On Bisectional Nonpositively Curved Compact Kähler Einstein Surfaces

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October 20, 2016

In this note we explain that the conjecture of the pinching of the bisectional curvature mentioned in [HGY] and [CHY] is proved by a combination of the arguments from the proofs of the Theorem 1.2 in [CHY], the Theorem 2 in [HGY] and the Proposition 4 in [SY]. Moreover, we prove that any compact Kähler-Einstein surface M is a quotient of the complex two dimensional unit ball or the complex two dimensional plane if (1) M has nonpositive Einstein constant and (2) at each point, the average holomorphic sectional curvature is closer to the minimal than to the maximal.

1 Introduction

In [SY] the authors conjectured that any compact Kähler surface with negative bisectional curvature is a quotient of the complex two dimensional unit ball. They proved that there is a number $a \in (1/3, 2/3)$ such that if at every point P, $K_{av} - K_{min} \leq a[K_{max} - K_{min}]$, then M is a quotient of the complex ball. Here, K_{min} (K_{max}, K_{av}) is the minimal (maximal, average) of the holomorphic sectional curvature. The number a they obtained is $a < \frac{2}{3[1+\sqrt{6/11}]}$ (almost 0.38, see [P2] page 398). In [HGY], Yi Hong[1] pointed out that this is also true if $a \leq \frac{2}{3[1+\sqrt{1/6}]} < 0.476$. We also observed in Theorem 2 that if $a \leq \frac{1}{2}$, then there is a ball-like point P. That is, at P, $K_{max} = K_{min}$. We notice that $\sqrt{1/6} > 1/3$. Therefore, we conjectured that M is a quotient of the complex ball if $a = \frac{1}{2}$. In general, we believe that we

Key Words and Phrases: Kähler-Einstein metrics, compact complex surfaces, bisectional curvature curvature, pinching of the curvatures.

Math. Subject Classifications: 53C21, 53C55, 32M15, 32Q20.

[1] For this part, it is due to Professor Hong. Notice that he was the first author there.

might not get a quotient of the complex ball if $a > \frac{1}{2}$. Around 1992, Hong Cang Yang almost proved this conjecture except some technical difficulties. See the argument of the Theorem 1.2 in [CHY]. In [P1, P2], the author used a different method and proved that a can be $(3+\frac{4\sqrt{3}}{3})/11$ (almost 0.48 according to [CHY] page 2628 right before Theorem 1.2), see [P1] page 669, or [P2] page 398. In [CHY], the authors improved the constant to $a < \frac{1}{2}$ that gave a proof of a weaker version of the conjecture.

In this note we first notice that in our proof of the Theorem 2 in [HGY] (for which this author was responsible), we actually proved that if $K_{av} - K_{min} = \frac{1}{2}[K_{max} - K_{min}]$ at P, then P must be a ball-like point (for this part, any negativity of the curvature is actually not needed except using the result from [SY] when $A = B$). See the remark after the Theorem 1 in [HGY]. According to [SY] p.485, Proposition 4, the subset of ball-like points is either the whole manifold or a real codimensional two real analytic subvariety. Since the function considered in Theorem 1.2 of [CHY] is bounded, it can be extended to the whole M and it is a constant and must be zero. We notice that we only need that the bisectional curvature is nonpositive. With this in mind, we can also have the possibility of the flat case. That is, the manifold could also be a quotient of the complex two dimensional complex plane if the Einstein constant is zero. This case should also be included in the Main Theorem of [SY] in page 472, and also for [HGY] Theorem A and Theorem 1.

Since [HGY] was only written in Chinese, we like to have a mostly self contained account here. Also, we notice that [P1, 2] had something more general than what we stated above. Therefore, we generalized our result to the case of nonpositive Einstein constant. We have:

THEOREM *Let M be a connected compact Kähler-Einstein surface with nonpositive scalar curvature, if we have*

$$K_{av} - K_{min} \leq \frac{1}{2}[K_{max} - K_{min}]$$

at every point, then M is a compact quotient of either the complex two dimensional unit ball or the complex two dimensional complex plane.

This note is written in a way that for those experts who are familiar with [HGY] and [CHY], the Introduction is enough for the conjecture in [HGY] and [CHY]. For those experts who are familiar with [CHY], next (the second) section is enough for that conjecture. We notice that we do not need the nonpositivity of the bisectional curvature except applying the result of [SY] or [CHY] for the case in which $A = B$. We shall give a complete proof of the conjecture in the third section with a simpler explanation than that of [CHY] for the last step, that also explains away the mystery of the negativity. In the last section, we should apply our methods for our Theorem.

