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Author
Zhao, Roy

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Results on Unlikely Intersection Problems

by

Roy Zhao

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Committee in charge:

Professor Xinyi Yuan, Co-chair
Professor Martin Olsson, Co-chair
Professor Kenneth Ribet
Associate Professor Prasad Raghavendra

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Abstract

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Roy Zhao

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University of California, Berkeley

Professor Xinyi Yuan, Co-chair

Professor Martin Olsson, Co-chair

This dissertation is concerned with problems related to unlikely intersections and is divided into three parts. The first part consists of background about unlikely intersection problems, with particular emphasis on the André–Oort conjecture and existential closedness problems. These problems will be the central focus of the subsequent parts. In the second part, we give an explicit formula for heights of special points on quaternionic Shimura varieties using Faltings heights of CM abelian varieties. Special points are associated to CM-fields E and partial CM-types $\phi \subset \mathrm{Hom}(E, \mathbb{C})$. We show that this quaternionic height is compatible with the canonical height of a partial CM-type given by Pila, Shankar, and Tsimerman [58]. By doing so, we give another proof showing that the height of partial CM-types is bounded in terms of the discriminant of E. This height bound is a crucial ingredient in proving the André–Oort conjecture for general Shimura varieties. The third part is about the intersection of algebraic varieties with the graph of transcendental functions. Let $q\colon \Omega \to S$ be the uniformization map of a Shimura variety. We prove two results that give geometric conditions for when an algebraic variety $V \subset \Omega \times S$ contains a Zariski dense subset of points of the form $(x, q(x))$.

Contents

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Chapter 1

Introduction

This dissertation is divided into three parts. In the first part, we give the background on the André–Oort conjecture and the existential closedness problem. In the second part we prove an explicit formula for heights on quaternionic Shimura varieties and compare our results with those of [58]. In the third part, we study the existential closedness problem for Shimura varieties, and prove a generalization of a conjecture initially proposed by Zilber for two different families of varieties.

1.1 Unlikely Intersection Problems

In this first part, we give the historical background behind unlikely intersection problems. We start with the history of the André–Oort conjecture.

Conjecture 1.1.1 (André–Oort Conjecture)**.** *Let S be a Shimura variety and $V \subset S$ be a subvariety. If V has a Zariski dense subset of CM points, then V is a Hecke translate of a Shimura subvariety.*

Definitions and properties for Shimura varieties can be found in [43, Sec. 4] and [16, Sec. 2]. The converse statement follows from the fact that if (G, X) is the Shimura datum associated with V, then $G(\mathbb{Q}) \cap G(\mathbb{R})^+$ is dense in $G(\mathbb{R})^+$. Philosophically, the conjecture states that CM points on a Shimura variety are distributed in such a way so that the only way for a variety $V \subset S$ to have large intersection with them is if V is the $G(\mathbb{R})^+$-orbit of a point, for some group G. This conjecture for Shimura varieties can be seen as a generalization the Manin–Mumford conjecture, as well as Lang's conjecture. We give precise statements of these conjectures as well as the history of their proofs in Section 2.1.

In the setting of Shimura varieties, the André–Oort conjecture was originally proven by André (see [3]) when S is the product of two modular curves and later for arbitrary products of modular curves by Pila (see [52]). By using what is known as the Pila–Zannier method, Tsimerman gave an unconditional proof (see [67]) of the conjecture for $\mathcal{A}_g$, the coarse moduli space of principally polarized abelian varieties of dimension g. Moreover, following results of [8], Pila, Shankar, and Tsimerman recently announced an unconditional proof of the conjecture for all Shimura varieties in [58], also using the Pila–Zannier method. We give a detailed description of the different steps involved in the Pila–Zannier method and all the contributions by various authors in providing the necessary ingredients for each step in Section 2.2.

Next, we turn our attention to a different class of unlikely intersection problems in which a generic variety V is expected to have a Zariski dense intersection with a set of points, and it is unlikely that V does not intersect them. Zilber conjectured that the complex exponential function satisfies a "Strong Exponential Closedness" property. A special case of this statement, considering two variables, states that if $p(x, y)$ is a polynomial in two variables, then generic solutions of $p(z, \exp(z)) = 0$ exist, where $\exp(x) = e^x$ denotes the exponential function for $x \in \mathbb{C}$. More specifically:

Conjecture 1.1.2 ([77])**.** *Let $k \subset \mathbb{C}$ be a finitely generated field and $p(x, y) \in k[x, y]$ an irreducible polynomial in which both x and y appear. Then, there exists $z \in \mathbb{C}$ such that $p(z, \exp(z)) = 0$ and $\mathrm{td}_k\, k(z, \exp(z)) = 1$, where $\mathrm{td}_k\, k(z, \exp(z))$ denotes the transcendental degree of $k(z, \exp(z))$ over k.*

A priori, a point $z \in \mathbb{C}$ satisfying $p(z, \exp(z)) = 0$ has $\mathrm{td}_k(z, \exp(z)) \leq 1$. Thus, the genericity condition implies that we can find a point so that both z and $\exp(z)$ are not algebraic over k. When $k = \overline{\mathbb{Q}}$, this says that there are solutions that are not special values for the exponential function.

This conjecture says that the exponential function should only satisfy the algebraic relation that $e^x e^y = e^{x+y}$. This conjecture has also been extended to other transcendental functions satisfying single algebraic relations, such as the exponential of abelian varieties (satisfying $\exp_A(x + y) = \exp_A(x) + \exp_A(y)$, where $\exp_A\colon \mathbb{C}^N \to A$ is the exponential of an abelian variety) and the modular j-function (satisfying $\Phi_N(j(x), j(Nx)) = 0$, where Φ_N is the modular polynomial of level N). In Section 2.3, we provide a brief history behind the partial results obtained towards Zilber's conjecture in these different settings.

1.2 Heights on Quaternionic Shimura Varieties

In Chapter 3, we present and prove an explicit formula for the height of a point on a quaternionic Shimura variety and demonstrate how this formula can be combined with the main result of [58].

Let E be a CM field, and F be its totally real subfield, so that $[E : F] = 2$. Set $g := [F : \mathbb{Q}]$. Let $\phi \subset \mathrm{Hom}(E, \mathbb{C})$ be a partial CM-type, meaning that $\phi \cap \overline{\phi} = \varnothing$. Write $\Sigma \subset \mathrm{Hom}(F, \mathbb{R})$ for the restriction of ϕ to F. Suppose that B/F is a quaternion algebra with the following properties:

1. There exists an embedding $E \hookrightarrow B$;
2. The ramification set of B at infinity is Σ^c;

3. If B is ramified at a finite prime $\mathfrak{p}$ of F, then E is also ramified over $\mathfrak{p}$.

We define the algebraic group G over $\mathbb{Q}$ as

$$G \coloneqq \operatorname{Res}_{F/\mathbb{Q}} B^\times.$$

For each compact open subgroup $U \subset G(\mathbb{A}_f)$, we obtain a (quaternionic) Shimura variety X_U defined over a number field E_X with the complex uniformization given by:

$$X_U(\mathbb{C}) = G(\mathbb{Q})\backslash(\mathcal{H}^\pm)^\Sigma \times G(\mathbb{A}_f)/U,$$

where $\mathcal{H}^\pm$ is the upper and lower complex half-planes. This is a Shimura variety of abelian type, and by utilizing ideas from [12] and [60], we can construct a regular integral model $\mathcal{X}_U$ for X_U over $\operatorname{Spec} \mathcal{O}_{E_X}$.

Let $\widehat{\mathcal{L}_U}$ be the arithmetic Hodge bundle of $\mathcal{X}_U$, which consists of a line bundle $\mathcal{L}_U$ on $\mathcal{X}_U$ and a Hermitian metric given by:

$$\left\| \bigwedge_{\sigma\in\Sigma} dz_\sigma \right\| \coloneqq \prod_{\sigma\in\Sigma} 2\mathrm{Im}(z_\sigma),$$

where the z_σ are given by the complex uniformization of X_U. When U is sufficiently small, the Hodge bundle is simply the canonical bundle $\mathcal{L}_U = \omega_{\mathcal{X}_U/\mathcal{O}_{E_X}}$. The precise definition of $\widehat{\mathcal{L}_U}$ is given in Section 3.6.

Let $P_U \subset X_U(\overline{\mathbb{Q}})$ be a special point arising from the embedding $E \hookrightarrow B$, and let $\overline{P_U}$ be the closure of this point in $\mathcal{X}_U$, which we will also denote by P_U by abuse of notation. The height of this point relative to $\widehat{\mathcal{L}_U}$ is the Arakelov height:

$$h_{\widehat{\mathcal{L}_U}}(P_U) \coloneqq \frac{1}{[\mathbb{Q}(P_U):\mathbb{Q}]}\widehat{\deg}(\widehat{\mathcal{L}_U}|_{P_U}).$$

Finally, let Φ be a full CM-type, and let $h(\Phi)$ be the Faltings height of an abelian variety with complex multiplication by $(\mathcal{O}_E, \Phi)$. Let $d_\phi, d_{\overline{\phi}}$ and $d_\Sigma \coloneqq d_{\phi\sqcup\overline{\phi}}$ be certain absolute discriminants of $\phi, \overline{\phi}$, and $\phi \sqcup \overline{\phi}$. These are defined in detail in Section 3.2.

There is a reflex norm $N_{F/E_X} \colon F \to E_X$ defined by $N_{F/E_X}(x) = \prod_{\sigma\in\Sigma}\sigma(x)$. Let $d_{E/F,\Sigma} \in \mathbb{Z}$ be the positive generator of $N_{E_X/\mathbb{Q}}(N_{F/E_X}(\mathfrak{d}_{E/F}))$. Let d_B be the positive generator of norm from F to $\mathbb{Q}$ of the product of all the finite places of $\mathcal{O}_F$ over which B ramifies.

With these definitions in place, we can now state our main theorem.

Theorem 1.2.1. *Suppose that $U = \prod_v U_v$ is a maximal compact subgroup of $G(\mathbb{A}_f)$. Then*

$$\begin{aligned}\frac{1}{2}h_{\widehat{\mathcal{L}_U}}(P_U) =& \frac{1}{2^{|\Sigma^c|}}\sum_{\Phi\supset\phi} h(\Phi) - \frac{|\Sigma^c|}{g2^g}\sum_{\Phi} h(\Phi) \\ &+ \frac{1}{8}\log d_{E/F,\Sigma} d_\Sigma^{-1} + \frac{1}{4}\log d_\phi d_{\overline{\phi}} + \frac{1}{4g}\log d_B d_\Sigma + \frac{|\Sigma|}{4g}\log d_F.\end{aligned}$$

The first summation is over all full CM-types which contain ϕ, and the second summation is over all full CM-types of E.

Additionally, if $|\phi| = 1$, then $E_X = F$, and we have that $d_\Sigma = d_{E/F} = d_{E/F,\Sigma}$ and $d_\phi = d_{\overline{\phi}} = 1$. As a result, the expression for $\frac{1}{2}h_{\widehat{\mathcal{L}_U}}(P_U)$ simplifies, and we recover [75, Thm. 1.6], where the factor of g is due to different normalizing factors of $h_{\widehat{\mathcal{L}_U}}(P_U)$.

In the Pila–Zannier method, an essential step involves bounding the height of a CM point. For general Shimura varieties, a CM point P is associated with a partial CM-type ϕ of E. In their work [58], a canonical height $h(\phi)$ is introduced for such ϕ, and they show that this height $h(\phi)$ is equal to the height $h(P)$ of the associated CM point P, up to a bounded constant. Furthermore, [58] provides a bound for $h(\phi)$ in terms of the discriminant d_E. The precise definition of $h(\phi)$ can be found in Section 3.8. Our second main result establishes the compatibility between $h_{\widehat{\mathcal{L}_U}}(P_U)$ and $h(\phi)$.

Theorem 1.2.2.

$$h(\phi) = \frac{1}{2}h_{\widehat{\mathcal{L}_U}}(P_U) + O(\log d_E).$$

With the combination of Theorem 1.2.1 and Theorem 1.2.2, we obtain the following corollary. This is an ingredient used in the Pila–Zannier method, and serves as one of the results of [58].

Corollary 1.2.3. *For all $\varepsilon > 0$, there exists a positive constant c depending only on $[E : \mathbb{Q}]$ such that*

$$h(\phi) \leq c \cdot d_E^\varepsilon$$

for all partial CM-types of E.

Proof. By [67, Cor. 3.3], the Faltings heights $h(\Phi)$ of full CM-types are bounded subpolynomially by d_E. Each of the discriminants $d_{E/F,\Sigma}, d_\Sigma, d_\phi, d_F$ are smaller than d_E, and $d_B \leq d_E$ since we specified that the ramification set of E contains the ramification set of B. Thus, each of the logarithm terms are bounded by $\log d_E$, which is also subpolynomial in d_E. □

We first give a high-level overview of the strategy of the proof of Theorem 1.2.1 in Section 3.1. Then, we recall from [75] the decomposition of a Faltings height in Section 3.2. We then describe three Shimura varieties that can be constructed from a quaternion algebra following [14] by describing their generic fiber in Section 3.3 and integral models in Section 3.4. We then describe some line bundles on these Shimura varieties in terms of Lie algebras of certain p-divisible groups described in Section 3.5. In Section 3.6, we define the Hodge bundle and relate it to the p-divisible groups defined previously. Finally, we prove our theorem for the height of partial CM-types in Section 3.7, and compare our height with the height introduced in [58] in Section 3.8.

1.3 Existential Closedness for Shimura Varieties

In Chapter 4, our aim is to present and prove partial results of Conjecture 1.1.2 in the general setting of the quotient map $q : \Omega \to S := \Gamma\backslash\Omega$ associated with a Shimura variety.

Let (G, Ω) be a connected Shimura datum. Let K be a compact open subgroup of $G(\mathbb{A}_f)$ and write $\Gamma := G(\mathbb{Q}) \cap K$. Then, the quotient

$$S := \Gamma \backslash \Omega$$

has the structure of an algebraic variety, and the quotient map $q : \Omega \to S$ is analytic and transcendental. Definitions and properties of Shimura varieties can be found in [43, Sec. 4].

The $G(\mathbb{R})$-conjugacy class Ω is not an algebraic set, but it is a Hermitian symmetric domain and can be realized as an open immersion into affine space through the Harish-Chandra embedding. Hence, we can represent Ω as a domain in $\mathbb{C}^N$, where $N = \dim \Omega = \dim S$. We will refer to this realization as $\Omega \subset \mathbb{C}^N$.

