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The 3D Vector Potential, Magnetic Field and Stored Energy in a Thin $\cos 2\theta$ Coil Array.*

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Abstract

The vector potential and the magnetic field have been derived for an arrays of quadrupole magnets with thin $\cos(2\theta)$ current sheet placed at $r=R$.^{bc} The field strength of each coil within the array, varies purely as a Fourier sinusoidal series of the longitudinal coordinate z in proportion to $\omega_m z$, where $\omega_m = \frac{(2m-1)\pi}{L}$, L denotes the half-period, and $m=1,2,3$ etc. The analysis is based on the expansion of the vector potential in the region external to the windings of a linear 3D quad, and a revision of that expansion by the application of the "Addition Theorem" from that around the coil center to that around any arbitrary point in space .

The expression of the current density J in a quad (a form that satisfies the conservation condition $\nabla \cdot \vec{J}_s = \frac{\partial J_z}{\partial z} + \frac{1}{R} \frac{\partial J_\theta}{\partial \theta} = 0$ as required), and its contribution to the vector-potential $\vec{A}$ and magnetic field $\vec{B}$ in an infinite array of quadrupole magnets, are listed below for the region $r \leq R$ (excluding the self field of the coil with its center at 0,0)

$$\begin{aligned} \vec{J}(\theta, z)|_{r=R} &= \sum_{m=1} J_{0z,m} \left[\left(\frac{\omega_m R}{2} \right) \sin 2\theta \sin \omega_m z \hat{e}_\theta + \cos 2\theta \cos \omega_m z \hat{e}_z \right] \\ J_{0z,m} &= -\frac{1}{\mu_0} \frac{8RG_{2,m}}{(\omega_m R)^3 K'_2(\omega_m R)} \text{ (A/m)} \quad \text{and} \quad G_{2,m} \text{ gradient at } z=0 \\ \omega_m &= \frac{(2m-1)\pi}{L} \quad \text{where } L \text{ denotes the half period.} \end{aligned}$$

$$\begin{aligned} \vec{A}_r &= \sum_{m=1} \sum_{k=1} \frac{\mu_0 J_{0z,m} R (4k-3)!}{2 \left(\frac{\omega_m R}{2} \right)^{2(2k-1)}} \left[C_{k,m}^+ I'_{2(2k-1)}(\omega_m r) + 2(2k-1) C_{k,m}^- \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \right] \cos 2(2k-1)\theta \sin \omega_m z \\ \vec{A}_\theta &= - \sum_{m=1} \sum_{k=1} \frac{\mu_0 J_{0z,m} R (4k-3)!}{2 \left(\frac{\omega_m R}{2} \right)^{2(2k-1)}} \left[C_{k,m}^- I'_{2(2k-1)}(\omega_m r) + 2(2k-1) C_{k,m}^+ \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \right] \sin 2(2k-1)\theta \sin \omega_m z \\ \vec{A}_z &= \sum_{m=1} \sum_{k=1} \frac{\mu_0 J_{0z,m} R (4k-3)!}{2 \left(\frac{\omega_m R}{2} \right)^{2(2k-1)}} C_{k,m}^+ I_{2(2k-1)}(\omega_m r) \cos 2(2k-1)\theta \cos \omega_m z \\ C_{k,m}^+ &= \sum_{i=1} \sum_{j=1} [K_{4(k-1)}(\omega_m S_{i,j}) \cos 4(k-1)\theta_{0i,j} + K_{4k}(\omega_m S_{i,j}) \cos 4k\theta_{0i,j}] \frac{16 I_2(\omega_m R)}{(4k-3)!} \left(\frac{\omega_m R}{2} \right)^{4k-2} \\ C_{k,m}^- &= \sum_{i=1} \sum_{j=1} [K_{4(k-1)}(\omega_m S_{i,j}) \cos 4(k-1)\theta_{0i,j} - K_{4k}(\omega_m S_{i,j}) \cos 4k\theta_{0i,j}] \frac{16 I'_2(\omega_m R)}{(4k-3)!} \left(\frac{\omega_m R}{2} \right)^{4k-1} \end{aligned}$$

Where I_n and K_n are the "modified" Bessel functions of the first and second kind of order n , and the prime denotes differentiation with respect to the argument. The summation i,j is over the infinite number of quads in the first octant of the array.

^b "Multipoles in $\cos(2\theta)$ Coil Arrays – Type I", SC-MAG-583, LBID-2203, May 1997.

^c "Multipoles in $\cos(2\theta)$ Coil Arrays – Type II", SC-MAG-596, June 1997.

The magnetic field components are,

$$\begin{aligned}
B_r &= \sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \left(\frac{2}{\omega_m R} \right)^{4k-3} (4k-3)! C_{k,m}^- I_{2(2k-1)}'(\omega_m r) \sin 2(2k-1)\theta \cos \omega_m z \\
B_\theta &= \sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \left(\frac{2}{\omega_m R} \right)^{4k-3} 2(2k-1)(4k-3)! C_{k,m}^- \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \cos 2(2k-1)\theta \cos \omega_m z \\
B_z &= - \sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \left(\frac{2}{\omega_m R} \right)^{4k-3} (4k-3)! C_{k,m}^- I_{2(2k-1)}(\omega_m r) \sin 2(2k-1)\theta \sin \omega_m z
\end{aligned}$$

The format used here for A and B was specifically chosen to avoid a singularity that may rise when L is large (e.g. when the 3d problem reduces to 2d). Appendix B contains plots of both C^+ and C^- .