To the author, the conjecture in [SY] is very important for the complex geometry. This work is heavily depended on the earlier works on this subject. Although we are able to prove the conjecture in [HGY], [CHY] and our main theorem, there are more work need to be done in the direction of compact complex surfaces with negative holomorphic bisectional or even real sectional curvatures. Therefore, the author think that it is proper to write this paper with an emphasis on the nonpositive holomorphic bisectional case instead of our main theorem. We thank the referee for useful comments and encouragements.

I thank Professors Poon, Wong and the Department of Mathematics, University of California at Riverside for their supports. I thank Professor Hong Cang Yang for showing me his work when I was a graduate student in Berkeley. I also thank Professor Paul Yang for telling me the work [P1].

2 Existence of Ball-like Points

Here, we repeat the argument in the proof of the Theorem 2 in [HGY]. In [HGY], we proved that:

Proposition 1.(Cf [HGY] p.597–599) *Suppose that*

$$K_{av} - K_{min} \leq \frac{1}{2}[K_{max} - K_{min}]$$

for every point on the compact Kähler Einstein surface with nonpositive holomorphic bisectional curvatures. There is at least one ball-like point.

Proof: Throughout this paper, as in [SY] and [CHY], we assume that $\{e_1, e_2\}$ be an unitary basis at a given point P with

$$R_{1\bar{1}1\bar{1}} = R_{2\bar{2}2\bar{2}} = K_{min},\ R_{1\bar{1}1\bar{2}} = R_{2\bar{2}2\bar{1}} = 0$$

$$A = 2R_{1\bar{1}2\bar{2}} - R_{1\bar{1}1\bar{1}} \geq 0,\ B = |R_{1\bar{2}1\bar{2}}|$$

As in [SY], we always have that $A \geq |B|$ and we assume that $B \geq 0$. This also implies if the sectional curvatures have a 1/4 pinching, i.e., the section curvature is inside an interval $[-\frac{1}{4}a(P), -a(P)]$ at every point P for a nonnegative function $a(P)$, then M is covered by a ball. This was pointed out in [CHY]. This is because if we let $a(P) = -R_{1\bar{1}1\bar{1}}$, $e_i = X_i + \sqrt{-1}Y_i$, then at least one of $R(X_1, X_2, X_1, X_2)$ and $R(X_1, Y_2, X_1, Y_2)$ is bigger or equal to $-\frac{1}{4}a(P)$. Same arguement works for the higher dimension case. Our Theorem is a kind of the generalization of the 1/4 pinching.

If P is not a ball-like point, according to [SY], we can do as above for a neighborhood $U(P)$ of P whenever $A > B$ (Case 1 in [SY] page 475). We should handle the case in which $A = B$ at the end of this proof. We write

$$\alpha = e_1 = \sum a_i \partial_i, \ \beta = e_2 = \sum b_i \partial_i$$

and

$$S_{1\bar{1}1\bar{1}} = R(e_1, \bar{e}_1, e_1, \bar{e}_1) = \sum R_{i\bar{j}k\bar{l}} a_i \bar{a}_j a_k \bar{a}_l$$

and so on. In particular, we have

$$S_{1\bar{1}1\bar{1}} = S_{2\bar{2}2\bar{2}} = K_{min}, \ S_{1\bar{1}1\bar{2}} = S_{2\bar{2}2\bar{1}} = 0$$

According to [SY], we have

$$K_{max} = K_{min} + \frac{1}{2}(A + B), \ K_{av} = K_{min} + \frac{1}{3}A$$

$$\frac{1}{3}[K_{max} - K_{min}] \leq K_{av} - K_{min} \leq \frac{2}{3}[K_{max} - K_{min}]$$

Our condition in the Proposition is therefore the same as $A \leq 3B$. As in [HGY], we let $\Phi_1 = \frac{|B|^2}{A^2} = \tau^2$.

If there is no ball-like point, by $1/3 \leq \tau \leq 1$, there is a minimal point.

We shall calculate the Laplace of Φ_1 at a minimal point, which is not a ball-like point. For example, when $A = 3B$, $\Phi_1 = 1/9$ achieves the minimum. The Laplace at that point should be nonnegative.