Let $E_q = \{(x, q(x)) \in \Omega \times S\}$ denote the graph of q. The existential closedness problem for q seeks to determine a minimal set of geometric conditions that an algebraic variety $V \subset \mathbb{C}^N \times S$ must satisfy to have a Zariski dense set of points in the graph of q. In this dissertation we explore this question for two families of varieties. We remark that, by the Ax–Schanuel Theorem [44], if $\dim V = N$, then we expect that the dimension of the irreducible components of $V \cap E_q$ (if there are any) is zero.

Our first main result of this chapter is as follows:

Theorem 1.3.1. *Let $V \subset \mathbb{C}^N \times S$ be an irreducible algebraic variety and let $\pi : \mathbb{C}^N \times S \to \mathbb{C}^N$ be the projection onto the first factor. If $\pi(V)$ is Zariski dense in $\mathbb{C}^N$, then $\pi(V \cap E_q)$ is Zariski dense in $\mathbb{C}^N$, and $V \cap E_q$ is Zariski dense in V.*

For our second main result of this chapter, instead of relying on a dominant projection to the first set of coordinates, we investigate varieties of the form $L \times W$, where L is a "subvariety" of Ω, and W is a subvariety of S. To state our second main result, we need to introduce some definitions. We follow the conventions of [68] for defining weakly special subvarieties and [69] for totally geodesic subvarieties (we will review some of these aspects in §4.4).

Definition 1.3.2. For every decomposition of $G^{\mathrm{ad}} = G_1 \times G_2$ into a product of two normal $\mathbb{Q}$-subgroups (possibly trivial), we obtain a splitting $\Omega = \Omega_1 \times \Omega_2$ and $S_1 \times S_2$ along with projections $p_i : \Omega \times S \to \Omega_i \times S_i$. A variety $V \subset \mathbb{C}^N \times S$ is *broad* if $\dim p_i(V) \geq \dim S_i$ for every such splitting.

A subvariety $W \subset S$ is said to be *Hodge-generic* if there does not exist a proper weakly special subvariety $S' \subset S$ such that $W \subset S'$. A subvariety $Z \subset \mathbb{C}^N$ is *Hodge-generic* if $\overline{q(Z \cap \Omega)}^{\mathrm{Zar}} = S$ (cf [69, Theorem 1.2]).

Furthermore, we say that $V \subseteq \mathbb{C}^N \times S$ is *Hodge-generic* if the projections of V to $\mathbb{C}^N$ and S are both Hodge-generic.

We can now state our second main result.

Theorem 1.3.3. *Let $L \subset \Omega$ be a totally geodesic subvariety and let $W \subset S$ be an algebraic variety such that $L \times W$ is a Hodge-generic and broad variety. Then the intersection $W \cap q(L)$ is Euclidean dense inside W.*

Given these results, we propose a generalization of Conjecture 1.1.2 to the setting of Shimura varieties.

Conjecture 1.3.4. *Let $V \subset \mathbb{C}^N \times S$ be an irreducible algebraic variety that is both broad and Hodge-generic. Then the intersection $V \cap E_q$ is Zariski dense in V.*

We note that the condition in Theorem 1.3.1, which requires the projection $\pi(V)$ to be Zariski dense in $\mathbb{C}^N$, is a stronger condition compared to broadness. This conjecture states that other than for dimension reasons (broadness) and Shimura variety group-theoretic reasons (Hodge-generic), the uniformization map q should satisfy no other algebraic relations and hence intersect any algebraic variety V.

In Section 4.1, we cover some basic facts about Hermitian symmetric domains and their various embeddings, as well as the metrics defined on them. Then, in Section 4.2, we prove results about the structure of the boundary of Ω, focusing on the Shilov boundary. We prove that the Γ-orbit of any point in Ω contains a Zariski dense subset of the Shilov boundary in its closure. With this result, we are ready to present the proof of Theorem 1.3.1 in Section 4.3.

In Section 4.4, we define weakly special and totally geodesic subvarieties of Ω, along with the notions of broad and Hodge-generic. We also present some consequences from Ratner theory. This will enable us to prove Theorem 1.3.3 in Section 4.4.

Chapter 2

History of Unlikely Intersection Problems

2.1 Origins of the André–Oort Conjecture

We start by providing an abstract formulation of the André–Oort conjecture. Subsequently, we explore how both the Manin–Mumford conjecture and Lang's conjecture can be regarded as specific instances of the André–Oort conjecture.

Let $S/\mathbb{C}$ be an irreducible complex algebraic variety. Let Σ be a set of irreducible algebraic subvarieties of S that satisfy the following three properties:

1. $S \in \Sigma$;
2. let $\Sigma_0 \subset \Sigma$ be the subset of points. Then for any $U \in \Sigma$, we have that $\overline{S' \cap \Sigma_0}^{\mathrm{Zar}} = S'$, the points are Zariski dense in S';
3. If $U, U' \in \Sigma$, then all the irreducible components of $U \cap U'$ also are in Σ.

The subvarieties Σ are called special subvarieties and the points are called special points. In this setting, a formulation of the André–Oort conjecture is the converse to the second condition.

Conjecture 2.1.1 (André–Oort Conjecture, Form 1)**.** *Let $V \subset S$ be an irreducible subvariety. If $\overline{V \cap \Sigma_0}^{\mathrm{Zar}} = V$, then $V \in \Sigma$.*

Oftentimes, the conjecture is stated in an equivalent form. If $V \subset S$ is a subvariety, we say that $U \subset V$ is a maximally special subvariety of V if $U \in \Sigma$ and whenever $U' \subset V$ is another special subvariety containing U, then $U = U'$.

Conjecture 2.1.2 (André–Oort Conjecture, Form 2)**.** *Let $V \subset S$ be an irreducible subvariety. There are finitely many maximally special subvarieties of V.*

We give a brief proof of why these two forms are equivalent.

Proposition 2.1.3. *Form 1 and 2 of the André–Oort conjecture are equivalent.*

Proof. ($1 \implies 2$) Let $Z = \overline{V \cap \Sigma_0}^{\mathrm{Zar}}$ be the Zariski closure of all the special points inside V. This is an algebraic variety and we can write it as $Z = \bigcup_{i=1}^n U_i$, where each U_i is an irreducible component of Z. Then $\overline{U_i \cap \Sigma_0}^{\mathrm{Zar}} = U_i$ for each U_i and so by Form 1, we have that each $U_i \in \Sigma$ is special. We claim that these U_i are the maximally special subvarieties of V. If $U' \subset V$ is a special subvariety, then $U' = \overline{U' \cap \Sigma_0}^{\mathrm{Zar}} \subset \overline{V \cap \Sigma_0}^{\mathrm{Zar}} = \bigcup_{i=1}^n U_i$. And since U' is irreducible, we must have $U' \subset U_i$ for some i, proving that the U_i are maximal.

($2 \implies 1$) Let $V \subset S$ be an irreducible subvariety such that $\overline{V \cap \Sigma_0}^{\mathrm{Zar}} = V$. By Form 2, there exists a set $\{U_i\}_{1 \leq i \leq n}$ of maximally special subvarieties of V. Each special point of V is special and hence must lie in one of these maximally special subvarieties U_i. Therefore, we get that $\overline{V \cap \Sigma_0}^{\mathrm{Zar}} = \bigcup_{i=1}^n U_i = V$. Each of the U_i is an irreducible subvariety and hence we must have one of the $U_i = V$, meaning V is special. □

For a concrete example, the Manin–Mumford conjecture is a version of the André–Oort conjecture when $S = A$ is an abelian variety. In this case, the special subvarieties Σ are translations of abelian subvarieties by torsion points, also called torsion cosets.

Theorem 2.1.4 (Manin–Mumford Conjecture)**.** *Let $A/\mathbb{C}$ be an abelian variety and let $V \subset A$ be an irreducible algebraic variety. Then V contains finitely many maximal torsion cosets.*

This was first proven by Raynaud (see [61]) in 1983 using p-adic methods. Later proofs were given by Hindry (see [25]) in 1988, Szpiro, Ullmo and Zhang (see [65]) in 1997, Hrushowski (see [26]) in 2001, Pink and Roessler (see [59]) in 2002, and Pila and Zannier (see [57]) in 2008.

Another version of the André–Oort conjecture is Lang's conjecture, which is formulated in the setting of $S = (\mathbb{C}^\times)^n$ as an algebraic tori. In this case, the special subvarieties Σ are translations of algebraic subtori by roots of unity, which we also refer to as torsion cosets.

Theorem 2.1.5 (Lang Conjecture)**.** *Let $V \subset (\mathbb{C}^\times)^n$ be an irreducible algebraic variety. Then V contains finitely many maximal torsion cosets.*

The special case of $n = 2$ was proven by Liardet (see [34]) in 1974. Then in 1983, the full conjecture was proven by Laurent (see [33]).

For this dissertation, our interest lies in the setting of Shimura varieties. In this context, the special subvarieties of a Shimura variety are Hecke translates of Shimura subvarieties, and the special points correspond to CM points. Definitions and properties for Shimura varieties can be found in [43, Sec. 4], and for special subvarieties in [16, Sec. 2]. In this setting, the André–Oort conjecture is stated as follows.

Theorem 2.1.6 (André–Oort for Shimura Varieties)**.** *Let S be a Shimura variety and let $V \subset S$ be an algebraic subvariety. If the CM points of S are dense in V, then V is a special subvariety of S.*

The conjecture was initially posed by André in 1989 (see [2]) in the following form: any curve of a Shimura variety containing an infinite number of CM points must be special. Independently, in 1997, Oort (see [47]) formulated the conjecture for $\mathcal{A}_g$, the moduli space of principally polarized abelian varieties of dimension g.

In the subsequent year, André (see [3]) proved the conjecture unconditionally for the case when $V \subset S = \mathcal{A}_1 \times \mathcal{A}_1$ is a curve within a product of two modular curves. Independently, Edixhoven (see [19]), assuming the Generalized Riemann Hypothesis (GRH), also gave a proof of the same result. In 2001, Yafaev (see [73]) successfully extended Edixhoven's strategy, still assuming GRH, to cover the situation where S is a product of two Shimura curves. Then, in 2005, Edixhoven (see [18]) further generalized his results, still under the assumption of GRH, to the case where $S = \mathcal{A}_1^n$ is an arbitrary product of modular curves. Finally, in 2006, Yafaev (see [72]) provided a proof, conditional on GRH, of André's original conjecture, addressing all cases where $V \subset S$ is a curve and S is an arbitrary Shimura variety.

The strategy employed by Edixhoven and Yafaev focuses on the Galois theory of special points and the geometry of Hecke translations. By incorporating equidistribution results from Clozel and Ullmo (see [13]), Klingler, Ullmo, and Yafaev (see [30, 70]) successfully provided a proof of the complete André–Oort conjecture for Shimura varieties in 2014, though it still relies on the assumption of GRH. Below, we provide a brief description of this this strategy, following [70].

Let (G, X) be a Shimura datum, and let S be the associated Shimura variety. We define a special subvariety U to be strongly special if, under any nontrivial splitting of the group $G^{\mathrm{ad}} = G_1 \times G_2$ and associated maps of Shimura varieties $S \to S_1 \times S_2$, the image of U in $S_1 \times S_2$ is not of the form $\{P\} \times S_2$, where P is a CM point of S_1. The equidistribution results of Clozel and Ullmo encompass the following two results:

Theorem 2.1.7 ([13, Thm. 1.1, 1.2]). *Consider a sequence $\{U_n\}_{n\in\mathbb{N}}$ of strongly special subvarieties of S, and let μ_n be the canonical probability measure of U_n. After taking a subsequence of the U_i, there exists is a special subvariety $U \subset S$ such that $U_i \subset U$, and μ_i weakly converges to μ_U, which is the canonical probability measure of U.*

Furthermore, if $V \subset S$ is any irreducible algebraic subvariety containing a Zariski dense subset of strongly special subvarieties, then V is special.

Let E_S be the reflex field of the Shimura variety S. For a special subvariety $U \subset S$, we define $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E_S) \cdot U)$ as the degree of the subvariety $\mathrm{Gal}(\overline{\mathbb{Q}}/E_S) \cdot U$ with respect to the Hodge bundle on S. Now, suppose that V has an infinite subset Σ_V of maximal special subvarieties. We may assume that they are all of the same dimension. If $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E_S) \cdot U_n)$ is bounded for all $U_n \in \Sigma_V$, then Ullmo and Yafaev extend the equidistribution result presented in [13].

Theorem 2.1.8 ([70, Thm. 3.8]). *If $\{U_i\}_{i\in\mathbb{N}}$ is a sequence of special varieties of S such that $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E_S) \cdot U_i)$ is bounded, then there exists a subsequence and a special subvariety $U \subset S$ such that $U_i \subset U$ for all i, and the canonical probability measures of the U_i weakly converge to the canonical probability measure of U.*

So, we can reduce the problem to the case when $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E_S) \cdot U_n) \to \infty$, which is solved using the methods of Edixhoven and Yafaev. Utilizing GRH, they show that we can find Hecke operators T_n with $\deg(T_n)$ bounded polynomially from above by $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E_S) \cdot U_n)$ such that $\mathrm{Gal}(\overline{\mathbb{Q}}/E_S) \cdot U_n \subset V \cap T_n V$. If $V \subset T_n V$ for all n, then results of Deligne and André show that $V = T_n V$, and V is special. Otherwise, we let V' be an irreducible component of $V \cap T_n V$ that contains U_n, thereby reducing the dimension of V by 1. Repeating this process eventually shows that V is itself special.

In terms of providing an unconditional proof, Pila and Zannier (see [57]) presented a new approach to the Manin–Mumford conjecture using the Pila–Wilkie point counting theorem in [51]. This theorem, derived from o-minimality, establishes bounds for the number of points with bounded degree and height on sets that are definable in an o-minimal structure. In 2009, Pila (see [54]) further generalized this strategy, offering another proof of André's original 1998 result for a curve lying inside a product of two modular curves. Subsequently, in 2011, Pila (see [52]) provided an unconditional proof of Edixhoven's 2005 result for a curve lying inside a product of an arbitrary product of modular curves.

