p}_J \leftrightarrow J$. Note that $\mathfrak{p}_J \subset \mathfrak{p}_{J'}$ if and only if $J \subset J'$, so this identification preserves the natural orders.

A subset of $\operatorname{Spec} \mathbb{N}^n$ is open if and only if the corresponding subset of $\mathcal{P}(\{1, \dots, n\})$ is downward closed. The basic open sets of $\operatorname{Spec} \mathbb{N}^n$ are the downward closed subsets of $\mathcal{P}(\{1, \dots, n\})$ with a greatest element. Thus $\mathsf{Open}^{\text{basic}}(\operatorname{Spec} \mathbb{N}^n) = \mathcal{P}(\{1, \dots, n\})$, where the bijection comes from sending a basic open set to the greatest element of the corresponding subset of $\mathcal{P}(\{1, \dots, n\})$.

Example 6.1.1.4. If $\pi_0(M)$ is finitely generated, then $\operatorname{Spec} M$ is finite by [Ogu18, Proposition 1.3.7(i)] (or alternatively by the fact that $\operatorname{Spec} M$ may be identified with a subset of $\operatorname{Spec} \mathbb{N}^n$ for some n). Thus we may understand $\operatorname{Spec} M$ in terms of the combinatorics of finite posets.

The topological spaces $\operatorname{Spec} \mathsf{M}$ behave in many ways like the spectra of local integral domains. For example, $\operatorname{Spec} \mathsf{M}$ always has a unique generic point and a unique closed point. Furthermore, $\operatorname{Spec} \mathsf{M}$ is always a spectral topological space (e.g. by [Ray20, Proposition 3.1]), and we can use this to understand sheaves on $\operatorname{Spec} \mathsf{M}$:

Proposition 6.1.1.5. *Let* M *be an* $\mathbb{E}_\infty$*-monoid, and let* $\mathcal{V}$ *be a complete category. Then the inclusion* $\mathsf{Open}^{\mathrm{basic}}(\operatorname{Spec} \mathsf{M}) \subset \mathsf{Open}(\operatorname{Spec} \mathsf{M})$ *induces an equivalence*

$$\mathsf{Sh}_{\mathcal{V}}(\operatorname{Spec} \mathsf{M}) = \mathsf{Fun}(\mathsf{Open}^{\mathrm{basic}}(\operatorname{Spec} \mathsf{M})^{\mathrm{op}}, \mathcal{V}).$$

6.1.2 $\mathbb{E}_\infty$-cones: definition and examples

We are now prepared to define $\mathbb{E}_\infty$-cones:

Definition 6.1.2.1. Let $\mathcal{C}$ be a grouplike SMC. The *(stacky)* $\mathbb{E}_\infty$*-cone* $\mathrm{Cone}(\mathcal{C})$ is the topological space $\operatorname{Spec} \mathrm{P}_0\mathcal{C}$ equipped with structure sheaf $\mathscr{O}_{\mathrm{Cone}(\mathcal{C})} \in \mathsf{Sh}_{\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})}(\operatorname{Spec} \mathrm{P}_0\mathcal{C})$ is defined (using Proposition 6.1.1.5) by

$$\mathscr{O}_{\mathrm{Cone}(\mathcal{C})}(\mathsf{D}(\mathfrak{m})) = \mathcal{C}[\mathfrak{m}^{-1}]$$

for all $\mathsf{D}(\mathfrak{m}) \in \mathsf{Open}^{\mathrm{basic}}(\operatorname{Spec} \mathsf{M})$.

We may use Proposition 6.1.1.5 to draw diagrams capturing the data of $\mathbb{E}_\infty$-cones.

Example 6.1.2.2. The topological space underlying Cone(0) has one point, and the structure sheaf has the value 0. As such, we will write pt = Cone(0). This will turn out to be the final (and initial!) object of a suitable category of $\mathbb{E}_\infty$-cones.

Example 6.1.2.3. The topological space underlying $\mathrm{Cone}(\mathsf{B}\mathbb{N})$ is $\operatorname{Spec} \mathbb{N}$ (i.e. the two-point space with one open point and one closed point). The structure sheaf (depicted using Proposition 6.1.1.5) is

$$\mathsf{B}\mathbb{N} \longrightarrow \mathsf{B}\mathbb{Z}.$$

Example 6.1.2.4. The topological space underlying $\mathrm{Cone}(\mathsf{B}\mathbb{N}^2)$ is $\operatorname{Spec} \mathbb{N}^2$. The structure sheaf is

$$\begin{tikzcd} \mathsf{B}\mathbb{N}^2 \arrow[r] \arrow[d] & \mathsf{B}\mathbb{N} \times \mathsf{B}\mathbb{Z} \arrow[d] \\ \mathsf{B}\mathbb{Z} \times \mathsf{B}\mathbb{N} \arrow[r] & \mathsf{B}\mathbb{Z}^2. \end{tikzcd}$$

We can generalize the preceding examples to any strongly convex rational polyhedral cone (instead of just $\mathbb{N} \subset \mathbb{Z}$ or $\mathbb{N}^2 \subset \mathbb{Z}^2$).

Example 6.1.2.5. Let $\mathsf{N} \cong \mathbb{Z}^n$ be a free finitely generated discrete abelian group, and let $\underline{\sigma} \subset \mathsf{N} \otimes_{\mathbb{Z}} \mathbb{R}$ be a strongly convex rational polyhedral cone (as in [CLS11, §1.2]). We may define a corresponding $\mathbb{E}_\infty$-cone $\mathsf{K}_{\underline{\sigma}}$ as follows. Let $\mathsf{M} = \mathrm{Hom}(\mathsf{N}, \mathbb{Z})$ be the dual lattice, and let

$$\underline{\sigma}^\vee = \{m \in \mathsf{M} \otimes_{\mathbb{Z}} \mathbb{R} \,|\, \langle m, n \rangle \geqslant 0 \text{ for all } n \in \underline{\sigma}\}.$$

Then $\mathsf{K}_{\underline{\sigma}} := \mathrm{Cone}(\mathsf{B}(\sigma^\vee \cap \mathsf{M}))$ is the $\mathbb{E}_\infty$-cone associated with σ.

