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RADIATIVE TRANSFER AXIOMS*

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ABSTRACT

Four axioms are stated from which the salient features of modern radiative transfer theory may be rigorously deduced. In particular, the various classical attenuating functions and the correct form of the classical homogeneous equation of transfer for radiance (specific intensity) are obtained. The contents of the axioms are culled from recurring themes which appear to be common to all discussions of the classical theory. In particular, the axioms summarize and abstract the notions of carrier space, radiative measure, radiative process, and transfer process. The phenomenon of polarization is included in the formalism. Besides introducing new analytical procedures into the radiative transfer theory, the axiomatic approach allows some novel connections with other branches of mathematical physics such as the Mueller phenomenological algebra, Maxwell's electromagnetic theory, neutron transport theory, and the theory of stochastic processes.

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INTRODUCTION

The equation of radiative transfer stands at about midlevel in the mathematical structure of radiative transfer theory with a detailed modern superstructure resting on it from above and a relatively undeveloped substructure supporting it from below. We intend to focus attention on this substructure of the theory and locate those underpinnings which actually support the equation of transfer and hence the entire superstructure. We then go on to indicate how these may form a mathematical foundation of modern radiative transfer theory. The predominantly phenomenological outlook of the theory keeps this complex task down to conceivable proportions, and limits the search to a handful of possibilities. As in the case of most problems, a careful, useful definition of the object under study is half the battle. For our present purposes, we may tentatively define the object under study, namely radiative transfer theory, as the quantitative study, on a phenomenological level, of the transfer of radiant energy through media that absorb, scatter, or emit radiant energy. The remaining task is then centered on the mathematical explication of the various terms in the definition, and finally, the deduction of their manifold consequences. This definition is the physical definition of radiative transfer theory and is quite what one would expect; but hidden in these lines are the germs of four

basic notions which have proven to be sufficient to initiate and sustain a chain of formal deductions leading to the development of the salient structure of radiative transfer theory up to and including the most general form of the equation of radiative transfer.

The definition singles out "radiant energy" as the basic observable, and the present problem requires first of all a precise quantitative description of this notion, namely an abstract means of measuring radiant energy. Such a procedure yields the idea of a radiative measure, i.e., a real- or vector- valued function on the subsets of some space. The term "media" occurring in the definition suggests first of all the idea of a geometrical setting for the radiant energy - a space which holds or carries the radiant energy - during its wanderings. Hence, there is need for an exact general description of the carrier space (by means of, say, the notion of a measure space in analysis), on which the radiative measures are to be defined. Next, the mechanisms of scattering, absorption, and emission of radiant energy on a phenomenological level require explication. This appears to entail the definition of some sort of radiative process which compactly and usefully describes these phenomena in terms of a semi-group of linear transformations. Finally, the notion of the "transfer of radiant energy" through the media hints at the need of some carefully defined transfer process which is generally independent of the radiative process, i.e., there is required a succinct formulation of those features of geometrical optics which are germain to radiative transfer theory. The discussion

which follows is expository in nature and dwells primarily on the physical and conceptual features of the theory. Those readers requiring a more detailed account in which the mathematical details have not been suppressed, are referred to the paper entitled, "A Mathematical Foundation for Radiative Transfer Theory."[1]

AXIOMS

Motivation

We now turn to some specific illustrations of the four basic notions extracted from the physical definition of radiative transfer theory. These will serve to supply some motivation for the axioms.

An example of a classical carrier space is given by the slab geometry in euclidean three space E_3. Let $X=\{\underset{\sim}{x}: z \geq 0\}$ and $\Xi=\{\underset{\sim}{x}: |\underset{\sim}{x}|\geq 1\}$, where $\underset{\sim}{x}=(x,y,z)$ is a point in E_3. Elements of Ξ (the unit sphere in E_3) will be denoted by $\underset{\sim}{\xi}$. The cartesian product $\Phi=X\times\Xi$ is the classical phase space, a point of Φ will be denoted by $(\underset{\sim}{x},\underset{\sim}{\xi})$. At each time t the radiance function N on Φ assigns to each $(\underset{\sim}{x},\underset{\sim}{\xi})$ in Φ a radiance (of radiant flux of some given fixed wavelength λ) $N(\underset{\sim}{x},\underset{\sim}{\xi},t)$. If F is a subset of Φ, then $\int_F (N/\nu)\, d\underset{\sim}{\rho}=U(F,t)$ is the radiant energy content of F at time t, where ν is the velocity of light function, and $\underset{\sim}{\rho}$ is the classical phase volume measure. Most commonly, F is of

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[1] Published thesis, "A Mathematical Foundation for Radiative Transfer Theory," (University of California at Los Angeles, May 1956), in J. Math. and Mech. 6, (1957).

the form $F = E \times \Xi$ where E is a subset of X. Let us denote by $\underset{\sim}{\Phi}$ the collection of all (measurable) subsets of Φ. Then the triple $(\Phi, \underset{\sim}{\Phi}, \underline{\rho})$ is the classical carrier space. In addition, introduce $\nu = \int n^3 d\underline{\rho}$ as the optical volume measure, where n is the index of refraction function $(n = c/v)$. The classical carrier space with general index of refraction is defined as $(\Phi, \underset{\sim}{\Phi}, \nu)$.

The set function U, defined for each F in $\underset{\sim}{\Phi}$ by $U(F,t) = \int_F (N/\nu) d\underline{\rho}$ is an example of a classical radiative measure. There are three properties that must be possessed by every radiative measure. First, if F_1 and F_2 are two sets of $\underset{\sim}{\Phi}$ with no point in common (i.e., are disjoint) then $U(F_1,t) + U(F_2,t) = U(F_1 \cup F_2, t)$. Further, if $\underline{\rho}(F) = 0$ then $U(F,t) = 0$; finally, $U(F,t) < \infty$ if $\underline{\rho}(F) < \infty$.

A recent work by Lehner and Wing[2] may be used to derive one of the few existing explicit examples of a radiative process. Let $N(\ ,t_1)$ denote the radiance function on a source-free Φ at time t_1. Then allowing the mechanisms of scattering and absorption to take their natural course, let the radiance function at time $t_2 \geq t_1$ be denoted by $N(\ ,t_2)$. It follows that $N(\ ,t_1)$ and $N(\ ,t_2)$ are related by a functional expression of the form*

$$N(\ ,t_2) = N(\ ,t_1)\, P_{t_1,t_2}\ , \tag{1}$$

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[2] J. Lehner and G. M. Wing, "Solution of the Linearized Boltzmann Transport Equation for the Slab Geometry," Duke Math. J. 23, 125-142 (1956).