Collborating with Tsimerman, in 2013, Pila (see [56]) gave an unconditional proof of the full André–Oort conjecture for $\mathcal{A}_2$, the moduli space of abelian surfaces. Following the same o-minimality point counting strategy, Tsimerman (see [67]) gave an unconditional proof of Oort's 1997 conjecture, the case when $S = \mathcal{A}_g$, in 2018. Recently, in 2021, in collaboration with Shankar, Pila and Tsimerman (see [58]) announced an unconditional proof for the André–Oort conjecture, encompassing all Shimura varieties. In this dissertation, we focus on exploring the Pila–Zannier strategy, which will be discussed in the next section.

Lastly, we mention that André formulated a version of the André–Oort conjecture for mixed Shimura varieties, such as the universal abelian variety $\mathbb{A}_g$ lying above $\mathcal{A}_g$. In 2016, Gao (see [22]) successfully reduced the mixed Shimura variety version of the conjecture to the case of pure Shimura varieties. Furthermore, it is essential to note that the André–Oort conjecture has been further generalized into the Zilber–Pink conjecture, which governs the behavior of the intersections of $U \in \Sigma$ with V, and has become an area of active research. A comprehensive summary of the conjecture and the partial results towards it can be found in [53].

2.2 The Pila–Zannier Method

Many of the recent unconditional results of the André–Oort conjecture have been proven using the Pila–Zannier method, which was initially employed by both authors to give a new proof of the Manin–Mumford conjecture (see [57]). We can reduce the André–Oort conjecture to the case where S is irreducible and takes the form of $\Gamma \backslash \Omega$, the quotient of a Hermitian symmetric domain by an arithmetic group. The Pila–Zannier method can be broken down into three main components.

1. The first component is the Pila–Wilkie point counting theorem from [51].

Theorem 2.2.1 ([51]). *Let $X \subset \mathbb{R}^n$ be a set definable in an o-minimal structure, and let X^{alg} be the union of all positive-dimensional algebraic sets in X. Then, the number of points of $X \setminus X^{\mathrm{alg}}$ with height at most N grows subpolynomially in N.*

This theorem provides an upper bound that is subpolynomial in height for the rational points of the transcendental component of definable sets. The o-minimal structure used for the Pila–Zannier method is $\mathbb{R}_{\mathrm{an,exp}}$, which is the model of $\mathbb{R}$ with restricted analytic functions and the exponential function (defined everywhere).

In the context of Lang's conjecture or the Manin–Mumford conjecture, the fundamental domain of the exponential function is tautologically definable. In the setting of Shimura varieties, let $q\colon \Omega \to S$ be the uniformization map, and $\mathcal{F}$ be a fundamental domain for the action of Γ on Ω. Then, $\mathcal{F}$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$.

In the case where $S = \mathbb{A}_1^n$ is a product of modular curves, the uniformization map q corresponds to the modular j-function. It is well-known that $j(z) = J(\exp(2\pi i z))$, where J is a meromorphic function defined on the open unit disc, showing that $\mathcal{F}$ is definable. The case of $S = \mathcal{A}_g$ was proven by Peterzil and Starchenko (see [50]) in 2013, while Klingler, Ullmo, and Yafaev (see [29]) extended the result to arbitrary Shimura varieties, also in 2013.

2. The second component is to bound the size of Galois orbits of special points from below by their height. If $P \in S$ is a special point, let $x \in \mathcal{F}$ be the preimage of P under the uniformization map restricted to the fundamental domain $\mathcal{F}$. The goal is to find positive constants c and ε such that for any P, x:

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| \geq c \cdot H(x)^{\varepsilon},$$

where $H(x)$ is the multiplicative height of x.

In the context of Lang's conjecture or the Manin–Mumford conjecture, special points correspond to torsion points, and their height is equal to their order. The bound can be interpreted as asserting that the number of points with order N is bounded below by N. When $S = (\mathbb{C}^{\times})^n$, and P is a point of order N, the number of points with order N is at least $\varphi(N)$, where φ is the Euler totient function. Elementary number theory then gives us the bound $\varphi(N) \geq c \cdot N^{\varepsilon}$

In the case of $S = A$, an abelian variety, a theorem by Masser provides the desired bound:

Theorem 2.2.2 ([39]). *Let A be an abelian variety defined over a number field. If $P \in A$ is a torsion point of order N, then there exist positive constants c and ε such that*

$$[\mathbb{Q}(P) : \mathbb{Q}] \geq c \cdot N^{\varepsilon}.$$

For Shimura varieties, when $S = \mathbb{A}_1^n$ is a product of modular curves, special points correspond to elliptic curves with complex multiplication. If $P \in S$ represents an elliptic curve with complex multiplication by the imaginary quadratic field E, the theory of complex multiplication shows that $[\mathbb{Q}(P) : \mathbb{Q}] = h(E)$, where $h(E)$ is the class number of E. The preimage in $\mathbb{H}$ of P is $\tau \in \mathcal{O}_E$, and the height of τ is bounded by the discriminant of E. Siegel's lower bound for class numbers of imaginary quadratic fields proves the bound.

Theorem 2.2.3 ([64]). *Let E be an imaginary quadratic field of discriminant d_E, and let $h(E)$ be the class number of E. Then, for any $\varepsilon > 0$, there exists a positive constant c_ε such that*

$$h(E) \geq c_\varepsilon \cdot d_E^{1/2-\varepsilon}.$$

For other Shimura varieties, proving the bound on the size of Galois orbits is done by splitting up the bound into two steps. The first step is to show that $h(x)$ is bounded above in terms of certain discriminants. Concretely, when $S = \mathcal{A}_g$, points $P \in S$ correspond to abelian varieties A. The following bound was proven by [56] in 2013:

Theorem 2.2.4 ([56, Thm. 3.1]). *Let A be an abelian variety with complex multiplication, and let $R \coloneqq Z(\mathrm{End}(A))$ be the center of the endomorphism algebra of A. Then, there exist positive constants c and ε such that*

$$H(x) \leq c \cdot \mathrm{disc}(R)^\varepsilon.$$

In 2012, Tsimerman (see [66]) provided the other half of the height bound for $\mathcal{A}_g$ when $g \leq 6$ and for all g assuming GRH.

Theorem 2.2.5. *Let A be an abelian variety of dimension $g \leq 6$, and let $x \in \mathcal{H}_n$ be a point in the Siegel upper half-space that parametrizes A. Then, there exist positive constants c' and ε' such that*

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| \geq c' \cdot \mathrm{disc}(R)^{\varepsilon'}.$$

Furthermore, if $g > 6$, then the same result holds under the assumption of GRH.

To provide an unconditional proof of the André–Oort conjecture for $\mathcal{A}_g$ when $g > 6$, a height function on Shimura varieties was utilized. For $S = \mathcal{A}_g$, the height $h(P)$ is taken to be the Faltings height of the abelian variety A, precisely defined in Section 3.2. Special points in $\mathcal{A}_g$ correspond to abelian varieties with complex multiplication of type (E, Φ). In this pair, $E/\mathbb{Q}$ is a totally imaginary quadratic extension of a totally real number field, and $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$ is a subset of places such that $\Phi \sqcup (\Phi \circ c) = \mathrm{Hom}(E, \mathbb{C})$, where $c \in \mathrm{Gal}(E/F)$ is the nontrivial automorphism. Colmez (see [14]) proved that the Faltings height $h(P) = h(\Phi)$ depends only on (E, Φ), the CM-type of the abelian variety, and not on the abelian variety itself.

A theorem of Bost (see [10]), combined with the averaged Colmez conjecture, proven independently by Andreatta, Goren, Howard, and Madapusi-Pera (see [4]), and by Yuan and Zhang (see [75]), provides a bound on the height $h(P)$. The theorem of Bost and the averaged Colmez conjecture are as follows:

Theorem 2.2.6 ([10, Thm. 1.1])**.** *There exists a constant c_g, depending only on g, such that if A is an abelian variety of dimension g, then $h(A) \geq c_g$.*

Theorem 2.2.7 ([4, Thm A], [75, Thm 1.1])**.** *Suppose E/F is an CM extension and let $\chi\colon \mathbb{A}_F^\times \to \{\pm 1\}$ the character corresponding to this extension, and $L(s,\chi)$ the corresponding Artin L-function. Let d_F be the absolute discriminant of F and $d_{E/F}$ the norm of the relative discriminent of E/F. Then*

$$\frac{1}{2^g}\sum_{\Phi} h(\Phi) = -\frac{1}{2}\frac{L'(0,\chi)}{L(0,\chi)} - \frac{1}{4}\log(d_{E/F}d_F),$$

where the sum on the left runs through the set of all CM-types of E.

Corollary 2.2.8. *For any positive choice of ε_1, there exists a constant c_1 so that*

$$h(\Phi) \leq c_1 \cdot d_E^{\varepsilon_1}$$

for all CM types with $|\Phi| = g$.

Proof. Using Theorem 2.2.6, we can express $h(\Phi)$ as:

$$h(\Phi) \leq (-2^g - 1)c_g + \sum_{\Phi'} h(\Phi'),$$

where the sum is taken over all CM-types Φ' of E. Thanks to the averaged Colmez conjecture, this sum is bounded in terms of the logarithmic derivative of $L(0,\chi)$. Moreover, the logarithmic derivative can be further bounded subpolynomially in terms of d_E by applying the Brauer–Siegel theorem. The completes the bound. □

In 2018, Tsimerman (see [67]) used this bound on $h(P)$ to give an unconditional proof that

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| \geq c \cdot \mathrm{disc}(R)^{\varepsilon}.$$

He achieved this by employing the following theorem of Masser and Wüstholz.

Theorem 2.2.9 ([40])**.** *Let A and B be abelian varieties of dimension g defined over a number field K. Suppose that they are isogenous over $\mathbb{C}$. Then, there exist positive constants c' and ε' such that there exists an isogeny between them over $\mathbb{C}$ of degree N, with*

$$N \leq c' \cdot \max(h_{\mathrm{Falt}}(A), [K:\mathbb{Q}])^{\varepsilon'}.$$

It follows from [67, Thm. 5.1] that proving the bound on the size of Galois orbits is equivalent to showing the following bound.

Theorem 2.2.10 ([67, Thm. 1.1])**.** *There exist positive constants c and ε such that*

$$[\mathbb{Q}(A) : \mathbb{Q}] \geq c \cdot d_E^{\varepsilon}.$$

Proof. Let $\Sigma(E, \Phi)$ be the set of abelian varieties with complex multiplication of type $(\mathcal{O}_E, \Phi)$. By the theory of complex multiplication (see [63]), we have that $|\Sigma(E, \Phi)| = h(E)$, the class number of E.

The Bruaer–Siegel theorem tells us that

$$|\Sigma(E, \Phi)| \geq c' \cdot d_E^{1/4-\varepsilon}.$$

Isogenies between two abelian varieties A and $B \in \Sigma(E, \Phi)$ correspond to ideals of $\mathcal{O}_E$, and the degree of the isogeny is equal to the norm of that ideal. As the number of ideals of norm n grows logarithmically in n, there exist two abelian varieties A and$B \in \Sigma(E, \Phi)$ such that any isogeny between them has a degree of at least $c'' d_E^{1/4-\varepsilon}$. Applying the theorem of Masser and Wüstholz yields

$$d_E \leq c \cdot \max(h(A), [\mathbb{Q}(A) : \mathbb{Q}])^{\varepsilon}.$$

Furthermore, the consequence of the averaged Colmez conjecture shows that $h(A)$ is bounded from above subpolynomially in terms of d_E. Hence, we conclude that $[\mathbb{Q}(A) : \mathbb{Q}]$ must be bounded below by d_E. □

When S is an arbitrary Shimura variety, special points $P \in S$ correspond to Shimura subdatum (T, P), where T is an algebraic torus defined over $\mathbb{Q}$. Let L be the splitting field of T, and d_L be the absolute discriminant of L. An analogous result bounding the height $H(x)$ for all Shimura varieties is given by [8].

Theorem 2.2.11 ([8, Prop. 7])**.** *There exist constants c and ε such that*

$$H(x) \leq c \cdot d_L^{\varepsilon}.$$

To establish bounds on the size of Galois orbits on general Shimura varieties, a similar strategy involving the use of a height function on the Shimura variety is used. The chosen height is a canonical height defined for Shimura varieties in [58], which we now describe.

Given a Shimura variety $\mathrm{Sh}_K(G, \Omega)$, fix a $\mathbb{Q}$-representation $G \to \mathrm{GL}(V)$ of G and a lattice $\Lambda \subset V$. Through the Riemann–Hilbert correspondence over p-adic local fields (given by [17]), a filtered automorphic vector bundle with connection $({}_{\mathrm{dR}}V, \mathrm{Fil}^\bullet, \nabla)$ is obtained, which is defined over the reflex field of $\mathrm{Sh}_K(G, \Omega)$ and all of its p-adic places.

The plan is then to define an adelic norm on $\mathrm{Gr}^{\bullet}_{\mathrm{dR}} V$, leading to an Arakelov height function on $\mathrm{Sh}_K(G, \Omega)$.

At the archimedean places, the representation admits a polarization $\psi\colon V \times V \to \mathbb{Q}$, and the norm can be defined as the Hodge norm $\psi(v, h(i)v)$. Over the finite places, the crystalline norm is used when the representation is crystalline, while an alternative intrinsic norm is used at the finitely many other places.

This height function is compatible, meaning that if $(G_1, \Omega_1) \to (G_2, \Omega_2)$ is a map of Shimura data, and ρ_i are representations of G_i compatible with this morphism, then the height of a point in $\mathrm{Sh}_{K_2}(G_2, \Omega_2)$ with respect to ρ_2 is equal to the height of a point in $\mathrm{Sh}_{K_1}(G_1, \Omega_1)$ with respect to ρ_1. Furthemore, this height recovers the Faltings height on $\mathcal{A}_g$.

Special points $P \in S$ correspond to partial CM-types (E, ϕ), where E is a CM-field, and $\phi \subset \Phi$ is a subset of a CM-type. In [58], a definition of the height of a partial CM-type $h(\phi)$ is provided, and it is shown that if $P \in S$ corresponds to the partial CM-type (E, ϕ), then $h(\phi)$ and $h(P)$ differ by a bound that depends on S and $\log d_E$.

The height of a partial CM-type $h(\phi)$ is shown to be bounded in terms of d_E in [58].

Theorem 2.2.12 ([58, Thm. 9.12])**.** *For any positive choice of ε_1, there exists a positive constant c_1 such that*

$$h(\phi) \leq c_1 \cdot d_E^{\varepsilon_1}$$

for all partial CM-types with $[E : \mathbb{Q}] = 2g$.