In calculating the values of the multipole coefficients we need to distinguish between coils placed on lines of symmetry and coils placed elsewhere.

$$\theta=45^\circ$$

On the 45 degree line of symmetry we use,

$$\omega_m S_{i,j} = (2m-1)2\sqrt{2}i \frac{\pi S}{L}$$

$$\begin{aligned}
C_{k,m}^+ &= -\frac{1}{2} \sum_{i=1} (-1)^k [-K_{4(k-1)}(\omega_m S_{i,j}) + K_{4k}(\omega_m S_{i,j}) \cos 4k\theta_{0i,j}] \frac{16I_2(\omega_m R)}{(4k-3)!} \left(\frac{\omega_m R}{2} \right)^{4k-2} \\
C_{k,m}^- &= -\frac{1}{2} \sum_{i=1} (-1)^{k+1} [K_{4(k-1)}(\omega_m S_{i,j}) + K_{4k}(\omega_m S_{i,j}) \cos 4k\theta_{0i,j}] \frac{16I_2'(\omega_m R)}{(4k-3)!} \left(\frac{\omega_m R}{2} \right)^{4k-1}
\end{aligned}$$

$$\theta=0^\circ$$

Summing on this line of symmetry requires ,

$$\omega_m S_{i,j} = 2(2m-1)i \frac{\pi S}{L}$$

and

$$\begin{aligned}
C_{k,m}^+ &= -\frac{1}{2} \sum_{i=1} (-1)^i [K_{4(k-1)}(\omega_m S_{i,j}) + K_{4k}(\omega_m S_{i,j})] \frac{16I_2(\omega_m R)}{(4k-3)!} \left(\frac{\omega_m R}{2} \right)^{4k-2} \\
C_{k,m}^- &= -\frac{1}{2} \sum_{i=1} (-1)^i [K_{4(k-1)}(\omega_m S_{i,j}) - K_{4k}(\omega_m S_{i,j})] \frac{16I_2'(\omega_m R)}{(4k-3)!} \left(\frac{\omega_m R}{2} \right)^{4k-1}
\end{aligned}$$

Coils with centers in the first octant but NOT on lines of symmetry

This part also requires,

$$\omega_m S_{i,j} = 2(2m-1) \sqrt{(i+j)^2 + j^2} \frac{\pi S}{L}$$

$$\theta_{0i,j} = \tan^{-1} \frac{j}{(i+j)}$$

and

$$C_{k,m}^+ = - \sum_{i=1} \sum_j (-1)^{i+2j} [K_{4(k-1)}(\omega_m S_{i,j}) \cos 4(k-1)\theta_{0i,j} + K_{4k}(\omega_m S_{i,j}) \cos 4k\theta_{0i,j}] \frac{16I_2(\omega_m R)}{(4k-3)!} \left(\frac{\omega_m R}{2}\right)^{4k-2}$$

$$C_{k,m}^- = - \sum_{i=1} \sum_j (-1)^{i+2j} [K_{4(k-1)}(\omega_m S_{i,j}) \cos 4(k-1)\theta_{0i,j} - K_{4k}(\omega_m S_{i,j}) \cos 4k\theta_{0i,j}] \frac{16I_2'(\omega_m R)}{(4k-3)!} \left(\frac{\omega_m R}{2}\right)^{4k-1}$$

Figure 1 Cross section showing current density arrangement

From Figures 6–8 in Appendix B we can estimate the effect the geometry has on the values of $C_{k,m}^+$ and $C_{k,m}^-$. If we just use the fundamental term $m=1$ and vary the value of k , we notice that for $k=1$, which is the contribution to quadrupole term, the C 's drop to 1/2 of their maximum value when S/R changes from 1.0 to 1.2. For the same change in S/R the dodecapole ($k=2$) and the 20's pole ($k=3$) will be reduced by a factor of 1/4 and 1/8 respectively. All harmonic terms do not vary significantly as long as $\frac{\pi R}{L} \leq 0.5$

Contributions from the self coil at i=0, r≤R

$$\begin{aligned}
 \vec{A}_r &= \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{4} (\omega_m R) [K_3(\omega_m R) I_3(\omega_m r) - K_1(\omega_m R) I_1(\omega_m r)] \cos 2\theta \sin \omega_m z \\
 \vec{A}_\theta &= \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{4} (\omega_m R) [K_3(\omega_m R) I_3(\omega_m r) + K_1(\omega_m R) I_1(\omega_m r)] \sin 2\theta \sin \omega_m z \\
 \vec{A}_z &= \sum_{m=1} \mu_0 J_{0z,m} R K_2(\omega_m R) I_2(\omega_m r) \cos 2\theta \cos \omega_m z \\
 \\
 B_r &= \sum_{m=1} \frac{\mu_0 J_{0z,m}}{2} (\omega_m R)^2 K_2'(\omega_m R) I_2'(\omega_m r) \sin 2\theta \cos \omega_m z \\
 B_\theta &= \sum_{m=1} \mu_0 J_{0z,m} (\omega_m R)^2 K_2'(\omega_m R) \frac{I_2(\omega_m r)}{(\omega_m r)} \cos 2\theta \cos \omega_m z \\
 B_z &= - \sum_{m=1} \frac{\mu_0 J_{0z,m}}{2} (\omega_m R)^2 K_2'(\omega_m R) I_2(\omega_m r) \sin 2\theta \sin \omega_m z
 \end{aligned}$$

Stored Energy

The energy stored in a single coil member of the array is,

$$E_{total} = -\frac{\pi R^2 L \mu_0}{8} \sum_{m=1} J_{0z,m}^2 (\omega_m R)^2 K_2'(\omega_m R) I_2'(\omega_m R) \left[1 + \frac{4C_{1,m}^-}{(\omega_m R)^3 K_2'(\omega_m R)} \right]$$

where the first term in the square bracket corresponds to the self coil energy and the second term arises from all other coils in the array.

Figure 2 Top view of a single periodic coil array of a single function (m=1) quad.

Figure 3 View of a 3x3 quadrupole array. The windings (of constant current) correspond to three terms $m=1,2,3$ which provide free space between coils.^d

^d This 3d CAD model was made on ProE by Ken Chow.

Analysis

Consider a quadrupole with its center located at $(S \cos \theta_0, S \sin \theta_0)$ as shown in Fig. 4. The expansion of the vector potential^e in the region outside the current sheet ($\rho > R$) and around that center is,

$$\begin{aligned}\vec{A}_\rho &= \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{4} (\omega_m R) [I_3(\omega_m R) K_3(\omega_m \rho) - I_1(\omega_m R) K_1(\omega_m \rho)] \cos 2\beta \sin \omega_m z \\ \vec{A}_\beta &= \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{4} (\omega_m R) [I_3(\omega_m R) K_3(\omega_m \rho) + I_1(\omega_m R) K_1(\omega_m \rho)] \sin 2\beta \sin \omega_m z \\ \vec{A}_z &= \sum_{m=1} \mu_0 J_{0z,m} R I_2(\omega_m R) K_2(\omega_m \rho) \cos 2\beta \cos \omega_m z\end{aligned}$$

The relations between the components of the vector around $(0,0) - (A_r, A_\theta, A_z)$ and the above components (A_ρ, A_β, A_z) around $(S \cos \theta_0, S \sin \theta_0)$ are,

$$\begin{aligned}A_r &= A_\rho \cos(\theta - \beta) + A_\beta \sin(\theta - \beta) \\ A_\theta &= -A_\rho \sin(\theta - \beta) + A_\beta \cos(\theta - \beta) \\ A_z &= A_z\end{aligned}$$

Figure 4 The geometry associated with the Addition Theorem — top, and components of the vector potential — bottom.

