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Vertical dynamics of a pantograph carbon-strip suspension under stochastic contact-force excitation

R.H. Huan · W.Q. Zhu · F. Ma · Z.G. Ying

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Abstract In electric trains, a pantograph is mounted on the roof of the train to collect power through contact with an overhead catenary wire. The carbon-strip suspension of a pantograph, along which contact with the catenary occurs, is subjected to harmonic and stochastic contact-force excitations. In this paper, vertical dynamics of the carbon-strip suspension is studied with an aim of improving the reliability and safety of running trains. A single-degree-of-freedom model of the carbon-strip suspension with nonlinear stiffness is developed using parameter values of the DSA X pantograph. Through stochastic averaging, a Fokker–Planck–Kolmogorov equation governing the stationary response of the carbon-strip suspension is set up. Based on the transition probability density of the stationary response, it is found that random jumps and bifurcations in the carbon-strip motion can occur. The possibility of motion bifurcations and the frequency of random jumps warrant consideration in advanced design of carbon-strip suspensions.

Keywords Carbon-strip suspension · Nonlinear model · Stochastic excitations · Stochastic averaging · Random jump

1 Introduction

Electric trains that collect their current from an overhead wire system utilize a pantograph, which typically consists of a head frame, carbon strips and a frame assembly as shown in Fig. 1. The pantograph is spring-loaded and pushes the carbon strips up against the catenary wire to draw the electricity needed to run the train. During operation, the carbon strips and the catenary interact through contact forces. Excessively small contact forces may lead to loss of contact, resulting in electric arcing and power interruptions. On the other hand, excessively large contact forces may cause rapid and uneven wear of the carbon strips. Furthermore, a pantograph may exhibit unexpected motions even when the contact-force variation is kept within a reasonable range. A detailed investigation of the dynamic behavior of a pantograph under contact-force excitation is thus much deserving.

There is fairly extensive literature on the dynamics of pantograph-catenary system. At low train speeds, it is usually assumed that the pantograph has small oscillations around an equilibrium position and nonlinear effects are negligible. This assumption allows a very simple approach. Wu and Brennan [1, 2] conducted an analytical study of the pantograph motion using a

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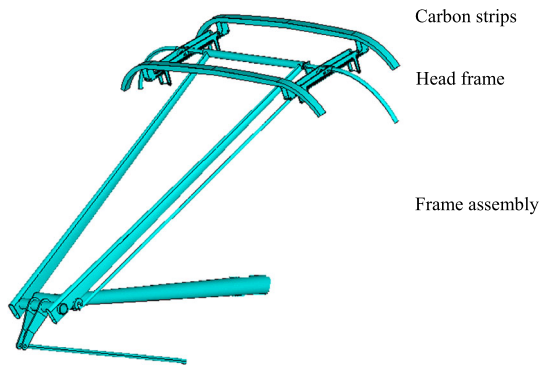


Fig. 1 CAD model of DSA X pantograph

linear single-degree-of-freedom model. In other studies, a linear mass–spring–damper model with three degrees of freedom was used [3–7]. As train speed increases, the effects of nonlinearity become significant due to larger pantograph displacements [8]. In two early papers [9, 10], the nonlinearity caused by the stops (a design in limiting pantograph motions) was examined. More recently, Poznic et al. [11] demonstrated that stiffness nonlinearity plays an important role in the dynamic response of a pantograph. A piecewise linear model of the carbon-strip suspension of the Schunk WBL88 pantograph was developed by Drugge et al. [12], and it was shown that nonlinear phenomena such as subharmonic motion and chaotic response can occur.

In this paper, a dynamic model of the carbon-strip suspension possessing nonlinear stiffness is developed using parameter values of the DSA X pantograph. The carbon-strip suspension is excited by contact force between the pantograph and catenary. In previous works, it was assumed that the contact force is deterministic and is composed of a static and a periodic component [12, 13]. However, a deterministic contact force cannot account for disturbances due to train-body vibration, ambient air flow, contact wire irregularities, etc., which are random in nature. To this end, the contact force is represented herein as a combination of static, periodic, and stochastic excitations. Using stochastic averaging [14–18], a Fokker–Planck–Kolmogorov (FPK) equation governing the stationary response of the carbon-strip suspension is derived. Based on the stationary probability densities of the amplitude and phase of the carbon-strip response, random jumps and bifurcations in the carbon-strip motion are examined for the first time. Finally, direct numerical simulations

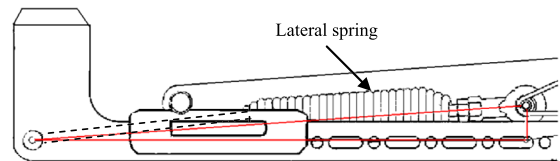


Fig. 2 CAD model of carbon-strip suspension

of the nonlinear carbon-strip suspension are performed to validate the theoretical analysis presented.

2 Modeling of carbon-strip suspension

In the DSA X pantograph, the two carbon strips are connected to the head frame via pre-tensioned lateral springs at each end of the strips as illustrated in Fig. 2. The lateral springs can rotate around the middle of the swivel heads, which brings about independent vertical motions of the two carbon strips. Due to symmetry, it is necessary to consider only one carbon strip. Suppose the carbon strip is excited at mid-point, the suspension can be modeled as a single-degree-of-freedom system as shown in Fig. 3, where $k(x)$ is the equivalent vertical stiffness obtained by using physical parameter values of the DSA X pantograph.

Let the constant stiffness coefficient of the lateral spring be k_0 . When the pantograph head is in a non-operating position, the deformation of the lateral spring is ΔL_0 , as shown in Fig. 4. When a contact force of magnitude F acts on the carbon strip, there is an elongation ΔL of the lateral spring such that

$$F = k_0 \Delta L \quad (1)$$

The deflection ΔL can be expressed as a function of the vertical displacement x of carbon strip so that

$$\Delta L = \sqrt{(x + H_0)^2 + B_0^2} - L_{g1} - L_{g2} - L_0 - \Delta L_0 \quad (2)$$

where L_0 is the original length of the lateral spring; B_0 , H_0 , L_{g1} , and L_{g2} are structural parameters of the suspension system, which can be obtained accurately through measurements of the pantograph motion. Parameter values for the carbon-strip suspension of the DSA X pantograph are given in Table 1.