* P_{t_1,t_2} has been written to the right of the operand for convenience in subsequent formulations.

where

$$P_{t_1,t_2} = \sum_{i=1}^{m} E_i \exp\{-\nu(\alpha-\beta_i)(t_2-t_1)\} + Z \,, \qquad (2)$$

in which

$$Z = \lim_{\omega\to\infty} \int_{\gamma-\omega i}^{\gamma+\omega i} \exp\{(\lambda-\nu\alpha)(t_2-t_1)\, R_\lambda(\cdot)\, d\lambda$$

and $R_\lambda(\cdot)$ is a certain linear operator (the resolvent) associated with the equation of transfer. Finally, $E_i = (\ , \psi_i^*)\psi_i$, α_{Λ} is the volume attenuation function, the β_i and ψ_i are eigenvalues and eigenfunctions associated with the equation of transfer. Above and beyond these mathematical details, the point to observe is that the operator P_{t_1,t_2} for each pair of times (t_1,t_2), $t_2 \geq t_1$ is a <u>linear transformation</u>, i.e., if we have $N_1(\ ,t_1)$ and $N_2(\ ,t_1)$ then formally

$$N_1(\ ,t_2) + N_2(\ ,t_2) = [N_1(\ ,t_1) + N_2(\ ,t_1)]P_{t_1,t_2} \qquad (3)$$

A further property of central importance arises from the uniqueness of the solutions of the equation of transfer. If (t_2,t_3), $t_3 \geq t_2$ is another time interval and $N(\ ,t_3)$ is the radiance function on Φ at time t_3, then as before,

$$N(\ ,t_3) = N(\ ,t_2)P_{t_2,t_3} = [N(\ ,t_1)P_{t_1,t_2}]P_{t_2,t_3} \,.$$

But by the same token,

$$N(\ ,t_3) = N(\ ,t_1)P_{t_1,t_3} \,,$$

so that necessarily (4)

$$P_{t_1,t_3} = P_{t_1,t_2}P_{t_2,t_3} \,, \quad t_1 \leq t_2 \leq t_3 \,,$$

which illustrates the so-called semi-group property of the transformations $P_{t,t'}$, $t' \geq t$. Last but not least, we observe that

$$P_{t,t} = I_{\phi}, \tag{5}$$

the identity operator. Consider $K = \int (\,/\nu)\, d\rho$. K is a linear (integral) operator; its inverse $K^{-1} = d(\,)/d\rho$ generally exists and is the Radon-Nikodym (derivative) operator. The points to be observed here are that the operator $R_{t_1,t_2} = K^{-1} P_{t_1,t_2} K$ is linear, enjoys the semi-group property, and that it works on radiative measures:

$$U(\,, t_2) = U(\,, t_1)\, R_{t_1,t_2}. \tag{6}$$

The collection $\mathcal{R}_{\phi} = \{ R_{t_1,t_2} \}$ of linear transformations is an example of a classical radiative process.

The idea of a transfer process is derived from geometrical optics as follows. Let p_1 be a given point in an isotropic phase space Φ at time t_1. Then there exists a unique ray through p_1 such that for each time $t_2 \geq t_1$ there is a point p_2 on the ray determined by a transformation $T_{t_1,t_2} : p_2 = T_{t_1,t_2}(p_1)$. T_{t_1,t_2} is a one to one mapping of Φ onto Φ with the additional property that it preserves the optical volume of subsets of Φ, i.e., if $F_2 = T_{t_1,t_2}(F_1)$ then $\nu(F_1) = \nu(F_2)$. This latter feature may be checked by using the fundamental Straubel invariant[3] or its generalization by Labussiere[4]. The general uniqueness

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[3] R. Straubel, "Über einen Allgemeinen Satz der Geometrischen Optik und einige Anwendung," Physik. Z. 4, 114-117 (1902).

[4] M. Labussiere, "Sur l'Existence Géométrique d'un Invariant Général des Faisceaux de Rayons se Refractant Suivant la loi de Descartes, et ses Applications a l'Optique Géométrique et au Rayonnment," Compt. rend. 174, 675-677 (1922).

of the solutions of the ray equations gives rise to the important semi-group property of the T-transformations:

$$T_{t_1,t_3} = T_{t_2,t_3}\, T_{t_1,t_2}\,, \quad t_1 \le t_2 \le t_3\,, \tag{7}$$

along with the property

$$T_{t,t} = I_\psi\,, \tag{8}$$

where I_ψ is the identity transformation.

Formulation

The transition from the classical to the abstract structure of radiative transfer theory is accomplished by stripping away from the existing substructure all those trappings which are really inessential to the mathematical form of the theory. One weeding process has already resulted in the singling out the four basic notions considered earlier. The next step is to sift through these, retain only their salient and essential properties, then test whether the result is the required, streamlined substructure. The final results are given below. Unfortunately, in the interests of brevity, some terminology had to be left undefined. On the other hand, any simplification of the wording of the axioms would vitiate the primary purpose of their formulation. However, with the preceding heuristic discussion in mind, the basic content of the axioms should be clear.

C_1: The Carrier Space Axiom.

There exists a carrier space $(X, \underset{\sim}{S}, \nu)$ where X is a completely arbitrary set, $\underset{\sim}{S}$ is a sigma-algebra of subsets of X, and ν is a totally sigma-finite measure on $\underset{\sim}{S}$ such that $\nu(X) > 0$.

M_1: The Radiative Measure Axiom.

Let $\mathcal{M}_\nu$ be the collection of all totally-finite, ν-continuous measures on $\underset{\sim}{S}$ associated with a fixed non-negative real number $\lambda \in \Lambda$ *. Then:

(i) For every $t \in T$** there is a non-empty subset $\mathcal{M}_{\nu,t} \subset \mathcal{M}_\nu$ which forms a non-negative cone.

(ii) If $\mu \in \mathcal{M}_\nu$ and if for every $E \in \underset{\sim}{S}$ there is a $\mu_1 \in \mathcal{M}_{\nu,t}$ and an $E_1 \in \underset{\sim}{S}$ such that $\mu(E) = \mu_1(E_1)$, then $\mu \in \mathcal{M}_{\nu,t}$.

(iii) If $\mu(\ , t, \lambda) \in \mathcal{M}_{\nu,t}$ and t is fixed, the corresponding function $\mu(E, t, \ )$ is Lebesgue integrable on Λ.

R_1: The Radiative Process Axiom.