We let

$$x_i = \nabla_i \Phi_1 = 2\frac{\tau}{A}[\mathrm{Re}\nabla_i S_{1\bar{2}1\bar{2}} + 3\tau \nabla_i S_{1\bar{1}1\bar{1}}]$$

As we pointed out earlier, we first assume that A does not equal to B always, then we can apply the argument in the case 1 of [SY] page 475 at the minimal point since $A > B$.

As in [SY]. [HGY], [CHY], we have:

$$\Delta R_{1\bar{1}1\bar{1}} = -AR_{1\bar{1}2\bar{2}} + B^2$$

$$\Delta R_{1\bar{2}1\bar{2}} = 3(R_{1\bar{1}2\bar{2}} - A)B.$$

At P we have $a_1 = b_2 = 1$ and $a_2 = b_1 = 0$, $\nabla a_1 = \nabla b_2 = 0$, $\nabla a_2 + \nabla \bar{b}_1 = 0$. Therefore, we write $y_{i1} = \nabla_i a_2$ and $y_{i2} = \nabla_i \bar{a}_2$. We also have:

$$\Delta(a_1 + \bar{a}_1) = -|\nabla a_2|^2, \Delta(a_2 + \bar{b}_2) = 0$$

$$\nabla_i R_{1\bar{1}1\bar{2}} = -Ay_{i1} - By_{i2}$$

since

$$0 = \nabla S_{1\bar{1}1\bar{2}} = \nabla R_{1\bar{1}1\bar{2}} + 2R_{2\bar{1}1\bar{2}} \nabla a_2 + B\nabla \bar{a}_2 + R_{1\bar{1}1\bar{1}} \nabla \bar{b}_1,$$

i.e.,

$$\nabla R_{1\bar{1}1\bar{2}} = -A\nabla a_2 - B\nabla \bar{a}_2.$$

This also gives a similar formula for $\nabla_{\bar{i}} R_{1\bar{1}1\bar{2}}$. Similarly,

$$\nabla S_{1\bar{1}1\bar{1}} = \nabla R_{1\bar{1}1\bar{1}}$$

$$\nabla S_{1\bar{2}1\bar{2}} = \nabla R_{1\bar{2}1\bar{2}}$$

$$\Delta S_{1\bar{1}1\bar{1}} = -2A \sum |y|^2 - 4B \mathrm{Re} \sum y_{i1} \bar{y}_{i2} - AR_{1\bar{1}2\bar{2}} + B^2$$

$$\mathrm{Re}\Delta S_{1\bar{2}1\bar{2}} = 4A \sum \mathrm{Re} y_{i1} \bar{y}_{i2} + 2B \sum |y|^2 + 3(R_{1\bar{1}2\bar{2}} - A)B.$$

$$\nabla_{\bar{1}} S_{1\bar{2}1\bar{2}} = -A\bar{y}_{22} - B\bar{y}_{21}$$

$$\nabla_2 S_{1\bar{2}1\bar{2}} = Ay_{11} + By_{12}$$

$$\nabla_1 S_{1\bar{2}1\bar{2}} = -A(6\tau^2 - 1)y_{22} - 5A\tau y_{21} + x_1$$

$$\nabla_{\bar{2}} S_{1\bar{2}1\bar{2}} = 5A\tau \bar{y}_{12} + A(6\tau^2 - 1)\bar{y}_{11} + \bar{x}_2$$

As in [HGY] p. 598, at P we have:

$$\begin{aligned}
\Delta\Phi_1 &= \frac{2\tau\Delta B}{A} + \frac{6\tau^2}{A}\Delta S_{1\bar{1}1\bar{1}} \\
&+ \frac{1}{A^2}\sum(|\nabla S_{1\bar{2}1\bar{2}}|^2 + |\bar{\nabla} S_{1\bar{2}1\bar{2}}|^2) + \frac{54\tau^2}{A^2}\sum |\nabla S_{1\bar{1}1\bar{1}}|^2 \\
&+ \frac{12\tau}{A^2}\sum \mathrm{Re}(\nabla_i S_{1\bar{1}1\bar{1}}(\nabla_{\bar{i}}(S_{1\bar{2}1\bar{2}} + S_{2\bar{1}2\bar{1}})) \qquad (1) \\
&= 2\tau[3A\tau(\tau^2 - 1) - 4\tau \sum |y|^2 + 4(1 - 3\tau^2)\sum \mathrm{Re}(y_{i1}\bar{y}_{i2})] \\
&+ |y_{22} + \tau y_{21}|^2 + |y_{11} + \tau y_{12}|^2 \\
&+ \frac{1}{A^2}[|x_1 + A[(1 - 6\tau^2)y_{22} - 5\tau y_{21}]|^2 + |x_2 + A[(6\tau^2 - 1)y_{11} + 5\tau y_{12}]|^2 \\
&- 18\tau^2[y_{12} + \tau y_{11}|^2 + |y_{21} + \tau y_{22}|^2] \\
&+ \frac{12\tau}{A}[\mathrm{Re}[(y_{21} + \tau y_{22})\bar{x}_1] - \mathrm{Re}[(y_{21} + \tau y_{11})\bar{x}_2]]
\end{aligned}$$