Proceeding, we shall make use of the following relationships,

The unit vector relations are:

$$\begin{aligned}\hat{e}_\rho &= -\cos(\beta - \pi)\hat{e}_x - \sin(\beta - \pi)\hat{e}_y = \cos \beta \hat{e}_x + \sin \beta \hat{e}_y \\ \hat{e}_\beta &= \sin(\beta - \pi)\hat{e}_x - \cos(\beta - \pi)\hat{e}_y = -\sin \beta \hat{e}_x + \cos \beta \hat{e}_y \\ \hat{e}_x &= \cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta \\ \hat{e}_y &= \sin \theta \hat{e}_r + \cos \theta \hat{e}_\theta\end{aligned}$$

^e “Combined Right and Left Hand Helical Function Magnets”, S.Caspi, SC-MAG-534, January 1996.

Therefore,

$$\begin{aligned}\hat{e}_\rho &= \cos(\theta - \beta)\hat{e}_r - \sin(\theta - \beta)\hat{e}_\theta \\ \hat{e}_\beta &= \sin(\theta - \beta)\hat{e}_r + \cos(\theta - \beta)\hat{e}_\theta\end{aligned}$$

or

$$\begin{aligned}\hat{e}_r &= \cos(\theta - \beta)\hat{e}_\rho + \sin(\theta - \beta)\hat{e}_\beta \\ \hat{e}_\theta &= -\sin(\theta - \beta)\hat{e}_\rho + \cos(\theta - \beta)\hat{e}_\beta\end{aligned}$$

Addition Theorem

In deriving the expansion around (0,0) we shall make use of the Addition Theorem as described in Reference^f

$$\begin{aligned}K_3(\omega_m \rho) \begin{Bmatrix} \cos 3\psi \\ \sin 3\psi \end{Bmatrix} &= \sum_{k=-\infty}^{\infty} K_{3+k}(\omega_m S) I_k(\omega_m r) \begin{Bmatrix} \cos k\phi \\ \sin k\phi \end{Bmatrix} \\ K_2(\omega_m \rho) \begin{Bmatrix} \cos 2\psi \\ \sin 2\psi \end{Bmatrix} &= \sum_{k=-\infty}^{\infty} K_{2+k}(\omega_m S) I_k(\omega_m r) \begin{Bmatrix} \cos k\phi \\ \sin k\phi \end{Bmatrix} \\ K_1(\omega_m \rho) \begin{Bmatrix} \cos \psi \\ \sin \psi \end{Bmatrix} &= \sum_{k=-\infty}^{\infty} K_{1+k}(\omega_m S) I_k(\omega_m r) \begin{Bmatrix} \cos k\phi \\ \sin k\phi \end{Bmatrix} \\ \rho &= \sqrt{S^2 + r^2 - 2Sr \cos \phi}\end{aligned}$$

for simplicity we have dropped the i,j index for the parameter S.

We proceed in the derivation of the vector potential using the above relations and,

$$\begin{aligned}\phi &= \theta - \theta_0 \\ \beta &= \pi + \theta_0 - \psi \\ \theta - \beta &= \phi + \psi - \pi \\ \cos(3\beta - \theta) &= -\cos(3\theta_0 - \theta) \cos 3\psi - \sin(3\theta_0 - \theta) \sin 3\psi \\ \sin(3\beta - \theta) &= -\sin(3\theta_0 - \theta) \cos 3\psi + \cos(3\theta_0 - \theta) \sin 3\psi \\ \cos(\beta + \theta) &= -\cos(\theta_0 + \theta) \cos \psi - \sin(\theta_0 + \theta) \sin \psi \\ \sin(\beta + \theta) &= -\sin(\theta_0 + \theta) \cos \psi + \cos(\theta_0 + \theta) \sin \psi \\ \cos 2\beta &= \cos 2\theta_0 \cos 2\psi + \sin 2\theta_0 \sin 2\psi\end{aligned}$$

$$\vec{A}_r$$

$$\vec{A}_r = A_\rho \cos(\theta - \beta) + A_\beta \sin(\theta - \beta)$$

$$\vec{A}_r = \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{4} (\omega_m R) [I_3(\omega_m R) K_3(\omega_m \rho) \cos(3\beta - \theta) - I_1(\omega_m R) K_1(\omega_m \rho) \cos(\beta + \theta)] \sin \omega_m z$$

^f "Theory of Bessel Functions", G.N.Eatson, Cambridge University Press, page 361

$$\vec{A}_r = \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{4} (\omega_m R) \sum_{k=-\infty}^{\infty} \left\{ \{-I_3(\omega_m R) K_{3+k}(\omega_m S) \cos[(k+3)\theta_0 - (k+1)\theta]\} + \right. \\ \left. + \{I_1(\omega_m R) K_{1+k}(\omega_m S) \cos[(k+1)\theta_0 - (k-1)\theta]\} \right\} I_k(\omega_m r) \sin \omega_m z$$

The above is a contribution from a single quad with its center at $Se^{-i\theta_0}$. If we wish to assume an arrangement of 8 quads with full symmetry around 0,0 and centers located at $Se^{-i(\pm\theta_0)}$, $Se^{-i(\pm\frac{\pi}{2}\pm\theta_0)}$, $Se^{-i(-\pi\pm\theta_0)}$ (quads on the symmetry line will require a weight factor of 1/2).

$$\sum_{\theta_0} \cos(k+3)\theta_0 = 2 \cos(k+3)\theta_0 \left[1 + 2 \cos \frac{(k+3)\pi}{2} + \cos(k+3)\pi \right] \\ \sum_{\theta_0} \sin(k+3)\theta_0 = 0 \\ \sum_{\theta_0} \cos(k+1)\theta_0 = 2 \cos(k+1)\theta_0 \left[1 + 2 \cos \frac{(k+1)\pi}{2} + \cos(k+1)\pi \right] \\ \sum_{\theta_0} \sin(k+1)\theta_0 = 0$$

