Title

The Geometry of Splitting Loci

ISBN

9798189279980

Author

Lin, Feiyang

Publication Date

2026-05-01

Peer reviewed|Thesis/dissertation

The Geometry of Splitting Loci

by

Feiyang Lin

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor David Eisenbud, Co-chair
Assistant Professor Hannah Kerner Larson, Co-chair
Professor Martin Olsson

Spring 2026

The Geometry of Splitting Loci

Abstract

The Geometry of Splitting Loci

by

Feiyang Lin

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor David Eisenbud, Co-chair

Assistant Professor Hannah Kerner Larson, Co-chair

Splitting loci are certain degeneracy loci that arise from the data of a family of vector bundles on $\mathbb{P}^1$. They can be viewed as closed substacks of the stack of vector bundles on $\mathbb{P}^1$, and are closely related to the Brill-Noether theory of general k-gonal curves, as well as opposite Schubert varieties in the affine Grassmannian for the general linear group. We construct a modular resolution of singularities for splitting loci using a relative flag Quot construction, and use this resolution to study their singularities. We identify the singular locus of $\overline{\mathbf{\Sigma}}_{\vec{e}}$, prove that tame splitting loci have rational singularities, and characterize exactly which splitting loci are Gorenstein or $\mathbb{Q}$-Gorenstein. The proof for rational singularities requires a cohomology vanishing result for tautological bundles on certain Quot schemes $\mathrm{Quot}^{r;d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N})$, which we push further into a Borel-Weil-Bott type theorem.

Contents

List of Figures

Acknowledgments

I could not have written this thesis without the support of many people.

Words cannot capture the gratitude I feel for my advisors David and Hannah. I feel very lucky that my time at Berkeley has been a happy one, and it would not have been if not for David and Hannah's continued mentorship and unwavering support.

I owe much of my algebraic geometry education to David. In particular, I would not have chosen my current research path if not for his curves course in my first year. In the years leading up to my qualifying exam, he answered what feels now like infinitely many annoying questions about various trivialities as I worked through his curves book and his commutative algebra book. I am perhaps a somewhat atypical Eisenbuddy - I could not tell you what the N_p property is, and am perhaps in the middle of a slow drift away from the Zariski topology. Regardless, David has been so generous with his time and advice, and lent a sympathetic ear to my research woes.

Among all the different things I want to thank Hannah for, I am most grateful that she has consistently been the one person that I can trust to be interested in my mathematics. (At least until I started thinking about derived categories recently!) She paid attention to what I had to say indiscriminately, even when I was in the trenches trying to prove involved combinatorial lemmas. She listened to my talks with attentiveness and only pointed out mistakes afterwards rather than interrupting. She invited me to hang out with her math friends who visited Berkeley to talk with her, treating me like a mathematical equal. I can tell that she held her mentorship to extremely high standards, and I am so lucky to have her as my advisor.

I am very thankful for all the people who constituted my community at Berkeley Math. In no particular order, many thanks to Ruoxi, Yuji, Alan, Swapnil, Forte, Franny, Yuki, and many others. I also feel extremely lucky to have found a community outside the math department, whose support has been so essential to my well-being. Rae, Victor, Chi, Jeffrey, Keshav, Mudit, thank you for your company, for the adventures that made the bay area feel like home, for making my world bigger.

Thank you to all of my FAB and PQ friends. This thesis would not have been possible without all the endorphins we produce together. Thank you for the quality time carpools, for the strat meetings where we rack our brains and plan for the future, for trying hard and not giving up, and in LP's words, for being a place to come to (specifically at 6pm every Tuesday). Thank you for all the intentional reflections about our sport and our team, and the energy you find to not only play well yourselves, but also lift up others.

I feel an enormous amount of gratitude to every single person who expressed interest in my research, perhaps by asking about it during a casual conversation, by stopping by my poster at a conference, or by coming to my talk. Thank you to my collaborators Shubham and Ajay for being confused with me about Schur functors while we were 9 hours apart. Thank you also to everyone who generously took time to answer my questions, including Martin Olsson, Dave Anderson, Frank-Olaf Schreyer, Claudiu Raicu, and Alina Marian. Thank you to Rita and Margarida for teaching me about surfaces, and Eric for introducing me to normal bundles.

Lastly, thank you to my family for being my harbor.

Chapter 1

Introduction

1.1 What are splitting loci?

Every vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles. This data can be packaged into a splitting type, which is a tuple $\vec{e} = (e_1, \ldots, e_r)$, where $e_1 \leq \cdots \leq e_r$ are integers. We say that $\vec{e}$ has rank r and degree $d = e_1 + \cdots + e_r$, and write $\mathcal{O}(\vec{e})$ to denote the vector bundle $\mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_r)$ on $\mathbb{P}^1$. Given a family of vector bundles on $\mathbb{P}^1$, which is a vector bundle $\mathcal{E}$ on $B \times \mathbb{P}^1$ for some base B, the points $b \in B$ can be decorated with the splitting type of the restriction $\mathcal{E}_b \cong \mathcal{O}(\vec{e}_b)$ (Figure 1.1).

Such a vector bundle $\mathcal{E}$ is flat over B, and as such, the degree and rank of $\vec{e}_b$ do not vary as we range over all points $b \in B$ as long as B is connected. However, perhaps somewhat surprisingly, the splitting type, which is a discrete invariant, could jump. There is a partial order on the set of all splitting types of fixed rank and degree. We say that $\vec{e}' \leq \vec{e}$ if $e'_1 + \cdots + e'_k \leq e_1 + \cdots + e_k$ for all $1 \leq k \leq r$. One can show that $\vec{e}' \leq \vec{e}$ if and only if there exists a family $\mathcal{E}$ over $B = \mathbb{A}^1$ such that $\vec{e}_0 = \vec{e}'$ and $\vec{e}_b = \vec{e}$ for all $b \neq 0$. When $\vec{e}' < \vec{e}$, we say that $\vec{e}'$ is more unbalanced, or more special, than $\vec{e}$. Splitting loci $\overline{\Sigma}_{\vec{e}}$ are closed subschemes

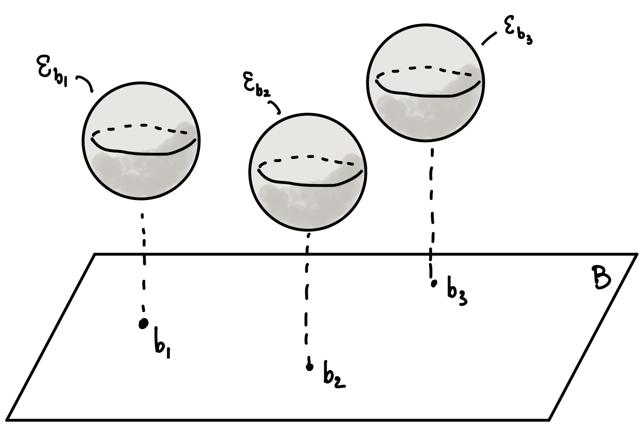


Figure 1.1: The restriction of $\mathcal{E}$ to different points of B

of the base B whose points consist of those $b \in B$ where $\vec{e}_b \leq \vec{e}$.

The closed subscheme structure on $\overline{\Sigma}_{\vec{e}}$ is defined using Fitting ideals for $R^1\pi_*\mathcal{E}(m)$ where $\pi : B \times \mathbb{P}^1 \to B$ is the projection. Roughly speaking, it imposes the rank condition that $h^1\mathcal{E}_b(m) \geq h^1\mathcal{O}(\vec{e})(m)$ for all $m \in \mathbb{Z}$. This is explained in more detail in Section 2.2. Since this closed subscheme structure is compatible with base change, in fact we have defined universal splitting loci $\overline{\mathbf{\Sigma}}_{\vec{e}}$ as closed substacks of the stack of vector bundles on $\mathbb{P}^1$. We denote by $\mathcal{B}_{r,d}$[1] the stack of vector bundles of rank r and degree d on $\mathbb{P}^1$ over a field k. This is a smooth algebraic stack, but it is in some ways quite a strange geometric object. Its closed points are in bijection with the poset of splitting types, and the residual gerbe at the closed point corresponding to the splitting type $\vec{e}$ is exactly $\mathbf{\Sigma}_{\vec{e}} = B(\mathrm{Aut}(\mathcal{O}(\vec{e})))$, which is an open substack of $\overline{\mathbf{\Sigma}}_{\vec{e}}$. The normal bundle of $\mathbf{\Sigma}_{\vec{e}}$ in $\mathcal{B}_{r,d}$ is the $\mathrm{Aut}(\mathcal{O}(\vec{e}))$-representation $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}))$, which is also the space of first order deformations of the vector bundle $\mathcal{O}(\vec{e})$. In particular, the codimension of $\overline{\mathbf{\Sigma}}_{\vec{e}}$ in $\mathcal{B}_{r,d}$ is exactly $u(\vec{e}) = h^1\mathcal{E}nd(\mathcal{O}(\vec{e}))$. Note that this computes the dimension of $\mathcal{B}_{r,d}$ to be

$$\dim \mathbf{\Sigma}_{\vec{e}} + \operatorname{codim} \mathbf{\Sigma}_{\vec{e}} = -h^0\mathcal{E}nd(\mathcal{O}(\vec{e})) + h^1\mathcal{E}nd(\mathcal{O}(\vec{e})) = -\chi(\mathcal{E}nd(\mathcal{O}(\vec{e}))) = -r^2.$$

As $\vec{e}$ gets more unbalanced, $\mathrm{Aut}(\mathcal{O}(\vec{e}))$ grows in dimension, and $\mathbf{\Sigma}_{\vec{e}}$ becomes stackier. Moreover, as splitting types can be arbitrarily unbalanced, this stack is not of finite type. See Figure 1.2 for an illustration of $\mathcal{B}_{3,0}$.

From the point of view of this stratification, the global geometry of $\overline{\mathbf{\Sigma}}_{\vec{e}}$ can seem mysterious. In fact, while $\mathcal{B}_{r,d}$ is smooth and each $\mathbf{\Sigma}_{\vec{e}'}$ is smooth, upon taking the closure, $\overline{\mathbf{\Sigma}}_{\vec{e}}$ can acquire singularities. The study of how the strata $\{\mathbf{\Sigma}_{\vec{e}'} : \vec{e}' \leq \vec{e}\}$ fit together into $\overline{\mathbf{\Sigma}}_{\vec{e}}$ is the main subject of this thesis.

Analogies

This type of geometric structure, where there is an algebraic stack whose closed points are given by elements of some poset, is actually a relatively common geometric situation in disguise. In studying spaces with group actions, we often encounter meaningful stratifications $X = \bigsqcup_{i \in I} X_i$ with the following properties:

1. the strata are orbits of some fixed subgroup;

2. there is a unique open stratum and each stratum is locally closed;

3. there is a natural poset indexing the closure order of the strata, i.e. $X_i \subseteq \overline{X_j}$ if and only if $i \leq j$;

4. each stratum has homogeneous geometry, e.g. the geometry of $\overline{X_i}$ at a point p only depends on the stratum X_j that p is contained in.

Then the closed strata $\overline{X_i}$ often bear interesting geometry, and are the object of intense study. We give two well-known examples of this situation.

The first is that of determinantal loci.

[1]The notation $\mathrm{Bun}_{\mathrm{GL}_r}(\mathbb{P}^1)$ is perhaps more familiar. Our $\mathcal{B}_{r,d}$'s are connected components of $\mathrm{Bun}_{\mathrm{GL}_r}(\mathbb{P}^1)$, and $\mathcal{B}_{r,0} = \mathrm{Bun}_{\mathrm{SL}_r}(\mathbb{P}^1)$.

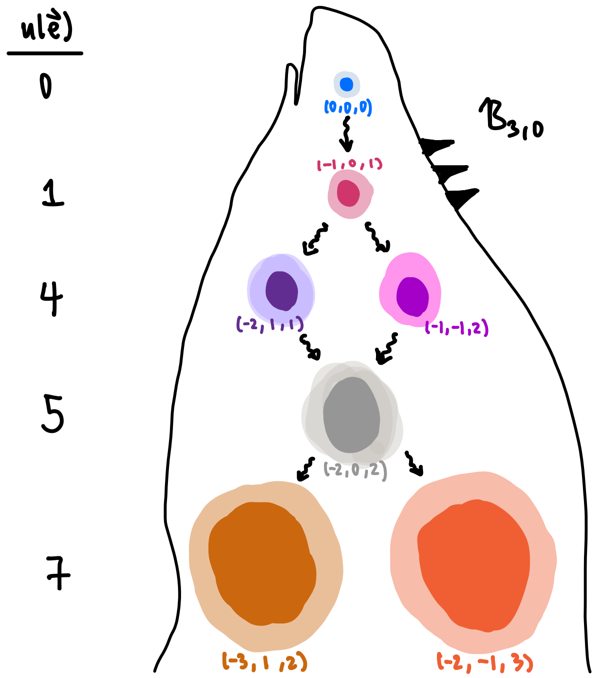


Figure 1.2: A cartoon of $\mathcal{B}_{3,0}$

Example 1.1.1. Let $m \leq n$ be positive integers, and let $X_{m,n} = \mathrm{Mat}_{m\times n}(k)$ be the affine space of $m \times n$ matrices over some field k. Then for each $0 \leq r \leq m$, appropriately sized minors cut out the variety D_r of matrices of rank at most r. In this case, the partially ordered set indexing the strata is the set of possible ranks, i.e. {0, 1, . . . , m}, and the partial order is the natural ordering of these integers.

Now consider instead the quotient $\mathfrak{X}_{m,n} = [\mathrm{Mat}_{m\times n} / \mathrm{GL}_m \times \mathrm{GL}_n]$, where $\mathrm{GL}_m \times \mathrm{GL}_n$ acts on $\mathrm{Mat}_{m\times n}$ by precomposition and post-composition. We can think of $\mathfrak{X}_{m,n}$ as the total space of the vector bundle $\mathcal{H}om(\mathcal{E}_{\mathrm{univ}}, \mathcal{F}_{\mathrm{univ}})$ on $B\,\mathrm{GL}_n \times B\,\mathrm{GL}_m$, where $\mathcal{E}_{\mathrm{univ}}, \mathcal{F}_{\mathrm{univ}}$ are the universal vector bundles on $B\,\mathrm{GL}_n$ and $B\,\mathrm{GL}_m$. From this perspective, it is clear that the stack $\mathfrak{X}_{m,n}$ parametrizes groupoids of vector bundles $\mathcal{E}, \mathcal{F}$ of ranks n, m respectively, plus a map $\mathcal{E} \to \mathcal{F}$. To take isomorphism classes of k-points of this stack is the same as asking for matrices up to invertible row and column operations. Therefore, $\pi_0(\mathfrak{X}(k))$ is naturally bijective to the set $\{0, 1, \ldots, m\}$. Each $r \in \{0, 1, \ldots, m\}$ corresponds to the isomorphism class of the $m \times n$ matrix consisting of an $r \times r$ identity block and zeros elsewhere. In $X_{m,n}$, each locally closed stratum $D_r \setminus D_{r-1}$ of matrices of rank exactly r is an orbit of $\mathrm{GL}_m \times \mathrm{GL}_n$, and maps onto a stacky point of $\mathfrak{X}_{m,n}$ under this quotient map.

Example 1.1.2. Let Fl_n be the complete flag variety on a n-dimensional vector space. Then the left action of B on $\mathrm{Fl}_n = \mathrm{GL}_n / B$ stratifies Fl_n into B-orbits, which are also called Schubert cells. The Bruhat decomposition for GL_n tells us that these orbits are indexed by the symmetric group S_n, which is a poset under the Bruhat order.

Instead of thinking about Fl_n and its stratification by Schubert cells, we can instead consider the fiber product $\mathcal{Y}_n = BB \times_{B\,\mathrm{GL}_n} BB \cong B\backslash \mathrm{GL}_n / B$, which is the stack parametrizing a vector bundle of rank n, plus two complete flags of subbundles for this vector bundle. The k-points $\pi_0(\mathcal{Y}_n(k))$ exactly correspond to the elements of S_n, because up to isomorphisms, the data of two complete flags of an n-dimensional vector space is exactly the relative position of these flags. However, each point has lots of automorphisms, corresponding to the subgroup of GL_n which preserves both flags. We see how these points fit together by considering the B-torsor over $\mathcal{Y}_n$ which frames one of the flags, which is precisely the flag variety $\mathrm{Fl}_n = \mathrm{GL}_n / B$.

In both cases, if we start by considering the stack $\mathfrak{X}_{m,n}$ or $\mathcal{Y}_n$, the stacky points corresponding to the locally closed strata fit together in a mysterious way, and it is unclear how to think about the closed strata. By considering the smooth cover which is the space of matrices $X_{m,n}$ or the flag variety Fl_n, we can see their geometry much more explicitly. The smooth cover is like a magnifying lens! Moreover, in both of these examples, there is an obvious choice of which magnifying lens to use, because the stack admits a description as a global quotient stack. However, the stack of vector bundles on $\mathbb{P}^1$ is not of finite type, and therefore it cannot be the quotient of any scheme of finite type. (Although, in fact, it does admit a description as a quotient of a scheme not of finite type, which is Kashiwara's thick affine Grassmannian.) Therefore, there are many choices of which magnifying lens to use to see how the points $\mathbf{\Sigma}_{\vec{e}'}$ fit together in the stack $\mathcal{B}_{r,d}$, or $\overline{\mathbf{\Sigma}}_{\vec{e}}$, and each has its advantages and disadvantages.

Note that a morphism from a smooth scheme to a smooth algebraic stack is smooth if and only if the map on $k[\epsilon]$-points is surjective at each k-point on the source. In practice, it is often convenient to work with the following condition on a family $(B, \mathcal{E})$, which is equivalent to the induced map $B \to \mathcal{B}_{r,d}$ being smooth.

Definition 1.1.3. Let $(B, \mathcal{E})$ be a family of vector bundles on $\mathbb{P}^1$ over B. Then we say that $(B, \mathcal{E})$ is a *complete family* if B is smooth and the Kodaira-Spencer map $T_bB \to \mathrm{Ext}^1(\mathcal{E}_b, \mathcal{E}_b)$ is surjective for all $b \in B$.

Different ways to see

I first learned about splitting loci in the context of Hurwitz-Brill-Noether theory [Lar21a; LLV25]. In their work, there is a particular base B which has geometric significance, namely the Picard scheme of a general k-gonal curve. Specifically, let $f : C \to \mathbb{P}^1$ be a general degree k map from a genus g curve C to $\mathbb{P}^1$. Then by pushing forward the universal Poincaré line bundle, we obtain a family of rank k, degree $d - g + 1 - k$ vector bundles on $\mathbb{P}^1$ over $\mathrm{Pic}^d(C)$. By degenerating to a chain of elliptic curves covering a chain of $\mathbb{P}^1$'s, [Lar21a] proves that this pushforward over $\mathrm{Pic}^d(C)$ is a complete family, inducing a smooth map $\mathrm{Pic}^d(C) \to \mathcal{B}_{k,d-g+1-k}$. As we fix k and $d - g$ and let g go to infinity, the union of $\mathrm{Pic}^d(C)$ forms a smooth cover of $\mathcal{B}_{k,d-g+1-k}$, allowing us to see arbitrarily deep into the poset of splitting types. To be able to say more about the splitting loci for this particular family of vector bundles is the main motivation for me to study splitting loci in general.

There are also many other complete families which we can use to "see" the geometry of $\overline{\mathbf{\Sigma}}_{\vec{e}}$. First I want to mention two different ways that I don't really use in my work. One way is due to Bolognesi-Vistoli [BV12, Section 4]. The idea there is that when a vector bundle E on $\mathbb{P}^1$ is globally generated, then there is a canonical short exact sequence

$$0 \to H^0(E(-1)) \otimes \mathcal{O}(-1) \to H^0(E) \otimes \mathcal{O} \to E \to 0.$$

This sequence can be upgraded to a family, and is called the Strømme sequence [Str87, Proposition 1.1]. This then gives rise to a description of certain open substacks of $\mathcal{B}_{r,d}$ as a quotient of an open subspace of affine space. The affine space corresponds to all possible maps of the form $H^0(E(-1)) \otimes \mathcal{O}(-1) \to H^0(E) \otimes \mathcal{O}$, and the open is the locus of maps which have locally free cokernel. This was the description used in [Lar23] to compute the Chow ring of the stack $\mathcal{B}_{r,d}$. Another way is to use Quot schemes on $\mathbb{P}^1$, as in the standard construction of the stack of vector bundles on any fixed curve [Wan11].

In practice, I have heavily relied on a specific smooth cover of $\mathcal{B}_{r,d}$, which I call successive extension space. For any $\vec{e}'$ of rank r and degree d, there is a family of vector bundles on $\mathbb{P}^1$ over the affine space $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$, which induces a smooth map to $\mathcal{B}_{r,d}$. This family of vector bundles has several nice properties.

1. $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ surjects smoothly onto $\bigcup_{\vec{e} \geq \vec{e}'} \mathbf{\Sigma}_{\vec{e}}$;

2. The functor of points for $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ has a modular description. If we let $\vec{e}' = (f_1^{s_1}, \ldots, f_\ell^{s_\ell})$, and let $Q_i = \mathcal{O}(f_i^{s_i})$, then a T-point of $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ is the data of vector bundles $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_\ell$ on $T \times \mathbb{P}^1$, the data of extensions

$$0 \to \mathcal{K}_i \to \mathcal{K}_{i+1} \to p^* Q_{i+1} \to 0,$$

plus an isomorphism $\mathcal{K}_1 \cong p^* Q_1$, where $p : T \times \mathbb{P}^1 \to \mathbb{P}^1$ [2];

3. This modular description allows us to think of $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ as built from $\mathcal{B}_{r,d}$ via the operations of taking open substacks, forming vector bundles, and forming GL_s-torsor. In particular, the splitting loci in $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ are irreducible of the expected codimension;

4. There is a torus $\mathbb{G}_m^{\ell-1}$ acting on $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$, so that the splitting loci are invariant under this action;

5. The splitting locus for $\vec{e}'$ is just the origin of this affine space, which is also the unique torus fixed point.

These properties make the spaces $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ very nice to work with. The specifics of this family of covers are laid out in Section 2.2. A cartoon of $H^1\mathcal{E}nd(\mathcal{O}(-2, 2))$ and the splitting loci therein can be found in Figure 1.3.

Remark 1.1.4. In theory, the fact that there exists a family of affine spaces forming a smooth cover of $\mathcal{B}_{r,d}$ reduces the study of $\overline{\mathbf{\Sigma}}_{\vec{e}}$ to an algebraic question. However, in practice, the ideals are difficult to write down and hard to work with. For example, it was a conjecture

[2] When $\vec{e}'$ has distinct parts, one can think of $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ as a fiber of the natural map $\mathrm{Bun}_B(\mathbb{P}^1) \to \mathrm{Bun}_T(\mathbb{P}^1)$ which takes the associated graded of a complete flag of vector bundles. This is denoted Bun_N^ω in geometric Langlands literature [NT25, Section 3.1], where ω corresponds to the data of $\vec{e}'$.

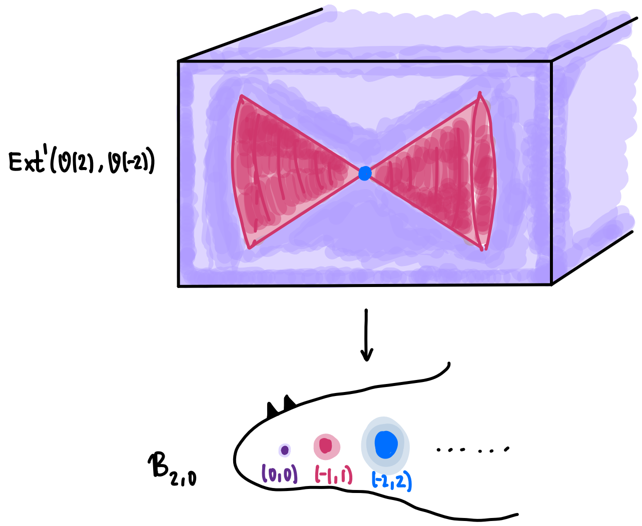


Figure 1.3: $H^1\mathcal{E}nd(\mathcal{O}(-2,2)) \cong \mathrm{Ext}^1(\mathcal{O}(2), \mathcal{O}(-2))$

of Eisenbud and Schreyer [ES08, Conjecture 5.1] that $\overline{\boldsymbol{\Sigma}}_{\vec{e}}$ equipped with the Fitting ideal scheme structure is reduced. While this is known through Hurwitz-Brill-Noether theory [LLV25], there is no algebraic proof of this fact that I am aware of. We discuss the generators of the ideal for $\overline{\Sigma}_{\vec{e}}$ in these affine spaces in more detail in Section 4.3.

Remark 1.1.5. When $\vec{e}'$ has rank 2, then $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}')) \cong \mathrm{Ext}^1(\mathcal{O}(e_2'), \mathcal{O}(e_1'))$. The ideals defining splitting loci in $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ are generated by minors of Hankel matrices, which are also called catalecticant matrices (studied in e.g. [Con98; Con+18]). The specific correspondence is explained in Example 2.2.12. Geometrically, it is well-known that Hankel determinantal ideals define the cone over secant varieties of rational normal curves, and that the tangent cone to $W^r_d(C)$ when C is a hyperelliptic curve can be described this way [SY22, Appendix A]. Beyond algebraic results, recently D. Brogan showed that the (projective) secant varieties of rational normal curves are all rational homology manifolds [Bro26]. Moreover, in the same paper, in the case of $\vec{e} = (-1, 1)$ and $\vec{e}' = (-d, d)$, in which case the splitting locus $\overline{\Sigma}_{(-1,1)}$ is a hypersurface in the affine space $\mathrm{Ext}^1(\mathcal{O}(d), \mathcal{O}(-d))$, certain nearby cycles and vanishing cycles were computed.

Lastly, I want to mention the affine Grassmannian Gr, which is an ind-scheme. The k-points of the affine Grassmannian (for GL_r) can be thought of as the data of a vector bundle on $\mathbb{P}^1$ plus a trivialization away from 0. As such, Gr admits a map to the stack $\bigsqcup_{d\in\mathbb{Z}} \mathcal{B}_{r,d}$, and the map is a $\mathrm{GL}_r(k[t^{-1}])$-torsor. The $\mathrm{GL}_r(k[t^{-1}])$-orbits in Gr, which are opposite Schubert varieties, are precisely preimages of $\boldsymbol{\Sigma}_{\vec{e}}$ under this map. While it is perhaps unclear what we mean by Gorenstein or Cohen-Macaulay for an ind-scheme, these notions make sense for algebraic stacks. In that sense, the connections between the quotient $\mathrm{GL}_r(k[t^{-1}])\backslash\,\mathrm{Gr}$ and Gr also make it possible to ask various geometric questions that would be otherwise standard, and indeed have been answered for the finite-dimensional Schubert varieties in Gr (see references in [Zhu16, Theorem 2.1.21, Remark 2.1.22]). Moreover, up to some work making precise identifications, results about the cohomology and K-theory classes of opposite Schubert varieties transport to splitting loci in the stack of vector bundles on $\mathbb{P}^1$. This connection is not be part of this thesis, but will hopefully be explained in forthcoming work of the author and collaborators.

1.2 Results

In this thesis, we construct a modular resolution of singularities for $\overline{\mathbf{\Sigma}}_{\vec{e}}$, and use this resolution to study the singularities of $\overline{\mathbf{\Sigma}}_{\vec{e}}$. Throughout, we work over an algebraically closed field of characteristic zero. We expect that many of our results also work over any field of arbitrary characteristic, and the precise field assumptions will be clarified in updated versions of the papers based on this thesis.

Chapters 2, 3, 5 in the body of this thesis correspond to the papers [Lin25a], [Lin25b], and [GLS25]. The last chapter is joint with with Ajay Gautam and Shubham Sinha. For the rest of this section, I will explain the results of the four ensuing chapters and the relationships between them.

At the beginning of my third year, I set out to construct a resolution of singularities for splitting loci. One way of thinking about $G^r_d(C)$, the scheme of g^r_d's on a curve, is as the result of applying the modular resolution of singularities for determinantal loci to $W^r_d(C)$. Then to figure out what the analogous object is for Hurwitz-Brill-Noether loci $W^{\vec{e}}(C)$, one could ask what is a modular resolution of singularities for splitting loci.

There are many benefits to having such a resolution. In the case of determinantal varieties, the following results were intricately connected to the resolution of singularities:

1. Porteous's formula for the class of determinantal loci is computed using its resolution of singularities, which in turn yields the class of $W^r_d(C)$ in $\mathrm{Pic}^d(C)$.

2. Kempf's proof that determinantal loci are normal, Cohen-Macaulay, and have rational singularities [Wey03, Propositions 6.1.5(a)-(b) and 6.2.3(b), Proposition 6.15(b)]. Because of our very good understanding of the cohomology of tautological bundles on the Grassmannian (the Borel-Weil-Bott theorem), one can pushforward a particular complex on a Grassmann bundle by hand to recover the minimal free resolution of determinantal ideals (Lascoux complexes).

3. An explicit understanding of the tangent spaces of $G^r_d(C)$ and $W^r_d(C)$, yielding the result that $W^r_d(C)$ is singular along exactly $W^{r-1}_d(C)$ [Arb+84, Chapter IV Proposition 4.2].

4. Brill-Noether existence and connectedness results, due to Fulton–Lazarsfeld [FL81].

When I first started working on this, we already guessed that a relative hyper-Quot construction was perhaps the answer, as relative Quot schemes were key to a universal inductive formula for the classes of splitting loci [Lar21b]. My first result is that this guess was correct.

To describe the resolution, for $\underline{r} = (r_1, \ldots, r_m), \underline{d} = (d_1, \ldots, d_m)$, let us define the hyper-Quot stack $\mathrm{FQuot}(\mathcal{B}_{r,d}, \mathcal{E}_{\mathrm{univ}}; \underline{r}, \underline{d}) \to \mathcal{B}_{r,d}$ to be the algebraic stack representing the following functor:

$$(T \to \operatorname{Spec} k) \mapsto \left\{ \begin{array}{l} \text{A vector bundle } \mathcal{E} \text{ of rank } r \text{ and fiberwise degree } d \\ \text{and a flag of subsheaves } \mathcal{E}_1 \hookrightarrow \mathcal{E}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_{m+1} = \mathcal{E} \text{ on } T \times \mathbb{P}^1 \\ \text{such that } \mathcal{E}_i \text{ is locally free of rank } r - r_i \text{ and fiberwise} \\ \text{degree } d - d_i, \text{ and each quotient } \mathcal{E}/\mathcal{E}_i \text{ is flat over } T \end{array} \right\} / \sim,$$

where two flags are equivalent if there are isomorphisms between the subsheaves forming a commutative diagram with the equality morphism on $\mathcal{E}$.

The generic splitting type of any rank and degree satisfies $|e_r - e_1| \leq 1$. Such a splitting type is called *balanced*.

Definition 1.2.1. Let $E = \mathcal{O}(\vec{e})$ be a vector bundle on $\mathbb{P}^1$, and let

$$E_1 \subset E_2 \subset \cdots \subset E_{m+1} = E$$

be a sequence of subbundles such that for all $1 \leq i \leq m$, each quotient sheaf E_{i+1}/E_i is a nonzero balanced vector bundle, and up to automorphisms of E_i, the inclusion $E_i \subset E_{i+1}$ is the unique injective map $E_i \hookrightarrow E_{i+1}$. Such a flag is called an *$\vec{e}$-admissible flag.*

Note that the Harder-Narasimhan flag of subbundles for $\mathcal{O}(\vec{e})$ is always $\vec{e}$-admissible, but in general it is possible to omit some subbundles in the Harder-Narasimhan flag while keeping it $\vec{e}$-admissible.

Theorem 1.2.2. *Let $E = \mathcal{O}(\vec{e})$ be a vector bundle on $\mathbb{P}^1$, and let $E_1 \subset E_2 \subset \cdots \subset E_{m+1} = E$ be an $\vec{e}$-admissible flag. Let $\underline{r} = (r_1, \ldots, r_m), \underline{d} = (d_1, \ldots, d_m)$ be the ranks and degrees of E/E_i. Then $\mathrm{FQuot}(\mathcal{B}_{r,d}, \mathcal{E}_{\mathrm{univ}}; \underline{r}, \underline{d})$ is a smooth and irreducible algebraic stack proper over $\mathcal{B}_{r,d}$. It maps surjectively onto $\overline{\mathbf{\Sigma}}_{\vec{e}}$ and it is an isomorphism over the dense open $\mathbf{\Sigma}_{\vec{e}}$.*

This theorem tells us that the splitting types $\vec{e}' \leq \vec{e}$ can be cut out by requiring flags of subsheaves with the same numerical data as an $\vec{e}$-admissible flag. Relying heavily on this resolution of singularities, we prove several different results about the singularties of $\overline{\mathbf{\Sigma}}_{\vec{e}}$. To state these results, let us make some definitions.

Definition 1.2.3.

- For $1 \leq s < r$, the gap $(s, s+1)$ is a *cut* for the pair $\vec{e}' \leq \vec{e}$ if $e'_1 + \cdots + e'_s = e_1 + \cdots + e_s$ and $e_s < e_{s+1}$. A cut is *admissible* if in addition, $e'_s < e'_{s+1}$.
- For $\vec{e}' \leq \vec{e}$, we say that $\vec{e}'$ lies in the smooth locus of $\vec{e}$ if $\mathbf{\Sigma}_{\vec{e}'}$ is contained in the smooth locus of $\overline{\mathbf{\Sigma}}_{\vec{e}}$.
- If there exists some $1 \leq k \leq r$ such that $\vec{f} = (e_1, \ldots, e_k)$ and $\vec{g} = (e_{k+1}, \ldots, e_r)$ are both balanced, then the splitting type $\vec{e} = (e_1, \ldots, e_r)$ is called *tame.*
- We say that $\vec{e}$ is a *block arithmetic progression* of block size s and difference t if it is a shift of the tuple $(0^s, t^s, \ldots, ((m-1)t)^s)$ (t^s is a short hand for s copies of the number t). As usual, if $s = 1$, we say that the tuple is an *arithmetic progression.* We say that $\vec{e}$ is *contiguous* if $\vec{e}$ is a shift of a tuple of the form $(\ldots, 0^a, 1^b, 2^a, 3^b, \ldots)$ for some positive integers a, b.

- We say that a Cohen-Macaulay scheme X is *N-Gorenstein* if $\omega_X^{\otimes N}$ is a line bundle, and *Gorenstein* if ω_X is a line bundle. Being Gorenstein and N-Gorenstein are smooth-local properties (see Section 3.2), so it makes sense to ask when an algebraic stack, such as a splitting locus $\overline{\boldsymbol{\Sigma}}_{\vec{e}}$, is Gorenstein or N-Gorenstein. We say that a Cohen-Macaulay algebraic stack is *$\mathbb{Q}$-Gorenstein* if it is N-Gorenstein for some $N \geq 1$.

Theorem 1.2.4. 1. Let $\vec{e}' \leq \vec{e}$. Then $\vec{e}'$ lies in the smooth locus of $\vec{e}$ if and only if there exists a collection of admissible cuts for the pair such that each block of $\vec{e}$ between two adjacent cuts is balanced.

2. Let $\vec{e}$ be a tame splitting type. Then $\overline{\boldsymbol{\Sigma}}_{\vec{e}}$ has rational singularities.

3. A splitting locus $\overline{\boldsymbol{\Sigma}}_{\vec{e}}$ is Gorenstein if and only if $\vec{e}$ is a block arithmetic progression of difference $0, 1$ or 2, or if $\vec{e}$ is contiguous. $\overline{\boldsymbol{\Sigma}}_{\vec{e}}$ is $\mathbb{Q}$-Gorenstein (but not Gorenstein) if and only if it is an arithmetic progression of difference $t \geq 3$. In this case, the smallest N for which $\overline{\boldsymbol{\Sigma}}_{\vec{e}}$ is N-Gorenstein is t if t is odd, and $t/2$ if t is even.

In Chapter 4, we describe explicitly the difference $\dim B - \dim T_b\overline{\Sigma}_{\vec{e}}$ for closed points $b \in \Sigma_{\vec{e}'}(B)$ and any $\vec{e}' \leq \vec{e}$. We also give a geometric description of the universal normal cone of $\overline{\boldsymbol{\Sigma}}_{\vec{e}}$ at $\vec{e}'$ for certain pairs of $\vec{e}$ and $\vec{e}'$.

Let $\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N})$ be the Quot scheme parametrizing quotients of $\mathcal{O}_{\mathbb{P}^1}^{\oplus N}$ of rank r and degree d [3]. In the proof of the rational singularities result above, I needed a cohomology vanishing statement for tautological bundles on $\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N})$, which is the first part of the theorem below. This is analogous to how in Kempf's proof that determinantal varieties have at most rational singularities, the Borel-Weil-Bott theorem for tautological bundles on Grassmannians is a critical ingredient. The study of the cohomology groups of tautological bundles on Quot schemes is the subject of Chapter 5.

In more detail, let
$$0 \to \mathcal{S} \to \mathcal{O}^{\oplus N} \to \mathcal{Q} \to 0$$
be the tautological sequence on $\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N}) \times \mathbb{P}^1$, and let $\pi : \mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N}) \times \mathbb{P}^1 \to \mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N})$, $p : \mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N}) \times \mathbb{P}^1 \to \mathbb{P}^1$. We will write $L_m = \mathcal{O}_{\mathbb{P}^1}(m)$. Given a line bundle L on $\mathbb{P}^1$, we denote $L^{[d]} = R\pi_*(\mathcal{Q} \otimes p^*L)$. When $\deg L \geq -1$, by cohomology and base change, $L^{[d]}$ is a vector bundle. When $\deg L < -1$, we think of $L^{[d]}$ as an object of the derived category of $\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N})$, which one can show can be represented by a two-term perfect complex. For λ a non-increasing sequence of (possibly negative) integers and $\mathcal{E}$ a vector bundle on a scheme X, we denote by $\mathbf{S}^\lambda\mathcal{E}$ the Schur functor applied to $\mathcal{E}$, which is a vector bundle on X. Schur functors can be appropriately generalized to objects in the derived category that can be represented as a two-term perfect complex.

Theorem 1.2.5. 1. Let $\lambda_{-1}, \ldots, \lambda_\ell$ be partitions, at least one of which is non-empty, then
$$H^{>D}(\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N}), \otimes_{m=-1}^{\ell}\mathbf{S}^{\lambda_m}(L_m^{[d]})^\vee) = 0,$$

[3] In this introduction, I shall state the results for $\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N})$, but all of them can be extended to $\mathrm{Quot}^{r,d}(V)$, where V is any vector bundle on $\mathbb{P}^1$, at the cost of d being sufficiently large, which is the version in Chapter 5. I also restrict to Schur functors associated to partitions here, but our results are stated more generally for Schur functors associated to nonincreasing sequences.

where $D = \sum_{m=-1}^{\ell} |\lambda_m|$.

2. Similarly, let $\lambda_{d-1}, \ldots, \lambda_\ell$ be partitions, at least one of which is non-empty, then

$$H^{>0}\left(\mathrm{Quot}_{\mathbb{P}^1}^{r,d}(\mathcal{O}^{\oplus N}), \bigotimes_{s=d-1}^{\ell} \mathbf{S}^{\lambda_s} L_s^{[d]}\right) = 0.$$

3. Let $M_1, M_2, \ldots, M_\ell$ be line bundles on $\mathbb{P}^1$. For any tuple of partitions $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ satisfying $|\lambda_1| + |\lambda_2| + \cdots + |\lambda_m| < (Nd+N)/(N-r)$, we have

$$H^D\left(\mathrm{Quot}_{\mathbb{P}^1}^{r,d}(\mathcal{O}^{\oplus N}), \bigotimes_{s=1}^{m} \mathbf{S}^{\lambda_s} M_s^{[d]}\right) \cong \bigotimes_{\deg M_s<0} \mathbf{S}^{\lambda_s^\dagger} H^1(\mathcal{O}_{\mathbb{P}^1}^{\oplus N} \otimes M_s) \otimes \bigotimes_{\deg M_s \geq 0} \mathbf{S}^{\lambda_s} H^0(\mathcal{O}_{\mathbb{P}^1}^{\oplus N} \otimes M_s),$$

where $D = \sum_{\deg L_s<0} |\lambda_s|$, and all other cohomology groups are zero.

The first two parts of the above theorem are of the same flavor, in that they show the vanishing of certain cohomology groups, without any size constraints on the partitions involved. Yet one has no control over the cohomology groups that are not shown to always vanish. Subject to a size constraint on the partitions involved, the third part of the theorem above pins down the cohomology groups exactly. This distinction can be viewed as a stabilization behavior: for fixed λ_s, M_s, as we let d tend to infinity, the cohomology groups of the corresponding tautological compelxes stabilize. As a corollary of the last part of the above theorem, we confirm [MOS22, Conjecture 1.3.1, 1.3.2].

Chapter 2

Resolving the singularities of splitting loci

2.1 Introduction

A vector bundle on a scheme $B \times \mathbb{P}^1$ leads to a stratification of the base B according to how the vector bundle splits when it is restricted to each fiber. The strata that arise are called splitting loci. The aim of this paper is to produce a modular resolution of singularities for these strata and use the resolution to study the singularities of splitting loci in the case of the universal family.

Let $\mathcal{B}_{r,d}$ be the moduli stack of rank r, degree d vector bundles on $\mathbb{P}^1$. The data of how a vector bundle of rank r and degree d splits is captured by a splitting type $\vec{e} = (e_1, \ldots, e_r)$, where $e_1 \leq e_2 \leq \cdots \leq e_r$ and $\sum_{i=1}^r e_i = d$. We define $\mathbf{\Sigma}_{\vec{e}}$ to be the locally closed substack in $\mathcal{B}_{r,d}$ parametrizing families of vector bundles each with splitting type $\vec{e}$, and define $\overline{\mathbf{\Sigma}}_{\vec{e}}$ as the closure of $\mathbf{\Sigma}_{\vec{e}}$ in $\mathcal{B}_{r,d}$. For convenience, we write $\mathcal{O}(\vec{e}) \coloneqq \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_r)$.

Definition 2.1.1. A splitting type $\vec{e} = (e_1, \ldots, e_r)$ is balanced if $|e_r - e_1| \leq 1$, or equivalently if $h^1 \mathcal{E}nd(\mathcal{O}(\vec{e})) = 0$. If there exists some $1 \leq k \leq r$ such that $\vec{f} = (e_1, \ldots, e_k)$ and $\vec{g} = (e_{k+1}, \ldots, e_r)$ are both balanced, then we say that the splitting type $\vec{e}$ is tame.

For example, all rank two splitting types are tame. $(1, 2, 5, 6)$ is tame, but $(0, 2, 4)$ is not. Recall that a scheme X has rational singularities if there exists a resolution of singularities $\rho : \widetilde{X} \to X$ such that $R^{>0}\rho_* \mathcal{O}_{\widetilde{X}} = 0$ and the natural map $\mathcal{O}_X \to \rho_* \mathcal{O}_{\widetilde{X}}$ is an isomorphism. One checks that having rational singularities is a smooth-local property, in the sense of [Stacks, Tag 0348]. So it makes sense to say that an algebraic stack has rational singularities.

Our main theorem is the following.

Theorem 2.1.2. Let k be an algebraically closed field of characteristic zero. Over $\operatorname{Spec} k$, for tame splitting types $\vec{e}$, $\overline{\mathbf{\Sigma}}_{\vec{e}}$ has rational singularities.

Combined with Hurwitz-Brill-Noether theory [Lar21a], [LLV25], our result implies the following corollary.

Corollary 2.1.3. Let C be a general k-gonal curve. Then components of $W^r_d(C)$ have rational singularities.

Proof. [Lar21a] proves that for a general degree k cover $f : C \to \mathbb{P}^1$, the morphism $\mathrm{Pic}_d(C) \to \mathcal{B}_{k,d-g+1-k}$ induced by pushing forward line bundles along f is smooth. The same paper plus the work of [Cop25] prove that the components of $W^r_d(C)$ correspond to the maximal splitting types with at least $r+1$ sections, which are tame. $\square$

The construction of a modular resolution of singularities for $\overline{\mathbf{\Sigma}}_{\vec{e}}$ is essential to the proof of Theorem 2.1.2. To describe the resolution, for $\underline{r} = (r_1, \ldots, r_m), \underline{d} = (d_1, \ldots, d_m)$, let us define the hyper-Quot stack $\mathrm{FQuot}(\mathcal{B}_{r,d}, \mathcal{E}_{\mathrm{univ}}; \underline{r}, \underline{d}) \to \mathcal{B}_{r,d}$ to be the algebraic stack representing the following functor:

$$(T \to \operatorname{Spec} k) \mapsto \left\{ \begin{array}{l} \text{A vector bundle } \mathcal{E} \text{ of rank } r \text{ and fiberwise degree } d \\ \text{and a flag of subsheaves } \mathcal{E}_1 \hookrightarrow \mathcal{E}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_{m+1} = \mathcal{E} \text{ on } T \times \mathbb{P}^1 \\ \text{such that } \mathcal{E}_i \text{ is locally free of rank } r - r_i \text{ and fiberwise} \\ \text{degree } d - d_i, \text{ and each quotient } \mathcal{E}/\mathcal{E}_i \text{ is flat over } T \end{array} \right\} / \sim .$$

Theorem 2.1.4. Let $E = \mathcal{O}(\vec{e})$ be a vector bundle on $\mathbb{P}^1$, and let

$$E_1 \subset E_2 \subset \cdots \subset E_{m+1} = E$$

be a sequence of subbundles such that for all $1 \leq i \leq m$, each quotient sheaf E_{i+1}/E_i is a nonzero balanced vector bundle, and up to automorphisms of E_i, the inclusion $E_i \subset E_{i+1}$ is the unique injective map $E_i \hookrightarrow E_{i+1}$. Let $\underline{r} = (r_1, \ldots, r_m), \underline{d} = (d_1, \ldots, d_m)$ be the ranks and degrees of E/E_i. Then $\mathrm{FQuot}(\mathcal{B}_{r,d}, \mathcal{E}_{\mathrm{univ}}; \underline{r}, \underline{d})$ is a smooth and irreducible algebraic stack proper over $\mathcal{B}_{r,d}$. It maps surjectively onto $\overline{\mathbf{\Sigma}}_{\vec{e}}$ and it is an isomorphism over the dense open $\mathbf{\Sigma}_{\vec{e}}$.

Note that the Harder-Narasimhan flag of subbundles of $\mathcal{O}(\vec{e})$ always satisfies the conditions of Theorem 2.1.4, but for any given $\vec{e}$, there may be more than one choice (see Example 2.3.3).

Combined with Hurwitz-Brill-Noether theory, Theorem 2.1.4 recovers the classical Gieseker-Petri theorem.

Corollary 2.1.5 (Gieseker-Petri)**.** Let C be a general curve of genus g. Then $G^r_d(C)$ is smooth.

Proof. Let k be sufficiently large and let $f : C \to \mathbb{P}^1$ be a general cover of degree k and genus g. Then by Hurwitz-Brill-Noether theory, $W^r_d(C) = W^{\vec{e}_{\max,r}}(C)$, where

$$\vec{e}_{\max,r} = ((-2)^{g-d+r}, (-1)^{k-g+d-2r-1}, 0^{r+1}),$$

and exponents mean repeated entry. Moreover, for any line bundle $L \in W^r_d(C)$, we know that $f_*L = \mathcal{O}(\vec{e})$, where

$$\vec{e} = ((-2)^{h^1(L)}, (-1)^{k-h^1(L)-h^0(L)}, 0^{h^0(L)}),$$

and $h^0(L) \geq r+1$. So to give an $(r+1)$-dimensional subspace $V \subseteq H^0(C, L)$ is the same as giving an inclusion $\mathcal{O}^{\oplus(r+1)} \hookrightarrow f_*L$. Therefore, the relative Quot construction corresponding to the one-step inclusion $\mathcal{O}^{\oplus(r+1)} \subseteq \mathcal{O}(\vec{e}_{\max,r})$, when base-changed to $\mathrm{Pic}_d(C)$, is precisely $G^r_d(C)$. Hence Theorem 2.1.4 implies that $G^r_d(C)$ is smooth. $\square$

Tame splitting loci admit a resolution of singularities by a relative Quot construction, and do not require a hyper-Quot construction, which is necessary in general. To show that the higher derived pushforward of the structure sheaf vanishes (and thus $\overline{\mathbf{\Sigma}}_{\vec{e}}$ has rational singularities), we embed this resolution inside a fiber bundle of Quot schemes $\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N})$, where it is cut out by the section of a vector bundle. This reduces the vanishing of higher pushforwards to a cohomology vanishing statement for certain tautological bundles on $\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N})$. While this approach parallels the classical proof that determinantal varieties have rational singularities, the lack of a Borel-Weil-Bott type statement for Quot schemes presents substantial challenges at this step. To access the desired cohomology vanishing statement, we appeal to Strømme's embedding of Quot schemes on $\mathbb{P}^1$ into a product of Grassmannians and further reduce the problem to a cohomology vanishing statement for certain tautological vector bundles on a product of Grassmannians. The desired statement then follows from an application of Borel-Weil-Bott and an involved combinatorial analysis.

Quot schemes on $\mathbb{P}^1$ were first systematically studied by Strømme [Str87], where they showed that $\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N})$ is a nonsingular, irreducible, projective, rational variety. More recently, motivated by analogies to the Hilbert scheme of points on a surface and computations of the quantum cohomology of the Grassmannian, there has been a lot of research towards understanding the cohomology of tautological bundles on Quot schemes on curves, such as [OS23], [MOS22], [MN24], [SZ24b]. However, most results in the literature focus on punctual Quot schemes (the case where $r = 0$), whereas our result applies to Quot schemes parametrizing higher rank quotients.

Here is the cohomology vanishing theorem we prove. Let $d > 0$ and $N \geq r \geq 0$. Let $\mathcal{Q}$ be the universal quotient on $\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N}) \times \mathbb{P}^1$, and let $p : \mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N}) \times \mathbb{P}^1 \to \mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N})$ be the projection to the first factor. Then for $m \geq -1$, $p_*\mathcal{Q}(m)$ is a vector bundle on $\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N})$. Let $\mathcal{E}_m = (p_*\mathcal{Q}(m))^\vee$. Given a partition λ, we write $\mathbf{S}_\lambda$ to denote the Schur functor associated to λ. When $\lambda = (1^k)$, $\mathbf{S}_\lambda\mathcal{E} = \wedge^k\mathcal{E}$; and when $\lambda = (k)$, $\mathbf{S}_\lambda\mathcal{E} = \mathrm{Sym}^k\mathcal{E}$.

Theorem 2.1.6. Let $m \geq -1$ be an integer. Let $\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_m$ be partitions, at least one of which is nonempty. Let $D = |\lambda_{-1}| + \cdots + |\lambda_m|$. Then

$$H^{>D}(\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N}), \otimes_{i=-1}^m \mathbf{S}_{\lambda_i}\mathcal{E}_i) = 0.$$

2.2 Splitting loci and vector bundles on $\mathbb{P}^1$

In this section, we review some facts about splitting loci that will be useful later.

Recall that $\mathcal{B}_{r,d}$ is defined by the following functor:

$$(T \to \operatorname{Spec} k) \mapsto \left\{ \begin{array}{l} \text{Vector bundle } \mathcal{E} \text{ on } T \times \mathbb{P}^1, \\ \text{such that } \mathcal{E}_t \text{ has rank } r \text{ and} \\ \text{degree } d \text{ for all } t \in T \end{array} \right\}.$$

When we have a scheme B equipped with a vector bundle $\mathcal{E}$ on $B \times \mathbb{P}^1$ of rank r and fiberwise degree d, this data gives rise to a morphism $\phi_{\mathcal{E}} : B \to \mathcal{B}_{r,d}$. We write $\Sigma_{\vec{e}}(\mathcal{E}) \subset \overline{\Sigma}_{\vec{e}}(\mathcal{E}) \subset B$ for the pullback of $\overline{\mathbf{\Sigma}}_{\vec{e}}$ and $\mathbf{\Sigma}_{\vec{e}}$ along $\phi_{\mathcal{E}}$. In general, $\overline{\Sigma}_{\vec{e}}(\mathcal{E})$ will no longer be the closure of $\Sigma_{\vec{e}}(\mathcal{E})$ in B, but it still has a nice description in terms of the dominance order between splitting types. Given two splitting types $\vec{e}, \vec{f}$, we say that $\vec{e} \geq \vec{f}$ if $e_1 + \cdots + e_j \geq f_1 + \cdots + f_j$ for each $1 \leq j \leq r$. Equivalently, $\vec{e} \geq \vec{f}$ if $e_j + \cdots + e_r \leq f_j + \cdots + f_r$ for each $1 \leq j \leq r$.

This partial order, also called the dominance order, turns out to capture when $\overline{\mathbf{\Sigma}}_{\vec{e}} \supset \mathbf{\Sigma}_{\vec{f}}$ in $\mathcal{B}_{r,d}$, as will be explained in Proposition 2.2.4. In terms of this partial order, we have $\overline{\Sigma}_{\vec{e}}(\mathcal{E}) = \bigcup_{\vec{f} \leq \vec{e}} \Sigma_{\vec{f}}(\mathcal{E})$. We write $u(\vec{e}) = h^1 \mathcal{E}nd(\mathcal{O}(\vec{e}))$, which is the expected codimension of $\overline{\Sigma}_{\vec{e}}$. Note also that given a fixed rank and degree, there is a unique balanced splitting type, which is maximal with respect to dominance. When the degree is divisible by rank, the unique balanced vector bundle of that rank and degree is called perfectly balanced. This corresponds to a splitting type $\vec{e} = (e_1, \ldots, e_r)$ where $e_1 = e_r$. Given a vector bundle E on $\mathbb{P}^1$, we will write $\mu(E) \coloneqq \frac{\deg E}{\operatorname{rk} E}$ for its slope.

Closure order on splitting types

We collect several equivalent conditions describing when one splitting locus lies in the closure of another.

Definition 2.2.1. Let $\vec{e}$, $\vec{f}$ be two splitting types of the same rank and degree. Let the Harder-Narasimhan flag of subbundles of $\mathcal{O}(\vec{e})$ be $E_1 \subset E_2 \subset \cdots \subset E_m \subset E_{m+1} = \mathcal{O}(\vec{e})$, and choose a splitting $\mathcal{O}(\vec{f}) = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_{n+1}$, where each Q_i is a perfectly balanced vector bundle and $\mu(Q_i) < \mu(Q_{i+1})$. We say that $\mathcal{O}(\vec{f})$ admits a flag of subbundles of type $\vec{e}$ if there exists a flag of subbundles $E'_1 \subset E'_2 \subset \cdots \subset E'_m \subset E'_{m+1} = \mathcal{O}(\vec{f})$ such that $\operatorname{rk} E'_i = \operatorname{rk} E_i$ and $\deg E'_i = \deg E_i$. An extension of type $\vec{f}$ is a sequence of extensions $0 \to K_i \to K_{i+1} \to Q_{i+1} \to 0$, where $1 \leq i \leq n$, and $K_1 = Q_1$. We say that $\mathcal{O}(\vec{e})$ can be realized as an extension of type $\vec{f}$ if there exists an extension of type $\vec{f}$ where $K_{n+1} \cong \mathcal{O}(\vec{e})$.

Example 2.2.2. Let $\vec{f} = (-2, 2)$. Then $Q_1 = \mathcal{O}(2)$, $Q_2 = \mathcal{O}(-2)$. For $\mathcal{O}(\vec{e})$ to be realizable as an extension of type $\vec{f}$ means there exists a short exact sequence $0 \to \mathcal{O}(-2) \to \mathcal{O}(\vec{e}) \to \mathcal{O}(2) \to 0$. This occurs if and only if $\vec{e} = (0, 0), (-1, 1), (-2, 2)$.

Lemma 2.2.3. Let $\vec{e}$, $\vec{f}$ be two splitting types of the same rank and degree. Let $\vec{e} = (\mathrm{Bal}, \vec{a})$, where Bal is a balanced splitting type with largest summand m_o and $a_1 > m_o$. Then $\vec{f} \leq \vec{e}$ if and only if there exists $\vec{a}' \leq \vec{a}$ and an inclusion $\mathcal{O}(\vec{a}') \hookrightarrow \mathcal{O}(\vec{f})$.

Proof. Let i be the length of Bal, and let $s = r - i$ be the length of $\vec{a}$.

($\Rightarrow$) Let $\vec{a}' = (\deg \vec{a} - \sum_{j=2}^{s} f_{i+j}, f_{i+2}, f_{i+3}, \ldots, f_r)$. Then since $\vec{f} \leq \vec{e}$, we see that $\vec{a}' \leq \vec{a}$ and $a'_1 = \deg \vec{a} - \sum_{j=2}^{s} f_{i+j} \leq f_{i+1}$. Since $a'_j \leq f_{i+j}$ for all j, there is an inclusion $\mathcal{O}(\vec{a}') \hookrightarrow \mathcal{O}(\vec{f})$.

($\Leftarrow$) Suppose that we have $\mathcal{O}(\vec{a}') \hookrightarrow \mathcal{O}(\vec{f})$. We claim that $a'_{s-j} \leq f_{r-j}$ for all $0 \leq j < s$. Suppose towards a contradiction that $a'_{s-j} > f_{r-j}$. Then there are no nonzero homomorphisms $\mathcal{O}(a'_{s-j}, \ldots, a'_s)$ to $\mathcal{O}(f_1, \ldots, f_{r-j})$. But then we must have $\mathcal{O}(a'_{s-j}, \ldots, a'_s) \hookrightarrow \mathcal{O}(f_{r-j+1}, \ldots, f_r)$. A rank $j+1$ vector bundle cannot be a subsheaf of a rank j vector bundle, so we get a contradiction.

Having proved the claim, it follows that for all $0 \leq j \leq s-1$,

$$f_{r-j} + \cdots + f_r \geq a'_{s-j} + \cdots + a'_s \geq a_{s-j} + \cdots + a_s = e_{r-j} + \cdots + e_r,$$

or equivalently $\sum_{j=1}^k f_j \le \sum_{j=1}^k e_j$ for all $k \ge i$. It remains to show that $f_1 + \cdots + f_k \le e_1 + \cdots + e_k$ for $k < i$. Let $\vec{g} = (f_1, \ldots, f_{i-1}, g_i)$, where $g_i = \deg \mathrm{Bal} - (f_1 + \cdots + f_{i-1})$. Then $\deg \vec{g} = \deg \mathrm{Bal}$, and $g_i \ge f_i$. Since Bal is balanced, $\vec{g} \le \mathrm{Bal}$. So for $k < i$,

$$f_1 + \cdots + f_k = g_1 + \cdots + g_k \le \mathrm{Bal}_1 + \cdots + \mathrm{Bal}_k = e_1 + \cdots + e_k.$$

□

Proposition 2.2.4. Let $E = \mathcal{O}(\vec{e}), F = \mathcal{O}(\vec{f})$ be vector bundles on $\mathbb{P}^1$ of the same degree and rank. Then the following are equivalent:

1. there exists a family of vector bundles on $\mathbb{P}^1_{k[t]}$ whose restriction to the fiber at 0 is $\mathcal{O}(\vec{f})$ and away from 0 is $\mathcal{O}(\vec{e})$;
2. $\overline{\mathbf{\Sigma}}_{\vec{e}} \supset \mathbf{\Sigma}_{\vec{f}}$;
3. $h^1\mathcal{O}(\vec{e})(m) \le h^1\mathcal{O}(\vec{f})(m)$ for all $m \in \mathbb{Z}$;
4. $\sum_{i=1}^k e_i \ge \sum_{i=1}^k f_i$ for all $1 \le k \le r$, i.e. $\vec{e} \ge \vec{f}$;
5. $\mathcal{O}(\vec{f})$ admits a flag of subbundles of type $\vec{e}$;
6. $\mathcal{O}(\vec{e})$ is realizable as an extension of type $\vec{f}$.

Proof. We will prove (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇔ (5) ⇒ (2), and delay the proof of (2) ⇒ (6) ⇒ (1) to the next subsection.

(1) ⇒ (2): clear.

(2) ⇒ (3): this follows from upper-semicontinuity.

(3) ⇒ (4): Suppose for contradiction that there exists some k such that $\sum_{i=1}^k f_i > \sum_{i=1}^k e_i$. Let k be the maximal index for which this occurs. Then $\sum_{i=k+1}^r f_i < \sum_{i=k+1}^r e_i$ but $\sum_{i=k+2}^r f_i \ge \sum_{i=k+2}^r e_i$. This implies that $f_{k+1} < e_{k+1}$. Now $h^0\mathcal{O}(\vec{f})(-f_{k+1} - 1) = f_{k+1} + \cdots + f_r - (r-k)f_{k+1} < e_{k+1} + \cdots + e_r - (r-k)f_{k+1} \le h^0\mathcal{O}(\vec{e})(-f_{k+1} - 1)$. Equivalently, $h^1\mathcal{O}(\vec{f})(-f_{k+1} - 1) \le h^1\mathcal{O}(\vec{e})(-f_{k+1} - 1)$, which contradicts (3).

(4) ⇔ (5): We proceed by induction on k. When $k = 1$, $\vec{e} = \vec{f}$ and both (4) and (5) are trivial. Now suppose that (4) ⇔ (5) is true for all ranks less than k. Write $\vec{e} = (m_o, \ldots, m_o, \vec{a})$ where $a_1 > m_o$. Then Lemma 2.2.3 shows that $\vec{e} \ge \vec{f}$ if and only if there exists a subbundle $\mathcal{O}(\vec{a}') \hookrightarrow \mathcal{O}(\vec{f})$ with $\vec{a}' \le \vec{a}$. By induction, $\vec{a}'$ admits a flag of subbundles of type $\vec{a}$ if and only if $\vec{a}' \le \vec{a}$. Adjoining the inclusion $\mathcal{O}(\vec{a}') \hookrightarrow \mathcal{O}(\vec{f})$ to this flag proves the desired equivalence.