Here we notice that $\Delta\Phi_1$ has two general terms. The first term has nothing to do with x and y, and therefore can be regarded as constant term to them. That term is always nonpositive since $\frac{1}{3} \leq \tau \leq 1$.

The second term can be regarded as a hermitian form h to x and y. We can separate x and y into two groups: x_1, y_{2j} in one group and x_2, y_{1j} in the other. These two groups of variables are orthogonal to each other with respect to this hermitian form. That is, $h = h_1 + h_2$ with h_1 (or h_2) only depends on the first (second) group of variables.

We need to check the nonpositivity for each of them.

For x_2, y_{11}, y_{12}, the corresponding matrix of h_2 is:

$$\begin{bmatrix} \frac{1}{A^2} & -\frac{1}{A} & -\frac{\tau}{A} \\ -\frac{1}{A} & 2(9\tau^2-1)(\tau^2-1) & 0 \\ -\frac{\tau}{A} & 0 & 0 \end{bmatrix}$$

And the matrix for h_1 of x_1, y_{21}, y_{22} is:

$$\begin{bmatrix} \frac{1}{A^2} & \frac{\tau}{A} & \frac{1}{A} \\ \frac{\tau}{A} & 0 & 0 \\ \frac{1}{A} & 0 & 2(9\tau^2-1)(\tau^2-1) \end{bmatrix}$$

When P is a critical point of Φ_1, then $x_1 = x_2 = 0$. The matrices on y is clearly semi negative. Therefore, if there is no ball-like point, then we have that at the minimal point of Φ_1, $\tau^2 = 1$ or $A = 0$ since $\tau \geq \frac{1}{3}$.

If $A = 0$, then we have a ball-like point. And we are done.

On the otherhand[2], if $\tau = 1$, we have $A = B$ at P. Since P is a minimal point, this implies that $A = B$ on the whole manifold. According to [SY] page 475 case 2, we have a smooth coordinates with $K_{max} = R_{1\bar{1}1\bar{1}}$ (this works fortunately when $A = B$ always. In general, the original argument might not always work since one might not have $A = B$ always nearby. However, as [SY] case 1 pointed out under our condition the directions for K_{max} are always isolated. Therefore, it might be better one chose K_{max} instead of K_{min} from the very beginning. But this is not in the scope of this paper). Using this new coordinate, we can define the similar function A_1 and B_1. In general, $B_1 = \frac{1}{2}(A - B)$ and $A_1 = -\frac{1}{2}(A + 3B)$. In our case, $B_1 = 0$ and $A_1 = -2A$. Using this new coordinate, one can do the calculation for any of the functions in [SY], [P1], [P2] (or [CHY], see the next section) that the set of ball-like points is the whole manifold. If one does not like Polombo's function Φ_α ([P2] page 418) with $\alpha = -\frac{8}{7}$ (e.g., [P2] page 417 Lemma), then one might simply use the function with $\alpha = -1$ (in

[P1, P2], not the vector we mentioned in this paper earlier), i.e., the new function is proportional to $\Phi_2 = (3B - A)A$. In our case, this is just $2A^2$. We can apply $\Phi_2^{\frac{1}{3}}$. This is relatively easy that we just leave it to the readers (or see (4) in the generalization). One can also use the function in [SY] page 477

$$3\gamma_2 - \gamma_1^2 = \frac{1}{2}(A^2 + 3B^2).$$

We can also still use the argument in [SY] case 1, in which the minimal vectors are not isolated any more but they are points in a smooth circle bundle over the manifold that we could just choose a smooth section instead.