With the help of the relations,

$$I_{4k-3} = I'_{2(2k-1)} + \frac{2(2k-1)}{\omega_m r} I_{2(2k-1)} \\ I_{4k-1} = I'_{2(2k-1)} - \frac{2(2k-1)}{\omega_m r} I_{2(2k-1)}$$

and additional algebraic manipulation and the introduction of a normalization factor,

$$\vec{A}_r = \sum_{m=1} \sum_{k=1} \frac{\mu_0 J_{0z,m} R (4k-3)!}{2 \left(\frac{\omega_m R}{2} \right)^{2(2k-1)}} \left[C_{k,m}^+ I'_{2(2k-1)}(\omega_m r) + 2(2k-1) C_{k,m}^- \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \right] \cos 2(2k-1)\theta \sin \omega_m z \\ C_{k,m}^+ = \sum_{i=1} \sum_{j=1} [K_{4(k-1)}(\omega_m S_{i,j}) \cos 4(k-1)\theta_{0i,j} + K_{4k}(\omega_m S_{i,j}) \cos 4k\theta_{0i,j}] \frac{16 I_2(\omega_m R)}{(4k-3)!} \left(\frac{\omega_m R}{2} \right)^{4k-2} \\ C_{k,m}^- = \sum_{i=1} \sum_{j=1} [K_{4(k-1)}(\omega_m S_{i,j}) \cos 4(k-1)\theta_{0i,j} - K_{4k}(\omega_m S_{i,j}) \cos 4k\theta_{0i,j}] \frac{16 I_2'(\omega_m R)}{(4k-3)!} \left(\frac{\omega_m R}{2} \right)^{4k-1}$$

The summation in i,j is carried out over all coils in the first octant.

$$\vec{A}_\theta$$

$$\vec{A}_\theta = -A_\rho \sin(\theta - \beta) + A_\beta \cos(\theta - \beta)$$

$$\vec{A}_\theta = \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{4} (\omega_m R) [I_3(\omega_m R) K_3(\omega_m \rho) \sin(3\beta - \theta) + I_1(\omega_m R) K_1(\omega_m \rho) \sin(\beta + \theta)] \sin \omega_m z$$

$$\vec{A}_\theta = \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{4} (\omega_m R) \sum_{k=-\infty}^{\infty} \left\{ I_3(\omega_m R) K_{3+k}(\omega_m S) \sin [(k+1)\theta - (k+3)\theta_0] + \right. \\ \left. + I_1(\omega_m R) K_{1+k}(\omega_m S) \sin [(k-1)\theta - (k+1)\theta_0] \right\} I_k(\omega_m r) \sin \omega_m z$$

The above expression is for the contribution from a single quad with its center at $Se^{-i\theta_0}$. As before if we now include contributions from 8 quads placed with full symmetry, we shall need to add the vector potential of coils with their center at $Se^{-i(\pm\theta_0)}$, $Se^{-i(\pm\frac{\pi}{2}\pm\theta_0)}$, $Se^{-i(-\pi\pm\theta_0)}$. Making use of the trigonometric relations as shown for A_r

$$\vec{A}_\theta = - \sum_{m=1} \sum_{k=1} \frac{\mu_0 J_{0z,m} R (4k-3)!}{2 \left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} \left[C_{k,m}^- I'_{2(2k-1)}(\omega_m r) + 2(2k-1) C_{k,m}^+ \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \right] \sin 2(2k-1)\theta \sin \omega_m z$$

$$\vec{A}_z$$

$$\vec{A}_z = \sum_{m=1} \mu_0 J_{0z,m} R I_2(\omega_m R) K_2(\omega_m \rho) \cos 2\beta \cos \omega_m z$$

$$K_2(\omega_m \rho) \cos 2\beta = \sum_{k=-\infty}^{\infty} K_{2+k}(\omega_m S) I_k(\omega_m r) [\cos 2\theta_0 \cos k(\theta - \theta_0) + \sin 2\theta_0 \sin k(\theta - \theta_0)]$$

The above expression is for the contribution from a single quad with its center at $Se^{-i\theta_0}$. As before if we now include contributions from 8 quads and make use of the relations as shown for A_r ,

$$\sum_{\theta_0} \cos 2\theta_0 \cos k(\theta - \theta_0) = 2 \cos 2\theta_0 \cos k\theta_0 \cos k\theta \left[1 + (-1)^k - 2 \cos \frac{k\pi}{2} \right]$$

$$\sum_{\theta_0} \sin 2\theta_0 \sin k(\theta - \theta_0) = -2 \sin 2\theta_0 \sin k\theta_0 \cos k\theta \left[1 + (-1)^k - 2 \cos \frac{k\pi}{2} \right]$$

$$K_2(\omega_m \rho) \cos 2\beta = 2 \sum_{k=-\infty}^{\infty} K_{2+k}(\omega_m S) I_k(\omega_m r) \cos (k+2)\theta_0 \left[1 + (-1)^k - 2 \cos \frac{k\pi}{2} \right] \cos k\theta$$

$$\vec{A}_z = \sum_{m=1} \sum_{k=1} \frac{\mu_0 J_{0z,m} R (4k-3)!}{2 \left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} C_{k,m}^+ I_{2(2k-1)}(\omega_m r) \cos 2(2k-1)\theta \cos \omega_m z$$

Check $\nabla \cdot A = 0$

A way of checking the results is to assure the divergence is zero anywhere in space.