(5) ⇒ (2): The proof of this part depends on results in future sections. We show that there exists an irreducible algebraic stack parametrizing families of vector bundles equipped with flags of subbundles of type $\vec{e}$, which maps properly to $\mathcal{B}_{r,d}$. The uniqueness of the Harder-Narasimhan flag for $\mathcal{O}(\vec{e})$ shows that this is an isomorphism over $\mathbf{\Sigma}_{\vec{e}}$, and the irreducibility of this stack shows that the image is exactly the closure $\overline{\mathbf{\Sigma}}_{\vec{e}}$. □

An explicit smooth cover of $\mathcal{B}_{r,d}$

In the literature, the algebraicity of the stack of vector bundles on a fixed projective scheme is usually proven by taking open loci in increasingly large Quot schemes. In the case of $\mathbb{P}^1$, there is a nice family of affine spaces which forms a smooth cover of $\mathcal{B}_{r,d}$. This smooth cover will be used in Section 2.3 to prove irreducibility and smoothness of the resolution of singularities we construct.

Let $U_{\vec{f}}$ be the affine space corresponding to the vector space $H^1\mathcal{E}nd(\mathcal{O}(\vec{f}))$. After fixing an isomorphism $\mathcal{O}(\vec{f}) \cong Q_1 \oplus Q_2 \oplus \cdots \oplus Q_{m+1}$, where each Q_i is perfectly balanced and $\mu(Q_1) \leq \mu(Q_2) \leq \cdots \leq \mu(Q_{m+1})$, we may identify $H^1\mathcal{E}nd(\mathcal{O}(\vec{f})) \cong \oplus_{j=2}^m \operatorname{Ext}^1_{\mathbb{P}^1}(Q_j, \oplus_{i=1}^{j-1} Q_i)$.

Lemma 2.2.5. Let A, B be vector bundles on $\mathbb{P}^1$, and let $0 \to A \to E \to B \to 0$ be an extension. Then $h^1E(m) \leq h^1(A \oplus B)(m)$ for all $m \in \mathbb{Z}$.

Proof. The long exact sequence in cohomology induced by

$$0 \to A(m) \to E(m) \to B(m) \to 0$$

shows that $h^1E(m) \leq h^1A(m) + h^1B(m) = h^1(A \oplus B)(m)$. $\square$

Combined with (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) from Proposition 2.2.4, this implies that

Lemma 2.2.6. Let $\vec{a}, \vec{b}$ be two splitting types of ranks s and $r-s$, let $\mathcal{O}(\vec{f}) = \mathcal{O}(\vec{a}) \oplus \mathcal{O}(\vec{b})$, and let $0 \to \mathcal{O}(\vec{a}) \to \mathcal{O}(\vec{e}) \to \mathcal{O}(\vec{b}) \to 0$ be an extension. Then $\vec{e} \geq \vec{f}$. $\square$

Lemma 2.2.7. Let $\vec{e}$ be realizable as an extension of type $\vec{f}$. Then $\vec{e} \geq \vec{f}$. In particular, if $\mathcal{O}(\vec{f}) = Q_1 \oplus \cdots \oplus Q_{m+1}$ is a decomposition into perfectly balanced bundles such that $\mu(Q_i) < \mu(Q_{i+1})$, then $e_r \leq \mu(Q_{m+1})$ and $\operatorname{Ext}^1(\mathcal{O}(\vec{e}), Q_{m+1}) = 0$. If Q is perfectly balanced with $\mu(Q) > \mu(Q_{m+1})$, then $\dim \operatorname{Ext}^1(Q, \mathcal{O}(\vec{e}))$ does not depend on $\vec{e}$.

Proof. The implications are clear from the statement that $\vec{e} \geq \vec{f}$. To prove this, we proceed by induction on m. If $m = 1$, then this is Lemma 2.2.6. Let $\vec{a} = Q_1 \oplus \cdots \oplus Q_m$. If $\vec{e}$ is realizable as an extension of type $\vec{f}$, then there exists an exact sequence

$$0 \to \mathcal{O}(\vec{b}) \to \mathcal{O}(\vec{e}) \to Q_{m+1} \to 0,$$

where $\mathcal{O}(\vec{b})$ is realizable as an extension of type $\vec{a}$. Lemma 2.2.6 implies that $\vec{e} \geq \vec{f}'$, where $\mathcal{O}(\vec{f}') = \mathcal{O}(\vec{b}) \oplus Q_{m+1}$. By induction, $\vec{b} \geq \vec{a}$ and the largest summand of both are less than $\mu(Q_{m+1})$. So $\vec{f}' \geq \vec{f}$. Putting the inequalities together, we get $\vec{e} \geq \vec{f}$. $\square$

Proposition 2.2.8. Let $\mathbf{U}_{\vec{f}}$ be the functor defined as follows. Let T be a k-scheme, and let $p : T \times \mathbb{P}^1 \to \mathbb{P}^1$. A morphism $T \to \mathbf{U}_{\vec{f}}$ is the data of vector bundles $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_{m+1}$ on $T \times \mathbb{P}^1$, the data of extensions

$$0 \to \mathcal{K}_i \to \mathcal{K}_{i+1} \to p^*Q_{i+1} \to 0,$$

plus an isomorphism $\mathcal{K}_1 \cong p^*Q_1$. Then there exist a sequences of stacks with maps $\mathbf{U}^{(0)}_{\vec{f}} \to \mathbf{U}^{(1)}_{\vec{f}} \to \cdots \to \mathbf{U}^{(m+1)}_{\vec{f}}$, so that $\mathbf{U}^{(0)}_{\vec{f}} \to \mathbf{U}^{(1)}_{\vec{f}}$ is a $\mathrm{GL}_{\mathrm{rk}\, Q_1}$-torsor, $\mathbf{U}^{(i)}_{\vec{f}} \to \mathbf{U}^{(i+1)}_{\vec{f}}$ where $i \geq 1$ is an open locus inside a vector bundle over $\mathbf{U}_{\vec{f}}^{(i+1)}$, $\mathbf{U}_{\vec{f}}^{(0)} \cong \mathbf{U}_{\vec{f}}$, and $\mathbf{U}_{\vec{f}}^{(m+1)} = \bigcup_{\vec{e}\geq\vec{f}} \mathbf{\Sigma}_{\vec{e}} \subset \mathcal{B}_{r,d}$. In particular, the map $\mathbf{U}_{\vec{f}} \to \mathcal{B}_{r,d}$ is surjective onto $\bigcup_{\vec{e}\geq\vec{f}} \mathbf{\Sigma}_{\vec{e}}$. Moreover, $\mathbf{U}_{\vec{f}}$ is an algebraic space.

Proof. First we explain the construction of $\mathbf{U}_{\vec{f}}^{(i)}$. We start with $\mathbf{U}_{\vec{f}}^{(m+1)} = \bigcup_{\vec{e}\geq\vec{f}} \mathbf{\Sigma}_{\vec{e}}$. Let T be a k-scheme with maps $p : T \times \mathbb{P}^1 \to \mathbb{P}^1$ and $\pi : T \times \mathbb{P}^1 \to T$. By induction, we assume that a morphism $T \to \mathbf{U}_{\vec{f}}^{(k)}$ is the data of extensions

$$0 \to \mathcal{K}_i \to \mathcal{K}_{i+1} \to p^*Q_{i+1} \to 0$$

where $k \leq i \leq m$, such that $R^1\pi_*\mathcal{H}om(\mathcal{K}_k, p^*Q_k) = 0$ and $\mathcal{K}_{m+1}$ splits as at worst $\vec{f}$. Then we define

$$\mathbf{U}_{\vec{f}}^{(k-1)} = \pi_*\mathcal{H}om(\mathcal{K}_k, p^*Q_k) \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2),$$

where $\mathcal{Z}_i$ are defined as follows. Let $\mathcal{K}_{k-1}$ be the kernel of the universal map $\mathcal{K}_k \to p^*Q_k$ on $\pi_*\mathcal{H}om(\mathcal{K}_k, p^*Q_k) \times \mathbb{P}^1$. Then $\mathcal{Z}_1 = \mathrm{Supp} R^1\pi_*\mathcal{H}om(\mathcal{K}_{k-1}, p^*Q_{k-1})$, and $\mathcal{Z}_2$ is the closed locus where the map $\mathcal{K}_{k-1} \to p^*Q_{k-1}$ fails to be surjective.

Continue this process until we have constructed $\mathbf{U}_{\vec{f}}^{(1)}$. Then $\mathbf{U}_{\vec{f}}^{(0)}$ is defined to be $\underline{\mathrm{Isom}}(\mathcal{K}_1, p^*Q_1)$ over the open locus in $\mathbf{U}_{\vec{f}}^{(1)}$ where $\mathcal{K}_1$ is isomorphic to Q_1. This is open because Q_1 is perfectly balanced. The functor of points of $\mathbf{U}_{\vec{f}}^{(0)}$ is that of $\mathbf{U}_{\vec{f}}$, with the extra conditions that $R^1\pi_*\mathcal{H}om(\mathcal{K}_i, p^*Q_i) = 0$ for all $2 \leq i \leq m+1$ and $\mathcal{K}_{m+1}$ splits as at worst $\vec{f}$. However, these extra conditions are also automatically satisfied by points of $\mathbf{U}_{\vec{f}}$ due to Lemma 2.2.7. So $\mathbf{U}_{\vec{f}}^{(0)} \cong \mathbf{U}_{\vec{f}}$. Since $\mathbf{U}_{\vec{f}} \to \mathcal{B}_{r,d}$ is flat and locally finitely presented, it is open. The modular nature of the image of $\mathbf{U}_{\vec{f}}$ shows that the image must be a union of $\mathbf{\Sigma}_{\vec{e}}$, where $\vec{e} \geq \vec{f}$ due to Lemma 2.2.7. We also know that its image contains $\mathbf{\Sigma}_{\vec{f}}$. So $\mathbf{U}_{\vec{f}} \to \mathcal{B}_{r,d}$ must be surjective onto $\bigcup_{\vec{e}\geq\vec{f}} \mathbf{\Sigma}_{\vec{e}}$.

To show that $\mathbf{U}_{\vec{f}}$ is an algebraic space, by Theorem 2.2.5 of [Con07], it suffices to check that the geometric points have no nontrivial automorphisms. Given an extension $k \to k'$, a k'-point of $\mathbf{U}_{\vec{f}}$ is the data of vector bundles K_i on $\mathbb{P}^1_{k'}$, the data of extensions

$$0 \to K_i \to K_{i+1} \to Q_{i+1} \otimes_k k' \to 0$$

for each $1 \leq i \leq m$, and an isomorphism $K_1 \cong Q_1 \otimes_k k'$. The automorphism group of this point is the product of the autormophism groups of each extension. So it suffices to show that each extension has no automorphisms. Lemma 2.2.9 below shows that since $\mathrm{Hom}_{\mathbb{P}^1_{k'}}(Q_{i+1} \otimes_k k', K_i) = 0$ for all $1 \leq i \leq m$, the extensions have no nontrivial automorphisms. ∎

Lemma 2.2.9. Let X be a proper variety over k, with coherent sheaves A, B, and an extension

$$\xi : 0 \to A \xrightarrow{i} E \xrightarrow{q} B \to 0$$

on X. Then there is a map $\mathrm{Hom}(B, A) \to \mathrm{Hom}(E, E)$, by precomposing with q and postcomposing with i. There is also a map $\mathrm{Aut}(\xi) \to \mathrm{Hom}(E, E)$ defined by identifying an automorphism of the extension with $\phi : E \to E$, and sending it to $\phi - \mathrm{id}_E$. Both maps are injective and maps onto the same image. In particular, if $\mathrm{Hom}(B, A) = 0$, then $\mathrm{Aut}(\xi)$ is trivial.

Proof.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & E & \xrightarrow{q} & B & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \phi} & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & E & \xrightarrow{q} & B & \longrightarrow & 0 \end{array}$$

Since $q\phi = q$, we have $q(\phi - \mathrm{id}_E) = 0$. Since $0 \to \mathrm{Hom}(E, A) \to \mathrm{Hom}(E, E) \to \mathrm{Hom}(E, B)$ is exact, we know that $q\phi = q$ if and only if $\phi - \mathrm{id}_E$ factors through $i : A \to E$. Similarly, $\phi i = i$ if and only if $\phi - \mathrm{id}_E$ also factors through $q : E \to B$. $\square$

Let X be proper over $\operatorname{Spec} k$, and let A, B be locally free sheaves on X such that $\mathrm{Hom}_X(B, A) = 0$ and $\mathrm{Ext}^i_X(B, A) = 0$ for $i \geq 2$. Let $p : X \times \mathrm{Ext}^1_X(B, A) \to X$. We claim that the T-points of $\mathrm{Ext}^1_X(B, A)$ is the set of extensions $0 \to A_T \to E \to B_T \to 0$ on $T \times_k X$ modulo isomorphisms of extensions. It suffices to show that a T-point of the affine space $\mathrm{Ext}^1_X(B, A)$ corresponds to an element of the vector space $\mathrm{Ext}^1_{X_T}(B_T, A_T)$. Under these hypotheses, if $p_T : X_T = X \times_k T \to X$, then by base change, $\mathrm{Ext}^1_{X_T}(B_T, A_T) = H^0(T, \mathcal{O}_T) \otimes \mathrm{Ext}^1_X(B, A)$. Now, if one has a map $T \to \mathrm{Ext}^1_X(B, A) \cong H^1(X, A \otimes B^\vee)$, then the map on global sections induces a map $H^1(X, A \otimes B^\vee)^\vee \to H^0(T, \mathcal{O}_T)$, or equivalently an element of $H^0(T, \mathcal{O}_T) \otimes \mathrm{Ext}^1_X(B, A)$.

The identity map $\mathrm{id} : \mathrm{Ext}^1_X(B, A) \to \mathrm{Ext}^1_X(B, A)$ induces a universal extension $0 \to p^*A \to \mathcal{E} \to p^*B \to 0$ on $X \times_k \mathrm{Ext}^1_X(B, A)$. Chasing through the correspondence above and using base change, we see that this extension is indeed universal in the sense that the extension corresponding to a T-point is the pullback of this universal extension. Applying this argument relatively and inductively gives us the following theorem.

Proposition 2.2.10. There is a natural isomorphism $\mathbf{U}_{\vec{f}} \to U_{\vec{f}}$.

Proof. Let $\mathcal{O}(\vec{f}) = Q_1 \oplus \cdots \oplus Q_{m+1}$ where Q_i is perfectly balanced and $\mu(Q_i) < \mu(Q_{i+1})$. We realize $U_{\vec{f}}$ as the top of a tower of affine spaces

$$U_{\vec{f}} = U_{\vec{f},m} \to U_{\vec{f},m-1} \to \cdots \to U_{\vec{f},1},$$

and inductively show that

- there is an identification $U_{\vec{f},i} \cong \oplus_{k=1}^i \mathrm{Ext}^1(Q_{k+1}, \oplus_{j=1}^k Q_j)$;
- Let $p : U_{\vec{f},i} \times \mathbb{P}^1 \to \mathbb{P}^1$. Then a T-point of $U_{\vec{f},i}$ is the data of isomorphism classes of extensions
$$0 \to \mathcal{K}_k \to \mathcal{K}_{k+1} \to p^*Q_{k+1} \to 0$$
on $T \times \mathbb{P}^1$, where $k \leq i$, $\mathcal{K}_1 \cong p^*Q_1$.

Let $U_{\vec{f},1} = \mathrm{Ext}^1_{\mathbb{P}^1}(Q_2, Q_1)$, which satisfies the claim by the argument directly preceding this proof. By induction, assume that the claim is true for $U_{\vec{f},i}$. Abusing notation, let

$$0 \to \mathcal{K}_k \to \mathcal{K}_{k+1} \to p^*Q_{k+1} \to 0$$

where $1 \leq k \leq i$ denote the isomorphism classes of extensions on $U_{\vec{f},i} \times \mathbb{P}^1$ corresponding to the identity map for $U_{\vec{f},i}$.

$$\begin{tikzcd}
T \times \mathbb{P}^1 \arrow[r, "\widetilde{q}"] \arrow[d, "\widetilde{\pi}"] & U_{\vec{f},i} \times \mathbb{P}^1 \arrow[r, "p"] \arrow[d, "\pi"] & \mathbb{P}^1 \\
T \arrow[r, "q"] & U_{\vec{f},i} &
\end{tikzcd}$$

Figure 2.1: Relevant maps for the proof of Proposition 2.2.10

Now we let $U_{\vec{f},i+1}$ be the total space of the vector bundle $\mathcal{V} = R^1\pi_*\mathcal{H}om(p^*Q_{i+2}, \mathcal{K}_{i+1})$. Note that by Lemma 2.2.7 and cohomology and base change, $\mathcal{V}$ is a vector bundle. A T-point of $U_{\vec{f},i+1}$ is the data of a morphism $q : T \to U_{\vec{f},i}$ and a global section of $q^*\mathcal{V} = R^1\widetilde{\pi}_*\widetilde{q}^*\mathcal{H}om(p^*Q_{i+2}, \mathcal{K}_{i+1})$. (The relevant maps are labelled in Figure 2.1.) Since

$$R\mathcal{H}om(\widetilde{q}^*p^*Q_{i+2}, \widetilde{q}^*\mathcal{K}_{i+1}) = \mathcal{H}om(\widetilde{q}^*p^*Q_{i+2}, \widetilde{q}^*\mathcal{K}_{i+1}),$$

and $R\widetilde{\pi}_*\mathcal{H}om(\widetilde{q}^*p^*Q_{i+2}, \widetilde{q}^*\mathcal{K}_{i+1})$ is only nonzero for $R^1\widetilde{\pi}_*$, we have that

$$H^0(T, q^*\mathcal{V}) \cong H^0(T, R^1\widetilde{\pi}_*\mathcal{H}om(\widetilde{q}^*p^*Q_{i+2}, \widetilde{q}^*\mathcal{K}_{i+1})) \cong \mathrm{Ext}^1_{T\times\mathbb{P}^1}(\widetilde{q}^*p^*Q_{i+2}, \widetilde{q}^*\mathcal{K}_{i+1}).$$

In particular, a section of $q^*\mathcal{V}$ is indeed an isomorphism class of an extension

$$0 \to \mathcal{K}_{i+1} \to \mathcal{K}_{i+2} \to \widetilde{q}^*p^*Q_{i+2} \to 0$$

on $T\times\mathbb{P}^1$. Combined with the interpretation of a T-point of $U_{\vec{f},i}$, this proves that the functor of points of $U_{\vec{f},i+1}$ is as claimed by the inductive hypothesis.

Moreover, since $U_{\vec{f},i}$ is in fact affine space, $\mathcal{V}$ is trivial and may be noncanonically identified with the fiber at the point of $U_{\vec{f},i}$ that corresponds to the split extensions, where the fiber is $\mathrm{Ext}^1(Q_{i+2}, \oplus_{j=1}^{i+1} Q_i)$. Hence $U_{\vec{f},i+1}$ is isomorphic to the affine space $\oplus_{k=1}^{i+1}\mathrm{Ext}^1(Q_{k+1}, \oplus_{j=1}^{k} Q_j)$.

Since $\mathbf{U}_{\vec{f}}$ is an algebraic space, the functor of points for $\mathbf{U}_{\vec{f}}$ is in fact valued in sets, i.e. $\mathbf{U}_{\vec{f}}(T) \simeq \pi_0(\mathbf{U}_{\vec{f}}(T))$. Comparing the functor of points for $\mathbf{U}_{\vec{f}}$ and $U_{\vec{f}} = U_{\vec{f},m}$, we see that $\mathbf{U}_{\vec{f}}$ and $U_{\vec{f}}$ represent exactly the same sheaf. Therefore there is an isomorphism $\mathbf{U}_{\vec{f}} \to U_{\vec{f}}$. □

Proof of (2) $\Rightarrow$ (6) $\Rightarrow$ (1) *of Proposition 2.2.4.* (6) $\Rightarrow$ (1): Recall that $U_{\vec{f}}$ is just an affine space with $\Sigma_{\vec{f}} = \{0\}$. (6) says that there is a point in $U_{\vec{f}}$ corresponding to the extension of type $\vec{f}$ realizing $\vec{e}$. By considering a line connecting this point to 0, we find the desired family in (1).

(2) $\Rightarrow$ (6): This follows from Proposition 2.2.8 and the discussion about the image of $\mathbf{U}_{\vec{f}}$. □

Fitting support scheme structure

We review in this subsection the Fitting support scheme structure on splitting loci, which is known to be reduced.

Let $\mathcal{F}$ be a finitely presented coherent sheaf on a locally Noetherian scheme X, and let

$$\mathcal{E}_1 \xrightarrow{\Phi} \mathcal{E}_0 \to \mathcal{F} \to 0$$

be one such presentation, where $\mathcal{E}_0, \mathcal{E}_1$ are locally free of rank n, m. Then $\mathrm{Fitt}_i(\mathcal{F}) \subseteq \mathcal{O}_X$ is the ideal sheaf generated by the $(n-i)$-minors of Φ. The Fitting ideal $\mathrm{Fitt}_i(\mathcal{F})$ is independent of the choice of presentation (e.g. Corollary 20.4 of [Eis95]).

The Fitting support scheme strucutre on $\overline{\Sigma}_{\vec{e}}$ is given by the ideal sheaf $\sum_{m\in\mathbb{Z}} \mathrm{Fitt}_{i_m}(R^1\pi_*\mathcal{E}(m))$, where $i_m = h^1\mathcal{O}(\vec{e})(m) - 1$ and $\pi : B \times \mathbb{P}^1 \to B$ is the projection. This is the most common scheme structure used in the literature. Condition (3) of Proposition 2.2.4 shows that this ideal cuts out the right closed points. The following result shows that the Fitting support scheme structure agrees with the reduced closed subscheme structure.

Theorem 2.2.11 ([LLV25]). The Fitting support scheme structure on splitting loci is normal and Cohen-Macaulay, and in particular reduced.

The Fitting support scheme structure is more amenable to computations. By working out a two-term complex of free modules representing the direct image $R\pi_*\mathcal{E}(m)$, Corollary 5.4 of [ES08] writes down explicit equations for splitting loci inside $U_{\vec{e}_d}$, where $\vec{e}_d = (0^{r-1}, d)$. Notably, in the rank 2 case, splitting loci are the affine cones over secant varieties of the rational normal curve.

Example 2.2.12 (Rank 2 splitting loci, [ES08]). Let $d > 1$ be a positive integer. Let $\mathrm{Spec}\, A$ be the affine space corresponding to the vector space $\mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}(d), \mathcal{O})$, with coordinates $a_0, \ldots, a_{d-2}$. Let $\mathcal{E}$ be the universal vector bundle on $\mathbb{P}^1_A$. Then for $1 \leq k \leq d-1$, the following complex represents $R\pi_*\mathcal{E}(-k-1)$:

$$0 \to A^{d-k} \xrightarrow{B_{k,d-k}} A^k \to 0,$$

where $B_{k,d-k}$ is the $k \times (d-k)$ Hankel matrix

$$B_{k,d-k} = \begin{bmatrix} a_0 & a_1 & \cdots & a_{d-k-1} \\ a_1 & a_2 & \cdots & a_{d-k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k-1} & a_k & \cdots & a_{d-2} \end{bmatrix}.$$

Let $\frac{d}{2} + 1 \leq e < d$, so that $(d-e, e)$ is an unbalanced splitting type that appears in $\mathrm{Ext}^1(\mathcal{O}(d), \mathcal{O})$ and is not $(0, d)$. Then one can check using the Fitting support definition that the ideal $I_{\overline{\Sigma}_{(d-e,e)}}$ is precisely the ideal of maximal minors $I_{d-e+1}(B_{d-e+1,e-1})$. It is well-known that the latter is the homogeneous ideal of $\mathrm{Sec}^{d-e-1}(C_{d-2})$, the $(d-e-1)$-th secant variety of the rational normal curve of degree $d-2$ in $\mathbb{P}^{d-2}$ (e.g. [Con+18]). In particular, when $d - e = 1$, $I_{\overline{\Sigma}_{(1,d-1)}} = I_2(B_{2,d-2})$ and $\overline{\Sigma}_{(1,d-1)} \subset \mathrm{Ext}^1(\mathcal{O}(d), \mathcal{O})$ is the cone over the rational normal curve of degree $d-2$. The case of $\overline{\Sigma}_{(1,3)} \subseteq \mathrm{Ext}^1(\mathcal{O}(4), \mathcal{O})$, along with the resolution of singularities we construct, is illustrated in Figure 2.2.

2.3 A resolution of singularities of $\overline{\Sigma}_{\vec{e}}$

We describe the construction of a resolution of singularities for $\overline{\Sigma}_{\vec{e}}$. This was initially inspired by the use of Quot schemes to compute the classes of splitting loci in [Lar21b].

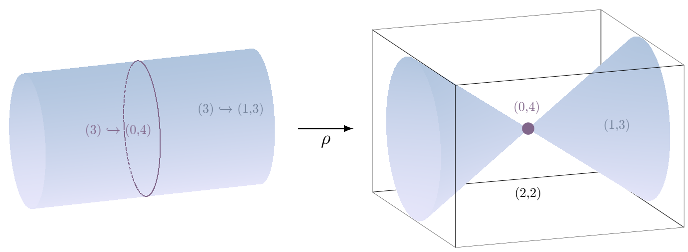


Figure 2.2: The splitting locus $\overline{\Sigma}_{(1,3)}$ in $\mathrm{Ext}^1(\mathcal{O}(4), \mathcal{O}) \cong \mathbb{A}^3$ and its resolution by the relative Quot scheme parametrizing $\mathcal{O}(3)$-subsheaves

As explained in the introduction, for any given $\vec{e}$, there may be multiple choices of resolutions. Specifically they are indexed by certain $\vec{e}$-admissible sets $I \subseteq [m]$, where m is the number of subbundles in the Harder-Narasimhan flag of $\mathcal{O}(\vec{e})$.

Definition 2.3.1. Let B an algebraic stack over k, and let $\mathcal{E}$ be a vector bundle of rank r on $B \times_k \mathbb{P}^1$. Let $\underline{r} = (r_1, \dots, r_m)$, $\underline{d} = (d_1, \dots, d_m)$, where $r \geq r_1 \geq r_2 \geq \cdots \geq r_m \geq 0$, $d_i \in \mathbb{Z}$. We define $\mathrm{FQuot}(B, \mathcal{E}; \underline{r}, \underline{d}) \to B$ to be the functor

$$(T \to \operatorname{Spec} k) \mapsto \left\{ \begin{array}{l} \text{A morphism } T \xrightarrow{f} B \text{ over } k\text{, and a flag of quotients} \\ f^*\mathcal{E} \twoheadrightarrow \mathcal{Q}_1 \twoheadrightarrow \mathcal{Q}_2 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{Q}_m \text{ on } T \times \mathbb{P}^1\text{, such that } \mathcal{Q}_i \text{ is} \\ \text{flat over } T\text{, and fiberwise of rank } r_i \text{ and degree } d_i \end{array} \right\} / \sim,$$

where we say that two flags are equivalent if there are isomorphisms $\mathcal{Q}_i \to \mathcal{Q}_i'$ making the following diagram commute:

$$\begin{tikzcd} f^*\mathcal{E} \arrow[r] \arrow[d, equal] & \mathcal{Q}_1 \arrow[r] \arrow[d] & \mathcal{Q}_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{Q}_m \arrow[d] \\ f^*\mathcal{E} \arrow[r] & \mathcal{Q}_1' \arrow[r] & \mathcal{Q}_2' \arrow[r] & \cdots \arrow[r] & \mathcal{Q}_m' \end{tikzcd} .$$

The abbreviation FQuot stands for flag Quot, also called hyper-Quot. The map $\mathrm{FQuot} \to B$ is representable by schemes due to the theory of Quot schemes. So if B is an algebraic stack, then for any choice of $\underline{r}, \underline{d}$, $\mathrm{FQuot}(B, \mathcal{E}; \underline{r}, \underline{d})$ is an algebraic stack. When $m = 1$ and $\underline{r} = (r_1), \underline{d} = (d_1)$, this is the usual relative Quot, which we denote by $\mathrm{Quot}^{r_1,d_1}(B, \mathcal{E})$, or $\mathrm{Quot}^{r_1,d_1}(\mathcal{E})$ if the base is clear.

Applying this definition to the pair $\mathcal{B}_{r,d}$ and the universal vector bundle $\mathcal{E}_{\mathrm{univ}}$ on $\mathcal{B}_{r,d} \times \mathbb{P}^1$, we get $\mathrm{FQuot}(\mathcal{B}_{r,d}, \mathcal{E}_{\mathrm{univ}}; \underline{r}, \underline{d}) \to \mathcal{B}_{r,d}$, which is the algebraic stack defined by the functor

$$(T \to \operatorname{Spec} k) \mapsto \left\{ \begin{array}{l} \text{A flag of quotients } \mathcal{E} \twoheadrightarrow \mathcal{Q}_1 \twoheadrightarrow \mathcal{Q}_2 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{Q}_m \text{ on } T \times \mathbb{P}^1\text{, flat over } T\text{,} \\ \text{such that } \mathcal{E} \text{ is a vector bundle of fiberwise rank } r \text{ and degree } d\text{,} \\ \text{and } \mathcal{Q}_i \text{ is a coherent sheaf fiberwise of rank } r_i \text{ and degree } d_i \end{array} \right\} / \sim$$

$$\cong \left\{ \begin{array}{l} \text{A flag of subsheaves } \mathcal{E}_1 \hookrightarrow \mathcal{E}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_{m+1} = \mathcal{E} \text{ on } T \times \mathbb{P}^1\text{, flat over } T\text{,} \\ \text{such that } \mathcal{E} \text{ is a vector bundle of fiberwise rank } r \text{ and degree } d\text{,} \\ \text{and } \mathcal{E}_i \text{ is locally free of rank } r - r_i \text{ and fiberwise degree } d - d_i \end{array} \right\} / \sim .$$

Here, the equivalence relation on flags of subsheaves is induced by the one on flags of quotients and the correspondence between flags of quotients and subsheaves. Let $\vec{e}$ be an arbitrary splitting type of rank r and degree d. Let the Harder-Narasimhan filtration of $\mathcal{O}(\vec{e})$ be

$$E_1 \subset E_2 \subset \cdots \subset E_{m+1} = \mathcal{O}(\vec{e}),$$

and let a corresponding flag of quotients be

$$\mathcal{O}(\vec{e}) \twoheadrightarrow Q_1 \twoheadrightarrow Q_2 \twoheadrightarrow \cdots \twoheadrightarrow Q_m.$$

Let $d_{i,HN} = \deg Q_i$, $r_{i,\mathrm{HN}} = \mathrm{rk}\, Q_i$, and let $\underline{r}_{\mathrm{HN}} = (r_{1,\mathrm{HN}}, \ldots, r_{m,\mathrm{HN}})$, $\underline{d}_{\mathrm{HN}} = (d_{1,\mathrm{HN}}, \ldots, d_{m,\mathrm{HN}})$.

Definition 2.3.2. Given a splitting type $\vec{e}$ and $I = \{i_1, \ldots, i_k\} \subseteq [m]$, let $\underline{r}_{I,\mathrm{HN}} = (r_{i_1,\mathrm{HN}}, \ldots, r_{i_k,\mathrm{HN}})$, and let $\underline{d}_{I,\mathrm{HN}} = (d_{i_1,\mathrm{HN}}, \ldots, d_{i_k,\mathrm{HN}})$. Let $E_{i_0} = 0, E_{i_{k+1}} = \mathcal{O}(\vec{e})$. We say that I is $\vec{e}$-admissible if for all $1 \le j \le k$, $E_{i_j}/E_{i_{j-1}}$ is balanced.

Example 2.3.3.

1. $I = \emptyset$ is $\vec{e}$-admissible if $\mathcal{O}(\vec{e})$ is balanced.
2. If the distinct entries of $\vec{e}$ are all at least 2 apart, then the only $\vec{e}$-admissible index set is $I = [m]$.
3. Consider $\vec{e} = (-2, -2, -1, -1, 0, 0)$. Then $\underline{r}_{\mathrm{HN}} = (4, 2), \underline{d}_{\mathrm{HN}} = (-6, -4)$, and the HN flag of subsheaves is

$$E_1 = \mathcal{O}(0, 0) \hookrightarrow E_2 = \mathcal{O}(-1, -1, 0, 0) \hookrightarrow \mathcal{O}(\vec{e}).$$

The empty set is not $\vec{e}$-admissible, but $\{1\}, \{2\}, \{1, 2\}$ are all $\vec{e}$-admissible.

Recall the definition of tame splitting types from the introduction. It's clear that $\vec{e}$ is tame if and only if there exists I which is $\vec{e}$-admissible and $|I| \le 1$.

Definition 2.3.4. We define

$$\widetilde{\mathbf{\Sigma}}_{\vec{e},I} = \mathrm{FQuot}(\mathcal{B}_{r,d}, \mathcal{E}_{\mathrm{univ}}; \underline{r}_{I,\mathrm{HN}}, \underline{d}_{I,\mathrm{HN}}),$$

and

$$\widetilde{\mathbf{\Sigma}}_{\vec{e},\mathrm{HN}} = \mathrm{FQuot}(\mathcal{B}_{r,d}, \mathcal{E}_{\mathrm{univ}}; \underline{r}_{\mathrm{HN}}, \underline{d}_{\mathrm{HN}}).$$

Theorem 2.3.5. Let $\vec{e}$ be a splitting type of rank r and degree d, and let I be $\vec{e}$-admissible. Then $\widetilde{\mathbf{\Sigma}}_{\vec{e},I}$ is a smooth and irreducible algebraic stack, proper over $\mathcal{B}_{r,d}$. The natural map $\widetilde{\mathbf{\Sigma}}_{\vec{e},I} \to \mathcal{B}_{r,d}$ is surjective onto $\overline{\mathbf{\Sigma}}_{\vec{e}}$, and is an isomorphism over $\mathbf{\Sigma}_{\vec{e}}$. In particular, $\dim \widetilde{\mathbf{\Sigma}}_{\vec{e},I} = -r^2 - u(\vec{e})$.

The stack $\widetilde{\mathbf{\Sigma}}_{\vec{e},I}$ is proper over $\mathcal{B}_{r,d}$ because relative Quot constructions are proper over the base. Let I be $\vec{e}$-admissible. We say that $\mathcal{O}(\vec{f})$ admits a flag of subbundles of type $(\vec{e}, I)$ if there exists $E'_{i_1} \subset \cdots \subset E'_{i_m} \subset \mathcal{O}(\vec{f})$ such that $\mathrm{rk}\, E'_{i_j} = r - r_{i_j}$, and $\deg E'_{i_j} = d - d_{i_j}$. We add one more equivalent condition to the list in Proposition 2.2.4.

Proposition 2.3.6. Let I be $\vec{e}$-admissible. Then $\vec{f} \le \vec{e}$ if and only if $\mathcal{O}(\vec{f})$ admits a flag of subbundles of type $(\vec{e}, I)$. (We call this condition (5').) Moreover, $\mathcal{O}(\vec{e})$ admits a unique flag of subbundles of type $(\vec{e}, I)$.

Proof. The same argument for (4) $\Leftrightarrow$ (5) in the proof of Proposition 2.2.4 applies here verbatim to show that (4) $\Leftrightarrow$ (5′), since Lemma 2.2.3 only requires that the first part of $\vec{e}$ is balanced.

Let $\vec{e} = (\vec{a}, \vec{b})$, where $\operatorname{rk} \vec{a} = s$ and $a_s < b_1$. Then up to automorphisms of $\mathcal{O}(\vec{b})$, we claim that there is a unique injection $\mathcal{O}(\vec{b}) \hookrightarrow \mathcal{O}(\vec{e})$. This is because after fixing a splitting $\mathcal{O}(\vec{e}) = \mathcal{O}(\vec{a}) \oplus \mathcal{O}(\vec{b})$, it's clear that every nonzero homomorphism $\mathcal{O}(\vec{b}) \to \mathcal{O}(\vec{e})$ factors through $\mathcal{O}(\vec{b})$. So the space of injections from $\mathcal{O}(\vec{b})$ to $\mathcal{O}(\vec{e})$ is an $\operatorname{Aut}(\mathcal{O}(\vec{b}))$-torsor. Now iteratively applying this claim shows that a flag of subbundles of type $(\vec{e}, I)$ of $\mathcal{O}(\vec{e})$ is unique. □

Assuming irreducibility of $\widetilde{\mathbf{\Sigma}}_{\vec{e},I}$, we have (2) $\Leftrightarrow$ (5) $\Leftrightarrow$ (5′) from Proposition 2.2.4. This implies that the natural map $\widetilde{\mathbf{\Sigma}}_{\vec{e},I} \to \mathcal{B}_{r,d}$ is surjective onto $\overline{\mathbf{\Sigma}}_{\vec{e}}$. The uniqueness statement from Proposition 2.3.6 shows that this map is an isomorphism over $\mathbf{\Sigma}_{\vec{e}}$. Thus to prove Theorem 2.3.5, it remains to show irreducibility and smoothness.

Irreducibility

We prove irreducibility for any hyper-Quot construction on $(\mathcal{B}_{r,d}, \mathcal{E}_{\text{univ}})$. The main idea is if $Z \subseteq X$ is a union of irreducible strata, such that only one stratum has codimension equal to the expected codimension of Z in X, then Z must be irreducible of the expected codimension and that particular stratum is dense in Z.

Definition 2.3.7. Let B be an irreducible scheme with a map $\phi_{\mathcal{E}} : B \to \mathcal{B}_{r,d}$, corresponding to $\mathcal{E}$ on $B \times \mathbb{P}^1$. We say that the pair $(B, \mathcal{E})$ is good, or the map $\phi_{\mathcal{E}}$ is good, if each splitting locus $\Sigma_{\vec{e}}(B)$ is irreducible of codimension $u(\vec{e})$.

Remark 2.3.8. Note that $\phi_{\mathcal{E}}$ may not be good even if it is smooth. Smoothness guarantees that the codimension of $\Sigma_{\vec{e}}$ is as expected, but $\Sigma_{\vec{e}}$ may become reducible after base change along a smooth map. However, note that the smooth covers $U_{\vec{f}} \to \mathcal{B}_{r,d}$ from Section 2.2 are good. Indeed, the description of $U_{\vec{f}}$ in Proposition 2.2.8 implies that the preimage of $\mathbf{\Sigma}_{\vec{e}}$ is irreducible. Note that since the codimension of $\overline{\Sigma}_{\vec{e}}$ is at most $u(\vec{e})$, when $\phi_{\mathcal{E}}$ is good, the closure of $\Sigma_{\vec{e}}$ is $\overline{\Sigma}_{\vec{e}}$.

Let B be a scheme of finite type with a good map $\phi_{\mathcal{E}} : B \to \mathcal{B}_{r,d}$. Let $0 \le s \le r$ and $d' \in \mathbb{Z}$. For this section, we are mostly thinking about Quot as parametrizing families of subsheaves with flat quotients. So it will be convenient to write $S_{s,d'}(B, \mathcal{E})$ to denote $\operatorname{Quot}^{r-s,d-d'}(B, \mathcal{E})$. When the base is clear, we abbreviate them as $S_{s,d'}(\mathcal{E})$ and $\operatorname{Quot}^{r-s,d-d'}(\mathcal{E})$. Since B is finite type, there exists $c \in \mathbb{Z}$, $m \in \mathbb{Z}$ such that $\mathcal{E}^\vee(c)$ and $\mathcal{E}(m)$ are relatively globally generated; equivalently, $R^1\pi_*\mathcal{E}^\vee(c-1) = 0$ and $R^1\pi_*\mathcal{E}(m-1) = 0$. For ease of notation, let $\mathcal{F} = \pi_*\mathcal{E}^\vee(c)$ and let $\mathcal{G} = \pi_*\mathcal{E}^\vee(c-1)$. Let $\mathcal{F}' = \pi_*\mathcal{E}(m)$, $\mathcal{G}' = \pi_*\mathcal{E}(m-1)$. Note that these are all vector bundles by cohomology and base change. The short exact sequence of Strømme (Proposition 1.1 [Str87]), applied to $\mathcal{E}^\vee(c)$ and $\mathcal{E}(m)$, after possibly dualizing and twisting, gives us

$$0 \to \mathcal{E} \to \pi^*\mathcal{F}(c) \to \pi^*\mathcal{G}(c+1) \to 0, \tag{2.1}$$

and

$$0 \to \pi^*\mathcal{G}'(-m-1) \to \pi^*\mathcal{F}'(-m) \to \mathcal{E} \to 0. \tag{2.2}$$

$$\begin{tikzcd} S_{s,d'}(\mathcal{E}) \arrow[r, hook, "\iota"] \arrow[dr] & S_{s,d'}(\pi^*\mathcal{F}(c)) \arrow[d] \\ & B \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Quot}^{r-s,d-d'}(\mathcal{E}) \arrow[r, hook, "j"] \arrow[dr] & \mathrm{Quot}^{r-s,d-d'}(\pi^*\mathcal{F}'(-m)) \arrow[d] \\ & B \end{tikzcd}$$

Figure 2.3: Two embeddings of a relative Quot scheme into a fiber bundle of Quot schemes

$$\begin{tikzcd} S_{s,d'}(\mathcal{E}) \times \mathbb{P}^1 \arrow[r, hook] \arrow[d] & S_{s,d'}(\pi^*\mathcal{F}(c)) \times \mathbb{P}^1 \arrow[r, "\widetilde{\gamma}"] \arrow[d, "f"] & B \times \mathbb{P}^1 \arrow[d, "\pi"] \\ S_{s,d'}(\mathcal{E}) \arrow[r, hook, "\iota"] \arrow[rr, bend right, "\sigma"'] & S_{s,d'}(\pi^*\mathcal{F}(c)) \arrow[r, "\gamma"] & B \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Quot}^{r-s,d-d'}(\mathcal{E}) \times \mathbb{P}^1 \arrow[r, hook] \arrow[d] & \mathrm{Quot}^{r-s,d-d'}(\pi^*\mathcal{F}'(-m)) \times \mathbb{P}^1 \arrow[r, "\widetilde{\gamma}"] \arrow[d, "f"] & B \times \mathbb{P}^1 \arrow[d, "\pi"] \\ \mathrm{Quot}^{r-s,d-d'}(\mathcal{E}) \arrow[r, hook, "j"] & \mathrm{Quot}^{r-s,d-d'}(\pi^*\mathcal{F}'(-m)) \arrow[r, "\gamma"] & B \end{tikzcd}$$

Figure 2.4: The relative Quot scheme and relavant maps. We abuse notation and use the same symbols for the two scenarios.

Using (2.1), by postcomposing, we can think of subsheaves of $\mathcal{E}$ with flat quotients as subsheaves of $\pi^*\mathcal{F}(c)$ with flat quotients. Similarly, using (2.2), we can think of quotients of $\mathcal{E}$ as quotients of $\pi^*\mathcal{F}'(-m)$ by precomposing. This gives rise to two different embeddings, as in Figure 2.3. For the discussion of irreducibility, we will only need the embedding ι. We label the relevant maps in Figure 2.4.

A subsheaf of $\pi^*\mathcal{F}(c)$ is a subsheaf of $\mathcal{E}$ if and only if its image in $\pi^*\mathcal{G}(c+1)$ is zero. So the image of the embedding ι in Figure 2.3 is closed, and is the zero locus of the section of the vector bundle $f_*\mathcal{H}om(\mathcal{S}, \widetilde{\gamma}^*\pi^*\mathcal{G}(c+1))$ corresponding to the composite $\mathcal{S} \to \widetilde{\gamma}^*\pi^*\mathcal{F}(c) \to \widetilde{\gamma}^*\pi^*\mathcal{G}(c+1)$, where $\mathcal{S}$ is the universal subsheaf on $S_{s,d'}(\pi^*\mathcal{F}(c)) \times \mathbb{P}^1$. Dually, the image of the closed embedding j in Figure 2.3 is the zero locus of the section of the vector bundle $f_*\mathcal{H}om(\widetilde{\gamma}^*\pi^*\mathcal{G}'(-m-1), \mathcal{Q})$, corresponding to the composite $\widetilde{\gamma}^*\pi^*\mathcal{G}'(-m-1) \to \widetilde{\gamma}^*\pi^*\mathcal{F}'(-m) \to \mathcal{Q}$, where $\mathcal{Q}$ is the universal quotient on $\mathrm{Quot}^{r-s,d-d'}(\pi^*\mathcal{F}'(-m)) \times \mathbb{P}^1$. Our calculations below will show, among other things, that these sections cut out $S_{s,d'}(\mathcal{E})$ and $\mathrm{Quot}^{r-s',d-d'}(\mathcal{E})$ as local complete intersections in their embeddings.

Lemma 2.3.9. Given $\vec{a}, r, d$, there is a unique splitting type $\vec{e}_b = \vec{e}_b(r, d, \vec{a})$ admitting a $\mathcal{O}(\vec{a})$-subsheaf which can be written as $(\mathrm{Bal}, \vec{a}_+)$, where Bal is balanced, and $\vec{a} = (\vec{a}_-, \vec{a}_+)$. This splitting type $\vec{e}_b$ is maximal among all $\vec{e}$ of rank r and degree d such that $\mathcal{O}(\vec{e})$ has a

$\mathcal{O}(\vec{a})$-subsheaf. Moreover, if $\vec{a}' \leq \vec{a}$, then $\vec{e}_b(r,d,\vec{a}') \leq \vec{e}_b(r,d,\vec{a})$.

Remark 2.3.10. For instance, if $\vec{a} = (0,0)$, and $(r,d) = (4,-1)$, then $\vec{e}_b = (-1,0,0,0)$. Notice that in this case, we can take $\vec{a}_+$ to be length $0, 1$ or 2. The claim of the lemma is that $\vec{e}_b$ is unique, but the decomposition into two parts may not be.

Proof of Lemma 2.3.9. Let $A_k = \sum_{j=0}^{k-1} a_{s-j}$. To form such an $\vec{e}_b$ amounts to choosing the length k of $\vec{a}_+$, and letting the first part be the balanced splitting type of the correct degree. The choice of a particular length would not work if the vector obtained is not a weakly increasing sequence, i.e. if $\left\lceil \frac{d-A_k}{r-k} \right\rceil > a_{s-k+1}$. It also would not work if the vector obtained, as a splitting type, corresponds to a vector bundle that does not admit a $\mathcal{O}(\vec{a})$-subsheaf. Recall that by Lemma 2.2.3, $\mathcal{O}(\vec{e}_b)$ admits an $\mathcal{O}(\vec{a})$-subsheaf if and only if $a_{s-j} \leq e_{b,r-j}$ for all $0 \leq j < s$.

There are two ways the lemma may be false. One is that no $\vec{a}_+$ works. We claim that in this case $\frac{d}{r} \geq a_{s-k+1}$ for all $1 \leq k \leq s$, so that it suffices to choose $k = 0$. Indeed, $k = s$ doesn't work if and only if $\left\lceil \frac{d-A_s}{r-s} \right\rceil > a_1$, which implies that $\frac{d}{r} \geq a_1$. Now since

$$\left\lceil \frac{d-A_{s-1}}{r-s+1} \right\rceil = \left\lceil \frac{d-A_s+a_1}{r-s+1} \right\rceil > a_1,$$

$k = s-1$ doesn't work if and only if $\left\lceil \frac{d-A_{s-1}}{r-s+1} \right\rceil > a_2$. The desired claim follows from induction like so.

Another way that this claim could fail is that there are two valid choices of k. Suppose that $k_1 > k_2$ both work, giving splitting types $\vec{e}_b$ and $\vec{e}'_b$. Since $k_1 > k_2$, $e'_{b,r-k_2} \leq e_{b,r-k_2}$. But in order for $\vec{e}'_b$ to admit an $\mathcal{O}(\vec{a})$-subsheaf, we must have $e'_{b,r-k_2} \geq a_{s-k_2} = e_{b,r-k_2}$. Therefore $e'_{b,r-k_2} = e_{b,r-k_2}$, which means that $\vec{e}_b$ must have been balanced up to the index $r-k_2$ to begin with. So in fact $\vec{e}_b = \vec{e}'_b$.

For the next claim, write $\vec{e}_b = (\mathrm{Bal}, \vec{a}_+)$, and let $k = \mathrm{len}(\vec{a}_+)$. (As explained in the remark above, there may be multiple ways of breaking $\vec{e}_b = (e_{b,1},\ldots,e_{b,r})$ up into two parts, but just choose one way.) Let $\vec{e}$ be some splitting type of rank r and degree d such that $\mathcal{O}(\vec{e})$ has a $\mathcal{O}(\vec{a})$-subsheaf. Then since $a_{s-j} \leq e_{r-j}$ for $0 \leq j < k$, we have for all $1 \leq i \leq k$,

$$\sum_{j=0}^{i-1} e_{b,r-j} = \sum_{j=0}^{i-1} a_{s-j} \leq \sum_{j=0}^{i-1} e_{r-j}.$$

Since the first part of $\vec{e}_b$ is balanced with $\deg \mathrm{Bal} \geq \sum_{j=1}^{r-k} e'_j$, we must have for $1 \leq i \leq r-k$,

$$\sum_{j=1}^{i} e_{b,j} \geq \sum_{j=1}^{i} e_j.$$

Together these inequalities show that $\vec{e}_b \geq \vec{e}$.

The second comparison follows along similar lines. Write $\vec{e}'_b = \vec{e}_b(r,d,\vec{a}') = (\mathrm{Bal}', \vec{a}'_+)$ and let $k' = \mathrm{len}(\vec{a}'_+)$. We divide into two cases: $k \geq k'$ and $k < k'$. First suppose $k \geq k'$. For $1 \leq i \leq k$, we have

$$\sum_{j=0}^{i-1} e_{b,r-j} = \sum_{j=0}^{i-1} a_{s-j} \leq \sum_{j=0}^{i-1} a'_{s-j} \leq \sum_{j=0}^{i-1} e'_{b,r-j}. \tag{2.3}$$

It remains to compare Bal and the first $r-k$ entries of Bal$'$. Both are balanced, with $\deg \mathrm{Bal} \geq \deg \mathrm{Bal}'|_{1,\dots,r-k}$. So for $1 \leq i \leq r-k$,

$$\sum_{j=1}^{i} e_{b,j} \geq \sum_{j=1}^{i} e'_{b,j}.$$

Therefore $\vec{e}_b(r,d,\vec{a}\,') \leq \vec{e}_b(r,d,\vec{a})$.

Now suppose $k < k'$. For $1 \leq i \leq k$, Equation 2.3 still holds, with the last inequality being in fact an equality in this case. Suppose that for some $k < i \leq k'$,

$$\sum_{j=0}^{i-1} e_{b,r-j} = |\vec{a}_+| + \sum_{j=k}^{r-1} e_{b,r-j} > \sum_{j=0}^{i-1} e'_{b,r-j} = \sum_{j=0}^{i-1} a'_{r-j}. \tag{2.4}$$

Since the inequality comparing the partial sums from the back changed as we went from summing the last k terms to summing the last i terms, it must be true that

$$e_{b,r-k} = \left\lceil \frac{d-|\vec{a}_+|}{r-k} \right\rceil > a'_{r-i+1} \geq e'_{b,r-k'} = \left\lceil \frac{d-|\vec{a}\,'_+|}{r-k'} \right\rceil.$$

In particular, $e_{b,j} \geq e'_{b,j}$ for all $1 \leq j \leq r-i$. However, that's impossible because together with Equation 2.4, this implies that $d = |\vec{e}_b| > |\vec{e}\,'_b| = d$. This shows that for $1 \leq i \leq k'$,

$$\sum_{j=0}^{i-1} e_{b,r-j} \leq \sum_{j=0}^{i-1} e'_{b,r-j}.$$

Now since the first $r-k'$ entries of $\vec{e}_b$ and $\vec{e}\,'_b$ are balanced with that of $\vec{e}_b$ having degree no less than that of $\vec{e}\,'_b$, we can conclude that $\vec{e}_b \geq \vec{e}\,'_b$. □

Lemma 2.3.11. Let $(\vec{a},\vec{e})$ be such that $\mathcal{O}(\vec{e})$ admits a $\mathcal{O}(\vec{a})$-subsheaf. Then

$$u(\vec{e}) - \mathrm{ext}^1(\mathcal{O}(\vec{a}),\mathcal{O}(\vec{e})) \geq 0,$$

with equality if and only if $\vec{e} = \vec{e}_b(r,d,\vec{a})$.

Proof. Write $\vec{e} = (e_1,\dots,e_r)$ and $\vec{a} = (a_1,\dots,a_s)$. We can write

$$\begin{aligned} u(\vec{e}) - \mathrm{ext}^1(\mathcal{O}(\vec{a}),\mathcal{O}(\vec{e})) &= \mathrm{ext}^1(\mathcal{O}(e_1,\dots,e_{r-s}),\mathcal{O}(\vec{e})) + \sum_{i=0}^{s-1}(h^1(\mathcal{O}(\vec{e})\otimes\mathcal{O}(e_{r-i})^\vee) - h^1(\mathcal{O}(\vec{e})\otimes\mathcal{O}(a_{s-i})^\vee)) \\ &= u(e_1,\dots,e_{r-s}) + \sum_{i=0}^{s-1}(h^1(\mathcal{O}(\vec{e})\otimes\mathcal{O}(e_{r-i})^\vee) - h^1(\mathcal{O}(\vec{e})\otimes\mathcal{O}(a_{s-i})^\vee)). \end{aligned}$$

Since $\mathcal{O}(\vec{e})$ has a $\mathcal{O}(\vec{a})$-subsheaf, we must have that $a_{s-i} \leq e_{r-i}$ for $0 \leq i < s$, so each difference in the sum above is nonnegative. It is equal to zero if and only if $u(e_1,\dots,e_{r-s}) = 0$, and

$$h^1(\mathcal{O}(\vec{e})\otimes\mathcal{O}(e_{r-i})^\vee) = h^1(\mathcal{O}(\vec{e})\otimes\mathcal{O}(a_{s-i})^\vee)$$

for all $0 \leq i < s$. The latter condition says that whenever there exists $1 \leq j \leq r$ such that $e_{r-i} - e_j \geq 2$, then $a_{s-i} = e_{r-i}$. In other words, $\vec{e}$ achieves equality if and only if $\vec{e}$ can be written as $(\mathrm{Bal},\vec{a}_+)$, where Bal is balanced, and $\vec{a} = (\vec{a}_-,\vec{a}_+)$. □

Definition 2.3.12. Let $\mathcal{S}$ be the universal subsheaf on $S_{s,d'}(B,\mathcal{E})\times\mathbb{P}^1$. Let $\sigma : S_{s,d'}(B,\mathcal{E}) \to B$. Define $X_{\vec{a},\vec{e}} = \Sigma_{\vec{a}}(\mathcal{S}) \cap \sigma^{-1}\Sigma_{\vec{e}}(\mathcal{E})$. Define $\mathrm{Subsheaves}(\vec{a},\vec{e}) = U(\vec{a},\vec{e})/\operatorname{Aut}(\mathcal{O}(\vec{a}))$, where $U(\vec{a},\vec{e}) \subset \operatorname{Hom}(\mathcal{O}(\vec{a}),\mathcal{O}(\vec{e}))$ is the open locus of injective maps, and $\operatorname{Aut}(\mathcal{O}(\vec{a}))$ acts on $U_{\vec{a},\vec{e}}$ by precomposition.

Lemma 2.3.13. The map $X_{\vec{a},\vec{e}} \to \Sigma_{\vec{e}}(\mathcal{E})$ is flat with fibers isomorphic to $\mathrm{Subsheaves}(\vec{a},\vec{e})$.

Proof. By taking a base change along a principal $\operatorname{Aut}(\mathcal{O}(\vec{e}))$-bundle, which is a faithfully flat morphism, we may without loss of generality assume that $\mathcal{E} \cong g^*\mathcal{O}(\vec{e})$, where $g : \Sigma_{\vec{e}}(\mathcal{E}) \times \mathbb{P}^1 \to \mathbb{P}^1$. Then we claim that $X_{\vec{a},\vec{e}} \cong \Sigma_{\vec{e}}(\mathcal{E}) \times \mathrm{Subsheaves}(\vec{a},\vec{e})$. By functoriality of Quot, $S_{s,d'}(B, g^*\mathcal{O}(\vec{e})) \cong \Sigma_{\vec{e}}(\mathcal{E}) \times S_{s,d'}(\operatorname{Spec} k, \mathcal{O}(\vec{e}))$. We can then take the splitting locus $\Sigma_{\vec{a}}(\mathcal{S})$ on both sides. So it remains to show that over a point, the splitting locus $\Sigma_{\vec{a}}(\mathcal{S}) \subset S_{s,d'}(\operatorname{Spec} k, \mathcal{O}(\vec{e}))$ is isomorphic to $\mathrm{Subsheaves}(\vec{a},\vec{e})$. The only difference between the functors they represent is that a priori, $\mathrm{Subsheaves}(\vec{a},\vec{e})$ does not require flatness for the family of quotients. So it remains to show that if

$$0 \to \mathcal{S} \to g^*\mathcal{O}(\vec{e}) \to \mathcal{Q} \to 0$$

is the universal short exact sequence on $\operatorname{Hom}(\mathcal{O}(\vec{a}),\mathcal{O}(\vec{e})) \times \mathbb{P}^1$, then $\mathcal{Q}$ is flat over exactly $U(\vec{a},\vec{e})$. To see this, we use the local criterion for flatness (Theorem 6.8 [Eis95]), which says that $\mathcal{Q}$ is flat at a closed point $\phi \in \operatorname{Hom}(\mathcal{O}(\vec{a}),\mathcal{O}(\vec{e}))$ if and only if $\operatorname{Tor}^1(\mathcal{O}_{\mathbb{P}^1_\phi}, \mathcal{Q}) = 0$. Tensoring the short exact sequence above with $\mathcal{O}_{\mathbb{P}^1_\phi}$, we see that since

$$\operatorname{Tor}^1(\mathcal{O}_{\mathbb{P}^1_\phi}, g^*\mathcal{O}(\vec{e})) = 0$$

for all ϕ, $\operatorname{Tor}^1(\mathcal{O}_{\mathbb{P}^1_\phi}, \mathcal{Q}) = 0$ if and only if the map $\phi : \mathcal{O}(\vec{a}) \to \mathcal{O}(\vec{e})$ is injective. □

Remark 2.3.14. The same argument of taking a faithfully flat base change along a frame bundle shows that $S_{s,d'}(\pi^*\mathcal{F}(c)) \to B$ is flat with fibers isomorphic to the absolute Quot scheme $S_{s,d'}(E)$, where $E = \mathcal{O}_{\mathbb{P}^1}(c)^{\oplus \operatorname{rk}\mathcal{F}}$.

The argument below follows the ideas of Lemma 6.1 of [Lar21b].

Proposition 2.3.15. Suppose that $\phi_{\mathcal{E}} : B \to \mathcal{B}_{r,d}$ is good. For each $\vec{a}$, let $\vec{e}_b = \vec{e}_b(r,d,\vec{a})$. Let c be such that $\mathcal{E}^\vee(c)$ is globally generated and take the embedding of $S_{s,d'}(\mathcal{E}) \hookrightarrow S_{s,d'}(\pi^*\mathcal{F}(c))$, defined above. Then $\operatorname{codim}(X_{\vec{a},\vec{e}_b} \subseteq S_{s,d'}(\pi^*\mathcal{F}(c))) = u(\vec{a}) + \operatorname{rk}(f_*\mathcal{H}om(\mathcal{S}, \widetilde{\gamma}^*\pi^*\mathcal{G}(c+1)))$. For all other tuples $(\vec{a}',\vec{e}')$ where $\vec{a}' \leq \vec{a}$ and $\mathcal{O}(\vec{e}')$ admits a $\mathcal{O}(\vec{a}')$-subsheaf, we have $\vec{e}' \leq \vec{e}_b$ and $\operatorname{codim}(X_{\vec{a}',\vec{e}'} \subseteq S_{s,d'}(\pi^*\mathcal{F}(c))) > u(\vec{a}) + \operatorname{rk}(f_*\mathcal{H}om(\mathcal{S}, \widetilde{\gamma}^*\pi^*\mathcal{G}(c+1)))$.