Also, this paragraph is not needed in the following Corollary 1 and Lemma 1 since in those two propositions, we already have $A = 3B$. With $A = B$, one could readily get that $A = B = 0$.

If $A = 0$, $K_{max} = K_{min}$ and P is a ball-like point. We have a contradiction. Therefore, the set of ball-like points is not empty.

Q. E. D.

Observe that if $A = 3B$ at P, then Φ_1 achieves the minimal value at P and $A \neq B$ unless P is a ball-like point. That is the first part of the proof of Proposition 1 goes through. That is, P must be a ball-like point.

Corollary 1. *Assume the above, if $K_{av} - K_{min} = \frac{1}{2}[K_{max} - K_{min}]$ at P, then P is a ball-like point.*

Therefore, we have:

Lemma 1. *If $K_{av} - K_{min} \leq \frac{1}{2}[K_{max} - K_{min}]$ on M, then we have $K_{av} - K_{min} < \frac{1}{2}[K_{max} - K_{min}]$ on $M - N$, where N is the subset of all the ball-like points.*

Therefore, we can apply the argument of [CHY]. To do that one need following Proposition 4 in [SY]:

Propositon 2.(Cf [SY], also [HGY Theorem 3]) *If $N \neq M$, then N is a real analytic subvariety and* $\text{codim} N \geq 2$.

As in [SY], Proposition 2 give us a way to the conjecture by finding a superharmonic function on M which was obtained by Hong Cang Yang around 1992. In [SY] and [HGY], the authors used $\Phi = 6B^2 - A^2$. In [P2], Polombo used $(11A - 3B)(B - A) + 16AB$, see [P2] page 417 Lemma. One might ask why do we need another function but do not use our Φ_1. The answer is that by a power of Φ_1, we can only correct the Laplace by $|\nabla\Phi_1|^2$.

[2] The paragraph is not needed for the proofs of Corollary 1 and Lemma 1. Also, in this special case, the original frame in [SY] actually work. So, one could simply apply [SY]

But that could only change the upper left coefficients of our matrices as it only provides $|x|^2$ terms. In the case of Φ_1, it does not work since $\frac{\tau}{A} \neq 0$ but the coefficients of $|y_{12}|^2, |y_{21}|^2$ are zeros.

Therefore, we need another function, which was provided by Hong Cang Yang.

Remark 1. Whenever there is a bounded continuous nonnegative function f on M such that (1) $f(N) = 0$, (2) f is real analytic on $M - N$ and (3) $\Delta f \leq 0$ on $M - N$, then $f = 0$. *Here N could be just a codimension two subset*. See [SY], [HGY] and [CHY]. The reason is that if we define $M_s = \{x \in M|_{\text{dist}(x,N) \geq s}\}$ and $h_s = \partial M_s$, then the measure of h_s is smaller than $O(s)$ when s tends to zero. Therefore,

$$0 \geq \ln 2 \int_{M_{2\delta}} \Delta f \omega^n \geq \int_{\delta}^{2\delta} [\int_{M_s} \Delta f \omega^n] s^{-1} ds = \int_{\delta}^{2\delta} [\int_{h_s} \frac{\partial f}{\partial n} d\tau] s^{-1} ds.$$

But by applying an integration by parts to the single variable integral, the last term is about $(\delta)^{-1} \int_{h_{2\delta}} (f - g) d\tau \to 0$ since f is bounded and $f - g$ tends to 0 near N, where g is the f value of the corresponding point on h_δ. For example, if $f = r^a$ with $a > 0$, then

$$\frac{\partial f}{\partial n} = a r^{a-1} = a s^{a-1}$$

and

$$\int_{h_s} \frac{\partial f}{\partial n} d\tau = O(s^a) \to 0.$$

Therefore, $\Delta f = 0$ on $M - N$. A similar arguments shows that $\int_{h_d} (f - g) d\tau = 0$, where g is the f value of the corresponding point on h_s for any given $s < d$. Let s tends to zero, we get $\int_{h_d} f d\tau = 0$. By $f \geq 0$ on h_d we obtained that $f = 0$ near N. Therefore f extends over N as a harmonic function. This implies that $f = 0$ on M.