The prime ideals of $\underline{\sigma}^\vee \cap \mathsf{M}$ are in bijection with the faces of $\underline{\sigma}$: a prime $\mathfrak{p} \subset \underline{\sigma}^\vee \cap \mathsf{M}$ corresponds to the face

$$\{n \in \underline{\sigma} \,|\, \langle m, n \rangle = 0 \text{ for all } m \in \mathfrak{p}\} \subset \underline{\sigma}.$$

Thus the underlying topological space of $\mathsf{K}_{\underline{\sigma}}$ may be identified with the poset $\mathrm{Face}(\underline{\sigma})$ of faces of $\underline{\sigma}$ (ordered by inclusion and considered with the topology where the open sets are the downward-closed sets). Every face $\underline{\sigma}' \subset \underline{\sigma}$ is the closed point (in the subspace topology) of a unique basic open

$$\mathsf{K}_{\underline{\sigma}'} = \mathrm{Cone}(\mathsf{B}((\underline{\sigma}')^\vee \cap \mathsf{M})) \subset \mathsf{K}_{\underline{\sigma}},$$

and every basic open in $\mathsf{K}_{\underline{\sigma}}$ arises in this way.

Now we discuss some truly "stacky" examples.

Example 6.1.2.6. Let $\mathcal{C} = \mathsf{A}$ be an $\mathbb{E}_\infty$-group. Then $\mathrm{Cone}(\mathsf{A})$ has a unique point, and the structure sheaf is defined on this unique point by $\mathscr{O}_{\mathrm{Cone}(\mathsf{A})}(*) = \mathsf{A}$.

Example 6.1.2.7. Consider $\mathcal{C} = \mathbb{Z}\vec{/}\mathbb{N}^2$ where $\mathbb{N}^2 \to \mathbb{Z}$ is the addition map. The topological space underlying $\mathrm{Cone}(\mathbb{Z}\vec{/}\mathbb{N}^2)$ is $\mathrm{Spec}\,\mathbb{N}^2$. The structure sheaf is

$$\begin{tikzcd}
\mathbb{Z}\vec{/}\mathbb{N}^2 \arrow[r] \arrow[d] & \mathsf{B}\mathbb{N} \arrow[d] \\
\mathsf{B}\mathbb{N} \arrow[r] & \mathsf{B}\mathbb{Z},
\end{tikzcd}$$

where we have identified $\mathbb{Z}\vec{/}(\mathbb{Z} \times \mathbb{N}) = \mathsf{B}\mathbb{N}$, $\mathbb{Z}\vec{/}(\mathbb{N} \times \mathbb{Z}) = \mathsf{B}\mathbb{N}$, and $\mathbb{Z}\vec{/}(\mathbb{Z} \times \mathbb{Z}) = \mathbb{Z}$. Here the bottommost horizontal map is induced by the positive inclusion $\mathbb{N} \to \mathbb{Z}$, while the rightmost vertical map is induced by the negative inclusion $\mathbb{N} \to \mathbb{Z}$.

6.2 $\mathbb{E}_\infty$-fans

Now that we have defined our local models (i.e. $\mathbb{E}_\infty$-cones) for derived toric geometry, we would like to turn our attention to more global questions. We introduce $\mathbb{E}_\infty$-*fans* here to track how the local data of $\mathbb{E}_\infty$-cones can be glued together.

6.2.1 GSM spaces

Schemes are typically understood as locally ringed spaces which are Zariski-locally covered by affine schemes. To produce an analogous definition / characterization of $\mathbb{E}_\infty$-fans, we first introduce the following definition:

Definition 6.2.1.1. A *locally grouplike symmetric monoidal space* (abbreviated *locally GSM space*) is a pair $(\Sigma, \mathscr{O}_\Sigma)$ where Σ is a topological space and $\mathscr{O}_\Sigma \in \mathsf{Sh}_{\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})}(\Sigma)$.

We would like to assemble these into a category LGS of locally GSM spaces satisfying:

- Objects of LGS are locally GSM spaces.
- 1-morphisms are pairs $(\mathsf{f}, \mathsf{f}^\#) : (\Sigma, \mathscr{O}_\Sigma) \to (\Sigma', \mathscr{O}_{\Sigma'})$ consisting of:
  - A continuous map $\mathsf{f} : \Sigma \to \Sigma'$ and
  - A map $\mathsf{f}^\# : \mathsf{f}^{-1}\mathscr{O}_{\Sigma'} \to \mathscr{O}_\Sigma$ in $\mathsf{Sh}_{\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})}(\Sigma)$.

  such that:

  - for all $\sigma \in \Sigma$, the induced map
  $$\mathsf{f}^\#_\sigma : (\mathsf{f}^{-1}\mathscr{O}_{\Sigma'})_\sigma = \mathscr{O}_{\Sigma',\mathsf{f}(\sigma)} \to \mathscr{O}_{\Sigma,\sigma}$$
  is conservative.[2]
- The composition law in LGS is formally identical to the composition law for morphisms of ringed spaces.