Let $D = \{(t_1, t_2) : t_2 \geq t_1\}$. Let $\mathcal{R}$ be the collection of all linear transformations from $\mathcal{M}_\nu$ into $\mathcal{M}_\nu$. Then there exists a transformation ϕ from D into $\mathcal{R}$ such that $\phi(D) = \mathcal{R}_\phi$ has the following properties. Let $\phi[(t_1, t_2)] = \mathcal{R}_{t_1,t_2}$, then:

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*the abstract counterpart to the notion of wavelength.

**the abstract counterpart to the notion of time.

(i) $\mu \in \mathcal{M}_{\nu,t_1}$ implies $\mu R_{t_1,t_2} \in \mathcal{M}_{\nu,t_2}$.

(ii) If t_1, t_2, t_3 are elements of T such that $t_1 \leq t_2 \leq t_3$, then

$$R_{t_1,t_3} = R_{t_1,t_2} R_{t_2,t_3} .$$

(iii) For every $t \in T$, $R_{t,t} = I_\phi$, the identity transformation on $\mathcal{M}_\nu$.

(iv) If $\{\mu_n\}$ is a sequence of measures in $\mathcal{M}_\nu$ and if $\mu_n \to \mu \in \mathcal{M}_{\nu,t}$ then for every $(t,t') \in D$, $\mu_n R_{t,t'} \to \mu R_{t,t'}$.

(v) If $\mu \in \mathcal{M}_{\nu,t}$, $\mu R_{t_1,t_2} \leq \mu(X)$, for every $(t_1,t_2) \in D$.

The collection $\mathcal{R}_\phi$ is the radiative process on $(X, \underset{\sim}{S}, \nu)$.

T_1: The Transfer Process Axiom.

Let $\mathcal{T}$ be the collection of all measure preserving transformations on $(X, \underset{\sim}{S}, \nu)$ onto $(X, \underset{\sim}{S}, \nu)$. Then there exists a transformation ψ from D into $\mathcal{T}$ such that $\psi(D) \equiv \mathcal{T}_\psi$ has the following properties. Let $\psi[(t_1,t_2)] \equiv T_{t_1,t_2}$, then:

(i) If t_1, t_2, t_3 are elements of T such that $t_1 \leq t_2 \leq t_3$, then

$$T_{t_1,t_3} = T_{t_2,t_3} T_{t_1,t_2} .$$

(ii) For every $t \in T$, $T_{t,t} = I_\psi$, the identity transformation on X.

The collection $\mathcal{T}_\psi$ is the transfer process on $(X, \underset{\sim}{S}, \nu)$.

These four axioms, known collectively as the structure axioms, supply all the mathematical machinery required for a rigorous formal synthesis of the principal mathematical components radiative transfer theory. The most far-reaching clauses occur in (i) of T_1 and (ii) of R_1. These summarize the semi-group properties of the radiative and transfer processes and are abstract variants of the major premise of

Huygens' Principle[5]. Semi-group type of formulations in modern radiative transfer theory promise to be the most natural and far-reaching formulations devised so far. For example, the principles of invariance in radiative transfer theory as originally formulated by Ambarzumian[6] and extensively developed by Chandrasekhar[7] have been subjected to fruitful preliminary analyses by means of the semi-group concept[8].

By means of the structure axioms, we may give a mathematical definition of radiative transfer theory. First we agree that a radiative transfer process is simply the pair $(\mathcal{R}_\phi, \mathcal{T}_\psi)$. Then we may say that radiative transfer theory is the study of radiative transfer processes $(\mathcal{R}_\phi, \mathcal{T}_\psi)$ on general carrier spaces (X, Σ, ν):

ABSTRACT TRANSFER EQUATIONS

The six topics presented below are selected to briefly illustrate the wide scope of, and the fine detail implicit in the axiomatic formula-

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[5] J. Hadamard, Lectures on Cauchy's Problem in Linear Partial Differential Equations. (Yale University Press, New Haven, 1923) pp. 53-55. See also E. Hille, Functional Analysis and Semi-groups (American Mathematical Society Colloquium Publications, New York, 1948) Vol.31, pp.388-390.

[6] V. A. Ambarzumian, "Diffuse Reflection of Light by a Foggy Medium," Compt. rend. (Doklady) Acad. Sci. U.R.S.S. 38, 229-232 (1943).

[7] S. Chandrasekhar, Radiative Transfer (Clarendon Press, Oxford, 1950, Chaps IV, VII.

[8] R. Bellman and R. Kalaba, "On the Principle of Invariant Embedding and Propagation through Inhomogeneous Media," Proc. Nat. Acad. Sci. 42, 629-632 (1956).

tions. The first three illustrate the scope; the last two, the detail.

Static Transition Relations for Measures

If $\mu(\ ,t)=\mu\in\mathcal{M}_{v,t}$ and $R_{t_1,t_2}\in\mathcal{R}_\phi$ then by (i)R_1, $\mu R_{t_1,t_2}\in\mathcal{M}_{v,t_2}$. For notational simplicity, we write the transformed measure in the form $\mu(\ ,t_2)=\mu(\ ,t_1)R_{t_1,t_2}$. Further let $E\in\underset{\sim}{S}$ be such that $\mu(E)>0$, then define μ_E as follows: $\mu_E(F)=\mu(E\cap F)/\mu(E)$ for every $F\in\underset{\sim}{S}$. The image of μ_E under R_{t_1,t_2} is written, $\mu(E,t_1;\ ,t_2)$, and a further abbreviation $\mu'=\mu_E R_{t_1,t_2}$ will be helpful in the sequel. From (ii) of $\ddot{M}_1$ it is clear that $\mu'\in\mathcal{M}_{v,t_2}$. From (iii) of R_1, we see that μ_E may also be written as $\mu(E,t_1\ ,t_1)$. Now the semi-group property of $\mathcal{R}_\phi$ yields the static transition relation for radiative measures:

$$\mu(E,t_1;\ \ ,t_3)=\sum_{j=1}^{n}\mu(E,t_1;F_j,t_2)\mu'(F_j,t_2;\ \ ,t_3), \qquad (9)$$

where $\{F_j : j=1,\cdots,n\}$ is a partition of X, i.e., the F_j are mutually disjoint elements of $\underset{\sim}{S}$ which together completely cover X. The physical significance of (9) is depicted in Figure 1.