Let F_v denote the equivalent vertical restoring force. Then

$$F_v = \frac{x + H_0}{\sqrt{(x + H_0)^2 + B_0^2}} F \quad (3)$$

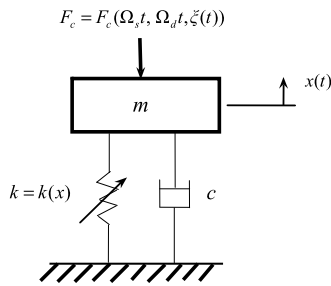


Fig. 3 Dynamic model of carbon-strip suspension

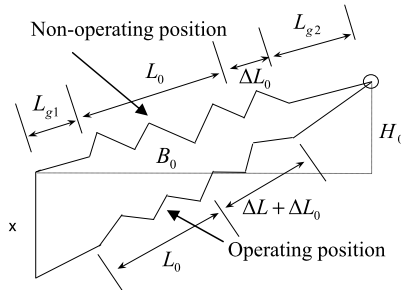


Fig. 4 Deformation of lateral spring

Table 1 Parameters values of DSA X pantograph

m/kg	B_0 /m	L_{g1} /m	L_{g2} /m	L_0 /m	k_0 /N m ⁻¹	ΔL_0 /m
1.5	0.27	0.12	0.12	0.14	16000	0.0093

Substitute Eq. (1) into Eq. (3) and combine the result with Eq. (2) to obtain

$$F_v = \frac{k_0(x + H_0)}{\sqrt{(x + H_0)^2 + B_0^2}} \left(\sqrt{(x + H_0)^2 + B_0^2} - L_{g1} - L_{g2} - L_0 - \Delta L_0 \right) \quad (4)$$

For low train speeds, the vertical displacement x of the contact strip is relatively small. As a result, Eq. (4) can be linearized, which permits a very simple approach [3–7]. As the train speed increases, the vertical displacement x also increases, and Eq. (4) can no longer be linearized. However, the nonlinear relationship between F_v and x in Eq. (4) may be approximated as

$$F_v = k_{v1}x + k_{v3}x^3 \quad (5)$$

A comparison of Eqs. (4) and (5) can be shown graphically. In Fig. 5, F_v – x curves based upon Eqs. (4) and (5) are plotted for different values of k_0 , in each case

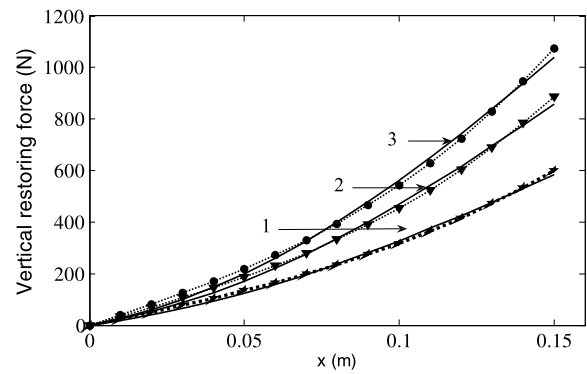


Fig. 5 Comparison of exact and approximate suspension stiffness for different k_0 . Solid lines are from Eq. (4) and dotted lines from Eq. (5). Line 1: $k_0 = e^4$; line 2: $k_0 = 1.6e^4$; line 3: $k_0 = 2e^4$

demonstrating close agreement between the two equations.

The equation of motion of the carbon-strip suspension can be formulated as

$$m\ddot{x} + c\dot{x} + k_{v1}x + k_{v3}x^3 = F_c \quad (6)$$

where m is half the mass of the carbon strip; c is the equivalent damping coefficient; and F_c denotes the contact-force excitation. The contact force consists of a static and a dynamic component. The static component is determined by a static pneumatic uplift force and a velocity-dependent aerodynamic uplift force. The dynamic component has a periodic term that is dependent on the distances between droppers and between support poles [12, 13]. In addition, it has a stochastic term to account for disturbances such as train-body vibration, contact wire irregularities, aerodynamic forces, and other random disturbances. In general, the contact force can be expressed as

$$F_c = F_0 - F_1 \cos(\Omega_s t) - F_2 \cos(\Omega_d t) + h_0 \xi(t) \quad (7)$$

where F_0 is a static lift force; F_1 and F_2 are the amplitudes of periodic excitations; and $h_0 \xi(t)$ represents a weak stochastic excitation. Let L_s and L_d denote the distances between support poles and between droppers, respectively. Then the frequencies of the periodic excitations can be expressed as [10, 12]:

$$\Omega_s = \frac{2\pi V}{L_s}, \quad \Omega_d = \frac{2\pi V}{L_d} \quad (8)$$

where V is the speed of train.

Substituting Eq. (7) into Eq. (6), the equation of motion of the suspension becomes

$$m\ddot{x} + c\dot{x} + k_{v1}x + k_{v3}x^3 = F_0 - F_1 \cos(\Omega_s t) - F_2 \cos(\Omega_d t) + h_0 \xi(t) \quad (9)$$

Define $\omega_n^2 = k_{v1}/m$ and $\tau = \omega_n t$. The above equation can be written in the non-dimensional form:

$$x'' + 2\zeta x' + x + \beta x^3 = f_0 - f_1 \cos(r_s \tau) - f_2 \cos(r_d \tau) + c_w \xi(\tau) \quad (10)$$

where $x' = dx/d\tau$, $x'' = d^2x/d\tau^2$, and

$$\begin{aligned} 2\zeta &= \frac{c}{m\omega_n}, & \beta &= \frac{k_{v3}}{m\omega_n^2}, & f_0 &= \frac{F_0}{m\omega_n^2}, \\ f_1 &= \frac{F_1}{m\omega_n^2}, & f_2 &= \frac{F_2}{m\omega_n^2}, & r_s &= \frac{\Omega_s}{\omega_n}, \\ r_d &= \frac{\Omega_d}{\omega_n}, & c_w &= \frac{h_0}{m\omega_n^2} \end{aligned} \quad (11)$$

Introduce $x = y + q$, where q is the non-dimensional static uplift of the carbon strip which, for $0 < q \ll 1$, satisfies

$$q + \beta q^3 = f_0 \quad (12)$$

In terms of y , Eq. (10) can be expressed as

$$\begin{aligned} y'' + 2\zeta y' + \omega_0^2 y + \alpha y^2 + \beta y^3 \\ = -f_1 \cos(r_s \tau) - f_2 \cos(r_d \tau) + c_w \xi(\tau) \end{aligned} \quad (13)$$

where both quadratic and cubic nonlinearities arise, and

$$\omega_0^2 = 1 + 3\beta q^2, \quad \alpha = 3\beta q \quad (14)$$

System (13) represents a nonlinear oscillator subjected to combined harmonic and stochastic excitations. When q is small, $\beta \gg \alpha$ and βy^3 is the dominant nonlinear term in Eq. (13). In Fig. 6, the coefficient β is plotted as a function of the ratio $\mu = H_0/B_0$ and it can be observed that β decreases rapidly as μ increases. In the next section, stochastic averaging involving generalized harmonic functions [14–18] will be applied to Eq. (13) to derive the statistical properties of response.