Proposition 6.2.1.2. *There exists a well-defined ∞-category LGS satisfying the above requirements.*

We write $\mathbb{E}_\infty\mathsf{Cone}$ for the full subcategory of $\mathbb{E}_\infty\mathsf{Fan}$ with objects given by the $\mathbb{E}_\infty$-cones. These satisfy a universal property analogous to that satisfied by affine schemes among locally ringed spaces:

Proposition 6.2.1.3. *Let $(\Sigma, \mathscr{O}_\Sigma)$ be a locally GSM space, and let $\mathcal{C}$ be a grouplike SMC. Then there is an equivalence (natural in Σ):*

$$\mathrm{Map}_{\mathsf{LGS}}(\Sigma, \mathrm{Cone}(\mathcal{C})) = \mathsf{Fun}^{\otimes}(\mathcal{C}, \Gamma(\Sigma, \mathscr{O}_\Sigma)).$$

Corollary 6.2.1.4. *There is an adjunction*

$$\Gamma(-, \mathscr{O}_{(-)}) : \mathsf{LGS} \rightleftarrows (\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat}))^{\mathrm{op}} : \mathrm{Cone}\,.$$

In particular Cone $: (\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat}))^{\mathrm{op}} \to \mathsf{LGS}$ *is functorial and limit-preserving. Furthermore,* Cone *is fully faithful, with image* $\mathbb{E}_\infty\mathsf{Cone}$.

[2] That is, if $\mathsf{f}^\#(\mathfrak{m})$ is an isomorphism, then $\mathfrak{m}$ is an isomorphism. By translation we may always assume $\mathfrak{m} \in \mathrm{P}_0\mathscr{O}_{\Sigma,\sigma}$.

6.2.2 $\mathbb{E}_\infty$-fans: definition and examples

We may now introduce $\mathbb{E}_\infty$-fans:

Definition 6.2.2.1. An $\mathbb{E}_\infty$-*fan* is a locally GSM space $(\Sigma, \mathscr{O}_\Sigma)$ which admits a finite open covering by $\mathbb{E}_\infty$-cones $\{\mathrm{Cone}(\mathcal{C}_i)\}_{i\in I}$ such that, for all $i_1, i_2 \in I$, we have

$$\mathrm{Cone}(\mathcal{C}_{i_1}) \cap \mathrm{Cone}(\mathcal{C}_{i_2}) = \mathrm{Cone}(\mathcal{C}_{i_3})$$

for some $i_3 \in I$. We will refer to these requirements (finiteness, consisting of open cones, and stability under finite intersections) as the *fan condition* for an open cover.

Let $\mathbb{E}_\infty\mathsf{Fan}$ be the full subcategory of LGS with objects $\mathbb{E}_\infty$-fans.

Remark 6.2.2.2. If we think of $\mathbb{E}_\infty$-cones as analogous to affine schemes, then $\mathbb{E}_\infty$-fans are analogous to quasicompact schemes with affine diagonal.

The fan condition actually implies that the intersection of any finite number of open cones in an $\mathbb{E}_\infty$-fan Σ is still an open cone:

Proposition 6.2.2.3. *Let Σ be an $\mathbb{E}_\infty$-fan, and let* $\mathrm{Cone}(\mathcal{C}_1), \mathrm{Cone}(\mathcal{C}_2) \subset \Sigma$ *be open cones. Then* $\mathrm{Cone}(\mathcal{C}_1) \cap \mathrm{Cone}(\mathcal{C}_2)$ *is an open cone.*

Thus the poset $\mathsf{Open}^{\mathrm{cone}}(\Sigma)$ of open cones in a fixed $\mathbb{E}_\infty$-fan Σ admits finite joins. We can describe sheaves on Σ in terms of $\mathsf{Open}^{\mathrm{cone}}(\Sigma)$:

Proposition 6.2.2.4. *Let Σ be a $\mathbb{E}_\infty$-fan, and let $\mathcal{V}$ be a complete category. Then the inclusion* $\mathsf{Open}^{\mathrm{cone}}(\Sigma) \subset \mathsf{Open}(\Sigma)$ *induces an equivalence* $\mathsf{Sh}_{\mathcal{V}}(\Sigma) = \mathsf{Fun}(\mathsf{Open}^{\mathrm{cone}}(\Sigma)^{\mathrm{op}}, \mathcal{V})$.

We can use Proposition 6.2.2.4 to give explicit descriptions of the structure sheaves on examples:

Example 6.2.2.5. We may glue two copies of $\mathrm{Cone}(\mathsf{B}\mathbb{N})$ (see Example 6.1.2.3) together along the open $\mathrm{Cone}(\mathsf{B}\mathbb{Z})$ (with the two copies of $\mathrm{Cone}(\mathsf{B}\mathbb{Z})$ identified using the additive inverse map $\mathsf{B}\mathbb{Z} \to \mathsf{B}\mathbb{Z}$) to form an $\mathbb{E}_\infty$-fan $\Sigma_{\mathbb{P}^1}$. The structure sheaf $\mathscr{O}_{\Sigma_{\mathbb{P}^1}}$ may be depicted as

$$\begin{tikzcd} & \mathsf{B}\mathbb{N} \arrow[d] \\ \mathsf{B}\mathbb{N} \arrow[r] & \mathsf{B}\mathbb{Z}. \end{tikzcd}$$

where the horizontal map comes from the positive inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$ and the vertical map comes from the negative inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$.

There is an obvious open immersion $\Sigma_{\mathbb{P}^1} \hookrightarrow \mathrm{Cone}(\mathbb{Z} \overrightarrow{/} \mathbb{N}^2)$ (see Example 6.1.2.7), so $\Sigma_{\mathbb{P}^1}$ is quasiconical. In Section 6.3.1, we will provide a general explanation for why this open immersion exists, generalizing the Cox construction in classical toric geometry.