Figure 1

(9) is the most primitive form of the equation of transfer on a set level. It is the radiative transfer counterpart to the Chapman-Kolmogoroff transition relation for probabilities[9] in the abstract theory of stochastic

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[9] A. Kolmogoroff, "Über die Analytischen Methoden in der Wahrscheinlichkeitsrechnung," Math, Ann. 104, 415-458 (1931).

processes. In this way the present axiomatic formulations may be seen to be conceptually linked to the diffusion process formulations by Feller[10].

General Density Functions

So far the abstract version of radiance (specific intensity) has not entered the picture. The existence of a general radiant density function, the abstract counterpart to the classical radiance function, follows immediately from M_1. In particular, if $\mu \in \mathcal{M}_{\nu,t}$ then the Radon-Nikodym derivative $d\mu/d\nu$ of μ with respect to ν is the required density, which we shall denote by $G(\ ,t)$. That is, for any $\mu \in \mathcal{M}_{\nu,t}$ it follows that from M_1 that there exists a function $G(\ ,t)$ on X such that for each $F \in \underset{\sim}{S}$,

$$\mu(F,t) = \int_F G(\ ,t)\, d\nu . \tag{10}$$

The significance of this density is best seen by returning to more familiar ground. In classical carrier spaces with general index of refraction we have

$$\int_F (N/v)\, d\underline{\rho} = U(F,t) = \int_F G\, d\nu = \int_F G n^3 d\underline{p} ,$$

so that the general density G and the radiance function N in this context are related as follows:

$$G = N/n^2 c . \tag{11}$$

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[10] W. Feller, "The General Diffusion Operator and Positivity Preserving Semi-groups in One Dimension," Ann. of Math. 60, 417-436 (1954).

It is well known that the function N/n^2 is constant along rays in optical media in which $\alpha \equiv 0$. Hence the advantage of considering the general density G is that any variation of G along a natural path must be due to the presence of the mechanisms of absorption and scattering in the medium. In passing, we note that the Radon-Nikodym derivative of the radiant energy measure function U with respect to the classical phase measure ρ is $dU/d\rho = N/\nu$.

Static Transition Relations for Densities

We denote the general radiant density of $\mu(E,t_1;\ ,t_2)$ by $G(E,t_1;\ ,t_2)$ Hence for every $F \in \underset{\sim}{\Sigma}$,

$$\mu(E,t_1;F,t_2) = \int_F G(E,t_1;\ ,t_2)\, d\nu. \tag{12}$$

Then, corresponding to (9) we have, by means of the structure axioms, the static transition relation for G-densities:

$$G(E,t_1;x_3,t_3) = \int_X G(E,t_1;\ ,t_2)\, dG'(\ ,t_2;x_3,t_3)\ , \tag{13}$$

which is the most primitive form of the equation of transfer on a point level, and is the radiative transfer counterpart to the Chapman-Kolmogoroff transition relation for probability densities. In the general theory there exist two further densities, the g- and the γ-density. These complement the G-density during the struggle to reconstruct the theory from the axioms. The transition relations they satisfy are given below. However, of the three densities, the G-density is the closest counterpart to the classical density N .

$$g(x_1,t_1;\ E_3,t_3) = \int_X dg\,(x_1,t_1;\ ,t_2)\,g'(\ ,t_2;\ E_3,t_3),$$

and

$$\gamma(x_1,t_1;\ x_3,t_3) = \int_X \gamma(x_1,t_1;\ x_2,t_2)\,\gamma'(x_2,t_2;\ x_3,t_3)\,d\nu(x_2)\ ,$$

where g and γ have the definitions

$$\mu(E,t_1;\ F,t_2) = \int_E g(\ ,t_1;\ F,t_2)\,d\nu$$

$$\mu(E,t_1;\ F,t_2) = \int_F \left[\int_E \gamma(\ ,t_1;\ ,t_2)\,d\nu\right] d\nu,$$

G', g' and γ' are the various Radon-Nikodym derivatives of μ'. The physical significance of each of these densities is shown in Figures 2 and 3.

Figure 2

Figure 3

Equivalence of Measures and Densities

Denote by η_ν the collection of all general radiant densities (i.e., all G-, g-, and γ-densities). Let K denote the linear (integral) operator $\int(\ )\,d\nu$. K has densities as inputs and measures as outputs, that is to say, the domain of K is η_ν and the range of K is $\mathcal{M}_\nu$. The inverse $K^{-1} = d/d\nu$ of K, the Radon-Nikodym (derivative) operator, has $\mathcal{M}_\nu$ as domain and η_ν as range. Both K and K^{-1} are one to one transformations. Hence if $G(\ ,t_1)$ is a general radiant density associated with time t_1, $G(\ ,t_1)K$ is a radiative measure $\mu(\ ,t_1)$ in $\mathcal{M}_{\nu,t_1}$. Further, $[G(\ ,t_1)K]R_{t_1,t_2}$, by (i)R_1, is a radiative measure $\ddot{\mu}(\ ,t_2)$ in $\mathcal{M}_{\nu,t_2}$.

Finally, $[G(,t_1)K]R_{t_1,t_2}K^{-1}$ is the general radiant density $G(,t_2)$ of $\mu(,t_2)$ and

$$G(,t_2) = G(,t_1)KR_{t_1,t_2}K^{-1} . \tag{14}$$

In this way we generate for each $t \in T$, (a) a subcollection $\mathcal{N}_{\nu,t}$ of $\mathcal{N}_\nu$ corresponding to $\mathcal{M}_{\nu,t}$ in $\mathcal{M}_\nu$, and (b) a linear transformation P_{t_1,t_2} from $\mathcal{N}_{\nu,t_1}$ into $\mathcal{N}_{\nu,t_2}$ of the form

$$P_{t_1,t_2} = KR_{t_1,t_2}K^{-1} . \tag{15}$$

The collection $\mathcal{P}_\phi = \{P_{t_1,t_2}\}$ is the <u>radiative process for densities</u>. An example of $\mathcal{P}_\phi$ in the classical context was given in (1).

Hence $\mathcal{R}_\phi$ gives rise to $\mathcal{P}_\phi$. Conversely, if $\mathcal{P}_\phi$ were initially postulated instead of $\mathcal{R}_\phi$, the existence of $\mathcal{R}_\phi$ would be demonstrated by following an analogous procedure to that given above. In a certain (well-definable) sense, $\mathcal{P}_\phi$ and $\mathcal{R}_\phi$ are isomorphic (identically structured) collections and, mathematically, neither is more general than the other. In other words, had we axiomatized the notion of radiance instead of radiant energy, we would eventually have arrived at an equivalent formulation of the mathematical foundations of radiative transfer theory. This equivalence may be given a more practical slant. It implies that: knowledge of the radiant energy content of all subsets of an optical medium determines the radiance at each point, and conversely.