$$\nabla \vec{A} = \frac{1}{r} \frac{\partial(rA_r)}{\partial r} + \frac{\partial A_\theta}{r \partial \theta} + \frac{\partial A_z}{\partial z} = 0$$

$$\frac{\partial(rA_r)}{\partial r} = \sum_{m=1} \sum_{k=1} \frac{\mu_0 J_{0z,m} R (4k-3)!}{2 \left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} \left\{ C_{k,m}^+ \left[I'_{2(2k-1)}(\omega_m r) + \omega_m r I''_{2(2k-1)}(\omega_m r) \right] + \right. \\ \left. + 2(2k-1) C_{k,m}^- I'_{2(2k-1)}(\omega_m r) \right\} \cos 2(2k-1)\theta \sin \omega_m z$$

$$\frac{\partial A_\theta}{\partial \theta} = - \sum_{m=1} \sum_{k=1} (2k-1) \frac{\mu_0 J_{0z,m} R (4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} \left[C_{k,m}^- I'_{2(2k-1)}(\omega_m r) + \right. \\ \left. + 2(2k-1) C_{k,m}^+ \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \right] \cos 2(2k-1)\theta \sin \omega_m z$$

$$\frac{\partial A_z}{\partial z} = - \sum_{m=1} \sum_{k=1} \frac{\mu_0 J_{0z,m} (\omega_m R) (4k-3)!}{2 \left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} C_{k,m}^+ I_{2(2k-1)}(\omega_m r) \cos 2(2k-1)\theta \sin \omega_m z$$

$$\nabla \vec{A} \propto \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} \left\{ - \frac{C_{k,m}^+}{\omega_m r} \left[I'_{2(2k-1)}(\omega_m r) + \omega_m r I''_{2(2k-1)}(\omega_m r) - \right. \right. \\ \left. \left. - [2(2k-1)]^2 \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \right] + C_{k,m}^+ I_{2(2k-1)}(\omega_m r) \right\} \cos 2(2k-1)\theta \sin \omega_m z$$

and since

$$\frac{1}{\omega_m r} \left[I'_{2(2k-1)}(\omega_m r) + \omega_m r I''_{2(2k-1)}(\omega_m r) - [2(2k-1)]^2 \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \right] = I_{2(2k-1)}(\omega_m r)$$

the divergence

$$\nabla \vec{A} = - \sum_m \mu_0 J_{0z,m} \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{4k-3}} \left\{ - C_{k,m}^+ I_{2(2k-1)}(\omega_m r) + \right. \\ \left. + C_{k,m}^+ I_{2(2k-1)}(\omega_m r) \right\} \cos 2(2k-1)\theta \sin \omega_m z = 0!$$

as it should be.

The Magnetic Field Components.

The r component of $\vec{B}$

From the vector potential we derive the radial magnetic field component in the region $r \leq R$,

$$B_r = (\nabla \times \vec{A})_r = \frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}$$

$$\frac{\partial A_z}{\partial \theta} = - \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{2} \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} 2(2k-1) C_{k,m}^+ I_{2(2k-1)}(\omega_m r) \sin 2(2k-1)\theta \cos \omega_m z$$

$$\frac{\partial A_\theta}{\partial z} = - \sum_{m=1} \frac{\mu_0 J_{0z,m} (\omega_m R)}{2} \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{2(2k-1)}} \left[C_{k,m}^- I'_{2(2k-1)}(\omega_m r) + \right. \\ \left. + 2(2k-1) C_{k,m}^+ \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \right] \sin 2(2k-1)\theta \cos \omega_m z$$

and B_r reduces to,

$$B_r = \sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \left(\frac{2}{\omega_m R} \right)^{4k-3} (4k-3)! C_{k,m}^- I_{2(2k-1)}'(\omega_m r) \sin 2(2k-1)\theta \cos \omega_m z$$

We need to add the self field contribution from the coil at $i=j=0$. For $r \leq R$ we add,

$$B_r = \sum_{m=1} \frac{2\mu_0 J_{0z,m}}{(\omega_m R)} I_2'(\omega_m r) \sin 2\theta \cos \omega_m z$$

and for $r \geq R$ we add,

$$B_r = \sum_{m=1} \frac{2\mu_0 J_{0z,m}}{(\omega_m R)} \frac{I_2'(\omega_m R)}{K_2'(\omega_m R)} K_2'(\omega_m r) \sin 2\theta \cos \omega_m z$$

The θ component of $\vec{B}$

$$B_\theta = (\nabla \times \vec{A})_\theta = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$\frac{\partial A_r}{\partial z} = \sum_{m=1} \mu_0 J_{0z,m} \left(\frac{\omega_m R}{2} \right) \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2} \right)^{2(2k-1)}} \left[C_{k,m}^+ I_{2(2k-1)}'(\omega_m r) + 2(2k-1) C_{k,m}^- \frac{I_{2(2k-1)}}{\omega_m r} \right] \cos 2(2k-1)\theta \cos \omega_m z$$

$$\frac{\partial A_z}{\partial r} = \sum_{m=1} \mu_0 J_{0z,m} \left(\frac{\omega_m R}{2} \right) \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2} \right)^{2(2k-1)}} C_{k,m}^+ I_{2(2k-1)}'(\omega_m r) \cos 2(2k-1)\theta \cos \omega_m z$$

$$B_\theta = \sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \frac{2(2k-1)(4k-3)!}{\left(\frac{\omega_m R}{2} \right)^{4k-3}} C_{k,m}^- \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \cos 2(2k-1)\theta \cos \omega_m z$$

In the region $r \leq R$ we add the self field,

$$B_\theta = \sum_{m=1} \frac{4\mu_0 J_{0z,m}}{(\omega_m R)} \frac{I_2(\omega_m r)}{\omega_m r} \cos 2\theta \cos \omega_m z$$

and similarly for $r \geq R$ we add,

$$B_\theta = \sum_{m=1} \frac{4\mu_0 J_{0z,m}}{(\omega_m R)} \frac{I_2'(\omega_m R)}{K_2'(\omega_m R)} \frac{K_2(\omega_m r)}{\omega_m r} \cos 2\theta \cos \omega_m z$$

The z component of $\vec{B}$

$r \leq R$

$$B_z = (\nabla \times \vec{A})_z = \frac{1}{r} \left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right]$$

$$\begin{aligned} \frac{\partial(rA_\theta)}{\partial r} &= - \sum_{m=1} \frac{\mu_0 J_{0z,m}}{\omega_m} \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{4k-3}} \left\{ C_{k,m}^- \left[I_{2(2k-1)}'(\omega_m r) + \omega_m r I_{2(2k-1)}''(\omega_m r) \right] + \right. \\ &\quad \left. + 2(2k-1) C_{k,m}^+ I_{2(2k-1)}'(\omega_m r) \right\} \sin 2(2k-1)\theta \sin \omega_m z \\ \frac{\partial A_r}{\partial \theta} &= - \sum_{m=1} \frac{\mu_0 J_{0z,m}}{\omega_m} \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{4k-3}} \left\{ 2(2k-1) C_{k,m}^+ I_{2(2k-1)}'(\omega_m r) + \right. \\ &\quad \left. + [2(2k-1)]^2 C_{k,m}^- \frac{I_{2(2k-1)}'(\omega_m r)}{\omega_m r} \right\} \sin 2(2k-1)\theta \sin \omega_m z \end{aligned}$$

but

$$\frac{1}{\omega_m r} \left[I_{2(2k-1)}'(\omega_m r) + \omega_m r I_{2(2k-1)}''(\omega_m r) - [2(2k-1)]^2 \frac{I_{2(2k-1)}'(\omega_m r)}{\omega_m r} \right] = I_{2(2k-1)}(\omega_m r)$$