Example 6.2.2.6. We continue with the notation of Example 6.1.2.5. Let $\underline{\Sigma} = \{\underline{\sigma} \subset \mathsf{N} \otimes_{\mathbb{Z}} \mathbb{R}\}$ be a fan in $\mathsf{N} \otimes_{\mathbb{Z}} \mathbb{R}$, as in [CLS11, §3.1]. We may define a corresponding $\mathbb{E}_\infty$-fan $\mathsf{F}_{\underline{\Sigma}}$ as follows. As a topological space, $\mathsf{F}_{\underline{\Sigma}}$ is $\underline{\Sigma}$ itself, viewed as a poset and equipped with the topology where the open sets are precisely the downward-closed subsets. Every $\underline{\sigma} \in \underline{\Sigma}$ is the closed point (in the subspace topology) of a unique open subset $\mathsf{U}_{\underline{\sigma}} = \mathsf{F}_{\underline{\Sigma}}$. We equip $\mathsf{F}_{\underline{\Sigma}}$ with the unique structure sheaf $\mathscr{O}_{\mathsf{F}_{\underline{\Sigma}}}$ such that $\mathscr{O}_{\mathsf{F}_{\underline{\Sigma}}}(\mathsf{U}_{\underline{\sigma}}) = \mathsf{B}(\sigma^\vee \cap \mathsf{M})$.

With this structure sheaf, each $\mathsf{U}_{\underline{\sigma}}$ is equivalent to the corresponding cone $\mathsf{K}_{\underline{\sigma}}$ constructed in Example 6.1.2.5. The closed points of $\mathsf{F}_{\underline{\Sigma}}$ correspond to the maximal cones of $\underline{\Sigma}$, while the generic point of $\mathsf{F}_{\underline{\Sigma}}$ corresponds to the minimal cone of $\underline{\Sigma}$.

6.2.3 Some nice properties of $\mathbb{E}_\infty\mathsf{Fan}$

In this subsection we discuss some useful properties of the category $\mathbb{E}_\infty\mathsf{Fan}$. Many (but not all) of these will be analogues of the corresponding properties for the category of derived schemes.

Recall that one of our goals was to construct a framework for derived toric geometry in which it is possible to take fiber products (so that we can compose spans). This will follow by combining Remark 6.2.4.3 below with:

Theorem 6.2.3.1. *The category $\mathbb{E}_\infty\mathsf{Fan}$ admits finite limits, and the inclusion $\mathbb{E}_\infty\mathsf{Fan} \subset \mathsf{LGS}$ preserves finite limits.*

Much like for schemes, we can construct a fully faithful embedding of $\mathbb{E}_\infty\mathsf{Fan}$ into the category of "Zariski sheaves" on $\mathbb{E}_\infty\mathsf{Cone}$. In fact, the Zariski topology on $\mathbb{E}_\infty\mathsf{Cone}$ is trivial,[3] so we can (and should) work entirely with *presheaves* on $\mathbb{E}_\infty\mathsf{Cone}$. We note that

$$\mathsf{PSh}_{\mathcal{S}}(\mathbb{E}_\infty\mathsf{Cone}) = \mathsf{Fun}(\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat}), \mathcal{S}),$$

so the presheaves in question may be defined purely in terms of grouplike SMCs.

Proposition 6.2.3.2. *The functor $\mathsf{y}' : \mathbb{E}_\infty\mathsf{Fan} \to \mathsf{PSh}_{\mathcal{S}}(\mathbb{E}_\infty\mathsf{Cone})$ defined by*

$$\mathsf{y}'(\Sigma)(\mathcal{C}) = \mathrm{Map}_{\mathbb{E}_\infty\mathsf{Fan}}(\mathrm{Cone}(\mathcal{C}), \Sigma)$$

is fully faithful and preserves finite limits.

It turns out that $\mathbb{E}_\infty$-cones may be characterized purely categorically among $\mathbb{E}_\infty$-fans: they are precisely the $\mathbb{E}_\infty$-monoid objects in $\mathbb{E}_\infty\mathsf{Fan}$. Furthermore, every morphism between $\mathbb{E}_\infty$-cones is naturally a morphism of $\mathbb{E}_\infty$-monoids in $\mathbb{E}_\infty$-fan. We get:

Proposition 6.2.3.3. *The inclusion $\mathbb{E}_\infty\mathsf{Cone} \hookrightarrow \mathbb{E}_\infty\mathsf{Fan}$ induces an equivalence*

$$\mathbb{E}_\infty\mathsf{Cone} = \mathsf{CMon}(\mathbb{E}_\infty\mathsf{Cone}) = \mathsf{CMon}(\mathbb{E}_\infty\mathsf{Fan}).$$

The $\mathbb{E}_\infty$-group objects in $\mathbb{E}_\infty\mathsf{Fan}$ are precisely those of the form $\mathrm{Cone}(\mathsf{A})$ with A an $\mathbb{E}_\infty$-group.

[3] All $\mathbb{E}_\infty$-cones $\mathrm{Cone}(\mathcal{C})$ have unique closed points, so every Zariski open cover of $\mathrm{Cone}(\mathcal{C})$ must contain the identity. Thus every covering sieve on $\mathrm{Cone}(\mathcal{C})$ is just $\mathbb{E}_\infty\mathsf{Cone}_{/\mathrm{Cone}(\mathcal{C})}$.

This, together with Proposition 2.3.2.8, allows us to present any $\mathbb{E}_\infty$-cone as a quotient:

Corollary 6.2.3.4. *Let $\mathcal{C}$ be a grouplike SMC. Then the $\mathbb{E}_\infty$-cone* $\mathrm{Cone}(\mathcal{C})$ *is equivalent to the quotient* $[\mathrm{Cone}(\mathsf{B}\,\mathrm{P}_0\mathcal{C})/\,\mathrm{Cone}(\iota_0\mathcal{C})]$ *computed in* $\mathbb{E}_\infty\mathsf{Fan}$.