Proof. Let $\vec{a}$, $\vec{e}$ be such that $\mathcal{O}(\vec{e})$ has an $\mathcal{O}(\vec{a})$-subsheaf. In particular $a_s \leq e_r \leq c$. Let $\gamma : S_{s,d'}(\pi^*\mathcal{F}(c)) \to B$. By Lemma 2.3.13,

$$\operatorname{codim}(X_{\vec{a},\vec{e}} \subseteq S_{s,d'}(\pi^*\mathcal{F}(c))) = u(\vec{e}) + \operatorname{fiberdim}(\gamma) - \dim(\mathrm{Subsheaves}(\mathcal{O}(\vec{a}),\mathcal{O}(\vec{e})))$$

Now, by restricting the exact sequence

$$0 \to \mathcal{E} \to \pi^*\mathcal{F}(c) \to \pi^*\mathcal{G}(c+1) \to 0$$

to some $p \in \Sigma_{\vec{e}}(\mathcal{E})$, we get that $\mathcal{O}(\vec{e})$ is a kernel like so:

$$0 \to \mathcal{O}(\vec{e}) \to \mathcal{O}(c)^{\oplus \operatorname{rk} \mathcal{F}} \to \mathcal{O}(c+1)^{\oplus \operatorname{rk} \mathcal{G}} \to 0. \tag{2.5}$$

Since $\operatorname{fiberdim}(\gamma) = \dim S_{s,d'}(\mathcal{O}(c)^{\oplus \operatorname{rk} \mathcal{F}})$, and the subsheaf generically splits as the balanced bundle $\mathcal{O}(\vec{a}_b)$, we can write

$$\operatorname{fiberdim}(\gamma) = h^0 \mathcal{H}om(\mathcal{O}(\vec{a}_b), \mathcal{O}(c)^{\oplus \operatorname{rk} \mathcal{F}}) - h^0 \mathcal{E}nd(\mathcal{O}(\vec{a}_b)).$$

Since $c \geq a_s$,

$$h^0 \mathcal{H}om(\mathcal{O}(\vec{a}_b), \mathcal{O}(c)^{\oplus \operatorname{rk} \mathcal{F}}) = h^0 \mathcal{H}om(\mathcal{O}(\vec{a}), \mathcal{O}(c)^{\oplus \operatorname{rk} \mathcal{F}}).$$

Similarly,

$$\dim(\operatorname{Subsheaves}(\mathcal{O}(\vec{a}), \mathcal{O}(\vec{e}))) = h^0 \mathcal{H}om(\mathcal{O}(\vec{a}), \mathcal{O}(\vec{e})) - h^0 \mathcal{E}nd(\mathcal{O}(\vec{a})).$$

Note that $\mathcal{E}nd(\mathcal{O}(\vec{a}))$ and $\mathcal{E}nd(\mathcal{O}(\vec{a}_b))$ have the same rank and degree, and therefore the same Euler characteristic. Therefore,

$$h^0(\mathcal{E}nd(\mathcal{O}(\vec{a}))) - h^0(\mathcal{E}nd(\mathcal{O}(\vec{a}_b))) = h^1(\mathcal{E}nd(\mathcal{O}(\vec{a}))) - h^1(\mathcal{E}nd(\mathcal{O}(\vec{a}_b))) = u(\vec{a}).$$

Moreover, we obtain a long exact sequence in Ext groups by applying $\operatorname{Hom}(\mathcal{O}(\vec{a}), -)$ to Equation 2.5. Again since $c \geq a_s$, $\operatorname{ext}^1(\mathcal{O}(\vec{a}), \mathcal{O}(c)^{\oplus \operatorname{rk} \mathcal{F}}) = 0$. So we get that

$$\begin{aligned} h^0 \mathcal{H}om(\mathcal{O}(\vec{a}), \mathcal{O}(c)^{\oplus \operatorname{rk} \mathcal{F}}) - h^0 \mathcal{H}om(\mathcal{O}(\vec{a}), \mathcal{O}(\vec{e})) &= h^0 \mathcal{H}om(\mathcal{O}(\vec{a}), \mathcal{O}(c+1)^{\oplus \operatorname{rk} \mathcal{G}}) - \operatorname{ext}^1(\mathcal{O}(\vec{a}), \mathcal{O}(\vec{e})) \\ &= \operatorname{rk} f_* \mathcal{H}om(\mathcal{S}, \tilde{\gamma}^* \pi^* \mathcal{G}(c+1)) - \operatorname{ext}^1(\mathcal{O}(\vec{a}), \mathcal{O}(\vec{e})). \end{aligned}$$

Combining everything, we get that

$$\operatorname{codim}(X_{\vec{a},\vec{e}} \subseteq S_{s,d'}(\pi^* \mathcal{F}(c))) = u(\vec{a}) + \operatorname{rk} f_* \mathcal{H}om(\mathcal{S}, \tilde{\gamma}^* \pi^* \mathcal{G}(c+1)) + u(\vec{e}) - \operatorname{ext}^1(\mathcal{O}(\vec{a}), \mathcal{O}(\vec{e})).$$

By Lemma 2.3.11, for a fixed $\vec{a}$, if $(\vec{a}', \vec{e}')$ is such that $\vec{a}' \leq \vec{a}$, and $\vec{e}'$ admits a $\mathcal{O}(\vec{a}')$-subsheaf, then

$$\begin{aligned} \operatorname{codim}(X_{\vec{a}',\vec{e}'} \subseteq S_{s,d'}(B, \mathcal{E})) &\geq u(\vec{a}') + \operatorname{rk} f_* \mathcal{H}om(\mathcal{S}, \tilde{\gamma}^* \pi^* \mathcal{G}(c+1)) \\ &\geq u(\vec{a}) + \operatorname{rk} f_* \mathcal{H}om(\mathcal{S}, \tilde{\gamma}^* \pi^* \mathcal{G}(c+1)) \end{aligned}$$

with equality if and only if $\vec{a}' = \vec{a}$ and $\vec{e}' = \vec{e}_b(r, d, \vec{a})$. By Lemma 2.3.9, $\vec{e}' \leq \vec{e}_b(r, d, \vec{a}') \leq \vec{e}_b(r, d, \vec{a})$. □

Proposition 2.3.16. Let B be an irreducible scheme of finite type with a good map $\phi_{\mathcal{E}} : B \to \mathcal{B}_{r,d}$. Then $S_{s,d'}(\mathcal{E})$ is irreducible and the induced map $\phi_{\mathcal{S}} : S_{s,d'}(\mathcal{E}) \to \mathcal{B}_{s,d'}$ is good. The image of $\overline{\Sigma}_{\vec{a}}(\mathcal{S})$ in B is exactly $\overline{\Sigma}_{\vec{e}_b}$, where $\vec{e}_b = \vec{e}_b(r, d, \vec{a})$.

Proof. Recall that $S_{s,d'}(\mathcal{E})$ is the vanishing locus of a section of the vector bundle

$$f_* \mathcal{H}om(\mathcal{S}, \tilde{\gamma}^* \pi^* \mathcal{G}(c+1)),$$

and that $\overline{\Sigma}_{\vec{a}}(\mathcal{S})$ is a splitting locus. Therefore the lower bound on codimension given in the previous proposition is also an upper bound on the codimension of a component of $\overline{\Sigma}_{\vec{a}}(\mathcal{S})$ in

$S_{s,d'}(\pi^*\mathcal{F}(c))$. Since $\phi_{\mathcal{E}}$ is good, $\Sigma_{\vec{e}}(\mathcal{E})$ is irreducible, which implies that $X_{\vec{a},\vec{e}}$ is irreducible. Thus Proposition 2.3.15 implies that $\overline{\Sigma}_{\vec{a}}(\mathcal{S})$ is irreducible of codimension $u(\vec{a})$ in $S_{s,d'}(\mathcal{E})$ and $X_{\vec{a},\vec{e}_b}$ is dense in $\overline{\Sigma}_{\vec{a}}(\mathcal{S})$. Specializing to the case where $\vec{a}$ is balanced proves that $S_{s,d'}(\mathcal{E})$ is irreducible.

By Proposition 2.3.15, the image of $\overline{\Sigma}_{\vec{a}}(\mathcal{S})$ in B is closed, contained in $\overline{\Sigma}_{\vec{e}_b}$, and contains $\Sigma_{\vec{e}_b}$. By the assumption that $\phi_{\mathcal{E}}$ is good, the closure of $\Sigma_{\vec{e}_b}$ is exactly $\overline{\Sigma}_{\vec{e}_b}$. So the image of $\overline{\Sigma}_{\vec{a}}(\mathcal{S})$ must be exactly $\overline{\Sigma}_{\vec{e}_b}$. □

Proposition 2.3.17. When $\phi_{\mathcal{E}} : B \to \mathcal{B}_{r,d}$ is good, the embeddings of Figure 2.3 defined earlier have codimension equal to $\operatorname{rk}\mathcal{R}$, where $\mathcal{R} = f_*\mathcal{H}om(\mathcal{S}, \widetilde{\gamma}^*\pi^*\mathcal{G}(c+1))$ in the first case, and $\mathcal{R} = f_*\mathcal{H}om(\widetilde{\gamma}^*\pi^*\mathcal{G}'(-m-1), \mathcal{Q})$ in the second case. In particular, $S_{s,d'}(\mathcal{E}) = \mathrm{Quot}^{r-s,d-d'}(\mathcal{E})$ sits as a local complete intersection under these two embeddings.

Proof. The proposition for the first case follows from the stratification of $S_{s,d'}(B,\mathcal{E})$ into irreducible strata $X_{\vec{a},\vec{e}}$ and our computation in Proposition 2.3.15, which says that the strata always have codimension at least $\operatorname{rk}\mathcal{R}$, and is equal to $\operatorname{rk}\mathcal{R}$ if and only if $\vec{a}$ is balanced and $\vec{e} = \vec{e}_b(r,d,\vec{a})$.

To see that the second embedding also has the expected codimension $\operatorname{rk}\mathcal{R}$, now it suffices to compute the difference in dimension between $X_{\vec{a}_b,\vec{e}_b}$ and $\mathrm{Quot}^{r-s,d-d'}(\pi^*\mathcal{F}'(-m))$, where $\vec{a}_b$ is balanced and $\vec{e}_b = \vec{e}_b(r,d,\vec{a}_b)$. Let us choose a k-point $\alpha \in X_{\vec{a}_b,\vec{e}_b} \subset \mathrm{Quot}^{r-s,d-d'}(\pi^*\mathcal{F}'(-m))$. Then α viewed as a point of $\mathrm{Quot}^{r-s,d-d'}(\pi^*\mathcal{F}'(-m))$ corresponds to an exact sequence

$$0 \to S \to E \to Q \to 0,$$

on $\mathbb{P}^1_k$, where $E \cong \mathcal{O}(-m)^{\oplus N}$, $N = h^0\mathcal{O}(\vec{e}_b)(m)$, and Q is a coherent sheaf of rank $r-s$, degree $d-d'$. Viewed as a point of $X_{\vec{a}_b,\vec{e}_b}$, it corresponds to an exact sequence

$$0 \to \mathcal{O}(\vec{a}_b) \to \mathcal{O}(\vec{e}_b) \to Q \to 0.$$

Let $K = \mathcal{O}(-m-1)^{\oplus M}$, where $M = h^0\mathcal{O}(\vec{e}_b)(m-1)$. Then these two exact sequences fit in a diagram as in Figure 2.5.

By deformation theory, the fiber dimension of the map $\gamma : \mathrm{Quot}^{r-s,d-d'}(\pi^*\mathcal{F}'(-m)) \to B$ is $\chi(\mathcal{H}om(S,Q))$. But

$$\begin{aligned}
\chi(\mathcal{H}om(S,Q)) &= \chi(\mathcal{H}om(K,Q)) + \chi(\mathcal{H}om(\mathcal{O}(\vec{a}_b),Q)) \\
&= \dim \mathrm{Hom}(K,Q) + \chi(\mathcal{H}om(\mathcal{O}(\vec{a}_b),\mathcal{O}(\vec{e}_b))) - \chi(\mathcal{E}nd(\mathcal{O}(\vec{a}_b))) \\
&= \dim \mathrm{Hom}(K,Q) + \dim \mathrm{Hom}(\mathcal{O}(\vec{a}_b),\mathcal{O}(\vec{e}_b)) - \mathrm{ext}^1(\mathcal{O}(\vec{a}_b),\mathcal{O}(\vec{e}_b)) - h^0\mathcal{E}nd(\mathcal{O}(\vec{a}_b)) \\
&= \dim \mathrm{Hom}(K,Q) - u(\vec{e}_b) + \dim \mathrm{Hom}(\mathcal{O}(\vec{a}_b),\mathcal{O}(\vec{e}_b)) - h^0\mathcal{E}nd(\mathcal{O}(\vec{a}_b)),
\end{aligned}$$

where the last equality is due to Lemma 2.3.11. It follows now that

$$\begin{aligned}
\operatorname{rk}\mathcal{R} &= \dim \mathrm{Hom}(K,Q) \\
&= \mathrm{fiberdim}(\gamma) + u(\vec{e}_b) - (\dim \mathrm{Hom}(\mathcal{O}(\vec{a}_b),\mathcal{O}(\vec{e}_b)) - h^0\mathcal{E}nd(\mathcal{O}(\vec{a}_b))) \\
&= \mathrm{codim}(X_{\vec{a}_b,\vec{e}_b} \subseteq \mathrm{Quot}^{r-s,d-d'}(\pi^*\mathcal{F}'(-m))).
\end{aligned}$$

□

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & S \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}(\vec{a}_b) \arrow[r] \arrow[d] & \mathcal{O}(\vec{e}_b) \arrow[r] \arrow[d] & Q \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Figure 2.5: Relationship between the two exact sequences that correspond to the same point $\alpha \in X_{\vec{a}_b,\vec{e}_b}$

Remark 2.3.18. Note that both Proposition 2.3.16 and Proposition 2.3.17 work for arbitrary numerical specifications s, d', r, d. In particular they hold when the map $S_{s,d'}(\mathcal{E}) \to B$ is not birational onto its image, though this is stronger than what we will need.

Theorem 2.3.19. Let B be an irreducible scheme of finite type with a good map $\phi_{\mathcal{E}} : B \to \mathcal{B}_{r,d}$. Let $\underline{r} = (r_1, \dots, r_\ell), \underline{d} = (d_1, \dots, d_\ell)$ be such that $r \geq r_1 \geq \cdots \geq r_\ell \geq 0$ and $r_i, d_i \in \mathbb{Z}$. Then $\mathrm{FQuot}(B, \mathcal{E}; \underline{r}, \underline{d})$ is irreducible. In particular, $\mathrm{FQuot}(\mathcal{B}_{r,d}, \mathcal{E}_{\mathrm{univ}}; \underline{r}, \underline{d})$ is irreducible.

Proof. We can build $\mathrm{FQuot}(B, \mathcal{E}; \underline{r}, \underline{d})$ as a tower of relative Quot schemes. Let $\mathcal{S}_2$ be the universal subbundle of $\mathcal{E}$ of rank $r - r_2$, degree $d - d_2$ on $Q := \mathrm{FQuot}(B, \mathcal{E}; (r_2, \dots, r_\ell), (d_2, \dots, d_\ell))$, then $\mathrm{FQuot}(B, \mathcal{E}; \underline{r}, \underline{d})$ can also be viewed as $S_{r-r_1, d-d_1}(Q, \mathcal{S}_2)$. Repeated applications of Proposition 2.3.16 along this tower shows that $\mathrm{FQuot}(B, \mathcal{E}; \underline{r}, \underline{d})$ is irreducible.

Let Z be a component of $\mathrm{FQuot}(\mathcal{B}_{r,d}, \mathcal{E}_{\mathrm{univ}}; \underline{r}, \underline{d})$. Then the restriction of $\mathcal{E}$ to Z must generically split as some $\vec{e}$. Thus, if $\mathrm{FQuot}(\mathcal{B}_{r,d}, \mathcal{E}_{\mathrm{univ}}; \underline{r}, \underline{d})$ has two components where $\mathcal{E}$ generically splits as $\vec{e}$ and $\vec{e}\,'$, then we will detect them when we base change to $U_{\vec{f}}$ where $\vec{f} \leq \vec{e}$ and $\vec{f} \leq \vec{e}\,'$. But $U_{\vec{f}} \to \mathcal{B}_{r,d}$ is good, so our argument above shows that the base change of $\mathrm{FQuot}(\mathcal{B}_{r,d}, \mathcal{E}_{\mathrm{univ}}; \underline{r}, \underline{d})$ to $U_{\vec{f}}$ is irreducible for all $\vec{f}$. $\square$

Smoothness

Now we will prove smoothness of $\widetilde{\boldsymbol{\Sigma}}_{\vec{e},I}$ when I is $\vec{e}$-admissible. The key ingredient is an exact sequence that allows us to count the dimension of the tangent space.

Proof of smoothness claim in Theorem 2.3.5. Recall from Section 2.2 and specifically Proposition 2.2.8 that there is an explicit smooth atlas $\{U_{\vec{e}\,'}\} = \{H^1 \mathcal{E}nd(\mathcal{O}(\vec{e}\,'))\}$ of $\mathcal{B}_{r,d}$, indexed by all possible splitting types of rank r and degree d, which satisfies the following properties:

- $\dim U_{\vec{e}\,'} = u(\vec{e}\,')$;

- $\Sigma_{\vec{e}'}(U_{\vec{e}'})$ is a single point, which we call $\{0\}$ (as it is 0 in an affine space);
- the map $U_{\vec{e}'} \to \mathcal{B}_{r,d}$ is good, in particular

$$\widetilde{U}_{\vec{e},\vec{e}'} = \widetilde{U}_{I,\vec{e},\vec{e}'} := \widetilde{\mathbf{\Sigma}}_{\vec{e},I} \times_{\mathcal{B}_{r,d}} U_{\vec{e}'} = \mathrm{FQuot}(U_{\vec{e}'}, \mathcal{E}_{U_{\vec{e}'}}; \underline{r}_{I,\mathrm{HN}}, \underline{d}_{I,\mathrm{HN}})$$

  is irreducible;
- $U_{\vec{e}'}$ maps surjectively onto $\bigcup_{\vec{f}\geq\vec{e}'} \mathbf{\Sigma}_{\vec{f}}$.

Using this smooth atlas, to prove that $\widetilde{\mathbf{\Sigma}}_{\vec{e},I}$ is smooth, it suffices to prove one of the following two equivalent claims for all $\vec{e}'$:

1. the restriction of $\widetilde{\mathbf{\Sigma}}_{\vec{e},I}$ to $\bigcup_{\vec{f}\geq\vec{e}'} \mathbf{\Sigma}_{\vec{f}}$ is smooth;
2. the scheme

$$\widetilde{U}_{\vec{e},\vec{e}'} = \widetilde{U}_{I,\vec{e},\vec{e}'} := \widetilde{\mathbf{\Sigma}}_{\vec{e},I} \times_{\mathcal{B}_{r,d}} U_{\vec{e}'} = \mathrm{FQuot}(U_{\vec{e}'}, \mathcal{E}_{U_{\vec{e}'}}; \underline{r}_{I,\mathrm{HN}}, \underline{d}_{I,\mathrm{HN}})$$

   is smooth.

The proof will proceed by induction on the dominance order for $\vec{e}'$. Since $\widetilde{\mathbf{\Sigma}}_{\vec{e},I}$ surjects onto $\overline{\mathbf{\Sigma}}_{\vec{e}}$, when $\vec{e}' > \vec{e}$, the restriction of $\widetilde{\mathbf{\Sigma}}_{\vec{e},I}$ to $\bigcup_{\vec{f}\geq\vec{e}'} \mathbf{\Sigma}_{\vec{f}}$ is empty.

Now assume that both (1) and (2) are true for all $\vec{e}'' > \vec{e}'$. Let $\pi : \widetilde{U}_{\vec{e},\vec{e}'} \to U_{\vec{e}'}$. Then since (1) is true for all $\vec{e}'' > \vec{e}'$, $U_{\vec{e},\vec{e}'}$ is smooth away from $\pi^{-1}(\{0\})$. By Theorem 2.3.19, $\widetilde{U}_{\vec{e},\vec{e}'}$ is irreducible, so

$$\dim \widetilde{U}_{\vec{e},\vec{e}'} = \dim \widetilde{U}_{\vec{e},\vec{e}'} \setminus \pi^{-1}(\{0\}) = \dim \widetilde{\mathbf{\Sigma}}_{\vec{e},I} - \dim \mathcal{B}_{r,d} + \dim U_{\vec{e}'} = u(\vec{e}') - u(\vec{e}).$$

Let $[\alpha]$ be a point in $\pi^{-1}(\{0\})$, corresponding to a flag $\alpha : E_1 \hookrightarrow E_2 \hookrightarrow \cdots \hookrightarrow E_k \hookrightarrow \mathcal{O}(\vec{e}')$. To show that $\widetilde{U}_{\vec{e},\vec{e}'}$ is everywhere smooth, it suffices to check that for all such $[\alpha] \in \pi^{-1}(\{0\})$, the dimension of the tangent space is

$$\dim \mathbb{T}_{[\alpha]}\widetilde{U}_{\vec{e},\vec{e}'} = u(\vec{e}') - u(\vec{e}).$$

Because the tangent space of $0 \in U_{\vec{e}'}$ is just $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$, the space of first-order deformations of $\mathcal{O}(\vec{e}')$, the tangent space of $U_{\vec{e},\vec{e}'}$ at $[\alpha]$ has a modular interpretation:

$$\mathbb{T}_{[\alpha]}\widetilde{U}_{\vec{e},\vec{e}'} = \left\{ \begin{array}{l} \text{A flag of subsheaves } \mathcal{E}_1 \hookrightarrow \mathcal{E}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_{k+1} = \mathcal{E} \text{ on } \mathbb{P}^1_{k[\varepsilon]/(\varepsilon^2)} \text{ specializing to } \alpha \\ \text{over } \mathbb{P}^1_k\text{, such that each quotient } \mathcal{E}/\mathcal{E}_i \text{ is flat over } k[\varepsilon]/(\varepsilon^2) \end{array} \right\} / \sim .$$

We claim that this tangent space fits inside an exact sequence

$$0 \to \oplus_{i=1}^k \mathrm{End}(E_i) \to \oplus_{i=1}^k \mathrm{Hom}(E_i, E_{i+1}) \to \mathbb{T}_{[\alpha]}\widetilde{U}_{\vec{e},\vec{e}'} \to \oplus_{i=1}^{k+1} H^1\mathcal{E}nd(E_i) \to \oplus_{i=1}^k \mathrm{Ext}^1(E_i, E_{i+1}) \to 0.$$

We will describe each of the maps and argue exactness going from right to left.

1. To specify the last map is the same as specifying maps $\phi_{i,j} = H^1\mathcal{E}nd(E_i) \to \mathrm{Ext}^1(E_j, E_{j+1})$ for all $1 \leq i \leq k+1, 1 \leq j \leq k$. We set $\phi_{i,j}$ to be zero if $i \neq j$ and $i \neq j+1$. Note that $\mathrm{Ext}^1(E_j, E_{j+1}) = H^1\mathcal{H}om(E_j, E_{j+1})$. The map $\phi_{i,i}$ is induced by the map of bundles $\mathcal{E}nd(E_i) \to \mathcal{H}om(E_i, E_{i+1})$ by postcomposing with the inclusion $E_i \hookrightarrow E_{i+1}$ in the flag. The map $\phi_{i,i-1}$ is the negation of the map induced by the map of bundles $\mathcal{E}nd(E_i) \to \mathcal{H}om(E_{i-1}, E_i)$, which is defined by precomposing with the inclusion $E_{i-1} \hookrightarrow E_i$. Let $i < k$. Then

$$0 \to \mathcal{H}om(E_{i+1}/E_i, E_{i+1}) \to \mathcal{E}nd(E_{i+1}) \to \mathcal{H}om(E_i, E_{i+1}) \to 0$$

   is exact, because E_{i+1} is a vector bundle. Since we are on a curve, the induced map $H^1\mathcal{E}nd(E_{i+1}) \to H^1\mathcal{H}om(E_i, E_{i+1}) = \mathrm{Ext}^1(E_i, E_{i+1})$ is surjective. This implies that the last map is surjective.

2. The kernel of the last map corresponds to those deformations of $E_1, \ldots, E_k$ that can be lifted to a deformation of the flag. To prove this, it suffices to prove it for a single map, namely if $E \hookrightarrow F$, then the kernel of $H^1\mathcal{E}nd(E) \oplus H^1\mathcal{E}nd(F) \to \mathrm{Ext}^1(E, F)$ corresponds to those pairs of first-order deformations of E and F that can be lifted to a deformation of $E \hookrightarrow F$. This is the content of Lemma 2.3.20 below. The map $\mathbb{T}_{[\alpha]}\widetilde{U}_{\vec{e},\vec{e}'} \to \oplus_{i=1}^{k+1} H^1\mathcal{E}nd(E_i)$ is the map that forgets the deformations of the inclusions, and only remembers the deformations of each bundle. It clearly maps onto the kernel of the last map, given our interpretation of the kernel.

3. The kernel of the third map corresponds to those deformations of the flag that do not deform the vector bundles. In other words, there is a map

$$\mathbb{T}_{[\alpha]} \prod_{i=1}^{m} \mathrm{Subsheaves}(E_i, E_{i+1}) \to \mathbb{T}_{[\alpha]}\widetilde{U}_{\vec{e},\vec{e}'},$$

   and the kernel of the third map is precisely the image of this map. In particular we can think about each $\mathrm{Subsheaves}(E_i, E_{i+1})$ individually. The tangent space of $\mathrm{Subsheaves}(E_i, E_{i+1})$ at $\phi : E_i \hookrightarrow E_{i+1}$ is the cokernel of $H^0\mathcal{E}nd(E_i) \to H^0(\mathcal{H}om(E_i, E_{i+1}))$, induced by post-composing with ϕ.

4. The first map is the direct sum of the maps $\mathrm{End}(E_i) \to \mathrm{Hom}(E_i, E_{i+1})$, which are all injective, as they are induced from the injective maps of sheaves $\mathcal{E}nd(E_i) \to \mathcal{H}om(E_i, E_{i+1})$.

Using this exact sequence, we can compute $\dim \mathbb{T}_{[\alpha]}\widetilde{U}_{\vec{e},\vec{e}'}$:

$$\begin{aligned}\dim \mathbb{T}_{[\alpha]}\widetilde{U}_{\vec{e},\vec{e}'} &= h^1(\mathcal{E}nd(\mathcal{O}(\vec{e}'))) - \sum_{i=1}^{m}\chi(\mathcal{E}nd(E_i')) + \sum_{i=1}^{m}\chi(\mathcal{H}om(E_i', E_{i+1}')) \\ &= u(\vec{e}') + \sum_{i=1}^{m}\chi(\mathcal{H}om(E_i', E_{i+1}'/E_i')) \\ &= u(\vec{e}') + \sum_{i=1}^{m}\chi(\mathcal{H}om(E_i, E_{i+1}/E_i)) \\ &= u(\vec{e}') - \sum_{i=1}^{m}h^1(\mathcal{H}om(E_i, E_{i+1}/E_i)) \\ &= u(\vec{e}') - u(\vec{e}).\end{aligned}$$

□

Lemma 2.3.20. Let A be an Artinian local ring with residue field k, and let I be a square-zero ideal in A. Given a map of vector bundles $\phi : E \to F$ on $X \times \operatorname{Spec} A/I$, the deformations of E, F to $\operatorname{Spec} A$ that can be extended to a deformation of ϕ is characterized by the kernel of a map $\operatorname{Ext}^1_{X_0}(E_0, E_0 \otimes_k I) \oplus \operatorname{Ext}^1_{X_0}(F_0, F_0 \otimes_k I) \to \operatorname{Ext}^1_{X_0}(E_0, F_0 \otimes_k I)$, which is the difference of the natural maps $\operatorname{Ext}^1_{X_0}(E_0, E_0 \otimes_k I) \to \operatorname{Ext}^1_{X_0}(E_0, F_0 \otimes_k I)$ and $\operatorname{Ext}^1_{X_0}(F_0, F_0 \otimes_k I) \to \operatorname{Ext}^1_{X_0}(E_0, F_0 \otimes_k I)$ induced by ϕ.

Proof. Corresponding to deformations $\widetilde{E}, \widetilde{F}$ of E, F over $\operatorname{Spec} A$, we have two extensions

$$\begin{aligned}0 \to E_0 \otimes_k I \to \widetilde{E} \to E \to 0, \\ 0 \to F_0 \otimes_k I \to \widetilde{F} \to F \to 0.\end{aligned}$$

To deform the map $\phi : E \to F$ into a map of $\widetilde{\phi} : \widetilde{E} \to \widetilde{F}$ is the same as writing down a map of these two short exact sequences such that the maps on the kernel and quotient are induced by ϕ. This is possible if and only if the pushout of the first sequence by the map $\phi_0 \otimes I : E_0 \otimes_k I \to F_0 \otimes_k I$ agrees with the pullback of the second sequence by the map $\phi : E \to F$. This is if and only if the difference of their classes in $\operatorname{Ext}^1_{X_{A/I}}(E, F_0 \otimes_k I) = \operatorname{Ext}^1_{X_0}(E_0, F_0 \otimes_k I)$ is zero. □

Remark 2.3.21. One can define Harder-Narasimhan strata (HN strata) inside the stack of vector bundles on a fixed curve C as those vector bundles with a fixed Harder-Narasimhan polygon. These HN strata are generalizations of $\mathbf{\Sigma}_{\vec{e}}$ to higher genus. One can also define an analogue of $\overline{\mathbf{\Sigma}}_{\vec{e}}$ as the stratum of vector bundles whose HN polygon is a specialization of some fixed polygon. In the literature, the standard way to show that the locally closed HN strata are smooth is via hyper-Quot constructions, similar to $\widetilde{\mathbf{\Sigma}}_{\vec{e},\mathrm{HN}}$. Proposition 15.4.2 of [Pot97] gives a general criterion for smoothness for such hyper-Quot constructions, and Corollary 15.4.3 uses this criterion to prove smoothness of HN strata for complete families of vector bundles. The main difference between the case of higher genus curves and $\mathbb{P}^1$ is that we no longer know if the hyper-Quot construction is irreducible, though it must contain one component which is a resolution of singularities for Harder-Narasimhan strata.

Fibers of the resolution

To get a better sense of the resolutions $\widetilde{\mathbf{\Sigma}}_{\vec{e},I}$, we give some examples of the fiber of $\widetilde{\mathbf{\Sigma}}_{\vec{e},\mathrm{HN}}$ over a point $p \in \mathbf{\Sigma}_{\vec{e}'}$ when $\vec{e}' < \vec{e}$. These examples show that the fibers can be reducible with components of different dimensions.

Example 2.3.22. Let $\vec{e} = (-2, 0, 2)$, and let $\vec{e}' = (-3, 0, 3)$. We can stratify the fiber by how each bundle in a flag splits, and there are two possibilities:

$$\alpha = [\mathcal{O}(2) \hookrightarrow \mathcal{O}(0, 2) \hookrightarrow \mathcal{O}(-3, 0, 3)],$$

$$\beta = [\mathcal{O}(2) \hookrightarrow \mathcal{O}(-1, 3) \hookrightarrow \mathcal{O}(-3, 0, 3)].$$

Let X_α, X_β denote the collection of flags with such splitting types. $X_\beta \cong \mathbb{P}^1 \times \mathbb{P}^1$ is closed. X_α really is only the data of the second inclusion, which can be identified with quotients $\mathcal{O}(0, 3) \twoheadrightarrow \kappa(p)$ which do not factor through the projection to $\mathcal{O}$. This is isomorphic to $\mathbb{P}(\mathcal{O}(0, 3))$ over $\mathbb{P}^1$ minus the directrix. The closure $\overline{X}_\alpha$ can be identified with $\mathbb{P}(\mathcal{O}(0, 3))$, and is glued to $X_\beta = \mathbb{P}^1 \times \mathbb{P}^1$ along the directrix of the former and the diagonal of the latter.

Example 2.3.23. Let $\vec{e} = (-1, 0, 1)$ and $\vec{e}' = (-2, -1, 3)$. In this case, we may have

$$\alpha = [\mathcal{O}(1) \hookrightarrow \mathcal{O}(-1, 2) \hookrightarrow \mathcal{O}(-2, -1, 3)],$$

$$\beta = [\mathcal{O}(1) \hookrightarrow \mathcal{O}(-2, 3) \hookrightarrow \mathcal{O}(-2, -1, 3)].$$

The dimensions are $\dim X_\alpha = 3$, $\dim X_\beta = 4$. The stratum X_β is closed and isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$. X_α is isomorphic to $\mathbb{P}^1 \times V$, where V is isomorphic to $\mathbb{P}(\mathcal{O}(-1, 3))$ minus the directrix. The intersection $\overline{X}_\alpha \cap X_\beta$ can be identified with those flags of type β

$$\mathcal{O}(1) \xrightarrow{\begin{bmatrix} 0 \\ Q \end{bmatrix}} \mathcal{O}(-2, 3) \xrightarrow{\begin{bmatrix} 0 & 0 \\ L & 0 \\ 0 & 1 \end{bmatrix}} \mathcal{O}(-2, -1, 3)$$

where L, Q are forms of degrees 1 and 2, and L divides Q. Since L and Q are defined up to scaling, this intersection is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

2.4 Rational singularities

In this section, we will use the resolution of singularities constructed in the previous section to show that tame splitting loci have rational singularities. Throughout this section, we will assume that we are working over an algebraically closed field of characteristic zero.

We will need to repeatedly apply Littlewood-Richardson rules and we also make heavy use of the Borel-Weil-Bott theorem. For more details on Littlewood-Richardson rules, we refer the reader to Section 2.3 of [Wey03], and Theorem 1.4 of [Kou91]. For details on Borel-Weil-Bott, the reader may refer to Corollary 4.1.9 of [Wey03]. Given partitions μ, α, β, we will use $c^\mu_{\alpha,\beta}$ to denote the Littlewood-Richardson coefficient, which, among other things, is the multiplicity of $\mathbf{S}_\alpha V \otimes \mathbf{S}_\beta W$ in the decomposition of $\mathbf{S}_\mu(V \oplus W)$.

$$\begin{tikzcd}
\widetilde{\Sigma}_{\vec{e}} \arrow[r, hook, "i"] \arrow[d, two heads, "\rho"] & F = \mathrm{Quot}^{r_1,d_1}(\pi^*\pi_*\mathcal{E}) \arrow[d, "\gamma"] & F \times \mathbb{P}^1 \arrow[l, "f"'] \arrow[d, "\widetilde{\gamma}"] \\
\overline{\Sigma}_{\vec{e}} \arrow[r, hook, "j"] & B & B \times \mathbb{P}^1 \arrow[l, "\pi"]
\end{tikzcd}$$

Figure 2.6: Embedding of $\widetilde{\Sigma}_{\vec{e}}$ into F

Let $\vec{e}$ be a tame splitting type. Then $\overline{\mathbf{\Sigma}}_{\vec{e}}$ admits a resolution of singularities by $\widetilde{\mathbf{\Sigma}}_{\vec{e},I}$, where $|I| = 1$. Let $\widetilde{\mathbf{\Sigma}}_{\vec{e}} = \widetilde{\mathbf{\Sigma}}_{\vec{e},I} = \mathrm{Quot}^{r_1,d_1}(\mathcal{B}_{r,d}, \mathcal{E}_{\mathrm{univ}})$.

Having rational singularities is a smooth local property, so we may assume that we are working with a smooth scheme B of finite type, equipped with a smooth map $B \to \mathcal{B}_{r,d}$ corresponding to a vector bundle $\mathcal{E}$ on $B \times \mathbb{P}^1$. The reader may refer to Figure 2.6 for our labelling of the relevant maps. We denote the pullback of $\widetilde{\mathbf{\Sigma}}_{\vec{e}}$ to B by $\widetilde{\Sigma}_{\vec{e}}$. Up to twisting, we may assume that the vector bundle $\mathcal{E}$ on $B \times \mathbb{P}^1$ is fiberwise globally generated. By Theorem 2.2.11, $\overline{\Sigma}_{\vec{e}}$ is normal, so the natural map $\mathcal{O}_{\overline{\Sigma}_{\vec{e}}} \to \rho_*\mathcal{O}_{\widetilde{\Sigma}_{\vec{e}}}$ is an isomorphism. To show that $\overline{\mathbf{\Sigma}}_{\vec{e}}$ has rational singularities, it remains to show that $R^i\rho_*\mathcal{O}_{\widetilde{\Sigma}_{\vec{e}}} = 0$ for $i > 0$. Recall from Proposition 2.3.17 that $\widetilde{\Sigma}_{\vec{e}}$ admits an embedding into $F = \mathrm{Quot}^{r_1,d_1}(\pi^*\pi_*\mathcal{E})$, and it is cut out as a local complete intersection by a section of the vector bundle $\mathcal{R} = f_*\mathcal{H}om(\widetilde{\gamma}^*\pi^*(\pi_*\mathcal{E}(-1))(-1), \mathcal{Q})$. The proposition below shows that in this situation, it suffices to prove the vanishing of certain higher pushforwards of the wedge powers of $\mathcal{R}^\vee$.

Proposition 2.4.1. Suppose that we have the following commutative diagram of schemes, where i and j are closed immersions, and i is a local complete intersection cut out by a section of the vector bundle $\mathcal{R}$ on F. Suppose that $R^{>i}\gamma_* \wedge^i \mathcal{R}^\vee = 0$ for $i \geq 0$. Then $R^{>0}\rho_*\mathcal{O}_{\widetilde{Z}} = 0$.

$$\begin{tikzcd}
\widetilde{Z} \arrow[r, hook, "i"] \arrow[d, "\rho"] & F \arrow[d, "\gamma"] \\
Z \arrow[r, hook, "j"] & B
\end{tikzcd}$$

Proof. The condition tells us that the Koszul complex $\wedge^\bullet\mathcal{R}^\vee \to i_*\mathcal{O}_{\widetilde{Z}}$ is a resolution of $i_*\mathcal{O}_{\widetilde{Z}}$ by locally free $\mathcal{O}_F$-modules. Let $\mathcal{K}_i = \ker(\wedge^{i-1}\mathcal{R}^\vee \to \wedge^{i-2}\mathcal{R}^\vee)$, so that for $i \geq 1$,

$$0 \to \mathcal{K}_{i+1} \to \wedge^i\mathcal{R}^\vee \to \mathcal{K}_i \to 0,$$

and

$$0 \to \mathcal{K}_1 \to \mathcal{O}_F \to i_*\mathcal{O}_{\widetilde{Z}} \to 0.$$

We claim that $R^{>1}\gamma_*\mathcal{K}_1 = 0$. Let $M = \mathrm{rk}\,\mathcal{R}$. By assumption $R^{>M}\gamma_*\mathcal{K}_M = R^{>M}\gamma_* \wedge^M \mathcal{R}^\vee = 0$. Assume that $R^{>i+1}\gamma_*\mathcal{K}_{i+1} = 0$. Then the first short exact sequence, combined with the fact that $R^{>i}\gamma_* \wedge^i \mathcal{R}^\vee = 0$, shows that $R^{>i}\gamma_*\mathcal{K}_i = 0$. The claim follows by induction.

Then using the second exact sequence, we see that $R^{>0}\gamma_*(i_*\mathcal{O}_{\widetilde{Z}}) = 0$. The diagram is commutative and pushforwards by closed immersions are exact, so $R^k\gamma_*(i_*\mathcal{O}_{\widetilde{Z}}) = j_*R^k\rho_*\mathcal{O}_{\widetilde{Z}}$ for all $k \geq 0$. Hence $j_*R^{>0}\rho_*\mathcal{O}_{\widetilde{Z}} = 0$, which implies that $R^{>0}\rho_*\mathcal{O}_{\widetilde{Z}} = 0$. $\square$

The same idea in the previous proof also proves the following proposition, which we will need later.

Proposition 2.4.2. Let $\mathcal{V}_\bullet \to \mathcal{F}$ be a finite exact sequence of coherent sheaves on a proper scheme X. Let k be a nonnegative integer. If for all $i \geq 0$, $H^{\geq i+k}(X, \mathcal{V}_i) = 0$, then $H^{\geq k}(X, \mathcal{F}) = 0$. $\square$

The map $q : F \to B$ is flat with fibers all isomorphic to $\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N})$, where $N = h^0\mathcal{O}(\vec{e})$, $\vec{e}$ is any splitting type showing up in B. By the theorem below and cohomology and base change, we know that $R^{>0}\gamma_*\mathcal{O}_F = 0$. So the condition of Proposition 2.4.1 is satisfied when $i = 0$.

Theorem 2.4.3. ([Str87], Theorem 2.1) $Q = \mathrm{Quot}^{r,d}(\mathcal{O}^{\oplus N}_{\mathbb{P}^1})$ is an irreducible, nonsingular, rational projective variety. In particular, $H^{>0}(Q, \mathcal{O}_Q) = 0$.

Therefore, to show the $\overline{\Sigma}_{\vec{e}}$ has rational singularities, by Proposition 2.4.1, it remains to show that $R^{>n}\gamma_* \bigwedge^n \mathcal{R}^\vee = 0$ for all $n > 0$, where

$$\begin{aligned}\mathcal{R} &= f_*\mathcal{H}om(\widetilde{\gamma}^*\pi^*(\pi_*\mathcal{E}(-1))(-1), \mathcal{Q}) \\ &= (\gamma^*\pi_*\mathcal{E}(-1))^\vee \otimes f_*\mathcal{Q}(1).\end{aligned}$$

In characteristic 0, Cauchy's formula says that if $\mathcal{E}, \mathcal{F}$ are vector bundles, then

$$\bigwedge^n(\mathcal{E} \otimes \mathcal{F}) \cong \oplus_{\mu \vdash n} \mathbf{S}_{\mu^\dagger}\mathcal{E} \otimes \mathbf{S}_\mu \mathcal{F},$$

where $\mu^\dagger$ is the conjugate partition of μ, and μ ranges over all partitions of n. Using this identity, we have

$$\begin{aligned}\bigwedge^n \mathcal{R}^\vee &= \bigwedge^n \left((\gamma^*\pi_*\mathcal{E}(-1)) \otimes (f_*\mathcal{Q}(1))^\vee\right) \\ &= \bigoplus_{\mu \vdash n} \mathbf{S}_{\mu^\dagger}(\gamma^*\pi_*\mathcal{E}(-1)) \otimes \mathbf{S}_\mu((f_*\mathcal{Q}(1))^\vee) \\ &= \bigoplus_{\mu \vdash n} \gamma^*\mathbf{S}_{\mu^\dagger}(\pi_*\mathcal{E}(-1)) \otimes \mathbf{S}_\mu((f_*\mathcal{Q}(1))^\vee).\end{aligned}$$

So by the projection formula,

$$R^k\gamma_* \bigwedge^n \mathcal{R}^\vee = \bigoplus_{\mu \vdash n} \mathbf{S}_{\mu^\dagger}(\pi_*\mathcal{E}(-1)) \otimes R^k\gamma_*(\mathbf{S}_\mu((f_*\mathcal{Q}(1))^\vee)).$$

Therefore, to prove Theorem 2.1.2, it suffices to show that when $k > n = |\mu|$, the higher pushforwards $R^k\gamma_*(\mathbf{S}_\mu((f_*\mathcal{Q}(1))^\vee))$ are zero. Since γ is flat with fibers all isomorphic to $\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N})$, by cohomology and base change, it suffices to show vanishing for the cohomology of corresponding bundles on $\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N})$.

Let $d > 0$ and $N \geq r \geq 0$. Let $\mathcal{Q}$ be the universal quotient on $\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N}) \times \mathbb{P}^1$, and let $p : \mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N}) \times \mathbb{P}^1 \to \mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N})$ be the projection to the first factor. Recall from the introduction that on $\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N})$, for all $m \geq -1$, we define tautological bundles $\mathcal{E}_m = (p_*\mathcal{Q}(m))^\vee$.

Theorem 2.4.4. Let λ be any nonempty partition. Then $H^{>|\lambda|}(\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N}), \mathbf{S}_\lambda \mathcal{E}_1) = 0$.

By our previous discussion, Theorem 2.4.4 implies Theorem 2.1.2. It follows from the more general statement below.

Theorem 2.4.5 (Restatement of Theorem 1.2.5)**.** Let $m \geq -1$ be an integer. Let $\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_m$ be partitions, at least one of which is nonempty. Let $D = |\lambda_{-1}| + \cdots + |\lambda_m|$. Then

$$H^{>D}(\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N}), \otimes_{i=-1}^m \mathbf{S}_{\lambda_i}\mathcal{E}_i) = 0.$$

We will prove Theorem 1.2.5 above by induction on the number of factors in the tensor product. For the base case, we will need the following proposition.

Proposition 2.4.6. Let $D > 0$ be a positive integer. Then for sufficiently large m, for any two partitions α, β such that $|\alpha| + |\beta| = D$,

$$H^{>D}(\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N}), \mathbf{S}_\alpha \mathcal{E}_{m-1} \otimes \mathbf{S}_\beta \mathcal{E}_m) = 0.$$

Proof of Theorem 1.2.5 assuming Proposition 2.4.6. On $\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N}) \times \mathbb{P}^1$, for any $n \in \mathbb{Z}$, we have an exact sequence

$$0 \to \mathcal{O}(n) \to \mathcal{O}(n+1)^{\oplus 2} \to \mathcal{O}(n+2) \to 0,$$

by pulling back the same sequence from $\mathbb{P}^1$. Tensoring this sequence with $\mathcal{Q}$, since $\mathcal{O}(n+2)$ is locally free, the sequence remains exact:

$$0 \to \mathcal{Q}(n) \to \mathcal{Q}(n+1)^{\oplus 2} \to \mathcal{Q}(n+2) \to 0.$$

For $n \geq -1$, $R^1 p_* \mathcal{Q}(n) = 0$, so the sequence remains exact after taking p_*, and each term is a vector bundle on $\mathrm{Quot}^{r,d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus N})$:

$$0 \to p_*\mathcal{Q}(n) \to p_*\mathcal{Q}(n+1)^{\oplus 2} \to p_*\mathcal{Q}(n+2) \to 0.$$

Dualizing, we have

$$0 \to \mathcal{E}_{n+2} \xrightarrow{\Phi_{n+1}} \mathcal{E}_{n+1}^{\oplus 2} \to \mathcal{E}_n \to 0. \tag{2.6}$$

We will prove Theorem 1.2.5 for a fixed $D > 0$ by induction on the index i of the smallest nonempty partition λ_i. Without loss of generality, we may take m to be sufficiently large. So Proposition 2.4.6 takes care of the base cases $i = m$ and $i = m - 1$. Now suppose that Theorem 1.2.5 is true when the smallest nonempty partition is λ_k. There is a complex $(\mathbf{S}_{\lambda_{k-1}}\Phi_k)_\bullet$ formed on the map $\Phi_k : \mathcal{E}_{k+1} \to \mathcal{E}_k^{\oplus 2}$, called the Schur complex, whose t-th term is

$$(\mathbf{S}_{\lambda_{k-1}}\Phi_k)_t = \oplus_{|\alpha|=t, c^{\lambda_{k-1}}_{\alpha,\beta} \neq 0} \mathbf{S}_{\alpha^\dagger}\mathcal{E}_{k+1} \otimes \mathbf{S}_\beta \mathcal{E}_k^{\oplus 2}.$$

Corollary V.1.15 of [ABW82] tells us that when the cokernel of Φ_k is locally free, this complex is a resolution of the Schur functor of the cokernel, which is the vector bundle $\mathbf{S}_{\lambda_{k-1}}\mathcal{E}_{k-1}$ in our case.

Tensoring this resolution with $\otimes_{i=k}^m \mathbf{S}_{\lambda_i}\mathcal{E}_i$ and applying Littlewood-Richardson rules, we find that the vector bundle $\otimes_{i=k-1}^m \mathbf{S}_{\lambda_i}\mathcal{E}_i$ has a resolution whose t-th term is the direct sum of bundles of the form $\otimes_{i=k+2}^{m}\mathbf{S}_{\lambda_i}\mathcal{E}_i \otimes \mathbf{S}_{\lambda'_{k+1}}\mathcal{E}_{k+1} \otimes \mathbf{S}_{\lambda'_k}\mathcal{E}_k$, where $|\lambda'_{k+1}| = |\lambda_{k+1}| + t$, and $|\lambda'_k| = |\lambda_k| + |\lambda_{k-1}| - t$. In particular, the sum of the sizes of the partitions involved is D for all bundles that show up in the resolution. The inductive hypothesis applies to show that all terms in the resolution have no cohomology in degree greater than D, which implies that $H^{>D}(\otimes_{i=k-1}^{m}\mathbf{S}_{\lambda_i}\mathcal{E}_i) = 0$. □

Proof of Proposition 2.4.6

For every integer $m \geq d-1$, we have a short exact sequence

$$0 \to p_*\mathcal{S}(m) \to p_*\mathcal{O}(m)^{\oplus N} \to p_*\mathcal{Q}(m) \to 0$$

of vector bundles on $\mathrm{Quot}_{\mathbb{P}^1}^{r,d}(\mathcal{O}^{\oplus N})$, inducing a map $g_m : \mathrm{Quot}_{\mathbb{P}^1}^{r,d}(\mathcal{O}^{\oplus N}) \to G_m$, where $G_m = \mathrm{Gr}(\mathrm{rk}\, p_*\mathcal{S}(m), N(m+1))$. Strømme proves that for $m \geq d$, the map $\iota = \iota_{m-1} = (g_{m-1}, g_m) : \mathrm{Quot}_{\mathbb{P}^1}^{r,d}(\mathcal{O}^{\oplus N}) \to G_{m-1} \times G_m$ is a closed embedding (Theorem 4.1, [Str87]). Let

$$0 \to \mathcal{A}_1 \to \mathcal{O}_{G_{m-1}}^{\oplus Nm} \to \mathcal{B}_1 \to 0 \tag{2.7}$$

$$0 \to \mathcal{A}_2 \to \mathcal{O}_{G_m}^{\oplus N(m+1)} \to \mathcal{B}_2 \to 0 \tag{2.8}$$

be the universal sequences on G_{m-1} and G_m, and let $W = H^0(\mathbb{P}^1, \mathcal{O}(1))$. For $i = 1, 2$, let $k_i = \mathrm{rk}\,\mathcal{A}_i, n_1 = Nm, n_2 = N(m+1)$, so that $G_{m-1} \times G_m = \mathrm{Gr}(k_1, n_1) \times \mathrm{Gr}(k_2, n_2)$. Strømme also proves that $\mathrm{Quot}_{\mathbb{P}^1}^{r,d}(\mathcal{O}^{\oplus N})$ is the zero locus of a global section of the vector bundle $\mathcal{A}_1^\vee \boxtimes (\mathcal{B}_2 \otimes W) \cong \mathcal{A}_1^\vee \boxtimes (\mathcal{B}_2 \oplus \mathcal{B}_2)$, whose rank is equal to the codimension of $\mathrm{Quot}_{\mathbb{P}^1}^{r,d}(\mathcal{O}^{\oplus N})$ in $G_{m-1} \times G_m$, exhibiting it as a local complete intersection (Theorem 4.1, [Str87]).

As a result, $\iota_*\mathcal{O}_{\mathrm{Quot}_{\mathbb{P}^1}^{r,d}(\mathcal{O}^{\oplus N})}$ admits a Koszul resolution as a $\mathcal{O}_{G_{m-1}\times G_m}$-module:

$$\wedge^\bullet (\mathcal{A}_1 \boxtimes (\mathcal{B}_2 \oplus \mathcal{B}_2)^\vee) \to \iota_*\mathcal{O}_{\mathrm{Quot}_{\mathbb{P}^1}^{r,d}(\mathcal{O}^{\oplus N})}.$$

By definition of g_{m-1} and g_m, we know that $\iota^*(\mathbf{S}_\alpha\mathcal{B}_1^\vee \boxtimes \mathbf{S}_\beta\mathcal{B}_2^\vee) = \mathbf{S}_\alpha\mathcal{E}_{m-1} \otimes \mathbf{S}_\beta\mathcal{E}_m$. So tensoring the Koszul resolution with $\mathbf{S}_\alpha\mathcal{B}_1^\vee \boxtimes \mathbf{S}_\beta\mathcal{B}_2^\vee$, we get a resolution of $\iota_*(\mathbf{S}_\alpha\mathcal{E}_{m-1} \otimes \mathbf{S}_\beta\mathcal{E}_m)$:

$$\wedge^\bullet (\mathcal{A}_1 \boxtimes (\mathcal{B}_2 \oplus \mathcal{B}_2)^\vee) \otimes (\mathbf{S}_\alpha\mathcal{B}_1^\vee \boxtimes \mathbf{S}_\beta\mathcal{B}_2^\vee) \to \iota_*(\mathbf{S}_\alpha\mathcal{E}_{m-1} \otimes \mathbf{S}_\beta\mathcal{E}_m).$$

For a given $D = |\alpha| + |\beta|$, we will choose m to be sufficiently large so that $D < n_1 - k_1$. In particular, any partition α with $|\alpha| \leq D$ must have $\alpha_{n_1-k_1} = 0$. Let $\mu = (\mu_1, \ldots, \mu_{k_1}), \alpha = (\alpha_1, \ldots, \alpha_{n_1-k_1})$ and β be partitions. Let

$$\mathcal{V}_{\mu,\alpha,\beta} = (\mathbf{S}_\mu\mathcal{A}_1 \otimes \mathbf{S}_\alpha\mathcal{B}_1^\vee) \boxtimes (\mathbf{S}_{\mu^\dagger}(\mathcal{B}_2^\vee \oplus \mathcal{B}_2^\vee) \otimes \mathbf{S}_\beta\mathcal{B}_2^\vee).$$

By Cauchy's formula,

$$\wedge^k (\mathcal{A}_1 \boxtimes (\mathcal{B}_2 \oplus \mathcal{B}_2)^\vee) \otimes (\mathbf{S}_\alpha\mathcal{B}_1^\vee \boxtimes \mathbf{S}_\beta\mathcal{B}_2^\vee) = \oplus_{\mu\vdash k}\mathcal{V}_{\mu,\alpha,\beta}.$$

By Proposition 2.4.2, the following result implies Proposition 2.4.6.

Proposition 2.4.7. For all partitions α, β with $|\alpha| + |\beta| > 0$ and $\alpha_{n_1-k_1} = 0$, and any partition μ,

$$H^{>|\mu|+|\alpha|+|\beta|}(\mathcal{V}_{\mu,\alpha,\beta}) = 0.$$

We will first deal with the case where $\mu \neq \emptyset$, which is harder. Suppose that there exist partitions $\gamma_1, \gamma_2, \nu', \nu = (\nu_1, \ldots, \nu_{n_2-k_2})$ such that $c^{\mu^\dagger}_{\gamma_1,\gamma_2} c^{\nu'}_{\gamma_1,\gamma_2} c^{\nu}_{\nu',\beta} \neq 0$. In other words,

$$\mathcal{W}_{\mu,\alpha,\nu} = (\mathbf{S}_\mu \mathcal{A}_1 \otimes \mathbf{S}_\alpha \mathcal{B}_1^\vee) \boxtimes \mathbf{S}_\nu \mathcal{B}_2^\vee$$

arises as a summand of $\mathcal{V}_{\mu,\alpha,\beta}$. By Borel-Weil-Bott and Künneth, if $\mathcal{W}_{\mu,\alpha,\nu}$ has cohomology, the vector bundles $\mathbf{S}_\mu \mathcal{A}_1 \otimes \mathbf{S}_\alpha \mathcal{B}_1^\vee$ and $\mathbf{S}_\nu \mathcal{B}_2^\vee$ must both have cohomology in some unique degrees D_1, D_2. Then $\mathcal{W}_{\mu,\alpha,\nu}$ has cohomology in degree $D = D_1 + D_2$. When this happens, Borel-Weil-Bott imposes some combinatorial conditions on the partitions involved. We would like to show that $D \leq |\mu| + |\alpha| + |\beta|$.

Before proceeding further with the proof, note the following example which shows that this bound is sharp in general.

Example 2.4.8. The Quot scheme $\mathrm{Quot}^{1,3}(\mathcal{O}^{\oplus 3})$ admits an embedding $\iota_2 : \mathrm{Quot}^{1,3}(\mathcal{O}^{\oplus 3}) \hookrightarrow \mathrm{Gr}(3,9) \times \mathrm{Gr}(5,12)$. We can take $\mu = (7,2), \alpha = (1), \beta = (2)$. Then $H^{12}(\mathcal{V}_{\mu,\alpha,\beta}) \neq 0$. Indeed, on the first factor, cohomology degree equals 7. On the second factor, nonzero cohomology arises from the partition $\nu = (6, 1^5)$, which has cohomology in degree 5. It comes from combining $\nu' = (4, 1^5)$ and $\beta = (2)$ using the Littlewood-Richardson rule. The possible pairs γ_1, γ_2 such that $c^{\mu^\dagger}_{\gamma_1,\gamma_2} c^{\nu'}_{\gamma_1,\gamma_2} \neq 0$ include $(\gamma_1 = (2), \gamma_2 = (2, 1^5)), (\gamma_1 = (2,1), \gamma_2 = (2, 1^4)), (\gamma_1 = (2,1,1), \gamma_2 = (2, 1^3))$, as well as the three tuples obtained from the above by switching γ_1 and γ_2. For all six pairs, $c^{\mu^\dagger}_{\gamma_1,\gamma_2} c^{\nu'}_{\gamma_1,\gamma_2} c^{\nu}_{\nu',\beta} = 1$.

Due to a lack of understanding of the map on cohomology induced by the Koszul resolution, we are unable to check whether this nonvanishing group implies that $H^3(\mathrm{Quot}, \mathcal{E}_2 \otimes \mathrm{Sym}^2 \mathcal{E}_3) \neq 0$.

Now let us delve into the combinatorics further. The vector bundle $\mathbf{S}_\nu \mathcal{B}_2^\vee$ has cohomology if and only if the following tuple obtained by pointwise addition

$$(0^{k_2}, \nu) + (n_2 - 1, n_2 - 2, \ldots, 1, 0)$$

has no repeated entries. Borel-Weil-Bott says that if this tuple has no repetitions, then $\mathbf{S}_\nu \mathcal{B}_2^\vee$ has cohomology in degree equal to the length of the permutation that sorts this vector in descending order, and no cohomology in other degrees. Since $|\mu| > 0$, $|\nu| = |\mu| + |\beta| > 0$. So the condition that the vector has no repetition is equivalent to the existence of some $1 \leq j \leq n_2 - k_2$ such that $\nu_j \geq k_2 + j$ and $\nu_{j+1} \leq j$. When such j exists, it is unique and $\mathbf{S}_\nu \mathcal{B}_2^\vee$ has cohomology in degree $D_2 = jk_2$.

The cohomology of the first vector bundle $\mathbf{S}_\mu \mathcal{A}_1 \otimes \mathbf{S}_\alpha \mathcal{B}_1^\vee$ is a bit more complicated to describe. Define

$$f_i = n_1 - k_1 + i - 1 - \mu_i,$$
$$g_j = n_1 - k_1 - j + \alpha_j.$$

Let $-\mu = (-\mu_{k_1}, \ldots, -\mu_2, -\mu_1)$. Then Borel-Weil-Bott says that $\mathbf{S}_\mu \mathcal{A}_1 \otimes \mathbf{S}_\alpha \mathcal{B}_1^\vee$ has cohomology if and only if the vector

$$(-\mu, \alpha) + (n_1 - 1, n_1 - 2, \ldots, 1, 0) = (f_{k_1}, f_{k_1-1}, \ldots, f_1, g_1, g_2, \ldots, g_{n_1-k_1})$$

does not have repeated entries. In this case, $\mathbf{S}_\mu \mathcal{A}_1 \otimes \mathbf{S}_\alpha \mathcal{B}_1^\vee$ has cohomology in degree equal to the length of the permutation that sorts this vector in strictly descending order.

Define indices $0 \leq i_1 \leq i_2 \leq \cdots \leq i_{n_1-k_1} \leq k_1$, where i_ℓ is the largest index such that $f_{i_\ell} < g_{n_1-k_1-\ell+1}$ for each $1 \leq \ell \leq n_1 - k_1$. When there doesn't exist such an index, we set $i_\ell = 0$. In terms of $i_1, \ldots, i_{n_1-k_1}$, the unique degree in which $\mathbf{S}_\mu \mathcal{A}_1 \otimes \mathbf{S}_\alpha \mathcal{B}_1^\vee$ has nonzero cohomology is

$$D_1 = \sum_{\ell=1}^{n_1-k_1} (i_\ell - i_{\ell-1})(n_1 - k_1 - \ell + 1),$$

where we let $i_0 = 0$. For $1 \leq \ell \leq n_1 - k_1$, the inequality $f_{i_\ell} < g_{n_1-k_1-\ell+1}$ is equivalent to

$$\mu_{i_\ell} - (n_1 - k_1 - \ell + 1) \geq i_\ell - \alpha_{n_1-k_1-\ell+1}. \tag{2.9}$$

Setting $\ell = 1$, we note that (2.9) says that

$$\mu_{i_1} \geq n_1 - k_1 + i_1 - \alpha_{n_1-k_1} = n_1 - k_1 + i_1. \tag{2.10}$$

When $i_\ell < i_{\ell+1}$, we also have $f_{i_\ell+1} > g_{n_1-k_1-\ell+1}$, which is equivalent to

$$i_\ell - \alpha_{n_1-k_1-\ell+1} > \mu_{i_\ell+1} - (n_1 - k_1 - \ell + 1). \tag{2.11}$$

Replacing ℓ with $\ell + 1$ in (2.9), we find

$$\mu_{i_{\ell+1}} - (n_1 - k_1 - \ell) \geq i_{\ell+1} - \alpha_{n_1-k_1-\ell}.$$

Combined with (2.11), we obtain

$$i_\ell - \alpha_{n_1-k_1-\ell+1} \geq \mu_{i_\ell+1} - (n_1 - k_1 - \ell) \geq \mu_{i_{\ell+1}} - (n_1 - k_1 - \ell) \geq i_{\ell+1} - \alpha_{n_1-k_1-\ell}.$$

So for $1 \leq \ell < n_1 - k_1$, whenever $i_\ell < i_{\ell+1}$,

$$\alpha_{n_1-k_1-\ell} - \alpha_{n_1-k_1-\ell+1} \geq i_{\ell+1} - i_\ell. \tag{2.12}$$

If $i_\ell = i_{\ell+1}$, then this inequality follows from α being a partition. So we've shown that (2.12) holds unconditionally for $1 \leq \ell < n_1 - k_1$.