An alternative proof is provided in Section 3.8 of this dissertation. We will now briefly describe the difference between these two proofs.

In [58], a set of disjoint CM-extensions $\{E_i\}_{1 \leq i \leq n}$ of a fixed totally real field F and a set of partial CM-types $\phi_i \subset \mathrm{Hom}(E_i, \mathbb{C})$ satisfying the condition that $\bigsqcup_{i=1}^{n} \phi_i|_F = \mathrm{Hom}(F, \mathbb{R})$ are considered. By employing a clever approach, they are able to express the height of ϕ_1 in terms of a sum of heights of full CM-types of $E_S = \prod_{i \in S} E_i$ for various subsets $S \subset \{1, 2, \ldots, n\}$. Each full CM-type height is individually bounded in terms of the discriminants of E_S.

Moreover, they show that it is possible to take $E_2, \ldots, E_n$ in such a way that their discriminants are bounded with respect to d_{E_1}, meaning that the discriminant of E_S is of bounded degree and relative discriminant over E_1. This establishes the desired height bound.

Instead of expressing the height $h(\phi)$ in terms of heights of CM-types of the different CM-fields E_S, we express $h(\phi)$ solely in terms of CM-types of E. We achieve this by giving an explicit formula for the height of a point on a quaternionic Shimura variety in terms of CM-types of E (see Theorem 1.2.1). Subsequently, we demonstrate that this quaternionic height is compatible with the definition of a partial CM-type given in [58] (see Theorem 1.2.2). The combination of these two results enables us to present a new proof of [58, Thm 9.12].

The last thing to do is to prove a lower bound for the size of Galois orbits. Again, by [67, Thm. 5.1], this can be reduced to the following bound:

Theorem 2.2.13 ([8, Thm. 1])**.** *There exist positive constants c and ε such that*

$$[\mathbb{Q}(P):\mathbb{Q}] \geq c \cdot d_E^{\varepsilon}.$$

Proof. Let $\Gamma_q \subset \Omega \times S$ be the set

$$\Gamma_q \coloneqq \{(x, q(x) : x \in \mathcal{F}\},$$

representing the graph of the uniformization map q, restricted to a fundamental domain of Ω. A sharpening of the Pila–Wilkie point counting theorem (see [8, Thm. 3]) states that

$$|\{(x,s) \in \Gamma_q : [\mathbb{Q}(x,s):\mathbb{Q}] \leq f, h(x,s) \leq h\}| = O(f,h),$$

meaning that the number of points in the product $\Omega \times S$ of bounded degree and height grows at most polynomially in the given bound.

Let $x \in \Omega$ be a preimage of P in $\mathcal{F}$. As we have seen before ([8, Prop. 7]), the height of x can be bounded in terms of the logarithm of the discriminant of $L = E^{\mathrm{Gal}}$, and therefore in terms of the logarithm of the discriminant of E. Now, for every $\varepsilon' > 0$, the height bound implies that there exists a constant c' such that $h(P) \leq c' \cdot d_E^{\varepsilon'}$. Consequently, we find that

$$h(x, q(x)) \leq h(x) + h(P)$$

can be bounded from above polynomially in terms of $d_E^{\varepsilon'}$.

Let $\Sigma(P) \subset S$ be the smallest zero-dimensional Shimura subvariety that contains P. According to [66, Thm. 2.1], the size of $\Sigma(P)$ is bounded from below by d_E^s, where s is a constant depending solely on S. Each of the points in $\Sigma(P)$ is defined over the same field and has the same height, leading to

$$d_E^s \leq \left|\{(x,s) \in \Gamma_q : [\mathbb{Q}(x,s):\mathbb{Q}] \leq [\mathbb{Q}(P):\mathbb{Q}], h(x,s) \leq d_E^{\varepsilon'}\}\right| = O([\mathbb{Q}(P):\mathbb{Q}], d_E^{\varepsilon'}).$$

By choosing ε' small enough, we obtain a polynomial lower bound of $[\mathbb{Q}(P):\mathbb{Q}]$ in terms of d_E. ∎

3. The third component of the Pila–Zannier method is the Ax–Lindemann theorem. To state it, we first need to define a weakly special subvariety. We say that $V \subset S$ is weakly special if the analytic connected components of $q^{-1}V \subset \Omega$ are algebraic. As expected, all special subvarieties are weakly special. Moreover, a weakly special subvariety is special if and only if it contains a special point.

Theorem 2.2.14 (Hyperbolic Ax–Lindemann)**.** *Let $V \subset S$ be an algebraic variety. Then, the irreducible algebraic subsets of $q^{-1}(V)$ come from the weakly special subvarieties of V.*

This result was initially proven by Pila (see [52]) when S is a product of modular curves. Later, it was extended to the case of $\mathcal{A}_g$ by Pila and Tsimerman (see [55]). Finally, Klingler, Ullmo, and Yafaev (see [29]) established the theorem for all Shimura varieties.

With these three components in place, we can now present a proof of the André–Oort conjecture using the Pila–Zannier method.

Proof. Let $V \subset S$ be an irreducible subvariety containing a Zariski dense subset of special points. As special points are defined over $\overline{\mathbb{Q}}$, we can assume that V is defined over some number field K. For each special point $P \in V$, let $x \in \mathcal{F} \subset \Omega$ be a preimage of P in Ω.

The second ingredient gives us a bound of the form:

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| \geq c \cdot h(P)^{\varepsilon},$$

Since V is defined over K, we know that $\mathrm{Gal}(\overline{\mathbb{Q}}/K) \cdot x \subset q^{-1}(V) \cap \mathcal{F}$. And since K is fixed, we find constants c', ε' such that

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/K) \cdot x| \geq c' \cdot h(P)^{\varepsilon'}.$$

The fundamental domain $\mathcal{F}$, the variety V, and the uniformization map q, are all definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$. Thus, the set $X = q^{-1}(V) \cap \mathcal{F}$ is also definable in an o-minimal structure. However, the height bound implies that the number of points of X with a height at most N grows polynomially in N. Consequently, the Pila–Wilkie point counting theorem asserts that X^{alg} contains positive-dimensional semi-algebraic sets.

Finally, the Ax–Lindemann theorem concludes that X^{alg} consists of weakly special subvarieties of V. But, as they contain the special point P, they must be special, thereby completing the theorem. □

2.3 Existential Closedness Problems

Chapter 4 addresses the existential closedness problem, a class of unlikely intersection problems. In 2005, Zilber (see [77]) conjectured that the complex exponential function satisfies a "Strong Exponential Closedness" property. A specific instance of this conjecture in one variable asks whether, for a polynomial $p(x, y)$ in two variables, generic solutions of $p(z, \exp(z)) = 0$ exist, where $\exp(x) = e^x$ is the exponential function for $x \in \mathbb{C}$. More specifically:

Conjecture 2.3.1 ([77]). *Let $k \subset \mathbb{C}$ be a finitely generated field and $p(x, y) \in k[x, y]$ be an irreducible polynomial in which both x and y appear. Then, there exists $z \in \mathbb{C}$ such that $p(z, \exp(z)) = 0$, and the transcendence degree of $k(z, \exp(z))/k$ is equal to 1.*

In the special case where $k = \bar{\mathbb{Q}}$, Marker demonstrated in [38] that the conjecture is a consequence of Schanuel's conjecture. Later, Mantova, in [36], extended this result to the case of general k, again showing that it follows from Schanuel's conjecture.

In geometric terms, this problem can be formulated as follows: Does an algebraic curve $C \subset \mathbb{C}^2$ have a generic point of the form $(z, \exp(z)) \in C$? The complete Strong Exponential Closedness property goes beyond the dimension 1 case and asks whether a generic variety $V \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$ of dimension n has generic solutions $(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)) \in V$. To state this more precisely, we need to define the terms "free" and "rotund".

Definition 2.3.2. A variety $V \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$ is *free* if its projection to the first factor of $\mathbb{C}^n$ does not lie in a translate of a $\mathbb{Q}$-linear subspace, i.e., it does not lie in a hyperplane given by the equation $\sum_{i=1}^n r_i x_i = c$, where $r_i \in \mathbb{Q}, c in \mathbb{C}$. Furthermore, its projection to the second factor of $(\mathbb{C}^\times)^n$ does not lie in a translate of an algebraic torus, i.e., it does not lie in a variety cut out by the equation $\prod_{i=1}^n y_i^{n_i} = b$, where $n_i \in \mathbb{Z}, b \in \mathbb{C}$.

For a matrix $M \in M_{m\times n}(\mathbb{Z})$ with coefficients $M = \{a_{ij}\}_{1\le i\le m, 1\le j\le n}$, define the map $f_M \colon \mathbb{C}^n \times (\mathbb{C}^\times)^n \to \mathbb{C}^m \times (\mathbb{C}^\times)^m$ as follows:

$$(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto \left(\sum_{j=1}^n a_{1j} x_j, \ldots, \sum_{j=1}^n a_{mj} x_j, \prod_{j=1}^n y_j^{a_{1j}}, \ldots, \sum_{j=1}^n y_j^{a_{mj}} \right).$$

A variety V is *broad* if $\dim(f_M(V)) \ge \operatorname{rank}(M)$.

Conjecture 2.3.3. *Let $V \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be a complex algebraic variety, and let $\Gamma_{\exp} \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be the set of points of the form $(z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n))$. If V is both free and broad, then $V \cap \Gamma \neq \varnothing$. Moreover, for any finitely generated field $k \subset \mathbb{C}$, we can find a point $(z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n)) \in V \cap \Gamma$ such that*

$$\operatorname{td}_k(z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n)) \ge n.$$

Note that when taking $M = I_n$, the $n \times n$ identity matrix, we have $\dim V \ge n$. The full conjecture still remains open, although some partial results have been proven. In 2017, Brownawell and Masser (see [11]) proved that under the stronger assumption that the projection of V to the first $\mathbb{C}^n$ factor is dominant, then $V \cap \Gamma$ is Zariski dense in V. Note that if the projection to $\mathbb{C}^n$ is dominant, then the variety V is automatically broad.

Regarding finding a generic point in the intersection, D'Aquino, Fornasiero, and Terzo (see [15]) were able to prove the following conditional result.

Theorem 2.3.4 ([15]). *Assuming Schanuel's conjecture, if $V \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$ is defined over $\overline{\mathbb{Q}}$, and the projections to $\mathbb{C}^n$ and $(\mathbb{C}^\times)^n$ are both dominant, then the set of generic points of $V \cap \Gamma$ is Zariski dense in V.*

In addition to the case of algebraic tori, the same problem can be considered in other contexts. For instance, we can take $\exp_A\colon \mathbb{C}^n \to A$ to be the exponential function associated to A, an abelian variety of dimension n. Aslanyan, Kirby, and Mantova (see [5]) were able to prove the conjecture true in this setting when the projection of $V \subset \mathbb{C}^n \times A$ to the first $\mathbb{C}^n$ factor is dominant. In the same article, they also provided a new proof of the Brownawell–Masser result.

Another context in which this problem can be formulated is by replacing the exponential function with the modular j-function $j\colon \mathbb{H} \to \mathbb{C}$. We present the analogues of the relevant definitions:

Definition 2.3.5. A variety $V \subset \mathbb{H}^n \times \mathbb{C}^n$ is *free* if:

1. The projection of V to the first factor of $\mathbb{H}^n$ does not lie in a set cut out by the equation $x_i = gx_j$ for some $1 \leq i \neq j \leq n$ and $g \in \mathrm{GL}_2(\mathbb{Q})$, or $x_i = c$ for some $c \in \mathbb{C}$ such that $[\mathbb{Q}(c) : \mathbb{Q}] = 2$;

2. The projection of V to the second factor of $\mathbb{C}^n$ does not lie in a set cut out by the equation $\Phi_N(y_i, y_j) = 0$, where Φ_N is the modular polynomial of level N, or $y_i = c$ where $c \in \overline{\mathbb{Z}}$ is an algebraic integer.

A variety V is *broad* if, for any subset $S \subset \{1, 2, \ldots, n\}$, the projection of V down to $\mathbb{H}^S \times \mathbb{C}^S$ has dimension at least $|S|$.

In this context, the existential closedness problem can be formulated as follows.

Conjecture 2.3.6. *Let $V \subset \mathbb{H}^n \times \mathbb{C}^n$ be a complex algebraic variety, and let $\Gamma_j \subset \mathbb{H}^n \times \mathbb{C}^n$ be the set of points of the form $(z_1, \ldots, z_n, j(z_1), \ldots, j(z_n))$. If V is both free and broad, then $V \cap \Gamma_j \neq \varnothing$. Moreover, for any finitely generated field $k \subset \mathbb{C}$, we can find a point $(z_1, \ldots, z_n, j(z_1), \ldots, j(z_n)) \in V \cap \Gamma$ such that*

$$\mathrm{td}_k(z_1, \ldots, z_n, j(z_1), \ldots, j(z_n)) \geq n.$$

In 2021, Eterović and Herrero (see [20]) proved the analogue of Brownawell and Masser's result; they showed that if $V \subset \mathbb{H}^n \times \mathbb{C}^n$ has a dominant projection to the first n variables, then the intersection $V \cap \Gamma_j$ is Zariski dense in V. In terms of the genericity of these solutions, they were able to conditionally prove an analogue to Conjecture 1.1.2. Assuming a modular Schanuel's conjecture, they prove that if $V \subset \mathbb{H} \times \mathbb{C}$ is a curve that is neither a vertical nor a horizontal line, then there exists a generic point in $V \cap \Gamma_j$.

In Chapter 4, we give a generalization of this conjecture to the setting of Shimura varieties. The results of [11, 5, 20] are extended, and we prove that the conjecture holds when the projection of $V \subset \Omega \times S$ to Ω is dominant.

Furthermore, in addition to requiring a dominant projection, partial results have been achieved in the setting where $V = L \times W$ is a product of two special subvarieties. In 2002, Zilber (see [78]) proved a special case of his conjecture.