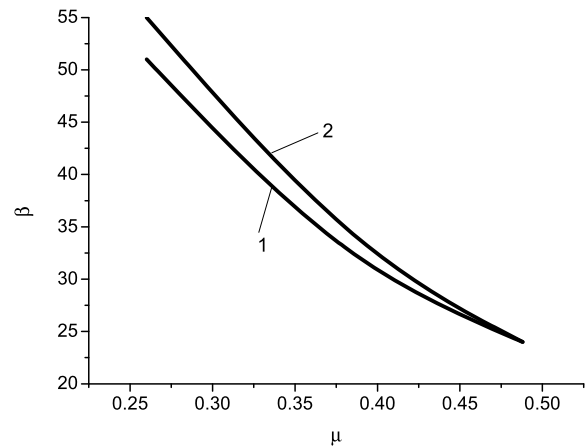


Fig. 6 The nonlinearity coefficient β as a function of $\mu = H_0/B_0$. Line 1: $k_0 = e^4$; line 2: $k_0 = 1.6e^4$

3 Analysis by stochastic averaging

In an actual pantograph, the damping coefficient c is usually very small, and $m\omega_n^2$ is generally much larger than F_1 , F_2 and c_w . That means the oscillator (13) is subjected to small damping and external excitations. Under this condition, it is useful to consider first the undamped free vibration of the oscillator (13), which is governed by

$$y'' + \omega_0^2 y + \alpha y^2 + \beta y^3 = 0 \quad (15)$$

When $\beta \gg \alpha$, periodic solutions of Eq. (15) are assumed to satisfy the relations [19]:

$$\begin{aligned} y(\tau) &= a \cos \theta(\tau), \\ y'(\tau) &= -av(a, \theta) \sin \theta(\tau), \\ \theta(\tau) &= \varphi(\tau) + \gamma \end{aligned} \quad (16)$$

where a is the amplitude, γ is the phase angle, $\cos \theta(\tau)$, $\sin \theta(\tau)$ are generalized harmonic functions, and

$$\begin{aligned} v(a, \theta) &= \frac{d\varphi}{d\tau} = \frac{1}{6} \left[36\omega_0^2 + 18\beta a^2 (1 + \cos^2 \theta) \right. \\ &\quad \left. + \frac{24\alpha a (1 - \cos^3 \theta)}{1 - \cos^2 \theta} \right]^{1/2} \end{aligned} \quad (17)$$

is the instantaneous frequency. The instantaneous frequency can be approximated by the finite sum

$$v(a, \theta) = b_0(a) + b_1(a) \cos \theta + b_2(a) \cos 2\theta$$

$$+ b_3(a) \cos 3\theta + b_4(a) \cos 4\theta \quad (18)$$

for which

$$\begin{aligned} b_0(a) &= \left(\frac{1}{6}T^4 - \frac{27}{4}\alpha\beta a^3 - \frac{9}{2}\alpha^2 a^2 + \frac{3}{4}\beta T^2 a^2 \right. \\ &\quad \left. + \frac{7}{4}\alpha a T^2 \right) / T^3, \\ b_1(a) &= \left(\frac{45}{4}\alpha\beta a^3 + 15\alpha^2 a^2 - \frac{11}{4}\alpha a T^2 \right) / T^3, \\ b_2(a) &= \left(-9\alpha\beta a^3 - 6\alpha^2 a^2 + \frac{3}{4}\beta T^2 a^2 \right. \\ &\quad \left. + 2\alpha a T^2 \right) / T^3, \end{aligned} \quad (19)$$

$$b_3(a) = \left(\frac{45}{8}\alpha\beta a^3 + \frac{15}{2}\alpha^2 a^2 - \frac{9}{8}\alpha a T^2 \right) / T^3,$$

$$b_4(a) = \left(-\frac{9}{4}\alpha\beta a^3 - \frac{3}{2}\alpha^2 a^2 + \frac{1}{4}\alpha a T^2 \right) / T^3,$$

$$T = \sqrt{36\omega_0^2 + 18\beta a^2 + 24\alpha a}$$

An average frequency can now be obtained by integrating Eq. (18), which gives

$$\omega(a) = \frac{1}{2\pi} \int_0^{2\pi} v(a, \theta) d\theta = b_0(a) \quad (20)$$

It follows from Eqs. (16) and (17) that

$$\theta(\tau) \cong \omega(a)\tau + \gamma \quad (21)$$

Assume that neither light damping nor weak excitations would destabilize system (13). The response of system (13) can be regarded as a random spread of periodic solutions of Eq. (15). As a consequence,

$$\begin{aligned} y(\tau) &= A \cos \Theta(\tau), \\ y'(\tau) &= -A\Omega(A, \Theta) \sin \Theta(\tau), \\ \Theta(\tau) &= \Phi(\tau) + \Gamma(\tau), \\ \Omega(A, \Theta) &= \frac{1}{6} \left[36\omega_0^2 + 18\beta A^2 (1 + \cos^2 \Theta) \right. \\ &\quad \left. + \frac{24\alpha A(1 - \cos^3 \Theta)}{1 - \cos^2 \Theta} \right]^{1/2} \end{aligned} \quad (22)$$

where A , Θ , Φ , Γ , and Ω are all random processes. The instantaneous and average frequencies of system (13) are of the same forms given by Eqs. (17) and (20).

Substituting Eq. (22) into Eq. (13) and treating Eq. (22) as a generalized van der Pol transformation from y , y' to A , Γ , the following equations for A and Γ can be obtained:

$$\begin{aligned} \frac{dA}{d\tau} &= F_1(A, \Theta, r_s \tau, r_d \tau) + G_1(A, \Theta) \xi(\tau), \\ \frac{d\Gamma}{d\tau} &= F_2(A, \Theta, r_s \tau, r_d \tau) + G_2(A, \Theta) \xi(\tau) \end{aligned} \quad (23)$$

In the above expressions,

$$\begin{aligned} F_1(A, \Theta, r_s \tau, r_d \tau) &= -\frac{1}{\omega_0^2 + \alpha A + \beta A^2} [2\zeta A \Omega(A, \Theta) \sin \Theta \\ &\quad - f_1 \cos(r_s \tau) \\ &\quad - f_2 \cos(r_d \tau)] \Omega(A, \Theta) \sin \Theta, \\ F_2(A, \Theta, r_s \tau, r_d \tau) &= -\frac{1}{A(\omega_0^2 + \alpha A + \beta A^2)} [2\zeta A \Omega(A, \Theta) \sin \Theta \\ &\quad - f_1 \cos(r_s \tau) \\ &\quad - f_2 \cos(r_d \tau)] \Omega(A, \Theta) \cos \Theta, \end{aligned} \quad (24)$$

$$G_1(A, \Theta) = -\frac{1}{\omega_0^2 + \alpha A + \beta A^2} c_w \Omega(A, \Theta) \sin \Theta,$$

$$\begin{aligned} G_2(A, \Theta) &= -\frac{1}{A(\omega_0^2 + \alpha A + \beta A^2)} c_w \Omega(A, \Theta) \cos \Theta \end{aligned}$$