To summarize,

Proposition 2.4.9. If $H^\bullet(\mathcal{W}_{\mu,\alpha,\nu}) \neq 0$, then there exist $1 \leq j \leq n_2 - k_2$, and $0 = i_0 \leq i_1 \leq \cdots \leq i_{n_1-k_1} \leq k_1$ such that the conditions (2.10) and (2.12) are satisfied, and the unique degree in which the cohomology of $\mathcal{W}_{\mu,\alpha,\nu}$ is nonzero is $D = jk_2 + \sum_{\ell=1}^{n_1-k_1}(i_\ell - i_{\ell-1})(n_1 - k_1 - \ell + 1)$. Moreover,

$$D \leq i_1(n_1 - k_1) + jk_2 + |\alpha|.$$

Proof. We've already proven the proposition except for the last inequality. Recall that m was chosen so that $\alpha_{n_1-k_1} = 0$. Now using Equation (2.12),

$$\begin{aligned} D &= jk_2 + \sum_{\ell=1}^{n_1-k_1}(i_\ell - i_{\ell-1})(n_1 - k_1 - \ell + 1) \\ &\leq jk_2 + i_1(n_1-k_1) + \sum_{\ell=2}^{n_1-k_1}(\alpha_{n_1-k_1-\ell+1} - \alpha_{n_1-k_1-\ell+2})(n_1-k_1-\ell+1) \\ &= jk_2 + i_1(n_1-k_1) + |\alpha| - \alpha_{n_1-k_1}(n_1-k_1) \\ &= jk_2 + i_1(n_1-k_1) + |\alpha|. \end{aligned}$$

□

Proposition 2.4.10. Suppose that after applying the Littlewood-Richardson decompositions, $\mathcal{V}_{\mu,\alpha,\beta}$ contains $\mathcal{W}_{\mu,\alpha,\nu}$ as a summand, i.e. there exist partitions $\gamma_1, \gamma_2, \nu', \nu$ such that $c^{\mu^\dagger}_{\gamma_1,\gamma_2} c^{\nu'}_{\gamma_1,\gamma_2} c^{\nu}_{\nu',\beta} \neq 0$. Suppose $\mathcal{W}_{\mu,\alpha,\nu}$ has cohomology with associated indices j, i_ℓ given by the previous proposition. Then

$$|\mu| + |\alpha| + |\beta| \geq jk_2 + i_1(n_1-k_1) + (i_1 - j)^2 + |\alpha|.$$

Proof. Write $\alpha+\beta$ for the partition with $(\alpha+\beta)_i = \alpha_i + \beta_i$, and write $\alpha \cup \beta$ for the partition obtained by stacking the Young diagrams of α and β and sorting the rows according to length. As explained in Proposition 3.2.1 of [MOS22], Littlewood-Richardson rules imply that ν' is dominated by $\alpha+\beta$, and $\mu^\dagger$ is dominated by $\alpha \cup \beta$. Therefore

$$\mu_1^\dagger + \cdots + \mu_{2j}^\dagger \geq \alpha_1 + \cdots \alpha_j + \beta_1 + \cdots + \beta_j \geq \nu_1' + \cdots + \nu_j'.$$

But since $\nu_j \geq k_2 + j$, we also have that

$$\nu_1' + \cdots + \nu_j' \geq \nu_1 + \cdots + \nu_j - |\beta| \geq j(k_2+j) - |\beta|.$$

Recall also from (2.10) that $\mu_{i_1} \geq n_1 - k_1 + i_1$. So

$$\begin{aligned} |\mu| &\geq \sum_{k=1}^{2j} \mu_k^\dagger + i_1(n_1 - k_1 + i_1 - 2j) \\ &\geq j(k_2+j) - |\beta| + i_1(n_1 - k_1 + i_1 - 2j) \\ &\geq jk_2 + i_1(n_1+k_1) + (i_1-j)^2 - |\beta|. \end{aligned}$$

Adding $|\alpha| + |\beta|$ to both sides yields the desired claim. □

Now Proposition 2.4.10 and Proposition 2.4.9 together show that $|\mu| + |\alpha| + |\beta| \geq D$, which proves Proposition 2.4.7 when $\mu \neq \emptyset$. It remains to prove Proposition 2.4.7 when $\mu = \emptyset$. In this case, $\mathcal{V}_{\emptyset,\alpha,\beta} = \mathbf{S}_\alpha \mathcal{B}_1^\vee \boxtimes \mathbf{S}_\beta \mathcal{B}_2^\vee$. The same analysis of $\mathbf{S}_\nu \mathcal{B}_2^\vee$ in the previous case applies. If α, β are both non-empty, the bundle $\mathcal{V}_{\emptyset,\alpha,\beta}$ has cohomology if and only if there exist $1 \leq i \leq n_1 - k_1$, $1 \leq j \leq n_2 - k_2$, such that $\alpha_i \geq k_1 + i$, $\alpha_{i+1} \leq i$, $\beta_j \geq k_2 + j$, $\beta_j \leq j$. If this is the case, then $\mathcal{V}_{\emptyset,\alpha,\beta}$ has cohomology in degree $D = ik_1 + jk_2$. Now $|\alpha| + |\beta| \geq i(k_1+i) + j(k_2+j) > D$. The argument in the case where only one of α, β is nonempty is completely analogous. This concludes the proof of Proposition 2.4.7 and the proof of Theorem 2.1.2.

Chapter 3

When are splitting loci Gorenstein?

3.1 Introduction

Let $\mathcal{B}_{r,d}$ be the stack of rank r, degree d vector bundles on $\mathbb{P}^1$. For every tuple of integers $\vec{e} = (e_1, \ldots, e_r)$ with $e_1 \leq e_2 \leq \cdots \leq e_r$ and $e_1 + \cdots + e_r = d$, there is a locally closed substack $\mathbf{\Sigma}_{\vec{e}} \subseteq \mathcal{B}_{r,d}$ parametrizing families of vector bundles on $\mathbb{P}^1$ that fiberwise are isomorphic to $\mathcal{O}(\vec{e})$. The closure of $\mathbf{\Sigma}_{\vec{e}}$ is a closed substack $\overline{\mathbf{\Sigma}}_{\vec{e}} \subseteq \mathcal{B}_{r,d}$, the *splitting locus* associated to $\vec{e}$, parametrizing vector bundles on $\mathbb{P}^1$ that split as $\mathcal{O}(\vec{e}) = \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_r)$ or a specialization of $\mathcal{O}(\vec{e})$. While $\mathbf{\Sigma}_{\vec{e}}$ is smooth, the closure $\overline{\mathbf{\Sigma}}_{\vec{e}}$ can be singular. In this paper, we completely characterize when splitting loci $\overline{\mathbf{\Sigma}}_{\vec{e}}$ are Gorenstein or $\mathbb{Q}$-Gorenstein. Our universal statement extends to splitting loci in every scheme that maps smoothly to $\mathcal{B}_{r,d}$ (i.e. if the Kodaira-Spencer map is surjective at every point). For example, our results exhibit the corresponding Hurwitz-Brill-Noether loci as having at most Gorenstein or $\mathbb{Q}$-Gorenstein singularities (see [Lar21a] for more details). In previous work on the singularities of splitting loci, it has been shown that splitting loci are normal and Cohen-Macaulay [LLV25], and that certain tame splitting loci have rational singularities (Theorem 2.1.2).

We say that $\vec{e}$ is a *block arithmetic progression* of block size s and difference t if it is a shift of the tuple $(0^s, t^s, \ldots, ((m-1)t)^s)$ (t^s is a short hand for s copies of the number t). As usual, if $s = 1$, we say that the tuple is an *arithmetic progression*. We say that $\vec{e}$ is *contiguous* if $\vec{e}$ is a shift of a tuple of the form $(\ldots, 0^a, 1^b, 2^a, 3^b, \ldots)$ for some positive integers a, b.

Example 3.1.1.

1. $\vec{e} = (-1, 0, 0, 0, 1)$ and $\vec{e} = (0, 1, 1, 1, 2, 3, 3, 3)$ are contiguous.
2. $\vec{e} = (-3, -3, 0, 0, 3, 3)$ is a block arithmetic progression of block size 2 and difference 3.

We say that a Cohen-Macaulay scheme X is N-Gorenstein if $\omega_X^{\otimes N}$ is a line bundle. Being Gorenstein and N-Gorenstein are smooth-local properties (see Section 3.2), so it makes sense to ask when an algebraic stack, such as a splitting locus $\overline{\mathbf{\Sigma}}_{\vec{e}}$, is Gorenstein or N-Gorenstein. We say that an algebraic stack is $\mathbb{Q}$-Gorenstein if it is N-Gorenstein for some $N \geq 1$.

Our main theorem is the following.

Theorem 3.1.2. A splitting locus $\overline{\mathbf{\Sigma}}_{\vec{e}}$ is Gorenstein if and only if $\vec{e}$ is a block arithmetic progression of difference $0, 1$ or 2, or if $\vec{e}$ is contiguous. $\overline{\mathbf{\Sigma}}_{\vec{e}}$ is $\mathbb{Q}$-Gorenstein (but not Gorenstein) if and only if it is an arithmetic progression of difference $t \geq 3$. In this case, the smallest N for which $\overline{\mathbf{\Sigma}}_{\vec{e}}$ is N-Gorenstein is t if t is odd, and $t/2$ if t is even.

To prove Theorem 3.1.2, we first exhibit a basis

$$a_1^{(1)}, \dots, a_1^{(\ell)}, b_2^{(1)}, \dots, b_2^{(\ell)}$$

for $A^1(\overline{\mathbf{\Sigma}}_{\vec{e}})$ in Proposition 3.3.2. The $a_1^{(i)}$'s lift generators for $A^1(\mathbf{\Sigma}_{\vec{e}})$. Setting $\vec{e} = (f_1^{s_1}, \dots, f_\ell^{s_\ell})$, let $\vec{e}^{(i)}$ be defined by replacing $f_i^{s_i}$ in $\vec{e}$ with $(f_i - 1, f_i^{s_i-2}, f_i + 1)$. Then the $b_2^{(i)}$'s are given by

$$b_2^{(i)} = \begin{cases} [\overline{\mathbf{\Sigma}}_{\vec{e}^{(i)}}] & \text{if } f_{i-1} + 1 < f_i < f_{i+1} - 1 \text{ and } s_i \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\overline{\mathbf{\Sigma}}_{\vec{e}^{(i)}}$ is of codimension 1 in $\overline{\mathbf{\Sigma}}_{\vec{e}}$. (When we say basis, we mean to include only those nonzero $b_2^{(i)}$'s.) In terms of these universal classes, we prove an adjunction-type formula for the class of the canonical module for splitting loci.

Theorem 3.1.3 (Theorem 3.3.8)**.** Let X be a scheme, and let $\mathcal{E}$ be a rank r, fiberwise degree d vector bundle on $X \times \mathbb{P}^1$, inducing a smooth map $X \to \mathcal{B}_{r,d}$ (so that X is necessarily smooth). Let $\vec{e} = (f_1^{s_1}, \dots, f_\ell^{s_\ell})$, where $f_1 < \cdots < f_\ell$, and let $\overline{\Sigma}_{\vec{e}}(X)$ denote the splitting locus corresponding to $\vec{e}$ in X. Write $\delta_i = \sum_{j<i} s_j - \sum_{j>i} s_j$. Then

$$[\omega_{\overline{\Sigma}_{\vec{e}}(X)}] = [\omega_X|_{\overline{\Sigma}_{\vec{e}}(X)}] + \sum_{1 \leq i \leq \ell} \left((\deg \vec{e} - r f_i + \delta_i) a_1^{(i)} + (r - s_i) b_2^{(i)} \right),$$

where the $a_1^{(i)}, b_2^{(i)}$ classes denote the pullbacks of the universal classes on $\overline{\mathbf{\Sigma}}_{\vec{e}}$ to $\overline{\Sigma}_{\vec{e}}(X)$.

Next, we consider a specific family of smooth covers of the stack $\mathcal{B}_{r,d}$ of rank r, degree d vector bundles on $\mathbb{P}^1$, namely the affine spaces $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}\,'))$ where $\vec{e}\,' = ((-M)^{r-1}, D)$, and D is large enough so that $\vec{e}\,' < \vec{e}$. Let $\overline{\Sigma}_{\vec{e}}$ denote the splitting locus corresponding to the splitting type $\vec{e}$ inside $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}\,'))$. As worked out in Section 2.2, for splitting types $\vec{e}\,' = ((-M)^{r-1}, D)$, the affine space $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}\,'))$ is a principal GL_{r-1}-bundle over an open substack of a vector bundle over $\mathcal{B}_{r,d}$. Using this interpretation of $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}\,'))$, we prove the following:

Theorem 3.1.4. Let $\vec{e}\,' = ((-M)^{r-1}, D)$ and let $\vec{e} = (f_1^{s_1}, \dots, f_\ell^{s_\ell})$, where $f_1 < \cdots < f_\ell$, such that $\vec{e}\,' < \vec{e}$. Then the class group of the splitting locus $\overline{\Sigma}_{\vec{e}} \subseteq H^1\mathcal{E}nd(\mathcal{O}(\vec{e}\,'))$ fits in the following right exact sequence:

$$\mathbb{Z}^2 \to A^1(\overline{\mathbf{\Sigma}}_{\vec{e}}) \to A^1(\overline{\Sigma}_{\vec{e}}) \to 0, \tag{3.1}$$

where the first map is given by sending the two basis elements to

$$\alpha_1 = \sum_{i=1}^{\ell} \left(-(f_i + M) a_1^{(i)} + b_2^{(i)} \right),$$
$$\alpha_2 = \sum_{i=1}^{\ell} \left((f_i + M + 1) a_1^{(i)} - b_2^{(i)} \right).$$

With some computation, the previous two theorems together tell us exactly when the class of the canonical module in $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ is trivial or N-torsion. This turns out to be enough. The affine spaces $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ in general admit an action by $\mathbb{G}_m^{k-1}$, where k is the number of perfectly balanced blocks in $\vec{e}'$, which makes the ideal for splitting loci therein $\mathbb{Z}^{k-1}$-graded (Section 3.2). When $\vec{e}'$ has the special shape of $((-M)^{r-1}, D)$, $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ admits a natural $\mathbb{G}_m$-action which makes the canonical module for $\overline{\Sigma}_{\vec{e}}$ standard-graded. As such, $\omega_{\overline{\Sigma}_{\vec{e}}}^{\otimes N}$ is a line bundle if and only if we have $N[\omega_{\overline{\Sigma}_{\vec{e}}}] = 0$ as a class in $A^1(\overline{\Sigma}_{\vec{e}})$ (Proposition 3.2.6).

Remark 3.1.5.

1. Note that the image of the first map in (3.1) is generated by the classes $\alpha_1 + \alpha_2 = \sum_{i=1}^{\ell} a_1^{(i)}$ and $\alpha_2 - (M+1)(\alpha_1+\alpha_2) = \sum_{i=1}^{\ell}(f_i a_1^{(i)} - b_2^{(i)})$, and in particular does not depend on M. One can check that these are precisely the classes that arise from restricting classes in $A^1(\mathcal{B}_{r,d})$. A priori, these classes must pull back to zero on $\overline{\Sigma}_{\vec{e}}$, since we can also factor the pullback through the affine space $H^1\mathcal{E}nd\mathcal{O}(\vec{e}')$, which has trivial class group. Our computation tells us that these are the only classes that become zero.

2. When $\vec{e}$ is rank 2, the ideals for splitting loci in $H^1\mathcal{E}nd\mathcal{O}(\vec{e}')$ are precisely Hankel determinantal ideals, as explained in [ES08, Corollary 5.4]. Thus our results recover the computation in [Con+18] for the class groups of Hankel determinantal rings and the classes of the canonical modules therein. Our previous remark gives an explanation for why the class group of Hankel determinantal rings only depends on the difference between the number of rows and columns of the Hankel matrix.

3. Our investigation is partly inspired by the results in the case of determinantal varieties, warranting a brief comparison here. Over a field, the class group of the determinantal variety of m-by-n matrices of rank at most $r > 0$ is $\mathbb{Z}$. It is freely generated by the class of one of two distinguished prime ideals P, Q, satisfying $[P] = -[Q]$, and the class of the canonical module is $m[P] + n[Q]$. In particular, the determinantal variety is Gorenstein (equivalently $\mathbb{Q}$-Gorenstein) precisely when the matrices are square, i.e. $m = n$ [BV88, Theorem 8.8].

4. Our work inches us one step closer to the question of whether $\overline{\mathbf{\Sigma}}_{\vec{e}}$ has rational singularities when $\vec{e} = (-2, 0, 2)$, which is the first non-tame case. Since $\vec{e} = (-2, 0, 2)$ is Gorenstein, it now suffices to check that $\overline{\mathbf{\Sigma}}_{\vec{e}}$ is log terminal. This seems out of reach to us, since our intersection-theoretic techniques are agnostic of any behavior in codimension 2 and beyond, which one would need a better handle of in order to compute the discrepancies.

3.2 Preliminaries

We refer the reader to [Lar21b] and Section 2.2 for background on splitting loci.

A Torus Action On $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}))$

Let $\vec{e} = (f_1^{s_1}, \dots, f_\ell^{s_\ell})$. Let $\vec{e} = (f_1^{s_1}, \dots, f_\ell^{s_\ell})$. We shall explain that there is a natural $\mathbb{G}_m^{\ell-1}$-action on $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}))$ under which splitting loci are invariant. In particular, the ideals for splitting loci in $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}))$ are naturally $\mathbb{Z}^{\ell-1}$-multigraded.

As explained in Subsection 2.2, a T-point of the affine space $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}))$ is the data of vector bundles $\mathcal{K}_1, \dots, \mathcal{K}_\ell$ on $T \times \mathbb{P}^1$, plus the data of an isomorphism $\mathcal{K}_1 \cong \mathcal{O}(f_1^{s_1})|_{T\times\mathbb{P}^1}$, plus the data of extensions

$$0 \to \mathcal{K}_{i-1} \xrightarrow{f_i} \mathcal{K}_i \xrightarrow{g_i} \mathcal{O}(f_i^{s_i}) \to 0$$

for each $2 \le i \le \ell$. Note that we will often write $\mathcal{O}(f_i^{s_i})$ to denote the pullback of this bundle along the second projection from $\mathbb{P}^1$ to $T \times \mathbb{P}^1$. In terms of these extensions, the action is easy to describe. An element $(\alpha_1, \dots, \alpha_{\ell-1}) \in \mathbb{G}_m^{\ell-1}$ acts on such a collection of data by keeping the vector bundles the same, but scaling the extensions to:

$$0 \to \mathcal{K}_{i-1} \xrightarrow{f_i} \mathcal{K}_i \xrightarrow{\beta_i g_i} \mathcal{O}(f_i^{s_i}) \to 0,$$

where $\beta_i = \alpha_1\alpha_2\cdots\alpha_{i-1}$. After fixing an isomorphism of $\mathcal{O}(\vec{e}) \cong \oplus_{i=1}^{\ell}\mathcal{O}(f_i^{s_i})$, we can identify $H^1\mathcal{E}nd(\mathcal{O}(\vec{e})) \cong \oplus_{1\le j<i\le\ell}\operatorname{Ext}^1(\mathcal{O}(f_i^{s_i}), \mathcal{O}(f_j^{s_j}))$. We will now work out what the action is in these coordinates by considering the action on a k-point.

Proposition 3.2.1. The isomorphism $H^1\mathcal{E}nd(\mathcal{O}(\vec{e})) \cong \oplus_{1\le j<i\le\ell}\operatorname{Ext}^1(\mathcal{O}(f_i^{s_i}), \mathcal{O}(f_j^{s_j}))$ is a decomposition of $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}))$ into weight spaces of the action of $\mathbb{G}_m^{\ell-1}$. For $1 \le j < i \le \ell$, if $v_{ij} \in \operatorname{Ext}^1(\mathcal{O}(f_i^{s_i}), \mathcal{O}(f_j^{s_j}))$, and $\alpha = (\alpha_1, \dots, \alpha_{\ell-1}) \in \mathbb{G}_m^{\ell-1}$, then $\alpha \cdot v_{ij} = \beta_j^{-1}\beta_i v_{ij} = \alpha_j\alpha_{j+1}\cdots\alpha_{i-1}v_{ij}$.

Proof. Let $K_1, \dots, K_\ell$ be the bundles involved in the data of a k-point of $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}))$, corresponding to the point $(v_{ij})_{1\le j<i\le\ell} \in \oplus_{1\le j<i\le\ell}\operatorname{Ext}^1(\mathcal{O}(f_i^{s_i}), \mathcal{O}(f_j^{s_j}))$. We will show that $\alpha = (\alpha_1, \dots, \alpha_{\ell-1}) \in \mathbb{G}_m^{\ell-1}$ acts by $v_{ij} \mapsto \beta_j^{-1}\beta_i v_{ij}$.

To prove this, we proceed by induction. Suppose that we have shown that $\alpha \cdot v_{i'j} = \beta_j^{-1}\beta_{i'}v_{i'j}$ whenever $i' < i$. By definition, α acts on $\operatorname{Ext}^1(\mathcal{O}(f_i^{s_i}), K_{i-1})$ by scaling the extensions with weight β_i for all $i \ge 2$. The identification of $\operatorname{Ext}^1(\mathcal{O}(f_i^{s_i}), K_{i-1})$ with $\oplus_{1\le j<i}\operatorname{Ext}^1(\mathcal{O}(f_i^{s_i}), \mathcal{O}(f_j^{s_j}))$ can be made step by step. Suppose we have identified

$$\operatorname{Ext}^1(\mathcal{O}(f_i^{s_i}), K_{i-1}) \cong \operatorname{Ext}^1(\mathcal{O}(f_i^{s_i}), K_{t-1}) \oplus \oplus_{t\le j<i}\operatorname{Ext}^1(\mathcal{O}(f_i^{s_i}), \mathcal{O}(f_j^{s_j})),$$

where $t \ge 3$. Then in order to identify $\operatorname{Ext}^1(\mathcal{O}(f_i^{s_i}), K_{t-1})$ with $\operatorname{Ext}^1(\mathcal{O}(f_i^{s_i}), \mathcal{O}(f_{t-1}^{s_{t-1}})) \oplus \operatorname{Ext}^1(\mathcal{O}(f_i^{s_i}), K_{t-2})$, we choose a (non-canonical) splittings of the following exact sequence:

$$0 \to \operatorname{Ext}^1(\mathcal{O}(f_i^{s_i}), K_{t-2}) \to \operatorname{Ext}^1(\mathcal{O}(f_i^{s_i}), K_{t-1}) \to \operatorname{Ext}^1(\mathcal{O}(f_i^{s_i}), \mathcal{O}(f_{t-1}^{s_{t-1}})) \to 0, \qquad (3.2)$$

Suppose by induction that α acts on $\operatorname{Ext}^1(\mathcal{O}(f_i^{s_i}), K_{t-1})$ by scaling by β_i, and that we have also shown the desired claim that $\alpha \cdot v_{ij} = \beta_j^{-1}\beta_i v_{ij}$ whenever $j \ge t$. The base case when $t = i$ is certainly true. We will show that α acts on $\operatorname{Ext}^1(\mathcal{O}(f_i^{s_i}), K_{t-2})$ by β_i as well, and that $\alpha \cdot v_{i,t-1} = \beta_{t-1}^{-1}\beta_i v_{i,t-1}$.

Let us first consider the second map in (3.2) above. The map is given by taking the pushout along the given map $K_{t-1} \xrightarrow{g_{t-1}} \mathcal{O}(f_{t-1}^{s_{t-1}})$:

$$\begin{tikzcd}
0 \arrow[r] & K_{t-1} \arrow[r, "f"] \arrow[d, "g_{t-1}"] & F_{t-1,i} \arrow[r, "g"] \arrow[d, "\psi"] & \mathcal{O}(f_i^{s_i}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}(f_{t-1}^{s_{t-1}}) \arrow[r, "\phi"] & E_{t-1,i} \arrow[r, "\gamma"] & \mathcal{O}(f_i^{s_i}) \arrow[r] & 0
\end{tikzcd}.$$

Since α acts on the first extension by sending g to $\beta_i g$ by our inductive hypothesis, the prescribed action on the original extensions would act on $\mathrm{Ext}^1(\mathcal{O}(f_i^{s_i}), \mathcal{O}(f_{i-1}^{s_{i-1}}))$ by scaling by $\beta_{t-1}^{-1}\beta_i$, as explained by the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & K_{t-1} \arrow[r, "f"] \arrow[d, "\beta_{t-1}g_{t-1}"] & F_{t-1,i} \arrow[r, "\beta_i g"] \arrow[d, "\beta_{t-1}\psi"] & \mathcal{O}(f_i^{s_i}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}(f_{t-1}^{s_{t-1}}) \arrow[r, "\phi"] & E_{t-1,i} \arrow[r, "\beta_{t-1}^{-1}\beta_i\gamma"] & \mathcal{O}(f_i^{s_i}) \arrow[r] & 0
\end{tikzcd}$$

We can also compute the restriction of the action to $\mathrm{Ext}^1(\mathcal{O}(f_i^{s_i}), K_{t-2})$, by interpreting the first map in (3.2). It is again given by taking a pushout along the map $g_{t-1} : K_{t-2} \to K_{t-1}$. So the action of α on $\mathrm{Ext}^1(\mathcal{O}(f_i^{s_i}), K_{t-1})$ with weight β_i restricts as the following diagram shows:

$$\begin{tikzcd}
0 \arrow[r] & K_{t-2} \arrow[r, "\phi'"] \arrow[d, "f_{t-1}"] & F_{t-2,i} \arrow[r, "\beta_i\gamma'"] \arrow[d, "\psi'"] & \mathcal{O}(f_i^{s_i}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_{t-1} \arrow[r, "f"] & F_{t-1,i} \arrow[r, "\beta_i g"] & \mathcal{O}(f_i^{s_i}) \arrow[r] & 0
\end{tikzcd}.$$

Thus $(\alpha_1, \ldots, \alpha_{\ell-1}) \in \mathbb{G}_m^{\ell-1}$ acts on $\mathrm{Ext}^1(\mathcal{O}(f_i^{s_i}), K_{t-2})$ also by scaling by β_i. □

Since the action above arises from scaling extension classes, it does not change the isomorphism class of the bundles $K_1, \ldots, K_\ell$. In particular, splitting loci corresponding to the isomorphism class of K_ℓ are invariant under this action.

Example 3.2.2. Let $\vec{e} = (-2, 0, 2)$. Then we can write the coordinate ring of $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}))$ as $k[a, b, c_1, c_2, c_3]$, where a, b are coordinates corresponding to the pairs $(-2, 0)$ and $(0, 2)$ respectively, and c_1, c_2, c_3 are coordinates of $\mathrm{Ext}^1(\mathcal{O}(2), \mathcal{O}(-2))$. Then there is a natural $\mathbb{G}_m^2$-action on this ring, giving rise to a $\mathbb{Z}^2$-grading. Namely, $\deg a = (1, 0), \deg b = (0, 1)$ and $\deg c_i = (1, 1)$. All splitting loci inside $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}))$ are homogeneous with respect to this grading.

Example 3.2.3. Of the most importance to us is the easiest case where $\vec{e} = (f_1^{s_1}, f_2^{s_2})$. In this case, $H^1\mathcal{E}nd(\mathcal{O}(\vec{e})) = \mathrm{Ext}^1(\mathcal{O}(f_2^{s_2}), \mathcal{O}(f_1^{s_1}))$ admits a $\mathbb{G}_m$-action that corresponds to scaling the extension class. The corresponding $\mathbb{Z}$-grading on the polynomial ring with $s_1s_2(f_2 - f_1 - 1)$ variables is the standard grading, and ideals for splitting loci are homogeneous with respect to this grading.

Gorenstein and N-Gorenstein Algebraic Stacks

Let us check that both being Gorenstein and being N-Gorenstein are smooth-local properties for locally Noetherian, Cohen-Macaulay schemes.

Lemma 3.2.4. Let X be a locally Noetherian Cohen-Macaulay scheme with a smooth cover by schemes U_α indexed by $\alpha \in I$. Then X is (N-)Gorenstein if and only if U_α is (N-)Gorenstein for every α.

Proof. Since a scheme is Gorenstein if and only if every local ring is Gorenstein, and since being locally Noetherian and Cohen-Macaulay are both smooth-local properties, we can reduce to the following claim: let A, B be Noetherian, Cohen-Macaulay local rings and let $f : A \to B$ be a smooth and local homomorphism. Then A is (N-)Gorenstein if and only if B is (N-)Gorenstein.

Since f is smooth, there exists a relative dualizing module ω_f which is a rank one projective B-module. Since B is local, in fact $\omega_f \cong B$. By [Stacks, Lemma 0E30], $\omega_B = \omega_f \otimes_B (B \otimes_A \omega_A)$ (the derived tensor product is just the regular tensor product because ω_f is projective, and the derived pullback of ω_A to $\operatorname{Spec} B$ is just the regular pullback because f is smooth and in particular flat). So in fact $\omega_B = \omega_A \otimes_A B$, and $\omega_B^{\otimes N} = \omega_A^{\otimes N} \otimes_A B$. Since $f : A \to B$ is smooth and in particular faithfully flat, given a finitely generated A-module M, $M \otimes_A B$ is free iff $M \otimes_A B$ is projective iff $M \otimes_A B$ is flat iff M is flat iff M is projective iff M is free. Therefore $\omega_A^{\otimes N} \cong A$ if and only if $\omega_B^{\otimes N} \cong B$.

Alternatively just for the Gorenstein claim, one could use [Stacks, Lemma 0BJL] ($B/\mathfrak{m}_B$ is Gorenstein because it's the fiber of a smooth morphism, so in particular should be regular). □

Thus as with all smooth-local properties, if $\mathfrak{X}$ is a locally Noetherian, Cohen-Macaulay algebraic stack, it makes sense to talk about whether $\mathfrak{X}$ is Gorenstein or N-Gorenstein. We say that $\mathfrak{X}$ is $\mathbb{Q}$-Gorenstein if there exists $N \geq 1$ such that $\mathfrak{X}$ is N-Gorenstein.

Reduction to a class calculation

In this subsection, we explain how to reduce the question of whether splitting loci are Gorenstein or $\mathbb{Q}$-Gorenstein to a class calculation.

Proposition 3.2.5. Let S be a $\mathbb{Z}$-graded Noetherian ring with $S_0 \cong k$. Let M be a graded finitely generated S-module. Then M is locally free if and only if it is a free S-module.

Proof. We shall show the forward direction. Suppose that M is a graded, finitely generated, locally free S-module. In particular, M is projective. By the graded Nakayama's lemma, there exists a graded surjection from a graded free S-module which is an isomorphism after tensoring with k. Let the kernel be K, so that we have a short exact sequence

$$0 \to K \to \oplus S(-d_i) \twoheadrightarrow M \to 0$$

Then the fact that M is projective implies that tensoring with k is exact, which implies that $K \otimes_S k = 0$. But by the graded Nakayama's lemma, this implies that $K = 0$. In particular, M is free. □

Recall that on an integral and normal scheme, the Weil class group can also be viewed as the group of rank 1 reflexive sheaves (for details see e.g. [Sch]). As such, given a rank 1 reflexive sheaf M on an integral and normal scheme X, we use $[M] \in A^1(X)$ to denote the corresponding class. Recall from the introduction that $\overline{\Sigma}_{\vec{e}}$ denotes the splitting locus inside $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}\,'))$, where $\vec{e}\,' = ((-M)^{r-1}, D)$ and $\vec{e}\,' > \vec{e}$. We will write $\omega_{\overline{\Sigma}_{\vec{e}}}$ for the canonical module of $\overline{\Sigma}_{\vec{e}}$.

Proposition 3.2.6. $\omega_{\overline{\Sigma}_{\vec{e}}}^{\otimes N}$ is locally free if and only if $N[\omega_{\overline{\Sigma}_{\vec{e}}}] \in A^1(\overline{\Sigma}_{\vec{e}})$ is zero.

Proof. The canonical module $\omega_{\overline{\Sigma}_{\vec{e}}}$ is a rank 1 reflexive sheaf (e.g. [BH98, Theorem 3.3.10(d), Proposition 3.3.13]). Therefore $N[\omega_{\overline{\Sigma}_{\vec{e}}}]$ is zero if and only if $\omega_{\overline{\Sigma}_{\vec{e}}}^{\otimes N}$ is the structure sheaf. But since $\omega_{\overline{\Sigma}_{\vec{e}}}^{\otimes N}$ is graded, by the previous proposition, it is the structure sheaf if and only if it is locally free. ∎

3.3 Some class calculations

Codimension One Chow Group of $\overline{\mathbf{\Sigma}}_{\vec{e}}$

Let $\vec{e} = (f_1^{s_1}, \dots, f_\ell^{s_\ell})$, where $f_1 < \dots < f_\ell$. Let $r = \sum_{i=1}^{\ell} s_i = \operatorname{rk}\vec{e}$, and let $d = \sum_{i=1}^{\ell} s_i f_i = \deg\vec{e}$. Recall from the proof of [LLV25, Proposition 9.1] that if $\vec{e}\,'$ is such that $u(\vec{e}\,') - u(\vec{e}) = 1$, then there exists $1 \leq i \leq \ell$ such that $f_{i-1} + 1 < f_i < f_{i+1} - 1$ (as far as i has neighbors), $s_i \geq 2$, and $\vec{e}\,'$ arises from replacing $f_i^{s_i}$ in $\vec{e}$ with $(f_i - 1, f_i^{s_i - 2}, f_i + 1)$. We denote by $\vec{e}^{\,(i)}$ the codimension one splitting type $\vec{e}\,'$ obtained by unbalancing $f_i^{s_i}$ when it exists. Let $\mathbf{Z}_{\vec{e}}$ be the union of splitting loci $\mathbf{\Sigma}_{\vec{e}}$ and all $\mathbf{\Sigma}_{\vec{e}\,'}$ where $u(\vec{e}\,') - u(\vec{e}) = 1$.

Let $\widetilde{\mathbf{\Sigma}}_{\vec{e}}$ be the algebraic stack whose functor is given by

$$(T \to \operatorname{Spec} k) \mapsto \left\{ \begin{array}{l} \text{A vector bundle } \mathcal{E} \text{ of rank } r \text{ and fiberwise degree } d \text{ and a flag of} \\ \text{quotients } \mathcal{E} = \mathcal{Q}_0 \twoheadrightarrow \mathcal{Q}_1 \twoheadrightarrow \mathcal{Q}_2 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{Q}_{\ell-1} \text{ on } T \times \mathbb{P}^1 \text{, such that } \mathcal{Q}_i \text{ is} \\ \text{flat over } T \text{, and fiberwise of rank } \sum_{j=1}^{\ell-i} s_j \text{ and degree } \sum_{j=1}^{\ell-i} s_j f_j \end{array} \right\} / \sim,$$

where two flags of quotients $\mathcal{E} = \mathcal{Q}_0 \twoheadrightarrow \mathcal{Q}_1 \twoheadrightarrow \mathcal{Q}_2 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{Q}_{\ell-1}$, $\mathcal{E} = \mathcal{Q}_0 \twoheadrightarrow \mathcal{Q}'_1 \twoheadrightarrow \mathcal{Q}'_2 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{Q}'_{\ell-1}$ are equivalent if there are isomorphisms $\mathcal{Q}_i \to \mathcal{Q}'_i$ which together with the identity morphism on $\mathcal{Q}_0$ form a commutative diagram. By Theorem 2.3.5, $\widetilde{\mathbf{\Sigma}}_{\vec{e}}$ is a resolution of singularities for $\overline{\mathbf{\Sigma}}_{\vec{e}}$. Moreover, for each $\vec{e}^{\,(i)}$, the only possible such flags are coarsenings of the Harder-Narasimhan flag for $\mathcal{O}(\vec{e}^{\,(i)})$, so the natural map $\widetilde{\mathbf{\Sigma}}_{\vec{e}} \to \overline{\mathbf{\Sigma}}_{\vec{e}}$ is not only an isomorphism over $\mathbf{\Sigma}_{\vec{e}}$, but also an isomorphism over $\mathbf{Z}_{\vec{e}}$. As such, on $\mathbf{Z}_{\vec{e}} \times \mathbb{P}^1$, there is a natural flag of quotients for the universal bundle

$$\mathcal{E} = \mathcal{Q}_0 \twoheadrightarrow \mathcal{Q}_1 \twoheadrightarrow \mathcal{Q}_2 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{Q}_{\ell-1},$$

where each $\mathcal{Q}_i$ is in fact a vector bundle. For $1 \leq i < \ell$, let $\mathcal{N}_{\ell-i+1} = \ker(\mathcal{Q}_{i-1} \to \mathcal{Q}_i)$, and let $\mathcal{N}_1 = \mathcal{Q}_{\ell-1}$. The sheaves $\mathcal{N}_i$ are vector bundles on $\mathbf{Z}_{\vec{e}} \times \mathbb{P}^1$, which split as $\mathcal{O}(f_i^{s_i})$ over $\mathbf{Z}_{\vec{e}} \setminus \mathbf{\Sigma}_{\vec{e}^{(i)}}$, and split as $\mathcal{O}(f_i - 1, f_i^{s_i-2}, f_i + 1)$ over $\mathbf{\Sigma}_{\vec{e}^{(i)}}$, if it exists.

Let z be the hyperplane class on $\mathbf{Z}_{\vec{e}} \times \mathbb{P}^1$, corresponding to $\mathcal{O}(1)$. For all $k \geq 1$, there exist classes $a_k^{(i)} \in A^k(\mathbf{Z}_{\vec{e}}), b_k^{(i)} \in A^{k-1}(\mathbf{Z}_{\vec{e}})$ such that

$$c_k(\mathcal{N}_i(-f_i)) = \pi^* a_k^{(i)} + \pi^* b_k^{(i)} z.$$

If $\vec{e}^{(i)}$ does not exist, then

$$\mathcal{N}_i(-f_i) \cong \pi^*\pi_*(\mathcal{N}_i(-f_i)), \tag{3.3}$$

and in particular $b_k^{(i)} = 0$ for all $k \geq 1$. If $\vec{e}^{(i)}$ exists, then away from $\mathbf{\Sigma}_{\vec{e}^{(i)}}$, we have (3.3) again, which shows that up to classes in the image of $A^*(\mathbf{\Sigma}_{\vec{e}^{(i)}})$, $b_k^{(i)} \equiv 0$. Note also that $b_1^{(i)}$ is the degree of $\mathcal{N}_i(-f_i)$ on fibers, which is zero.

Lemma 3.3.1. For all $M \in \mathbb{Z}$,

$$\begin{aligned} c_1(R\pi_*\mathcal{N}_i(-f_i+M)) &= c_1(\pi_*\mathcal{N}_i(-f_i)) + M\pi_*(c_1(\mathcal{N}_i).z) \\ &= (M+1)a_1^{(i)} - b_2^{(i)}. \end{aligned}$$

Proof. Twisting changes the Chern classes as follows:

$$\begin{aligned} c_1(\mathcal{N}_i(-f_i+M)) &= \pi^*a_1^{(i)} + Ms_iz, \\ c_2(\mathcal{N}_i(-f_i+M)) &= \pi^*a_2^{(i)} + \pi^*b_2^{(i)}z + (s_i-1)M\pi^*a_1^{(i)}z. \end{aligned}$$

Recall that the relative Todd class of a trivial $\mathbb{P}^1$-bundle is $1+z$. So by Grothendieck-Riemann-Roch,

$$\begin{aligned} c_1(R\pi_*\mathcal{N}_i(-f_i+M)) &= \pi_*\left(\frac{1}{2}(c_1(\mathcal{N}_i(-f_i+M))^2 - 2c_2(\mathcal{N}_i(-f_i+M))) + c_1(\mathcal{N}_i(-f_i+M)).z\right) \\ &= \pi_*\left(\frac{1}{2}(2Ms_i\pi^*a_1^{(i)}.z - 2\pi^*b_2^{(i)}.z - 2M(s_i-1)\pi^*a_1^{(i)}.z) + \pi^*a_1^{(i)}.z\right) \\ &= (M+1)a_1^{(i)} - b_2^{(i)}. \end{aligned}$$

□

Proposition 3.3.2. Whenever the codimension one splitting type $\vec{e}^{(i)}$ exists, $[\mathbf{\Sigma}_{\vec{e}^{(i)}}] = b_2^{(i)} \in A^1(\mathbf{Z}_{\vec{e}})$. In particular, $A^1(\overline{\mathbf{\Sigma}}_{\vec{e}})$ has a basis given by $a_1^{(1)}, \ldots, a_1^{(\ell)}$ and those $b_2^{(i)}$'s for which $\vec{e}^{(i)}$ exists.

Proof. By Lemma 3.3.1,

$$\begin{aligned} c_1(\pi_*\mathcal{N}_i(-f_i)) &= a_1^{(i)} - b_2^{(i)}, \\ c_1(\pi_*\mathcal{N}_i(-f_i+1)) &= 2a_1^{(i)} - b_2^{(i)}. \end{aligned}$$

Moreover, as explained in the proof of [LLV25, Proposition 9.1], $\mathbf{\Sigma}_{\vec{e}^{(i)}}$ is Cartier in $\mathbf{Z}_{\vec{e}}$ and cut out by a section of $\wedge^{2s_i}\pi_*(\mathcal{N}_i(-f_i) \otimes H^0(\mathbb{P}^1, \mathcal{O}(1)))^\vee \otimes \wedge^{2s_i}\pi_*\mathcal{N}_i(-f_i+1)$. In particular,

$$[\mathbf{\Sigma}_{\vec{e}^{(i)}}] = c_1(\pi_*\mathcal{N}_i(-f_i+1)) - 2c_1(\pi_*\mathcal{N}_i(-f_i)) = b_2^{(i)}$$

as desired.

By Lemma 3.1 of [CL24], we know that $A^1(\mathbf{\Sigma}_{\vec{e}})$ is freely generated by $c_1(\pi_*\mathcal{N}_i(-f_i))|_{\mathbf{\Sigma}_{\vec{e}}}$. By excision, it follows that $A^1(\overline{\mathbf{\Sigma}}_{\vec{e}})$ is generated by lifts of $c_1(\pi_*\mathcal{N}_i(-f_i))|_{\mathbf{\Sigma}_{\vec{e}}}$ from $\mathbf{\Sigma}_{\vec{e}}$ to $\overline{\mathbf{\Sigma}}_{\vec{e}}$ and the classes of these codimension one strata. The $a_1^{(i)}$ classes certainly lift $c_1(\pi_*\mathcal{N}_i(-f_i))$, and we just saw that when $\vec{e}^{(i)}$ exists, then $b_2^{(i)}$ is exactly the class of that stratum. Thus it

$$\begin{tikzcd}
U_{\vec{e}'}^{(0)} \arrow[d, "GL_{r-1}\text{-bundle}"] & \\
U_{\vec{e}'}^{(1)} \arrow[r, hook] & V = \pi_*\mathcal{H}om(\mathcal{E}, \mathcal{O}(D)) \arrow[d, "\text{vector bundle}"] & \\
& U_{\vec{e}'}^{(2)} = \bigcup_{\vec{e}'' \geq \vec{e}'} \mathbf{\Sigma}_{\vec{e}''} \arrow[r, hook] & \mathcal{B}_{r,d}
\end{tikzcd}$$

Figure 3.1: Building $\mathrm{Ext}^1(O(D), \mathcal{O}(-M)^{\oplus r-1})$ from $\mathcal{B}_{r,d}$

remains to show that there are no relations between the $b_2^{(i)}$'s. Consider the map $\mathcal{B}_{s_i,0} \to \overline{\mathbf{\Sigma}}_{\vec{e}}$ induced by the vector bundle $\mathcal{N}(f_i) \oplus \oplus_{j \neq i} \mathcal{O}(f_j^{s_j})$, where $\mathcal{N}$ is the universal vector bundle on $\mathcal{B}_{s_i,0} \times \mathbb{P}^1$. Under this map, the classes $a_1^{(j)}, b_2^{(j)}$ all pull back to zero for $j \neq i$, and $a_1^{(i)}, b_2^{(i)}$ pull back to the classes a_1, a_2' as defined in [Lar23], which are free generators of the (rational) Picard group of $\mathcal{B}_{s_i,0}$. In particular, if there were any relation $\sum_{j=1}^{\ell}(c_j b_2^{(j)} + d_j a_1^{(j)}) = 0$ in $A^1(\overline{\mathbf{\Sigma}}_{\vec{e}})$, then it must be the case that $c_i a_2' + d_i a_1 = 0$ in the rational Chow ring of $\mathcal{B}_{s_i,0}$ for all $1 \leq i \leq \ell$, but that implies that $c_i = d_i = 0$. So in fact there cannot be any relations and the $a_1^{(i)}$ and $b_2^{(i)}$'s form a basis. □

Proof of Theorem 3.1.4

Let $\vec{e}' = ((-M)^{r-1}, D)$, such that $D - (r-1)M = d$, and D is large enough so that $\vec{e}' < \vec{e}$. Let $\mathcal{E}$ be the universal bundle on $\mathcal{B}_{r,d} \times \mathbb{P}^1$. We will also use $\mathcal{E}$ to denote the pullback of the universal bundle on $\mathcal{B}_{r,d} \times \mathbb{P}^1$ to various spaces $U \times \mathbb{P}^1$ when there is an obvious map $U \to \mathcal{B}_{r,d}$. When there is an obvious map $U \to \overline{\mathbf{\Sigma}}_{\vec{e}}$, we also omit the pullback from our notation, writing $a_1^{(i)}, b_2^{(i)}$ to denote the pullbacks of the universal classes on $\overline{\mathbf{\Sigma}}_{\vec{e}}$. We will write $U_{\vec{e}'} = H^1\mathcal{E}nd(\mathcal{O}(\vec{e}')) = \mathrm{Ext}^1(O(D), \mathcal{O}(-M)^{\oplus r-1})$. Let us recall from Section 2.2 how to construct the affine space $U_{\vec{e}'}$ from $\mathcal{B}_{r,d}$:

1. First we restrict to the open $U_{\vec{e}'}^{(2)}$ in $\mathcal{B}_{r,d}$ where $\mathcal{E}$ splits as $\vec{e}'' \geq \vec{e}'$.

2. Over this open, we form the pushforward $V = \pi_*\mathcal{H}om(\mathcal{E}, \mathcal{O}(D))$, which is a vector bundle.

3. We excise the locus where the corresponding map $\phi : \mathcal{O}(\vec{e}'') \to \mathcal{O}(D)$ is not surjective, and the locus where the kernel is not perfectly balanced, and obtain an open locus $U_{\vec{e}'}^{(1)}$.

4. On $U_{\vec{e}'}^{(1)} \times \mathbb{P}^1$, there is a universal surjection $\mathcal{E} \to \mathcal{O}(D)$ so that fiberwise the kernel is perfectly balanced. Let $\mathcal{K} = \ker(\mathcal{E} \to \mathcal{O}(D))$. The last step now is to form the principal GL_{r-1}-bundle $U_{\vec{e}'} = U_{\vec{e}'}^{(0)} = \mathrm{Isom}(\mathcal{O}(-M)^{r-1}, \mathcal{K})$ which frames $\mathcal{K}$.

After base change from $U^{(2)}_{\vec{e}'}$ to $\overline{\mathbf{\Sigma}}^{\circ}_{\vec{e}} = \cup_{\vec{e}' \leq \vec{f} \leq \vec{e}} \mathbf{\Sigma}_{\vec{f}}$, we note that this construction is also a recipe for building $\overline{\Sigma}_{\vec{e}}$ from $\overline{\mathbf{\Sigma}}^{\circ}_{\vec{e}}$.

Luckily, this construction is pretty amenable to tracking how $A^1(\overline{\Sigma}_{\vec{e}})$ compares to $A^1(\overline{\mathbf{\Sigma}}_{\vec{e}})$. Forming a vector bundle does not affect Chow, and as far as A^1 is concerned, forming a GL-bundle amounts to setting c_1 of the corresponding vector bundle to zero. We have also excised two closed loci in this process, and we must take note of whether they contain codimension one components, and if so, we must remember to excise them in the class group computation. The rest of this subsection will follow this recipe to prove Theorem 3.1.4.

Let Z_{ns} denote the closed locus in $V = \pi_* \mathcal{H}om(\mathcal{E}, \mathcal{O}(D))$ of those maps $\mathcal{O}(\vec{f}) \to \mathcal{O}(D)$ which are not surjective. We label the relevant maps as follows:

$$\begin{array}{ccc} V \times \mathbb{P}^1 & \xrightarrow{\tilde{p}} & \overline{\mathbf{\Sigma}}^{\circ}_{\vec{e}} \times \mathbb{P}^1 \\ \Big\downarrow{\scriptstyle \pi'} & & \Big\downarrow{\scriptstyle \pi} \\ V & \xrightarrow{p} & \overline{\mathbf{\Sigma}}^{\circ}_{\vec{e}} \end{array}$$

Lemma 3.3.3. The closed locus Z_{ns} is irreducible of codimension $r-1$. In particular, Z_{ns} is of codimension 1 if and only if $r = 2$. In that case,

$$[Z_{\text{ns}}] = \pi'_* c_2(\mathcal{H}om(\mathcal{E}, \mathcal{O}(D))) = \begin{cases} -(f_1+M)a_1^{(1)} - (f_2+M)a_1^{(2)} & \text{if } \vec{e} = (f_1, f_2), f_1 < f_2; \\ -(f+M)a_1 + b_2 & \text{if } \vec{e} = (f, f). \end{cases}$$

Note that in the latter case, since $m = 1$, we use a_1, b_2 as a shorthand for $a_1^{(1)}, b_2^{(1)}$.

Proof. Let Z' be the locus of points in $V \times \mathbb{P}^1$ where the universal map $\mathcal{E} \to \mathcal{O}(D)$ is zero. By our choice of D, the evaluation map

$$\text{ev} : \pi^* V = \pi^* \pi_* \mathcal{H}om(\mathcal{E}, \mathcal{O}(D)) \to \mathcal{H}om(\mathcal{E}, \mathcal{O}(D))$$

is surjective, and Z' is in fact the total space of the vector bundle $W = \ker(\text{ev})$ on $\overline{\mathbf{\Sigma}}^{\circ}_{\vec{e}} \times \mathbb{P}^1$. In particular Z' must be irreducible of codimension $r = \operatorname{rk} \mathcal{H}om(\mathcal{E}, \mathcal{O}(D))$ in $V \times \mathbb{P}^1$. A map $\mathcal{O}(\vec{f}) \to \mathcal{O}(D)$ is not surjective if it is zero at some point of $\mathbb{P}^1$. Therefore, the nonsurjective locus $Z_{\text{ns}} \subset V$ is the image of Z' in V. Since a generic nonsurjective map $\mathcal{O}(\vec{f}) \to \mathcal{O}(D)$ drops rank at exactly one point, $[Z_{\text{ns}}] = \pi'_* c_r(\mathcal{H}om(\mathcal{E}, \mathcal{O}(D)))$. As this class has codimension $r-1$, it has codimension 1 if and only if $r = 2$.

Now suppose that $\vec{e} = (f_1, f_2)$ where $f_1 < f_2$. Then $\mathbf{Z}_{\vec{e}} = \mathbf{\Sigma}_{\vec{e}}$ and we have the short exact sequence over $\mathbf{Z}_{\vec{e}}$

$$0 \to \mathcal{N}_2 \to \mathcal{E} \to \mathcal{N}_1 \to 0.$$

Note that in this case, $D = f_1 + f_2 + M$. By Lemma 3.3.1,

$$\begin{aligned} \pi'_* c_2(\mathcal{H}om(\mathcal{E}, \mathcal{O}(D))) &= \pi'_* \Big((c_1(\mathcal{N}_1^\vee(f_1)) + (D - f_1)z)(c_1(\mathcal{N}_2^\vee(f_2)) + (D - f_2)z) \Big) \\ &= \pi'_* \Big((-\pi^* a_1^{(1)} + (f_2 + M)z)(-\pi^* a_1^{(2)} + (f_1 + M)z) \Big) \\ &= -(f_1 + M)a_1^{(1)} - (f_2 + M)a_1^{(2)}. \end{aligned}$$

Lastly, suppose that $\vec{e} = (f, f)$. Then $m = 1$, $\mathcal{E} = \mathcal{N}_1$, $M = D - 2f$, and we get

$$\begin{aligned}\pi'_* c_2(\mathcal{H}om(\mathcal{E}, \mathcal{O}(D))) &= \pi'_*\Big(c_2(\mathcal{E}^\vee(f)) + c_1(\mathcal{E}^\vee(f))(D-f)z\Big) \\ &= \pi'_*\left((\pi^* a_2 + \pi^* b_1.z) - \pi^* a_1.(f+M)z)\right) \\ &= -(f+M)a_1 + b_1.\end{aligned}$$

□

Let Z_{ubal} denote the closed locus in $\pi_*\mathcal{H}om(\mathcal{E}, \mathcal{O}(D)) \setminus Z_{\text{ns}}$ of those surjective maps $\mathcal{O}(\vec{f}) \to \mathcal{O}(D)$ whose kernel is not isomorphic to $\mathcal{O}(-M)^{\oplus(r-1)}$. Note that Z_{ubal} is empty if $r = 2$.

Lemma 3.3.4. Suppose that $r \geq 3$. Then Z_{ubal} is of pure codimension 1 and

$$[Z_{\text{ubal}}] = \sum_{i=1}^{\ell}((-f_i - M)a_1^{(i)} + b_2^{(i)}) \in A^1(V \setminus Z_{\text{ns}}).$$

Proof. Since we are always working with a finite type stack, there exists S sufficiently large such that $\mathcal{K}(S)$ is globally generated. The Strømme sequence [Str87, Proposition 1.1] tells us that

$$0 \to \pi^*(\pi_*\mathcal{K}(S-1))(-1) \to \pi^*\pi_*\mathcal{K}(S) \to \mathcal{K}(S) \to 0.$$

Twisting this down by $\mathcal{O}(-S+M-1)$ and pushing forward, we see that

$$\pi_*\mathcal{K}(S-1) \otimes H^1\mathcal{O}_{\mathbb{P}^1}(-S+M-2) \to \pi_*\mathcal{K}(S) \otimes H^1\mathcal{O}_{\mathbb{P}^1}(-S+M-1) \to R^1\pi_*\mathcal{K}(M-1) \to 0$$

is exact. The locus where $\mathcal{K}$ is not perfectly balanced is precisely the support of $R^1\pi_*\mathcal{K}(M-1)$.

In the resolution above of $R^1\pi_*\mathcal{K}(M-1)$ by a map of vector bundles, the two vector bundles involved both have rank $(r-1)(S-M)(S-M+1)$. So the support of $R^1\pi_*\mathcal{K}(M-1)$ is cut out by the section of a line bundle. Since it is possible for $\mathcal{K}$ to split as perfectly balanced, this section cannot be zero, and since we are working with an integral stack, this shows that Z_{ubal} is of pure codimension one. The class of Z_{ubal} is just $(S-M)c_1(\pi_*\mathcal{K}(S)) - (S-M+1)c_1(\pi_*\mathcal{K}(S-1))$.

Now, recall that on $(V \setminus Z_{\text{ns}}) \times \mathbb{P}^1$,

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{O}(D) \to 0,$$

and as such we have $c_1(\pi_*\mathcal{K}(S)) = c_1(\pi_*\mathcal{E}(S))$ and $c_1(\pi_*\mathcal{K}(S-1)) = c_1(\pi_*\mathcal{E}(S-1))$. By Lemma 3.3.1,

$$\begin{aligned}[Z_{\text{ubal}}] &= (S-M)c_1(\pi_*\mathcal{K}(S)) - (S-M+1)c_1(\pi_*\mathcal{K}(S-1)) \\ &= (S-M)c_1(\pi_*\mathcal{E}(S)) - (S-M+1)c_1(\pi_*\mathcal{E}(S-1)) \\ &= \sum_{i=1}^{\ell}\left((S-M)c_1(\pi_*\mathcal{N}_i(S)) - (S-M+1)c_1(\pi_*\mathcal{N}_i(S-1))\right) \\ &= \sum_{i=1}^{\ell}\Big((S-M)((f_i+S+1)a_1^{(i)} - b_2^{(i)}) - (S-M+1)((f_i+S)a_1^{(i)} - b_2^{(i)})\Big) \\ &= \sum_{i=1}^{\ell}((-f_i-M)a_1^{(i)} + b_2^{(i)}).\end{aligned}$$

□

Lemma 3.3.5. Z_{ubal} is irreducible.

Proof. We have already shown that Z_{ubal} is of pure codimension 1. We can stratify $V \setminus Z_{\text{ns}}$ by the splitting type $\vec{a}$ of $\mathcal{K}$ and the splitting type $\vec{f}$ of $\mathcal{E}$. Each of these strata is isomorphic to $[U_{\vec{a},\vec{f}}/\operatorname{Aut}(\mathcal{O}(\vec{a})) \times \operatorname{Aut}(\mathcal{O}(\vec{f}))]$, where $U_{\vec{a},\vec{f}} \subset \operatorname{Hom}(\mathcal{O}(\vec{a}), \mathcal{O}(\vec{f}))$ is the locus of maps with locally free cokernel. Note that each stratum is irreducible. Note also that $\pi_*\mathcal{H}om(\mathcal{O}(\vec{e}^{(i)}), \mathcal{O}(D))$ is irreducible of codimension 1 in V, and over this space the kernel is generically balanced. So when $\vec{a}$ is not balanced and $\vec{f} = \vec{e}^{(i)}$ for some i, the stratum associated to $\vec{a}$ and $\vec{f}$ must be of codimension at least 2 in V. In particular, it suffices to restrict our attention to the case where $\vec{f} = \vec{e}$, and check that the only unbalanced $\vec{a}$ such that $[U_{\vec{a},\vec{e}}/\operatorname{Aut}(\mathcal{O}(\vec{a})) \times \operatorname{Aut}(\mathcal{O}(\vec{e}))]$ achieves codimension 1 in $V \setminus Z_{\text{ns}}$ is $\vec{a} = (-M-1, -M, \ldots, -M, -M+1)$.

Note that in order for $(-M, \ldots, -M, D) < (e_1, \ldots, e_r)$, we must have that $D \geq e_r + \sum_{i=2}^{r-1}(e_i - e_1)$. So if $U_{\vec{a},\vec{e}} \neq \emptyset$, then $\deg \vec{a} \leq (r-1)e_1$. In the special case where $\vec{e}$ itself is (f_1^{r-1}, f_2), we must have that $D > f_2$. So if $U_{\vec{a},\vec{e}} \neq \emptyset$ in this case, then $\deg \vec{a} < (r-1)e_1$.

We have

$$\dim[U_{\vec{a},\vec{e}}/\operatorname{Aut}(\mathcal{O}(\vec{a})) \times \operatorname{Aut}(\mathcal{O}(\vec{e}))] = \hom(\mathcal{O}(\vec{a}), \mathcal{O}(\vec{e})) - h^0\mathcal{E}nd(\mathcal{O}(\vec{a})) - h^0\mathcal{E}nd(\mathcal{O}(\vec{e})),$$

and the dimension of $V \setminus Z_{\text{ns}}$ is precisely

$$\chi(\mathcal{H}om(\mathcal{O}(\vec{a}), \mathcal{O}(\vec{e}))) - \chi(\mathcal{E}nd(\mathcal{O}(\vec{a}))) - h^0\mathcal{E}nd(\mathcal{O}(\vec{e})).$$

So the codimension of $[U_{\vec{a},\vec{e}}/\operatorname{Aut}(\mathcal{O}(\vec{a}))]$ is $u(\vec{a}) - h^1\mathcal{H}om(\mathcal{O}(\vec{a}), \mathcal{O}(\vec{e}))$.

As explained by Hong and Larson in [Hon23, Theorem A.1.1], in order for there to exist an injective map $\mathcal{O}(\vec{a}) \to \mathcal{O}(\vec{e})$ with locally free cokernel, we could have the following two scenarios:

1. $a_i \leq e_i$ for $1 \leq i \leq r-1$; or
2. there exists some $1 \leq m \leq r$ such that $a_k = e_{k+1}$ for $k \geq m$, and $a_k < e_k$ for $k < m$.

In the first case,

$$u(\vec{a}) - h^1\mathcal{H}om(\mathcal{O}(\vec{a}), \mathcal{O}(\vec{e})) = \sum_{i=1}^{r-1}\sum_{j=1}^{i-1}\Big(h^1\mathcal{H}om(\mathcal{O}(a_i), \mathcal{O}(a_j)) - h^1\mathcal{H}om(\mathcal{O}(a_i), \mathcal{O}(e_j))\Big)$$

where each term

$$\delta_{ij} = h^1\mathcal{H}om(\mathcal{O}(a_i), \mathcal{O}(a_j)) - h^1\mathcal{H}om(\mathcal{O}(a_i), \mathcal{O}(e_j))$$

is nonnegative because $a_j \leq e_j$. If all $\delta_{ij} \leq 1$ and $\delta_{ij} = 1$ exactly once, then whenever $a_i \geq a_j + 3$, we have $e_j - a_j \leq 1$ and $e_j = a_j + 1$ exactly once. Moreover, when $e_j = a_j + 1$, there can be at most one i with $a_i - a_j \geq 2$.

Now suppose that $a_{r-1} - a_1 \geq 3$. Then we must have that $e_1 - a_1 \leq 1$. We cannot have $a_1 = e_1$, because then $\deg \vec{a} \geq (r-1)e_1 + 3$ would be too large. If $a_1 = e_1 - 1$, as we said before then it must be the case that $a_i - a_1 \leq 1$ for $i < r-1$. But then $a_{r-1} - a_i \geq 2$ for $1 < i < r-1$, which implies that $e_i = a_i$ for $1 < i < r-1$ (we've used up our one chance to have $e_i = a_i + 1$). This is now again impossible because $\deg \vec{a}$ is now too large.

Now suppose that $a_{r-1} - a_1 = 2$. Then again because $\deg \vec{a} \leq (r-1)e_1$, we need that $e_1 - a_1 \geq 1$. But this implies that $a_i - a_1 \leq 1$ for $1 < i < r-1$. Thus up to a shift, $\vec{a}$ is of the form $((-1)^k, 0^s, 1)$, where $k > 0, s \geq 0$, $k + s = r-1$, and $e_1 \geq 0$. In particular, $h^1\mathcal{H}om(\mathcal{O}(\vec{a}), \mathcal{O}(\vec{e})) = 0$. So $u(\vec{a}) = 1$, which implies that $k = 1$ and up to a shift, we have $\vec{a} = (-1, 0^{r-2}, 1)$, which is the expected splitting type.