Therefore :

$$B_z = - \sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \left(\frac{2}{\omega_m R} \right)^{4k-3} (4k-3)! C_{k,m}^- I_{2(2k-1)}(\omega_m r) \sin 2(2k-1)\theta \sin \omega_m z$$

in the region $r \leq R$ we supplement the field with the self field,

$$B_z = - \sum_{m=1} \frac{2\mu_0 J_{0z,m}}{(\omega_m R)} I_2(\omega_m r) \sin 2\theta \sin \omega_m z$$

and in the outer region $r \geq R$ we add

$$B_z = - \sum_{m=1} \frac{2\mu_0 J_{0z,m}}{(\omega_m R)} \frac{I_2'(\omega_m R)}{K_2'(\omega_m R)} K_2(\omega_m r) \sin 2\theta \sin \omega_m z$$

Check $\nabla \cdot B = 0$

As was done for vector potential we make sure the divergence of B is zero,

$$\nabla \vec{B} = \frac{1}{r} \frac{\partial(rB_r)}{\partial r} + \frac{\partial B_\theta}{r \partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\begin{aligned} \frac{\partial(rB_r)}{\partial r} &= \sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{4k-3}} C_{k,m}^- \left[I'_{2(2k-1)}(\omega_m r) + \omega_m r I''_{2(2k-1)}(\omega_m r) \right] \sin 2(2k-1)\theta \cos \omega_m z \\ \frac{\partial B_\theta}{\partial \theta} &= - \sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{4k-3}} [2(2k-1)]^2 C_{k,m}^- \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \sin 2(2k-1)\theta \cos \omega_m z \\ \frac{\partial B_z}{\partial z} &= - \sum_{m=1} \mu_0 J_{0z,m} \omega_m \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{4k-3}} C_{k,m}^- I_{2(2k-1)}(\omega_m r) \sin 2(2k-1)\theta \cos \omega_m z \\ \nabla \vec{B} &\propto \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{4k-3}} \left\{ - \frac{C_{k,m}^-}{\omega_m r} \left[I'_{2(2k-1)}(\omega_m r) + \omega_m r I''_{2(2k-1)}(\omega_m r) - [2(2k-1)]^2 \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \right] + C_{k,m}^- I_{2(2k-1)}(\omega_m r) \right\} \sin 2(2k-1)\theta \cos \omega_m z \end{aligned}$$

and since

$$\frac{1}{\omega_m r} \left[I'_{2(2k-1)}(\omega_m r) + \omega_m r I''_{2(2k-1)}(\omega_m r) - [2(2k-1)]^2 \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \right] = I_{2(2k-1)}(\omega_m r)$$

$$\nabla \vec{B} = - \sum_m \mu_0 J_{0z,m} \omega_m \sum_{k=1} \frac{(4k-3)!}{\left(\frac{\omega_m R}{2}\right)^{4k-3}} \left[- C_{k,m}^- I_{2(2k-1)}(\omega_m r) + C_{k,m}^- I_{2(2k-1)}(\omega_m r) \right] \sin 2(2k-1)\theta \cos \omega_m z = 0!$$

The stored energy

In calculating the stored energy we start from,

$$E = \frac{1}{2} \int \int \int \vec{J} \cdot \vec{A} dv$$

and integrate the product on the surface current only over a full period⁸ :

$$E = \frac{1}{2} \int_0^{2\pi} \int_{-L}^L \vec{J} \cdot \vec{A} d\sigma = \frac{1}{2} \int_0^{2\pi} \int_{-L}^L \vec{J} \cdot \vec{A} R d\theta dz$$

(the current density is per unit length and the unit of energy is $J = T \cdot A \cdot m^2$).

⁸ "Forces and Stored Energy in Thin Cosine($n\theta$) Accelerator magnets." S.Caspi, SC-MAG-546, LBL-38500, March 1996.

We shall calculate the contributions arising from all external coil arrays followed by the self coil contribution. The product of the current density and the vector potential is :

$$\vec{J} \cdot \vec{A}|_{r=R} = J_\theta A_\theta + J_z A_z$$

In performing the integration we shall make use of the following orthogonality relations:

$$\begin{aligned} \int_0^{2\pi} \sin 2\theta \sin 2(2k-1)\theta d\theta &= \begin{cases} 0 & k \neq 1 \\ \pi & k = 1 \end{cases} \\ \int_0^{2\pi} \cos 2\theta \cos 2(2k-1)\theta d\theta &= \begin{cases} 0 & k \neq 1 \\ \pi & k = 1 \end{cases} \\ \int_{-L}^L \sin \omega_m z \sin \omega_j z dz &= \begin{cases} 0 & m \neq j \\ L & m = j \end{cases} \\ \int_{-L}^L \cos \omega_m z \cos \omega_j z dz &= \begin{cases} 0 & m \neq j \\ L & m = j \end{cases} \end{aligned}$$

As a result the stored energy is reduced to,

$$E_{ext.} = \frac{1}{2} \int_0^{2\pi} \int_{-L}^L J \cdot A R d\theta dz = \begin{cases} -\frac{\pi R^2 L \mu_0}{2} \sum_{m=1} \frac{J_{0z,m}^2}{\omega_m R} \left[C_{1,m}^- I_2'(\omega_m R) + 2C_{1,m}^+ \frac{I_2(\omega_m R)}{\omega_m R} \right] + \\ + \pi R^2 L \mu_0 \sum_{m=1} \frac{J_{0z,m}^2}{(\omega_m R)^2} C_{1,m}^+ I_2(\omega_m R) \end{cases}$$

or,

$$E_{ext.} = -\frac{\pi R^2 L \mu_0}{2} \sum_{m=1} \frac{J_{0z,m}^2}{\omega_m R} C_{1,m}^- I_2'(\omega_m R)$$

where,

$$C_{1,m}^- = \sum_{i=1} \sum_{j=1} [K_0(\omega_m S_{i,j}) - K_4(\omega_m S_{i,j}) \cos 4\theta_{0i,j}] 16 I_2'(\omega_m R) \left(\frac{\omega_m R}{2} \right)^3$$

the stored energy of the self coil is :

$$E_{self} = -\frac{\pi R^2 L \mu_0}{8} \sum_m J_{0z,m}^2 (\omega_m R)^2 K_2'(\omega_m R) I_2'(\omega_m R)$$

and the total energy is,

$$E_{total} = -\frac{\pi R^2 L \mu_0}{8} \sum_{m=1} J_{0z,m}^2 (\omega_m R)^2 K_2'(\omega_m R) I_2'(\omega_m R) \left[1 + \frac{4C_{1,m}^-}{(\omega_m R)^3 K_2'(\omega_m R)} \right]$$