Now, let $f = (3B - A)^a$, this is natural after the proof of Proposition 1, we will show in the next section that $\Delta f \leq 0$ for $a \leq \frac{1}{3}$ (see also a proof in [CHY]). Therefore, $A = 3B$ always. By the Corollary 1, we have $A = B = 0$. This function is also related to the functions in [P2] page 417 with $a_1 = a_3 = 0$. In [P2] Polombo had to pick up functions with $a_1 = a_2$ to avoid a complication of the singularities. See [P2] page 406 and the first paragraph in page 418 (see also [P1], the last paragraph of page 668). While we shall completely resolve the difficulty in the next section.

3 Hong Cang Yang's Function

Let $\Psi = 3B - A$. About 1992, Hong Cang Yang considered $f = \Psi^{\frac{1}{3}}$.

Lemma 2. ([CHY] p.2630 (13)) *We have:*

$$\begin{aligned}\Delta(3B - A) &= 3[\Psi R_{1\bar{1}2\bar{2}} + B(B - 3A)] \\ &+ \frac{3}{B}|\nabla(\mathrm{Im} R_{1\bar{2}1\bar{2}})|^2 + 6(B - A)\sum |y_{i1} - y_{i2}|^2.\end{aligned}$$

Let $z_i = \nabla_i \Psi$. Then

$$z_1 = \nabla_1(3B - A) = \frac{3}{2}\nabla_1(R_{1\bar{2}1\bar{2}} + R_{2\bar{1}2\bar{1}} + 2R_{1\bar{1}1\bar{1}})$$

$$\begin{aligned}\sqrt{-1}\nabla_1(\mathrm{Im} R_{1\bar{2}1\bar{2}}) &= \frac{1}{2}\nabla_1(R_{1\bar{2}1\bar{2}} - R_{2\bar{1}2\bar{1}}) \\ &= \frac{1}{3}z_1 - \nabla_1 R_{2\bar{1}2\bar{1}} - \nabla_1 R_{1\bar{1}1\bar{1}} \\ &= \frac{1}{3}z_1 - \nabla_2 \bar{R}_{1\bar{1}1\bar{2}} + \nabla_2 R_{1\bar{1}1\bar{2}} \\ &= \frac{1}{3}z_1 + (A - B)y_{22} + (B - A)y_{21}\end{aligned}$$

$$z_2 = \nabla_2(3B - A) = \frac{3}{2}\nabla_2(R_{2\bar{1}2\bar{1}} + R_{1\bar{2}1\bar{2}} + 2R_{1\bar{1}1\bar{1}})$$

$$\begin{aligned}\sqrt{-1}\nabla_2(\mathrm{Im} R_{1\bar{2}1\bar{2}}) &= \frac{1}{2}\nabla_2(R_{1\bar{2}1\bar{2}} - R_{2\bar{1}2\bar{1}}) \\ &= -\frac{1}{3}z_2 + \nabla_2 R_{1\bar{1}1\bar{1}} + \nabla_2 R_{1\bar{2}1\bar{2}} \\ &= -\frac{1}{3}z_2 + \nabla_1 R_{2\bar{1}1\bar{1}} - \nabla_1 R_{1\bar{1}1\bar{2}} \\ &= -\frac{1}{3}z_2 + (B - A)y_{12} + (A - B)y_{11}\end{aligned}$$

we can write the formula in the Lemma 2 as:

$$\begin{aligned}\Delta\Psi &= 3[\Psi R_{1\bar{1}2\bar{2}} + B(B - 3A)] \\ &- 3\frac{A - B}{B}\Psi\sum |y_{i1} - y_{i2}|^2 \\ &+ 2\frac{A - B}{B}\mathrm{Re}[(y_{12} - y_{11})\bar{z}_2 + (y_{22} - y_{21})\bar{z}_1] + \sum\frac{1}{3B}|z|^2\end{aligned} \tag{2}$$

Similar to what we have in the last section, we have two general terms, the first is negative as the constant term of z and y. The second is a hermitian form on z and y. We can actually let $w_i = y_{i^*1} - y_{i^*2}$ with $i^* \neq i$. Then the second term is a sum of two hermitian forms. One of them is on w_1, z_1 and the other is on w_2, z_2. We notice that the second term is also nonpositive on y (or nonpositive on w, if we assume that $z = 0$). We can modify the coeficient of $|z|^2$ (only) by taking the power of Ψ. More precisely, if we let $g = \Psi^a$, to make sure that $\Delta g < 0$, after taking out a factor $3\frac{A-B}{B}$ we need

$$\begin{vmatrix} -\Psi & 1/3 \\ 1/3 & \frac{1-3\Psi^{-1}(1-a)B}{9(A-B)} \end{vmatrix} \geq 0$$

That is,

$$A - 3B + 3(1-a)B - A + B = (3(1-a) - 2)B \geq 0.$$

We have $1 - 3a \geq 0$. So, $a \leq 1/3$.