Lastly, we will show that in the second case, the stratum $[U_{\vec{a},\vec{e}}/\operatorname{Aut}(\mathcal{O}(\vec{a})) \times \operatorname{Aut}(\mathcal{O}(\vec{e}))]$ cannot be of codimension 1. Let us write $\vec{e} = (\vec{e}_-, \vec{e}_+)$, $\vec{a} = (\vec{a}_-, \vec{e}_+)$, where $\vec{e}_+ = (e_{m+1}, e_{m+2}, \ldots, e_r) = (a_m, \ldots, a_{r-1})$. Note that $\vec{e}_-$ has length m whereas $\vec{a}_-$ has length $m-1$.

$$\begin{aligned} &u(\vec{a}) - h^1\mathcal{H}om(\mathcal{O}(\vec{a}), \mathcal{O}(\vec{e})) \\ = \ &\Big(u(\vec{a}_-) - h^1\mathcal{H}om(\mathcal{O}(\vec{a}_-), \mathcal{O}(\vec{e}_-)\Big) + \Big(\operatorname{ext}^1(\mathcal{O}(\vec{e}_+), \mathcal{O}(\vec{a}_-)) - \operatorname{ext}^1(\mathcal{O}(\vec{e}_+), \mathcal{O}(\vec{e}_-))\Big). \end{aligned}$$

The first part is nonnegative for the same reason as in the first case. The second part can be computed directly as follows:

$$\begin{aligned} &\operatorname{ext}^1(\mathcal{O}(\vec{e}_+), \mathcal{O}(\vec{a}_-)) - \operatorname{ext}^1(\mathcal{O}(\vec{e}_+), \mathcal{O}(\vec{e}_-)) \\ = \ &-\chi(\mathcal{H}om(\mathcal{O}(\vec{e}_+), \mathcal{O}(\vec{a}_-))) + \chi(\mathcal{H}om(\mathcal{O}(\vec{e}_+), \mathcal{O}(\vec{e}_-))) \\ = \ &-((r-m)(m-1) + (r-m)\deg \vec{a}_- - (m-1)\deg(\vec{e}_+)) + ((r-m)m + (r-m)\deg \vec{e}_- - m\deg(\vec{e}_+)) \\ = \ &(r-m)(1+D) - \deg \vec{e}_+. \end{aligned}$$

But since $1 + D > e_r$, the quantity $(r-m)(1+D) - \deg \vec{e}_+$ must be positive and it equals 1 if and only if $m = r-1$ and $e_r = D$. But this is only possible if $-M = e_1 = e_2 = \cdots = e_{r-1}$, which just means that $\vec{e}' = \vec{e}$, which is a contradiction. ☐

If we let $\alpha_1 = \sum_{i=1}^{\ell} \left(-(f_i + M)a_1^{(i)} + b_2^{(i)}\right)$, then we see that

$$\alpha_1 = \begin{cases} [Z_{\mathrm{ns}}] & \text{if } r = 2, \\ [Z_{\mathrm{ubal}}] & \text{if } r \geq 3. \end{cases}$$

As a result of the previous lemmas, α_1 is the only class that gets excised prior to the last step of framing the kernel bundle.

Next, by framing the universal kernel $\mathcal{K}$, we introduce another relation α_2 in A^1.

Lemma 3.3.6. We have that

$$\mathbb{Z} \to A^1(U_{\vec{e}'}^{(1)}) \to A^1(U_{\vec{e}'}^{(0)}) \to 0,$$

where the first map sends 1 to

$$\alpha_2 = c_1(\pi_*(\mathcal{K}(M))) = \sum_{i=1}^{\ell} \left((f_i + M + 1)a_1^{(i)} - b_2^{(i)}\right).$$

Proof. Recall that forming $\operatorname{Isom}(\mathcal{K}, \mathcal{O}(-M)^{\oplus(r-1)})$ is the same as first taking $\pi_* \operatorname{Hom}(\mathcal{K}, \mathcal{O}(-M)^{\oplus(r-1)})$, and then taking the open corresponding to isomorphisms. At this stage of the construction, $\mathcal{K}$ is fiberwise isomorphic to $\mathcal{O}(-M)^{\oplus(r-1)}$, so $\mathcal{K} \cong \pi^*(\pi_*\mathcal{K}(M))(-M)$. The closed locus which one excises is cut out by the determinant, which can be understood as a section of $\wedge^{r-1}\mathcal{K}^\vee \otimes \wedge^{r-1}\mathcal{O}(-M)^{\oplus(r-1)} = \pi^* \wedge^{r-1} (\pi_*\mathcal{K}(M))^\vee$, so the locus in $\pi_* \operatorname{Hom}(\mathcal{K}, \mathcal{O}(-M)^{\oplus(r-1)})$ corresponding to maps which are not isomorphisms has class $-c_1(\pi_*\mathcal{K}(M))$. As in our computation for $[Z_{\mathrm{ubal}}]$, we find that

$$\begin{aligned} c_1(\pi_*\mathcal{K}(M)) &= c_1(\pi_*\mathcal{E}(M)) \\ &= \sum_{i=1}^{\ell} c_1(\pi_*\mathcal{N}_i(M)) \\ &= \sum_{i=1}^{\ell} \left((f_i + M + 1)a_1^{(i)} - b_2^{(i)}\right). \end{aligned}$$

□

The Class of the Canonical Module

We give a compact description of the normal bundle for the smooth closed substack $\mathbf{Z}_{\vec{e}}$ in $\mathcal{B}^\circ_{r,d} = \mathcal{B}_{r,d} \setminus \cup_{\vec{f}<\vec{e}, \vec{f}\neq\vec{e}^{(i)}} \mathbf{\Sigma}_{\vec{f}}$ using filtered Ext and relative analogs. We refer to [Ill71, Section V.2.2.2] for more details on the functors $R\operatorname{Hom}_\pm(-,-)$, $R\mathcal{H}om_\pm(-,-)$.

Proposition 3.3.7. The normal bundle of $\mathbf{Z}_{\vec{e}}$ in $\mathcal{B}^\circ_{r,d} = \mathcal{B}_{r,d} \setminus \cup_{\vec{f}<\vec{e}, \vec{f}\neq\vec{e}^{(i)}} \mathbf{\Sigma}_{\vec{f}}$ is $R^1\pi_*\mathcal{E}nd_+(\mathcal{E})$, which is filtered with associated graded isomorphic to $\oplus_{1\leq j<i\leq \ell} R^1\pi_*\mathcal{H}om(\mathcal{N}_i, \mathcal{N}_j)$. In particular, for all smooth maps $U \to \mathcal{B}^\circ_{r,d}$, there is an exact sequence

$$0 \to \mathcal{T}_{\mathbf{Z}_{\vec{e}} \times_{\mathcal{B}^\circ_{r,d}} U} \to \mathcal{T}_U \to R^1\pi_*\mathcal{E}nd_+(\mathcal{E}) \to 0,$$

where $\mathcal{T}_{(-)}$ denotes the tangent bundle of the specified space.

Proof. Let $\rho : \widetilde{\mathbf{\Sigma}}_{\vec{e}} \to \overline{\mathbf{\Sigma}}_{\vec{e}}$ be the resolution of singularities recalled at the beginning of Section 3.3. Thus, the first map in

$$0 \to \mathcal{T}_{\mathbf{Z}_{\vec{e}}} \to \mathcal{T}_{\mathcal{B}^\circ_{r,d}}|_{\mathbf{Z}_{\vec{e}}} \to \mathcal{N}_{\mathbf{Z}_{\vec{e}} \subset \mathcal{B}^\circ_{r,d}} \to 0$$

can be naturally identified with the natural map $\mathcal{T}_{\widetilde{\mathbf{\Sigma}}_{\vec{e}}} \to \rho^*\mathcal{T}_{\mathcal{B}^\circ_{r,d}}$, restricted to $\rho^{-1}\mathbf{Z}_{\vec{e}}$. Let us consider this situation after the base change along a smooth map $B \to \mathcal{B}_{r,d}$, where B is a scheme. Let $\mathcal{E}$ be the vector bundle on $B \times \mathbb{P}^1$ inducing the map $B \to \mathcal{B}_{r,d}$, and let $Z_{\vec{e}} \subset U \subset B$ be the base change of the stacks $\mathbf{Z}_{\vec{e}} \subset \mathcal{B}^\circ_{r,d} \subset \mathcal{B}_{r,d}$. We would like to show that the normal bundle of $Z_{\vec{e}}$ in U is $R^1\pi_*\mathcal{E}nd_+\mathcal{E}$, where $\mathcal{E}nd_+(\mathcal{E})$ is taken with respect to the filtration on $\mathcal{E}$ induced by $\widetilde{\mathbf{\Sigma}}_{\vec{e}}$.

Recall that the Kodaira-Spencer map $\mathcal{T}_B \to R^1\pi_*\mathcal{E}nd(\mathcal{E})$ exists globally. Since $B \to \mathcal{B}_{r,d}$ is smooth, this map is surjective. Moreover, the natural surjection $\mathcal{E}nd(\mathcal{E}) \to \mathcal{E}nd_+(\mathcal{E})$ induces a surjection $R^1\pi_*\mathcal{E}nd(\mathcal{E}) \to R^1\pi_*\mathcal{E}nd_+(\mathcal{E})$. The composition of these two surjections gives rise to a surjective map $\mathcal{T}_B \to R^1\pi_*\mathcal{E}nd_+\mathcal{E}$. We claim that the composition

$$0 \to \mathcal{T}_{Z_{\vec{e}}} \to \mathcal{T}_U|_{Z_{\vec{e}}} \to R^1\pi_*\mathcal{E}nd_+\mathcal{E} \to 0 \tag{3.4}$$

is exact in the middle.

This follows from Proposition 1.5 in [DL85]. The content of their proposition is the following. Suppose that $\mathcal{E}$ is a vector bundle on $B \times X$ where X is a curve and the Kodaira-Spencer map $T_bB \to \operatorname{Ext}^1_X(\mathcal{E}_b, \mathcal{E}_b)$ is surjective for all $b \in B$. Then whenever we form a flag Quot construction associated to such $\mathcal{E}$, then there is in general an exact sequence

$$0 \to \operatorname{Ext}^0_+(\mathcal{E}_b, \mathcal{E}_b) \to T_t \operatorname{FQuot}(\mathcal{E}) \to T_bB \xrightarrow{\omega_+} \operatorname{Ext}^1_+(\mathcal{E}_b, \mathcal{E}_b) \to 0, \tag{3.5}$$

where $t \in \operatorname{FQuot}(\mathcal{E})$ is a point mapping to $b \in B$, corresponding to the data of a filtration on $\mathcal{E}_b$. The $\operatorname{Ext}^i_+(\mathcal{E}_b, \mathcal{E}_b)$'s in the sequence are taken with respect to the filtration given by t, and ω_+ is the composition of the Kodaira-Spencer map $T_bB \to \operatorname{Ext}^1(\mathcal{E}_b, \mathcal{E}_b)$ and the natural map $\operatorname{Ext}^1(\mathcal{E}_b, \mathcal{E}_b) \to \operatorname{Ext}^1_+(\mathcal{E}_b, \mathcal{E}_b)$.

Applied to our situation, we can take the flag Quot construction which corresponds to base-changing $\widetilde{\mathbf{\Sigma}}_{\vec{e}}$ to B, which is an isomorphism over $Z_{\vec{e}}$. We know that we must have $\operatorname{Ext}^0_+(\mathcal{E}_b, \mathcal{E}_b) = 0$ for all $b \in Z_{\vec{e}}$ since $\widetilde{\mathbf{\Sigma}}_{\vec{e}} \to \overline{\mathbf{\Sigma}}_{\vec{e}}$ is an isomorphism over $\mathbf{Z}_{\vec{e}}$. This fact can be seen independently by using a spectral sequence to compute Ext^0_+. The associated graded pieces arising from the filtration on $\mathcal{E}$ are just the $\mathcal{N}_i$'s, which are vector bundles. This implies that over $Z_{\vec{e}}$, $R\mathcal{H}om_+(\mathcal{E}, \mathcal{E}) \cong \mathcal{H}om_+(\mathcal{E}, \mathcal{E})$. Since $R\operatorname{Hom}_+ = R\Gamma \circ R\mathcal{H}om_+$ ([Ill71] V.2.2.10), this shows that $(R^1\pi_*\mathcal{E}nd_+\mathcal{E})_b \cong H^1(\mathbb{P}^1, \mathcal{E}nd_+(\mathcal{E}_b, \mathcal{E}_b)) \cong \operatorname{Ext}^1_+(\mathcal{E}_b, \mathcal{E}_b)$. Thus the exactness of (3.5) shows that the base change of (3.4) to any closed point $b \in B$ is exact in the middle.

Lastly, since $\mathcal{E}nd_+(\mathcal{E})$ is filtered with associated graded isomorphic to $\oplus_{1\le j<i\le \ell}\mathcal{H}om(\mathcal{N}_i, \mathcal{N}_j)$, and since $\pi_*\mathcal{H}om(\mathcal{N}_i, \mathcal{N}_j) = 0$ for all $1 \le j < i \le \ell$, the conormal bundle $R^1\pi_*\mathcal{E}nd_+(\mathcal{E})$ is filtered with associated graded isomorphic to $\oplus_{1\le j<i\le \ell} R^1\pi_*\mathcal{H}om(\mathcal{N}_i, \mathcal{N}_j)$, as claimed. □

Even though $\overline{\mathbf{\Sigma}}_{\vec{e}}$ is not smooth in general, if we only care about the class of the canonical module, we may excise codimension ≥ 2 closed strata and work over a smooth open, where we have an explicit understanding of the normal bundle as given by the proposition above.

Theorem 3.3.8. Let X be a scheme, and let $\mathcal{E}$ be a rank r, fiberwise degree d vector bundle on $X \times \mathbb{P}^1$, inducing a smooth map $X \to \mathcal{B}_{r,d}$ (so that X is necessarily smooth). Let $\vec{e} = (f_1^{s_1}, \dots, f_\ell^{s_\ell})$, where $f_1 < \cdots < f_\ell$, and let $\overline{\Sigma}_{\vec{e}}(X)$ denote the splitting locus corresponding to $\vec{e}$ in X. Write $\delta_i = \sum_{j<i} s_j - \sum_{j>i} s_j$. Then

$$[\omega_{\overline{\Sigma}_{\vec{e}}(X)}] = [\omega_X|_{\overline{\Sigma}_{\vec{e}}(X)}] + \sum_{1\le i\le \ell} \left((\deg \vec{e} - rf_i + \delta_i)a_1^{(i)} + (r - s_i)b_2^{(i)}\right).$$

Proof. Let $X^\circ = X \setminus \cup_{\vec{f}<\vec{e}, \vec{f}\ne \vec{e}^{(i)}} \Sigma_{\vec{f}}$ and let $Z_{\vec{e}} = \mathbf{Z}_{\vec{e}} \times_{\mathcal{B}^\circ_{r,d}} X^\circ$. Using Proposition 3.3.7, we find that

$$\begin{aligned}
[\omega_{\overline{\Sigma}_{\vec{e}}}] &= [\omega_{Z_{\vec{e}}}] \\
&= -c_1(\mathcal{T}_{Z_{\vec{e}}}) \\
&= -c_1(\mathcal{T}_{X^\circ}|_{Z_{\vec{e}}}) + c_1(\mathcal{N}_{Z_{\vec{e}} \subset X^\circ}) \\
&= [\omega_X|_{\overline{\Sigma}_{\vec{e}}}] + \sum_{1\le j<i\le \ell} c_1(R^1\pi_*\mathcal{H}om(\mathcal{N}_i, \mathcal{N}_j)).
\end{aligned}$$

We claim that when $1 \leq j < i \leq \ell$,

$$c_1(R^1\pi_*\mathcal{H}om(\mathcal{N}_i, \mathcal{N}_j)) = (f_i - f_j - 1)(s_i a_1^{(j)} - s_j a_1^{(i)}) + s_i b_2^{(j)} + s_j b_2^{(i)},$$

which would yield the desired result.

First, we can consider the class of this vector bundle after we excise $\Sigma_{\vec{e}^{(i)}}$. Then $\mathcal{N}_i$ is perfectly balanced, and we have $\mathcal{N}_i = \pi^*\mathcal{M}_i(f_i)$, where we define $\mathcal{M}_i = \pi_*\mathcal{N}_i(-f_i)$.

Now,

$$\begin{aligned} R^1\pi_*\mathcal{H}om(\mathcal{N}_i, \mathcal{N}_j) &= R^1\pi_*\mathcal{H}om(\pi^*\mathcal{M}_i(f_i), \mathcal{N}_j) \\ &= \mathcal{M}_i^\vee \otimes R^1\pi_*\mathcal{N}_j(-f_i). \end{aligned}$$

By Lemma 3.3.1, we have $c_1(R^1\pi_*\mathcal{N}_j(-f_i)) = -(f_j - f_i + 1)a_1^{(j)} + b_2^{(j)}$, and $c_1(\mathcal{M}_i^\vee) = -a_1^{(i)}$. So if we excise $\Sigma_{\vec{e}^{(i)}}$, then we see that modulo $b_2^{(i)}$,

$$\begin{aligned} c_1 R^1\pi_*\mathcal{H}om(\mathcal{N}_i, \mathcal{N}_j) &= s_i c_1(R^1\pi_*\mathcal{N}_j(-f_i)) + s_j(f_i - f_j - 1)c_1(\mathcal{M}_i^\vee) \\ &= -s_i(f_j - f_i + 1)a_1^{(j)} - s_j(f_i - f_j - 1)a_1^{(i)} + s_i b_2^{(j)} \\ &= (f_i - f_j - 1)(s_i a_1^{(j)} - s_j a_1^{(i)}) + s_i b_2^{(j)}. \end{aligned}$$

Similarly, if we excise $\Sigma_{\vec{e}^{(j)}}$, then

$$R^1\pi_*\mathcal{H}om(\mathcal{N}_i, \mathcal{N}_j) = \mathcal{M}_j \otimes R^1\pi_*(\mathcal{N}_i^\vee(f_j)) = \mathcal{M}_j \otimes \pi_*(\mathcal{N}_i(-f_j - 2))^\vee.$$

Again by Lemma 3.3.1, $c_1(\mathcal{M}_j) = a_1^{(j)}$ and $c_1(\pi_*(\mathcal{N}_i(-f_j - 2))^\vee) = -(f_i - f_j - 1)a_1^{(i)} + b_2^{(i)}$. So modulo $b_2^{(j)}$,

$$c_1 R^1\pi_*\mathcal{H}om(\mathcal{N}_i, \mathcal{N}_j) = (f_i - f_j - 1)(s_i a_1^{(j)} - s_j a_1^{(i)}) + s_j b_2^{(i)}.$$

Since $b_2^{(i)}$ and $b_2^{(j)}$ are part of a free basis, the two calculations above together shows the desired formula. $\square$

Proof of Theorem 3.1.2

By Theorem 3.3.8, the class of the canonical module for $\overline{\Sigma}_{\vec{e}} \subset H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ is

$$[\omega_{\overline{\Sigma}_{\vec{e}}}] = \sum_{1 \leq i \leq \ell} \left((\deg \vec{e} - rf_i + \delta_i)a_1^{(i)} + (r - s_i)b_2^{(i)}\right).$$

Recall also that the two classes that generate the kernel of $A^1(\overline{\mathbf{\Sigma}}_{\vec{e}}) \to A^1(\overline{\Sigma}_{\vec{e}})$ are

$$\begin{aligned} \alpha_1 &= \sum_{i=1}^{\ell} \left(-(f_i + M)a_1^{(i)} + b_2^{(i)}\right), \\ \alpha_2 &= \sum_{i=1}^{\ell} \left((f_i + M + 1)a_1^{(i)} - b_2^{(i)}\right). \end{aligned}$$

For convenience, given two vectors $\vec{p}, \vec{q}$ both of length ℓ, we use the notation $(\vec{p} \mid \vec{q})$ to denote the element $\sum_{i=1}^{\ell}(p_i a_1^{(i)} + q_i b_2^{(i)})$. The subgroup generated by α_1, α_2 is free of rank 2, which we think of in terms of the following basis:

$$(\mathbf{1}_{\vec{a}} \mid \mathbf{0}) = \sum_{i=1}^{\ell} a_1^{(i)},$$

$$(\vec{f} \mid -\mathbf{1}_{\vec{b}}) = \sum_{i=1}^{\ell} \left(f_i a_1^{(i)} - b_2^{(i)}\right).$$

We can rewrite the formula for $[\omega_{\overline{\Sigma}_{\vec{e}}}]$ from Theorem 3.3.8 in terms of this basis:

$$[\omega_{\overline{\Sigma}_{\vec{e}}}] = \deg \vec{e}(\mathbf{1}_{\vec{a}} \mid \mathbf{0}) - r(\vec{f} \mid -\mathbf{1}_{\vec{b}}) + (\vec{\delta} \mid -\vec{s}),$$

where $\vec{\delta} = (\delta_1, \ldots, \delta_\ell)$ and $\vec{s} = (s_1, \ldots, s_\ell)$. So $[\omega_{\overline{\Sigma}_{\vec{e}}}]$ is zero or N-torsion if and only if $(\vec{\delta} \mid -\vec{s})$ is zero or N-torsion. We now explain exactly when the canonical class is trivial or torsion.

Let $\vec{\delta} = (\delta_1, \ldots, \delta_\ell)$. Let $\Delta(\vec{\delta}) = (\delta_2 - \delta_1, \ldots, \delta_\ell - \delta_{\ell-1}) = (s_1 + s_2, \ldots, s_{\ell-1} + s_\ell)$, and similarly let $\Delta(\vec{f}) = (f_2 - f_1, \ldots, f_\ell - f_{\ell-1})$. Let $\vec{e}^{(i_1)}, \ldots, \vec{e}^{(i_k)}$ be those splitting types which are codimension one in $\overline{\mathbf{\Sigma}}_{\vec{e}}$.

First suppose there is at least one such codimension one splitting type. Then by observing the generators, we see that if $N(\vec{\delta} \mid -\vec{s}) \in \operatorname{span}\langle \alpha_1, \alpha_2 \rangle$ for some $N \geq 1$, then we must have that $s_{i_1} = s_{i_2} = \cdots = s_{i_k} = s$ for some $s \geq 2$, and there exist some A, B such that

$$(N\vec{\delta} \mid -Ns\mathbf{1}_{\vec{b}}) = (N\vec{\delta} \mid -N\vec{s}) = A(\mathbf{1}_{\vec{a}} \mid \mathbf{0}) + B(\vec{f} \mid -\mathbf{1}_{\vec{b}}),$$

But then we are forced to take $B = Ns$. Looking only at the $\vec{a}$ part of the basis, we see that

$$N\vec{\delta} = A \cdot \mathbf{1}_{\vec{a}} + Ns\vec{f}.$$

Equivalently, $N\Delta(\vec{\delta}) = Ns\Delta(\vec{f})$. But if this is true, then we also have that $\Delta(\vec{\delta}) = s\Delta(\vec{f})$. Let $i \in \{i_1, \ldots, i_k\}$ so that $s_i = s$. Then since $s_{i-1} + s_i = \Delta(\vec{\delta})_{i-1}$ is divisible by s, s_{i-1} must also be divisble by s. Similarly, since $s_i + s_{i+1} = \Delta(\vec{\delta})_i$ is divisible by s, s_{i+1} must be divisible by s. Hence we can write $s_i = k_i s$ for each $1 \leq i \leq \ell$, where $k_i > 0$. The condition tells us that $f_{i+1} - f_i = \Delta(\vec{f})_i = \frac{1}{s}\Delta(\vec{\delta})_i = k_i + k_{i+1} \geq 2$. Moreover, $s_i = k_i s \geq s \geq 2$. So actually every $1 \leq i \leq \ell$ contributes a codimension 1 class. It follows that $s_i = s$ for all $1 \leq i \leq \ell$, and $f_{i+1} - f_i = 2$ for all $1 \leq i < \ell$. In sum, we have shown that if there exists a codimension one splitting type $\vec{e}^{(i)}$ and $\overline{\mathbf{\Sigma}}_{\vec{e}}$ is N-Gorenstein, then $\vec{e}$ is a block arithmetic progression of difference 2 and $\overline{\mathbf{\Sigma}}_{\vec{e}}$ is actually Gorenstein.

Now suppose that there are no codimension 1 splitting loci in $\overline{\mathbf{\Sigma}}_{\vec{e}}$. In this case, $\overline{\mathbf{\Sigma}}_{\vec{e}}$ is N-Gorenstein if and only if $N\Delta(\vec{\delta}) = B\Delta(\vec{f})$ for some $B \geq 1$. For each $1 \leq i \leq \ell$, there are two possibilities:

1. $f_{i+1} = f_i + 1$ or $f_{i-1} = f_i - 1$, and s_i can be anything;

2. $f_i \geq f_{i-1} + 2$ and $f_i \leq f_{i+1} - 2$, and $s_i = 1$.

When only the first possibility occurs, we claim that in fact for every $1 \leq i < \ell$, $f_{i+1} = f_i + 1$. This is trivial if $\ell \leq 3$, so let us assume that $\ell \geq 4$. Suppose towards the contrary that $f_{i+1} - f_i \geq 2$ for some i, which can only happen for $2 \leq i \leq \ell - 2$. Then we must have $f_{i-1} = f_i - 1$ and $f_{i+2} = f_{i+1} + 1$. So the relevant entries of $\Delta(\vec{f})$ and $\Delta(\vec{\delta})$ are $(\dots, 1, f_{i+1} - f_i, 1, \dots)$ and $(\dots, s_{i-1} + s_i, s_i + s_{i+1}, s_{i+1} + s_{i+2}, \dots)$. Since $\Delta(\vec{f})$ and $\Delta(\vec{\delta})$ are proportional, we must have

$$\begin{aligned} s_{i-1} + s_i &= s_{i+1} + s_{i+2} \\ s_i + s_{i+1} &= (f_{i+1} - f_i)(s_{i-1} + s_i). \end{aligned}$$

In particular,

$$2(s_i + s_{i+1}) = (f_{i+1} - f_i)(s_{i-1} + s_i + s_{i+1} + s_{i+2}),$$

which is impossible. Thus we conclude that $\Delta(\vec{f}) = (1, 1, \dots, 1)$, and there exist N, B such that $N\Delta(\vec{\delta}) = B\Delta(\vec{f})$ if and only if there exists $B \geq 1$ such that $\Delta(\vec{\delta}) = B\Delta(\vec{f})$. The latter condition is saying that $s_{i-1} + s_i = s_i + s_{i+1}$ for all $1 < i < \ell$. This is exactly the case where $\vec{e}$ is contiguous. In sum, if a splitting type $\vec{e}$ is N-Gorenstein, does not dominate any splitting type of codimension 1, and only possibility (1) occurs, then $\overline{\mathbf{\Sigma}}_{\vec{e}}$ is contiguous and it is Gorenstein.

If only the second possibility occurs, we have $\Delta(\vec{\delta}) = (2, 2, \dots, 2)$. So in this case if $\overline{\mathbf{\Sigma}}_{\vec{e}}$ is N-Gorenstein for any $N \geq 1$, then $\vec{e}$ is an arithmetic progression. If $\vec{e}$ is an arithmetic progression with difference t, then $\overline{\mathbf{\Sigma}}_{\vec{e}}$ is Gorenstein if and only if $t = 0, 1, 2$. When $t > 2$, if $t = 2t'$, then the smallest N such that $\overline{\mathbf{\Sigma}}_{\vec{e}}$ is N-Gorenstein is $N = t'$. If t is odd, then the smallest N such that $\overline{\mathbf{\Sigma}}_{\vec{e}}$ is N-Gorenstein is t.

Lastly, we claim that the two possibilities cannot both occur for an N-Gorenstein splitting type $\vec{e}$ with no codimension 1 strata. It suffices to show that it is not possible to have i, j such that $|i - j| = 1$, (1) occurs at i and (2) occurs at j. We make the argument for $j = i - 1$, as the other case is entirely analogous. In this case, $\vec{e} = (\dots, f_{i-1}, f_i^{s_i}, f_{i+1}^{s_{i+1}}, \dots)$, where $f_{i+1} = f_i + 1$. But then $\Delta(\vec{f})_{i-1} = f_i - f_{i-1} > 1 = \Delta(\vec{f})_i$. This implies that $1 + s_i = \Delta(\vec{\delta})_{i-1} > \Delta(\vec{\delta})_i = s_i + s_{i+1}$, which is not possible.

Chapter 4

The Singular Locus

4.1 Statement of Results

In this chapter, we will prove a characterization of the singular locus of $\overline{\mathbf{\Sigma}}_{\vec{e}}$ and compute the codimension of the tangent space of $\overline{\mathbf{\Sigma}}_{\vec{e}}$ at each $\mathbf{\Sigma}_{\vec{e}'}$. First we explain what these words mean for algebraic stacks.

Definition 4.1.1. Let $\mathfrak{X}$ be an algebraic stack locally of finite type over a field, with $U \twoheadrightarrow \mathfrak{X}$ a smooth cover by a scheme. Let $R = U \times_{\mathfrak{X}} U$. Since U is locally of finite type over a field, the regular locus is open [Stacks, Lemma 07R5]. We define $\mathrm{Reg}(U)$ to be the regular locus of U, and $\mathrm{Sing}(U)$ to be the complement of $\mathrm{Reg}(U)$ with the reduced closed subscheme structure. Since $\mathrm{pr}_1, \mathrm{pr}_2 : R \to U$ are smooth, $\mathrm{pr}_1^{-1}(\mathrm{Sing}(U)) = \mathrm{pr}_2^{-1}(\mathrm{Sing}(U)) = \mathrm{Sing}(R)$. In particular, $\mathrm{Sing}(U)$ is an R-invariant closed subscheme of U, and defines a closed substack of $\mathfrak{X}$ by [Stacks, Lemma 0505, Definition 044F]. Similarly, if $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ is a smooth local homomorphism of Noetherian local rings, then A is regular if and only if B is regular. This shows that $\mathrm{Reg}(U)$ is also an R-invariant open subscheme of U, defining an open substack $\mathrm{Reg}(\mathfrak{X}) \subset \mathfrak{X}$.

Using the same algebraic fact above, one can show that this definition of $\mathrm{Reg}(\mathfrak{X})$ and $\mathrm{Sing}(\mathfrak{X})$ is independent of the choice of U. Since reducedness is a smooth-local property, $\mathrm{Sing}(\mathfrak{X})$ is by construction reduced.

Recall that a lower set of a partially ordered set is one which is closed under taking $\leq$.

Lemma 4.1.2. Reduced closed substacks of $\overline{\mathbf{\Sigma}}_{\vec{e}}$ are all of the form $\bigcup_{\vec{e}' \in I} \mathbf{\Sigma}_{\vec{e}'}$ for a lower set I of the poset of splitting types so that $\vec{e}' \in I$ implies $\vec{e}' \leq \vec{e}$.

Proof. By [Stacks, Lemma 0509], reduced closed substacks of an algebraic stack $\mathfrak{X}$ are in one-to-one correspondence with closed subsets of $|\mathfrak{X}|$. In the case of $\mathfrak{X} = \overline{\mathbf{\Sigma}}_{\vec{e}}$, $|\overline{\mathbf{\Sigma}}_{\vec{e}}|$ is the poset $\{\vec{e}' : \vec{e}' \leq \vec{e}\}$. It follows from the definition of the topology on $|\mathfrak{X}|$ [Stacks, 04XG, 04XL] and the properties of the complete families $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ that the closed subsets of $|\overline{\mathbf{\Sigma}}_{\vec{e}}|$ are exactly the lower sets $I \subseteq |\overline{\mathbf{\Sigma}}_{\vec{e}}|$. The unique reduced closed substack $\mathcal{Z}$ of $\overline{\mathbf{\Sigma}}_{\vec{e}}$ such that $|\mathcal{Z}| = I$ is precisely $\bigcup_{\vec{e}' \in I} \mathbf{\Sigma}_{\vec{e}'}$ with the reduced closed substack structure. $\square$

In particular, in the universal setting, the singular locus of $\overline{\mathbf{\Sigma}}_{\vec{e}}$ is a union of $\mathbf{\Sigma}_{\vec{e}'}$'s, which warrants the following definition.

Definition 4.1.3. We say that $\vec{e}'$ is in the singular locus of $\vec{e}$, denoted $\vec{e}' \in \mathrm{Sing}(\vec{e})$ if $\mathbf{\Sigma}_{\vec{e}'}$ is contained in the singular locus of $\overline{\mathbf{\Sigma}}_{\vec{e}}$, and similarly we write $\vec{e}' \in \mathrm{Reg}(\vec{e})$ if $\mathbf{\Sigma}_{\vec{e}'}$ is contained in the regular locus of $\overline{\mathbf{\Sigma}}_{\vec{e}}$.

Remark 4.1.4. A different, but also natural, closed substack structure on $\mathrm{Sing}(\overline{\mathbf{\Sigma}}_{\vec{e}})$ would be to use the Fitting ideal scheme structure on the sheaf of differentials, such as explained in [Stacks, Section 0C3H]. These two scheme structures differ in general, as can already be seen in the case of determinantal varieties. In this chapter, we choose to focus on the reduced closed substack structure.

Next we explain what we mean by the codimension of the tangent space of $\overline{\mathbf{\Sigma}}_{\vec{e}}$ at $\mathbf{\Sigma}_{\vec{e}'}$.

Lemma 4.1.5. Let $(B, \mathcal{E})$ be a complete family, and let $b \in \Sigma_{\vec{e}'}(B)$ be any closed point. Then the quantity $\dim B - \dim T_b\overline{\Sigma}_{\vec{e}}(B)$ only depends on $\vec{e}$ and $\vec{e}'$.

Proof. In the case where $B = H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$, $\Sigma_{\vec{e}'}(B)$ is just the closed point at the origin, and we can compute the quantity $\dim B - \dim T_b\overline{\Sigma}_{\vec{e}}(B) = u(\vec{e}') - \dim T_0\overline{\Sigma}_{\vec{e}}$.

We note that by the exact sequence of sheaves of differentials associated to a smooth morphism, if $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a smooth local homomorphism of Noetherian local rings, then $\dim_{S/\mathfrak{n}} \mathfrak{n}/\mathfrak{n}^2 - \dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = \dim S - \dim R$. Now consider the fiber product

$$\begin{array}{ccc} F = B \times_{\mathcal{B}_{r,d}} H^1\mathcal{E}nd(\mathcal{O}(\vec{e}')) & \longrightarrow & H^1\mathcal{E}nd(\mathcal{O}(\vec{e}')) \\ \downarrow & & \downarrow \\ B & \longrightarrow & \mathcal{B}_{r,d} \end{array} .$$

Note that all spaces and morphisms in this diagram are smooth. Let $b \in B$ be any closed point in $\Sigma_{\vec{e}'}(B)$. Since $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}')) \to \mathcal{B}_{r,d}$ surjects onto $\bigcup_{\vec{f} \geq \vec{e}'} \mathbf{\Sigma}_{\vec{f}}$, upon base change, F surjects onto $\bigcup_{\vec{f} \geq \vec{e}'} \Sigma_{\vec{f}}(B)$ as well. So we can always choose some closed point $b' \in F$ mapping to $b \in \Sigma_{\vec{e}'}(B)$. Under the map $F \to H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$, $b' \in F$ necessarily maps to $0 \in H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$. Now the algebraic argument above shows that

$$\dim B - \dim T_b\overline{\Sigma}_{\vec{e}}(B) = \dim F - \dim T_{b'}\overline{\Sigma}_{\vec{e}}(F) = u(\vec{e}') - \dim T_0\overline{\Sigma}_{\vec{e}}$$

as desired. ∎

Definition 4.1.6. Let $T(\vec{e}, \vec{e}')$ be the quantity $\dim B - \dim T_b\overline{\Sigma}_{\vec{e}}(B)$, whenever $(B, \mathcal{E})$ is a complete family and $b \in \Sigma_{\vec{e}'}(B)$ is any closed point.

Note that $T(\vec{e}, \vec{e}') \leq u(\vec{e})$, and achieves equality if and only if $\vec{e}' \in \mathrm{Reg}(\vec{e})$. For the rest of this section, we shall state our results characterizing $\mathrm{Sing}(\vec{e})$ and $T(\vec{e}, \vec{e}')$ for arbitrary $\vec{e}' \leq \vec{e}$.

Definition 4.1.7. For $1 \leq s < r$, the gap $(s, s+1)$ is a *cut* for the pair $\vec{e}' \leq \vec{e}$ if $e'_1 + \cdots + e'_s = e_1 + \cdots + e_s$ and $e_s < e_{s+1}$. A cut is *admissible* if in addition, we have $e'_s < e'_{s+1}$. We also view the positions $(0, 1)$ and $(r, r+1)$ as admissible cuts.

Theorem 4.1.8. Let $\vec{e}' \leq \vec{e}$. Then $\vec{e}' \in \mathrm{Reg}(\vec{e})$ if and only if there exist admissible cuts for the pair such that each block of $\vec{e}$ between two adjacent cuts is balanced.

In other words, $\vec{e}'$ lies in the regular locus of $\vec{e}$ if and only if $\vec{e}$ can be obtained by cutting $\vec{e}'$ into blocks such that $e'_s < e'_{s+1}$ at each gap, and balancing each block of the partition. A special case of this theorem was obtained in [Cop25, Proposition 7].

Definition 4.1.9. Let $\vec{e}' \leq \vec{e}$. Then we say that the pair $(\vec{e}', \vec{e})$ is *nice* if there exists a collection of inadmissible cuts for the pair so that $\vec{e}$ is balanced between two adjacent cuts.

Nice pairs $(\vec{e}, \vec{e}')$ are an important base case where our understanding of the geometry of $\overline{\mathbf{\Sigma}}_{\vec{e}}$ along $\mathbf{\Sigma}_{\vec{e}'}$ is quite explicit. The following description of $T(\vec{e}, \vec{e}')$ may be viewed as a flow chart for computing $T(\vec{e}, \vec{e}')$. One of the exit states for this flow chart is the case of a nice pair $(\vec{e}, \vec{e}')$.

Theorem 4.1.10. If $(\vec{e}, \vec{e}')$ share a nontrivial admissible cut, then we can partition $(\vec{e}, \vec{e}')$ simultaneously according to the cut, so that $\vec{e} = (\vec{c}_1, \vec{c}_2)$, and $\vec{e}' = (\vec{c}'_1, \vec{c}'_2)$. Then

$$T(\vec{e}, \vec{e}') = T(\vec{c}_1, \vec{c}'_1) + T(\vec{c}_2, \vec{c}'_2) + \mathrm{ext}^1(\vec{c}'_2, \vec{c}'_1).$$

If $(\vec{e}, \vec{e}')$ do not share any nontrivial cuts, then $T(\vec{e}, \vec{e}') = 0$.

If $(\vec{e}, \vec{e}')$ share only inadmissible cuts, then we can write $\vec{e} = (\vec{c}_m, \dots, \vec{c}_1)$, $\vec{e}' = (\vec{c}'_m, \dots, \vec{c}'_1)$, where the blocks witness all the inadmissible cuts for the pair $(\vec{e}, \vec{e}')$. Then if $\vec{f} = (\vec{b}_m, \dots, \vec{b}_1)$, where $\vec{b}_i$ is the result of balancing $\vec{c}_i$, then $T(\vec{e}, \vec{e}') = T(\vec{f}, \vec{e}')$.

If $(\vec{e}, \vec{e}')$ is a nice pair, then using the inadmissible cuts exhibiting $(\vec{e}, \vec{e}')$ as a nice pair, we can write $\vec{e} = (\vec{c}_m, \vec{c}_{m-1}, \dots, \vec{c}_1)$, and

$$\begin{aligned} \vec{e}' &= (\vec{c}'_m, \dots, \vec{c}'_1) \\ &= (\vec{b}_m, g_{m-1}^{\rho_{m-1}}, g_{m-1}^{\ell_{m-1}}, \vec{b}_{m-1}, g_{m-2}^{\rho_{m-2}}, g_{m-2}^{\ell_{m-2}}, \dots, g_2^{\rho_2}, g_2^{\ell_2}, \vec{b}_2, g_1^{\rho_1}, g_1^{\ell_1}, \vec{b}_1). \end{aligned}$$

where the change in color signifies a cut in between. Note that each $\vec{c}_i$ is balanced, $g_i \in \mathbb{Z}$, $\rho_i, \ell_i > 0$, and $\mathrm{len}(\vec{c}_1) = \mathrm{len}(\vec{c}'_1) = \mathrm{len}(\vec{b}_1) + \ell_1$, $\mathrm{len}(\vec{c}_2) = \mathrm{len}(\vec{c}'_2) = \mathrm{len}(\vec{b}_2) + \ell_2 + \rho_1$, etc. Let $t_i = \rho_i + \ell_i$. Then

$$T(\vec{e}, \vec{e}') = u(\vec{e}') - \sum_{i=1}^{m} h^1 \mathcal{E}nd(\mathcal{O}(g_i^{t_i}, \vec{b}_i, g_{i-1}^{t_{i-1}}))$$

To see that Theorem 4.1.10 implies Theorem 4.1.8, it suffices to check that in the case of nice pairs, $T(\vec{e}, \vec{e}') < u(\vec{e})$. This is actually a somewhat nontrivial computation, which we do at the end of Section 4.4. In Remark 4.4.8, we explain how to adapt our proof of Theorem 4.1.10 to obtain Theorem 4.1.8 in all characteristics.

We give a few small examples of our theorems above.

Example 4.1.11.

1. If $\vec{e}, \vec{e}'$ are of rank 2 and $\vec{e}$ is not balanced, then $\vec{e}' < \vec{e}$ is always in the singular locus for $\vec{e}$ and $T(\vec{e}', \vec{e}) = 0$.

2. The splitting type $\vec{e}' = (-2, 0, 0, 2)$ is in the singular locus of $\vec{e} = (-1, -1, 1, 1)$. Note that $(\vec{e}, \vec{e}')$ is a nice pair, and using Theorem 4.1.10, we compute that $T(\vec{e}, \vec{e}') = 7 - 4 = 3$. On the other hand, the codimension of $\overline{\mathbf{\Sigma}}_{\vec{e}}$ is $u(\vec{e}) = 4$. So the tangent space dimension of $\overline{\mathbf{\Sigma}}_{\vec{e}}$ at $\vec{e}'$ is one larger than the dimension of $\overline{\mathbf{\Sigma}}_{\vec{e}}$.

3. The splitting type $\vec{e}' = (-1,-1,0,0,0,2)$ is in the regular locus of $\vec{e} = (-1,-1,0,0,1,1)$. One would take the cut $(2,3)$ to satisfy the conditions of Theorem 4.1.8. Since $\vec{e}'$ is in the regular locus of $\vec{e}$, we have $T(\vec{e},\vec{e}') = u(\vec{e}) = 4$. We can also compute the same number using the algorithm of Theorem 4.1.10. Since $\vec{c}_1, \vec{c}_2$ are both balanced, we have that $T(\vec{e},\vec{e}') = \mathrm{ext}^1(\vec{c}_2', \vec{c}_1') = 4$.

Lastly, in Section 4.5, we give a description of the normal cone of $\mathbf{\Sigma}_{\vec{e}'}$ in $\overline{\mathbf{\Sigma}}_{\vec{e}}^{\geq \vec{e}'}$ when $(\vec{e},\vec{e}')$ is a nice pair. To be more precise about what we mean, let $\mathcal{B}_{r,d}^{\geq \vec{e}'} = \bigcup_{\vec{f} \geq \vec{e}'} \mathbf{\Sigma}_{\vec{f}}$ be the open substack of $\mathcal{B}_{r,d}$ which $U_{\vec{e}'}$ surjects onto. For any $\vec{e} \geq \vec{e}'$, denote $\overline{\mathbf{\Sigma}}_{\vec{e}}^{\geq \vec{e}'} = \overline{\mathbf{\Sigma}}_{\vec{e}} \cap \mathcal{B}_{r,d}^{\geq \vec{e}'}$.

Since $\mathcal{B}_{r,d}^{\geq \vec{e}'}$ and $\mathbf{\Sigma}_{\vec{e}'}$ are both smooth, it makes sense to talk about the normal bundle of $\mathbf{\Sigma}_{\vec{e}'}$ in $\mathcal{B}_{r,d}^{\geq \vec{e}'}$. Let $\mathbf{N}_{\vec{e}'}$ be the normal bundle of $\mathbf{\Sigma}_{\vec{e}'}$ in $\mathcal{B}_{r,d}^{\geq \vec{e}'}$. We abuse notation and also use $\mathbf{N}_{\vec{e}'}$ to denote the total space of this normal bundle. Now let $\mathbf{N}_{\vec{e},\vec{e}'}$ denote the normal cone associated to the closed embedding $\mathbf{\Sigma}_{\vec{e}'} \subset \overline{\mathbf{\Sigma}}_{\vec{e}}^{\geq \vec{e}'}$. As in the case of schemes, $\mathbf{N}_{\vec{e},\vec{e}'}$ admits a natural closed embedding into $\mathbf{N}_{\vec{e}'}$. In Proposition 4.2.2, we show that $\mathbf{N}_{\vec{e}'}$ is the $\mathrm{Aut}(\mathcal{O}(\vec{e}'))$-representation $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$, where $\mathrm{Aut}(\mathcal{O}(\vec{e}'))$ acts on $\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ by conjugation, inducing an action on this cohomology group. In Section 4.5, we describe the normal cone $\mathbf{N}_{\vec{e},\vec{e}'}$ as a closed substack of $\mathbf{N}_{\vec{e}'} = [H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))/\mathrm{Aut}(\mathcal{O}(\vec{e}'))]$ by describing the closed subscheme of $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ which descends to $\mathbf{N}_{\vec{e},\vec{e}'}$.

4.2 Successive extension spaces, revisited

We clarify the isomorphism $\mathbf{\Sigma}_{\vec{e}} \cong B\,\mathrm{Aut}(\mathcal{O}(\vec{e}))$ and compute the normal bundle for $\mathbf{\Sigma}_{\vec{e}} \hookrightarrow \mathcal{B}_{r,d}^{\geq \vec{e}}$. Let $N_{\vec{e},\vec{e}'}$ be the normal cone for the closed embedding $\{0\} \hookrightarrow \overline{\Sigma}_{\vec{e}}$ in $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$. We shall also illucidate the relationship between $\mathbf{N}_{\vec{e},\vec{e}'}$ and $N_{\vec{e},\vec{e}'}$.

Proposition 4.2.1. Let $\mathbf{\Sigma}_{\vec{e}}$ be the locally closed substack of $\overline{\mathbf{\Sigma}}_{\vec{e}} \subseteq \mathcal{B}_{r,d}$, defined by Fitting ideal conditions. In other words, let $\mathbf{\Sigma}_{\vec{e}}$ be the algebraic stack representing the following functor:

$$(T \to \operatorname{Spec} k) \mapsto \left\{ \begin{array}{l} \text{A vector bundle } \mathcal{E} \text{ of rank } r \text{ and fiberwise degree } d \\ \text{on } T \times \mathbb{P}^1 \text{ such that } R^1\pi_*\mathcal{E}(m) \text{ is a vector bundle of} \\ \text{rank } h^1\mathcal{O}(\vec{e})(m) \text{ for all } m \in \mathbb{Z} \end{array} \right\},$$

where $\pi : T \times \mathbb{P}^1 \to T$. Then $\mathbf{\Sigma}_{\vec{e}} \cong B\,\mathrm{Aut}(\mathcal{O}(\vec{e}))$.

Proof. We can think of the functor of points for $B\,\mathrm{Aut}(\mathcal{O}(\vec{e}))$ as

$$(T \to \operatorname{Spec} k) \mapsto \left\{ \begin{array}{l} \text{A vector bundle } \mathcal{E} \text{ of rank } r \text{ and fiberwise degree } d \\ \text{on } T \times \mathbb{P}^1 \text{ such that there exists an étale cover } \{U_i\} \text{ of } T, \\ \text{so that } \mathcal{E}_{U_i} \cong \mathcal{O}(\vec{e})_{U_i} \end{array} \right\}.$$

The argument in [Lar21b, Lemma 4.3] for $T = \operatorname{Spec} k[\varepsilon]$ generalizes verbatim to arbitrary T to show that these two functors are equivalent. ☐

Note that in fact this argument shows that $\mathcal{E}$ can be Zariski-locally trivialized.

Proposition 4.2.2. The normal bundle $\mathbf{N}_{\vec{e}}$ of $\mathbf{\Sigma}_{\vec{e}}$ in $\mathcal{B}_{r,d}^{\geq \vec{e}}$ is the $\mathrm{Aut}(\mathcal{O}(\vec{e}))$-representation $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}))$, where $\mathrm{Aut}(\mathcal{O}(\vec{e}))$ acts via conjugation on $\mathcal{E}nd(\mathcal{O}(\vec{e}))$, inducing an action on $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}))$.

Proof. Suppose that $T \to \mathcal{B}_{r,d}^{\geq \vec{e}}$ is smooth and surjective. First note that by Proposition 3.3.7, the normal bundle for $\Sigma_{\vec{e}} = \overline{\Sigma}_{\vec{e}}$ in T is $R^1\pi_*\mathcal{E}nd(\mathcal{E})$, where $\mathcal{E}$ is the family of vector bundles on $\Sigma_{\vec{e}} \times \mathbb{P}^1$ inducing the composite $\Sigma_{\vec{e}} \to T \to \mathcal{B}_{r,d}^{\geq \vec{e}}$. By descent, this description of the normal bundle is also true in the universal setting, i.e., on $\mathbf{\Sigma}_{\vec{e}}$, the normal bundle $\mathcal{N}_{\vec{e}}$ can be written as $R^1\pi_*\mathcal{E}nd(\mathcal{E}_{\text{univ}})$. When pulled back from $\mathbf{\Sigma}_{\vec{e}}$ to $\operatorname{Spec} k$, the universal $\mathrm{Aut}(\mathcal{O}(\vec{e}))$-torsor over $\mathbf{\Sigma}_{\vec{e}}$, this vector bundle becomes the vector space $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}))$ by cohomology and base change. It remains to check that the action of $\mathrm{Aut}(\mathcal{O}(\vec{e}))$ on $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}))$ which makes it descend to $R^1\pi_*\mathcal{E}nd(\mathcal{E}_{\text{univ}})$ is precisely the one claimed above.

To show this, let us consider an arbitrary T-point of $\mathcal{B}_{r,d}^{\geq \vec{e}}$ again. Let $\alpha \in \mathrm{Aut}(\mathcal{O}(\vec{e}))$, let $q : \Sigma_{\vec{e}} \times \mathbb{P}^1 \to \mathbb{P}^1$, $\pi_{\vec{e}} : \Sigma_{\vec{e}} \times \mathbb{P}^1 \to \Sigma_{\vec{e}}$, and let $\mathcal{P} = \underline{\mathrm{Isom}}(\mathcal{E}, q^*\mathcal{O}(\vec{e}))$. We will often write $\mathcal{O}(\vec{e})_S$ to denote the pullback of $\mathcal{O}(\vec{e})$ from $\mathbb{P}^1$ to $S \times \mathbb{P}^1$, and if $f : X \to Y$, then we will use $\widetilde{f} : X \times \mathbb{P}^1 \to Y \times \mathbb{P}^1$.

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\gamma_\alpha"] \arrow[dr, "h"'] & \mathcal{P} \arrow[r, "g"] \arrow[d, "h"] & \operatorname{Spec} k \arrow[d] \\
& \Sigma_{\vec{e}} = \overline{\Sigma}_{\vec{e}} \arrow[r] \arrow[d] & \mathbf{\Sigma}_{\vec{e}} \arrow[d] \\
& T \arrow[r] & \mathcal{B}_{r,d}^{\geq \vec{e}}
\end{tikzcd}$$

Then α induces an automorphism $\gamma_\alpha : \mathcal{P} \to \mathcal{P}$ by post-composition. On $\mathcal{P} \times \mathbb{P}^1$, we have a universal identification $\mu : \widetilde{h}^*\mathcal{E} \to \mathcal{O}(\vec{e})_{\mathcal{P}}$, so that when pulled back along $\widetilde{\gamma_\alpha}$, this becomes $\widetilde{\gamma_\alpha}^*\mu : \widetilde{h}^*\mathcal{E} \xrightarrow{\mu} \mathcal{O}(\vec{e})_{\mathcal{P}} \xrightarrow{\alpha} \mathcal{O}(\vec{e})_{\mathcal{P}}$. As such,the following diagram commutes:

$$\begin{tikzcd}
\widetilde{h}^*\mathcal{E} \arrow[r] & \widetilde{\gamma_\alpha}^*\widetilde{h}^*\mathcal{E} \arrow[d, "\widetilde{\gamma_\alpha}^*\mu"] \\
\mathcal{O}(\vec{e})_{\mathcal{P}} \arrow[u, "\mu^{-1}"] \arrow[r, "\alpha"] & \mathcal{O}(\vec{e})_{\mathcal{P}}
\end{tikzcd}$$

The isomorphism $h^*R^1\pi_{\vec{e}*}\mathcal{E}nd(\mathcal{E}) \cong \gamma_\alpha^* h^* R^1\pi_{\vec{e}*}\mathcal{E}nd(\mathcal{E})$ can be identified as an automorphism of $R^1\pi_{\mathcal{P}*}\mathcal{E}nd(\mathcal{O}(\vec{e})_{\mathcal{P}})$. The latter is induced by the automorphism α, which acts by conjugation on $\mathcal{E}nd(\mathcal{O}(\vec{e})_{\mathcal{P}})$, as desired. $\square$

We are interested in the normal cone $\mathbf{N}_{\vec{e},\vec{e}'}$ as defined in Section 4.1, which is a closed substack of $\mathbf{N}_{\vec{e}'} = [H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))/\operatorname{Aut}(\mathcal{O}(\vec{e}'))]$. Since the map from $U_{\vec{e}'} \to \mathcal{B}_{r,d}^{\geq \vec{e}'}$ is smooth and surjective, the normal bundle for $\operatorname{Spec} k \hookrightarrow U_{\vec{e}'}$ is the base change of the normal bundle $\mathbf{N}_{\vec{e}'}$. Similarly, the normal cone for $\mathbf{\Sigma}_{\vec{e}'} \subseteq \mathbf{\Sigma}_{\vec{e}}^{\geq \vec{e}'}$ admits a smooth surjection from the normal cone $N_{\vec{e},\vec{e}'}$ of $\{0\}$ in $\overline{\Sigma}_{\vec{e}} \subseteq U_{\vec{e}'}$. The following diagram illustrates this. Every square is a fiber product.

$$\begin{tikzcd}
U_{\vec{e}'} \arrow[d, two heads] & \operatorname{Spec} k \arrow[l, hook', "/" marking] \arrow[d, two heads] & H^1\mathcal{E}nd(\mathcal{O}(\vec{e}')) \arrow[l] \arrow[d] & N_{\vec{e},\vec{e}'} \arrow[l, hook', "/" marking] \arrow[d] \\
\mathcal{B}_{r,d}^{\geq \vec{e}'} & \mathbf{\Sigma}_{\vec{e}'} \arrow[l, hook', "/" marking] & \mathbf{N}_{\vec{e}'} \arrow[l] & \mathbf{N}_{\vec{e},\vec{e}'} \arrow[l, hook', "/" marking]
\end{tikzcd}$$

In Section 4.5, we describe $N_{\vec{e},\vec{e}'}$ when $(\vec{e}, \vec{e}')$ is a nice pair. This description then descends to a description of $\mathbf{N}_{\vec{e},\vec{e}'}$ as a closed substack of $\mathbf{N}_{\vec{e}'}$.

4.3 Section Matrices

Lemma 4.3.1. Every class $\xi \in H^1\mathcal{E}nd(\mathcal{O}(\vec{e}))$ can be uniquely identified as an $r \times r$ strictly upper triangular matrix $T = (T_{ij})$ with entries in $k[t^{\pm 1}]$, where $T_{ij} = \sum_{s=e_i+1}^{e_j-1} a_{i,j,s} t^s$ if $1 \leq i < j \leq r$.

Proof. Consider the Čech complex computing $H^i\mathcal{E}nd(\mathcal{O}(\vec{e}))$, namely

$$0 \to \operatorname{Mat}_r(k[t]) \oplus \operatorname{Mat}_r(k[t^{-1}]) \to \operatorname{Mat}_r(k[t^{\pm 1}]) \to 0,$$

where t is the local coordinate at $[0 : 1] \in \mathbb{P}^1$. Let $\mathbb{A}^1_\infty$ and $\mathbb{A}^1_0$ denote the affine charts $\mathbb{P}^1 \setminus \{0\}$ and $\mathbb{P}^1 \setminus \{\infty\}$ respectively. Recall that the transition function for $\mathcal{O}_{\mathbb{P}^1}(r)$ from $\mathbb{A}^1_\infty$ to $\mathbb{A}^1_0$ is given by multiplication by t^r. Therefore let

$$\alpha = \begin{bmatrix} t^{e_1} & & & \\ & t^{e_2} & & \\ & & \cdots & \\ & & & t^{e_r} \end{bmatrix}$$

be the transition function from $\mathbb{A}^1_\infty$ to $\mathbb{A}^1_0$ for $\mathcal{O}(\vec{e})$. Then in terms of α, the map in the Čech complex above is given by

$$(\Phi, \Psi) \mapsto \Phi - \alpha\Psi\alpha^{-1}.$$

Examining this map, we find that only the strictly upper triangular entries do not all lie in the image. Moreover, for $i < j$, the (i, j)-th of any element of the cokernel can be uniquely represented by a linear combination of the monomials $t^{e_i+1}, \ldots, t^{e_j-1}$. □

Suppose that $\vec{e} = (e_1, \ldots, e_r) = (f_1^{s_1}, \ldots, f_\ell^{s_\ell})$. After choosing an isomorphism $\mathcal{O}(\vec{e}) = \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_r)$, where $e_1 \leq \cdots \leq e_r$, we can identify

$$H^1\mathcal{E}nd(\mathcal{O}(\vec{e})) = \oplus_{1 \leq i < j \leq r} \operatorname{Ext}^1(\mathcal{O}(e_j), \mathcal{O}(e_i)).$$

Then each $\operatorname{Ext}^1(\mathcal{O}(e_j), \mathcal{O}(e_i)) = H^1\mathcal{H}om(\mathcal{O}(e_j), \mathcal{O}(e_i))$ precisely corresponds to the (i, j)-th entry of the strictly upper triangular matrix of Lemma 4.3.1. This description of classes in $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}))$ gives rise to a distinguished basis for this vector space, which gives us linear coordinates for this affine space. Let $H^1\mathcal{E}nd(\mathcal{O}(\vec{e})) = \operatorname{Spec} A$, where

$$A = k[a_{i,j,s} : 1 \leq i < j \leq r, e_i < s < e_j].$$

Then there is a universal element of $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}))$ over this affine space, namely $T = (T_{ij})$, where T_{ij} is only nonzero if $i < j$, in which case

$$T_{ij} = \sum_{s=e_i+1}^{e_j-1} a_{i,j,s}t^s.$$

As explained in Section 3.2, the polynomial ring A is naturally $\mathbb{Z}^{\ell-1}$-graded. If we let $\delta_i \in \mathbb{Z}^{\ell-1}$ denote the vector which is 1 in the i-th entry and zero elsewhere, then the degree of $a_{i,j,s}$ is

$$\deg a_{i,j,s} = \delta_i + \delta_{i+1} + \cdots + \delta_{j-1} \in \mathbb{Z}^{\ell-1}.$$

Proposition 4.3.2. The universal bundle on $H^1\mathcal{E}nd(\mathcal{O}(\vec{e})) \times \mathbb{P}^1$ is the one whose transition function from $H^1\mathcal{E}nd(\mathcal{O}(\vec{e})) \times \mathbb{A}^1_\infty$ to $H^1\mathcal{E}nd(\mathcal{O}(\vec{e})) \times \mathbb{A}^1_0$ is given by $\alpha + T$.

Proof. In general, if $\mathcal{F}, \mathcal{G}$ are vector bundles on a scheme X, then the isomorphism $H^1\mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \operatorname{Ext}^1(\mathcal{F}, \mathcal{G})$ can be described as follows. Take a collection of opens $\{U_i\}$ together with trivialization $\mathcal{F}_{U_i} \cong \mathcal{O}^r_{U_i}, \mathcal{G}_{U_i} \cong \mathcal{O}^s_{U_i}$. Let P_{ij}, Q_{ij} be the transition functions for $\mathcal{G}$ and $\mathcal{F}$ respectively, and let $M_{ij} \in \operatorname{Hom}(\mathcal{F}_{U_{ij}}, \mathcal{G}_{U_{ij}})$ represent a chosen class in $H^1\mathcal{H}om(\mathcal{F}, \mathcal{G})$. Then the corresponding class in $\operatorname{Ext}^1(\mathcal{F}, \mathcal{G})$ is given by the vector bundle $\mathcal{E}$ whose transition function from U_i to U_j is

$$\begin{bmatrix} P_{ij} & M_{ij} \\ 0 & Q_{ij} \end{bmatrix}.$$

Note that $\mathcal{E}$ admits an inclusion from $\mathcal{G}$ and a quotient to $\mathcal{F}$, forming an extension.