Simulation of Current density and flow lines

To generate flow lines we make use of a technic first demonstrated by J.Laslett and W. Fawley of this laboratory. The character of the flow lines for a single function magnet n, will follow from the differential equation,

$$\frac{Rd\theta}{dz} = \frac{J_\theta}{J_z}$$

Assuming the current density for magnet type n=2 as,

$$\vec{J} = \sum_{m=1} J_{0z,m} \left[\left(\frac{\omega_m R}{2} \right) \sin 2\theta \sin \omega_m z \hat{e}_\theta + \cos 2\theta \cos \omega_m z \hat{e}_z \right]$$

it follows that,

$$\frac{Rd\theta}{dz} = \frac{\sum_{m=1} J_{0z,m} (\omega_m R) \sin 2\theta \sin \omega_m z}{2 \sum_{m=1} J_{0z,m} \cos 2\theta \cos \omega_m z}$$

and

$$\frac{\cos 2\theta}{\sin 2\theta} d\theta - \frac{\sum_{m=1} J_{0z,m} (\omega_m R) \sin \omega_m z}{2R \sum_{m=1} J_{0z,m} \cos \omega_m z} dz = 0$$

or,

$$\ln(\sin 2\theta) + \ln \left(2 \sum_{m=1} J_{0z,m} \cos \omega_m z \right) = \text{const.}$$

So that,

$$\sin 2\theta = \frac{\text{const.}}{2 \sum_{m=1} J_{0z,m} \cos \omega_m z}$$

and the flow lines are therefore,

$$\sin 2\theta = \frac{\sum_{m=1} J_{0z,m}}{\sum_{m=1} J_{0z,m} \cos \omega_m z} \sin 2\theta_0$$

where θ_0 denotes the value of θ at $z=0$.

In a special case, we may choose special values for $J_{0z,m}$ such that,

$$J_{0z,m} = \frac{J_{02}}{2^{2(M-1)}} \binom{2M-1}{M-m} = J_{02} \frac{1}{2^{2(M-1)}} \frac{(2M-1)!}{(M+m-1)!(M-m)!}$$

where M is the number of m terms used in a particular case and J_{02} is a constant.

We note that in this particular case

$$\begin{aligned} \frac{1}{2^{2(M-1)}} \sum_{m=1}^M \frac{(2M-1)!}{(M+m-1)!(M-m)!} \cos \omega_m z &= \cos^{2M-1} \omega_1 z \\ \frac{1}{2^{2(M-1)}} \sum_{m=1}^M \frac{(2M-1)!}{(M+m-1)!(M-m)!} &= 1 \end{aligned}$$

and therefore the current density is,

$$\sum_{m=1} J_{02,m} \cos \omega_m z = J_{02} \sum_{m=1}^M \frac{1}{2^{2(M-1)}} \frac{(2M-1)!}{(M+m-1)!(M-m)!} \cos \omega_m z = J_{02} \cos^{2M-1} \omega_1 z$$

With that, the flow lines reduce to the simple expression,

$$\sin 2\theta = \frac{1}{\cos^{2M-1} \omega_1 z} \sin 2\theta_0$$

$$\omega_1 = \frac{\pi}{L}$$

and the components of current density are,

$$\vec{J}(\theta, z)|_{r=R} = J_{02} \left\{ \begin{array}{l} 0\hat{e}_r \\ \frac{\pi R}{2L} (2M-1) \cos^{2(M-1)} \frac{\pi z}{L} \sin \frac{\pi z}{L} \sin 2\theta \hat{e}_\theta \\ \cos^{2M-1} \frac{\pi z}{L} \cos 2\theta \hat{e}_z \end{array} \right\}$$

m=1

$$J_\theta = \frac{\pi R}{2L} \sin \frac{\pi z}{L} \sin 2\theta$$

$$J_z = \cos \frac{\pi z}{L} \cos 2\theta$$

$$\text{Flow lines : } \sin 2\theta = \frac{\sin 2\theta_0}{\cos \frac{\pi z}{L}}$$

m=1 and m=2

$$J_\theta = 3 \frac{\pi R}{2L} \left(\cos \frac{\pi z}{L} \right)^2 \sin \frac{\pi z}{L} \sin 2\theta$$

$$J_z = \left(\cos \frac{\pi z}{L} \right)^3 \cos 2\theta$$

$$\text{Flow lines : } \sin 2\theta = \frac{\sin 2\theta_0}{\left(\cos \frac{\pi z}{L} \right)^3}$$

m=1, m=2 and m=3

$$J_{\theta} = 5 \frac{\pi R}{2L} \left(\cos \frac{\pi z}{L} \right)^4 \sin \frac{\pi z}{L} \sin 2\theta$$

$$J_z = \left(\cos \frac{\pi z}{L} \right)^5 \cos 2\theta$$

$$\text{Flow lines : } \sin 2\theta = \frac{\sin 2\theta_0}{\left(\cos \frac{\pi z}{L} \right)^5}$$

Figure 5 View of a full period array of a quad with $m=1$ only, a summation over $m=1,2$, and $m=1,2,3$. These special cases reveal the reduction in crowding between magnets at the expense of an increased non-linear field.