Theorem 2.3.7 ([78])**.** *Let $L \subset \mathbb{C}^n$ be an $\mathbb{R}$-linear subspace that is not $\mathbb{Q}$-linear and let $W \subset (\mathbb{C}^\times)^n$ be any variety such that $V = L \times W$ is free and broad. Then $\exp(L) \cap W$ is Euclidean dense in W.*

In 2021, Gallinaro (see [21]) extended the previous result to the setting of the exponential of an abelian variety, and of the j-function. In the setting of the j-function, the subvariety L is taken to be a Möbius subvariety, a variety cut out by equations of the form $x_i = gx_j$ where $1 \leq i \neq j \leq n$ and $g \in \mathrm{SL}_2(\mathbb{R}) \setminus \mathrm{SL}_2(\mathbb{Q})$.

Theorem 2.3.8 ([21])**.** *Let $L \subset \mathbb{H}^n$ be a variety cut out by equations of the form $x_i = gx_j$ for $1 \leq i \neq j \leq n$ and $g \in \mathrm{GL}_2(\mathbb{R}) \setminus \mathrm{GL}_2(\mathbb{Q})$. Let $W \subset \mathbb{C}^n$ be any variety such that $V = L \times W$ is free and broad. Then $j(L) \cap W$ is Euclidean dense in W.*

A Möbius subvariety of $\mathbb{H}^n$ can be viewed as the $F(\mathbb{R})^+$-orbit of a point, where $F(\mathbb{R})$ is a real semi-simple algebraic subgroup of $\mathrm{SL}_2(\mathbb{R})^n$. For example, the Möbius subvariety of $\mathbb{H}^2$ defined by the equation $\tau_2 = g\tau_1$ for $g \in \mathrm{SL}_2(\mathbb{R})$ is the orbit of (i, gi) by the group $F(\mathbb{R}) = \{(h, ghg^{-1}) : h \in \mathrm{SL}_2(\mathbb{R})\} \subset \mathrm{SL}_2(\mathbb{R})^2$. The condition that the subvariety be free, namely without any constant coordinates and not defined by elements of $\mathrm{GL}_2(\mathbb{Q})$, is equivalent to the condition that $F(\mathbb{R})$ is Hodge-generic; there is no proper $\mathbb{Q}$-subgroup of SL_2^n whose real points contain $F(\mathbb{R})$.

Given this, we are motivated to define the generalization of Möbius subvarieties in the setting of Shimura varieties in Section 4.4. Then, we prove the analogous result of Zilber's and Gallinaro's result in Section 4.4.

Chapter 3

Heights on Quaternionic Shimura Varieties

3.1 Overview of the Proof

The idea is similar to that of [75]. We use their decomposition of Faltings heights of a CM abelian variety $h(\Phi)$ into constituent parts $h(\Phi, \tau)$, one for each archimedean place $\tau \in \Phi$. The constituent parts are related to the full CM-type by the formula

$$h(\Phi) - \sum_{\tau \in \Phi} h(\Phi, \tau) = \frac{-1}{4[E_\Phi : \mathbb{Q}]} \log(d_\Phi d_{\overline{\Phi}}),$$

where $d_\Phi, d_{\overline{\Phi}}$ are discriminants associated with $\Phi, \overline{\Phi}$ respectively, and E_Φ is the reflex field of Φ. Moreover, if (Φ_1, Φ_2) are nearby CM-types of E in that they differ only at a single place τ_i, then [75] proves that the quantity

$$h(\Phi_1, \tau_1) + h(\Phi_2, \tau_2)$$

is the same across any choice of nearby CM-types.

We define the group

$$G'' \coloneqq \mathrm{Res}_{F/\mathbb{Q}}(B^\times \times E^\times)/F^\times,$$

where F embeds diagonally as $a \mapsto (a, a^{-1})$. We can construct a norm $N \colon G'' \to \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ and define the group

$$G' \coloneqq G'' \times_{\mathbb{G}_m} \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$$

consisting of elements G'' with norm lying in $\mathbb{Q}^\times$. Then if ϕ is a partial CM-type and ϕ' is a complementary partial CM-type in that $\phi \sqcup \phi'$ constitute a full CM-type, we can construct morphisms $h' \colon \mathbb{C}^\times \to G'(\mathbb{R})$ and $h'' \colon \mathbb{C}^\times \to G''(\mathbb{R})$. They give rise to Shimura datum and Shimura varieties $X'_{U'}$ and $X''_{U''}$ for compact open subgroups $U' \subset G'(\mathbb{A}_f)$ and $U'' \subset G''(\mathbb{A}_f)$ with complex uniformizations

$$X'_{U'}(\mathbb{C}) = G'(\mathbb{Q}) \backslash (\mathcal{H}^\pm)^\Sigma \times G'(\mathbb{A}_f)/U'$$

and

$$X''_{U''}(\mathbb{C}) = G''(\mathbb{Q})\backslash(\mathcal{H}^{\pm})^{\Sigma} \times G''(\mathbb{A}_f)/U''$$

They have canonical models defined over the same reflex field $E_{X'} = E_{X''}$.

The Shimura variety $X'_{U'}$ is of PEL type and has an integral model $\mathcal{X}'_{U'}$ by [12] and [60]. The pair (ϕ, ϕ') gives rise to a point $P'_{U'} \in X'_{U'}$ which parametrizes an abelian variety isogenous to a product $A_1 \times A_2$ of abelian varieties, one with complex multiplication of type $\phi \sqcup \phi'$ and the other with complex multiplication of type $\overline{\phi} \sqcup \phi'$. After defining a suitable metric on $\omega_{\mathcal{X}'_{U'}/\mathcal{O}_{E_{X'}}}$, the Kodaira–Spencer isomorphism on $X'_{U'}$ gives us an equality of heights

$$h_{\widehat{\omega_{\mathcal{X}'_{U'}/\mathcal{O}_{E_{X'}}}}}(P'_{U'}) = \sum_{\tau\in\phi}\big(h(\phi\sqcup\phi',\tau) + h(\overline{\phi}\sqcup\phi',\overline{\tau})\big).$$

Now the idea is to relate $\omega_{\mathcal{X}_U/\mathcal{O}_{E_X}}$ and $\omega_{\mathcal{X}'_{U'}/\mathcal{O}_{E_{X'}}}$. We do this by mapping both X_U and $X'_{U'}$ into the third Shimura variety $X''_{U''}$ so that the points P_U and $P'_{U'}$ have the same image $P''_{U''} \in X''_{U''}(\overline{\mathbb{Q}})$. We represent both canonical bundles in terms of deformations of p-divisible groups $\mathcal{H}_U$ and $\mathcal{H}'_{U'}$ over $\mathcal{X}_U$ and $\mathcal{X}'_{U'}$ respectively, and then relate those p-divisible groups to a p-divisible group $\mathcal{H}''_{U''}$ over $\mathcal{X''}_{U''}$. After showing all this, we get that

$$h_{\widehat{\mathcal{L}_U}}(P_U) = h_{\widehat{\omega_{\mathcal{X}'_{U'}/\mathcal{O}_{E_{X'}}}}}(P'_{U'}) = \sum_{\tau\in\phi}\big(h(\phi\sqcup\phi',\tau) + h(\overline{\phi}\sqcup\phi',\overline{\tau})\big).$$

Of note is that this formula does not depend on the choice of complementary partial CM-type ϕ', because the Shimura variety X_U was defined independently of the choice of ϕ'. We utilize this by summing over all possible complementary CM-type, which will express $h_{\widehat{\mathcal{L}_U}}(P_U)$ in terms of heights of full CM-types containing ϕ as well as nearby CM-types. The sum of heights of nearby CM-types is shown to be constant in [75], and equal to the averaged height of all CM-types of E. Combining these two, we are able to express the height in terms of CM-types containing ϕ and an average of all possible CM-types.

3.2 CM-types and Faltings Heights

Faltings Height

We first define the Faltings height of an abelian variety. It will be defined as the degree of a metrized line bundle. Let A be an abelian variety of dimension g defined over a number field K and let $\mathcal{A}$ be the Néron model over $\mathcal{O}_K$ and let the identity section be $s\colon \operatorname{Spec}\mathcal{O}_K \to \mathcal{A}$. Let $\Omega_{\mathcal{A}/\mathcal{O}_K}$ be the sheaf of relative differentials. The *Hodge bundle* of A is the vector bundle $\Omega(\mathcal{A}) := s^*\Omega_{\mathcal{A}/\mathcal{O}_K}$ over $\mathcal{O}_K$. This is canonically isomorphic to the pushforward $\pi_*\Omega_{\mathcal{A}/\mathcal{O}_K}$, where $\pi\colon \mathcal{A} \to \mathcal{O}_K$ is the structure sheaf morphism.

The Hodge bundle $\Omega(\mathcal{A})$ is a vector bundle over $\mathcal{O}_K$ of rank g and taking the determinant $\omega(\mathcal{A}) := \Omega(\mathcal{A})^{\wedge g}$ is now a line bundle over $\mathcal{O}_K$. We will make this into a metrized line bundle, and to do so, we need to define a norm for each archimedean place of K. We have that

$$\omega(\mathcal{A}) \otimes_{\mathcal{O}_K} K \cong s^*\omega_{A/K} = H^0(A, \omega_{A/K})$$

and so for each archimedean place v of K, we put the norm as

$$\|\alpha\|_v := \left| \frac{1}{(2\pi)^g} \int_{A_v(\mathbb{C})} \alpha \wedge \overline{\alpha} \right|^{\frac{1}{2}}$$

for each $\alpha \in \omega(\mathcal{A}) \otimes_{\mathcal{O}_K} K_v \cong H^0(A_v, \omega_{A_v/K_v})$. In this way, we get a metrized line bundle $\widehat{\omega(\mathcal{A})}$.

Definition 3.2.1. The *Faltings height* of the abelian variety A/K is the Arakelov height

$$h(A) := \frac{1}{[K : \mathbb{Q}]} \widehat{\deg} \widehat{\omega(\mathcal{A})} = \frac{1}{[K : \mathbb{Q}]} \left(\log |\omega(\mathcal{A})/\mathcal{O}_K s| - \sum_{\sigma : K \to \mathbb{C}} \log \|s\|_\sigma \right),$$

for a choice of $s \in \omega(\mathcal{A}) \backslash \{0\}$. This is well defined independent of the choice of s by the Product Formula.

If A has semistable reduction over K, then the Faltings height is invariant under finite field extensions. In general, we can define the stable Faltings height as the height after base change to a finite extension K'/K such that A has semistable reduction over K'. Such a K' always exists.

CM-types

A *CM field extension* is an extension E/F of number fields such that $F/\mathbb{Q}$ is a totally real field and E/F is a quadratic totally imaginary extension. We say E is a *CM field* and F is its *totally real subfield*.

A *(full) CM-type* is a subset $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$ such that $\Phi \sqcup \overline{\Phi} = \mathrm{Hom}(E, \mathbb{C})$, where $\overline{\Phi} = \{\overline{\sigma} : \sigma \in \Phi\}$. A *partial CM-type* is a subset $\phi \subset \mathrm{Hom}(E, \mathbb{C})$ such that $\phi \cap \overline{\phi} = \varnothing$. We say that ϕ' is a *complementary partial CM-type to* ϕ if $\phi \sqcup \phi'$ is a CM-type.

We say that a complex abelian variety A has complex multiplication of type $(\mathcal{O}_E, \Phi)$ if there exists is an embedding $\iota \colon \mathcal{O}_E \to \mathrm{End}(A)$ and an isomorphism $\mathrm{Lie}(A) \cong \mathbb{C}^g \overset{\Phi}{\cong} E \otimes_{\mathbb{Q}} \mathbb{R}$ of $\mathcal{O}_E$ modules.

Let E be a CM field with degree $[E : \mathbb{Q}] = 2g$ and let $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$ be a CM-type. Let A_Φ be an abelian variety of CM-type $(\mathcal{O}_E, \Phi)$. Then there is a number field K over which A_Φ is defined and has a smooth projective integral model $\mathcal{A}/\mathcal{O}_K$. Colmez proved the following theorem

Theorem 3.2.2 ([14, Thm 0.3])**.** *The Faltings height $h(A_\Phi)$ depends only on the CM-type (E, Φ).*

We write $h(\Phi) \coloneqq h(A_\Phi)$. Colmez conjectured a formula about $h(\Phi)$ in terms of logarithmic derivatives of Artin L-functions related to Φ. This conjecture has been proven when $E/\mathbb{Q}$ is an abelian extension by Obus and Colmez (see [46]) and when F is a real quadratic field by Yang (see [74]). An averaged version was proven in[75], and independently in [4].

Theorem 3.2.3 ([4, Thm A], [75, Thm 1.1]). *Suppose E/F is an CM extension and let $\chi\colon \mathbb{A}_F^\times \to \{\pm 1\}$ the character corresponding to this extension, and $L(s,\chi)$ the corresponding Artin L-function. Let d_F be the absolute discriminant of F and $d_{E/F}$ the norm of the relative discriminent of E/F. Then*

$$\frac{1}{2^g}\sum_{\Phi} h(\Phi) = -\frac{1}{2}\frac{L'(0,\chi)}{L(0,\chi)} - \frac{1}{4}\log(d_{E/F}d_F),$$

where the sum on the left runs through the set of all CM-types of E.

Decomposition of Heights

We recall the results of [75] decomposing the Faltings height of a CM-type Φ into its constituent embeddings $\tau \in \Phi$. To decompose the height, we first decompose the Hodge bundle into its eigenspaces.

Assume that A has complex multiplication of type $(\mathcal{O}_E, \Phi)$. Then we define

$$\Omega(A)_\tau \coloneqq \Omega(A) \otimes_{E,\tau} \mathbb{C},$$

where E acts on $\mathbb{C}$ through the embedding $\tau\colon E \to \mathbb{C}$. This gives us a decomposition of the Hodge bundle as

$$\Omega(A) \cong \bigoplus_{\tau\colon E\to\mathbb{C}} \Omega(A)_\tau \cong \bigoplus_{\tau\in\Phi} \Omega(A)_\tau.$$

The latter isomorphism holds because $\Omega(A)_\tau = 0$ for $\tau \notin \Phi$.

Let A^t be the dual abelain variety of A. Then we have canonical isomorphisms

$$\Omega(A^t) = \mathrm{Lie}(A^t)^\vee \cong H^1(A, \mathcal{O}_A)^\vee \cong H^{0,1}(A)^\vee = \overline{\Omega(A)}^\vee,$$

and such that if A is of CM-type $(\mathcal{O}_E, \Phi)$, then A^t is of CM-type $(\mathcal{O}_E, \overline{\Phi})$. From this isomorphism, we also get a perfect Hermitian pairing $\Omega(A^t) \otimes \Omega(A) \to \mathbb{C}$.