This gives the reason for the unpleasant fact that it is just as difficult (or entails just as much effort) to solve the exact equation of transfer for scalar irradiance (or energy density) as it is to solve the equation of transfer for radiance.

Attenuating Functions

The classical volume (or mass) absorption function a, volume scattering function σ, and the volume attenuation function α ($= a+\mathcal{A}$ where $\mathcal{A} = \int_{\Xi} \sigma \, d\Omega$) are all latent in the axiomatic formulations. The formal steps required to uncover these functions are easily taken and simply entail a methodical comparison of the values of a given radiative measure μ and its transform $\mu R_{t_1,t_2}$. To this end, let $\mu \in \mathcal{M}_{\nu,t}$ and $E \in \underset{\sim}{S}$; then define the radiative measure $\bar{\mu}_E$ by

$$\bar{\mu}_E(F) = \mu(E \cap F) \qquad \text{for all } F \in \underset{\sim}{S}.$$

By (ii) M_1, $\bar{\mu}_E$ is clearly in $\mathcal{M}_{\nu,t}$.

Let Δ be any positive real number, then consider the image

$$\bar{\mu}_E R_{t,t+\Delta}$$

of $\bar{\mu}_E$, which may be written as $\bar{\mu}(E,t;\ ,t+\Delta)$. The difference quotient

$$\mathcal{A}_{*\Delta}(E,F,t) = \left[\bar{\mu}_E(F) - \bar{\mu}(E,t;F,t+\Delta)\right]/\Delta \tag{16}$$

is of central interest in what follows. In a certain (rather rough, but informative) sense $\mathcal{A}_{*\Delta}(E,F,t)$ is the average rate of loss (attenuation) from the quantity $\mu(E \cap F)$ of radiant energy starting from E and traveling to F during the time interval $(t,t+\Delta)$.

The limit

$$\lim_{\Delta \to 0} \mathcal{A}_{*\Delta}(E,F,t) = \mathcal{A}_{*}(E,F,t) \tag{17}$$

is the instantaneous rate of loss at time t .

The set function $\mathcal{A}_{*}(\ ,\ ,t)$ is the fountainhead of all the attenuating functions; for by judiciously choosing E and F , the value $\mathcal{A}_{*}(E,F,t)$ may be interpreted as the result of a scattering or absorption mechanism (or both). By using a technical device known as "restricting" a function, i.e., limiting a function to certain proper subsets of its domain of definition, new functions are formed. The new functions $A_{*}(\ ,t)$, $\Sigma_{*}(\ ,\ ,t)$, $S_{*}(\ ,t)$, and $Q_{*}(\ ,t)$ are obtained by suitably restricting $\mathcal{A}_{*}(\ ,\ ,t)$. In particular, $A_{*}(\ ,t)$ is obtained by permanently setting $F = X$, and then defining

$$A_{*}(E,t) = \mathcal{A}_{*}(E,X,t). \tag{18}$$

$\Sigma_{*}(\ ,\ ,t)$ is obtained by restricting $\mathcal{A}_{*}(\ ,\ ,t)$ to pairs (E,F) such that E and F are disjoint. If $E \cap F = \phi$ (i.e., E and F are disjoint), then

$$\Sigma_{*}(E,F,t) = -\mathcal{A}_{*}(E,F,t). \tag{19}$$

Further, if in addition to $E \cap F = \phi$, we have $E \cup F = X$ (i.e., E and F partition X, so that F is the complement, denoted by E', of E) then we define $S_{*}(\ ,t)$ as follows:

$$S_{*}(E,t) = -\mathcal{A}_{*}(E,E',t). \tag{20}$$

Finally, if $E = F$, we write

$$Q_{*}(E,t) = \mathcal{A}_{*}(E,E,t). \tag{21}$$

It turns out that A_* is the abstract (set theoretic) counterpart to the volume absorption function a ; Σ_* corresponds to σ ; S_* to s , the total scattering function; and G_* to α . It is easily seen from the various definitions, that

$$G_*(E,t) = A_*(E,t) + S_*(E,t) , \tag{22}$$

which corresponds to the classical relation,

$$\alpha(x,t) = a(x,t) + s(x,t).$$

By means of the Radon-Nikodym derivatives (with respect to μ) of $A_*(\,,t)$, etc., we arrive at the generalizations $a_*, \sigma_*, s_*, \alpha_*$ of the classical attenuating functions a, σ, s, α .

Dynamic Transition Relations

Dynamic transition relations are distinguished from their static correspondents by the presence of time derivatives of the measure or density in question. In the present theory there is only one dynamic transition relation for measures:

$$d\mu(E,t';F,t)/dt = -\lambda(*(X,F,t) = -G_*(F,t) + S_*(F,t), \tag{23}$$

and three relations for densities, the dynamic transition relation for G-densities being of the form:

$$dG(E,t';x,t)/dt = -\alpha_*(x,t)\,G(E,t';x,t) + \tag{24}$$

$$+ \int_X \sigma_*(x,x',t)\,G(E,t';x',t)\,d\nu(x') .$$

The other two are of the same general form. Despite the generality of (23) and (24), the physical significance of the terms on the right of each equation is readily evident from the correspondences given above and their striking general similarity to the classical transfer equation. In the derivation of these equations formal use was made only of the structure axioms C_1, M_1, and R_1. T_1 played no explicit part whatsoever in the deductions; it is used in the present theory only to drive the most general form of the classical transfer equation. (23) and (24) thus hold on a completely general carrier space $(X, \underset{\sim}{\Sigma}, \nu)$.

CLASSICAL TRANSFER EQUATION

It is a long,tortuous trail from the structure axioms to the classical form of the equation of transfer for radiance (or specific intensity). The eventual termination of the trail in the correct transfer (equation (25)below) testifies, on the one hand, that the axiomatic formulation has captured the essence of a radiative transfer process. The seemingly large number of steps required to traverse the trail brings to light, on the other hand, the manifold assumptions that are unsuspectedly and implicitly embedded in the mathematics of the classical theory. If any one of these assumptions is omitted during the derivation procedure, the result will fall somewhere between (9) and (25).