The limiting 2 dimensional case

As a farther simplification and check, we reduce the results obtained for the 3D array by extending the periodicity to infinity, $\lim_{L \rightarrow \infty} \omega_m = 0$, and compare those with more familiar 2D cases of multipole magnets as cited in the abstract.

$$\begin{aligned}
 & s \rightarrow 0 \\
 & L \rightarrow \infty \\
 & \left\{ \begin{array}{ll} \lim_{s \rightarrow 0} I_2(s) \rightarrow \frac{s^2}{8} & \lim_{s \rightarrow 0} I'_2(s) \rightarrow \frac{s}{4} \\ \lim_{s \rightarrow 0} K_2(s) \rightarrow \frac{2}{s^2} & \lim_{s \rightarrow 0} K'_2(s) \rightarrow -\frac{4}{s^3} \\ \lim_{s \rightarrow 0} \frac{I'_2(s)}{K'_2(s)} \rightarrow -\frac{s^4}{16} & \\ \lim_{s \rightarrow 0} I_{2(2k-1)}(s) \rightarrow \frac{1}{[2(2k-1)]!} \left(\frac{s}{2}\right)^{2(2k-1)} & \lim_{s \rightarrow 0} I'_{2(2k-1)}(s) \rightarrow \frac{1}{2(4k-3)!} \left(\frac{s}{2}\right)^{4k-3} \\ \lim_{s \rightarrow 0} K_{4(k-1)}(s) \rightarrow \frac{(4k-5)!}{2} \left(\frac{s}{2}\right)^{-4(k-1)} & \lim_{s \rightarrow 0} K_{4k}(s) \rightarrow \frac{(4k-1)!}{2} \left(\frac{s}{2}\right)^{-4k} \end{array} \right. \\
 & \vec{J}_z = J_0 \cos 2\theta ; \quad J_0 = \frac{2RG_{2,m}}{\mu_0}
 \end{aligned}$$

The 3D vector-potential reduces to the 2D expression (including self coil):

$$\vec{A}_r = 0$$

$$\vec{A}_\theta = 0$$

$$\vec{A}_z = \sum_m \frac{\mu_0 J_{0z,m}}{4} \left(\frac{r}{R}\right)^2 \cos 2\theta + 2 \sum_{m=1} \mu_0 J_{0z,m} R \sum_{k=1} (4k-1) \cos 4k\theta_{0i,j} \left(\frac{R}{S_{i,j}}\right)^{4k} \left(\frac{r}{R}\right)^{4k-2} \cos 2(2k-1)\theta$$

where the first term corresponds to the self field contribution from coil $i=j=0$. The field-not including the self field, is :

$$\begin{aligned}
 \vec{B}_r &= -4 \sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} (2k-1)(4k-1) \cos 4k\theta_{0i,j} \left(\frac{R}{S_{i,j}}\right)^{4k} \left(\frac{r}{R}\right)^{4k-3} \sin 2(2k-1)\theta \\
 \vec{B}_\theta &= -4 \sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} (2k-1)(4k-1) \cos 4k\theta_{0i,j} \left(\frac{R}{S_{i,j}}\right)^{4k} \left(\frac{r}{R}\right)^{4k-3} \cos 2(2k-1)\theta \\
 \vec{B}_z &= 0
 \end{aligned}$$

in agreement with the references cited in the abstract. The self field is :

$$\begin{aligned}
 \vec{B}_r &= - \sum_{m=1} \frac{\mu_0 J_{0z,m}}{2} \left(\frac{r}{R}\right) \sin 2\theta \\
 \vec{B}_\theta &= - \sum_{m=1} \frac{\mu_0 J_{0z,m}}{2} \left(\frac{r}{R}\right) \cos 2\theta \\
 \vec{B}_z &= 0
 \end{aligned}$$

The array 2D multipole coefficients are equal to their contribution to the stored energy.

$$E_{total}^{2D} = \frac{\pi R^2 L \mu_0}{8} \sum_m J_0^2 (1 - C_m^{-2D}) = \frac{\pi R^2 L \mu_0}{8} \sum_m J_0^2 \sum_i \sum_j \left[1 + 24 \left(\frac{R}{S_{i,j}}\right)^4 \cos 4\theta_{0i,j} \right]$$

Appendix A Region outside the coil $r \geq R$

The contribution of the self field (centered at 0,0) in the region outside the current sheet is listed below,

Contributions from the coil at $i=0, j=0$ in the region $r \geq R$

$$\vec{A}_r = \sum_{m=1} \frac{\mu_0 J_{0z,m}}{4} (\omega_m R) [I_3(\omega_m R) K_3(\omega_m r) - I_1(\omega_m R) K_1(\omega_m r)] \cos 2\theta \sin \omega_m z$$

$$\vec{A}_\theta = \sum_{m=1} \frac{\mu_0 J_{0z,m}}{4} (\omega_m R) [I_3(\omega_m R) K_3(\omega_m r) + I_1(\omega_m R) K_1(\omega_m r)] \sin 2\theta \sin \omega_m z$$

$$\vec{A}_z = \sum_{m=1} \mu_0 J_{0z,m} R I_2(\omega_m R) K_2(\omega_m r) \cos 2\theta \cos \omega_m z$$

$$B_r = \sum_{m=1} \frac{J_{0z,m} \mu_0}{2} (\omega_m R)^2 I_2'(\omega_m R) K_2'(\omega_m r) \sin 2\theta \cos \omega_m z$$

$$B_\theta = \sum_{m=1} \frac{J_{0z,m} \mu_0}{2} (\omega_m R)^2 I_2'(\omega_m R) \frac{2K_2(\omega_m r)}{\omega_m r} \cos 2\theta \cos \omega_m z$$

$$B_z = - \sum_{m=1} \frac{J_{0z,m} \mu_0}{2} (\omega_m R)^2 I_2'(\omega_m R) K_2(\omega_m r) \sin 2\theta \sin \omega_m z$$

$$G_{2,m} = - \frac{J_{0z,m} \mu_0}{8R} (\omega_m R)^3 K_2'(\omega_m R)$$

Appendix B 3 Dimensional Coefficients

Figure 6 The values associated with the multipole coefficients for $m=1$ and the quad term $k=1$. Values at $\pi \cdot R/L = 0$ correspond to the 2 dimensional case.

Figure 7 The values associated with the multipole coefficients for $m=1$ and the dodecapole term $k=2$. Values at $\pi R/L = 0$ correspond to the 2 dimensional case.

Figure 8 The values associated with the multipole coefficients for $m=1$ and the 20's pole $k=3$. Values at $\pi R/L = 0$ correspond to the 2 dimensional case.

Figure 9 Ratio of stored energy contributed by external coils for $m=1$ and the quad term $k=1$. Values at $\pi \cdot R/L = 0$ correspond to the 2 dimensional case.

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