3.1 Response to Gaussian white noise

Suppose the stochastic excitation $\xi(\tau)$ is weak Gaussian white noise with power spectral density $2D$. Then Eq. (23) can be modeled as Stratonovich stochastic differential equations and transformed into a set of Itô equations by adding the Wong–Zakai correction terms [14] so that

$$\begin{aligned} dA &= m_1(A, \Theta, r_s \tau, r_d \tau) d\tau + \sigma_1(A, \Theta) dB(\tau), \\ d\Gamma &= m_2(A, \Theta, r_s \tau, r_d \tau) d\tau + \sigma_2(A, \Theta) dB(\tau) \end{aligned} \quad (25)$$

where $B(\tau)$ is the unit Wiener process, and the drift and diffusion coefficients are given by

$$\begin{aligned}
m_1(A, \Theta, r_s \tau, r_f \tau) &= F_1(A, \Theta, r_s \tau, r_d \tau) + D \left(G_1 \frac{\partial G_1}{\partial A} + G_2 \frac{\partial G_1}{\partial \Theta} \right), \\
m_2(A, \Theta, r_s \tau, r_f \tau) &= F_2(A, \Theta, r_s \tau, r_d \tau) + D \left(G_1 \frac{\partial G_2}{\partial A} + G_2 \frac{\partial G_2}{\partial \Theta} \right), \\
b_1(A, \Theta) &= \sigma_1^2(A, \Theta) = 2DG_1^2, \\
b_2(A, \Theta) &= \sigma_2^2(A, \Theta) = 2DG_2^2
\end{aligned} \tag{26}$$

Since system (13) is subjected to harmonic excitations, there is the possibility of resonance. Since large response in the carbon-strip suspension may cause malfunction in power collection, the case of primary resonance will be emphasized. In the actual catenary system, the support poles distance L_s and droppers distance L_d often lie in the range of 45–65 m and 7–9 m, respectively. Assume that the highest operational train speed is less than 500 km/h. From Eq. (11), it can be deduced that the frequency ratios $r_s \leq 0.4$ and $r_d \leq 2.6$. Obviously, primary resonance is more likely to be caused by the harmonic term $-f_2 \cos(r_d \tau)$ and the harmonic term $-f_1 \cos(r_s \tau)$ may be ignored. Assume that in primary resonance there exists

$$\frac{r_d}{\omega(A)} = 1 + \varepsilon \eta \tag{27}$$

where $\omega(A)$ is the average frequency of system (13) and $\varepsilon \eta \ll 1$ is the small detuning parameter. Multiply

Eq. (27) by τ and utilize the approximate relation (21) to obtain

$$r_d \tau = \Theta + \varepsilon \eta \omega \tau - \Gamma \tag{28}$$

Introduce a new variable $\Delta = \varepsilon \eta \omega \tau - \Gamma$ so that Eq. (28) can be rewritten as

$$r_d \tau = \Theta + \Delta \tag{29}$$

Substitute Eq. (29) into Eq. (25) to result in

$$\begin{aligned}
dA &= m_1(A, \Theta, \Theta + \Delta) d\tau + \sigma_1(A, \Theta) dB(\tau), \\
d\Delta &= \left[-m_2(A, \Theta, \Theta + \Delta) \right. \\
&\quad \left. + \left(\frac{r_d}{\omega(A)} - 1 \right) \Omega(A, \Theta) \right] d\tau \\
&\quad - \sigma_2(A, \Theta) dB(\tau)
\end{aligned} \tag{30}$$

The coefficients in Eq. (30) are functions of the slowly-varying processes A , Δ and the rapidly-varying process Θ . Take average of Eq. (30) with respect to the rapidly-varying process Θ from 0 to 2π to generate the following averaged Itô equations:

$$\begin{aligned}
dA &= \bar{m}_1(A, \Delta) d\tau + \bar{\sigma}_1(A) dB(\tau), \\
d\Delta &= \bar{m}_2(A, \Delta) d\tau + \bar{\sigma}_2(A) dB(\tau)
\end{aligned} \tag{31}$$

where the averaged drift and diffusion coefficients are

$$\begin{aligned}
\bar{m}_1(A, \Delta) &= -\frac{\zeta \omega_0^2 A + 4/3 \zeta \alpha A^2 + 5/8 \zeta \beta A^3}{(\omega_0^2 + \alpha A + \beta A^2)} + \frac{f_2[2b_0(A) - b_2(A)]}{4(\omega_0^2 + \alpha A + \beta A^2)} \sin \Delta \\
&\quad - \frac{c_w^2 D(1/2 \omega_0^4 - 1/3 \alpha^2 A^2 + 1/3 \alpha \omega_0^2 A + 3/16 \omega_0^2 \beta A^2 - 5/8 \alpha \beta A^3 + 1/16 \beta^2 A^4)}{A(\omega_0^2 + \alpha A + \beta A^2)^3}, \\
\bar{m}_2(A, \Delta) &= (r_d - b_0(A)) + \frac{f_2[2b_0(A) + b_2(A)]}{4A(\omega_0^2 + \alpha A + \beta A^2)} \cos \Delta, \\
\bar{b}_1(A) &= \bar{\sigma}_1^2(A) = \frac{c_w^2 D(\omega_0^2 + 4/3 \alpha A + 5/8 \beta A^2)}{(\omega_0^2 + \alpha A + \beta A^2)^2}, \\
\bar{b}_2(A) &= \bar{\sigma}_2^2(A) = \frac{c_w^2 D(\omega_0^2 - 4/3 \alpha A + 7/8 \beta A^2)}{A^2(\omega_0^2 + \alpha A + \beta A^2)^2}
\end{aligned} \tag{32}$$