Just as before, we can decompose

$$\Omega(A^t) \cong \bigoplus_{\tau\in\overline{\Phi}} \Omega(A^t)_\tau.$$

The Hermitian pairing from before decomposes into a sum of orthogonal pairings $\Omega(A)_\tau \otimes \Omega(A^t)_{\overline{\tau}} \to \mathbb{C}$. Taking the determinant gives a Hermitian norm on the line bundle

$$N(A,\tau) \coloneqq \det \Omega(A)_\tau \otimes \det \Omega(A^t)_{\overline{\tau}}.$$

We can extend all of this to an integral model of A. If $\mathcal{A}$ is the Néron model over $\mathcal{O}_K$ as before, with K including all embeddings of $E \to \overline{\mathbb{Q}}$, define

$$\Omega(\mathcal{A})_\tau \coloneqq \Omega(\mathcal{A}) \otimes_{\mathcal{O}_K \otimes \mathcal{O}_E, \tau} \mathcal{O}_K$$

for each $\tau\colon E \to K$. We define $\Omega(\mathcal{A}^t)_\tau$ analogously. For each archimedean place of K, we use the aforementioned Hermitian norm $\|\cdot\|$ on the generic fiber of $\det \Omega(\mathcal{A})_\tau \otimes \det \Omega(\mathcal{A}^t)_{\overline{\tau}}$, and thus we get a metrized line bundle

$$\widehat{\mathcal{N}(\mathcal{A}, \tau)} \coloneqq (\det \Omega(\mathcal{A})_\tau \otimes \det \Omega(\mathcal{A}^t)_{\overline{\tau}}, \|\cdot\|).$$

Definition 3.2.4. If A is an abelian variety of CM-type (E, Φ) and $\tau\colon E \to \mathbb{C}$, then the τ-part of the Faltings height of A is

$$h(A, \tau) \coloneqq \frac{1}{2[K : \mathbb{Q}]} \widehat{\deg} \widehat{\mathcal{N}(\mathcal{A}, \tau)}.$$

Note that if $\tau \notin \Phi$, then $\mathcal{N}(\mathcal{A}, \tau) = 0$ and so the height contribution is 0 as well.

Just as with the Faltings height, this τ-component is independent of the abelian variety itself. Thus, we will write $h(\Phi, \tau)$ for $h(A, \tau)$.

Theorem 3.2.5 ([75, Thm 2.2])**.** *If A has CM of type $(\mathcal{O}_E, \Phi)$, the height $h(A, \tau)$ depends only on the pair (Φ, τ).*

We call a pair of CM-types (Φ_1, Φ_2) *nearby* if $|\Phi_1 \cap \Phi_2| = g - 1$. Let $\tau_i = \Phi_i \backslash (\Phi_1 \cap \Phi_2)$ be the place where they differ. Then the sum of the τ_i-components of $h(\Phi_i)$ is independent of the choice of nearby CM-type.

Theorem 3.2.6 ([75, Thm. 2.7])**.** *The quantity $h(\Phi_1, \tau_1) + h(\Phi_2, \tau_2)$ is independent of the choice of nearby CM-type (Φ_1, Φ_2).*

Finally, we compare $h(\Phi)$ with its constituents $h(\Phi, \tau)$.

Definition 3.2.7. Let $\Psi \subset \mathrm{Hom}(E, \mathbb{C})$ be any subset, not necessarily a (partial) CM-type. The reflex field $E_\Psi \subset E^{\mathrm{Gal}}$ is the subfield of the Galois closure of E fixed by all automorphisms that fix Ψ. The trace map $\mathrm{Tr}_\Psi\colon E \to E_\Psi$ is given by $\mathrm{Tr}_\Psi(x) = \sum_{\tau \in \Psi} \tau(x)$.

We can decompose $E_\Psi \otimes_{\mathbb{Q}} E \cong E \cong \widetilde{E_\Psi} \times \widetilde{E_{\Psi^c}}$ where the trace of the action of E on $\widetilde{E_\Psi}$ is Tr_Ψ and the trace of the action on $\widetilde{E_{\Psi^c}}$ is Tr_{Ψ^c}. Let $\mathfrak{d}_\Psi$ be the relative discriminant of the image of $\mathcal{O}_{E_\Psi} \otimes_{\mathbb{Z}} \mathcal{O}_E$ in $\widetilde{E_\Psi}$ over $\mathcal{O}_{E_\Psi}$ and let d_Ψ be the positive generator of the $N_{E_\Psi/\mathbb{Q}}(\mathfrak{d}_\Psi)$.

Theorem 3.2.8 ([75, Thm 2.3])**.**

$$h(\Phi) - \sum_{\tau \in \Phi} h(\Phi, \tau) = \frac{-1}{4[E_\Phi : \mathbb{Q}]} \log(d_\Phi d_{\overline{\Phi}}).$$

3.3 Quaternionic Shimura Varieties

We fix a totally real field $F/\mathbb{Q}$ of degree g. Let $\Sigma \subset \mathrm{Hom}(F,\mathbb{R})$ be the subset of places of F and let B/F be a quaternion algebra over F that is split at infinity precisely at Σ, which means that

$$B \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{\tau\in\Sigma} M_2(\mathbb{R})_\tau \oplus \prod_{\sigma\notin\Sigma} \mathbb{H}_\sigma.$$

From this quaternion algebra, we will construct three related quaternionic Shimura varieties and relate their heights.

We are primarily interested in the Shimura variety X associated with the group $G = \mathrm{Res}_{F/\mathbb{Q}} B^\times$. However, this Shimura datum does not parametrize abelian varieties and is of abelian type. We will follow the approach of [12] by finding a unitary Shimura datum G' that has the same derived group as G, and is of PEL type which will give us a nice description of the integral models of X' in terms of abelian varieties. Then following [28, 27], we will give an integral model for X by taking the connected components of X'.

We start with the primary Shimura variety of study, the one associated to the group $G = \mathrm{Res}_{F/\mathbb{Q}}(B^\times)$. Take a cocharacter $h\colon \mathbb{C}^\times \to G(\mathbb{R})$ as

$$h(a+bi) = \left(\prod_{\tau\in\Sigma}\begin{pmatrix} a & b \\ -b & a\end{pmatrix}_\tau, \prod_{\sigma\notin\Sigma} 1_\sigma\right) \in \prod_{\tau\in\Sigma} M_2(\mathbb{R})_\tau \times \prod_{\sigma\notin\Sigma}\mathbb{H}_\sigma.$$

We can identify the $G(\mathbb{R})$-conjugacy class of h with $(\mathcal{H}^\pm)^{|\Sigma|}$, where $\mathcal{H}^\pm \coloneqq \mathbb{C}\backslash\mathbb{R}$, by sending $ghg^{-1} \mapsto \prod_{\tau\in\Sigma} g_\tau(i_\tau)$, where $g_\tau \in M_2(\mathbb{R})_\tau$ is the τ component of g. From the Shimura datum $(G,(\mathcal{H}^\pm)^\Sigma)$, we get a Shimura variety X_U for each open compact subgroup $U \subset G(\mathbb{A}_f)$ that has a complex uniformization

$$X_U(\mathbb{C}) = G(\mathbb{Q})\backslash(\mathcal{H}^\pm)^\Sigma \times G(\mathbb{A}_f)/U.$$

The reflex field $E_X \coloneqq E(G,(\mathcal{H}^\pm)^\Sigma)$ of X is the subfield of $\mathbb{C}$ fixed by the automorphisms of $\mathbb{C}$ that fix $\Sigma \subset \mathrm{Hom}(F,\mathbb{R})$. The Shimura variety X_U has a canonical model over E_X whose complex points have the above uniformization (see [42]).

Let $N_{B/F}\colon B^\times \to F^\times$ be the reduced norm on B. Then the derived group is $G^{\mathrm{der}} = \ker[N_{B/F}\colon \mathrm{Res}_{F/\mathbb{Q}} B^\times \to \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m]$, the elements of B with norm $1\in F$, and its adjoint group is $\mathrm{Res}_{F/\mathbb{Q}} B^\times/F^\times$.

We now introduce two auxiliary Shimura data that have the same derived group and the same adjoint group as G. Thus their associated Shimura varieties have isomorphic connected components to X. Let E/F be a CM extension such that there is an embedding $E \hookrightarrow B$. Let

$$G'' \coloneqq \mathrm{Res}_{F/\mathbb{Q}}(B^\times \times E^\times)/F^\times,$$

where $F^\times \hookrightarrow B^\times \times E^\times$ by $a \mapsto (a, a^{-1})$. Let $\Phi : E\otimes_{\mathbb{Q}}\mathbb{R} \xrightarrow{\sim} \mathbb{C}^g$ be a full CM-type of E. Split Φ into partial CM-types by

$$\phi = \{\sigma \in \Phi : \sigma|_F \in \Sigma\},$$

and

$$\phi' = \{\sigma \in \Phi : \sigma|_F \notin \Sigma\},$$

such that ϕ and ϕ' are complementary partial CM-types. These partial CM-types give maps $\phi\colon E \to \mathbb{C}^\Sigma$ and $\phi'\colon E \to \mathbb{C}^{\Sigma^c}$. Identify $E \otimes_\mathbb{Q} \mathbb{R}$ with $\mathbb{C}^g$ through the CM-type Φ. Define the cocharacter $h_E\colon \mathbb{C}^\times \to E \otimes_\mathbb{Q} \mathbb{R}$ to be

$$h_E(z) = (\phi(1), \phi'(z)) \in \mathbb{C}^g.$$

We can now define $h''\colon \mathbb{C}^\times \to G''(\mathbb{R})$ as the image of $(h(z), h_E(z))$ after quotienting by the $F^\times$-action. As before, the $G''(\mathbb{R})$-conjugacy class of h'' can be identified with $(\mathcal{H}^\pm)^\Sigma$.

There exists a well defined norm $\nu\colon G'' \to \operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ given by mapping

$$(b, e) \mapsto N_{B/F}(b) N_{E/F}(e).$$

We use this norm to define an algebraic subgroup

$$G' \coloneqq G'' \times_{\operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_m} \mathbb{G}_m,$$

which consists of elements of G'' whose norm lies in $\mathbb{Q}^\times \subset F^\times$. For our chosen cocharacter $\nu(h''(a+bi)) = a^2 + b^2 \in \mathbb{R}$ and hence h'' factors through a map $h'\colon \mathbb{C}^\times \to G'(\mathbb{R})$. The $G'(\mathbb{R})$ conjugacy class of h' can be identified with $(\mathcal{H}^\pm)^{|\Sigma|}$ as well. For open and compact subgroups $U' \subset G'(\mathbb{A}_f)$ and $U'' \subset G''(\mathbb{A}_f)$, we get Shimura varieties $X'_{U'}$ and $X''_{U''}$ with complex uniformizations

$$X'_{U'}(\mathbb{C}) = G'(\mathbb{Q}) \backslash (\mathcal{H}^\pm)^\Sigma \times G'(\mathbb{A}_f)/U',$$

and

$$X''_{U''}(\mathbb{C}) = G''(\mathbb{Q}) \backslash (\mathcal{H}^\pm)^\Sigma \times G''(\mathbb{A}_f)/U''.$$

The reflex fields of these Shimura varieties $E_{X'} \coloneqq E(G', (\mathcal{H}^\pm)^\Sigma)$ and $E_{X''} \coloneqq E(G'', (\mathcal{H}^\pm)^\Sigma)$ are both equal to the subfield of $\mathbb{C}$ fixed by all automorphisms of $\mathbb{C}$ fixing $\phi' \subset \operatorname{Hom}(E, \mathbb{C})$. If an automorphism of $\mathbb{C}$ fixes ϕ', then it fixes Σ as well. Therefore, the reflex field E_X of X is a subfield of $E_{X'} = E_{X''}$.

We now describe the abelian varieties which X' parametrizes. Let $V = B$ viewed as a $\mathbb{Q}$-vector space with a natural left action by E and choose $\gamma \in E \subset B$ so that $\overline{\gamma} = -\gamma$. Then we define $\psi\colon V \times V \to \mathbb{Q}$ by

$$\psi(v, w) = \operatorname{Tr}_{F/\mathbb{Q}} \operatorname{Tr}_{B/F}(\gamma v \overline{w}),$$

where $\overline{w}$ is conjugation on B. This is a nondegenerate alternating form and $\psi(ev, w) = \psi'(v, e^* w)$ for all $v, w \in V$ and $e \in E$, where the involution $e^* = \overline{e}$ is just conjugation on E. We define an action of $(B \times E)^\times / F^\times$ on V by setting $(b, e) \cdot v = ev\overline{b}$. In this way we can identify G' with E-linear automorphisms of V with rational norm

$$G' = \{g \in \operatorname{GL}_E(V) : \psi(gv, gw) = \nu(g) \cdot \psi(v, w) \text{ for some } \nu(g) \in \mathbb{G}_m\}.$$

The action of $\mathbb{C}$ on $V_{\mathbb{R}}$ through the morphism h' induces a Hodge structure on V of weight 1, and we can choose γ such that ψ induces a polarization satisfying $\psi(v, h'(i)v) \geq 0$ for all $v \in V_{\mathbb{R}}$ and hence (V, ψ) is a symplectic $(E, *)$-module.

Thus by [43, Thm 8.17], the pair $(G', (\mathcal{H}^{\pm})^{\Sigma})$ is PEL Shimura datum and $X'_{U'}$, for U' small enough, represents the functor which for any test scheme S over $E_{X'}$, the S-points are isomorphism classes of quadruples $(A, \iota, \theta, \kappa)$ where

1. A/S is an abelian scheme of relative dimension $2g$ up to isomorphism;
2. $\iota\colon E \to \mathrm{End}(A/S) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an injection such that the action of $\iota(E)$ on $\mathrm{Lie}(A/S)$ has trace given by
$$\mathrm{Tr}(\ell, \mathrm{Lie}(A/S)) = \mathrm{Tr}_{\phi \sqcup \phi'}(\ell) + \mathrm{Tr}_{\overline{\phi} \sqcup \phi'}(\ell)$$
for all $\ell \in E$, where for a CM-type (E, Φ) the trace map $\mathrm{Tr}_{\Phi}\colon E \to \mathbb{C}$ is defined as
$$\mathrm{Tr}_{\Phi}(e) = \sum_{\sigma \in \Phi} \sigma(e);$$
3. $\theta\colon A \to A^t$ is a polarization whose Rosati involution on $\mathrm{End}(A/S)_{\mathbb{Q}}$ induces the involution $\gamma \mapsto \gamma^*$ on E;
4. and $\kappa\colon H_1(A, \mathbb{A}_f) \simeq V_{\mathbb{A}_f}$ is a U'-orbit of $\mathbb{A}_{E,f}$-modules that respects the bilinear forms on both factors up to an element in $\mathbb{A}_f^{\times}$.