Now the desired claim follows from iterating this argument. For the base case where $\ell = 2$, the vector bundle on $\operatorname{Ext}^1(\mathcal{O}(f_2^{s_2}), \mathcal{O}(f_1^{s_1})) \times \mathbb{P}^1$ arising from the universal extension can be described using a transition function precisely in this manner. For the inductive step, we consider the extension

$$0 \to \mathcal{K}_{\ell-1} \to \mathcal{K}_\ell \to \mathcal{O}(f_\ell^{s_\ell}) \to 0.$$

In Proposition 2.2.10, we argued that $R^1\pi_*\mathcal{H}om(\mathcal{O}(f_\ell^{s_\ell}), \mathcal{K}_{\ell-1})$ is a vector bundle over the affine space $H^1\mathcal{E}nd(\mathcal{O}(f_1^{s_1}, \dots, f_{\ell-1}^{s_{\ell-1}}))$. But as the latter is just affine space, this vector bundle must be trivial, and can be identified non-canonically with the fiber over $0 \in H^1\mathcal{E}nd(\mathcal{O}(f_1^{s_1}, \dots, f_{\ell-1}^{s_{\ell-1}}))$. This non-canonical identification precisely corresponds to filling the top right block of the transition function for $\mathcal{K}_\ell$ with the same entries as in the case of $\mathcal{K}_{\ell-1} \cong \mathcal{O}(f_1^{s_1}, \dots, f_{\ell-1}^{s_{\ell-1}})$. □

Let $\mathcal{E}$ be the universal vector bundle on $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}\,'))\times\mathbb{P}^1$. We say that ℓ is a *critical twist* for $\vec{e}\,'$ if $R^1\pi_*\mathcal{E}(-\ell)$ is not a vector bundle, where $\pi : H^1\mathcal{E}nd(\mathcal{O}(\vec{e}\,')) \times \mathbb{P}^1 \to H^1\mathcal{E}nd(\mathcal{O}(\vec{e}\,'))$. Let i be the largest index such that $e'_i < \ell$. Then there is a distinguished short exact sequence

$$0 \to \mathcal{K}_\ell \to \mathcal{E} \to \mathcal{Q}_\ell \to 0,$$

where $\mathcal{K}_\ell$ is the subbundle glued using the top left $i \times i$ block of the transition function for $\mathcal{E}$, and $\mathcal{Q}_\ell$ is the quotient bundle glued using the bottom right $(r-i) \times (r-i)$ block. Note that by construction, $\pi_*\mathcal{Q}_\ell(-\ell)$ and $R^1\pi_*\mathcal{K}_\ell(-\ell)$ are vector bundles of rank $h^0\mathcal{O}(\vec{e}\,')(-\ell)$ and

$h^1\mathcal{O}(\vec{e}')(-\ell)$ respectively, and the long exact sequence for derived pushforward gives rise to a resolution of $R^1\pi_*\mathcal{E}(-\ell)$ by a map of vector bundles:

$$\pi_*\mathcal{Q}_\ell(-\ell) \to R^1\pi_*\mathcal{K}_\ell(-\ell) \to R^1\pi_*\mathcal{E}(-\ell) \to 0.$$

After identifying $\pi_*\mathcal{Q}_\ell(-\ell) \cong A^{\oplus h^0}$, $R^1\pi_*\mathcal{K}_\ell(-\ell) \cong A^{\oplus h^1}$, where $h^i = h^i\mathcal{O}(\vec{e}')(-\ell)$, the map of vector bundles is simply a matrix with entries in A. [New97] calls these matrices *section matrices*, and their Lemma 4.1.2 describes these section matrices in an inductive manner, from which we extract the following proposition, which is a key ingredient of our proof.

We say that a homogeneous element $f \in A$ with respect to the $\mathbb{Z}^{\ell-1}$-grading is *curvilinear* if at least one of the terms of f is linear in the usual sense.

Proposition 4.3.3 ([New97, Lemma 4.1.2])**.** When ℓ is a critical twist, the A-module $R^1\pi_*\mathcal{E}(-\ell)$ admits a resolution of the form

$$A^{\oplus h^0} \xrightarrow{M_\ell} A^{\oplus h^1} \to R^1\pi_*\mathcal{E}(-\ell) \to 0,$$

where $h^i = h^i\mathcal{O}(\vec{e}')(-\ell)$, and M_ℓ is a matrix whose entries are all curvilinear forms in A. □

Remark 4.3.4. In fact, though we don't need this result here, it is not so hard to read off the linear part of the matrix M_ℓ from New's lemma.

4.4 Proof of Theorem 4.1.8 and Theorem 4.1.10

We will proceed by analyzing the tangent space to $\overline{\Sigma}_{\vec{e}}$ at the origin of $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$.

Lemma 4.4.1. There is a natural surjection from the set of critical twists ℓ where $h^1\mathcal{O}(\vec{e}')(-\ell) = h^1\mathcal{O}(\vec{e})(-\ell)$ to the set of cuts for $(\vec{e}, \vec{e}')$.

Proof. Suppose that ℓ is a critical twist such that $h^1\mathcal{O}(\vec{e}')(-\ell) = h^1\mathcal{O}(\vec{e})(-\ell)$. We can write $\vec{e}' = (e'_1, \ldots, e'_m, e'_{m+1}, \ldots, e'_r)$, where $e'_m < \ell \le e'_{m+1}$. There also exists a unique s such that $e_s < \ell \le e_{s+1}$.

First note that it must be the case that $s \le m$. Indeed, if $s > m$, on one hand by dominance, $e_{s+1} + \cdots + e_r \le e'_{s+1} + \cdots + e'_r$, but on the other hand it's supposed to contribute the same amount of h^0 as $e'_{m+1}, \ldots, e'_s, e'_{s+1}, \ldots, e'_r$ after twisted down by ℓ, which is not possible.

Next we claim that $e_{s+1} + \cdots + e_r = e'_{s+1} + \cdots + e'_r$. We have $\le$ by dominance. (For this computation, assume that $\ell = 0$, so $e_{s+1} \ge 0 > e_s$, and $e'_{m+1} \ge 0 > e_m$.) But at the same time,

$$e'_{m+1} + \cdots + e'_r = h^0(\mathcal{O}(\vec{e}')) - (r-m) = h^0\mathcal{O}(\vec{e}) - (r-m) = e_{s+1} + \cdots + e_r + (m-s).$$

Since $e'_m < 0$, we also have

$$e'_{s+1} + \cdots + e'_r \le e'_{m+1} + \cdots + e'_r - (m-s) = e_{s+1} + \cdots + e_r.$$

So in fact, if $m > s$, then $e'_k = \ell - 1$ for all $s+1 \le k \le m$.

To see that this map is surjective, given a cut, then one critical twist that would satisfy $h^1\mathcal{O}(\vec{e}')(-\ell) = h^1\mathcal{O}(\vec{e})(-\ell)$ is $\ell = e'_{s+1} + 1$. □

First, suppose that there exists an admissible cut $(s, s+1)$ for $(\vec{e}, \vec{e}')$. Then we can do induction. Let $\vec{b} = (\vec{b}_1, \vec{b}_2)$ be the splitting type obtained by balancing $(e_1, \ldots, e_s)$ and $(e_{s+1}, \ldots, e_r)$ respectively. Then the splitting locus $\overline{\Sigma}_{\vec{b}}$ in $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ is the product $H^1\mathcal{E}nd(\mathcal{O}((e_1', \ldots, e_s'))) \times H^1\mathcal{E}nd(\mathcal{O}(e_{s+1}', \ldots, e_r'))$ (e.g. because $\overline{\Sigma}_{\vec{b}}$ is an irreducible smooth subvariety of codimension equal to the codimension of this product). Moreover, $\overline{\Sigma}_{\vec{e}}$ is precisely the product of the splitting loci for $(e_1, \ldots, e_s)$ and $(e_{s+1}, \ldots, e_r)$ in these two extension spaces. This shows the first part of Theorem 4.1.10.

From now on we assume that there are no admissible cuts for $(\vec{e}, \vec{e}')$.

Proposition 4.4.2. If there are no nontrivial cuts for $(\vec{e}, \vec{e}')$, then $T(\vec{e}, \vec{e}') = 0$.

Proof. As explained in Lemma 4.4.1, if there are no cuts, then there is no critical twist where $h^1\mathcal{O}(\vec{e}')(-\ell) = h^1\mathcal{O}(\vec{e})(-\ell)$. The latter is precisely when the rank condition requires us to take 1×1 minors of the section matrix M_ℓ to cut out $\overline{\Sigma}_{\vec{e}}$. So if there are no cuts, then all the rank conditions are either trivial, or would require us to take minors of size 2 or more of the section matrices. In particular, either $\vec{e}$ is balanced, or the ideal cutting out $\vec{e}$ is contained in $\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal for the multigraded polynomial ring whose spectrum is $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$. $\square$

Next, let us suppose that there are nontrivial cuts for $(\vec{e}, \vec{e}')$. Then $\vec{e}$ cannot be balanced. By assumption, none of the cuts is admissible. We shall compute $T(\vec{e}, \vec{e}')$ in this case, and prove that $\vec{e}'$ is always in the singular locus of $\overline{\Sigma}_{\vec{e}}$.

After applying all the cuts for the pair, we can write $\vec{e} = (\vec{c}_m, \vec{c}_{m-1}, \ldots, \vec{c}_1)$, and

$$\vec{e}' = (\vec{b}_m, g_{m-1}^{\rho_{m-1}}, g_{m-1}^{\ell_{m-1}}, \vec{b}_{m-1}, g_{m-2}^{\rho_{m-2}}, g_{m-2}^{\ell_{m-2}}, \ldots, g_2^{\rho_2}, g_2^{\ell_2}, \vec{b}_2, g_1^{\rho_1}, g_1^{\ell_1}, \vec{b}_1),$$

where $g_i \in \mathbb{Z}$, $\operatorname{len}(\vec{c}_1) = \operatorname{len}(\vec{b}_1) + \ell_1$, $\operatorname{len}(\vec{c}_2) = \operatorname{len}(\vec{b}_2) + \ell_2 + \rho_1$, etc. We have used colors to highlight the location of the cuts. (If it helps, the letters ℓ, ρ are chosen to signify left and right.) We will also use $\vec{c}_i' = (g_i^{\ell_i}, \vec{b}_i, g_{i-1}^{\rho_{i-1}})$, where $\rho_0 = \ell_m = 0$. We also know that $\deg(\vec{c}_i) = \ell_i g_i + \deg(\vec{b}_i) + \rho_{i-1} g_{i-1}$, $\rho_i, \ell_i > 0$, and $\vec{c}_{i,\text{last}} < \vec{c}_{i-1,1}$. The reason we index in reverse is because when we take the corresponding flag for $\mathcal{O}(\vec{e})$, we have

$$F_\bullet : \mathcal{O}(\vec{c}_1) \subseteq \mathcal{O}(\vec{c}_2, \vec{c}_1) \subseteq \cdots \subseteq \mathcal{O}(\vec{e}),$$

so that

$$\operatorname{gr}_i F_\bullet = E_i/E_{i-1} \cong \mathcal{O}(\vec{c}_i).$$

First, we claim that we may reduce to the case where $\vec{c}_i$ is balanced for all i. For the moment, let us write $\vec{f} = (bal(\vec{c}_m), \ldots, bal(\vec{c}_1))$. Let $\rho : \widetilde{\Sigma}_{\vec{f}} \twoheadrightarrow \overline{\Sigma}_{\vec{f}} \subseteq H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ be the resolution of singularities parametrizing flags of numerical type specified by the blocks in $\vec{f} = (bal(\vec{c}_m), \ldots, bal(\vec{c}_1))$. On $\widetilde{\Sigma}_{\vec{f}} \times \mathbb{P}^1$, we have a universal flag of subbundles

$$\mathcal{F}_\bullet : \mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \mathcal{S}_m \cong \mathcal{E}.$$

Every splitting type present in $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ is at least $\vec{e}'$. As such, each $\operatorname{gr}_i \mathcal{F}_\bullet = \mathcal{S}_i/\mathcal{S}_{i-1}$ is a vector bundle of rank $\operatorname{rk} \vec{c}_i$ and fiberwise degree $\deg \vec{c}_i$.

In fact more is true:

Lemma 4.4.3. Let $x = F_\bullet \in \widetilde{\Sigma}_{\vec{f}}$ be a flag, where $F_\bullet$ is the flag $S_1 \subset S_2 \subset \cdots \subset S_m \cong \mathcal{E}_x$. Then

$$\begin{aligned}\operatorname{Ext}^1_+(\mathcal{E}_x, \mathcal{E}_x) &\cong \bigoplus_{1 \le i < j \le m} \operatorname{Ext}^1(S_i/S_{i-1}, S_j/S_{j-1}),\\ \operatorname{Ext}^1_-(\mathcal{E}_x, \mathcal{E}_x) &\cong \bigoplus_{1 \le i \le m} \operatorname{Ext}^1(S_i/S_{i-1}, S_i/S_{i-1}),\end{aligned}$$

and the sequence

$$0 \to \operatorname{Ext}^1_-(\mathcal{E}_x, \mathcal{E}_x) \to \operatorname{Ext}^1(\mathcal{E}_x, \mathcal{E}_x) \to \operatorname{Ext}^1_+(\mathcal{E}_x, \mathcal{E}_x) \to 0. \tag{4.1}$$

is exact. Note that all filtered Ext's are taken with respect to the flag $F_\bullet$.

Proof. There are spectral sequences computing $\operatorname{Ext}^i_\pm(\mathcal{E}_p, \mathcal{E}_p)$. In general, given a coherent sheaf $\mathcal{E}$ with a filtration, let

$$E_1^{p,q} = \begin{cases} \oplus_i \operatorname{Ext}^{p+q}(\operatorname{gr}_i \mathcal{E}, \operatorname{gr}_{i-p} \mathcal{E}) & p \ge 0 \\ 0 & \text{otherwise} \end{cases},$$

then the spectral sequence whose E_1-page is given by these terms computes $\operatorname{Ext}^{p+q}_-(\mathcal{E}, \mathcal{E})$ in degree $p+q$. Similarly, if we let

$$E_1^{p,q} = \begin{cases} \oplus_i \operatorname{Ext}^{p+q}(\operatorname{gr}_i \mathcal{E}, \operatorname{gr}_{i-p} \mathcal{E}) & p < 0 \\ 0 & \text{otherwise} \end{cases},$$

then the spectral sequence whose E_1-page is given by these terms computes $\operatorname{Ext}^{p+q}_+(\mathcal{E}, \mathcal{E})$ in degree $p+q$ [Pot97, Section 15.3].

Let us now specialize these spectral sequences to our situation. Since every splitting type present in $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ is at least $\vec{e}'$, $\operatorname{gr}_i := \operatorname{gr}_i F_\bullet = S_i/S_{i-1}$ is a vector bundle of rank $\operatorname{rk} \vec{c}_i$ and degree $\deg \vec{c}_i$. For both spectral sequences, since we are working over $\mathbb{P}^1$, there are only nonzero terms when $p+q=0$ or $p+q=1$.

For $\operatorname{Ext}^\bullet_+(\mathcal{E}_x, \mathcal{E}_x)$, the only nonzero terms are $E_1^{-p,p+1}$ where $p<0$ and $E_1^{-1,1}$. So the spectral sequence abuts on the E_1-page, showing that

$$\operatorname{Ext}^0_+(\mathcal{E}_x, \mathcal{E}_x) = \oplus_i \operatorname{Hom}(\operatorname{gr}_i, \operatorname{gr}_{i+1}),$$

and

$$\operatorname{Ext}^1_+(\mathcal{E}_x, \mathcal{E}_x) = \oplus_{1 \le i < j \le m} \operatorname{Ext}^1(\operatorname{gr}_i, \operatorname{gr}_j).$$

Note that only when $\rho(x) = 0$, $\operatorname{Ext}^0_+(\mathcal{E}_x, \mathcal{E}_x)$ can be nonzero.

Similarly, for $\operatorname{Ext}^\bullet_-(\mathcal{E}_x, \mathcal{E}_x)$, the only nonzero terms are $E_1^{p,-p}$ and $E_1^{0,1}$. As such, this spectral sequence also abuts on the E_1-page, showing that

$$\operatorname{Ext}^1_-(\mathcal{E}_x, \mathcal{E}_x) = \oplus_i \operatorname{Ext}^1(\operatorname{gr}_i, \operatorname{gr}_i),$$

$$\operatorname{Ext}^0_-(\mathcal{E}_x, \mathcal{E}_x) = \oplus_{1 \le j \le i \le m} \operatorname{Hom}(\operatorname{gr}_j, \operatorname{gr}_i).$$

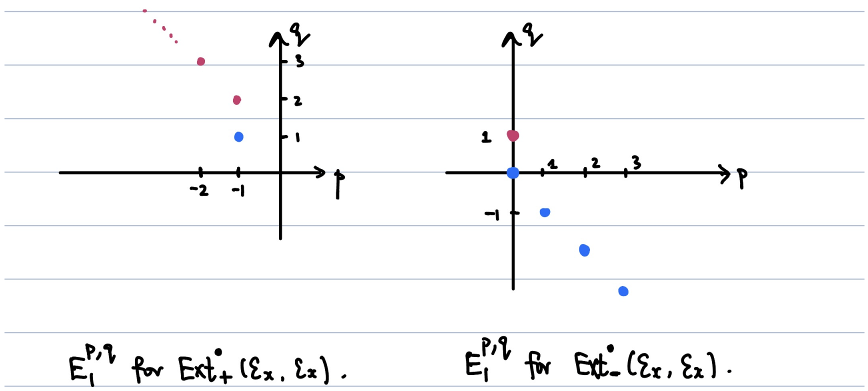


Figure 4.1: E_1-page of the spectral sequences for $\mathrm{Ext}^\bullet_\pm(\mathcal{E}_x, \mathcal{E}_x)$; dots denote nonzero terms

The spectral sequences are illustrated in Figure 4.1.

Using these descriptions of $\mathrm{Ext}^1_\pm(\mathcal{E}_x, \mathcal{E}_x)$, we see that $\mathrm{Ext}^1(\mathcal{E}_x, \mathcal{E}_x)$ can be naturally written as a direct sum of $\mathrm{Ext}^1_-(\mathcal{E}_x, \mathcal{E}_x)$ and $\mathrm{Ext}^1_+(\mathcal{E}_x, \mathcal{E}_x)$, so that the natural maps in the sequence (4.1) are simply the inclusion of and projection onto a summand, making the sequence evidently exact. $\square$

Lemma 4.4.4. When $\rho(p) = 0 \in H^1\mathcal{E}nd(\mathcal{O}(\vec{e}\,'))$, the image of the tangent map is $\rho_*(T_p\widetilde{\Sigma}_{\vec{f}}) = \mathrm{Ext}^1_-(\mathcal{O}(\vec{e}\,'), \mathcal{O}(\vec{e}\,'))$.

Proof. As explained in Proposition 3.3.7, there is a right exact sequence

$$T_p\widetilde{\Sigma}_{\vec{f}} \xrightarrow{\rho_*} \mathrm{Ext}^1(\mathcal{O}(\vec{e}\,'), \mathcal{O}(\vec{e}\,')) \to \mathrm{Ext}^1_+(\mathcal{O}(\vec{e}\,'), \mathcal{O}(\vec{e}\,')) \to 0,$$

where the second map is the natural map from Ext^1 to Ext^1_+. We proved in the previous lemma that the kernel of this map, which is the image of $T_p\widetilde{\Sigma}_{\vec{f}}$, is precisely $\mathrm{Ext}^1_-(\mathcal{O}(\vec{e}\,'), \mathcal{O}(\vec{e}\,'))$. $\square$

Lemma 4.4.5. The map $\widetilde{\Sigma}_{\vec{f}} \to \mathcal{B}_{\mathrm{rk}\,\vec{c}_i, \deg \vec{c}_i}$ induced by the bundle $\mathrm{gr}_i \mathcal{F}_\bullet = \mathcal{S}_i/\mathcal{S}_{i-1}$ is smooth.

Proof. Since $\widetilde{\Sigma}_{\vec{f}}$ is smooth, it suffices to show that at each $p = F_\bullet \in \widetilde{\Sigma}_{\vec{f}}$, $T_p\widetilde{\Sigma}_{\vec{f}}$ surjects onto $\mathrm{Ext}^1(\mathrm{gr}_i F_\bullet, \mathrm{gr}_i F_\bullet)$. This follows from the previous lemma, where we showed that the tangent space surjects onto $\mathrm{Ext}^1_-(\mathcal{E}_p, \mathcal{E}_p) \cong \oplus_{i=1}^m \mathrm{Ext}^1(\mathrm{gr}_i F_\bullet, \mathrm{gr}_i F_\bullet)$. $\square$

Lemma 4.4.6. In the case of a nice pair $(\vec{f}, \vec{e}\,')$, let $\rho : \widetilde{\Sigma}_{\vec{f}} \to \overline{\Sigma}_{\vec{f}} \subseteq H^1\mathcal{E}nd(\mathcal{O}(\vec{e}\,'))$. Let $E = \rho^{-1}(0)$, and let $N_{E,\widetilde{\Sigma}_{\vec{f}}}, T_{0,\overline{\Sigma}_{\vec{f}}}$ denote the normal cone and tangent cone for the closed embeddings $E \hookrightarrow \widetilde{\Sigma}_{\vec{f}}$ and $\{0\} \hookrightarrow \overline{\Sigma}_{\vec{f}}$ respectively. Then the induced map $\rho' : N_{E,\widetilde{\Sigma}_{\vec{f}}} \to T_{0,\overline{\Sigma}_{\vec{f}}}$ is surjective and a rational resolution.

Proof. Let V be the image of ρ'. Note that in this case, E is a smooth subvariety, and so the normal cone is the total space of the normal bundle for $E \hookrightarrow \widetilde{\Sigma}_{\vec{f}}$. By [Kem73, Proposition

1], it suffices to show that $N_{E,\widetilde{\Sigma}_{\vec{f}}} \to V$ is a rational resolution, i.e. a proper birational map exhibiting V as having rational singularities, and in particular normal. This follows from our explicit description in Section 4.5, which says that the map ρ' is a collapsing of homogeneous bundles, and a theorem of Kempf [Kem76, Theorem 0], which says that such maps are rational resolutions. □

Lemma 4.4.7. $T(\vec{e},\vec{e}') = T(\vec{f},\vec{e}')$.

Proof. Let us work in the setting of $B = H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$, so that it is equivalent to show that $\dim T_0\overline{\Sigma}_{\vec{e}} = \dim T_0\overline{\Sigma}_{\vec{f}}$. Since $\overline{\Sigma}_{\vec{e}} \subseteq \overline{\Sigma}_{\vec{f}}$, we certainly have $\dim T_0\overline{\Sigma}_{\vec{e}} \leq \dim T_0\overline{\Sigma}_{\vec{f}}$. It suffices to show the other inequality.

We have a factorization

$$\widetilde{\Sigma}_{\vec{e}} \xrightarrow{\psi} \widetilde{\Sigma}_{\vec{f}} \xrightarrow{\rho} B = H^1\mathcal{E}nd(\mathcal{O}(\vec{e}')),$$

so that the image $\psi(\widetilde{\Sigma}_{\vec{e}}) \subseteq \widetilde{\Sigma}_{\vec{f}}$ contains $\rho^{-1}(0)$, and it is precisely the intersection of the splitting loci $\overline{\Sigma}_{\vec{c}_i}(\mathcal{S}_i/\mathcal{S}_{i-1})$. Since there are no cuts for $\vec{c}_i \geq \vec{c}_i'$, by Proposition 4.4.2, $T_p\overline{\Sigma}_{\vec{c}_i}(\mathcal{S}_i/\mathcal{S}_{i-1}) = T_p\widetilde{\Sigma}_{\vec{f}}$ for each i and all $p \in \rho^{-1}(0)$. So $T_p\widetilde{\Sigma}_{\vec{e}} = T_p \bigcap_{i=1}^m \overline{\Sigma}_{\vec{c}_i}(\mathcal{S}_i/\mathcal{S}_{i-1}) = T_p\widetilde{\Sigma}_{\vec{f}}$ for all $p \in \rho^{-1}(0)$. In particular,

$$T_0\overline{\Sigma}_{\vec{e}} \supseteq \operatorname{span}\{\rho_*T_p\psi(\widetilde{\Sigma}_{\vec{e}}) : p \in \rho^{-1}(0)\} = \operatorname{span}\{\rho_*T_p\widetilde{\Sigma}_{\vec{f}} : p \in \rho^{-1}(0)\} = T_0\overline{\Sigma}_{\vec{f}},$$

where the last equality uses Lemma 4.4.6. We conclude that $\dim T_0\overline{\Sigma}_{\vec{e}} \geq \dim T_0\overline{\Sigma}_{\vec{f}}$ as desired. □

Remark 4.4.8. Without using Kempf's theorem, i.e. without assuming characteristic zero, we only know that $\dim T_0\overline{\Sigma}_{\vec{e}} \geq \dim \operatorname{span}\{\rho_*T_p\widetilde{\Sigma}_{\vec{f}} : p \in \rho^{-1}(0)\}$. The argument below will show that

$$\dim \operatorname{span}\{\rho_*T_p\widetilde{\Sigma}_{\vec{f}} : p \in \rho^{-1}(0)\} > \dim \overline{\Sigma}_{\vec{f}} \geq \dim \overline{\Sigma}_{\vec{e}},$$

which also completes the proof of Theorem 4.1.8.

So from now on we assume that $(\vec{e},\vec{e}')$ is a nice pair, i.e. each $\vec{c}_i$ is balanced.

Proposition 4.4.9. Let $t_i = \rho_i + \ell_i$, with $t_0 = t_{m+1} = 0$. Then the tangent space of $\overline{\Sigma}_{\vec{e}} \subseteq H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ at $\vec{e}'$ is equal to $\oplus_{i=1}^m H^1\mathcal{E}nd\ \mathcal{O}(g_i^{t_i}, \vec{b}_i, g_{i-1}^{t_{i-1}})$.

Proof. Over the special fiber $\rho^{-1}(0)$, the degree constraints makes it so that $\mathcal{S}_i/\mathcal{S}_{i-1}$ always splits as $\vec{c}_i'$. Therefore, to choose the subbundle $F_i \subset F_{i+1}$ is the same as choosing $\mathcal{O}(g_i^{\ell_i}) \subseteq \mathcal{O}(g_i^{t_i})$, which is parametrized by the $\operatorname{Gr}(\ell_i, t_i)$. Therefore in fact the special fiber $\rho^{-1}(0) \cong \Pi_{i=1}^{m-1}\operatorname{Gr}(\ell_i, t_i)$. For a fixed flag $p = F_\bullet \in \rho^{-1}(0)$, Lemma 4.4.4 shows that

$$\rho_*(T_p\widetilde{\Sigma}_{\vec{e}}) = \operatorname{Ext}^1_-(\mathcal{E}_p, \mathcal{E}_p) \cong \oplus_{i=1}^m H^1\mathcal{E}nd\ \mathcal{O}(g_i^{\ell_i}, \vec{b}_i, g_{i-1}^{\rho_{i-1}}).$$

As we range over all $p \in \rho^{-1}(0)$, the span of all $\rho_*(T_p\widetilde{\Sigma}_{\vec{e}})$ is exactly $\oplus_{i=1}^m H^1\mathcal{E}nd\ \mathcal{O}(g_i^{t_i}, \vec{b}_i, g_{i-1}^{t_{i-1}})$. Lastly, by Lemma 4.4.6, the span is in fact all of $T_0\overline{\Sigma}_{\vec{e}}$. □

At this point, we have proven Theorem 4.1.10. To complete the proof of Theorem 4.1.8, it remains to show that when $(\vec{e}, \vec{e}')$ is a nice pair, we have the inequality

$$u(\vec{e}) > T(\vec{e}, \vec{e}').$$

In the following computation, we will write $\mathrm{ext}^i_{\pm}(\mathcal{O}(\vec{e}), \mathcal{O}(\vec{e}))$ to denote the dimension of filtered Exts taking with respect to the unique filtration corresponding to the inadmissible cuts for $(\vec{e}, \vec{e}')$. We will also write $\mathrm{ext}^i_{\pm}(\mathcal{O}(\vec{e}'), \mathcal{O}(\vec{e}'))$ to denote the dimension of filtered Exts taking with respect to any filtration in $\rho^{-1}(0)$. Note that the dimensions do not depend on the choice of a point in $\rho^{-1}(0)$.

Let $D = \oplus_{i=1}^m H^1\mathcal{E}nd\ \mathcal{O}(g_i^{t_i}, \vec{b}_i, g_{i-1}^{t_{i-1}})$. Then

$$\begin{aligned}
&u(\vec{e}) - T(\vec{e}, \vec{e}') \\
&= u(\vec{e}) - u(\vec{e}') + D \\
&= \mathrm{ext}^1_+(\mathcal{O}(\vec{e}), \mathcal{O}(\vec{e})) - \left[\mathrm{ext}^1_+(\mathcal{O}(\vec{e}'), \mathcal{O}(\vec{e}')) + \mathrm{ext}^1_-(\mathcal{O}(\vec{e}'), \mathcal{O}(\vec{e}'))\right] + D \\
&= \left[\mathrm{ext}^1_+(\mathcal{O}(\vec{e}'), \mathcal{O}(\vec{e}')) - \mathrm{ext}^0_+(\mathcal{O}(\vec{e}'), \mathcal{O}(\vec{e}'))\right] - \mathrm{ext}^1_+(\mathcal{O}(\vec{e}'), \mathcal{O}(\vec{e}')) - \mathrm{ext}^1_-(\mathcal{O}(\vec{e}'), \mathcal{O}(\vec{e}')) + D \\
&= D - \mathrm{ext}^1_-(\mathcal{O}(\vec{e}'), \mathcal{O}(\vec{e}')) - \mathrm{ext}^0_+(\mathcal{O}(\vec{e}'), \mathcal{O}(\vec{e}')).
\end{aligned}$$

In other words, it suffices to show that

$$\oplus_{i=1}^m H^1\mathcal{E}nd\ \mathcal{O}(g_i^{t_i}, \vec{b}_i, g_{i-1}^{t_{i-1}}) - \oplus_{i=1}^m H^1\mathcal{E}nd\ \mathcal{O}(g_i^{\ell_i}, \vec{b}_i, g_{i-1}^{\rho_{i-1}}) > \mathrm{ext}^0_+(\mathcal{O}(\vec{e}'), \mathcal{O}(\vec{e}')).$$

The right hand side can be simplified further:

$$\begin{aligned}
\mathrm{ext}^0_+(\mathcal{O}(\vec{e}'), \mathcal{O}(\vec{e}')) &= \sum_{1 \le i < j \le m} \mathrm{hom}(\mathcal{O}(g_i^{\ell_i}, \vec{b}_i, g_{i-1}^{\rho_{i-1}}), \mathcal{O}(g_j^{\ell_j}, \vec{b}_j, g_{j-1}^{\rho_{j-1}})) \\
&= \sum_{i=1}^{m-1} \ell_i \rho_i.
\end{aligned}$$

Recall that

$$(\vec{c}'_{i+1}, \vec{c}'_i) = (g_{i+1}^{\ell_{i+1}}, \vec{b}_{i+1}, g_i^{\rho_i}, g_i^{\ell_i}, \vec{b}_i, g_{i-1}^{\rho_{i-1}}).$$

Now for all $1 \le i < m$, define

$$F_{i,i+1} = \mathrm{ext}^1(g_i^{\ell_i}, (g_{i+1}^{\ell_{i+1}}, \vec{b}_{i+1})) + \mathrm{ext}^1((\vec{b}_i, g_{i-1}^{\rho_{i-1}}), g_i^{\rho_i}) - \rho_i \ell_i.$$

Then our computation above shows that

$$u(\vec{e}) - T(\vec{e}, \vec{e}') \ge \sum_{i=1}^{m-1} F_{i,i+1}.$$

Let us assume that $g_i = 0$, which we can achieve up to a shift, so that

$$(\vec{c}'_{i+1}, \vec{c}'_i) = (g_{i+1}^{\ell_{i+1}}, \vec{b}_{i+1}, 0^{\rho_i}, 0^{\ell_i}, \vec{b}_i, g_{i-1}^{\rho_{i-1}}).$$

Let $d = -(\deg \vec{b}_{i+1} + \ell_{i+1} g_{i+1})$, $e = \deg \vec{b}_i + \rho_{i-1} g_{i-1}$. Let $r_i = \mathrm{rk}\, \vec{b}_i + \rho_{i-1}$, $r_{i+1} = \mathrm{rk}\, \vec{b}_{i+1} + \ell_{i+1}$.

Lemma 4.4.10. $F_{i,i+1} \geq 0$, with equality if and only if up to a shift, either

1. $(\vec{c}_{i+1}, \vec{c}_i) = ((-1)^{r_{i+1}+\rho_i}, 0^{\ell_i}, 1^{r_i})$, and $(\vec{c}'_{i+1}, \vec{c}'_i)$ results from unbalancing $(-1)^{r_{i+1}+\rho_i}$, or

2. $(\vec{c}_{i+1}, \vec{c}_i) = ((-1)^{r_{i+1}}, 0^{\rho_i}, 1^{r_i+\ell_i})$, $(\vec{c}'_{i+1}, \vec{c}'_i)$ results from unbalancing $1^{r_i+\ell_i}$.

Proof. Note that by construction, $d \geq r_{i+1}$ and $e \geq r_i$. Then

$$\begin{aligned} F_{i,i+1} &= -\ell_i \chi(\mathcal{O}(g_{i+1}^{\ell_{i+1}}, \vec{b}_{i+1})) - \rho_i \chi(-(\vec{b}_i, g_{i-1}^{\rho_{i-1}})) - \rho_i \ell_i \\ &= -\ell_i(-d + r_{i+1}) - \rho_i(-e + r_i) - \rho_i \ell_i \\ &= \ell_i(d - r_{i+1} - \rho_i) + \rho_i(e - r_i) \\ &= \ell_i(d - r_{i+1}) + \rho_i(e - r_i - \ell_i). \end{aligned}$$

If $e \geq r_i + \ell_i$ and $d \geq r_{i+1} + \rho_i$, then this difference is strictly positive and $\overline{\Sigma}_{\vec{e}}$ is singular at $\vec{e}'$.

Now suppose that $r_i \leq e < r_i + \ell_i$, in which case $\vec{c}_i = (0, \ldots, 0, 1^e)$. Then in fact since $c_{i+1,\text{last}} < c_{i,1} = 0$, we must have that $d \geq r_{i+1} + \rho_i$. If $d > r_{i+1} + \rho_i$, then again $F_{i,i+1} > 0$. But if $d = r_{i+1} + \rho_i$, we have $\vec{c}_{i+1} = ((-1)^{r_{i+1}+\rho_i})$, and $F_{i,i+1} = \rho_i(e - r_i)$, which is zero if and only if $e = r_i$. This is the case where $(\vec{c}_{i+1}, \vec{c}_i) = ((-1)^{r_{i+1}+\rho_i}, 0^{\ell_i}, 1^{r_i})$, and $\vec{e}'$ results from unbalancing $(-1)^{r_1+\ell}$.

Similarly, if we consider the possibility where $r_{i+1} \leq d < r_{i+1} + \rho_i$, we will find that $\vec{e}'$ is in the regular locus only if $e = r_i + \ell_i$ and $d = r_{i+1}$. □

However, the special cases of the lemma above cannot in fact occur. Indeed, in the special cases detailed by the lemma, there is an admissible cut that is not accounted for. For example, in case (1), the gap between 0^{ℓ_i} and 1^{r_i} is an admissible cut. Therefore, we actually always have $F_{i,i+1} > 0$. This concludes the proof of Theorem 4.1.8.

4.5 Normal Cone Description

When $(\vec{e}, \vec{e}')$ is a nice pair, we shall give a description of $N_{\vec{e},\vec{e}'}$, which is the normal cone of $\{0\}$ in $\overline{\Sigma}_{\vec{e}}$ for the family over $H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$. As explained in Section 4.2, the universal normal cone $\mathbf{N}_{\vec{e},\vec{e}'}$ is the quotient of $N_{\vec{e},\vec{e}'}$ by the action of $\mathrm{Aut}(\mathcal{O}(\vec{e}'))$.

Let $\vec{e}, \vec{e}'$ be a nice pair, and inherit the notation from the statement of Theorem 4.1.10:

$$\begin{aligned} \vec{e}' &= (\vec{c}'_m, \ldots, \vec{c}'_1) \\ &= (\vec{b}_m, g_{m-1}^{\rho_{m-1}}, g_{m-1}^{\ell_{m-1}}, \vec{b}_{m-1}, g_{m-2}^{\rho_{m-2}}, g_{m-2}^{\ell_{m-2}}, \ldots, g_2^{\rho_2}, g_2^{\ell_2}, \vec{b}_2, g_1^{\rho_1}, g_1^{\ell_1}, \vec{b}_1). \end{aligned}$$

For $1 \leq i \leq m-1$, let V_{t_i} denote a t_i-dimensional vector space.

Lemma 4.5.1. Let $\rho : \widetilde{\Sigma}_{\vec{e}} \to \overline{\Sigma}_{\vec{e}} \subseteq H^1\mathcal{E}nd(\mathcal{O}(\vec{e}'))$ be the resolution of singularities defined using the $\vec{e}$-admissible flag corresponding to the inadmissible cuts of $(\vec{e}, \vec{e}')$, exhibiting them as a nice pair. Then $\rho^{-1}(0) \cong \prod_{i=1}^{m-1} \mathrm{Gr}(\ell_i, V_{t_i})$.

Proof. $\rho^{-1}(0)$ is the hyper-Quot scheme parametrizing flags

$$\mathcal{O}(\vec{c}'_1) \subseteq \mathcal{O}(\vec{c}'_2, \vec{c}'_1) \subseteq \cdots \subseteq \mathcal{O}(\vec{e}').$$

For $1 \leq i \leq m$, let

$$M_i = \mathcal{O}(\vec{b}_i, g_i^{\rho_{i-1}}, g_i^{\ell_{i-1}}, \ldots, g_1^{\ell_1}, \vec{b}_1) \subseteq \mathcal{O}(\vec{e}'),$$

and for $1 \leq j \leq m-1$, let

$$N_j = \mathcal{O}(g_j^{\rho_j}, g_j^{\ell_j}, \ldots, g_1^{\ell_1}, \vec{b}_1) \subseteq \mathcal{O}(\vec{e}'),$$

which are distinguished subbundles of $\mathcal{O}(\vec{e}')$. Then the choice of a flag as above is equivalent to $m-1$ independent choices of subbundles of the form

$$\mathcal{O}(g_i^{\ell_i}) \subseteq \mathcal{O}(g_i) \otimes V_{t_i} \cong \operatorname{coker}(M_{i-1} \to N_i),$$

which is equivalent to a point of $\mathrm{Gr}(\ell_i, V_{t_i})$. $\square$

Let $0 \to \mathcal{S}_i \to V_{t_i} \otimes \mathcal{O}_{\rho^{-1}(0)} \to \mathcal{Q}_i \to 0$ be the pullback to $\rho^{-1}(0)$ of the universal short exact sequence on $\mathrm{Gr}(\ell_i, V_{t_i})$. Let $F_\bullet \in \rho^{-1}(0)$. Then recall from Proposition 4.4.9 that the tangent map factors as

$$T_{F_\bullet}\widetilde{\Sigma}_{\vec{e}} \to \mathrm{Ext}^1_-(\mathcal{O}(\vec{e}'), \mathcal{O}(\vec{e}')) \hookrightarrow \oplus_{i=1}^m H^1\mathcal{E}nd(\mathcal{O}(g_i^{t_i}, \vec{b}_i, g_{i-1}^{t_{i-1}})) \hookrightarrow H^1\mathcal{E}nd(\mathcal{O}(\vec{e}')),$$

where the second map is a direct sum of the maps

$$H^1\mathcal{E}nd(\mathcal{O}(g_i^{\ell_i}, \vec{b}_i, g_{i-1}^{\rho_{i-1}})) \to H^1\mathcal{E}nd(\mathcal{O}(g_i^{t_i}, \vec{b}_i, g_{i-1}^{t_{i-1}}))$$

for $1 \leq i \leq m$.

After choosing a compatible splitting of the domain and codomain above, we may break this map down further as the direct sum of three maps:

$$\mathrm{Ext}^1(\mathcal{O}(g_{i-1}^{\rho_{i-1}}), \mathcal{O}(g_i^{\ell_i})) \to \mathrm{Ext}^1(\mathcal{O}(g_{i-1}^{t_{i-1}}), \mathcal{O}(g_i^{t_i})),$$

$$\mathrm{Ext}^1(\mathcal{O}(g_{i-1}^{\rho_{i-1}}), \mathcal{O}(\vec{b}_i)) \to \mathrm{Ext}^1(\mathcal{O}(g_{i-1}^{t_{i-1}}), \mathcal{O}(\vec{b}_i)),$$

$$\mathrm{Ext}^1(\mathcal{O}(\vec{b}_i), \mathcal{O}(g_i^{\ell_i})) \to \mathrm{Ext}^1(\mathcal{O}(\vec{b}_i), \mathcal{O}(g_i^{t_i})).$$

Note that each of these maps can be described over all of $\rho^{-1}(0)$ as built from the tautological inclusions and quotients.

Let $d_i = \mathrm{ext}^1(\mathcal{O}(g_{i-1}), \mathcal{O}(g_i))$, $e_i = \mathrm{ext}^1(\mathcal{O}(g_{i-1}), \mathcal{O}(\vec{b}_i))$, $f_i = \mathrm{ext}^1(\mathcal{O}(\vec{b}_i), \mathcal{O}(g_i))$. Let

$$\mathcal{R}_i = (\mathcal{S}_i \otimes \mathcal{Q}_{i-1}^\vee)^{\oplus d_i} \oplus (\mathcal{Q}_{i-1}^\vee)^{\oplus e_i} \oplus \mathcal{S}_i^{\oplus f_i},$$

which is a vector bundle on $\rho^{-1}(0)$. Then there are natural maps $\varphi_i : \mathrm{Tot}(\mathcal{R}_i) \to T_i$, where

$$T_i = \mathrm{Hom}(V_{t_{i-1}}, V_{t_i})^{\oplus d_i} \oplus (V_{t_{i-1}}^\vee)^{\oplus e_i} \oplus V_{t_i}^{\oplus f_i}$$

is an affine space over $\operatorname{Spec} k$. For example, we have a natural map of vector bundles over $\rho^{-1}(0)$: $\mathcal{H}om(\mathcal{Q}_{i-1}, \mathcal{S}_i) \to \mathrm{Hom}(V_{t_{i-1}}, V_{t_i}) \otimes \mathcal{O}_{\rho^{-1}(0)}$, which we can then compose with the projection to the affine space $\mathrm{Hom}(V_{t_{i-1}}, V_{t_i})$.

In fact, this is an example of a collapsing of a homogeneous bundle in the sense of Kempf [Kem76, Theorem 0]. To apply their result, the group G is $\prod \mathrm{GL}(V_{t_i})$, which acts on $\mathbb{W} = \oplus_{i=1}^m T_i$, P is $\prod P_i$, so that $G/P = \prod_i \mathrm{Gr}(\ell_i, V_{t_i})$. $\mathbb{V}$ is the P-invariant subspace of $\mathbb{W}$ which pulls back to $\oplus_{i=1}^m \mathcal{R}_i$ along $G/P \to BP$. Note that their $G \times^P \mathbb{V}$ is $\mathrm{Tot}(\oplus_{i=1}^m \mathcal{R}_i)$ in our case. By the theorem of Kempf cited above, to complete the proof of Lemma 4.4.6, it suffices to demonstrate that the map $\mathrm{Tot}(\oplus_{i=1}^m \mathcal{R}_i) \to \oplus_{i=1}^m T_i$ is birational onto its image. Since $\mathrm{Tot}(\oplus_{i=1}^m \mathcal{R}_i)$ is irreducible, the image is irreducible as well. So it suffices to notice that, first, the fiber over each point of $\rho^{-1}(0)$ maps isomorphically onto its image. Second, for a general point of the image, the maps in $\mathrm{Hom}(V_{t_{i-1}}, V_{t_i})$ are all of maximal rank, i.e. rank $\rho_{i-1}\ell_i$, so that there is only one choice of a point in $\rho^{-1}(0)$, namely one must take the point of $\mathrm{Gr}(\ell_{i-1}, V_{t_{i-1}})$ to be the kernel of some (equivalently any) map in a $\mathrm{Hom}(V_{t_{i-1}}, V_{t_i})$ summand.

As the result of the discussion above and Lemma 4.4.6, we have the following theorem:

Theorem 4.5.2. $N_{\vec{e},\vec{e}'}$ is the image of $\mathrm{Tot}(\oplus_{i=1}^m \mathcal{R}_i) \to \oplus_{i=1}^m T_i$. □

In general, it is possible to write down determinantal equations for this image. Indeed, a point of $\oplus_{i=1}^m T_i$ lies in the image of $\mathrm{Tot}(\oplus_{i=1}^m \mathcal{R}_i)$ if and only if there exists a collection of subspaces $W_i \subset V_{t_i}$, such that the f_i vectors in V_{t_i} all lie in W_i, the e_i functionals on $V_{t_{i-1}}$ all have W_i in their kernels, and the d_i maps $V_{t_{i-1}} \to V_{t_i}$ all factor through $V_{t_{i-1}}/W_{i-1}$ and map into W_i. We illustrate this in two simple examples.

Example 4.5.3. Let $\vec{e}' = (-2, 0, 0, 2)$, and let $\vec{e} = (-1, -1, 1, 1)$. Then $\rho^{-1}(0) \cong \mathbb{P}^1$. Let

$$0 \to \mathcal{S} \to V \to \mathcal{Q} \to 0$$

be the universal sequence on $\mathbb{P}^1$, where $V = V_{t_1}$ is a 2-dimensional vector space. Then the normal cone $N_{\vec{e},\vec{e}'}$ is the image of $\mathrm{Tot}(\mathcal{S} \oplus \mathcal{Q}^\vee) \to V \oplus V^\vee$. A point of the target is a pair (v, f) where $v \in V$ and $f \in V^\vee$, which lie in the image of this map if and only if $f(v) = 0$. So in this case, $N_{\vec{e},\vec{e}'}$ is the cone over a smooth quadric in $\mathbb{P}^3$.

Example 4.5.4. Let $\vec{e}' = (-3, -1, -1, 1, 1, 3)$, $\vec{e} = (-2, -2, 0, 0, 2, 2)$. Then $u(\vec{e}) - u(\vec{e}') = 5$, so we expect the normal cone to be a 5-fold.

The fiber over 0 is $\rho^{-1}(0) \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $V = V_{t_1}$ and let $V' = V_{t_2}$. Let the two universal sequences be

$$0 \to \mathcal{S}_1 \to V \to \mathcal{Q}_1 \to 0,$$
$$0 \to \mathcal{S}_2 \to V' \to \mathcal{Q}_2 \to 0.$$

Then the tangent cone is the image of

$$\mathcal{S}_1 \oplus \mathcal{S}_2 \otimes \mathcal{Q}_1^\vee \oplus \mathcal{Q}_2^\vee \to V \oplus \mathrm{Hom}(V, V') \oplus V'^\vee.$$

If we let a point of the target be $(v \in V, \psi : V \to V', f : V' \to k)$, then the image is cut out by the equations $\mathrm{rk}\,\psi \leq 1$, $\psi(v) = 0$, $f \circ \psi = 0$. Over the open locus where $\mathrm{rk}\,\psi = 1$, this image is the total space of the rank two vector bundle which is the direct sum of the kernel and the dual of the quotient for the universal map $V \to V'$.

Chapter 5

A Borel–Weil–Bott theorem for Quot schemes on $\mathbb{P}^1$

5.1 Introduction

Let C be a smooth projective curve and V a vector bundle of rank n on C. Let $\mathsf{Quot}_d(C, V, r)$ denote the Quot scheme that parameterizes short exact sequences of sheaves

$$0 \to S \to V \to Q \to 0 \quad \text{with } \deg Q = d, \text{ rank } Q = r.$$

Consider the universal exact sequence over the product $\mathsf{Quot}_d(C, V, r) \times C$,

$$0 \to \mathcal{E} \to p^*V \to \mathcal{F} \to 0 \tag{5.1}$$

where p and π denote the projection maps to C and $\mathsf{Quot}_d(C, V, r)$, respectively. Given vector bundles K and M on C, the associated tautological complexes on $\mathsf{Quot}_d(C, V, r)$ are defined as the pushforwards

$$K^{\{d\}} := R\pi_*(p^*K \otimes \mathcal{E}), \qquad M^{[d]} := R\pi_*(p^*M \otimes \mathcal{F}).$$

When $K^{\{d\}}$ and $M^{[d]}$ are vector bundles, which occurs when the bundles K, M are sufficiently positive, we call them tautological bundles.

The study of the Quot scheme $\mathsf{Quot}_d(C, V, r)$ comes in two distinct flavors. When $r = 0$, i.e. punctual Quot schemes, $\mathsf{Quot}_d(C, V, 0)$ is a smooth projective variety of dimension nd, and $L^{[d]}$ is a vector bundle of rank d for all line bundles L on C. In this case, the geometry of the Quot scheme has close analogies with the Hilbert scheme of points on surfaces, manifesting, for instance, in the cohomological behavior of tautological bundles. A closed formula for the Euler characteristic of the exterior powers of $L^{[d]}$ was computed in [OS23], and a description of their cohomology groups was conjectured. This conjecture was proved in [MOS22] for $C \cong \mathbb{P}^1$. For arbitrary genus, it was partially proved in [Kru24], and fully resolved in [MN24] by studying the derived category of the Quot scheme.

When $r > 0$, many of the techniques available in the punctual case no longer apply, and not much is known about the cohomology of tautological bundles, except for some Euler characteristic calculations in [OS23; SZ24a; SZ24b]. In this article, we study the cohomology of tautological complexes on $\mathsf{Quot}_d(\mathbb{P}^1, V, r)$ for any vector bundle V and any r. Our theorems may be viewed as Borel–Weil–Bott–type results for Quot schemes on $\mathbb{P}^1$ in direct analogy with the classical theorem for Grassmannians.

Cohomology of tautological bundles

We begin by fixing notation. For a partition $\lambda = (\lambda_1, \dots, \lambda_k)$, let $|\lambda|$ denote the sum of its parts, and let $\lambda^\dagger$ be its conjugate partition. We write $\mathbf{S}^\lambda$ for the Schur functor associated to λ; for example,

$$\mathbf{S}^{(\ell)} V = \operatorname{Sym}^\ell V \qquad \text{and} \qquad \mathbf{S}^{(1)^\ell} V = \wedge^\ell V,$$

where $(1)^\ell = (1, \dots, 1)$ with ℓ parts.

Let V be a vector bundle on $\mathbb{P}^1$ of rank n; fix $r < n$. By [PR03], there exists an integer $d_0 = d_0(V, r)$ such that for all $d \geqslant d_0$, the Quot scheme

$$\mathsf{Quot}_d(\mathbb{P}^1, V, r) \quad (\text{abbreviated } \mathsf{Quot}_d)$$

is irreducible and generically smooth of dimension $nd + rb + r(n-r)$, where $b = -\deg V$. In the special case $V = \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$, one may take $d_0 = 0$ and $\mathsf{Quot}_d(\mathbb{P}^1, V, r)$ is smooth and irreducible for all $d \geqslant 0$. In the remainder of the introduction, we assume $d \geqslant d_0(V, r)$.

We now state our main result.

Theorem 5.1.1. Let K and M be vector bundles on $\mathbb{P}^1$, and let μ, λ be partitions satisfying

$$|\mu| + |\lambda| \; < \; \frac{nd + rb + n}{n - r}.$$

(i) If $\mu \neq \emptyset$, then

$$H^i\Big(\mathsf{Quot}_d(\mathbb{P}^1, V, r),\, \mathbf{S}^\mu K^{\{d\}} \otimes \mathbf{S}^\lambda M^{[d]}\Big) = 0 \qquad \text{for all } i \geqslant 0.$$

(ii) If $\mu = \emptyset$, there is a natural isomorphism of graded vector spaces

$$H^\bullet\Big(\mathsf{Quot}_d(\mathbb{P}^1, V, r),\, \mathbf{S}^\lambda M^{[d]}\Big) \;\cong\; \mathbf{S}^\lambda H^\bullet(V \otimes M).$$

Remark 5.1.2. When M splits as a direct sum of nonnegative degree line bundles, the complex $M^{[d]}$, and hence $\mathbf{S}^\lambda M^{[d]}$, is a vector bundle. The isomorphism in part (ii) of Theorem 5.1.1

$$H^0(\mathsf{Quot}_d(\mathbb{P}^1, V, r), \mathbf{S}^\lambda M^{[d]}) \cong \mathbf{S}^\lambda H^0(V \otimes M)$$

is induced geometrically, by considering the morphism

$$R\pi_* p^*(V \otimes M) \;\longrightarrow\; M^{[d]}$$

arising from the universal sequence (5.1), taking the associated morphism of the Schur functor and then applying the global sections functor.

Theorem 5.1.1 greatly generalizes the results in [MOS22], even for punctual Quot schemes, and in the positive-rank setting, proves [MOS22, Conjecture 1.3.1, 1.3.2] concerning the exterior and symmetric powers of the tautological bundles. Specifically,

Corollary 5.1.3. Let L be a line bundle on $\mathbb{P}^1$. For all $k < (nd+rb+n)/(n-r)$, we have the following isomorphisms of graded vector spaces

(i) $H^\bullet\Big(\mathsf{Quot}_d, \wedge^k L^{[d]}\Big) \cong \wedge^k H^\bullet(V \otimes L)$;

(ii) $H^\bullet\Big(\mathsf{Quot}_d, \operatorname{Sym}^k L^{[d]}\Big) \cong \operatorname{Sym}^k H^\bullet(V \otimes L)$.

Remark 5.1.4. In Theorem 5.5.5, we prove a more general result analogous to Theorem 5.1.1, for the tensor product of Schur functors applied to multiple vector bundles $K_1, K_2, \dots, K_s$ and $M_1, M_2, \dots, M_t$. Note that while Theorem 5.5.5 is stated for line bundles, it is easy to upgrade the theorem to allow vector bundles since every vector bundle splits on $\mathbb{P}^1$. Below, we state the result in the case of trivial V, describing each cohomology group explicitly.

Corollary 5.1.5. Let $L_1, L_2, \dots, L_t$ be line bundles on $\mathbb{P}^1$. For any tuple of partitions $\lambda^1, \lambda^2, \dots, \lambda^t$ satisfying $|\lambda^1| + |\lambda^2| + \cdots + |\lambda^t| < (nd+n)/(n-r)$, we have

$$H^D\left(\mathsf{Quot}_d(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus n}, r), \bigotimes_{j=1}^{t} \mathbf{S}^{\lambda^j} L_j^{[d]}\right) \cong \bigotimes_{\deg L_j < 0} \mathbf{S}^{(\lambda^j)^\dagger} H^1(L_j^{\oplus n}) \otimes \bigotimes_{\deg L_j \geq 0} \mathbf{S}^{\lambda^j} H^0(L_j^{\oplus n}),$$

where $D = \sum_{\deg L_j < 0} |\lambda^j|$, and all other cohomology groups are zero.

Fixing the partitions and line bundles on $\mathbb{P}^1$, we may view the corresponding tautological bundles on Quot_d for any d, and study the asymptotic behavior of their cohomology groups as d becomes large. With this perspective, we can view Theorem 5.1.1 as a stabilization result. In particular, as predicted in [MOS22],

Corollary 5.1.6. For line bundles $L_1, L_2, \dots, L_t$ and partitions $\lambda^1, \lambda^2, \dots, \lambda^t$

$$\sum_{d=0}^{\infty} q^d \chi\left(\mathsf{Quot}_d(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus n}, r), \bigotimes_{j=1}^{t} \mathbf{S}^{\lambda^j} L_j^{[d]}\right)$$

is given by a rational function with simple pole only possibly at $q = 1$.

The vanishing of higher cohomology in Corollary 5.1.5 for line bundles of nonnegative degree can be established without any restriction on the sizes of the partitions, at the expense of imposing degree constraints on the line bundles involved. The next result is analogous to Theorem 2.1.6; see Theorem 5.5.7 for general V.

Theorem 5.1.7. Let $L_1, L_2, \dots, L_t$ be line bundles on $\mathbb{P}^1$ of degrees at least d. For any tuple of partitions $\lambda^1, \lambda^2, \dots, \lambda^t$, we have

$$H^i\left(\mathsf{Quot}_d(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus n}, r), \bigotimes_{j=1}^{t} \mathbf{S}^{\lambda^j} L_j^{[d]}\right) = 0 \quad \text{for all } i \geqslant 1.$$

The space of global sections in Theorem 5.1.7 does not, in general, follow the description given in Corollary 5.1.5. The size constraint on the partition λ appearing in Theorem 5.1.1 is quite close to optimal; see [OS23, Theorem 12] and Example 5.5.8. Theorem 5.1.7 implies that the dimension of the space of global sections is equal to the Euler characteristic. The latter can be computed by other means, such as torus localization (see [OS23, Theorem 3] and [SZ24b, Corollary 4.10] for exterior powers) or via wall-crossing to other moduli spaces [HF23].

Extension groups

When the degree of a line bundle L is sufficiently large, our method also allows the computation of the global extension groups for Schur functors of the vector bundle $L^{[d]}$. The answer is expressed in terms of Ext-groups of Schur functors of the universal quotient bundle on a Grassmannian (see Proposition 5.4.6). The Borel–Weil–Bott theorem for the Grassmannian completely determines these Ext-groups and has the following implications.

Theorem 5.1.8. Assume V splits as a direct sum of line bundles of nonpositive degrees. Let L be a line bundle of degree at least $d+b$. Then

(i) For any nontrivial partition ν with $|\nu| < (nd+rb+n)/r$ and $\nu_1 < n-r$,

$$H^i\left(\mathsf{Quot}_d, (\mathbf{S}^\nu L^{[d]})^\vee\right) = 0 \quad \text{for all } i \geqslant 0. \tag{5.2}$$

(ii) Let $p_1, p_2, \ldots, p_t$ and k be nonnegative integers, and set $|p| = p_1+p_2+\cdots+p_t$. Assume $k < (nd+rb+n)/(n-r)$, $|p| \leqslant (nd+rb+n)/r$, and $t < n-r$. Then

$$\operatorname{Ext}^i\left(\bigwedge^{p_1} L^{[d]} \otimes \cdots \otimes \bigwedge^{p_t} L^{[d]}, \bigwedge^{k} L^{[d]}\right) = \begin{cases} \bigwedge^{k-|p|} L^{[d]} & i=0,\ k \geqslant |p|; \\ 0 & \text{otherwise.} \end{cases}$$

A similar statement holds upon replacing all $\bigwedge$ with Sym, and replacing the condition $t < n-r$ with $|p| < n-r$.

(iii) For any partition λ, with $|\lambda| \leqslant d$ and $\lambda_1 < n-r$,

$$\operatorname{Ext}^i(\mathbf{S}^\lambda L^{[d]}, \mathbf{S}^\lambda L^{[d]}) = \begin{cases} \mathbb{C} & i=0; \\ 0 & i \geqslant 1. \end{cases}$$

(iv) More generally, the set $\{\mathbf{S}^\lambda(L^{[d]}) : \ |\lambda| \leqslant d,\ \lambda_1 < n-r\}$ forms an exceptional collection, ordered by the sizes of the partitions.

Remark 5.1.9. Note that the exceptional collection above generates only a subcategory of the derived category of coherent sheaves on Quot_d. A full exceptional collection built from Schur functors of universal quotients on Grassmannian was studied by Kapranov [Kap84]. For Quot schemes of rank-zero quotients, a full exceptional collection (and, in higher genus, a semiorthogonal decomposition) of the derived category was obtained in [Tod22] and [MN24]. However, these methods do not directly extend to the case of higher-rank quotients.

Part (ii) of Theorem 5.1.8 recovers the results on extension groups in [MOS22] for Quot schemes of rank-zero quotients, and partially proves [MOS22, Conjecture 1.3.3] in the higher-rank setting. Moreover, in Theorem 5.4.1, we establish a mixed version of Proposition 5.4.6 and Theorem 5.1.1 for two line bundles of consecutive degrees; this may be regarded as our main technical result. We state a consequence for the duals of tautological bundles below.

Theorem 5.1.10. Assume V splits as a direct sum of line bundles of nonpositive degrees. Let $L_{m-1} = \mathcal{O}_{\mathbb{P}^1}(m-1)$ and $L_m = \mathcal{O}_{\mathbb{P}^1}(m)$ on $\mathbb{P}^1$, with $m \geqslant d+b$. For any pair of partitions ν and μ satisfying

$$|\nu| + |\mu| \leqslant (nd + rb + n)/r \quad \text{and} \quad 0 < \nu_1 + \mu_1 < n - r,$$

all cohomology groups of the vector bundle $\left(\mathbf{S}^{\nu} L_{m-1}^{[d]}\right)^{\vee} \otimes \left(\mathbf{S}^{\mu} L_{m}^{[d]}\right)^{\vee}$ vanish.

Relation to quantum K-theory of Grassmannians

When V is the trivial vector bundle of rank n, the Quot scheme $\mathsf{Quot}_d(\mathbb{P}^1, V, r)$ is a smooth moduli space (see [Str87]) that compactifies the space of morphisms from $\mathbb{P}^1$ to the Grassmannian $\mathrm{Gr}(r,n)$ of degree d. This viewpoint was taken up by Bertram [Ber97] to study the quantum cohomology ring of $\mathrm{Gr}(r,n)$, and later to prove the Vafa–Intriligator formula [Ber94; MO07] for intersection numbers on $\mathsf{Quot}_d(C, V, r)$, which counts the number of maps from a fixed curve C of genus g to $\mathrm{Gr}(r,n)$ satisfying some incidence conditions.