Therefore, we have:

Lemma 3. *$\Delta g < 0$ for $a \leq 1/3$ on $M - N$.*

This is exactly the same as what they had in [CHY]. Actually, the number 1/6 was already in [SY], [HGY], [P1, 2] for those quadratic functions.

So, finally we have:

Theorem 1. *If $K_{av} - K_{min} \leq \frac{1}{2}[K_{max} - K_{min}]$, then M has a constant holomorphic sectional curvature.*

Remark 2. The reason we did not get this earlier was that there was a difficulty when $A = B$. In that case, the argument in [SY] page 475 case 2 seems not working. Polombo resolved the problem by using a function which is symmetric to $\lambda_1 = -\frac{A}{3}$ and $\lambda_2 = \frac{A-3B}{6}$ (see [P2] page 418 first paragraph and the end of page 397). However, Hong Cang Yang's function Ψ is only $-6\lambda_2$ and therefore is not symmetric after all. To overcome the difficulty, we let $\Omega = \{x \in M|_{A=B}\}$. Then according to [SY], all our calculation are good on $M - \Omega$ since $N \subset \Omega$. In [CHY] page 2632, there was a suggestion to prove that codim $\Omega \leq 2$, although it was not very well explained. Then everything went through. The relation was that if we use the argument in [SY] page 475 case 2, using the maximal instead of the minimal, we let $B_1 = |R_{1\bar{2}1\bar{2}}|$ then $2B_1 = A - B$. That is $\Omega = \{x \in M|_{B_1=0}\}$. The argument goes as follows:

Case 1: If Ω is a closed region, we have:

$$0 \geq \int_{M-\Omega} \Delta g$$

$$
\begin{aligned}
&= a\int_{-\partial\Omega}\Psi^{a-1}\frac{\partial(-A_1-3B_1)}{\partial n}\\
&\geq a\int_{-\partial\Omega}(2A)^{a-1}\frac{\partial(-A_1)}{\partial n}\\
&= -\int_{\Omega}\Delta F_1 \geq 0
\end{aligned}
$$

where F_1 can be chosen from one of the functions in [P2] which satisfies the symmetric condition on M, e.g., a power of Φ_2 in the proof of Proposition 1, or one of our functions with a calculation using the new smooth coordinate in [SY] page 475 with $R_{1\bar{1}1\bar{1}} = K_{max}$ (e.g., see (4) in the next section). Actually, A_1 itself is proportional to the λ_2 in [P2] and is symmetric in the sense of Polombo. On Ω, F_1 is just our g since $B_1 = 0$. We notice that there is a sign difference for the Laplace operator in [P2]. Again, on Ω, since $A = B$ on a neighborhood, the set of minimal directions is a S^1 bundle over Ω, therefore, one might choose a smooth section of it locally that the calculation of [SY] still works in our case. That is, one could simply choose F_1 to be g.

Case 2: If Ω is a hypersurface. Same argument went through except that $\int_{\partial(M-\Omega)}(A)^{a-1}\frac{\partial A}{\partial n} = 0$ since $A \neq 0$ outside a codimension one subset and on $\Omega_1 = \{x \in \Omega|_{A\neq 0}\}$ the integral is integrated from both sides.

Therefore, Ω is a subset of codimension two and we can apply Remark 1. By the calculation in Remark 1, we see that g is harmonic on $M - \Omega$. Now, by Lemma 2, that implies that $B(B-3A) = 0$ and hence $A = B = 0$ by our assumptions.

4 The Generalization

Actually, in the first section of [SY], the authors did not require any negativity. We also see that in our second section, we do not really need any negativity except when we applied the formula in the Lemma 2 in the third section.