3.4 Integral Models

To construct integral models for these Shimura varieties, we first use the PEL structure of X' to get an integral model $\mathcal{X}'$ which parametrizes abelian schemes. Then we will transfer the integral model of X' to construct integral models for X and X'', as done in [28, 27].

PEL Type $\mathcal{X}'$

We construct $\mathcal{X}'$ following [60, 48]. Let $p \in \mathbb{Z}$ be a prime number. Let $\mathfrak{p}$ be a prime of F lying above p. Set $\mathcal{O}_{E,\mathfrak{p}} = \mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_{F,\mathfrak{p}}$.

- If B is unramified at $\mathfrak{p}$, then $B_{\mathfrak{p}} \cong M_2(F_{\mathfrak{p}})$. Choose an isomorphism such that $\mathcal{O}_{E,\mathfrak{p}} \subset M_2(\mathcal{O}_{F,\mathfrak{p}})$ and set $\Lambda_{\mathfrak{p}} = M_2(\mathcal{O}_{F,\mathfrak{p}})$.
- If B is ramified at $\mathfrak{p}$, then $B_{\mathfrak{p}}$ is a division algebra over $F_{\mathfrak{p}}$ and there is a unique choice of a maximal order $\mathcal{O}_{B,\mathfrak{p}}$, which must contain $\mathcal{O}_{E,\mathfrak{p}}$. We set $\Lambda_{\mathfrak{p}} = \mathcal{O}_{B,\mathfrak{p}}$.

From this choice of $\mathcal{O}_{E,\mathfrak{p}}$-lattice $\Lambda_{\mathfrak{p}}$, construct a chain of lattices by taking
$$\mathcal{L}_{\mathfrak{p}} = \{\cdots \subset \omega_{\mathfrak{q}}\Lambda_{\mathfrak{p}} \subset \Lambda_{\mathfrak{p}} \subset \omega_{\mathfrak{q}}^{-1}\Lambda_{\mathfrak{p}} \subset \cdots\},$$

where $\omega_{\mathfrak{q}}$ is a uniformizer of $E_{\mathfrak{q}}$, taken to be a uniformizer of $F_{\mathfrak{p}}$ if $\mathfrak{q}$ is unramified over $\mathfrak{p}$. From these chains, we can construct a multichain $\mathcal{L}_p$ of $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$-lattices which consist of all lattices Λ_p which can be written as

$$\Lambda_p = \oplus_{\mathfrak{p}|p} \Lambda_{\mathfrak{p}}, \quad \Lambda_{\mathfrak{p}} \in \mathcal{L}_{\mathfrak{p}}.$$

We now require that E is ramified above all finite primes $\mathfrak{p}$ of F that also ramify in B. Let $\delta_{\mathfrak{p}/p} \in F_{\mathfrak{p}}$ be a generator for the different ideal $\mathfrak{d}_{F_{\mathfrak{p}}/\mathbb{Q}_p}$ of $F_{\mathfrak{p}}/\mathbb{Q}_p$.

Lemma 3.4.1. *We can choose $\gamma \in E^\times$ such that*

- $\gamma = -\overline{\gamma}$;
- $\gamma \in \delta_{\mathfrak{p}/p}^{-1} \mathcal{O}_{E,\mathfrak{p}}^\times$;
- *and* $\psi(v, h'(i)v) > 0$ *for all* $v \in V_{\mathbb{R}} \backslash \{0\}$.

Moreover, under this choice of γ, the multichain of $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$-lattices $\mathcal{L}_p$ is self-dual with respect to the alternating form $\psi_p(u, v)$.

Proof. The anti-symmetric elements of E are dense in the anti-symmetric elements of $(E \otimes_{\mathbb{Q}} \mathbb{Q}_p) \oplus (E \otimes_{\mathbb{Q}} \mathbb{R})$ and the conditions given are all open and non-empty, meaning that we can find such a γ.

To show that $\mathcal{L}_p$ is a self-dual multichain, it suffices to look locally at each prime $\mathfrak{p}$ of F. The alternating form tensored with $\mathbb{Q}_p$ becomes a sum over all primes $\mathfrak{p}$ of

$$\psi_{\mathfrak{p}}(v, w) = \operatorname{Tr}_{F_{\mathfrak{p}}/\mathbb{Q}_p} \operatorname{Tr}_{B_{\mathfrak{p}}/F_{\mathfrak{p}}}(\gamma v \overline{w}).$$

Thus for the lattice $\Lambda_{\mathfrak{p}}$, its dual with respect to $\psi_{\mathfrak{p}}$ is

$$\Lambda_{\mathfrak{p}}^\vee = \{w \in V_{\mathfrak{p}} : \psi_{\mathfrak{p}}(v, w) \in \mathbb{Z}_p \forall v \in \Lambda_{\mathfrak{p}}\} = \{w \in V_{\mathfrak{p}} : \operatorname{Tr}_{B_{\mathfrak{p}}/F_{\mathfrak{p}}}(\gamma v \overline{w}) \in \delta_{\mathfrak{p}/p}^{-1} \mathcal{O}_{F,\mathfrak{p}}\}.$$

If $\mathfrak{p}$ is unramified in B, to check that $\Lambda_{\mathfrak{p}}$ is self dual, it suffices to take $\delta = \delta_{\mathfrak{p}/p}^{-1}$. Under the isomorphism $B_{\mathfrak{p}} \cong M_2(F_{\mathfrak{p}})$, conjugation is given by $\overline{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ and the trace is

$$\operatorname{Tr}_{B_{\mathfrak{p}}/F_{\mathfrak{p}}}\left(\delta \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \overline{\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)}\right) = \delta_{\mathfrak{p}/p}^{-1}(ad' + bc' + a'd + b'c).$$

From this, we see that $\Lambda_{\mathfrak{p}}^\vee = \Lambda_{\mathfrak{p}} = M_2(\mathcal{O}_{F,\mathfrak{p}})$.

If $\mathfrak{p}$ is ramified in B, as before it suffices to take $\delta = \delta_{\mathfrak{p}/p}^{-1}$. In this setting, we required that E is also ramified at $\mathfrak{p}$ meaning that we can find our element $j \in B_{\mathfrak{p}}$ such that $j^2 \in \mathcal{O}_{F,\mathfrak{p}}^\times$. For this choice of j, the unique maximal order $\mathcal{O}_{B,\mathfrak{p}}$ is $\mathcal{O}_{E,\mathfrak{p}} + \mathcal{O}_{E,\mathfrak{p}} j$. Then for $a + bj \in E_{\mathfrak{p}} + E_{\mathfrak{p}} j = B_{\mathfrak{p}}$, the trace is $\operatorname{Tr}_{B_{\mathfrak{p}}/F_{\mathfrak{p}}}(a + bj) = \operatorname{Tr}_{E_{\mathfrak{p}}/F_{\mathfrak{p}}}(a)$. We thus have

$$\operatorname{Tr}_{B_{\mathfrak{p}}/F_{\mathfrak{p}}}\left(\gamma(a + bj)(\overline{a' + b'j})\right) = \delta_{\mathfrak{p}/p}^{-1} \operatorname{Tr}_{E_{\mathfrak{p}}/F_{\mathfrak{p}}}(a\overline{a'} - b\overline{b'}j^2).$$

From this, we get that $\Lambda_{\mathfrak{p}}^\vee = \omega_{\mathfrak{q}}^{-1} \Lambda_{\mathfrak{p}} \in \mathcal{L}_{\mathfrak{p}}$. $\square$

Let $U'_p := U'_p(0) \subset G'(\mathbb{Q}_p)$ be the parahoric subgroup of elements fixing the multichain $\mathcal{L}_p$. Let $U'^p \subset G'(\mathbb{A}_f^p)$ and set $U' := U'_p U'^p \subset G'(\mathbb{A}_f)$. Choose a place v of $\mathcal{O}_{E_X}$ lying above p. The field extension $E_{X'}/E_X$ is imaginary quadratic and splits at every prime of $\mathcal{O}_{E_X}$ above p. Choose a place $v' \mid v$ of $\mathcal{O}_{E_{X'}}$. From this integral data, consider the functor $\mathcal{F}_{U'_p U'^p}$ which associates to a locally Noetherian scheme S over $\mathcal{O}_{E_{X'},v'}$ the set of isomorphism classes of quadruples $(\mathcal{A}, \iota, \theta, \kappa)$ where:

1. $\mathcal{A}/\mathcal{S}$ is an abelian scheme of relative dimension $2g$ up to isogeny of order prime to p;
2. $\iota \colon \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to \mathrm{End}(\mathcal{A}/\mathcal{S}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a homomorphism satisfying the Kottwitz condition. There is an identity of polynomial functions
$$\det_{\mathcal{O}_S}(\iota(\ell); \mathrm{Lie}\, \mathcal{A}/S) = \prod_{\varphi \in \phi} \varphi(\ell)\overline{\varphi(\ell)} \prod_{\varphi' \in \phi'} \varphi'(\ell)^2;$$
3. $\theta \colon \mathcal{A} \to \mathcal{A}^t$ is a principle polarization whose Rosati involution on $\mathrm{End}(\mathcal{A}/S) \otimes \mathbb{Z}_{(p)}$ induces complex conjugation on $\mathcal{O}_{E,(p)}$;
4. and $\kappa \colon H_1(\mathcal{A}, \mathbb{A}_f^p) \simeq V_{\mathbb{A}_f^p}$ is a U'^p-orbit of skew $\mathcal{O}_{E,(p)} \otimes \mathbb{A}_f^p$-modules that respects the bilinear forms up to a constant in $(\mathbb{A}_f^p)^\times$.

Theorem 3.4.2. *If U'^p is sufficiently small, then the functor $\mathcal{F}_{U'}$ is represented by a quasi-projective scheme $\mathcal{M}_{U'}$ over $\mathcal{O}_{E_{X'},v'}$ whose generic fiber is $X'_{U'}$. Moreover, we have:*

1. *If p is unramified in F and B, the scheme $\mathcal{M}_{U'}$ is smooth over $\mathcal{O}_{E_{X'},v'}$;*
2. *The p-adic completion of $\mathcal{M}_{U'}$ along the basic locus has a p-adic uniformization by a Rapoport–Zink space*

Proof. This is the PEL moduli problem studied by [31] and [60]. The first case is covered by [31, Sec. 5] and the second case is covered by [60, Thm 6.50]. □

If p is ramified in F, then $\mathcal{M}_{U'}$ is not necessarily flat. We explain how to construct a flat model following [48]. We first construct the corresponding local model for $\mathcal{A}_{U'}$ following [60, Def. 3.27]. Let $\mathbb{M}^{\mathrm{naive}}$ be the functor which associates to a locally Noetherian scheme S over $\mathcal{O}_{E_{X'},v'}$ the set of $\mathcal{O}_{E,p} \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-submodules $t_\Lambda \subset \Lambda_p \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ such that

1. t_Λ is a finite locally free $\mathcal{O}_S$ module;
2. For all $\ell \in \mathcal{O}_{E,p}$, there is an identity
$$\det_{\mathcal{O}_S}(\ell; t_\Lambda) = \prod_{\varphi \in \phi} \varphi(\ell)\overline{\varphi(\ell)} \prod_{\varphi' \in \phi'} \varphi'(\ell)^2;$$
3. and t_Λ is totally isotropic under the nondegenerate alternating pairing
$$\psi_{p,\mathcal{O}_S} : (\Lambda_p \otimes_{\mathbb{Z}_p} \mathcal{O}_S) \times (\Lambda_p \otimes_{\mathbb{Z}_p} \mathcal{O}_S) \to \mathcal{O}_S.$$

This functor is represented by a closed subscheme of a Grassmannian. Let $\mathcal{P}/\operatorname{Spec}(\mathbb{Z}_p)$ be the group scheme whose S points are $\operatorname{Aut}(\mathcal{L}\otimes_{\mathbb{Z}_p}\mathcal{O}_S)$ automorphisms of the multichain $\mathcal{L}$ that respects the similitude ψ_p. Then by [60, p. 3.30], there is a smooth morphism of algebraic stacks of relative dimension $\dim G' = 5g$.

$$\mathcal{M}_{U'} \to \left[\mathbb{M}^{\mathrm{naive}}/\mathcal{P}_{\mathcal{O}_{E_{X',v'}}}\right]$$

From this, we see that $\mathbb{M}^{\mathrm{naive}}$ controls the structure of $\mathcal{M}_{U'}$. While conjectured to be flat, it has since been shown by [49] that when the prime p is ramified in E, the scheme is no longer flat. So instead, take $\mathbb{M}^{\mathrm{loc}}$ to be the flat scheme theoretic closure of $\mathbb{M}^{\mathrm{naive}}\otimes_{\mathcal{O}_{E_{X',v'}}} E_{X',v'}$ in $\mathbb{M}^{\mathrm{naive}}$.

Proposition 3.4.3 ([48, Thm. 9.1])**.** *The scheme* $\mathbb{M}^{\mathrm{loc}}$ *is normal and Cohen–Macaulay with reduced special fiber. It also admits an action by* $\mathcal{P}_{\mathcal{O}_{E_{X',v'}}}$ *such that the natural inclusion* $\mathbb{M}^{\mathrm{loc}} \to \mathbb{M}^{\mathrm{naive}}$ *is* $\mathcal{P}_{\mathcal{O}_{E_{X',v'}}}$*-equivariant.*

With this flat local model, we can define a flat integral model $\mathcal{X}'_{U'}$ for $X'_{U'}$ by pulling back $\mathcal{M}_{U'}$ to $\mathbb{M}^{\mathrm{loc}}$ to get the following picture.

$$\begin{tikzcd}
\mathcal{X}'_{U'} \arrow[r] \arrow[d] & \left[\mathbb{M}^{\mathrm{loc}}/\mathcal{P}_{\mathcal{O}_{E_{X',v'}}}\right] \arrow[d] \\
\mathcal{M}_{U'} \arrow[r] & \left[\mathbb{M}^{\mathrm{naive}}/\mathcal{P}_{\mathcal{O}_{E_{X',v'}}}\right]
\end{tikzcd}$$

The schemes $\mathcal{X}'_{U'}$ and $\mathcal{M}_{U'}$ have generic fiber equal to $X'_{U'}$ and since $\mathbb{M}^{\mathrm{loc}}$ is flat, our integral model $\mathcal{X}'_{U'}$ of $X'_{U'}$ is flat as well.