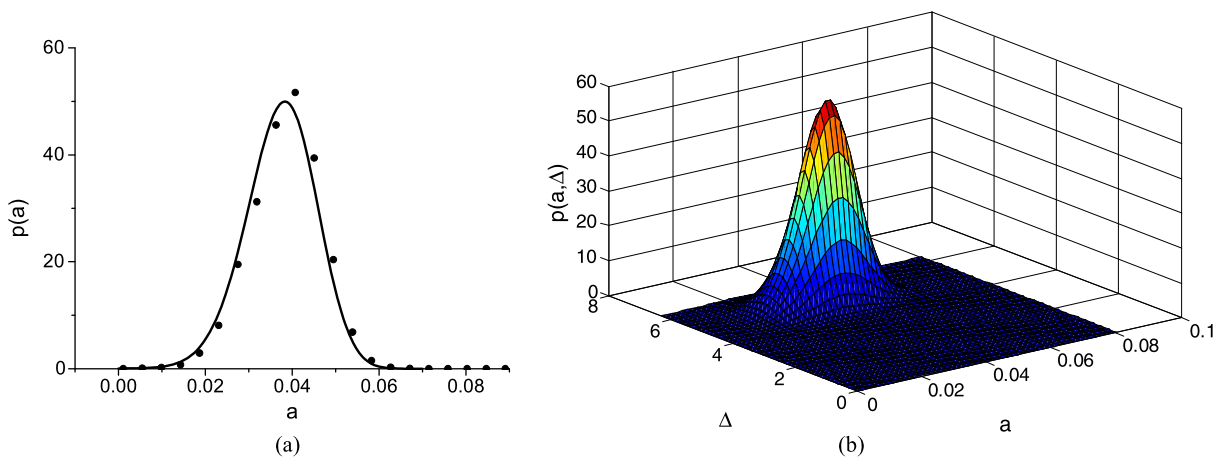


Fig. 7 Stationary probability densities of system (13) under white noise excitation obtained from FPK equation. **(a)** Stationary probability density $p(a)$ of the amplitude a . **(b)** Joint sta-

tionary probability density $p(a, \Delta)$ of amplitude a and phase Δ . Dotted data are from direct numerical simulations of system (13)

The FPK equation associated with the averaged Itô equations (31) is

$$\begin{aligned} \frac{\partial p}{\partial \tau} = & -\frac{\partial}{\partial a}(\bar{m}_1 p) - \frac{\partial}{\partial \Delta}(\bar{m}_2 p) + \frac{1}{2} \frac{\partial^2}{\partial a^2}(\bar{b}_1 p) \\ & + \frac{1}{2} \frac{\partial^2}{\partial \Delta^2}(\bar{b}_2 p) \end{aligned} \quad (33)$$

where $p = p(a, \Delta, \tau)$ is the probability density of amplitude a and phase Δ . The initial condition for Eq. (33) is

$$p = p(a_0, \Delta_0) \quad (34)$$

and boundary conditions for Eq. (33) are

$$\begin{aligned} p &= \text{finite}, \quad a = 0; \\ p, \frac{\partial p}{\partial a} &\rightarrow 0, \quad a \rightarrow \infty; \\ p(a, \Delta + 2n\pi, \tau) &= p(a, \Delta, \tau) \end{aligned} \quad (35)$$

The nonlinear three-dimensional parabolic problem as given in Eqs. (33)–(36) does not admit an easy solution, either analytically or numerically. In practical applications, however, one is more interested in the stationary solution of the FPK equation. In this case, Eq. (33) can be simplified by putting

$$\frac{\partial p}{\partial \tau} = 0 \quad (36)$$

The joint stationary probability density $p(a, \Delta)$ is obtained readily by using the finite-difference method. The stationary probability density $p(a)$ of the amplitude can be obtained from $p(a, \Delta)$ by

$$p(a) = \int_0^{2\pi} p(a, \Delta) d\Delta \quad (37)$$

Numerical results for $p(a)$ and $p(a, \Delta)$ are obtained for $\zeta = 1.45e^{-3}$, $\alpha = 1.7$, $\beta = 50.48$, $f_2 = 3.25e^{-3}$, $r_d = 1.0$, $c_w = 0.075$, $f_0 = 0.01$, $D = 5.31e^{-4}$ and shown in Fig. 7. First, it is observed that the stationary probability density $p(a)$ obtained from the FPK equation closely matches data obtained from direct simulations of system (13), thus validating the analysis presented herein. Second, the stationary probability density $p(a)$ is unimodal, which implies that the carbon strip moves randomly about an equilibrium position.

If the train speed V is increased so that $r_d = 1.1$, the stationary probability density $p(a)$ becomes bimodal, as illustrated in Fig. 8. There are two probable motions of the carbon strip and random jumps may occur, which is not desirable for power collection. If the nonlinearity βy^3 in system (13) is decreased from $\beta = 50.48$ for Fig. 8 to $\beta = 20.48$ for Fig. 9, the stationary probability density $p(a)$ becomes unimodal again and no random jumps can occur. That means bifurcations in the carbon-strip motion and random jumps may occur as the system nonlinearity is varied.

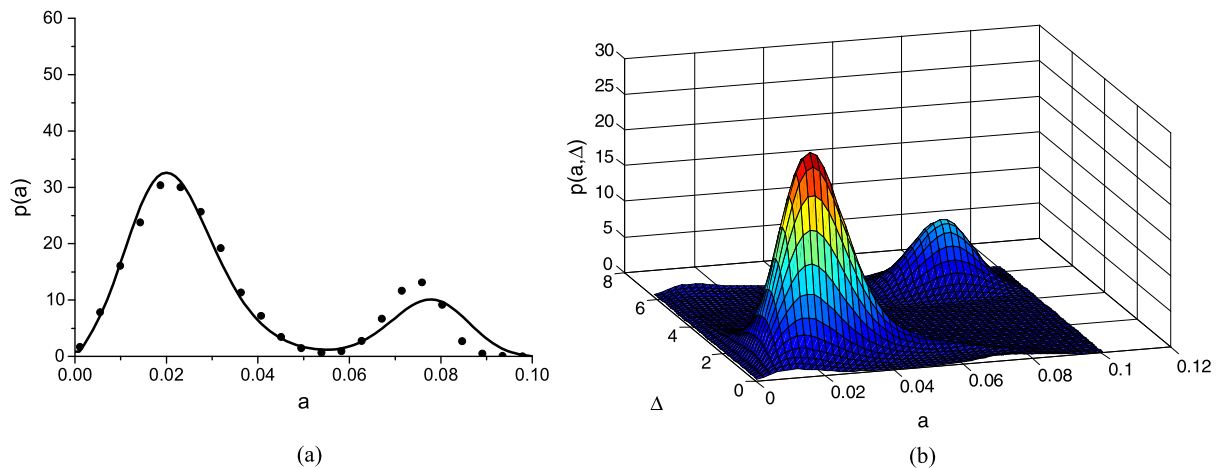


Fig. 8 Stationary probability densities of system (13) under white noise excitation obtained from FPK equation using the same parameters as in Fig. 7, except $r_d = 1.1$. (a) Stationary

probability density $p(a)$ of the amplitude a . (b) Joint stationary probability density $p(a, \Delta)$ of amplitude a and phase Δ . Dotted data are from direct numerical simulations of system (13)