In the first section of [SY], they also considered the coordinate in which $R_{1\bar{1}1\bar{1}}$ achieves the maximal instead of the minimal. We let $C = R_{1\bar{1}2\bar{2}}$ in the earlier sections and C_1 be the bisectional curvature for the maximal case. Then

$$K_{min} + C = K_{max} + C_1$$

be the Einstein constant Q.

$$C_1 - C = K_{min} - K_{max} = -\frac{1}{2}(A+B)$$

and

$$C_1 = C - \frac{1}{2}(A+B) = \frac{1}{2}(R_{1\bar{1}1\bar{1}} - B) = \frac{1}{2}(Q - C_1 - \frac{1}{2}(A+B) - B).$$

Therefore

$$3C_1 = Q - \frac{1}{2}(A+B) - B \leq 0$$

always. Also, $C_1 = 0$ implies $A = B = Q = 0$.

The constant term in the Lemma 2 is

$$\begin{aligned} 3[(3B-A)C - B(3A-B)] &= 3[(3B-A)(C_1 + \frac{1}{2}(A+B)) - B(3A-B)] \\ &= \frac{3}{2}[2\Psi C_1 - (A-B)(A+5B)] \leq 0 \qquad (3) \end{aligned}$$

always. Therefore, we have the same result only if $Q \leq 0$ unless $C_1 = 0$. As above if $C_1 = 0$ we have $A = B = 0$, then $C = 0$ and therefore, $K_{min} = Q = 0$. The manifold is flat.

Now, with $C_1 \leq 0$, we could also easily cover the arguments in both at the end of the proof of Proposition 1 and in Remark 2 in the case of $A = B$. If we denote the maximal direction by e_{1^*} and use * in the notation of the corresponding terms to the minimal case, then similar to the calculation in section 2 we obtain:

$$\Delta R_{1^*\bar{1}^*1^*\bar{1}^*} = -A_1 C_1 + B_1^2 = 2AC_1 \leq 0.$$

See also [MZ] page 27 for a good calculation for this Laplacian at a maximal direction for any complex dimension.

We also have:

$$\nabla R_{1^*\bar{1}^*1^*\bar{2}^*} = -A_1 \nabla a_{2^*} - B_1 \nabla \bar{a}_{2^*} = 2A \nabla a_{2^*},$$

$$\Delta S_{1^*\bar{1}^*1^*\bar{1}^*} = 4A \sum |y^*|^2 + 2AC_1.$$

$$\nabla_{1^*} A_1 = -3\nabla S_{1^*\bar{1}^*1^*\bar{1}^*} = -3A_1 y^*_{21} = 6Ay^*_{21},$$

$$\nabla_{2^*} A_1 = -6Ay^*_{12}.$$

$$\nabla_{\bar{1}^*} R_{1^*\bar{2}^*1^*\bar{2}^*} = -A_1 \bar{y}^*_{22} = 0,$$

$$\nabla_{2^*} R_{1^*\bar{2^*}1^*\bar{2^*}} = A_1 y^*_{11} = 0.$$

$$\begin{aligned}2^a\Delta(A^a) = \Delta(|A_1|^a) = 3a|A_1|^{a-1}\Delta S_{1^*\bar{1^*}1^*\bar{1^*}} + a(a-1)|A_1|^{a-2}|\nabla A_1|^2 \\ = \quad & 3a \times (2A)^{a-1}(4A\sum |y^*|^2 + 2AC_1) \\ + \quad & 9a(a-1)(2A)^a \sum |y^*|^2 \\ = \quad & 3a(2A)^a[(2-3(a-1))\sum |y^*|^2 + C_1]\end{aligned} \tag{4}$$

is nonpositive when $a \leq 1/3$. This is same as in the Lemma 3 and that in [CHY].

Therefore, we concluded the general case. One might conjecture that our Theorem is also true in the higher dimensional cases.

Remark 3. Notice that this generalization basically covers the results in [P1] and [P2] for the Kähler-Einstein case (see [P2] page 398 Corollary). See also [De] page 415 Proposition 2 for the W^+ for a Kähler surface. One might ask whether our result could be generalized to the Riemannian manifolds with closed half Weyl curvature tensors. This is out of the scope of this paper although a similar result is true, i.e., if $\lambda_2 \leq 1$ at every point. To make the relation between this paper and [P1], [P2] clearer to the readers, we just mention that any one of the half Weyl tensors is harmonic if and only if it is closed since the tensor is dual to either itself or the negative of itself. The Remark (i) in [P2] page 397 says that if M is Riemannian-Einstein, the second Bianchi identity says that the half Weyl tensors are closed (see also [De] page 408 formula (9) and page 411 remark 1).

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