In order to obtain the classical form of the equation of transfer, the carrier space must be endowed with a great amount of structure,

that is, with the rich analytical and topological structure of euclidean three space E_3 . Formally, however, the classical equation may be obtained by starting with (9) and introducing the following conditions; (a) T_1 holds, (b) X is factorable into the cartesian product of a "location space" X_1 , and a "direction space" X_2 (cf. with Φ), (c) the radiative process is restricted on X_1 . This latter assumption reflects the usually adopted viewpoint that scattering of light cannot take place instantaneously between two spatially remote points in phase space Φ , but can take place if the two points are only directionally remote. When these data are fed into the axiomatic machinery, out comes the required classical form of the equation of transfer:

$$\frac{n^2(\underset{\sim}{x},t)}{v(\underset{\sim}{x},t)} D\left[N(\underset{\sim}{x},\underset{\sim}{\xi},t)/n^2(\underset{\sim}{x},t)\right]/Dt = -\alpha(\underset{\sim}{x},t)N(\underset{\sim}{x},\underset{\sim}{\xi},t) + \int_{\Xi} \sigma(\underset{\sim}{x},\underset{\sim}{\xi},\underset{\sim}{\xi}',t)N(\underset{\sim}{x},\underset{\sim}{\xi}',t)\,d\Omega(\underset{\sim}{\xi}'), \tag{25}$$

or, in abbreviated form:

$$(n^2/v)\, D[N/n^2]/Dt = -\alpha N + N_* , \tag{26}$$

where $D[\,]/Dt$ is the lagrangian, or mobile derivative operator. The relation $G = N/n^2c$ was used to go from the G-density to the radiance function N . Ω is the measure function defined on Ξ . Observe that if $\alpha \equiv 0$ on X_1 , then (25) reduces to

$$D[N/n^2]/Dt \equiv 0 , \tag{27}$$

so that N/n^2 is constant along rays in X (i.e., the paths defined by the transformations $T_{t,t'}$).

Since the outlook of radiative transfer theory is predominantly phenomenological[11], the present theory may be rounded out to take into account the presence of sources by including a properly phrased local source axiom S_1 which authorizes the additive adjunction of an emission function N_η to (25).

POLARIZATION

A perusal of the wording of the structure axioms would show that the radiative measures can have, among other forms, the structure of a vector measure, i.e., the explicit form of $\mu \in \mathfrak{M}_\nu$ may be that of a quadruple of real-valued measures: $(\mu_1, \mu_2, \mu_3, \mu_4)$; and to emphasize the specific adoption of such a form we would write μ in vector type: $\underset{\sim}{\mu}$. Now the formal development of the theory from (1) to (25) goes through with very few basic changes if the vector form of radiative measures is adopted. In particular, the general radiant density of $\underset{\sim}{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4)$ is $\underset{\sim}{G} = (G_1, G_2, G_3, G_4)$, where G_i is the usual real-valued density of μ_i , and $R_{t,t'}$ now has the form of a 4 x 4 matrix:

$$\underset{\sim}{R}_{t,t'} = \left({}^{ij}R_{t,t'} \right).$$

The principal advantage arising from these possible alternate forms of the radiative measures, their densities, and the radiative

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11 R. Wildt, "Radiative Transfer and Thermodynamics," Astrophys. J. 123, 107-116 (1956)

process, is that the phenomenon of polarization may be included in the abstract formalism. In particular, $\underset{\sim}{G}$ is the abstract counterpart to the classical Stokes vector (N_l, N_r, N_U, N_V) (or (I_l, I_r, U, V) as used by Chandrasekhar) for which, as in the scalar case, we have connections between the corresponding G's and N's by means of the relation $G = N/n^2$. Hence all that has been said in the scalar theory has its vector echo, and the trail from the vector form of the structure axioms to the vector form of the transfer equation parallels the corresponding trail in the scalar theory. Again, to attain the correct form of the vector transfer equation, the rich structure of the classical carrier space must ultimately be adopted; and in addition, certain special consideration must be given the attenuating matrices $\underset{\sim}{\alpha}$ and $\underset{\sim}{\sigma}$. For example, the general theory does not limit the structure of $\underset{\sim}{\alpha}$, but the classical theory requires that $\underset{\sim}{\alpha}$ be simply the product of a real-valued function and the identity matrix $\underset{\sim}{I}$. By adopting the required regularity properties, the resulting equation is:

$$(n^2/v)\, D[\underset{\sim}{N}/n^2]/Dt = -\alpha \underset{\sim}{N} + \int \underset{\equiv}{\sigma}\, \underset{\sim}{N}\, d\Omega \,. \qquad (28)$$

RADIATIVE TRANSFER AND THE MUELLER ALGEBRA

The possibility of including the phenomenom of polarization within the abstract formalism gives rise to several novel connections of radiative transfer theory with its neighboring disciplines in mathematical

physics. We will limit the discussion to perhaps the most striking connection, namely that with the Mueller phenomenological algebra. An exposition of this algebra along with further references to the original work of Mueller may be found in some reports by Parke[12].

In essence, the Mueller algebra provides an analytical means of computing the relation between the input and output of a general optical instrument in terms of the inherent optical properties of the instrument. The input and output are represented by Stokes vectors $\underset{\sim}{L}(\omega)$ and $\underset{\sim}{L}'(\omega)$, respectively; and the instrument by a so-called Mueller matrix $\underset{\sim}{M}(\omega)$, where ω is the radian frequency of the radiation considered. The basic relation in the algebra is

$$\underset{\sim}{L}'(\omega) = \underset{\sim}{M}(\omega)\,\underset{\sim}{L}(\omega) . \tag{29}$$

The general similarity of (29) to the vector counterpart of (14) is clear:

$$\underset{\sim}{G}(\,,t') = \underset{\sim}{G}(\,,t)\,\underset{\sim}{P}_{t,t'} . \tag{30}$$

Thus, by virtue of this correspondence, we may say that a general carrier space $(X, \underset{\sim}{S}, \nu)$ (and hence any classical carrier space) over a time interval (t, t') may be considered as a generalized optical instrument

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[12] N. G. Parke, Matric Algebra of Electromagnetic Waves, Research Laboratory of Electronics Technical Report No. 70, Mass.Inst.of Tech., June 30, 1948; Statistical Optics: I. Radiation, Technical Report No. 95, January 31, 1949; Statistical Optics: II. Mueller Phenomenological Algebra, Technical Report No. 119, June 15, 1949.

whose input and output are, respectively, the general vector radiant density functions $G(\ ,t)$ and $G(\ ,t')$. Thus, the members $\underset{\sim}{P}_{t,t'}$ of the radiative process $\mathcal{P}_\phi$ may be considered abstract versions of the Mueller matrices.