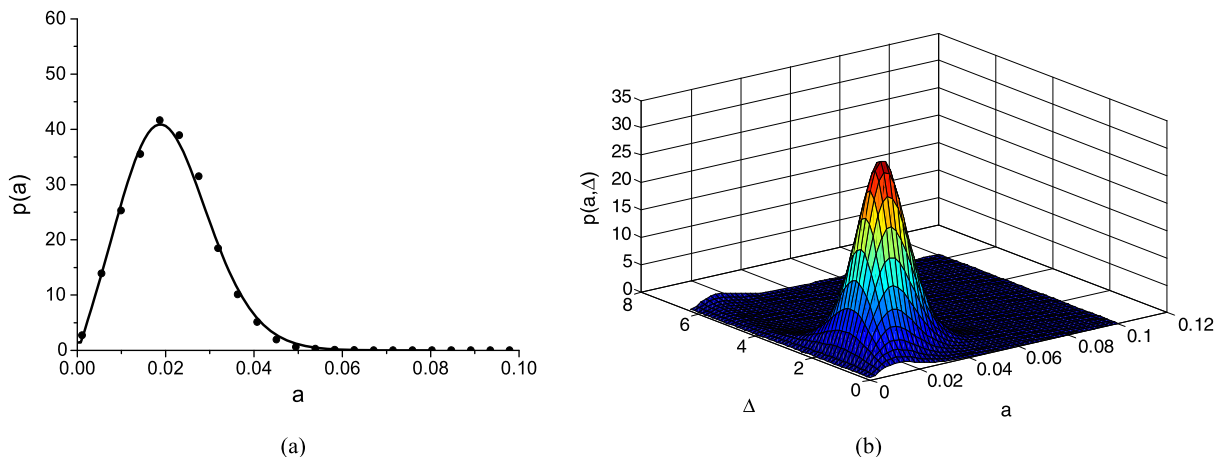


Fig. 9 Stationary probability densities of system (13) under white noise excitation obtained from FPK equation using the same parameters as in Fig. 8, except $\beta = 20.48$. (a) Stationary

probability density $p(a)$ of the amplitude a . (b) Joint stationary probability density $p(a, \Delta)$ of amplitude a and phase Δ . Dotted data are from direct numerical simulations of system (13)

3.2 Response to stationary and ergodic excitation

In the previous section, random jumps and bifurcations in the carbon-strip motion are investigated when the excitation is Gaussian white noise. The response of the carbon strip to more general excitation will now be taken up. Suppose the stochastic disturbance is stationary and ergodic with zero mean and power spectral density

$$S(\omega) = \frac{D\omega + E}{A\omega^2 + B\omega + C} \quad (38)$$

where A, B, C, D, E are constants. In this case, by adding the Wong–Zakai correction terms [20, 21], the Itô equations associated with Eq. (23) have the same form as Eq. (25) but with new drift and diffusion coefficients given by

$$\begin{aligned} m_1(A, \Theta, r_s \tau, r_d \tau) &= F_1(A, \Theta, r_s \tau, r_d \tau) \\ &+ \int_{-\infty}^0 \left(\frac{\partial G_1}{\partial A} \Big|_t G_1|_{t+\tau} + \frac{\partial G_1}{\partial \Theta} \Big|_t G_2|_{t+\tau} \right) \\ &\times R(\tau) d\tau, \end{aligned}$$

$$\begin{aligned}
m_2(A, \Theta, r_s \tau, r_d \tau) \\
= F_2(A, \Theta, r_s \tau, r_d \tau) \\
+ \int_{-\infty}^0 \left(\frac{\partial G_2}{\partial A} \Big|_t G_1|_{t+\tau} + \frac{\partial G_2}{\partial \Theta} \Big|_t G_2|_{t+\tau} \right) \\
\times R(\tau) d\tau,
\end{aligned} \quad (39)$$

$$b_1(A, \Theta) = \sigma_1^2(A, \Theta) = \int_{-\infty}^{\infty} (G_1|_t G_1|_{t+\tau}) R(\tau) d\tau,$$

$$b_2(A, \Theta) = \sigma_2^2(A, \Theta) = \int_{-\infty}^{\infty} (G_2|_t G_2|_{t+\tau}) R(\tau) d\tau$$

where $R(\tau)$ is the autocorrelation function of the stochastic excitation. To investigate primary resonance, utilize Eq. (29) and take average of Eq. (25) with respect to the rapidly-varying process Θ from 0 to 2π to generate averaged Itô equations of the same form as Eq. (31) but with new averaged drift and diffusion coefficients given by

$$\begin{aligned}
\bar{m}_1(A, \Delta) \\
= -\frac{\zeta \omega_0^2 A + 4/3 \zeta \alpha A^2 + 5/8 \zeta \beta A^3}{(\omega_0^2 + \alpha A + \beta A^2)} \\
+ \frac{f_2[2b_0(A) - b_2(A)]}{4(\omega_0^2 + \alpha A + \beta A^2)} \sin \Delta \\
+ f_{11} S(\omega(A)) + f_{12} S(2\omega(A)) \\
+ f_{13} S(3\omega(A)) + f_{14} S(4\omega(A)), \\
\bar{m}_2(A, \Delta) \\
= (r_d - b_0(A)) + \frac{f_2[2b_0(A) + b_2(A)]}{4A(\omega_0^2 + \alpha A + \beta A^2)} \cos \Delta, \\
\bar{b}_1(A) = \bar{\sigma}_1^2(A) \\
= \frac{c_w^2(b_0 - 1/2b_2)^2\pi}{(\omega_0^2 + \alpha A + \beta A^2)^2} S(\omega) \\
+ \frac{c_w^2(b_1 - b_3)^2\pi}{4(\omega_0^2 + \alpha A + \beta A^2)^2} S(2\omega) \\
+ \frac{c_w^2(b_2 - b_4)^2\pi}{(\omega_0^2 + \alpha A + \beta A^2)^2} S(3\omega) \\
+ \frac{c_w^2 b_3^2 \pi}{4(\omega_0^2 + \alpha A + \beta A^2)^2} S(4\omega),
\end{aligned} \quad (40)$$

$$\begin{aligned}
\bar{b}_2(A) = \bar{\sigma}_2^2(A) \\
= \frac{2c_w^2 b_1^2 \pi}{A^2(\omega_0^2 + \alpha A + \beta A^2)^2} S(0) \\
+ \frac{c_w^2(b_0 + 1/2b_2)^2\pi}{A^2(\omega_0^2 + \alpha A + \beta A^2)^2} S(\omega) \\
+ \frac{c_w^2(b_1 + b_3)^2\pi}{4A^2(\omega_0^2 + \alpha A + \beta A^2)^2} S(2\omega) \\
+ \frac{c_w^2(b_2 + b_4)^2\pi}{4A^2(\omega_0^2 + \alpha A + \beta A^2)^2} S(3\omega) \\
+ \frac{c_w^2 b_3^2 \pi}{4A^2(\omega_0^2 + \alpha A + \beta A^2)^2} S(4\omega)
\end{aligned}$$