6.2.4 Geometrization of $\mathbb{E}_\infty$-fans

Of course, the theory of $\mathbb{E}_\infty$-fans is essentially worthless without a way to construct genuine geometric objects (e.g. derived stacks, spectral stacks, etc.) out of $\mathbb{E}_\infty$-fans. We can do this using the Tannakian approach of Chapter 3.

Definition 6.2.4.1. A *geometrization context* (abbreviated *g-context*) $(\mathcal{X}, \tau, \mathsf{D})$ consists of a subcanonical site $(\mathcal{X}, \tau)$ together with a sheaf of (typically large) symmetric monoidal categories $\mathsf{D} \in \mathsf{Sh}_{\mathsf{CAlg}(\widehat{\mathsf{Cat}})}(\mathcal{X})$. If D takes values in $\mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})$, we refer to $(\mathcal{X}, \tau, \mathsf{D})$ as a *presentable g-context.*

Definition 6.2.4.2. Given a g-context $(\mathcal{X}, \tau, \mathsf{D})$, we can define a *geometrization* functor $\mathsf{X}^{\mathsf{D}}_{(-)} : \mathbb{E}_\infty\mathsf{Cone} \to \mathsf{Stk}(\mathcal{X}) := \mathsf{Sh}_{\mathcal{S}}(\mathcal{X})$ by the formula $\mathsf{X}^{\mathsf{D}}_{\mathrm{Cone}(\mathcal{C})}(\mathsf{U}) = \mathsf{Fun}^{\otimes}(\mathcal{C}, \mathsf{D}(\mathsf{U}))$. Using Proposition 6.2.3.2, we may left Kan extend $\mathsf{X}^{\mathsf{D}}_{(-)}$ to a functor $\mathsf{X}^{\mathsf{D}}_{(-)} : \mathbb{E}_\infty\mathsf{Fan} \to \mathsf{Stk}(\mathcal{X})$.

Remark 6.2.4.3. Using an argument similar to Remark 3.2.1.4, one can show that geometrization functors $\mathsf{X}^{\mathsf{D}}_{(-)}$ preserve all limits which exist in $\mathbb{E}_\infty\mathsf{Fan}$.

Most of our examples of g-contexts will fit into the following framework:

Definition 6.2.4.4. Let $(\mathcal{V}, \otimes_{\mathcal{V}})$ be a presentable symmetric monoidal category. Let $\mathsf{Aff}(\mathcal{V}) = \mathsf{CAlg}(\mathcal{V})^{\mathrm{op}}$, and write $\mathrm{Spec}_{\mathcal{V}}\,\mathsf{R}$ for the object of $\mathsf{Aff}(\mathcal{V})$ corresponding to $\mathsf{R} \in \mathsf{CAlg}(\mathcal{V})$. Let $\tau_{\max}$ be the finest topology on $\mathsf{Aff}(\mathcal{V})$ such that the functor

$$\begin{aligned} \mathsf{QC}_{\mathcal{V}} : \mathsf{Aff}(\mathcal{V})^{\mathrm{op}} &\to \mathsf{Pr}^{\mathsf{L}}_{\mathcal{V}} \\ \mathrm{Spec}_{\mathcal{V}}\,\mathsf{R} &\mapsto \mathsf{Mod}_{\mathsf{R}}(\mathcal{V}) \end{aligned}$$

is a sheaf. Then $(\mathsf{Aff}(\mathcal{V}), \tau_{\max}, \mathsf{QC}_{\mathcal{V}})$ is a g-context, which we call the *Tannakian g-context* associated with $\mathcal{V}$. More generally, we will say that any g-context $(\mathsf{Aff}(\mathcal{V}), \tau, \mathsf{QC}_{\mathcal{V}})$ with $\tau \subset \tau_{\max}$ is *sub-Tannakian.*

Example 6.2.4.5. Taking $(\mathcal{V}, \otimes_{\mathcal{V}}) = (\mathsf{Mod}^{\heartsuit}_{\mathsf{R}}, \otimes^{0}_{\mathsf{R}})$ to be the discrete abelian category of modules over a discrete commutative ring R, we see that $\mathsf{Aff}(\mathcal{V})$ is the category of (classical) affine schemes over $\mathrm{Spec}\,\mathsf{R}$. The corresponding Tannakian g-context allows us to realize $\mathbb{E}_\infty$-fans as classical algebraic stacks. This realizes affine $\mathbb{E}_\infty$-fans as classical schemes.

Example 6.2.4.6. Taking $(\mathcal{V}, \otimes_{\mathcal{V}})$ to be the category of simplicial abelian groups, we see that $\mathsf{CAlg}(\mathcal{V})$ is the category of simplicial commutative rings, so $\mathsf{Aff}(\mathcal{V})$ is the category of affine derived schemes. The corresponding Tannakian g-context allows us to realize $\mathbb{E}_\infty$-fans (resp. affine $\mathbb{E}_\infty$-fans) as *derived algebraic stacks* (resp. derived schemes).

Example 6.2.4.7. Taking $(\mathcal{V}, \otimes_{\mathcal{V}}) = (\mathsf{Sp}^{\leqslant 0}, \otimes_{\mathbb{S}})$ to be the category of connective spectra (with its usual smash product), we see that $\mathsf{CAlg}(\mathcal{V})$ is the category of connective $\mathbb{E}_\infty$-rings. The corresponding Tannakian g-context allows us to realize $\mathbb{E}_\infty$-fans (resp. affine $\mathbb{E}_\infty$-fans) as (connective) *spectral algebraic stacks* (resp. spectral algebraic schemes).