We conclude by observing that this conceptual connection between the elements of the radiative process $\mathcal{P}_\phi$ and the Mueller matrices ties together two main lines of study originated at the turn of the century by Schuster, and at the same time supplies a new link between electromagnetic theory and radiative transfer theory. The first line of study has its roots in the paper[13] on the two-flow analysis of the light field. This study contained the germ of present day radiative transfer theory as subsequently developed by a host of workers. The other paper[14] dealt with the periodogram method of analysis of optical phenomena and contained the germ of present day generalized harmonic analysis as developed principally by Wiener[15]. Parke, in the references cited above, used the notions of generalized harmonic analysis to construct a "statistical bridge" between Maxwell's electromagnetic theory and Mueller's phenomenological algebra. The preceding discussion supplies

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[13] A. Schuster, "Radiation Through a Foggy Atmosphere," Astrophys.J., 21, 1-22 (1905).

[14] A. Schuster, "The Periodogram and its Optical Analogy, " Proc.Roy.Soc.77, 136-140 (1906).

[15] N. Wiener, "Generalized Harmonic Analysis," Acta Math.55,117-258 (1930)

the conceptual link between Mueller's algebra and radiative transfer theory. Thus the statistical bridge used in series with the present axiomatic formulation supplies an apparently novel path from Maxwell's electromagnetic theory to radiative transfer theory.

These connections are summarized in Figure 4.

Figure 4

CAPTIONS

Fig. 1. Illustrating the physical significance of the transition relation for radiative measures (see text).

Fig. 2. The dual roles of the G- and g-densities are illustrated in (a) and (b). The G-density assigns to the point x_1 at time t_2, the general radiant density induced by the radiant energy of the set E at time t_1. On the other hand, the g-density gives the general radiant density at x_1 at time t_1, which contributes to the radiant energy content of the set F at time t_2.

Fig. 3. The G-density assigns to x_2 at time t_2 the general radiant density induced by the general radiant density at x_1 at time t_1.

Fig. 4. Two investigations by Schuster, namely the periodogram analysis and the two-flow analysis of light initiated the lines of investigation shown. These lines, through the efforts of many workers not all shown here, culminate in present day general harmonic analysis and radiative transfer theory. The connection of radiative transfer theory with Maxwell's electromagnetic theory and related branches of mathematical physics, according to the present scheme, is indicated.

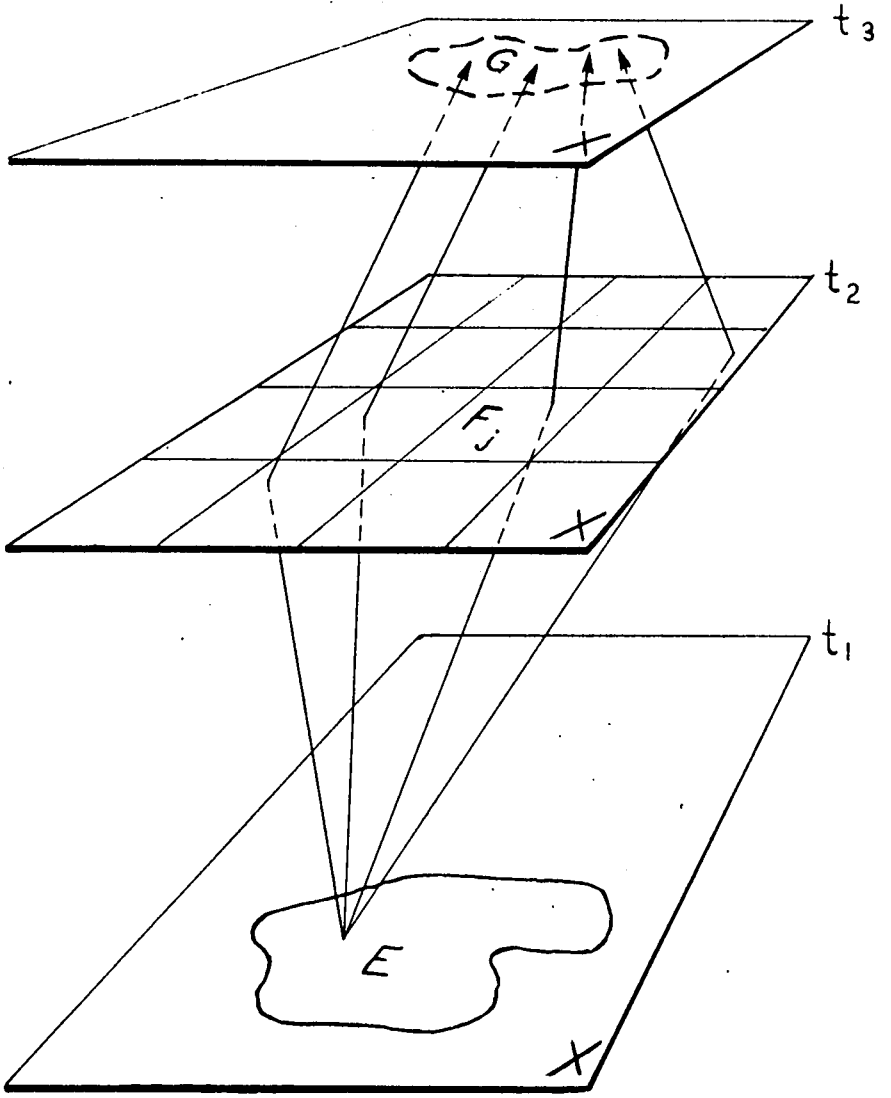
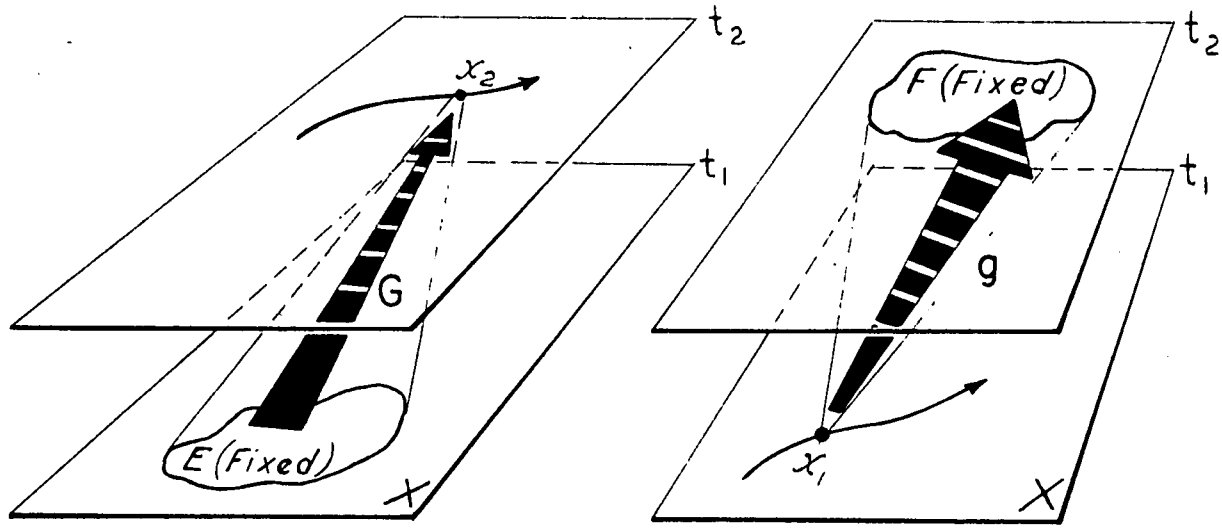
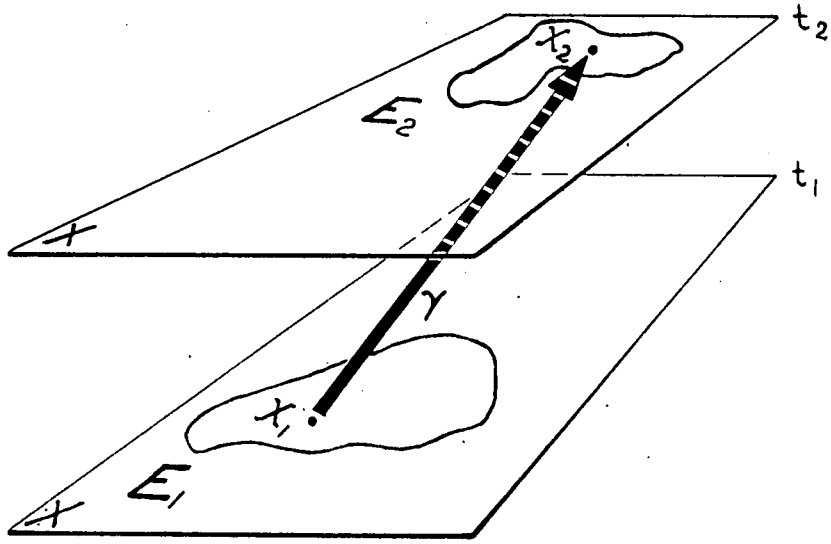