Recently, [SZ24a; SZ24b] studied K-theoretic invariants on $\mathsf{Quot}_d(\mathbb{P}^1, V, r)$ in the context of the quantum K-ring of $\mathrm{Gr}(r,n)$ (cf. [BM11]). Fix a point $x \in \mathbb{P}^1$ and set

$$\mathcal{E}_x := \mathcal{E}|_{\mathsf{Quot}_d \times \{x\}},$$

a rank $n-r$ vector bundle on Quot_d. By [SZ24a, Theorem 1.12], the Euler characteristics of Schur functors of $\mathcal{E}_x$ compute K-theoretic Gromov–Witten invariants: For partitions μ^1, μ^2, μ^3 whose first parts are at most r,

$$\chi\left(\overline{\mathcal{M}}_{0,3}(\mathrm{Gr}(r,n), d), \bigotimes_{i \leqslant 3} ev_{p_i}^{*} \mathbf{S}^{\mu^i}(\mathcal{B})\right) = \chi\left(\mathsf{Quot}_d(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus n}, r), \bigotimes_{i \leqslant 3} \mathbf{S}^{\mu^i}(\mathcal{E}_x)\right) \tag{5.3}$$

Here $\mathcal{B}$ denotes the universal quotient bundle on $\mathrm{Gr}(r,n)$, and $\overline{\mathcal{M}}_{0,3}(\mathrm{Gr}(r,n), d)$ is the moduli space of 3-pointed genus-0 stable maps to $\mathrm{Gr}(r,n)$ of degree d with ev_{p_i} the evaluation map at the i-th marking. In this paper, we prove the following consequence of Theorem 5.1.1.

Theorem 5.1.11. Let $\mu^1, \mu^2, \ldots \mu^t$ be partitions satisfying $0 \neq |\mu^1| + \cdots + |\mu^t| < (nd + n)/(n-r)$, then

$$H^i\left(\mathsf{Quot}_d(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus n}, r), \bigotimes_{j=1}^{t} \mathbf{S}^{\mu^j} \mathcal{E}_x\right) = 0 \qquad \text{for all } i \geqslant 0.$$

The Euler characteristics of Schur functors of $\mathcal{E}_x$ were computed in [SZ24b, Theorem 1.6 and 1.9]. The vanishing of Euler characteristics of these bundles was shown to provide enough relations to completely determine the quantum K-ring of $\mathrm{Gr}(r, n)$. Theorem 5.1.11 partially answers [SZ24b, Question 1.11] by upgrading a subset of the vanishing statements for Euler characteristics to statements about individual cohomology groups. Using our results, one can already prove that the K-theoretic Gromov–Witten invariant in (5.3) vanishes for all sufficiently large degrees d. We expect that our methods may in future shed light on positivity phenomena in the quantum K-theory of the Grassmannian; see, for instance, [BM11; Buc+22].

Analogy with Hilbert scheme of points on surfaces

Let V be a vector bundle on a smooth projective curve C. The punctual Quot scheme, denoted $\mathsf{Quot}_d(C, V) = \mathsf{Quot}_d(C, V, 0)$, parametrizes quotients of V supported at zero dimensional scheme of length d. Punctual Quot schemes have been extensively studied; we mention only a few relevant works here, including computations of cohomology groups [MN23], descriptions of nef cones for divisors [GS21], positivity results for tautological bundles [Opr22], and cohomology of tangent bundle [BGS24].

For a line bundle L on C, the description of the cohomology groups for the tautological bundle $L^{[d]}$ and their exterior powers is explicitly given by the formula (see [MN24, Theorem 3] for the statement involving extension groups)

$$H^\bullet(\mathsf{Quot}_d(C, V), \wedge^k L^{[d]}) \cong \wedge^k H^\bullet(V \otimes L) \otimes \mathrm{Sym}^{d-k} H^\bullet(\mathcal{O}_C).$$

There is a parallel story for the Hilbert schemes $X^{[d]}$ of d points on a smooth projective surface X. For a line bundle L on X, the tautological bundle $L^{[d]}$, defined similarly, has rank d. Indeed, the cohomology groups are given by the formula [Sca09; Kru18]

$$H^\bullet(X^{[d]}, \wedge^k L^{[d]}) \cong \wedge^k H^\bullet(L) \otimes \mathrm{Sym}^{d-k} H^\bullet(\mathcal{O}_X).$$

The proofs of the above statement in both cases rely on studying the bounded derived categories of sheaves $D^b(X^{[d]})$ and $D^b(\mathsf{Quot}_d(C, V))$.

The description of cohomology groups of the symmetric powers $\mathrm{Sym}^k L^{[d]}$, and general Schur functors, is not known to admit a simple formula, for both $X^{[d]}$ and $\mathsf{Quot}_d(C, V)$. In special case, Euler characteristics [Arb21] and the space of global sections [Dan04; Sca15] of the symmetric products of tautological bundles $L^{[d]}$ on $X^{[d]}$ admits a simple form. For instance, let L be a positive degree line bundle on $X = \mathbb{P}^2$, then for all $k < d + 1$,

$$H^0\left(X^{[d]}, \mathrm{Sym}^k\ L^{[d]}\right) = \mathrm{Sym}^k H^0(L)$$

while all higher cohomology groups vanish. We predict that an anolgue of formulas in Corollary 5.1.5 also hold for Hilbert scheme of points on $X \cong \mathbb{P}^2$. We note here that tautological bundles on the Quot schemes on surfaces have also been studied in [OP21; Arb+21].

Proof strategy

For a fixed sufficiently large integer m, the Quot scheme on $\mathbb{P}^1$ admits an embedding into a product of Grassmannians, constructed by Strømme [Str87],

$$\iota_m :\ \mathsf{Quot}_d \hookrightarrow \mathrm{Gr}(k_1, N_1) \times \mathrm{Gr}(k_2, N_2), \tag{5.4}$$

as the zero locus of a section of a vector bundle $\mathcal{K}$, expressed in terms of the universal bundles on the two Grassmannians. In the rank-zero setting, [MOS22] uses the Koszul resolution of $\mathcal{O}_{\mathsf{Quot}_d}$ to reduce the study of the cohomology of exterior and symmetric powers of $L_m^{[d]}$ to the cohomology of universal bundles on the product of Grassmannians, where the Borel-Weil-Bott theorem applies.

We follow the approach of [MOS22] to study the case of arbitrary rank, but the combinatorial analysis of the vector bundles that show up in the Koszul resolution becomes rather involved. To address this combinatorial proliferation, in Section 5.3 we provide a streamlined combinatorial criterion for detecting the non-vanishing of the cohomology of universal bundles on a Grassmannian, which may be of independent interest. A second ingredient is provided by Horn's inequalities, which give criteria for the nonvanishing of the Littlewood–Richardson coefficients appearing in our expansions. By applying these two sets of criteria carefully, we show that all cohomology groups vanish for every term in the Koszul resolution except the first. We then conclude by computing the cohomology of the first term explicitly.

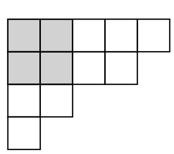
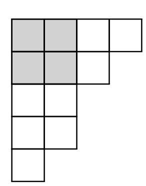
Our methods compute the cohomology of all Schur functors (subject to the relevant size constraints) of the tautological bundles associated to consecutive-degree line bundles, $L_m^{[d]}$ and $L_{m-1}^{[d]}$ (new even in the rank-zero case). This is especially advantageous because Schur functors of any tautological complex $M^{[d]}$ can be expressed in terms of the Schur functors of the tautological bundles $L_{m-1}^{[d]}$ and $L_m^{[d]}$ in the derived category of sheaves on Quot_d. We then leverage the explicit results about $\mathbf{S}^\alpha L_{m-1}^{[d]} \otimes \mathbf{S}^\beta L_m^{[d]}$ to obtain results for the Schur complexes $\mathbf{S}^\lambda M^{[d]}$, as well as their tensor products.

Assumptions on characteristic. Our results are valid for all algebraically closed fields of characteristic zero. For ease of notation, we use $\mathbb{C}$ throughout. The main dependence on the characteristic comes from the use of the Borel-Weil-Bott theorem for Grassmannians. The fact that Schur functors of complexes respect quasi-isomorphisms, and therefore are well-defined on objects in the derived category, also depends on the characteristic zero assumption.

5.2 Combinatorial preliminaries

Notations

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a non-increasing sequence of non-negative integers. The size of the partition is defined as $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_k$. We can represent λ graphically by its Young diagram, which contains k rows of boxes with λ_i boxes in the ith row. For example, the Young diagram of the partition $\lambda = (5, 4, 2, 1)$, which has size $|\lambda| = 12$, is shown below:

$\lambda =$  $\lambda^\dagger =$ 

The *conjugate partition* $\lambda^\dagger$ is obtained by transposing the Young diagram of λ. In the above example, $\lambda^\dagger = (4, 3, 2, 2, 1)$.

The *Durfee square* of a partition λ is the largest square contained in the Young diagram of λ, and the side length of the Durfee square is called the *rank* of the partition. In the above example, λ has rank two.

Schur functors

Let $W \cong \mathbb{C}^k$ be a complex vector space. For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we denote by $\mathbb{S}^\lambda(W)$ the Schur functor, which corresponds to the irreducible polynomial $\mathrm{GL}(W)$-representation with highest weight λ. The character of the representation $\mathbb{S}^\lambda(W)$ is the Schur polynomial $s_\lambda(x_1, x_2, \ldots, x_k)$, and its dimension is given by the hook-content formula:

$$\dim \mathbf{S}^\lambda(W) = s_\lambda(\underbrace{1, 1, \ldots, 1}_{k}) = \prod_{c \in \lambda} \frac{k + \mathrm{cont(c)}}{\mathrm{hook(c)}} \tag{5.5}$$

where for each cell $c = (i, j)$ in λ is in the ith row and jth column, $\mathrm{cont}(c) = j - i$ and $\mathrm{hook}(c) = 1 + (\lambda_i - j) + (\lambda_j^\dagger - i)$ is the hook length of c.

Every polynomial $\mathrm{GL}(W)$-representation can be expressed uniquely as a direct sum of Schur functors. Given two partitions λ and μ, one can decompose the tensor product as

$$\mathbf{S}^\lambda(W) \otimes \mathbf{S}^\mu(W) = \bigoplus_\gamma \mathbf{S}^\gamma(W)^{\oplus c^\gamma_{\lambda,\mu}}, \tag{5.6}$$

where the multiplicities $c^\gamma_{\lambda,\mu}$ are the *Littlewood–Richardson coefficients*. The *Littlewood–Richardson rule* provides a combinatorial description of these coefficients in terms of counting Littlewood–Richardson tableaux. In the next subsection, we list the properties of Littlewood–Richardson coefficients that we shall need in this article.

Let V be another complex vector space. Schur functors naturally appear in the context of *Cauchy's formula*: for any non-negative integer t, there is an isomorphism of vector spaces

$$\bigwedge^t (V \otimes W) \cong \bigoplus_{|\mu| = t} \mathbf{S}^\mu(V) \otimes \mathbf{S}^{\mu^\dagger}(W), \tag{5.7}$$

where the direct sum runs over all partitions μ of size t.

We will also require the following decomposition formula for the Schur functor of the direct sum of two vector spaces

$$\mathbf{S}^\gamma(V \oplus W) \cong \bigoplus_{\alpha,\beta} \left(\mathbf{S}^\alpha(V) \otimes \mathbf{S}^\beta(W)\right)^{\oplus c^\gamma_{\alpha,\beta}}. \tag{5.8}$$

Here the direct sum is taken over all partitions α and β such that $|\alpha| + |\beta| = |\gamma|$.

Schur functors associated to highest weights

Irreducible rational representations of $\mathrm{GL}(W)$ are indexed by highest weights

$$\eta = (\eta_1, \eta_2, \ldots, \eta_k),$$

which are non-increasing sequences of integers $\eta_1 \geqslant \eta_2 \geqslant \cdots \geqslant \eta_k$. Note that each η_i is allowed to be negative. For any highest weight η, we denote the corresponding irreducible $\mathrm{GL}(W)$-representation by $\mathbf{S}^\eta(W)$. For any $m \geqslant -\eta_k$, we can express $\mathbf{S}^\eta(W)$ in terms of the usual Schur functor using the following isomorphism of $\mathrm{GL}(W)$-representations

$$\mathbf{S}^\eta W \cong \det(W)^{-m} \otimes \mathbf{S}^{\eta+(m)^k} W,$$

where $\det(W)$ is the determinant representation and $\eta + (m)^k = (\eta_1 + m, \eta_2 + m, \ldots, \eta_k + m)$ is a partition. For any highest weight η, the dual of the corrresponding representation is given by $\mathbf{S}^\eta(W)^\vee = \mathbf{S}^{-\eta}(W)$, where

$$-\eta = (-\eta_k, -\eta_{k-1}, \ldots, -\eta_1).$$

The tensor product of two rational representations with highest weights η and ρ can also be described using the Littlewood-Richardson rule

$$\mathbf{S}^\eta W \otimes \mathbf{S}^\rho W = \bigoplus_\chi (\mathbf{S}^\chi W)^{\oplus c^\chi_{\eta,\rho}}, \tag{5.9}$$

where χ runs over all the highest weights with $|\chi| = |\eta| + |\rho|$, and the multiplicity is given using the identity

$$c^\chi_{\eta,\rho} = c^{\chi+(m+k)^r}_{\eta+(m)^r,\rho+(k)^r}, \tag{5.10}$$

where m and k are any two integers, such that $\eta + (m)^r, \rho + (k)^r$ and $\chi + (m+k)^r$ are partitions.

Given a highest weight $\eta = (\eta_1, \eta_2, \ldots, \eta_k)$, it is sometimes useful to partition it into two parts, where one consists of nonnegative integers and one consists of nonpositive ones. For this, we use the notation $\eta = (\gamma, -\delta)$, where γ, δ are partitions. Note that due to the presence of zeros, there isn't necessarily a unique way to write η in this form.

Remark 5.2.1. Schur functors $\mathbf{S}^\eta$ can be defined in the generality of vector bundles on a scheme. The identities in the above discussion, such as (5.6), (5.7), and (5.8), all generalize verbatim. This is explained in e.g. [Wey03, Chapter 2]. Schur functors associated to a partition can be generalized even further to perfect complexes, and it is well-defined up to quasi-isomorphisms. We explain the case of length two perfect complexes in Section 5.5.

Horn's inequalities

There is a vast literature on the study of Littlewood–Richarson coefficients. In particular, a lot of it has been devoted to answering the following fundamental question:

What are the conditions on the partitions α, β and γ such that $c^\gamma_{\alpha,\beta} \neq 0$?

Knutson and Tao in their proof of the saturation conjecture [KT99] showed that $c^\gamma_{\alpha,\beta} \neq 0$ if and only if α, β, and γ are the eigenvalues of Hermitian matrices A, B, and C with $C = A + B$, respectively. We assume that the rank N of the matrices to be greater than the number of parts in each of the partitions α, β, and γ. Then using the identity for traces, $\operatorname{tr} A + \operatorname{tr} B = \operatorname{tr} C$, we get

$$\sum_{i=1}^{N} \alpha_i + \sum_{i=1}^{N} \beta_i = \sum_{i=1}^{N} \gamma_i. \tag{5.11}$$

In general, given A and B be $N \times N$ Hermitian matrices, define $C = A + B$. Let $\alpha = (\alpha_1 \geqslant \cdots \geqslant \alpha_N)$, $\beta = (\beta_1 \geqslant \cdots \geqslant \beta_N)$ and $\gamma = (\gamma_1 \geqslant \cdots \geqslant \gamma_N)$ denote the eigenvalues of the matrices A, B and C written in non-increasing order.

Let $I = (i_1 \leqslant \cdots \leqslant i_s)$ and $J = (j_1 \leqslant \cdots \leqslant j_s)$ be increasing sequence at most N positive integers such that $i_s + j_s \leqslant N + s$. Then we have

$$\sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \geqslant \sum_{k \in K} \gamma_k \tag{5.12}$$

where $K = (k_1 \leqslant \cdots \leqslant k_s)$ with $k_p = i_p + j_p - p$ for $1 \leqslant p \leqslant s$; see [Ful00] for a detailed exposition.

In the following proposition, we enumerate various properties of Littlewood–Richardson coefficients that will be used later in the proofs.

Proposition 5.2.2. Consider three partitions α, β, and γ such that $c^{\gamma}_{\alpha,\beta} \neq 0$. Then the following hold:

1. (Size constraint)
$$|\alpha| + |\beta| = |\gamma|;$$

2. (Symmetry)
$$c^{\gamma}_{\alpha,\beta} = c^{\gamma^\dagger}_{\alpha^\dagger,\beta^\dagger};$$

3. (Weyl inequality) For any positive integers i and j, we have
$$\alpha_i + \beta_j \geqslant \gamma_{i+j-1}; \tag{5.13}$$

4. (Dominance inequality I) We have $\gamma \leqslant \alpha + \beta$, that is, for all s,
$$\sum_{i=1}^{s} \gamma_i \leqslant \sum_{i=1}^{s} \alpha_i + \sum_{i=1}^{s} \beta_i; \tag{5.14}$$

5. (Dominance inequality II) We have $\alpha \cup \beta \leqslant \gamma$, that is, for all t,
$$\sum_{i=1}^{t} \alpha_i + \sum_{i=1}^{t} \beta_i \leqslant \sum_{i=1}^{2t} \gamma_i.$$

Note that part (iii) and (iv) also holds when α, β and γ are highest weights.

Proof. Statements (i) and (ii) are standard facts about Littlewood–Richardson coefficients. Statements (iii) and (iv) both follow from (5.12). Specifically, statement (iii) can be obtained by letting $I = \{i\}, J = \{j\}$. Statement (iv) can be obtained by letting $I = J = \{1, \ldots, s\}$. We now explain the proof of (v). Taking rank of matrices, N, to be sufficiently large, the inequality in (5.12) implies

$$\sum_{p=t+1}^{t+s} \alpha_p + \sum_{p=t+1}^{t+s} \beta_p \geqslant \sum_{p=2t+1}^{2t+s} \gamma_p$$

for any $2s + t \leqslant N + t$. Now take s larger than the number of parts in α, β, and γ. Then (v) follows by subtracting the above inequality from the equality in (i). □

5.3 Indices of partitions

In this section, we focus on certain cases of the Borel-Weil-Bott theorem, and pictorially describe the combinatorial conditions on the partitions involved when a tautological bundle has a nontrivial cohomology group. The notions of the t-index and the $(t;\eta)$-index introduced in this section will be key ingredients in our proofs later.

Borel-Weil-Bott

Let $\mathrm{Gr}(k, N)$ denote the Grassmannian of k-dimensional subspaces of the vector space $\mathbb{C}^N$. Consider the universal exact sequence over $\mathrm{Gr}(k, N)$:

$$0 \to \mathcal{A} \to \mathcal{O}_{\mathrm{Gr}(k,N)}^{\oplus N} \to \mathcal{B} \to 0,$$

where $\mathcal{A}$ is the universal subbundle of rank k and $\mathcal{B}$ is the universal quotient bundle of rank $N - k$. The sheaf cohomology of Schur functors applied to $\mathcal{A}$ and $\mathcal{B}$ is described by the Borel–Weil–Bott theorem [Bot57]. We state below the version presented in [Wey03, Corollary 4.1.9].

Theorem 5.3.1 (Borel-Weil-Bott)**.** For any two non-increasing sequences of integers

$$\rho = (\rho_1, \ldots, \rho_k) \quad \text{and} \quad \chi = (\chi_1, \ldots, \chi_{N-k}),$$

the vector bundle $\mathbb{S}^\rho \mathcal{A}^\vee \otimes \mathbb{S}^\chi \mathcal{B}^\vee$ on $\mathrm{Gr}(k, N)$ has at most one non-vanishing cohomology group. Let

$$\omega := (\rho, \chi) + (N-1, N-2, \ldots, 0)$$

be the component-wise sum. If ω contains a repetition, then all cohomology groups of $\mathbb{S}^\rho(\mathcal{A}^\vee) \otimes \mathbb{S}^\chi(\mathcal{B}^\vee)$ vanish. Otherwise, let σ be the permutation that sorts ω into a strictly decreasing sequence, and

$$\gamma := \sigma \circ \omega - (N-1, N-2, \ldots, 0),$$

be a non-increasing sequence, then

$$H^D\left(\mathrm{Gr}(k, N),\ \mathbb{S}^\rho \mathcal{A}^\vee \otimes \mathbb{S}^\chi \mathcal{B}^\vee\right) = \mathbb{S}^\gamma (\mathbb{C}^N)^\vee$$

where the cohomological degree $D = \ell(\sigma)$ is the length of the permutation σ, and all other cohomology groups are zero.

Remark 5.3.2. When we consider Schur functors of the quotient bundle independently, Theorem 5.3.1 implies that for any partition λ, the morphism induced from the tautological sequence

$$\mathbf{S}^{\lambda}\mathcal{O}^{\oplus N}_{\mathrm{Gr}(k,N)} \to \mathbf{S}^{\lambda}\mathcal{B}$$

gives an isomporhism on the space of global sections

$$H^0\Big(\mathrm{Gr}(k,N),\ \mathbf{S}^{\lambda}\mathcal{B}\Big) \cong \mathbf{S}^{\lambda}(\mathbb{C}^N).$$

Similar statement holds for $\mathbf{S}^{\lambda}\mathcal{A}^{\vee}$.

t-Index

The notion of the t-index of a partition ν was introduced in [MOS22] to systematically study the conditions which determine when all the cohomology groups of $\mathbf{S}^{\nu}\mathcal{B}^{\vee}$ vanish. Here we recall, and later generalize this notion which plays a crucial role in the sections that follow.

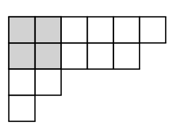
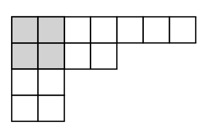
Definition 5.3.3. Fix a positive integer t. We say that a non-increasing sequence of integers $\chi = (\chi_1, \chi_2, \ldots, \chi_m)$ has a **well-defined t-index** if there exists a nonnegative integer j such that

$$\chi_j \geqslant j + t \quad \text{and} \quad \chi_{j+1} \leqslant j.$$

In this case, the t-index of χ is j. Note that the t-index of χ is 0 (resp. m) if $\chi_1 \leqslant 0$ (resp. $\chi_m \geqslant m + t$).

Remark 5.3.4. Note that every integer partition λ has a well-defined 0-index, which equals the rank of the partition. If λ has a well-defined t-index j, then j must also be the 0-index (i.e., the rank) of λ. Furthermore, if j is the 0-index of λ, then λ has a well-defined t-index if and only if $t \leqslant \lambda_j - j$.

Example 5.3.5. Fix $t = 3$. Consider the integer partitions $\lambda = (6, 5, 2, 1)$ and $\mu = (7, 4, 2, 2)$. The Durfee square of both partitions λ and μ has side length 2. We observe that λ has a well-defined t-index, whereas μ does not.

$\lambda =$  $\mu =$ 

Lemma 5.3.6. Let $\chi = (\chi_1, \ldots, \chi_{N-k})$ be a non-decreasing sequence of integers. The vector bundle $\mathbf{S}^{\chi}(\mathcal{B}^{\vee})$ on $\mathrm{Gr}(k, N)$ has a nontrivial cohomology group

$$H^D\Big(\mathrm{Gr}(k,N), \mathbf{S}^{\chi}(\mathcal{B}^{\vee})\Big) \neq 0 \tag{5.15}$$

if and only if χ has a well-defined k-index j, and the cohomological degree is $D = kj$.

Proof. It is a straightforward application of Theorem 5.3.1. Indeed, (5.15) is satisfied if and only if

$$\begin{aligned}\omega &= (0, \ldots, 0, \chi_1, \ldots, \chi_{N-k}) + (N-1, \ldots, N-k,\ N-k-1, \ldots, 0)\\ &= (f_k, \ldots, f_1,\ g_1, \ldots, g_{N-k})\end{aligned}$$

has no repetitions, and D is the length of the permutation that sorts ω. Observe that $f_k > \cdots > f_1$ are consecutive integers, and thus we may choose j such that

$$g_1 > \cdots > g_j > f_k > \cdots > f_1 > g_{j+1} > \cdots > g_{N-k}.$$

Hence the cohomological degree is $D = kj$. The required conditions $\chi_j \geqslant j+k$ and $\chi_{j+1} \leqslant j$ follow from the inequalities $g_j > f_k$ and $g_{j+1} < f_1$, respectively.

□

$(t;\eta)$-Index

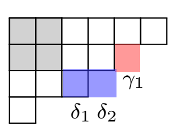
To pictorially describe the conditions for the vanishing of cohomology groups of the vector bundle $\mathbb{S}^\mu(\mathcal{A}) \otimes \mathbb{S}^\eta(\mathcal{B})$, we introduce the following generalization of the notion of index.

Definition 5.3.7. Fix a positive integer t and a tuple of non-increasing integers $\eta = (\eta_1, \ldots, \eta_r)$. Write $\eta = (\gamma, -\delta)$, where γ and δ are partitions describing the nonnegative and negative entries of η. We say that a partition μ has a **well-defined** $(t;\eta)$**-index** if there exists a non-negative integer i such that

$$\mu_{i+1-s} \geqslant i + t - \gamma_s^\dagger \quad \text{and} \quad \mu_{i+s} \leqslant i + \delta_s^\dagger \quad \text{for all } s \geqslant 1.$$

We call such an integer i the $(t;\eta)$-index of μ.

Example 5.3.8. Fix $t = 3$ and $\eta = (1, -1, -1)$. Consider the partition $\nu = (6, 4, 3, 1)$, and observe that ν also has a well-defined $(t;\eta)$-index, equal to 2. We have $\delta = (1, 1)$ and $\gamma = (1)$ in blue and red, respectively.

$\nu =$ 

However, ν itself does not have a well-defined t-index. This example also illustrates that the $(t;\eta)$-index of a partition is not necessarily equal to the rank of the partition.

Lemma 5.3.9. Let $\mu = (\mu_1, \ldots, \mu_k)$ be a partition and $\eta = (\eta_1, \eta_2, \ldots, \eta_{N-k})$ be tuple of non-increasing integers. Write $\eta = (\gamma, -\delta)$ for partitions γ and δ. Suppose that the vector bundle $\mathbf{S}^\mu(\mathcal{A}) \otimes \mathbf{S}^\eta(\mathcal{B})$ on $\mathrm{Gr}(k, N)$ has a (unique) nonzero cohomology group, namely

$$H^D(\mathrm{Gr}(k, N), \mathbf{S}^\mu(\mathcal{A}) \otimes \mathbf{S}^\eta(\mathcal{B})) \neq 0.$$

Then the following hold.

(i) The partition μ has a well-defined $(N-k;\eta)$-index, i.e., there exists a natural number i such that

$$\mu_{i+1-s} \geqslant i + N - k - \gamma_s^\dagger \quad \text{and} \quad \mu_{i+s} \leqslant i + \delta_s^\dagger \quad \text{for all } s \geqslant 1. \tag{5.16}$$

(ii) The cohomological degree satisfies the bound

$$D \leqslant |\delta| + (\mu_1 + \mu_2 + \cdots + \mu_i) - i^2.$$

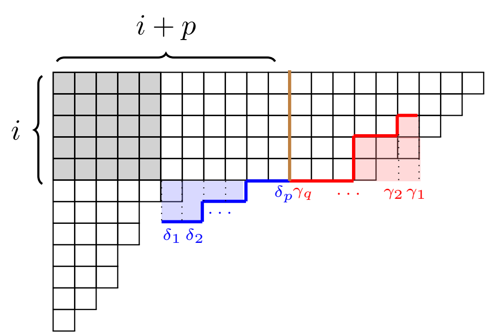
Proof. Let $\eta = (\gamma_1, \ldots, \gamma_q, -\delta_p, \ldots, -\delta_1)$ be a tuple of non-increasing integers, where γ and δ are partitions with at most q and p parts respectively, and $p + q = N - k$. We will show that μ has a well-defined $(N-k;\eta)$-index i given by the condition

$$\mu_i \geqslant i + p \quad \text{and} \quad \mu_{i+1} < i + p. \tag{5.17}$$

Note that $\mathbf{S}^\mu \mathcal{A} \cong \mathbf{S}^{-\mu}\mathcal{A}^\vee$ and $\mathbf{S}^\eta \mathcal{B} \cong \mathbf{S}^{-\eta}(\mathcal{B}^\vee)$, where $-\eta = (\delta, -\gamma)$. Theorem 5.3.1 implies that the vector

$$\omega = (-\mu_k, \ldots, -\mu_1, \delta_1, \ldots, \delta_p, -\gamma_q, \ldots, -\gamma_1) + (N-1, \ldots, 1, 0)$$

has no repetition, and that D is the length of the permutation that sorts ω. We split the analysis into two parts, first looking at the entries of ω that are strictly less than q, and then at those that are at least q.

$\mu =$ 

By the definition of i in (5.17), the entries strictly less than q in ω consist of

$$p + q + (i-1) - \mu_i, \ \ldots, \ p + q - \mu_1 \quad \text{and} \quad q - 1 - \gamma_q, \ \ldots, \ -\gamma_1.$$

Subtracting $q-1$ from each of the entries above and negating, we may assume that there are no repetitions among the following $i + q$ nonnegative integers

$$\alpha_i, \alpha_{i-1} + 1, \ \ldots, \ \alpha_1 + i - 1 \quad \text{and} \quad \gamma_q, \gamma_{q-1} + 1, \ \ldots, \ \gamma_1 + q - 1,$$

where $\alpha = (\mu_1 - p - i, \ldots, \mu_i - p - i)$. Lemma 5.3.10 implies that $\alpha_{i-s} \geqslant q - \gamma_s^\dagger$ for all $s \geqslant 0$, which proves the first set of inequalities in (5.16). Moreover, the length of the permutation that sorts the entries above (in the stated order) is at most $|\alpha|$, hence the contribution to the length of the permutation that sorts ω is at most $ip + |\alpha|$ (extra ip for moving the i entries containing $\mu_1, \ldots, \mu_i$ to the right of entries containing $\delta_1, \ldots, \delta_p$ in ω).

Now we consider the entries of ω that are at least q, namely

$$\delta_p, \delta_{p-1}+1, \ \ldots, \ \delta_1+p-1 \quad \text{and} \quad \beta_{k-1}, \beta_{k-2}+1, \ \ldots, \ \beta_1+(k-i)-1,$$

where $\beta = (i+p-\mu_k, \ldots, i+p-\mu_{i+1})$. Lemma 5.3.10 implies $\beta_{i-s} \geqslant p - \delta_s$, proving the second set of inequalities in (5.16). Furthermore, the length of the permutation that sorts the entries above contributes at most $|\delta|$ to the length of the permutation that sorts ω. Hence $D \leqslant |\delta| + ip + |\alpha| = |\delta| + (\mu_1 + \cdots + \mu_i) - i^2$.

□

The following result is elementary; due to the lack of a reference, we provide a short proof for completeness. We use the abacus notation for partitions (also called beta-sets in the literature cf. [JK81]). We adapt the proof from [Mac95, (1.7)] to our setting.

Lemma 5.3.10. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_i)$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_q)$ be partitions with at most i and q parts respectively. If the string of natural numbers

$$\omega = (\alpha_i, \alpha_{i-1}+1, \ldots, \alpha_1+i-1, \lambda_q, \lambda_{q-1}+1, \ldots, \lambda_1+q-1)$$

has no repitition, then the permutation σ that sorts ω has length $\ell(\sigma) \leqslant |\alpha|$, and

$$\alpha_{i-s} \geqslant q - \lambda_s^\dagger \quad \text{for all } 0 \leqslant s < i.$$

Proof. We define a sequence $u = (u_0, u_1, u_2, \ldots)$ of 0's and 1's, by

$$u_m = \begin{cases} 1 & \text{if } m \in \{\alpha_i, \ldots, \alpha_1+i-1\} \\ 0 & \text{otherwise.} \end{cases}$$

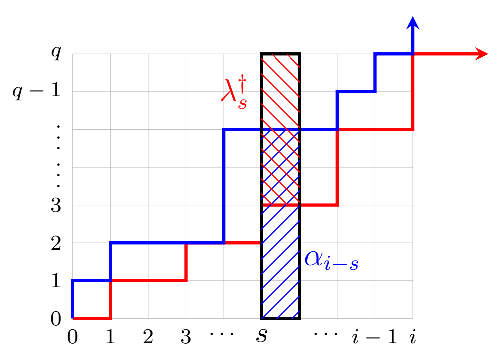
Similarly define $v = (v_0, v_1, v_2, \ldots) \in \{0,1\}^{\mathbb{N}}$ with 1's placed exactly at the positions $(\lambda_q, \ldots, \lambda_1+q-1)$. Note that ω has no repetition if and only if $u + v \in \{0,1\}^{\mathbb{N}}$. The length of permutation, by definition, equals

$$\begin{aligned} \ell(\sigma) &= \#\{(m,n) \in \mathbb{N}^2 : n < m, \ v_n = 1, \ u_m = 1\} \\ &\leqslant \#\{(m,n) \in \mathbb{N}^2 : n < m, \ u_n = 0, \ u_m = 1\} = |\alpha|. \end{aligned}$$

Let $\widehat{v}$ be the sequence obtained by flipping 1's and 0's in v. To observe the inequalities, we will draw the lattice path diagrams for u and $\widehat{v}$, such that 0's denote increment in the y-coordinate by one, and 1's denote right step by increasing x-coordinate by one. The condition $u + v \in \{0,1\}^{\mathbb{N}}$ implies that for any $m \geqslant 0$,

$$\text{number of 0's in } (u_0, \ldots, u_m) \geqslant \text{number of 0's in } (\widehat{v}_0, \ldots, \widehat{v}_m),$$

and hence the lattice path of $\widehat{v}$ lies always below that of u.



By construction, the lattice path of u (denoted in blue color) carves out α over x-axis, with α_{i-s} be the y-coordinate for the s-th right step. Similarly, the lattice path of $\widehat{v}$ (colored in red) carves out λ over y-axis. The inequality follows easily by considering the rectangle $[s, s+1] \times [0, q]$ in the $i \times q$ grid as showing in the sketch.

$\square$

5.4 Two line bundles of consecutive degrees

Our main goal in this section is to study the cohomology groups of tensor products of Schur functors applied to the tautological vector bundles $L_{m-1}^{[d]}$ and $L_m^{[d]}$ on $\mathsf{Quot}_d(\mathbb{P}^1, V, r)$, for line bundles

$$L_{m-1} := \mathcal{O}_{\mathbb{P}^1}(m-1) \quad \text{and} \quad L_m := \mathcal{O}_{\mathbb{P}^1}(m)$$

when m is sufficiently large. We will prove slightly more general statement than what is stated in the introduction for Schur functors corresponding to highest weights (see Section 5.2) instead of partitions. The statement is of independent interest, and we state it below.

Let us first recall the notations. We consider Quot scheme $\mathsf{Quot}_d(\mathbb{P}^1, V, r)$, Quot_d for short, for the vector bundle V on $\mathbb{P}^1$, which splits as

$$V \cong \mathcal{O}(-b_1) \oplus \mathcal{O}(-b_2) \oplus \cdots \oplus \mathcal{O}(-b_n)$$

where $0 = b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$, and thus the sum $b = \sum_{i=1}^{n} b_i$ is a nonnegative number. Furthermore, we will assume that $d \geqslant d_0(V, r)$ such that Quot_d is an irreducible variety of dimension $nd + rb + r(n-r)$ for the rest of this article.

Theorem 5.4.1. Let $\eta = (\gamma, -\delta)$ and $\rho = (\lambda, -\nu)$ be non-increasing sequences of lengths rank $L_{m-1}^{[d]}$ and rank $L_m^{[d]}$ respectively. Suppose $\gamma, \lambda, \nu, \delta$ are partitions satisfying

$$(n-r)(|\lambda| + |\gamma|) + r(|\nu| + |\delta|) < nd + rb + n; \qquad \nu_1 + \delta_1 < n - r.$$

(i) If either $|\delta| \neq 0$ or $|\nu| \neq 0$, and $m \geqslant b + d$, then

$$H^i\left(\mathsf{Quot}_d, \mathbf{S}^\eta L_{m-1}^{[d]} \otimes \mathbf{S}^\rho L_m^{[d]}\right) = 0 \quad \text{for all } i \geqslant 0.$$

(ii) If $\delta = \nu = \emptyset$, then, for m sufficiently large,

$$H^0\left(\mathsf{Quot}_d, \mathbf{S}^\gamma L_{m-1}^{[d]} \otimes \mathbf{S}^\lambda L_m^{[d]}\right) \cong \mathbf{S}^\gamma H^0(V \otimes L_{m-1}) \otimes \mathbf{S}^\lambda H^0(V \otimes L_m) \tag{5.18}$$

and all higher cohomology groups vanish.

The condition for m to be sufficiently large in part(ii) of Theorem 5.4.1 will be removed in the next section. The proof of Theorem 5.4.1 is quite involved, its components will be split in several subsections below. Using the same method, we will also obtain a bound on the cohomological degree of $\mathbf{S}^\eta L_{m-1}^{[d]} \otimes \mathbf{S}^\rho L_m^{[d]}$ in Proposition 5.4.7 that holds with no restrictions on ν and ρ.

Remark 5.4.2. The condition $b_1 = 0$ is not an extra assumption, since there is an isomorphism

$$\mathsf{Quot}_d(\mathbb{P}^1, V, r) \;\cong\; \mathsf{Quot}_{\,d+rb_1}\Big(\mathbb{P}^1, V(b_1), r\Big),$$

which takes $L_m^{[d]}$ to $L_{m-b_1}^{[d+rb_1]}$ on the right, which reduces the analysis to our cases.

Remark 5.4.3. The special case when η and ρ are trivial in Theorem 5.4.1 implies that

$$H^0(\mathsf{Quot}_d, \mathcal{O}_{\mathsf{Quot}_d}) = \mathbb{C},$$

while all higher cohomology groups vanish. This is consistent with the fact that $\mathsf{Quot}_d(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus n}, r)$ is a rational variety [Str87].

Strømme embedding

There is an embedding of the Quot scheme into a product of two Grassmannians (see [Str87] for details):

$$\iota : \mathsf{Quot}_d(\mathbb{P}^1, V, r) \hookrightarrow \mathrm{Gr}(k_1, N_1) \times \mathrm{Gr}(k_2, N_2). \tag{5.19}$$

The explicit values of k_1, k_2, N_1, and N_2 depend on the choice of an integer $m \geqslant b + d$, and are given as follows:

$$\begin{aligned} N_1 &= H^0(V(m-1)) = nm - b, & N_2 &= H^0(V(m)) = n(m+1) - b, \\ r_1 &= \mathrm{rank}(L_{m-1}^{[d]}) = rm + d, & r_2 &= \mathrm{rank}(L_m^{[d]}) = r(m+1) + d, \\ k_1 &= \mathrm{rank}(L_{m-1}^{\{d\}}) = (n-r)m - b - d, & k_2 &= \mathrm{rank}(L_m^{\{d\}}) = (n-r)(m+1) - b - d. \end{aligned} \tag{5.20}$$

We give a short overview of the construction. Let us recall the universal exact sequence

$$0 \to \mathcal{E} \to p^*V \to \mathcal{F} \to 0,$$

on $\mathsf{Quot}_d(\mathbb{P}^1, V, r) \times \mathbb{P}^1$ where π and p are projections to $\mathsf{Quot}_d(\mathbb{P}^1, V, r)$ and $\mathbb{P}^1$ respectively. Twisting the above short exact sequence by p^*L_m and pushing forward, we get an exact sequence of vector bundles (since $m \geqslant b + d$)

$$0 \to \pi_*(\mathcal{E}(m)) \to H^0(V(m)) \otimes \mathcal{O}_{\mathsf{Quot}_d} \to \pi_*(\mathcal{F}(m)) \to 0. \tag{5.21}$$

This gives us a morphism from $\iota_m\colon \mathsf{Quot}_d(\mathbb{P}^1, V, r) \to \mathrm{Gr}(k_2, N_2)$. Similarly, twisting by p^*L_{m-1} yields an analogous sequence of vector bundles as in (5.21) and hence a morphism $\iota_{m-1}\colon \mathsf{Quot}_d(\mathbb{P}^1, V, r) \to \mathrm{Gr}(k_1, N_1)$. The Strømme embedding in (5.19) is defined as $\iota := (\iota_{m-1}, \iota_m)$.

We have universal short exact sequences

$$0 \to \mathcal{A}_i \to W_i \to \mathcal{B}_i \to 0 \tag{5.22}$$

on each $\mathrm{Gr}(k_i, N_i)$ for $i = 1, 2$ where

$$W_1 := H^0(V(m-1)) \otimes \mathcal{O}_{\mathrm{Gr}(k_1,N_1)} \quad \text{and} \quad W_2 = H^0(V(m)) \otimes \mathcal{O}_{\mathrm{Gr}(k_2,N_2)}.$$

By a slight abuse of notation, we use the same notation for the pullbacks of these bundles to the product $\mathrm{Gr}(k_1, N_2) \times \mathrm{Gr}(k_2, N_2)$ along the projection maps.

Consider the section $\mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$ defining the diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$. Taking the tensor product with the pullback of $V(m-1)$ along the first factor of $\mathbb{P}^1$ and passing to cohomology, we get an inclusion map

$$W_1 \to W_2 \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1)).$$

Consider the following diagram on $\mathrm{Gr}(k_1, N_1) \times \mathrm{Gr}(k_2, N_2)$,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}_1 \arrow[r] & W_1 \arrow[r] \arrow[d] & \mathcal{B}_1 \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{A}_2 \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \arrow[r] & W_2 \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \arrow[r] & \mathcal{B}_2 \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \arrow[r] & 0.
\end{tikzcd}$$

The composition

$$\mathcal{A}_1 \to W_1 \to W_2 \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \to \mathcal{B}_2 \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))$$

gives us a section s of the vector bundle

$$\mathcal{K} := \mathcal{A}_1^\vee \otimes \mathcal{B}_2 \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1)).$$

Following the proof of [Str87, Thm. 4.1] with minor modifications, one shows that ι is a closed embedding. Moreover, the image $\iota(\mathsf{Quot}_d)$ is precisely the zero locus of the section s defined above. We note here that

$$L_{m-1}^{[d]} \cong \iota^*(\mathcal{B}_1) \quad \text{and} \quad L_m^{[d]} \cong \iota^*(\mathcal{B}_2).$$

Similarly, we have

$$L_{m-1}^{\{d\}} \cong \iota^*(\mathcal{A}_1) \quad \text{and} \quad L_{m-1}^{\{d\}} \cong \iota^*(\mathcal{A}_2).$$

Proof of Theorem 5.4.1

Let $\iota : \mathsf{Quot}_d(\mathbb{P}^1, V, r) \to \mathrm{Gr}(k_1, N_1) \times \mathrm{Gr}(k_2, N_2)$ be the Strømme embedding associated to the positive integer m, as in Section 5.4. As discussed in the previous section, $\iota(\mathsf{Quot}_d)$ is the vanishing locus of a regular section s of the vector bundle $\mathcal{K}$. The assumption $d \geqslant d_0(V, r)$ implies that Quot_d is a local complete intersection of expected dimension

$$\dim(\mathrm{Gr}(k_1, N_1) \times \mathrm{Gr}(k_2, N_2)) - \operatorname{rank} \mathcal{K} = nd + rb + r(n-r).$$

We thus have the Koszul resolution

$$\cdots \longrightarrow \textstyle\bigwedge^2 \mathcal{K}^\vee \longrightarrow \mathcal{K}^\vee \longrightarrow \mathcal{O}_{\mathrm{Gr}(k_1,N_1)\times \mathrm{Gr}(k_2,N_2)} \longrightarrow \iota_* \mathcal{O}_{\mathsf{Quot}_d} \longrightarrow 0. \tag{5.23}$$

We can apply formulas (5.7) and (5.8) to decompose $\bigwedge^t \mathcal{K}^\vee$ as

$$\begin{aligned}\bigwedge^t \mathcal{K}^\vee &= \bigwedge^t (\mathcal{A}_1 \boxtimes (\mathcal{B}_2^\vee \oplus \mathcal{B}_2^\vee)) \\ &= \bigoplus_{|\mu|=t} \mathbf{S}^\mu \mathcal{A}_1 \boxtimes \mathbf{S}^{\mu^\dagger}(\mathcal{B}_2^\vee \oplus \mathcal{B}_2^\vee) \\ &= \bigoplus_{|\mu|=t} \mathbf{S}^\mu \mathcal{A}_1 \boxtimes \left(\bigoplus_{\alpha,\beta,\sigma} (\mathbf{S}^\sigma \mathcal{B}_2^\vee)^{\oplus c^{\mu^\dagger}_{\alpha,\beta} \cdot c^\sigma_{\alpha,\beta}} \right).\end{aligned}$$

This shows that the direct summands of $\bigwedge^t \mathcal{K}^\vee$ are of the form

$$\mathbf{S}^\mu \mathcal{A}_1 \boxtimes \mathbf{S}^\sigma \mathcal{B}_2^\vee \quad \text{with } |\mu| = t \text{ and there exist } \alpha, \beta \text{ such that } c^{\mu^\dagger}_{\alpha,\beta} \cdot c^\sigma_{\alpha,\beta} \neq 0.$$

Proof of Theorem 5.4.1. The tautological bundles in consideration can be expressed as a pullback via (5.19)

$$\mathbf{S}^\eta L^{[d]}_{m-1} = \iota^* \mathbf{S}^\eta \mathcal{B}_1 \quad \text{and} \quad \mathbf{S}^\rho L^{[d]}_m = \iota^* \mathbf{S}^\rho \mathcal{B}_2.$$

Tensoring the Koszul resolution in (5.23) with $S^\eta \mathcal{B}_1 \boxtimes \mathbf{S}^\rho \mathcal{B}_2$, we obtain

$$\cdots \longrightarrow \mathcal{V}_2 \longrightarrow \mathcal{V}_1 \longrightarrow \mathcal{V}_0 \xrightarrow{\phi} \iota_* \left(\mathbf{S}^\eta L^{[d]}_{m-1} \otimes \mathbf{S}^\rho L^{[d]}_m \right) \longrightarrow 0. \tag{5.24}$$

where $\mathcal{V}_t := (\mathbf{S}^\eta \mathcal{B}_1 \boxtimes \mathbf{S}^\rho \mathcal{B}_2) \otimes \bigwedge^t \mathcal{K}^\vee$.

We will show below in Lemma 5.4.4 that for all $t \geqslant 1$, all cohomology groups of $\mathcal{V}_t$ vanish (with the given conditions on m and the partitions involved). This implies that ϕ induces natural isomorphisms of cohomology groups

$$H^i(\mathrm{Gr}(k_1, N_1) \times \mathrm{Gr}(k_2, N_2),\, \mathbf{S}^\eta \mathcal{B}_1 \boxtimes \mathbf{S}^\rho \mathcal{B}_2) \;\cong\; H^i\Big(\mathsf{Quot}_d,\, \mathbf{S}^\eta L^{[d]}_{m-1} \otimes \mathbf{S}^\rho L^{[d]}_m\Big)$$

for all $i \geqslant 0$.

When $\eta = \gamma$ and $\rho = \lambda$ are partitions, Borel–Weil–Bott theorem (see Remark 5.3.2) tells us that the cohomology groups of $\mathbf{S}^\gamma \mathcal{B}_1$ and $\mathbf{S}^\lambda \mathcal{B}_2$ are nonzero only in degree 0, where they are naturally isomorphic to $\mathbf{S}^\gamma H^0(V \otimes L_{m-1})$ and $\mathbf{S}^\lambda H^0(V \otimes L_m)$, respectively. Hence, proving part (ii).

If δ (resp. ν) is nontrivial, then by the Borel–Weil–Bott theorem (Theorem 5.3.1), all cohomology groups of $\mathbf{S}^\eta \mathcal{B}_1$ (resp. $\mathbf{S}^\rho \mathcal{B}_2$) vanish whenever

$$\delta_1 \leqslant k_1 \quad (\text{resp. } \nu_1 \leqslant k_2).$$

Using the assumptions $m \geqslant d + b$ and $\delta_1 + \nu_1 < n - r$, we have

$$\delta_1 \leqslant n - r - 1 \leqslant (n - r - 1)(b + d) \leqslant (n - r)m - (b + d) = k_1,$$

and similarly $\nu_1 \leqslant k_2$. Note that $d + b \geqslant 0$ (considering the degree of the subsheaf $S \subset V \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$). The extremal case $d + b = 0$ implies S is trivial, and thus $b = d = 0$, which corresponds to the case when Quot_d is the Grassmannian; there the claim follows directly from Borel–Weil–Bott.

∎

Technical vanishing lemma

Lemma 5.4.4. For all $t \geqslant 1$, all cohomology groups of $\mathcal{V}_t$ in (5.24) vanish, if

(a) either δ or ν is a non-empty partition; or

(b) $\delta = \nu = \emptyset$ and m is sufficiently large.

Proof. Following the discussion in Section 5.4, $\bigwedge^t \mathcal{K}^\vee$ splits as a direct sum

$$\bigwedge^t \mathcal{K}^\vee = \bigoplus_{|\mu|=t} \left(\mathbf{S}^\mu \mathcal{A}_1 \boxtimes \mathbf{S}^\sigma (\mathcal{B}_2^\vee)^{\oplus c^{\mu^\dagger}_{\alpha,\beta} \cdot c^\sigma_{\alpha,\beta}} \right)$$

where μ runs over all partitions of size t and α, β and σ runs over partitions. We see that $\mathcal{V}_t$ has direct summands of the form

$$(\mathbf{S}^\mu \mathcal{A}_1 \otimes \mathbf{S}^\eta \mathcal{B}_1) \boxtimes (\mathbf{S}^\chi \mathcal{B}_2^\vee).$$

In order for such a summand to exist and have a nonzero cohomology group, the following conditions must hold simultaneously:

- The Littlewood-Richardson coefficients are non-zero, i.e.,

  $$c^{\mu^\dagger}_{\alpha,\beta} \cdot c^\sigma_{\alpha,\beta} \cdot c^\chi_{\sigma,-\rho} \neq 0.$$

  Here $c^\chi_{\sigma,-\rho}$ is the generalized Littlewood-Richardson coefficient (5.9) for $\mathrm{GL}_{r_2}(\mathbb{C})$-representations and $-\rho = (\nu, -\lambda)$.

- Not all cohomology groups of $\mathbf{S}^\mu \mathcal{A}_1 \otimes \mathbf{S}^\eta \mathcal{B}_1$ vanish over $\mathrm{Gr}(k_1, N_1)$. Lemma 5.3.9 implies that we have a well-defined

  $$i := (r_1; \eta)\text{-index of } \mu.$$

  Recall $\eta = (\gamma, -\delta)$ and note that $\gamma_1^\dagger + \delta_1^\dagger \leqslant r_1$. The index i satisfies the following

  $$\mu_{i+1-s} \geqslant i + r_1 - \gamma_s^\dagger \quad \text{and} \quad \mu_{i+s} \leqslant i + \delta_s^\dagger \quad \text{for all } s \geqslant 1.$$

  This is illustrated in Figure 5.1.

- Not all cohomology groups of $\mathbf{S}^\chi \mathcal{B}_2^\vee$ vanish over $\mathrm{Gr}(k_2, N_2)$. In such a case, Lemma 5.3.6 implies that we have a well-defined

  $$j := k_2\text{-index of } \chi.$$

  The first inequality in Lemma 5.3.6 implies that first j entries $\chi_1, \dots \chi_j \geqslant k_2 + j$. The second inequality in Lemma 5.3.6 and rank of $\mathcal{B}_2$ implies $\chi_{j+1}, \dots, \chi_{r_2} \leqslant j$. This is illustrated in Figure 5.2.

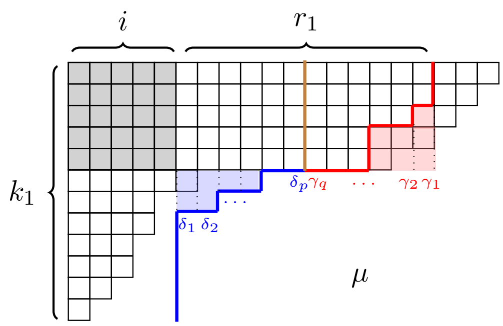


Figure 5.1: A schematic diagram for the shape of the partition μ, whose (r_1, η)-index equals i. The first i parts of μ must lie to the right of bold red outline by γ and the last $k_1 - i$ parts of μ lie left of blue line defined by δ.

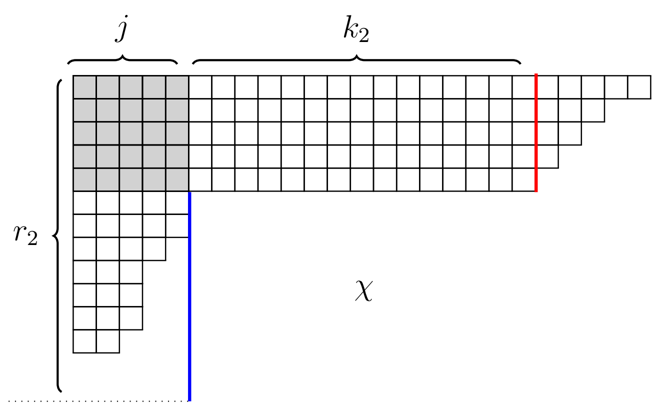


Figure 5.2: A schematic diagram for the shape of χ when it is a partition: The k_2-index equals j. The first j entries of χ must lie to the right of bold red line and the last $r_2 - j$ entries lie left of blue line.

We will apply the properties of Littlewood–Richardson coefficients, listed in Proposition 5.2.2, to arrive at a contradiction on the existence of the indices j and i, which would then show that the conditions above cannot simultaneously be satisfied, whence all the cohomology groups of $\mathcal{V}_t$ must be zero. We divide this into four exhaustive cases:

Case 1: Suppose $j > i$. By Weyl's inequality (5.13), the condition $c^{\sigma}_{\alpha,\beta} \neq 0$ implies

$$\alpha_j + \beta_1 \geqslant \sigma_j.$$

Similarly, from $c^{\chi}_{\sigma,-\rho} \neq 0$ we obtain (recall $-\rho = (\nu, -\lambda)$)

$$\sigma_j + \nu_1 \geqslant \chi_j \geqslant j + k_2.$$

Note that $\beta_1 \leqslant \mu^{\dagger}_1 \leqslant k_1$ (since $c^{\mu}_{\alpha^{\dagger},\beta^{\dagger}} \neq 0$) and $k_2 - k_1 = n - r$. Combining the two inequalities above yields,

$$\alpha_j \geqslant \sigma_j - \beta_1 \geqslant j + k_2 - \nu_1 - k_1 \geqslant j + n - r - \nu_1.$$

Hence the conjugate of α satisfy

$$j \leqslant \alpha^{\dagger}_{j+(n-r-\nu_1)}. \tag{5.25}$$

Using $c^{\mu}_{\alpha^{\dagger},\beta^{\dagger}} \neq 0$ and the assumption $j > i$, we have

$$\alpha^{\dagger}_{j+(n-r-\nu_1)} \leqslant \mu_{j+(n-r-\nu_1)} \leqslant \mu_{i+(n-r-\nu_1)} \leqslant i + \delta^{\dagger}_{(n-r-\nu_1)} \tag{5.26}$$

The assumption $\delta_1 + \nu_1 \leqslant n - r - 1$ implies that $\delta^{\dagger}_{(n-r-\nu_1)} = 0$. Hence we obtain $j \leqslant i$ from (5.25) and (5.26), which is a contradiction.

Case 2: Suppose $j < i \leqslant 2j$. Applying dominance inequalities in Proposition 5.2.2 for

$$\sigma \leqslant \alpha + \beta \quad \text{and} \quad \alpha \cup \beta \leqslant \mu^{\dagger},$$

arising from $c^{\sigma}_{\alpha,\beta}$ and $c^{\mu^{\dagger}}_{\alpha,\beta}$, respectively, we obtain:

$$\begin{aligned} \sigma_1 + \cdots + \sigma_j &\leqslant \alpha_1 + \cdots + \alpha_j + \beta_1 + \cdots + \beta_j \\ &\leqslant \mu^{\dagger}_1 + \cdots + \mu^{\dagger}_{2j} \\ &\leqslant ik_1 + (2j - i)i + |\delta|. \end{aligned} \tag{5.27}$$

The last inequality is evident by analyzing the shape of μ in Figure 5.1. Similarly, using the dominance inequalities for

$$\mu \leqslant \alpha^{\dagger} + \beta^{\dagger} \quad \text{and} \quad \alpha^{\dagger} \cup \beta^{\dagger} \leqslant \sigma^{\dagger},$$

and analyzing Figure 5.1 we obtain:

$$\begin{aligned} i(r_1 + i) - |\gamma| &\leqslant \mu_1 + \cdots + \mu_i \\ &\leqslant \alpha^{\dagger}_1 + \cdots + \alpha^{\dagger}_i + \beta^{\dagger}_1 + \cdots + \beta^{\dagger}_i \\ &\leqslant \sigma^{\dagger}_1 + \cdots + \sigma^{\dagger}_{2i} \end{aligned} \tag{5.28}$$

The non-vanishing of generalized Littlewood-Richardson coefficient $c^{\chi}_{\sigma,-\rho} \neq 0$ implies the following two identities:

$$\sigma_1 + \cdots + \sigma_j \geqslant \chi_1 + \cdots + \chi_j - |\nu| \tag{5.29}$$

$$\sigma_1^\dagger + \cdots + \sigma_{2i}^\dagger \leqslant \chi_1^\dagger + \cdots + \chi_{2i}^\dagger + |\lambda| \tag{5.30}$$

The first inequality (5.29) simply follows from the dominance inequality (5.14), while the second inequality (5.30) requires an argument, that we provide in Lemma 5.4.5. For the tuple χ, we define $\chi_s^\dagger$ for the number of entries in χ that are at least s.

Combining the inequalities (5.27) and (5.29), and analyzing the shape of χ in Figure 5.2, we obtain

$$j(k_2 + j) - |\nu| \leqslant \chi_1 + \cdots + \chi_j - |\nu| \leqslant ik_1 + (2j - i)i + |\delta|. \tag{5.31}$$

Similarly, by combining (5.28) and (5.30) and analyzing Figure 5.2, we have

$$i(r_1 + i) - |\gamma| \leqslant \chi_1^\dagger + \cdots + \chi_{2i}^\dagger + |\lambda| \leqslant jr_2 + (2i - j)j + |\lambda| \tag{5.32}$$

These two inequalities (5.31) and (5.32) can be rearranged into the following useful form:

$$j(k_2 - k_1) \leqslant (i - j)k_1 - (i - j)^2 + |\delta| + |\nu|, \tag{5.33}$$

$$j(r_2 - r_1) \geqslant (i - j)r_1 + (i - j)^2 - |\gamma| - |\lambda|. \tag{5.34}$$

Recall that $k_2 - k_1 = n - r$ and $r_2 - r_1 = r$. Multiplying the first inequality by r and the second by $n - r$, we get:

$$\begin{aligned} r(|\delta| + |\nu|) + (n - r)(|\gamma| + |\lambda|) &\geqslant (i - j)\left((n - r)r_1 - rk_1\right) + n(i - j)^2 \\ &= (nd + rb)(i - j) + n(i - j)^2 \\ &\geqslant nd + rb + n, \end{aligned}$$

where we used the inequality $i - j \geqslant 1$ in the final step. This contradicts the hypothesis of the lemma.

Case 3: Suppose $2j < i$. In the previous case, the assumption $i \leqslant 2j$ was used only in (5.31), so using our new assumption $2j < i$, we can replace (5.31) by

$$j(k_2 + j) - |\nu| \leqslant ik_1, \tag{5.35}$$

while (5.32) stays the same. We can rewrite (5.34) and (5.35) as

$$j(n - r) = j(k_2 - k_1) \leqslant (i - j)k_1 - j^2 + |\nu|, \tag{5.36}$$

$$jr = j(r_2 - r_1) \geqslant (i - j)r_1 + (i - j)^2 - |\gamma| - |\lambda|. \tag{5.37}$$

Multiplying (5.36) and (5.37) by r and $n - r$ respectively, we obtain

$$\begin{aligned} r|\nu| + (n - r)(|\gamma| + |\lambda|) &\geq (i - j)((n - r)r_1 - rk_1) + rj^2 + (i - j)^2(n - r) \\ &> (nd + rb)(i - j) + rj^2 + (n - r)(i - j)^2 \end{aligned} \tag{5.38}$$

When $j = 0$, we have $\chi_1 \leqslant 0$. By Littewood-Richardson rule, we have $\nu_1 \leqslant \chi_1$, so $|\nu| = 0$. Let us assume $j > 0$ or $|\delta| > 0$, i.e. the hypothesis of part (a). Then using (5.38), we obtain the inequality

$$r(|\delta| + |\nu|) + (n - r)(|\gamma| + |\lambda|) \geq nd + rb + n,$$

which is a contradiction.

So let us suppose that $j = 0$ and $|\delta| = 0$, i.e. the hypothesis of part (b). Using $j = 0$, (5.37) gives us $|\lambda| + |\gamma| \geqslant ir_1 + i^2 \geqslant r_1 = rm + d$, so we get a contradiction for m sufficiently large.