Example 6.2.4.8. Taking $(\mathcal{V}, \otimes_{\mathcal{V}}) = (\mathsf{QC}^{\leqslant 0}(\mathsf{S}), \otimes_{\mathscr{O}_\mathsf{S}})$ for some derived algebraic stack S, we see that $\mathsf{Aff}(\mathcal{V})$ is the category of affine morphisms to S. The corresponding Tannakian g-context lets us realize $\mathbb{E}_\infty$-fans as stacks together with a morphism to S. Any stack we obtain in this way will be "globally trivial," i.e. it can be obtained as a fiber product $\mathsf{S} \times_{\operatorname{Spec}\mathbb{Z}} \mathsf{X}$, where X is the geometrization of the corresponding $\mathbb{E}_\infty$-fan over $\operatorname{Spec}\mathbb{Z}$.

To get a full theory of "relative derived toric geometry over S" (allowing for bundles of toric stacks over S *et cetera*), we should likely use a variant of the above constructions, replacing individual $\mathbb{E}_\infty$-fans by cosheaves of $\mathbb{E}_\infty$-fans over S. This may also be useful in constructing derived toric stacks for non-split tori.

There are also interesting examples which don't obviously relate to stable / prestable ∞-categories:

Example 6.2.4.9. Let $(\mathcal{V}, \otimes_{\mathcal{V}}) = (\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat}), \times)$. Then $\mathsf{Aff}(\mathcal{V}) = \mathbb{E}_\infty\mathsf{Cone}$, and geometrization via the corresponding Tannakian g-context just gives the inclusion $\mathbb{E}_\infty\mathsf{Fan} \subset \mathsf{PSh}_{\mathcal{S}}(\mathbb{E}_\infty\mathsf{Cone})$.

Example 6.2.4.10. Let $(\mathcal{V}, \otimes_{\mathcal{V}}) = (\mathsf{Cat}_\mathsf{k}^{\mathrm{perf}}, \otimes)$ for some $\mathbb{E}_\infty$-ring spectrum k, so $\mathsf{Aff}(\mathcal{V})$ is the opposite of the category of symmetric monoidal perfect k-linear categories. Writing $\mathsf{X}_{(-)}$ for geometrization with respect to the corresponding Tannakian g-context, we see by [AG14, Theorem 3.15] that, for any $\mathbb{E}_\infty$-algebra R over k,

$$\mathsf{X}_{\mathrm{Cone}(\mathbb{F}^{\mathrm{gp}})}(\mathsf{Perf}(\mathsf{R})) = \mathsf{Fun}^{\otimes}(\mathbb{F}^{\mathrm{gp}}, \mathsf{Mod}_{\mathsf{Perf}(\mathsf{R})} = \mathrm{dBr}(\mathsf{R}),$$

the *derived Brauer group of* R. Thus the stack $\mathsf{X}_{\mathrm{Cone}(\mathbb{F}^{\mathrm{gp}})}(\mathsf{Perf}(-)) : \mathsf{Aff}(\mathsf{CAlg}_{\mathsf{k}/})^{\mathrm{op}} \to \mathcal{S}$ may be identified with the classifying stack $\mathsf{B}^2\mathrm{GL}_1$, allowing us to approach a theory of *higher* (or *categorified*) derived toric geometry.

6.3 Projected applications

Here we mention some work in progress on applications of the theory of $\mathbb{E}_\infty$-fans. We caution readers that these are still preliminary.

6.3.1 The Cox construction

Recall that the Cox construction from classical toric geometry says that every smooth toric variety without torus factors embeds as an open substack of some quotient stack $\mathbb{A}^n/\mathsf{G}$ (with

G commutative and reductive). This can be understood entirely at the level of fans, and so it makes sense to ask whether an analogous construction works for nice enough $\mathbb{E}_\infty$-fans.

By Corollary 6.2.3.4, we may think of any $\mathbb{E}_\infty$-cone as something like a quotient stack of an affine variety by an abelian group. This motivates the following definition:

Definition 6.3.1.1. An $\mathbb{E}_\infty$-fan Σ is *quasiconical* if there exists an open immersion $\Sigma \hookrightarrow \mathrm{Cone}(\mathcal{C})$ for some $\mathbb{E}_\infty$-cone $\mathrm{Cone}(\mathcal{C})$.

Remark 6.3.1.2. In general, geometrization functors X^{D} do not preserve colimits, so the quotient description of an $\mathbb{E}_\infty$-cone $\mathrm{Cone}(\mathcal{C})$ does not immediately imply a corresponding description of $\mathsf{X}^{\mathsf{D}}_{\mathrm{Cone}(\mathcal{C})}$ as a quotient stack. However, such a description of $\mathsf{X}^{\mathsf{D}}_{\mathrm{Cone}(\mathcal{C})}$ *is* possible in some favorable cases – see e.g. Proposition 3.2.3.6.