Figure 1 Weight Preserver



(a) (b)


t_2
x_2
E_2
t_1
γ
x_1
E_1

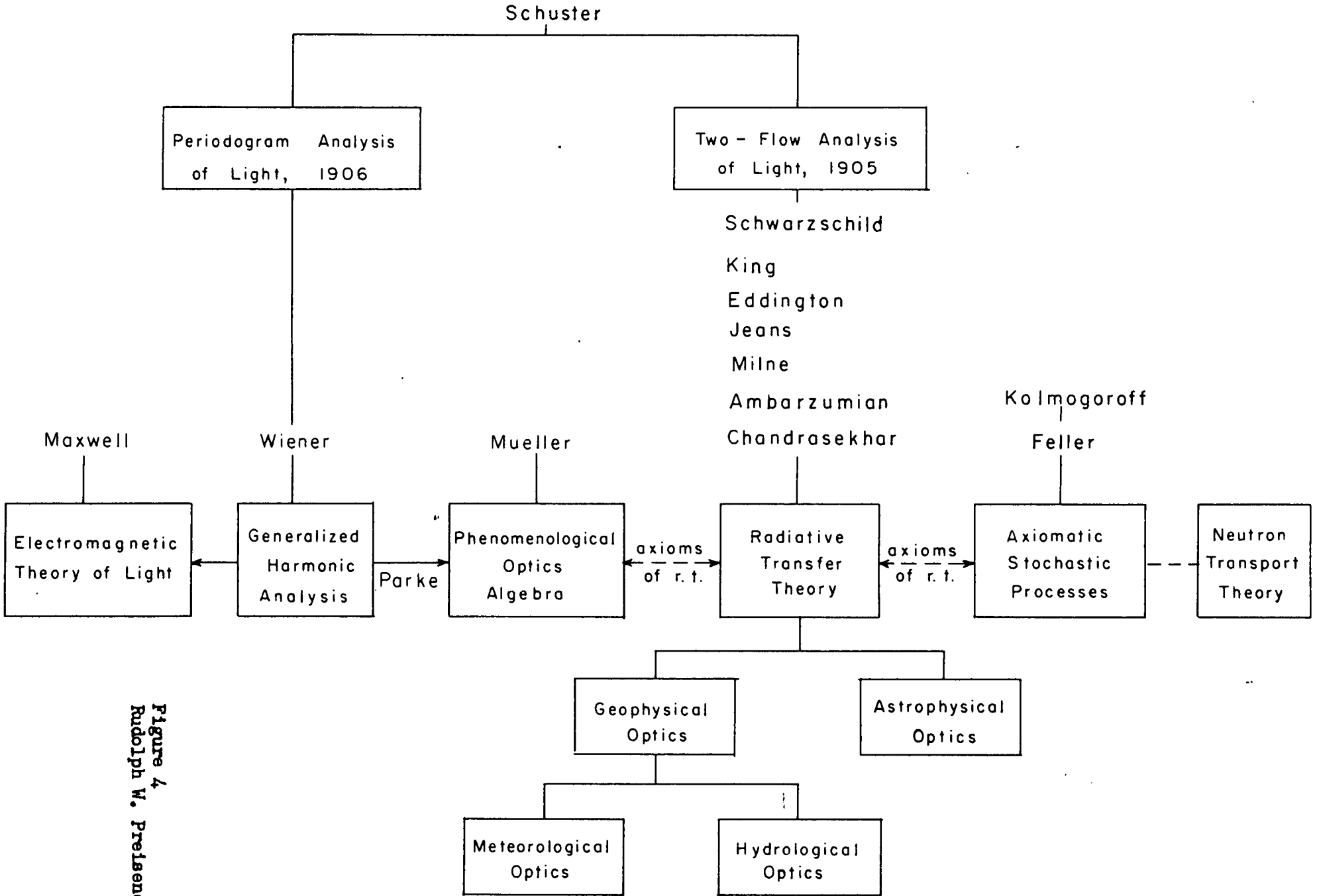


Figure 4
Rudolph W. Preisendorfer