In the above expressions,

$$\begin{aligned}
f_{11} &= \left[\frac{\partial b_0/\partial A - \partial b_2/\partial A}{(\omega_0^2 + \alpha A + \beta A^2)^2} - \frac{(2b_0 - b_2)(2\beta A + \alpha)}{2(\omega_0^2 + \alpha A + \beta A^2)^3} \right] \\
&\times \frac{c_w^2(2b_0 - b_2)\pi}{4} + \frac{c_w^2(4b_0^2 - b_2^2)\pi}{8A(\omega_0^2 + \alpha A + \beta A^2)^2}, \\
f_{12} &= \left[\frac{\partial b_2/\partial A - \partial b_2/\partial A}{(\omega_0^2 + \alpha A + \beta A^2)^2} - \frac{(b_1 - b_3)(2\beta A + \alpha)}{(\omega_0^2 + \alpha A + \beta A^2)^3} \right] \\
&\times \frac{c_w^2(2b_1 - b_3)\pi}{8} + \frac{c_w^2(b_1^2 - b_3^2)\pi}{4A(\omega_0^2 + \alpha A + \beta A^2)^2}, \\
f_{13} &= \left[\frac{\partial b_2/\partial A - \partial b_4/\partial A}{(\omega_0^2 + \alpha A + \beta A^2)^2} - \frac{(b_2 - b_4)(2\beta A + \alpha)}{(\omega_0^2 + \alpha A + \beta A^2)^3} \right] \\
&\times \frac{c_w^2(b_2 - b_4)\pi}{8} + \frac{3c_w^2(b_2^2 - b_4^2)\pi}{8A(\omega_0^2 + \alpha A + \beta A^2)^2}, \\
f_{14} &= \frac{c_w^2 b_3 \partial b_3/\partial A \pi}{8(\omega_0^2 + \alpha A + \beta A^2)^2} - \frac{c_w^2 b_3^2 (2\beta A + \alpha) \pi}{8(\omega_0^2 + \alpha A + \beta A^2)^3} \\
&+ \frac{c_w^2 b_3^2 \pi}{2A(\omega_0^2 + \alpha A + \beta A^2)^2}
\end{aligned} \quad (41)$$

By solving the stationary FPK equation associated with the averaged Itô equations, the stationary probability densities of the carbon strip under stationary-ergodic excitation are obtained for $\zeta = 1.35e^{-3}$, $\alpha = 1.7$, $\beta = 65.48$, $f_2 = 2.6e^{-3}$, $r_d = 1.1$, $c_w = 0.075$, $f_0 = 0.01$, $D = 8e^{-4}$. In Fig. 10, the stationary probability densities are shown when the nonlinearity βy^3 increases. For $\beta = 20.48$, the stationary probability

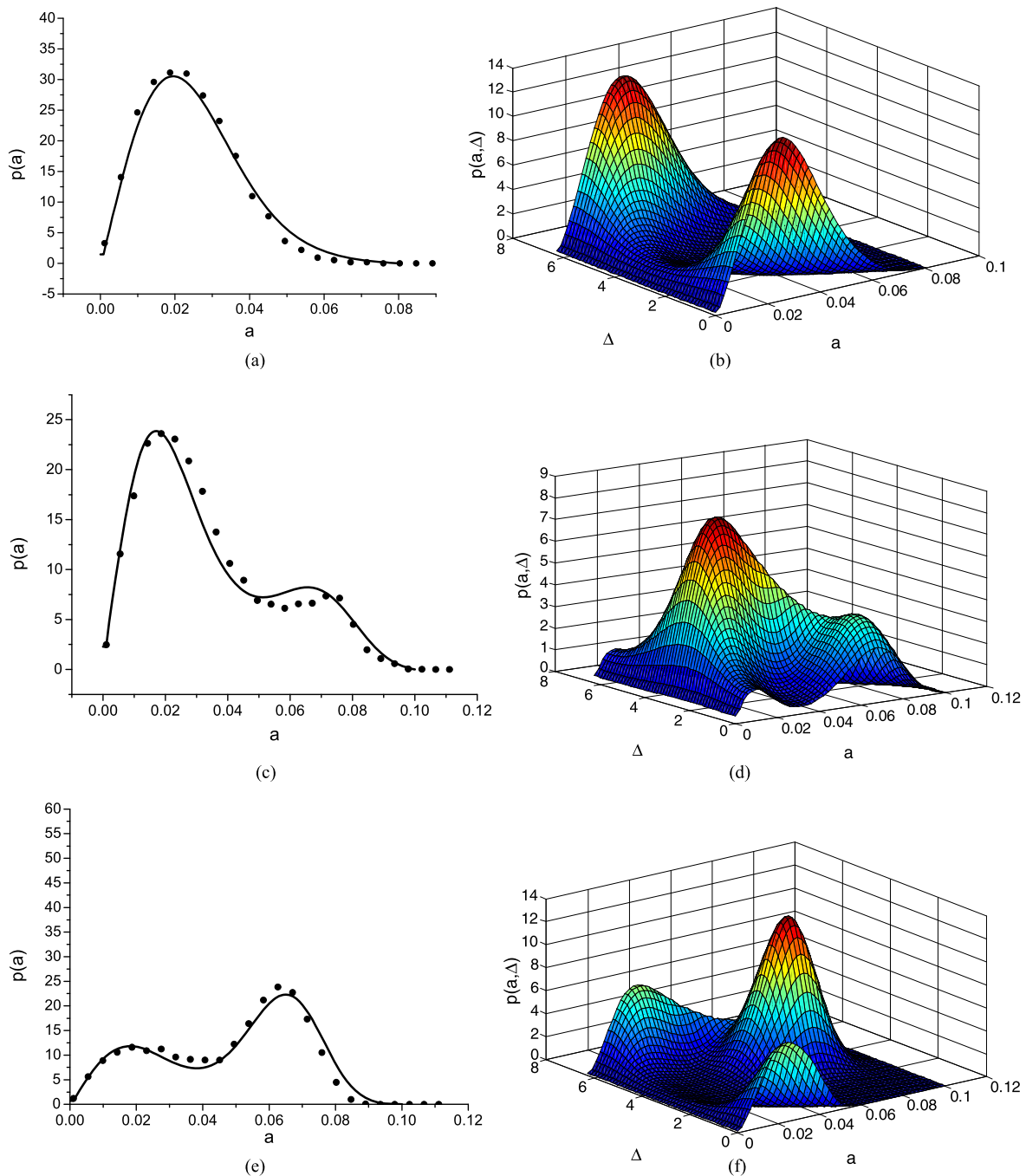


Fig. 10 Stationary probability densities of system (13) under stationary and ergodic excitation obtained from FPK equation. (a), (b) $\beta = 20.48$; (c), (d) $\beta = 50.48$; (e), (f) $\beta = 65.48$. Dotted data are from direct numerical simulations of system (13)

density $p(a)$ is unimodal as plotted in Fig. 10(a). The stationary probability density $p(a)$ becomes bimodal when $\beta = 50.48$ as shown in Fig. 10(c), implying that there are two probable motions of the car-

bon strip and random jumps may occur. As the non-linearity is further increased to $\beta = 65.48$, the stationary probability density $p(a)$ remains bimodal as plotted in Fig. 10(e). However, there is subtle differ-