Quasiconicality, much like quasi-affineness in scheme theory, is a *property* rather than a piece of extra structure. In fact, if Σ is quasiconical, then there always exists a *canonical* open immersion from Σ into an $\mathbb{E}_\infty$-cone:

Proposition 6.3.1.3. *An $\mathbb{E}_\infty$-fan Σ is quasiconical if and only if the natural map $\mathsf{j} : \Sigma \to \mathrm{Cone}(\Gamma(\Sigma, \mathscr{O}_\Sigma))$ is an open immersion.*

The Cox construction therefore translates into the claim that any smooth fan is quasiconical when viewed as an $\mathbb{E}_\infty$-fan. We may generalize this as follows:

Theorem 6.3.1.4 (Cox construction (work in progress))**.** *Fix a sharp $\mathbb{E}_\infty$-monoid N with $\pi_0(\mathsf{N}) = \mathbb{N}$. Let Σ be an $\mathbb{E}_\infty$-fan which admits a cover by $\mathbb{E}_\infty$-cones $\mathrm{Cone}(\mathcal{C}_\mathsf{i})$ such that there exist equivalences $\mathrm{P}_0\mathcal{C}_\mathsf{i} \simeq \mathsf{N}^{\mathsf{n}_\mathsf{i}}$ for all i (and for some n_i). Then Σ is quasiconical, and $\mathrm{P}_0\Gamma(\Sigma, \mathscr{O}_\Sigma) = \mathsf{N}^{\mathsf{n}}$ for some n.*

6.3.2 Composing spans

As mentioned before, we are interested in having a notion of derived toric geometry so that we may use toric language to discuss compositions of spans. Many interesting functors between derived categories of toric varieties arise via push-pull operations along spans. For example, this is true for:

- Derived equivalences of toric varieties induced by toric flops (see [BO95, Theorem 3.6 and the remark after it]),
- The McKay correspondence for cyclic subgroups of SL_2 acting on $\mathbb{A}^2$ (see [BKR01]),
- Certain window functors (e.g. by [BDF17] or the discussion in Section 5.1.2), and
- The extended convolution products on toric varieties of Bondal–Ruan type constructed in this thesis.

These functors may have surprising relationships with each other, and we can hope to witness these relationships (and present new toric models of interesting functors) using the geometry of $\mathbb{E}_\infty$-fans.

Example 6.3.2.1. If

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X_- & & X_+ \end{tikzcd}$$

is a toric flop, then the autoequivalence $p_*q^*q_*p^* : \mathrm{QC}(X_-) \to \mathrm{QC}(X_-)$ typically is not the identity – instead, it is the inverse of a spherical twist. (See [Bar20, Theorem 1.0.2] for a general statement.)

Using base change, we may also understand $q_*q^*q_*p^*$ as the push-pull functor induced by the span

$$\begin{tikzcd} & \widetilde{X} \times_{X_+} \widetilde{X} \arrow[dl] \arrow[dr] & \\ X_- & & X_-. \end{tikzcd}$$

If X_- (resp. X_+, $\widetilde{X}$) is the geometrization of the $\mathbb{E}_\infty$-fan Σ_- (resp. Σ_+, $\widetilde{\Sigma}$), then our span in question is the geometrization of the span (in $\mathbb{E}_\infty\mathsf{Fan}$)

$$\begin{tikzcd} & \widetilde{\Sigma} \times_{\Sigma_+} \widetilde{\Sigma} \arrow[dl] \arrow[dr] & \\ \Sigma_- & & \Sigma_-. \end{tikzcd}$$

Thus *understanding the inverse spherical twist here reduces to studying the geometry of a certain fiber product of* $\mathbb{E}_\infty$*-fans.*

Of course, to say anything computationally meaningful, we need an effective strategy for computing fiber products of $\mathbb{E}_\infty$-fans. This reduces to computing pushouts of grouplike SMCs, which itself reduces to computing pushouts in CMon. We have not seen this last problem treated in the necessary generality in the literature, so we need to develop a method for doing so.

We propose the following strategy for computing a pushout $N \oplus_M P$ in CMon:

1. Write N as a colimit of copies of the regular right M-space M in the topos $\mathcal{S}^M$, say $N = \operatorname{colim}_i M_i$. Because the forgetful functor $\mathcal{S}^M \to \mathcal{S}$ is conservative and colimit-preserving, to check that $N = \operatorname{colim}_i M^i$ for some diagram $M_- : I \to \mathcal{S}^M$, it suffices to construct a map $\operatorname{colim}_i M_i \to N$ in $\mathcal{S}^M$ and then check that this map is an equivalence in $\mathcal{S}$. This latter task is feasible because colimits in $\mathcal{S}$ are computable using standard model-categorical techniques (e.g. cofibrant replacement).

2. The pushout $\mathsf{N} \oplus_{\mathsf{M}} \mathsf{P}$ in CMon is the coproduct in $\mathsf{CMon}_{\mathsf{M}/} = \mathsf{CAlg}(\mathcal{S}^{\mathsf{M}})$, which agrees with the monoidal structure $\star$ on $\mathcal{S}^{\mathsf{M}}$ corresponding to Day convolution in $\mathsf{PSh}_{\mathcal{S}}(\mathsf{BM})$. At the level of underlying spaces, we obtain

$$\mathsf{N} \oplus_{\mathsf{M}} \mathsf{P} = \mathsf{N} \star \mathsf{P} = \operatorname*{colim}_{i} \mathsf{M}_i \star \mathsf{P} = \operatorname*{colim}_{i} \mathsf{P}_i,$$

where $\mathsf{P}_{-} : \mathrm{I} \to \mathcal{S}^{\mathsf{M}}$ is the diagram with transition maps defined as in M_{-}, but all objects taken to be P.

3. The preceding step gives a formula for the M-space underlying $\mathsf{N} \oplus_{\mathsf{M}} \mathsf{P}$. Upgrading this to an equivalence of $\mathbb{E}_\infty$-monoids requires some level of guesswork. One needs to construct an $\mathbb{E}_\infty$-monoid Q together with a commutative square (in CMon)

$$\begin{tikzcd} \mathsf{M} \arrow[r] \arrow[d] & \mathsf{N} \arrow[d] \\ \mathsf{P} \arrow[r] & \mathsf{Q}. \end{tikzcd}$$

To check that the natural map $\mathsf{N} \oplus_{\mathsf{M}} \mathsf{P} \to \mathsf{Q}$ is an isomorphism, it suffices to check this at the level of underlying spaces, which can be done using the presentation obtained in step (2).