Case 4: Suppose $i = j$. Since $|\mu| = t \geqslant 1$, $i \neq 0$. We can slightly modify the inequality (5.31) as follows:

$$\begin{aligned} j(k_2 + j - \nu_1) &\leq \sigma_1 + \cdots + \sigma_j \\ &\leq \mu_1^\dagger + \cdots + \mu_{2j}^\dagger \\ &\leqslant ik_1 + i(2j - i + \delta_1). \end{aligned}$$

But since $i = j$, we get $\nu_1 + \delta_1 \geq n - r + i \geqslant n - r$, which is a contradiction. $\square$

Lemma 5.4.5. Suppose the generalized Littlewood–Richardson coefficient $c^{\chi}_{\sigma,(\nu.-\lambda)} \neq 0$, for $\mathrm{GL}_{r_2}(\mathbb{C})$ representations, where σ is a partition. Then for any positive integer s,

$$\sigma_1^\dagger + \cdots + \sigma_s^\dagger - |\lambda| \leqslant \chi_1^\dagger + \cdots + \chi_s^\dagger,$$

Proof. Let $\chi = (\chi_1, \ldots, \chi_{r_2}) = (\omega, -\theta)$, and note that $\chi_j^\dagger = \omega_j^\dagger$ for all $j > 0$. The size constraint implies the equality

$$|\sigma| + |\nu| - |\lambda| = |\omega| - |\theta|. \tag{5.39}$$

It suffices to prove the inequality

$$\sigma_{s+1}^\dagger + \cdots + \sigma_M^\dagger + |\nu| \geqslant \chi_{s+1}^\dagger + \cdots + \chi_M^\dagger \geqslant \chi_{s+1}^\dagger + \cdots + \chi_M^\dagger - |\theta|,$$

and the the result follows after subtracting it from equation (5.39).

Fix an integer $\ell \geqslant \max\{\theta_1, \lambda_1\}$, then we have the classical Littlewood-Richardson coefficient $c^{\phi}_{\sigma,\tau}$, where $\tau = (\nu, -\lambda) + (\ell)^{r_2}$ and $\phi = \chi + (\ell)^{r_2}$ are partitions. Apply the Horn's inequality (5.12) coming from $c^{\phi^\dagger}_{\sigma^\dagger,\tau^\dagger} \neq 0$, we obtain

$$\sigma_{s+1}^\dagger + \cdots + \sigma_M^\dagger + \tau_{\ell+1}^\dagger + \cdots + \tau_{\ell+M}^\dagger \geqslant \phi_{\ell+s+1}^\dagger + \cdots + \phi_{\ell+M}^\dagger$$

Note that $\tau_{\ell+j}^\dagger = \nu_j^\dagger$ and $\phi_{\ell+s+j}^\dagger = \chi_{s+j}^\dagger$ for all $j > 0$. Hence we finish the proof by letting M large enough. $\square$

Extension groups

We will now specialize Theorem 5.4.1 to calculate the extension groups and prove all the conclusions in Theorem 5.1.8. Note that the cohomology groups of Schur functors of two dual tautological bundles in Theorem 5.1.10 immediately follow from Theorem 5.4.1.

Proposition 5.4.6. Assume V splits as a direct sum of line bundles of nonpositive degrees. Let L be a line bundle with degree at least $d+b$. Let ν and λ be two partitions such that $r|\nu|+(n-r)|\lambda|<nd+rb+n$ and $\nu_1<n-r$. Then

$$\mathrm{Ext}^i\left(\mathbf{S}^\nu L^{[d]},\mathbf{S}^\lambda L^{[d]}\right)=\begin{cases}\mathrm{Hom}\left(\mathbf{S}^\nu\mathcal{B},\mathbf{S}^\lambda\mathcal{B}\right) & i=0\\ 0 & i\geqslant 1.\end{cases}\tag{5.40}$$

where $\mathcal{B}$ is the universal quotient bundle on the Grassmannian $\mathrm{Gr}(k,N)$, where $N=\dim H^0(V\otimes L)$ and $k=N-\mathrm{rank}\,L^{[d]}$.

Proof. We follow the notation in the proof of Theorem 5.4.1. We may express

$$\mathbf{S}^\nu(L^{[d]})^\vee\otimes\mathbf{S}^\lambda L^{[d]}=\bigoplus_\eta\left(\mathbf{S}^\eta(L^{[d]})^\vee\right)^{\oplus c^\eta_{\nu,-\lambda}},$$

where $\eta=(\eta_1,\eta_2,\dots,\eta_{N_1-k_1})=(\gamma,-\delta)$ runs over tuples of non-increasing integers, and $c^\eta_{\nu,-\lambda}$ are the modified Littlewood–Richardson coefficients (5.10). Horn's inequalities imply that

$$\eta_j\leqslant\nu_j+(-\lambda)_1\leqslant\nu_j\quad\text{for all } j\geqslant 1,$$

hence $|\gamma|\leqslant|\nu|$ and $\eta_1\leqslant\nu_1<n-r$. Similarly, we obtain $|\delta|\leqslant|\lambda|$, and thus Theorem 5.4.1 implies that all higher cohomology groups vanish and the zeroth cohomology group is expressed in terms of the cohomology groups of $\mathcal{B}_1$ as stated. □

Proof of Theorem 5.1.8. Proposition 5.4.6 immediately implies part (i), and a straightforward application of the Borel–Weil–Bott theorem yields part (ii). The statement about the exceptional collection follows by noting that $\lambda_1<n-r$ and $m\geqslant b+d$ imply $\lambda_1\leqslant k_1$, and by invoking [Kap84] to observe that

$$\{\mathbf{S}^\lambda\mathcal{B}_1\ :\ |\lambda|\leqslant d,\ \lambda_1<n-r\}$$

is contained in a full exceptional collection for $\mathrm{Gr}(k_1,N_1)$. □

Cohomological degree of the tautological bundles

In Section 5.4, we showed that, except for the space of global sections of $\mathbf{S}^\eta L^{[d]}_{m-1}\otimes\mathbf{S}^\rho L^{[d]}_m$, all cohomology groups vanish, and we explicitly computed H^0 under size constraints on η and ρ. Here we remove the constraints on η and ρ and establish the vanishing of the higher cohomology groups, strengthening Theorem 2.1.6.

Proposition 5.4.7. Assume V splits as a direct sum of line bundles of nonpositive degrees. Let $\eta=(\gamma,-\delta)$, $\rho=(\lambda,-\nu)$. Then for all $m\geq d+b$,

$$H^D(\mathsf{Quot}_d,\mathbf{S}^\eta L^{[d]}_{m-1}\otimes\mathbf{S}^\rho L^{[d]}_m)=0\quad\text{for all } D>|\delta|+|\nu|.$$

In particular, when $\eta=\gamma$ and $\rho=\lambda$ are partitions,

$$H^D(\mathsf{Quot}_d,\mathbf{S}^\gamma L^{[d]}_{m-1}\otimes\mathbf{S}^\lambda L^{[d]}_m)=0\quad\text{for all } D>0.$$

Proof. First, recall the Koszul resolution (5.24), and note that it suffices to show

$$H^D\Big(\operatorname{Gr}(k_1,n_1)\times\operatorname{Gr}(k_2,n_2),\mathcal{V}_t\Big)=0\quad\text{for all } t\geqslant 0,\ \text{and } D>t+|\delta|+|\nu|.$$

Indeed, assume there exist partitions $\mu,\alpha,\beta,\sigma,\chi$ such that

$$H^D\Big(\operatorname{Gr}(k_1,n_1)\times\operatorname{Gr}(k_2,n_2),\Big(\mathbf{S}^\mu\mathcal{A}_1\otimes\mathbf{S}^\eta\mathcal{B}_1\Big)\boxtimes\Big(\mathbf{S}^\chi\mathcal{B}_2^\vee\Big)\Big)\neq 0,$$

with $c^{\mu^\dagger}_{\alpha,\beta}\cdot c^{\sigma}_{\alpha,\beta}\cdot c^{\chi}_{\sigma,-\rho}\neq 0$. Recall the indices i and j from the proof of Lemma 5.4.4. Analyzing the non-vanishing of the cohomology groups and the Littlewood–Richardson coefficients, in particular, the inequalities (5.27), (5.29), and (5.31), implies

$$j(k_2+j)-|\nu|\ \leqslant\ \mu_1^\dagger+\cdots+\mu_{2j}^\dagger.$$

By Lemma 5.3.9(ii) the cohomological degree satisfies

$$D\leqslant k_2j+|\delta|+(\mu_1+\cdots+\mu_i)-i^2=k_2j+|\delta|+ip+\sum_{s\geqslant i+p+1}\mu_s^\dagger,$$

where k_2j is the cohomological degree contributed by $\mathbf{S}^\chi\mathcal{B}_2^\vee$. Adding the two inequalities above yields

$$|\delta|+|\nu|+|\mu|\ \geqslant\ D+\begin{cases} j^2-ip+\displaystyle\sum_{s=2j+1}^{i+p}\mu_s^\dagger, & \text{if } 2j\leqslant i+p,\\ j^2-ip, & \text{if } 2j>i+p.\end{cases}$$

In both cases on the right, the expression is nonnegative, which is precisely what we need. Indeed, if $2j>i+p$, then $j^2-ip\geqslant(i+p)^2/4-ip\geqslant 0$, and if $2j\leqslant i+p$, then

$$j^2-ip+\sum_{s=2j+1}^{i+p}\mu_s^\dagger\ \geqslant\ j^2-ip+i(i+p-2j)=(i-j)^2\ \geqslant\ 0.$$

□

5.5 Multiple line bundles and vector bundles

In the previous section, we calculated the cohomology groups of $\mathbf{S}^\eta L^{[d]}_{m-1}\otimes\mathbf{S}^\rho L^{[d]}_m$ for sufficiently positive m. We required m to be large enough so that these tautological bundles are pulled back along a Strømme embedding. In order to access those degrees inaccessible through a Strømme embedding, we work with Schur complexes, introduced in Subsection 5.5 below. This will allow us to bootstrap Theorem 5.4.1 to prove Theorem 5.1.1 in full generality.

Schur functor of perfect complexes

For a partition λ of n, let π_λ be the irreducible representation of S_n corresponding to λ. Then Schur functors can be defined for a perfect complex A of arbitrary length as $\mathbf{S}^\lambda(A) = (\pi_\lambda \otimes A^{\otimes n})^{S_n}$ (see for instance Page 302 between Theorems 9.11.4 and 9.11.5 in [Eti+17]). From this definition, it is also clear that in characteristic zero, Schur functors respect quasi-isomorphisms: taking tensor products respect quasi-isomorphisms, and taking S_n invariants is exact in characteristic zero.

When we write $\mathbf{S}^\lambda M^{[d]}$ for $M^{[d]}$ is an object in the derived category of the Quot scheme, we mean to represent $M^{[d]}$ first as a perfect complex and take the Schur functor. In our case, we can actually always represent $M^{[d]}$ and $M^{\{d\}}$ as a map of vector bundles. Below we recall definitions and basic properties of Schur functors associated to a partition and a map of vector bundles. The construction in this case is called Schur complexes in the literature. See [Wey03, Chapter 2] for details.

Let X be a locally Noetherian scheme and let

$$\Phi : E_1 \to E_2$$

be a map of vector bundles on X.

Definition 5.5.1. Let λ be a partition. We write $\mathbf{S}^\lambda(\Phi)^\bullet$ for the Schur complex of length $|\lambda|$, so that

$$\mathbf{S}^\lambda(\Phi)^q = \bigoplus_{\nu\subseteq\lambda, |\nu|=q} \mathbf{S}^{\lambda/\nu} E_1 \otimes \mathbf{S}^{\nu^\dagger} E_2. \tag{5.41}$$

is placed in q-th degree. The beginning of $\mathbf{S}^\lambda(\Phi)^\bullet$ looks like

$$\mathbf{S}^\lambda E_1 \to \mathbf{S}^{\lambda/(1)} E_1 \otimes E_2 \to \mathbf{S}^{\lambda/(2)} E_1 \otimes \wedge^2 E_2 \oplus \mathbf{S}^{\lambda/(1^2)} E_1 \otimes \operatorname{Sym}^2 E_2 \to \cdots$$

Sometimes it is useful to index homologically as well. We write $\mathbf{S}^\lambda(\Phi)_\bullet$ for the Schur complex of length $|\lambda|$ associated to the map Φ, so that

$$\mathbf{S}^\lambda(\Phi)_q = \bigoplus_{\nu\subseteq\lambda, |\nu|=q} \mathbf{S}^{\nu^\dagger} E_1 \otimes \mathbf{S}^{\lambda/\nu} E_2. \tag{5.42}$$

is placed in $(-q)$-th degree. The beginning of $\mathbf{S}^\lambda(\Phi)_\bullet$ looks like

$$\cdots \to \operatorname{Sym}^2 E_1 \otimes \mathbf{S}^{\lambda/(1^2)} E_2 \oplus \wedge^2 E_1 \otimes \mathbf{S}^{\lambda/(2)} E_2 \to E_1 \otimes \mathbf{S}^{\lambda/(1)} E_2 \to \mathbf{S}^\lambda E_2.$$

Note that $\mathbf{S}^\lambda(\Phi)_\bullet = \mathbf{S}^{\lambda^\dagger}(\Phi)^\bullet[\ell]$, where $\ell = |\lambda|$.

Proposition 5.5.2. If we have a short exact sequence

$$0 \to E_1 \xrightarrow{\Psi} E_2 \xrightarrow{\Phi} E_3 \to 0, \tag{5.43}$$

where each E_i is a vector bundle, then $\mathbf{S}^\lambda(\Psi)_\bullet \to \mathbf{S}^\lambda E_3 \to 0$ is exact, and $0 \to \mathbf{S}^\lambda E_1 \to \mathbf{S}^\lambda(\Phi)^\bullet$ is exact.

Proof. Since E_3 is locally free, the short exact sequence (5.43) is locally split. Acyclicity can be checked locally, so [ABW82, Corollary V.1.15] implies that $(\mathbf{S}^\lambda\Psi)_\bullet$ is acyclic. The same result also shows that the cokernel locally agrees with $\mathbf{S}^\lambda E_3$, and the functoriality of Schur complexes shows that the isomorphisms must glue, so, globally, the cokernel is also exactly $\mathbf{S}^\lambda E_3$.

The first statement applied to the dual of (5.43) shows that the complex $S^\lambda(\Phi^\vee : E_3^\vee \to E_2^\vee) \to S^\lambda E_1^\vee \to 0$ is exact. Dualizing this complex shows the second statement. $\square$

Remark 5.5.3. In the theorem statements, such as Theorem 5.1.1, we sometimes write $\mathbf{S}^\lambda H^\bullet(V \otimes M)$, where V, M are vector bundles on $\mathbb{P}^1$. The definition of the Schur functor is rather trivial in this case, but requires caution. For instance, $\mathbf{S}^\lambda H^\bullet(\mathcal{O}(-k))$, where $k > 1$, is the vector space $\mathbf{S}^{\lambda^\dagger} H^1(\mathcal{O}(-k))$, concentrated in degree $|\lambda|$.

Vanishing results for two insertions

In Theorem 5.4.1, we computed the cohomology groups of Schur functors of $L_{m-1}^{[d]}$ and $L_m^{[d]}$. We shall now use the properties of Schur complex to conclude results about vector bundles $L_{m-1}^{\{d\}}$ and $L_m^{\{d\}}$ and their Schur functors.

Lemma 5.5.4. Let L_{m-1} and L_m line bundles with m sufficiently large. Let $\mu, \nu, \gamma, \lambda$ be partitions satisfying

$$|\mu| + |\nu| + |\gamma| + |\lambda| < (nd + rb + n)/(n - r).$$

Then all cohomology groups of the vector bundles

$$\mathbf{S}^\mu L_{m-1}^{\{d\}} \otimes \mathbf{S}^\nu L_m^{\{d\}} \otimes \mathbf{S}^\gamma L_{m-1}^{[d]} \otimes \mathbf{S}^\lambda L_m^{[d]}$$

vanish unless $\mu = \nu = \emptyset$.

Proof. Let m be sufficiently large such that part (ii) of Theorem 5.4.1 holds. Now consider the exact sequence of vector bundles

$$0 \longrightarrow L_m^{\{d\}} \longrightarrow W_m \xrightarrow{\Phi_m} L_m^{[d]} \longrightarrow 0$$

where $W_m := \pi_*(p^*(V \otimes L_m))$.

We will first prove the case $\mu = \emptyset$ and $\nu \neq \emptyset$. Applying the Schur functor $\mathbf{S}^\nu$ to the above, then tensoring with the vector bundles $F := \mathbf{S}^\gamma L_{m-1}^{[d]} \otimes \mathbf{S}^\lambda L_m^{[d]}$ we obtain a complex

$$0 \longrightarrow F \otimes \mathbf{S}^\nu L_m^{\{d\}} \longrightarrow F \otimes \mathbf{S}^\nu(\Phi_m)^\bullet$$

which is exact by applying Proposition 5.5.2. Therefore, the cohomology groups of the vector bundle $F \otimes \mathbf{S}^\nu L_m^{\{d\}}$ equals the hypercohomology groups of the complex $F \otimes \mathbf{S}^\nu(\Phi_m)^\bullet$. Now consider the hypercohomolgy spectral sequence

$$E_1^{p,q} \;\Rightarrow\; H^{p+q}(\mathsf{Quot}_d, F \otimes \mathbf{S}^\nu(\Phi_m)^\bullet) \tag{5.44}$$

whose first page is given by

$$E_1^{p,q} := H^q\left(\mathsf{Quot}_d, F \otimes \mathbf{S}^\nu(\Phi_m)^p\right).$$

By the definition of Schur complex, we note that $F \otimes \mathbf{S}^\nu(\Phi_m)^p$ splits as a direct sum with summands of the form $F \otimes \mathbf{S}^\alpha W_m \otimes \mathbf{S}^\beta L_m^{[d]}$, such that $|\alpha|+|\beta| = |\nu|$ and $|\beta| = p$. Note that W_m is a trivial bundle. Applying Littlewood-Richardson rule, we further split $F \otimes \mathbf{S}^\alpha W_m \otimes \mathbf{S}^\beta L_m^{[d]}$ into direct sum, with summands of form

$$\mathbf{S}^\gamma L_{m-1}^{[d]} \otimes \mathbf{S}^\eta L_m^{[d]} \tag{5.45}$$

where

$$|\gamma| + |\eta| = |\gamma| + |\lambda| + |\beta| \leqslant |\gamma| + |\lambda| + |\nu| < (nd + rb + n)/(n - r).$$

Theorem 5.4.1 implies that q-th cohomology groups of each of these summand vanish for all $q \geqslant 1$. Therefore, $H^q(\mathsf{Quot}_d, F \otimes \mathbf{S}^\alpha W_m \otimes \mathbf{S}^\beta L_m^{[d]}) = 0$, and thus $E_1^{p,q} = 0$ for all $q \geqslant 1$. We will now show that the first row corresponding to $q = 0$

$$0 \to E_1^{0,0} \to E_1^{1,0} \to \cdots \to E_1^{|\nu|,0} \to 0 \tag{5.46}$$

in page one is exact. Consider the diagram

$$\begin{array}{ccc} W_m & \xrightarrow{id} & W_m \\ \downarrow & & \downarrow \\ W_m & \xrightarrow{\Phi_m} & L_m^{[d]} \end{array}$$

and the induced morphisms of complexes

$$\tilde{F} \otimes \mathbf{S}^\nu(id)^\bullet \longrightarrow F \otimes \mathbf{S}^\nu(id)^\bullet \longrightarrow F \otimes \mathbf{S}^\nu(\Phi_m)^\bullet.$$

where $\tilde{F} = \mathbf{S}^\gamma W_{m-1} \otimes \mathbf{S}^\lambda W_m$. Theorem 5.4.1 implies that the induced morphism for each summand (5.45) in the p-th degree

$$\mathbf{S}^\gamma W_{m-1} \otimes \mathbf{S}^\eta W_m \to \mathbf{S}^\gamma L_{m-1}^{[d]} \otimes \mathbf{S}^\eta L_m^{[d]}$$

induces isomorphisms for the space of global sections, and hence induces isomorphisms

$$H^0(\mathsf{Quot}_d, \tilde{F} \otimes \mathbf{S}^\nu(id)^p) \cong H^0(\mathsf{Quot}_d, F \otimes \mathbf{S}^\nu(\Phi_m)^p) = E_1^{p,0}.$$

Note that the complex of trivial bundles $\tilde{F} \otimes \mathbf{S}^\nu(id)^\bullet$ (equivalently of their space of global sections) is exact by Proposition 5.5.2, and hence (5.46) is exact. Therefore the spectral sequence (5.44) degenetates in page two with all the terms zero, so we conclude that

$$H^D(\mathsf{Quot}_d, \mathbf{S}^\nu L_m^{\{d\}} \otimes \mathbf{S}^\gamma L_{m-1}^{[d]} \otimes \mathbf{S}^\lambda L_m^{[d]}) \cong H^D(\mathsf{Quot}_d, F \otimes \mathbf{S}^\nu(\Phi_m)^\bullet) = 0$$

for all $D \geqslant 0$. The same proof works when $\nu = \emptyset$ and $\mu \neq \emptyset$, and it gives us

$$H^D(\mathsf{Quot}_d, \mathbf{S}^\mu L_{m-1}^{\{d\}} \otimes \mathbf{S}^\gamma L_{m-1}^{[d]} \otimes \mathbf{S}^\lambda L_m^{[d]}) = 0 \quad \text{for all } D \geqslant 0. \tag{5.47}$$

When both μ and ν are non-trivial, we again consider the exact sequence,

$$0 \longrightarrow F \otimes \mathbf{S}^\nu L_m^{\{d\}} \longrightarrow F \otimes \mathbf{S}^\nu (\Phi_m)^\bullet$$

where $F := \mathbf{S}^\mu L_{m-1}^{\{d\}} \otimes \mathbf{S}^\gamma L_{m-1}^{[d]} \otimes \mathbf{S}^\lambda L_m^{[d]}$. Running the same argument as above, we note that, for each p, the bundle $F \otimes \mathbf{S}^\nu(\Phi_m)^p$ splits as a direct sum with summands of the form

$$\mathbf{S}^\mu L_{m-1}^{\{d\}} \otimes \mathbf{S}^\gamma L_{m-1}^{[d]} \otimes \mathbf{S}^\eta L_m^{[d]}$$

for partitions η with $|\eta| \leqslant |\lambda| + |\nu|$. Since μ is nontrivial, all cohomology of the above summands vanish by (5.47), and hence

$$H^q(\mathsf{Quot}_d, F \otimes \mathbf{S}^\nu(\Phi_m)^p) = 0$$

for all p and q. This implies that each object in the first page of the corresponding hypercohomology spectral sequence is zero, and thus $H^D(\mathsf{Quot}_d, F \otimes \mathbf{S}^\nu L_m^{\{d\}}) = 0$ for all $D \geqslant 0$. $\square$

We now prove Theorem 5.1.11 about the vanishing of the cohomology of $\mathcal{S}_x$, restriction of the universal subsheaf $\mathcal{S}$ to $\mathsf{Quot}_d \times \{x\}$, using Lemma 5.5.4.

Proof of Theorem 5.1.11. Let $L_m = \mathcal{O}(m)$ and $L_{m-1} = \mathcal{O}(m-1)$ for the same choice of m as in part (ii) of Theorem 5.4.1 and Lemma 5.5.4. For a point $x \in \mathbb{P}^1$, the short exact sequence

$$0 \to L_{m-1} \to L_m \to \mathcal{O}_x \to 0$$

induces the short exact sequence involving tautological bundles

$$0 \longrightarrow L_{m-1}^{\{d\}} \xrightarrow{\Psi} L_m^{\{d\}} \longrightarrow \mathcal{E}_x \longrightarrow 0.$$

on Quot_d. Proposition 5.5.2 provides a resolution

$$\mathbf{S}^\lambda(\Psi)_\bullet \longrightarrow \mathbf{S}^\lambda \mathcal{E}_x \longrightarrow 0$$

for a partition λ. When $0 \neq |\lambda| < (nd+rb+n)/(n-r)$, Lemma 5.5.4 applies to each term in the resolution, and hence proving that all cohomology groups of $\mathbf{S}^\lambda\mathcal{E}_x$ vanish. The statement involving multiple partitions is a simple consequence of Littlewood-Richardson rule. $\square$

Proof of Theorem 5.1.1

As noted in Remark 5.1.4, every vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles, so Theorem 5.1.1 follows from the following statement concerning multiple line bundles.

Theorem 5.5.5. Let $K_1, K_2, \dots K_s$ and $M_1, M_2, \dots, M_t$ be line bundles on $\mathbb{P}^1$. For any tuples of partitions $\mu^1, \mu^2, \dots, \mu^s$ and $\lambda^1, \lambda^2, \dots, \lambda^t$ satisfying

$$|\mu^1| + \cdots + |\mu^s| + |\lambda^1| + \cdots + |\lambda^t| < (nd + rb + n)/(n-r),$$

(a) If μ^i is a nontrivial partition for some $1 \leqslant i \leqslant s$, then

$$H^\bullet\left(\mathsf{Quot}_d, \bigotimes_{i=1}^{s} \mathbf{S}^{\mu^i} K_i^{\{d\}} \otimes \bigotimes_{j=1}^{t} \mathbf{S}^{\lambda^j} M_j^{[d]}\right) = 0.$$

(b) If $\mu^1 = \cdots = \mu^s = \emptyset$, then

$$H^\bullet\left(\mathsf{Quot}_d, \bigotimes_{j=1}^{t} \mathbf{S}^{\lambda^j} M_j^{[d]}\right) \cong \bigotimes_{j=1}^{t} \mathbf{S}^{\lambda^j} H^\bullet(V \otimes M_j).$$

Remark 5.5.6. The universal quotient $p^*V \to \mathcal{Q}$ induces natural maps $H^\bullet(V \otimes M_j) \to H^\bullet(M_j^{[d]})$ for each j. The proof below will additionally show that the isomorphism in part (b) is given by taking Schur functors and tensoring these natural maps.

Proof of Theorem 5.5.5. Fix sufficiently large integer m such that Theorem 5.4.1(ii) and Lemma 5.5.4 hold for Schur functors of consecutive line bundles $L_{m-1} = \mathcal{O}(m-1)$ and $L_m = \mathcal{O}(m)$. The key idea here is to represent each $K_i^{\{d\}}$ and $M_j^{[d]}$ as two-term complexes involving only $L_{m-1}^{[d]}$ and $L_m^{[d]}$, so that we may analyze Schur functors of the latter instead.

Note that $R^1\pi_*(\mathcal{F} \otimes p^*L_{m-1}) = 0$, where π and p are the projections maps (5.1) for $\mathsf{Quot}_d \times \mathbb{P}^1$. The diagonal sequence for $\mathbb{P}^1$ gives an exact sequence [Str87, Proposition 1.1]

$$0 \longrightarrow \pi^* L_{m-1}^{[d]} \otimes p^*\mathcal{O}(-1) \longrightarrow \pi^* L_m^{[d]} \longrightarrow \mathcal{F} \otimes p^* L_m \longrightarrow 0$$

on $\mathsf{Quot}_d \times \mathbb{P}^1$. For any line bundle M, twist the above sequence by $p^*(M \otimes L_m^\vee)$, and push forward via π to obtain an exact triangle

$$L_{m-1}^{[d]} \otimes R\pi_* p^*(M \otimes L_{m-1}^\vee) \xrightarrow{\Phi_M} L_m^{[d]} \otimes R\pi_* p^*(M \otimes L_m^\vee)) \longrightarrow M^{[d]} \xrightarrow{+1} \tag{5.48}$$

in the derived category of coherent sheaves on Quot_d. Explicitly, $R\pi_* p^*(M \otimes L_m^\vee) = H^\bullet(M \otimes L_m^\vee) \otimes \mathcal{O}_{\mathsf{Quot}_d}$ is a trivial vector bundle supported in degree 0 or 1; the complex is identically zero if $\deg M \otimes L_m^\vee = -1$. This exhibits $M^{[d]}$ as quasi-isomorphic to a two-term complex of vector bundles, denoted Φ_M, which is

$$\Phi_M = \begin{cases} L_{m-1}^{[d]} \otimes H^0(M \otimes L_{m-1}^\vee) \to L_m^{[d]} \otimes H^0(M \otimes L_m^\vee) & \text{if } \deg(M \otimes L_m^\vee) \geq 0, \\ L_{m-1}^{[d]} \otimes H^1(M \otimes L_{m-1}^\vee) \to L_m^{[d]} \otimes H^1(M \otimes L_m^\vee) & \text{if } \deg(M \otimes L_m^\vee) \leq -1, \end{cases}$$

where the first complex $(\Phi_M)_\bullet$ is viewed as concentrated in degrees $[-1, 0]$, and the second $(\Phi_M)^\bullet$ is concentrated in degrees $[0, 1]$. Similarly, replacing $\mathcal{F}$ by $\mathcal{E}$ in the above argument, we obtain the exact triangle

$$L_{m-1}^{\{d\}} \otimes R\pi_* p^*(K \otimes L_{m-1}^\vee) \xrightarrow{\Psi_K} L_m^{\{d\}} \otimes R\pi_* p^*(K \otimes L_m^\vee)) \longrightarrow K^{\{d\}} \xrightarrow{+1} \tag{5.49}$$

for any line bundle K.

Thus, our problem is reduced to computing the hypercohomology groups of the complex

$$G^\bullet := \bigotimes_{i=1}^{s} \mathbf{S}^{\mu^i}(\Psi_{K_i})^\bullet \otimes \bigotimes_{j=1}^{t} \mathbf{S}^{\lambda^j}(\Phi_{M_j})^\bullet.$$

Note that we write $\mathbf{S}^{\mu^i}(\Psi_{K_i})^\bullet$, $\mathbf{S}^{\lambda^j}(\Phi_{M_j})^\bullet$ instead of $\mathbf{S}^{\mu^i}(\Psi_{K_i})_\bullet$, $\mathbf{S}^{\lambda^j}(\Phi_{M_j})_\bullet$ for all i, j for ease of presentation, but apply the corresponding construction in Definition 5.5.1 depending on whether Ψ_{K_i}, Φ_{M_j} are concentrated in degrees $[0,1]$ or $[-1,0]$. Now the degree q term G^q in the complex $G^\bullet$ splits as direct sum of vector bundles of the form

$$\mathbf{S}^\alpha L_{m-1}^{\{d\}} \otimes \mathbf{S}^\beta L_m^{\{d\}} \otimes \mathbf{S}^\gamma L_{m-1}^{[d]} \otimes \mathbf{S}^\delta L_m^{[d]}$$

for some partitions $\alpha, \beta, \gamma, \delta$ satisfying

$$|\alpha| + |\beta| = |\mu^1| + \cdots + |\mu^s| \quad \text{and} \quad |\gamma| + |\delta| = |\lambda^1| + \cdots + |\lambda^t|.$$

If μ^i is nontrivial for some i, then either α or β is nontrivial, and thus, all its cohomology groups vanish by Lemma 5.5.4. Therefore, the first page of the hypercohomology spectral sequence for $G^\bullet$ is identically zero, so we get $H^D(\mathsf{Quot}_d, G^\bullet) = 0$ for all $D \geqslant 0$. This concludes the proof of part (a).

To prove part (b), we will use the exact triangle

$$M_j^{\{d\}} \xrightarrow{\Theta_{M_j}} R\pi_*(p^*(V \otimes M_j)) \longrightarrow M_j^{[d]} \xrightarrow{+1} \tag{5.50}$$

where $R\pi_*(p^*(V \otimes M_j)) \cong H^\bullet(V \otimes M_j) \otimes \mathcal{O}_{\mathsf{Quot}_d}$. Let us first observe that, when $\deg M_j \geqslant d + b$, it is a vector bundle supported in degree zero, and whenever $\deg M_j < 0$, the complex $M_j^{\{d\}}$ is vector bundle $R^1\pi_*(p^*M_j \otimes \mathcal{S})$ supported in degree one.

When $\deg M_j \geqslant d + b$, we apply Proposition 5.5.2 to see that $\mathbf{S}^{\lambda^j} M_j^{[d]}$ is quasi-isomorphic to $\mathbf{S}^{\lambda^j}(\Theta_{M_j})_\bullet$, supported in degrees $[-|\lambda^j|, 0]$, with $(-q)$-th term given by

$$\mathbf{S}^{\lambda^j}(\Theta_{M_j})_q = \bigoplus_{\alpha \subset \lambda^j, |\alpha| = s} \mathbf{S}^{\alpha^\dagger} M_j^{\{d\}} \otimes \mathbf{S}^{\lambda^j/\alpha} \left(H^0(V \otimes M_j) \otimes \mathcal{O}_{\mathsf{Quot}_d} \right)$$

On the other hand, when $\deg M_j < 0$, we see that $\mathbf{S}^{\lambda^j} M_j^{[d]}$ is quasi-isomorphic to $\mathbf{S}^{\lambda^j}(\Theta_{M_j})^\bullet$, supported in degrees $[0, |\lambda^j|]$, with q-th term given by

$$\mathbf{S}^{\lambda^j}(\Theta_{M_j})^q = \bigoplus_{\alpha \subset \lambda^j, |\alpha| = s} \mathbf{S}^{\lambda^j/\alpha}(M_j^{\{d\}}[1]) \otimes \mathbf{S}^{\alpha^\dagger} \left(H^1(V \otimes M_j) \otimes \mathcal{O}_{\mathsf{Quot}_d} \right),$$

where $M_j^{\{d\}}[1] = R^1\pi_*(p^*M_j \otimes \mathcal{S})$. Now consider the complex

$$E^\bullet = \bigotimes_{j=1}^{t} \mathbf{S}^{\lambda^j}(\Theta_{M_j})^\bullet,$$

where we use cohomology notation for ease of presentation again. Observe that all cohomology groups of each summand of each term E^q vanish by part(a), except for the D-th term, with $D = \sum_{\deg M_j < 0} |\lambda^j|$, which contains the summand

$$\bigotimes_{\deg M_j < 0} \mathbf{S}^{(\lambda^j)^\dagger} \left(H^1(V \otimes M_j) \otimes \mathcal{O}_{\mathsf{Quot}_d} \right) \otimes \bigotimes_{\deg M_j \geq 0} \mathbf{S}^{\lambda^j} \left(H^0(V \otimes M_j) \otimes \mathcal{O}_{\mathsf{Quot}_d} \right). \tag{5.51}$$

Therefore, the hypercohomology spectral sequence degenerates on page two, and gives us

$$H^D(\mathsf{Quot}_d, E^\bullet) = \bigotimes_{\deg M_j<0} \mathbf{S}^{(\lambda^j)^\dagger} H^1(V\otimes M_j) \otimes \bigotimes_{\deg M_j\geq 0} \mathbf{S}^{\lambda^j} H^0(V\otimes M_j).$$

for $D = \sum_{\deg M_j<0} |\lambda^j|$, while all other cohomology groups vanish. Note that in computing H^D from (5.51), we use the fact that $\mathcal{O}_{\mathsf{Quot}_d}$ has no higher cohomology groups, which is a corollary of Theorem 5.4.1, as explained in Remark 5.4.3.

Now assume $0 \leqslant \deg M_j \leqslant d+b-1 \leqslant m-1$. In this case $M_j^{\{d\}}$ is supported in degrees $[0,1]$, and is quasi-isomorphic to $(\Psi_{M_j})^\bullet$ using the triangle (5.49). The latter complex is supported in degrees $[0,1]$ and is given by

$$L_{m-1}^{\{d\}} \otimes H^1(M_j\otimes L_{m-1}^\vee) \xrightarrow{\Psi_{M_j}} L_m^{\{d\}} \otimes H^1(M_j \otimes L_m^\vee)).$$

Thus, using the triangle (5.50), we get a quasi-isomorphism of $M_j^{[d]}$ with the complex of vector bundle concentrated in degrees $[-1,0]$ given by

$$L_{m-1}^{\{d\}} \otimes H^1(M_j\otimes L_{m-1}^\vee) \longrightarrow L_m^{\{d\}} \otimes H^1(M_j \otimes L_m^\vee)) \oplus H^0(V\otimes M_j)\otimes \mathcal{O}_{\mathsf{Quot}_d},$$

and argue similarly. $\square$

Vanishing without size constraints

We now turn to the setting when there is no restrictions on the sizes of partitions. We will extend the result for two consecutive line bundles in Proposition 5.4.7 to showing vanishing of higher cohomology groups for multiple line bundle.

Theorem 5.5.7. Assume V splits as a direct sum of line bundles of nonpositive degrees. Let $M_1, M_2, \dots, M_t$ be line bundles on $\mathbb{P}^1$ of degrees at least $d+b$. For any tuple of partitions $\lambda^1, \lambda^2, \dots, \lambda^t$, we have

$$H^i\left(\mathsf{Quot}_d(\mathbb{P}^1, V, r), \bigotimes_{j=1}^t \mathbf{S}^{\lambda^j} M_j^{[d]}\right) = 0 \quad \text{for all } i \geqslant 1.$$

Proof. Let $m = d+b$, and thus Proposition 5.4.7 applies whenever two consective line bundles are involved. Let $M_1, M_2, \dots, M_t$ be line bundles with degrees at least m, and consider the short exact sequences (5.48) of vector bundles ,

$$0 \longrightarrow L_{m-1}^{[d]} \otimes H^0(M_j\otimes L_{m-1}^\vee) \xrightarrow{\Phi_{M_j}} L_m^{[d]} \otimes H^0(M_j\otimes L_m^\vee)) \longrightarrow M_j^{[d]} \longrightarrow 0$$

for each $1 \leqslant j \leqslant t$. For any partitions $\lambda^1, \lambda^2, \dots, \lambda^t$, we consider the complex of vector bundles

$$G^\bullet = \bigotimes_{j=1}^t \mathbf{S}^{\lambda^j}(\Phi_{M_j})_\bullet$$

Note that the complex is supported in degrees $[-\ell, 0]$ where $\ell = (|\lambda^1| + \cdots + |\lambda^t|)$. Proposition 5.5.2 implies that the complex

$$0 \to G^{-\ell} \to G^{-\ell+1} \to \cdots \to G_0 \to \bigotimes_{j=1}^{t} \mathbf{S}^{\lambda^j} M_j$$

is exact. The $(-q)$-th term G^{-q} splits as a direct sum with each summand of the form

$$\mathbf{S}^{\gamma} L_{m-1}^{[d]} \otimes \mathbf{S}^{\delta} L_m^{[d]},$$

which by Proposition 5.4.7 can only have zeroth cohomology. In particular, $H^i(\mathsf{Quot}_d, G^{-q}) = 0$ for all $0 \leqslant q \leqslant \ell$ and $i \geqslant 1$, so we conclude that

$$H^i\left(\mathsf{Quot}_d, \bigotimes_{j=1}^{t} \mathbf{S}^{\lambda^j} M_j\right) = 0 \qquad \text{for all } i \geqslant 1.$$

$\square$

We now provide some concrete examples explaining why the size constraint is necessary in the statement of Theorem 5.1.1, even though all higher cohomology groups vanish for positive enough line bundles.

Example 5.5.8. Let $V = \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$, $r = 1$ and $d = 2$. For the line bundle $L_4 = \mathcal{O}_{\mathbb{P}^1}(4)$, the natural map

$$\wedge^6 H^0(V \otimes L_4) \to H^0(\mathsf{Quot}_d, \wedge^6 L_4^{[2]})$$

is not an isomorphism. This shows that the bound $|\lambda| < (dn + n)/(n - r)$ is sharp in Theorem 5.1.5.

Our computation uses the Strømme embedding of $\iota : \mathsf{Quot}_d \hookrightarrow \mathrm{Gr}(3, 10) \times \mathrm{Gr}(4, 12)$, corresponding to the twists $m = 4, 5$. We have $\wedge^6 L_4^{[2]} = \iota^* \mathbf{S}^{(1^6)} \mathcal{B}_1$, which yields a resolution of $\iota_* \wedge^6 L_4^{[2]}$ as in (5.24). This then gives rise to a spectral sequence $H^k(\mathcal{V}_t) \Rightarrow H^{k-t}(\mathsf{Quot}_d, \wedge^6 L_4^{[2]})$. By using the computer to run an exhaustive search, we find that there are exactly two nonzero terms on the initial page of the spectral sequence: in addition to the term $H^0(\mathcal{V}_0) = \wedge^6 H^0(V \otimes L_4)$, which has dimension $\dim H^0(\mathcal{V}_0) = 210$, the cohomology group $H^{23}(\mathcal{V}_{24})$ is also nonzero.

Specifically, let $W_1 = H^0(V \otimes L_4)$, $W_2 = H^0(V \otimes L_5)$, and let $\mu = (10, 10, 4)$, $\sigma = (6, 6, 2^6)$. Then the multiplicity

$$\sum_{\alpha,\beta} c^{\mu^\dagger}_{\alpha,\beta} c^{\sigma}_{\alpha,\beta} = 28$$

is nonzero, and

$$\begin{aligned} H^{23}(\mathrm{Gr}(3,10) \times \mathrm{Gr}(4,12), \mathcal{V}_{24}) &= H^{23}((\mathbf{S}^{\mu} \mathcal{A}_1 \otimes \mathbf{S}^{(1^6)} \mathcal{B}_1) \boxtimes \mathbf{S}^{\sigma} \mathcal{B}_2^{\vee})^{\oplus 28} \\ &= (\mathbf{S}^{(3^{10})} W_1 \otimes \mathbf{S}^{((2)^{12})} W_2^{\vee})^{\oplus 28} \\ &\cong \left((\det W_1)^{\otimes 3} \otimes (\det W_2^{\vee})^{\otimes 2}\right)^{\oplus 28}. \end{aligned}$$

One possible choice of α, β would be

$$\alpha = (3,3),$$
$$\beta = (3,3,2^6),$$

for which the multiplicity is $c^{\mu^\dagger}_{\alpha,\beta} c^{\sigma}_{\alpha,\beta} = 1$.

Since there are only two nonzero terms on the initial page and $\wedge^6 L_4^{[2]}$ cannot have cohomology in a negative degree, the induced map $H^{23}(\mathcal{V}_{24}) \to H^0(\mathcal{V}_0)$ must be injective. In fact, there is a short exact sequence

$$0 \to \left((\det W_1)^{\otimes 3} \otimes (\det W_2^\vee)^{\otimes 2}\right)^{\oplus 28} \to \wedge^6 W_1 \to H^0(\mathsf{Quot}_d, \wedge^6 L_4^{[2]}) \to 0.$$

Example 5.5.9. Let $V = \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}, r = 1, d = 3$. Then

$$H^2(\mathsf{Quot}_d, \mathrm{Sym}^2(L_2^{[d]})^\vee) = H^2(\mathsf{Quot}_d, \mathbf{S}^{(2)}(L_2^{[d]})^\vee) \neq 0.$$

Note that the partition $\nu = (2)$ satisfies the constraint that $|\nu| < (nd+n)/r = 12$, but fails the condition of part (i) of Theorem 5.1.8 because $\nu_1 = n - r$ is too large. This example also shows that the same theorem cannot be upgraded to the statement that for $|\nu| < (nd+n)/r$,

$$H^{\geq \nu_1 - (n-r-1)}(\mathsf{Quot}_d(\mathbb{P}^1, V, r), (\mathbf{S}^\nu L_m^{[d]})^\vee) = 0.$$

To see the non-vanishing of H^2, we consider the Strømme embedding $\iota : \mathsf{Quot}_d \hookrightarrow \mathrm{Gr}(3,9) \times \mathrm{Gr}(5,12)$, corresponding to the twists $m = 2,3$. Then we run an exhaustive search of all vector bundles

$$\mathcal{V}^\mu = \mathbf{S}^\mu \mathcal{A}_1 \otimes \mathrm{Sym}^2 \mathcal{B}_1^\vee \boxtimes \mathbf{S}^{\mu^\dagger}(\mathcal{B}_2^\vee \oplus \mathcal{B}_2^\vee)$$

on $\mathrm{Gr}(3,9) \times \mathrm{Gr}(5,12)$, where $|\mu| \leq \mathrm{codim}(\mathsf{Quot}_d(\mathbb{P}^1, V, r) \subseteq \mathrm{Gr}(3,9) \times \mathrm{Gr}(5,12)) = 42$. Let $\mu_1 = (8,2,2)$ and $\mu_2 = (7,2,2)$, and let $W_1 = H^0(V \otimes L_2)$, $W_2 = H^0(V \otimes L_3)$. We find that the only vector bundles with nonzero cohomology groups arise from these two partitions:

$$H^{13}(\mathcal{V}^{\mu_1}) = \mathbf{S}^{((-1)^8, -2)}(W_1^\vee) \otimes \mathbf{S}^{(1^{12})}(W_2^\vee)^{\oplus 7} = (W_1 \otimes \det W_1 \otimes \det W_2^\vee)^{\oplus 7},$$
$$H^{13}(\mathcal{V}^{\mu_2}) = \mathbf{S}^{((-1)^9)}(W_1^\vee) \otimes \mathbf{S}^{(1^{11})}(W_2^\vee)^{\oplus 6} = (\det W_1 \otimes \wedge^{11} W_2^\vee)^{\oplus 6}.$$

The spectral sequence computing the cohomology of $\mathrm{Sym}^2(L_2^{[d]})^\vee$ in this case reduces to an exact sequence

$$0 \to H^1(\mathrm{Sym}^2(L_2^{[d]})^\vee) \to (W_1 \otimes \det W_1 \otimes \det W_2^\vee)^{\oplus 7} \to (\det W_1 \otimes \wedge^{11} W_2^\vee)^{\oplus 6} \to H^2(\mathrm{Sym}^2(L_2^{[d]})^\vee) \to 0.$$

While we don't know how to describe this middle map explicitly, the cokernel $H^2(\mathrm{Sym}^2(L_2^{[d]})^\vee)$ must be nonzero for dimension reasons.

Bibliography

[ABW82] Kaan Akin, David A. Buchsbaum, and Jerzy Weyman. "Schur functors and Schur complexes". In: *Adv. in Math.* 44.3 (1982), pp. 207–278. ISSN: 0001-8708. DOI: 10.1016/0001-8708(82)90039-1. URL: https://doi.org/10.1016/0001-8708(82)90039-1.

[Arb+21] Noah Arbesfeld et al. "The virtual K-theory of Quot schemes of surfaces". In: *J. Geom. Phys.* 164 (2021), Paper No. 104154, 36. ISSN: 0393-0440,1879-1662. DOI: 10.1016/j.geomphys.2021.104154. URL: https://doi.org/10.1016/j.geomphys.2021.104154.

[Arb+84] E. Arbarello et al. *Geometry of Algebraic Curves*. 1st ed. Vol. I. Grundlehren der mathematischen Wissenschaften. Springer New York, NY, Dec. 1984.

[Arb21] Noah Arbesfeld. "K-theoretic Donaldson-Thomas theory and the Hilbert scheme of points on a surface". In: *Algebr. Geom.* 8.5 (2021), pp. 587–625. ISSN: 2313-1691. DOI: 10.14231/ag-2021-018. URL: https://doi.org/10.14231/ag-2021-018.

[Ber94] Aaron Bertram. "Towards a Schubert calculus for maps from a Riemann surface to a Grassmannian". In: *Internat. J. Math.* 5.6 (1994), pp. 811–825. ISSN: 0129-167X,1793-6519. DOI: 10.1142/S0129167X94000401. URL: https://doi.org/10.1142/S0129167X94000401.

[Ber97] Aaron Bertram. "Quantum Schubert calculus". In: *Adv. Math.* 128.2 (1997), pp. 289–305. ISSN: 0001-8708,1090-2082. DOI: 10.1006/aima.1997.1627. URL: https://doi.org/10.1006/aima.1997.1627.

[BGS24] Indranil Biswas, Chandranandan Gangopadhyay, and Ronnie Sebastian. "Infinitesimal deformations of some Quot schemes". In: *Int. Math. Res. Not. IMRN* 9 (2024), pp. 8067–8100. ISSN: 1073-7928,1687-0247. DOI: 10.1093/imrn/rnae033. URL: https://doi.org/10.1093/imrn/rnae033.

[BH98] Winfried Bruns and H. Jürgen Herzog. *Cohen-Macaulay Rings*. 2nd ed. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1998.

[BM11] Anders S. Buch and Leonardo C. Mihalcea. "Quantum K-theory of Grassmannians". In: *Duke Math. J.* 156.3 (2011), pp. 501–538. ISSN: 0012-7094,1547-7398. DOI: 10.1215/00127094-2010-218. URL: https://doi.org/10.1215/00127094-2010-218.

[Bot57] Raoul Bott. “Homogeneous vector bundles”. In: *Ann. of Math. (2)* 66 (1957), pp. 203–248. ISSN: 0003-486X. DOI: 10.2307/1969996. URL: https://doi.org/10.2307/1969996.

[Bro26] Daniel Brogan. “Invariants of the singularities of secant varieties of curves”. In: *Journal de l’École polytechnique - Mathématiques* 13 (2026), pp. 1–39.

[Buc+22] Anders S. Buch et al. “Positivity of minuscule quantum K-theory”. In: (May 2022). arXiv: 2205.08630 [math.AG].

[BV12] Michele Bolognesi and Angelo Vistoli. “Stacks of Trigonal Curves”. In: *Transactions of the American Mathematical Society* 364.7 (2012), pp. 3365–3393. ISSN: 00029947. URL: http://www.jstor.org/stable/41524970 (visited on 03/12/2026).

[BV88] Winfried Bruns and Udo Vetter. *Determinantal Rings*. Lecture Notes in Mathematics. Springer Berlin, Heidelberg, 1988.

[CL24] Samir Canning and Hannah Larson. “The Chow Rings of the Moduli Spaces of Curves of Genus 7, 8, and 9”. In: *Journal of Algebraic Geometry* 33 (2024), pp. 55–116.

[Con+18] Aldo Conca et al. “Hankel determinantal rings have rational singularities”. In: *Advances in Mathematics* 335 (2018), pp. 111–129. ISSN: 0001-8708. DOI: 10.1016/j.aim.2018.06.011.

[Con07] Brian Conrad. “Arithmetic moduli of generalized elliptic curves”. In: *Journal of the Institute of Mathematics of Jussieu* 6.2 (2007), pp. 209–278. DOI: 10.1017/S1474748006000089.

[Con98] Aldo Conca. “Straightening Law and Powers of Determinantal Ideals of Hankel Matrices”. In: *Advances in Mathematics* 138.2 (1998), pp. 263–292. ISSN: 0001-8708. DOI: https://doi.org/10.1006/aima.1998.1740. URL: https://www.sciencedirect.com/science/article/pii/S0001870898917406.

[Cop25] Marc Coppens. *A picture of the irreducible components of $W^r_d(C)$ for a general k-gonal curve C*. 2025. arXiv: 2504.21141 [math.AG].

[Dan04] Gentiana Danila. “Sur la cohomologie de la puissance symétrique du fibré tautologique sur le schéma de Hilbert ponctuel d’une surface”. In: *J. Algebraic Geom.* 13.1 (2004), pp. 81–113. ISSN: 1056-3911,1534-7486. DOI: 10.1090/S1056-3911-03-00372-2. URL: https://doi.org/10.1090/S1056-3911-03-00372-2.

[DL85] J.-M. Drezet and J. Le Potier. “Fibrés stables et fibrés exceptionnels sur $\mathbb{P}_2$”. In: *Annales scientifiques de l’École Normale Supérieure*. 4th ser. 18.2 (1985), pp. 193–243.

[Eis95] David Eisenbud. *Commutative Algebra: with a View Toward Algebraic Geometry*. 1st ed. Vol. 150. Graduate Texts in Mathematics. Published: 30 March 1995. Springer New York, NY, 1995. ISBN: 978-0-387-94268-1. DOI: 10.1007/978-1-4612-5350-1.

[ES08] David Eisenbud and Frank-Olaf Schreyer. “Relative Beilinson monad and direct image for families of coherent sheaves”. In: *Transactions of the American Mathematical Society* 360.10 (Oct. 2008), pp. 5367–5396.

[Eti+17] P.I. Etingof et al. *Tensor Categories*. Mathematical surveys and monographs. American Mathematical Society, 2017. ISBN: 9781470437411.

[FL81] W. Fulton and R. Lazarsfeld. "On the connectedness of degeneracy loci and special divisors". In: *Acta Mathematica* 146.1 (1981), pp. 271–283. DOI: 10.1007/BF02392466.

[Ful00] William Fulton. "Eigenvalues, invariant factors, highest weights, and Schubert calculus". In: *Bull. Amer. Math. Soc. (N.S.)* 37.3 (2000), pp. 209–249. ISSN: 0273-0979,1088-9485. DOI: 10.1090/S0273-0979-00-00865-X. URL: https://doi.org/10.1090/S0273-0979-00-00865-X.

[GLS25] Ajay Gautam, Feiyang Lin, and Shubham Sinha. *A Borel–Weil–Bott theorem for Quot schemes on* $\mathbb{P}^1$. 2025. arXiv: 2511.03519 [math.AG].

[GS21] Chandranandan Gangopadhyay and Ronnie Sebastian. "Nef cones of some Quot schemes on a smooth projective curve". In: *C. R. Math. Acad. Sci. Paris* 359 (2021), pp. 999–1022. ISSN: 1631-073X,1778-3569. DOI: 10.5802/crmath.245. URL: https://doi.org/10.5802/crmath.245.

[HF23] Daniel Halpern-Leistner and Andres Fernandez Herrero. *The structure of the moduli of gauged maps from a smooth curve*. 2023. arXiv: 2305.09632 [math.AG].

[Hon23] Serin Hong. "Classification of quotient bundles on the Fargues–Fontaine curve". In: *Selecta Mathematica* 29.20 (Jan. 2023).

[Ill71] Luc Illusie. *Complexe Cotangent et Deformations I*. Springer Berlin, Heidelberg, 1971.

[JK81] Gordon James and Adalbert Kerber. *The representation theory of the symmetric group*. Vol. 16. Encyclopedia of Mathematics and its Applications. With a foreword by P. M. Cohn, With an introduction by Gilbert de B. Robinson. Addison-Wesley Publishing Co., Reading, MA, 1981, pp. xxviii+510. ISBN: 0-201-13515-9.

[Kap84] M. M. Kapranov. "Derived category of coherent sheaves on Grassmann manifolds". In: *Izv. Akad. Nauk SSSR Ser. Mat.* 48.1 (1984), pp. 192–202. ISSN: 0373-2436.

[Kem73] George Kempf. "On the Geometry of a Theorem of Riemann". In: *Annals of Mathematics* 98.1 (1973), pp. 178–185. ISSN: 0003486X, 19398980. URL: http://www.jstor.org/stable/1970910.

[Kem76] George R. Kempf. "On the collapsing of homogeneous bundles". In: *Inventiones mathematicae* 37.3 (Oct. 1976), pp. 229–239. ISSN: 1432-1297. DOI: 10.1007/BF01390321. URL: https://doi.org/10.1007/BF01390321.

[Kou91] Frank M Kouwenhoven. "Schur and Weyl functors". In: *Advances in Mathematics* 90.1 (1991), pp. 77–113. ISSN: 0001-8708. DOI: https://doi.org/10.1016/0001-8708(91)90020-8. URL: https://www.sciencedirect.com/science/article/pii/0001870891900208.

[Kru18] Andreas Krug. “Remarks on the derived McKay correspondence for Hilbert schemes of points and tautological bundles”. In: *Math. Ann.* 371 (2018), pp. 461–486. ISSN: 0025-5831. DOI: 10.1007/s00208-018-1660-5. URL: https://doi.org/10.1007/s00208-018-1660-5.

[Kru24] Andreas Krug. “Extension groups of tautological bundles on punctual Quot schemes of curves”. In: *J. Math. Pures Appl. (9)* 189 (2024), Paper No. 103600, 30. ISSN: 0021-7824,1776-3371. DOI: 10.1016/j.matpur.2024.103600. URL: https://doi.org/10.1016/j.matpur.2024.103600.

[KT99] Allen Knutson and Terence Tao. “The honeycomb model of $\mathrm{GL}_n(\mathbf{C})$ tensor products. I. Proof of the saturation conjecture”. In: *J. Amer. Math. Soc.* 12.4 (1999), pp. 1055–1090. ISSN: 0894-0347,1088-6834. DOI: 10.1090/S0894-0347-99-00299-4. URL: https://doi.org/10.1090/S0894-0347-99-00299-4.

[Lar21a] Hannah K. Larson. “A refined Brill–Noether theory over Hurwitz spaces”. In: *Inventiones mathematicae* 224.3 (June 2021), pp. 767–790. DOI: 10.1007/s00222-020-01023-z.

[Lar21b] Hannah K. Larson. “Universal degeneracy classes for vector bundles on $\mathbb{P}^1$ bundles”. In: *Advances in Mathematics* 380.107563 (Mar. 2021).

[Lar23] Hannah Larson. “The intersection theory of the moduli stack of vector bundles on $\mathbb{P}^1$”. In: *Canadian Mathematical Bulletin* 66.2 (2023), pp. 359–379.

[Lin25a] Feiyang Lin. *Resolving the Singularities of Splitting Loci*. 2025. arXiv: 2507.01233 [math.AG].

[Lin25b] Feiyang Lin. *When are splitting loci Gorenstein?* 2025. arXiv: 2510.26044 [math.AG].

[LLV25] Eric Larson, Hannah K. Larson, and Isabel Vogt. “Global Brill-Noether theory over the Hurwitz space”. In: *Geometry & Topology* 29.1 (Jan. 2025), pp. 193–257.

[Mac95] I G Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford University Press, Mar. 1995. ISBN: 9780198534891. DOI: 10.1093/oso/9780198534891.001.0001.

[MN23] Alina Marian and Andrei Neguţ. *The cohomology of the Quot scheme on a smooth curve as a Yangian representation*. 2023. arXiv: 2307.13671 [math.AG].

[MN24] Alina Marian and Andrei Neguţ. *Derived categories of Quot schemes on smooth curves and tautological bundles*. 2024. arXiv: 2411.08695 [math.AG].

[MO07] Alina Marian and Dragos Oprea. “Virtual intersections on the Quot scheme and Vafa-Intriligator formulas”. In: *Duke Math. J.* 136.1 (2007), pp. 81–113. ISSN: 0012-7094,1547-7398. DOI: 10.1215/S0012-7094-07-13613-5.

[MOS22] Alina Marian, Dragos Oprea, and Steven V Sam. *On the cohomology of tautological bundles over Quot schemes of curves*. Nov. 2022. DOI: 10.48550/arXiv.2211.03901. arXiv: 2211.03901 [math.AG].

[New97] Stephen J. H. New. “Deformations of Bundles Over the Riemann Sphere”. PhD thesis. McGill University, July 1997.

[NT25] David Nadler and Jeremy Taylor. "The Whittaker Functional is a Shifted Microstalk". In: *Transformation Groups* 30 (2025), pp. 1425–1450.

[OP21] Dragos Oprea and Rahul Pandharipande. "Quot schemes of curves and surfaces: virtual classes, integrals, Euler characteristics". In: *Geom. Topol.* 25.7 (2021), pp. 3425–3505. ISSN: 1465-3060,1364-0380. DOI: 10.2140/gt.2021.25.3425. URL: https://doi.org/10.2140/gt.2021.25.3425.

[Opr22] Dragos Oprea. "Big and nef tautological vector bundles over the Hilbert scheme of points". In: *SIGMA Symmetry Integrability Geom. Methods Appl.* 18 (2022), Paper No. 061, 21. ISSN: 1815-0659. DOI: 10.3842/SIGMA.2022.061. URL: https://doi.org/10.3842/SIGMA.2022.061.

[OS23] Dragos Oprea and Shubham Sinha. "Euler characteristics of tautological bundles over Quot schemes of curves". In: *Adv. Math.* 418 (2023), Paper No. 108943, 45. ISSN: 0001-8708. DOI: 10.1016/j.aim.2023.108943. URL: https://doi.org/10.1016/j.aim.2023.108943.

[Pot97] Joseph Le Potier. *Lectures on vector bundles*. Vol. 54. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1997.

[PR03] Mihnea Popa and Mike Roth. "Stable maps and Quot schemes". In: *Invent. Math.* 152 (2003), pp. 625–663. ISSN: 0020-9910. DOI: 10.1007/s00222-002-0279-y. URL: https://doi.org/10.1007/s00222-002-0279-y.

[Sca09] Luca Scala. "Cohomology of the Hilbert scheme of points on a surface with values in representations of tautological bundles". In: *Duke Math. J.* 150 (2009), pp. 211–267. ISSN: 0012-7094. DOI: 10.1215/00127094-2009-050. URL: https://doi.org/10.1215/00127094-2009-050.

[Sca15] Luca Scala. *Higher symmetric powers of tautological bundles on Hilbert schemes of points on a surface*. 2015. DOI: 10.48550/ARXIV.1502.07595. URL: https://arxiv.org/abs/1502.07595.

[Sch] Karl Schwede. *Generalized Divisors and Reflexive Sheaves*.

[Stacks] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu. 2018.

[Str87] Stein Arild Strømme. "On parametrized rational curves in Grassmann varieties". In: *Space curves (Rocca di Papa, 1985)*. Vol. 1266. Lecture Notes in Math. Springer, Berlin, 1987, pp. 251–272. ISBN: 3-540-18020-6. DOI: 10.1007/BFb0078187.

[SY22] Christian Schnell and Ruijie Yang. *A log resolution for the theta divisor of a hyperelliptic curve*. 2022.

[SZ24a] Shubham Sinha and Ming Zhang. "Quantum K-invariants via Quot schemes I". To appear in: *J. Math. Pures Appl.*, arXiv:2406.12191 (June 2024).

[SZ24b] Shubham Sinha and Ming Zhang. "Quantum K-invariants via Quot schemes II". In: arXiv:2410.23486 (Oct. 2024).

[Tod22] Yukinobu Toda. *Derived categories of Quot schemes of zero-dimensional quotients on curves*. July 2022. arXiv: 2207.09687 [math.AG].

[Wan11] Jonathan Wang. *The Moduli Stack of G-Bundles*. 2011.

[Wey03] Jerzy Weyman. *Cohomology of Vector Bundles and Syzygies*. Cambridge Tracts in Mathematics. Cambridge University Press, 2003.

[Zhu16] Xinwen Zhu. *An introduction to affine Grassmannians and the geometric Satake equivalence*. 2016. arXiv: 1603.05593 [math.AG].