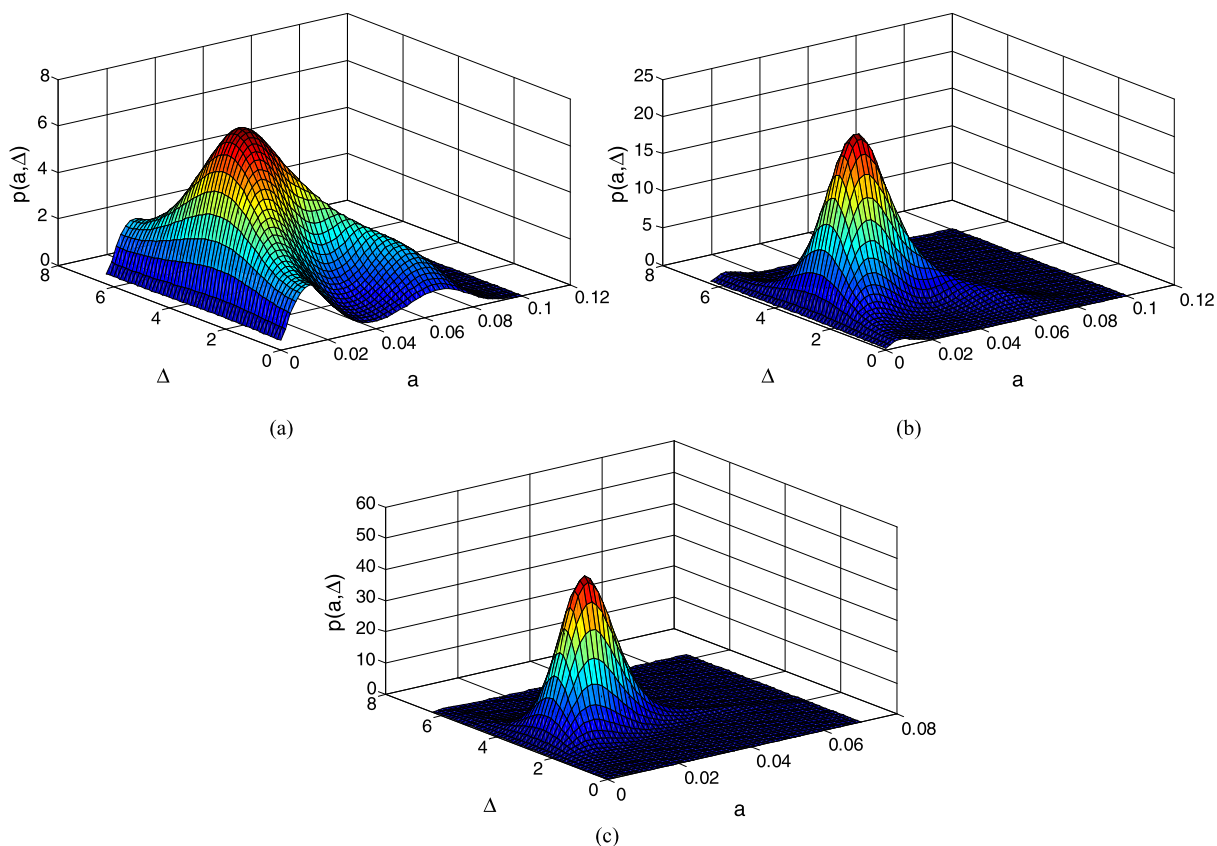


Fig. 11 Stationary probability densities of system (13) under stationary and ergodic excitation obtained from FPK equation using the same parameters as in Fig. 10, except (a) $f_2 = 1.2e^{-3}$;

(b) $c_w = 0.035$; (c) $\zeta = 2.8e^{-2}$. Dotted data are from direct numerical simulations of system (13)

ence between the two bimodal probability densities. For smaller nonlinearity, the two modes are easily distinguishable; the lower-amplitude mode has a much larger probability as shown in Fig. 10(c). That means random jumps occur infrequently and the most probable motion is around the lower-amplitude mode. For larger nonlinearity, the two modes are more comparable in probability and their peaks are less separated, as is evident in Fig. 10(e). In this case, random jumps occur more frequently and the most probable motion is around the larger-amplitude mode.

Furthermore, the effect of changing the amplitude of harmonic excitation f_2 , the strength of stochastic excitation c_w , and the damping factor ζ is shown in Fig. 11. The stationary probability density of the carbon strip changes as these parameters are varied, and random jumps and motion bifurcations could also occur. However, the effect of these parameters seems less pronounced when compared with changes in non-

linearity. As an important observation, the possibility of motion bifurcations and the frequency of random jumps should be taken into consideration in advanced design of carbon-strip suspensions in order to enhance the reliability and safety of electric trains.

Finally, direct numerical simulations of the nonlinear model (13) are also performed to generate $p(a)$. Data from direct numerical simulations closely match those generated by the stationary FPK equation, thus validating the theoretical analysis presented.

4 Conclusions

The carbon-strip suspension of a pantograph and the catenary wire interact through contact forces. In earlier investigations, the contact force was taken as a deterministic excitation. However, a deterministic excitation cannot account for forces due to train-body vibration, ambient air flow, contact wire irregularities,

and other random disturbances. In the present paper, vertical dynamics of the carbon-strip suspension subjected to harmonic and stochastic contact-force excitation has been explored. A single-degree-of-freedom model of the carbon-strip suspension with nonlinear stiffness has been developed using parameter values of the DSA X pantograph. Through stochastic averaging, a FPK equation governing the stationary response of the carbon-strip suspension has been derived. Based on the stationary probability density of the carbon-strip response, random jumps and bifurcations in the carbon-strip motion are examined for the first time.

Analytical results indicate that the effect of nonlinearity amplifies under harmonic and stochastic excitations. It is found that random jumps and bifurcations in the carbon-strip motion can occur for either white-noise or stationary-ergodic excitation. These random jumps and motion bifurcations may remain or disappear as various system parameters are changed. The random jumps may cause a transition from one probable motion to another probable motion of the carbon strip, causing malfunction and interruptions in power collection. The possibility of motion bifurcations and the frequency of random jumps warrant consideration in advanced design of carbon-strip suspensions in order to improve the reliability and safety of running trains. Finally, direct numerical simulations of the nonlinear carbon-strip suspension have been performed to validate the theoretical analysis presented herein.

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