Remark 6.3.2.2. Note the close relationship between this method and the standard method for computing pushouts of cdgas. Indeed, in both cases, we take a “free resolution” of one of the factors, tensor that free resolution with the other factor, and then attempt to determine the commutative algebra structure on the resulting object.

We can think of our colimit presentations in $\mathcal{S}^{\mathsf{M}}$ as *non-abelian free resolutions*. It is plausible that these can be systematized using some version of the “double-arrow complexes” of [Flo15, §4].

Remark 6.3.2.3. Step (1) above is always doable in principle: under the equivalence $\mathcal{S}^{\mathsf{M}} \simeq \mathsf{PSh}_{\mathcal{S}}(\mathsf{BM})$, M corresponds to the unique representable functor, and every object of $\mathsf{PSh}_{\mathcal{S}}(\mathsf{BM})$ is a colimit of representable functors. However, in practice, the universal colimit presentation of N is rather unwieldy, and there are often much smaller presentations we can use.

Here's an easy example of the above method in practice:

Example 6.3.2.4. Let's compute the fiber product $\mathrm{pt} \times_{\mathrm{Cone}(\mathsf{B}\mathbb{N})} \mathrm{pt}$ in $\mathbb{E}_\infty\mathsf{Fan}$ (or equivalently in $\mathbb{E}_\infty\mathsf{Cone}$). This is equivalent to computing the pushout $0 \oplus_{\mathsf{B}\mathbb{N}} 0 = \mathsf{B}0 \oplus_{\mathsf{B}\mathbb{N}} \mathsf{B}0$ in $\mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})$. Because the functor $\mathsf{B} : \mathsf{CMon} \to \mathsf{CAlg}^{\mathrm{gplike}}(\mathsf{Cat})$ preserves colimits, it suffices to compute the pushout $0 \oplus_{\mathbb{N}} 0$ in CMon. We follow the steps outlined above:

1. We can write 0 as the coequalizer of the diagram $\mathbb{N} \rightrightarrows \mathbb{N}$ in $\mathcal{S}^{\mathbb{N}}$, where one map is $\phi(n) = n + 1$ and the other map is the identity.[4] Indeed, the coequalizer of this

[4] A practical explanation of how to compute (homotopy) coequalizers can be found at `https://ncatlab.org/nlab/show/homotopy+coequalizer`.

diagram can be rewritten as the pushout of the diagram

$$\begin{tikzcd} \mathbb{N} \sqcup \mathbb{N} \arrow[r, "(\phi, \mathrm{id}_{\mathbb{N}})"] \arrow[d, "(\mathrm{id}_{\mathbb{N}}, \mathrm{id}_{\mathbb{N}})"] & \mathbb{N} \\ \mathbb{N} & \end{tikzcd}$$

Replacing $(\mathrm{id}_{\mathbb{N}}, \mathrm{id}_{\mathbb{N}}) : \mathbb{N} \sqcup \mathbb{N} \to \mathbb{N}$ with the cofibration of topological spaces $\mathbb{N} \sqcup \mathbb{N} \hookrightarrow \mathbb{N} \times [0,1]$ (given by sending one copy of $\mathbb{N}$ to $\mathbb{N} \times \{0\}$ and the other to $\mathbb{N} \times \{1\}$), we can compute the pushout here at the level of topological spaces as

$$(\mathbb{N} \times [0,1])/\big((n+1,0) \sim (n,1) \text{ for all } n\big) = \mathbb{R}_{\geqslant 0}.$$

This is clearly homotopy equivalent to the contractible $\mathbb{N}$-space 0.

2. It follows that the underlying space of $0 \oplus_{\mathbb{N}} 0$ is the coequalizer of $0 \rightrightarrows 0$. Arguing using cofibrant replacements as above, we see that this coequalizer is

$$(\mathrm{pt} \times [0,1])/(0 \sim 1) = \mathsf{S}^1.$$

3. There is a commutative diagram of $\mathbb{E}_\infty$-monoids

$$\begin{tikzcd} \mathbb{N} \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \mathsf{B}\mathbb{Z}, \end{tikzcd}$$

where the commutativity is witnessed by a generator of $\mathbb{Z} = \Omega\mathsf{B}\mathbb{Z}$. (The simplest way to see this rigorously is by viewing this diagram as living in the 2-category of symmetric monoidal 1-groupoids.) Thus we get a natural homomorphism of $\mathbb{E}_\infty$-monoids $0 \oplus_{\mathbb{N}} 0 \to \mathsf{B}\mathbb{Z}$. By step (2), we saw that $0 \oplus_{\mathbb{N}} 0 = \mathsf{S}^1 = \mathsf{B}\mathbb{Z}$, so the homomorphism we constructed is an equivalence.

Thus $0 \oplus_{\mathbb{N}} 0 = \mathsf{B}\mathbb{Z} = \mathsf{S}^1$.[5] It follows that $\mathrm{pt} \times_{\mathrm{Cone}(\mathsf{B}\mathbb{N})} \mathrm{pt} = \mathrm{Cone}(\mathsf{S}^1)$.

To apply this to examples like Example 6.3.2.1, we need to be able to construct more examples of non-abelian free resolutions. This is the subject of ongoing research.

Question 6.3.2.5. *Which standard free resolutions of toric subvarieties can be upgraded to non-abelian free resolutions (i.e. can be defined in $\mathcal{S}^{\mathsf{M}}$ for some M)? Does this hold for Koszul resolutions? What about HHL resolutions ([HHL24])?*

[5] Of course, the equivalence $0 \oplus_{\mathbb{N}} 0 = \mathsf{B}\mathbb{Z}$ also follows directly from the group completion theorem, but the method presented here is much more general.

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