We have defined integral models $\mathcal{X}'_{U'}$ when $U' = U'_pU'^p$ where $U'_p = U'_p(0)$ is maximally parahoric and U'^p is sufficiently small. In order to get an integral model over all of $\mathcal{O}_{E_{X'}}$, we show how to construct these integral models when U'^p is big.

Suppose that the moduli problem above p is unramified, meaning that E (and hence B) is unramified above p. For any $m \geq 0$, let $U_p(m)$ denote

$$U'_p(m) := \{g \in G'(\mathbb{Q}_p) : g\Lambda_p = \Lambda_p, g|_{p^{-m}\Lambda_p/\Lambda_p} \equiv 1\}$$

the subgroup of U'_p that acts as the identity on $\Lambda_p/p^m\Lambda_p$. The nondegenerate alternating form ψ_p on Λ_p gives rise to a nondegenerate $*$-hermitian alternating form

$$\langle , \rangle_{p,m} : p^{-m}\Lambda_p/\Lambda_p \times p^{-m}\Lambda_p/\Lambda_p \to p^{-m}\mathbb{Z}_p/\mathbb{Z}_p$$

given by

$$\langle x, y\rangle_{p,m} = \psi_p(p^m x, y).$$

Then for U'^p small enough, Mantovan defines an integral model of $\mathcal{X}'_{U'_p(m)U'^p}$ over $\mathcal{O}_{E_{X'},v'}$ by using the notion of a full set of sections. Let $\mathcal{F}_{U'_p(m)U'^p}$ be the functor over $\mathcal{F}_{U'_pU'^p} = \mathcal{F}_{U'_p(0)U'^p}$ which associates to a locally Noetherian scheme S over $\mathcal{O}_{E_{X'},v'}$ the set of isomorphism classes of data $(\mathcal{A}, \iota, \theta, \kappa, \alpha)$ where $(\mathcal{A}, \iota, \theta, \kappa)$ are as in the functor $\mathcal{F}_{U'_p(0)U'^p}$ and

$$\alpha : p^{-m}\Lambda_p/\Lambda_p \to \mathcal{A}[p^m](S)$$

is an $\mathcal{O}_{E,p}$-linear homomorphism such that $\{\alpha(x) : x \in p^{-m}\Lambda/\Lambda\}$ is a full set of sections of $\mathcal{A}[p^m]$ and α maps the pairing $\langle,\rangle_{p,m}$ to the Weil pairing on $\mathcal{A}[p^m]$, up to a scalar multiple in $(\mathbb{Z}/p^m\mathbb{Z})^\times$.

Theorem 3.4.4 ([37, Prop. 15])**.** *The functor $\mathcal{F}_{U'_p(m)U'^p}$ is represented by a smooth scheme $\mathcal{X}_{U'_p(m)U'^p}$ over $\mathcal{O}_{E_{X'},v'}$.*

To make the notion of U'^p small enough explicit, fix a lattice $\Lambda \subset V$ over $\mathbb{Z}$ such that $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \Lambda_p$. For $N \in \mathbb{N}$, let

$$U'(N) := \{g \in G'(\mathbb{A}_f) : g|_{\Lambda/N\Lambda} \equiv 1\}.$$

We now remove our restriction that the moduli problem above p is unramified.

Proposition 3.4.5. *If $U'_p(m)U'^p \subset U'(N)$ is a normal subgroup such that $N \geq 3$ and $m = 0$ when p is ramified in $E_{X'}$, then the functor $\mathcal{F}_{U'_p(m)U'^p}$ is representable by a normal scheme with reduced special fiber. Moreover, if p is unramified in $E_{X'}$, then the functor $\mathcal{F}_{U'_p(m)U'^p}$ is representable by a smooth scheme.*

Proof. Choose a normal subgroup $U_0'^p \subset U'^p$ sufficiently small so that the functor $\mathcal{F}_{U'_p(m)U_0'^p}$ is represented by the scheme $\mathcal{X}'_{U'_p(m)U_0'^p}$. There is an action of $U'^p \subset U'(N)^p$ on this scheme and it suffices to show that $U'(N)^p$ acts freely on $\mathcal{X}'_{U'_p(m)U_0'^p}$. A point $x \in \mathcal{X}'_{U'_p(m)U_0'^p}$ corresponds to a quintuple $(\mathcal{A}, \iota, \theta, \kappa, \alpha)$, where the full set of sections α is trivial when $m = 0$. Suppose that $g \in U'(N)^p$ fixes x. We may choose our $\mathcal{A}$ and κ so that $H_1(\mathcal{A}, \mathbb{A}_f^p) \cong \Lambda_{\mathbb{A}_f^p}$ and κ induces an isomorphism between the two. The element g acts by sending $(\mathcal{A}, \iota, \theta, \kappa, \alpha)g = (\mathcal{A}, \iota, \theta, g^{-1} \circ \kappa, \alpha)$. Thus, there exist some isomorphism f of $\mathcal{A}$ and element $g' \in U_0'^p$ such that $(gg')^{-1} \circ \kappa = \kappa \circ f_* : H_1(\mathcal{A}, \mathbb{A}_f^p) \to \Lambda_{\mathbb{A}_f^p}$. Now since $gg' \in U'(N)^p$ acts on the identity on $\Lambda/N\Lambda$, we get that f_* must act as the identity on $\mathcal{A}[N]$, meaning that f is the identity since $N \geq 3$. Thus $gg' = 1$ and $g^{-1} \circ \kappa$ is in the same $U_0'^p$ orbit as κ. □

Transferring Integral Models

Now we can use the integral model for X' to get integral models for X and X'' as done in [28, 27] by extending the adjoint group G^{ad} action on the neutral component of $\mathcal{X}'$. We briefly recall how this is done because we will use the same idea to transfer the p-divisible group on $\mathcal{X}'$ to p-divisible groups over $\mathcal{X}$ and $\mathcal{X}''$. Set $U_p(m) := (1 + p^m\mathcal{O}_{B,p})^\times$ and

$U''_p := U''_p(m) := (1 + p(\mathcal{O}_{B,p} \times \mathcal{O}_{E,p}))^\times \subset G''(\mathbb{Q}_p)$. Then $U_p := U_p(0)$ and $U''_p := U''_p(0)$ are the $\mathbb{Z}_p$ points of parahoric subgroups over $\mathbb{Z}_{(p)}$ which fix the lattices $\mathcal{O}_{B,p}$ and $\mathcal{O}_{B,p} \times \mathcal{O}_{E,p}$ respectively, and over the generic fiber they are isomorphic to G and G''. Denote these models G_p, G''_p so that $G_p(\mathbb{Z}_p) = U_p$ and $G''_p(\mathbb{Z}_p) = U''_p$. For $S = X, X', X''$, let $U_{S,p} = U_p, U'_p, U''_p$ and $U^p_S = U^p, U'^p, U''^p$ and $G_S = G, G', G''$ and $G_{S,p} = G_p, G'_p, G''_p$ respectively. Let Z_S be the center of G_S. For each choice of S, take the limit over all choices of S^p_U to get

$$S_{U_{S,p}} = \varprojlim_{U^p_S} S_{U_{S,p}U^p_S} = G_S(\mathbb{Q})\backslash(\mathcal{H}^\pm)^\Sigma \times G_S(\mathbb{A}_f)/U_{S,p},$$

and let the entire projective limit be

$$S = \varprojlim_{U_{S,p}U^p_S} S_{U_{S,p}U^p_S} = G_S(\mathbb{Q})\backslash(\mathcal{H}^\pm)^\Sigma \times G_S(\mathbb{A}_f).$$

We recall the star product notation of [28, 27]. Suppose that a group Δ acts on a group H and suppose that $\Gamma \subset H$ is Δ-stable. Let Δ act on itself by left conjugation and suppose there is a group homomorphism $\varphi : \Gamma \to \Delta$ that respects Δ-action. We also impose for all $\gamma \in \Gamma$ that the $\varphi(\gamma)$-action on H is by left conjugation by γ. Then the subgroup $\{(\gamma, \varphi(\gamma)^{-1}) : \gamma \in \Gamma\}$ is a normal subgroup of $H \rtimes \Delta$ and we let $H *_\Gamma \Delta$ be the quotient.

Let $G^{\mathrm{ad}}_S(\mathbb{R})^+$ be the neutral component and let $G_S(\mathbb{R})_+$ be the preimage of $G^{\mathrm{ad}}_S(\mathbb{R})^+$ under the map $G_S(\mathbb{R}) \to G^{\mathrm{ad}}_S(\mathbb{R})$. Let $G_S(\mathbb{Q})_+ = G_S(\mathbb{R})_+ \cap G_S(\mathbb{Q})$ and let $G_{S,p}(\mathbb{Z}_{(p)})_+ = G_{S,p}(\mathbb{Z}_{(p)}) \cap G_S(\mathbb{Q})_+$. Let $G^{\mathrm{ad}}_{S,p}(\mathbb{Z}_{(p)})^+ = G^{\mathrm{ad}}_{S,p}(\mathbb{Z}_{(p)}) \cap G^{\mathrm{ad}}_S(\mathbb{R})^+$. There is a natural right action of $G_S(\mathbb{A}_f)$ on S by right multiplication on the $G_S(\mathbb{A}_f)$ factor, on which $Z_S(\mathbb{Q})$ acts trivially. There is also a right action of $G^{\mathrm{ad}}_S(\mathbb{Q})^+$ on S where $\gamma \in G^{\mathrm{ad}}_S(\mathbb{Q})^+$ acts on a representative $[x, g]$ as $[x, g]\gamma = [\gamma^{-1}x, \gamma^{-1}g\gamma]$. Let $Z_S(\mathbb{Q})^-$ be the closure of $Z_S(\mathbb{Q})$ in $G_S(\mathbb{A}_f)$ so we get an action of

$$G_S(\mathbb{A}_f)/Z_S(\mathbb{Q})^- \rtimes G^{\mathrm{ad}}_S(\mathbb{Q})^+$$

on S. The subgroup $G_S(\mathbb{Q})_+/Z_S(\mathbb{Q})$ embeds into to both $G_S(\mathbb{A}_f)/Z_S(\mathbb{Q})^-$ and $G^{\mathrm{ad}}_S(\mathbb{Q})^+$ and its action on S is the same through both embeddings. Thus, we get an action of

$$\mathscr{A}(G_S) := G_S(\mathbb{A}_f)/Z_S(Q)^- *_{G_S(\mathbb{Q})_+/Z_S(\mathbb{Q})} G^{\mathrm{ad}}_S(\mathbb{Q})^+$$

on X. We also define a subgroup

$$\mathscr{A}(G_S)^\circ := G_S(\mathbb{Q})^-_+/Z_S(\mathbb{Q})^- *_{G_S(\mathbb{Q})_+/Z_S(\mathbb{Q})} G^{\mathrm{ad}}_S(\mathbb{Q})^+,$$

where $G_S(\mathbb{Q})^-_+$ is the closure of $G_S(\mathbb{Q})_+$ in $G_S(\mathbb{A}_f)$.

After taking the quotient of S by $U_{S,p} = G_{S,p}(\mathbb{Z}_p)$, we a right action of

$$\mathscr{A}(G_{S,p}) = G_S(\mathbb{A}^p_f)/Z_{S,p}(\mathbb{Z}_{(p)})^- *_{G_{S,p}(\mathbb{Z}_{(p)})_+/Z_{S,p}(\mathbb{Z}_{(p)})} G^{\mathrm{ad}}_{S,p}(\mathbb{Z}_{(p)})^+$$

on $S_{U_{S,p}}$, where $Z_{S,p}(\mathbb{Z}_{(p)})^-$ is the closure inside $G_S(\mathbb{A}^p_f)$. We also define a subgroup

$$\mathscr{A}(G_{S,p})^\circ = G_{S,p}(\mathbb{Z}_{(p)})^-_+/Z_{S,p}(\mathbb{Z}_{(p)})^- *_{G_{S,p}(\mathbb{Z}_{(p)})_+/Z_{S,p}(\mathbb{Z}_{(p)})} G^{\mathrm{ad}}_{S,p}(\mathbb{Z}_{(p)})^+.$$

Fix a geometrically connected component $S^+ \subset S$ as the image of the product of upper half planes $(\mathcal{H}^+)^\Sigma \times 1$ in the complex uniformization of S. Then take

$$S^+ = \varprojlim_{U_{S,p}U_S^p} S^+_{U_{S,p}U_S^p} = G_S^{\mathrm{der}}(\mathbb{Q})\backslash(\mathcal{H}^+)^\Sigma \times G_S^{\mathrm{der}}(\mathbb{A}_f),$$

and

$$S^+_{U_{S,p}} = \varprojlim_{U_S^p} S^+_{U_{S,p}U_S^p} = G_{S,p}^{\mathrm{der}}(\mathbb{Z}_{(p)})^-\backslash(\mathcal{H}^+)^\Sigma \times G_S^{\mathrm{der}}(\mathbb{A}_f^p).$$

Let E_S be the reflex field of S and let $E_S^p \subset \overline{E_S}$ be the maximal extension of E that is unramified over all primes dividing p. The connected component S^+ is defined over $\overline{E_S}$ and $S^+_{U_{S,p}}$ is defined over E_S^p. Let

$$\mathscr{E}(G_S) \subset \mathscr{A}(G_S) \times \mathrm{Gal}(\overline{E_S}/E_S)$$

be the stabilizer of S^+ and let

$$\mathscr{E}(G_{S,p}) \subset \mathscr{A}(G_{S,p}) \times \mathrm{Gal}(E_S^p/E_S)$$

be the stabilizer of $S^+_{U_{S,p}}$. Then, we have the following.

Proposition 3.4.6 ([28, Lem. 3.3.7])**.** *The stabilizer $\mathscr{E}(G_S)$ (resp. $\mathscr{E}(G_{S,p})$) depends only on G_S^{der} (resp. $G_{S,p}^{\mathrm{der}}$) and X^{ad} and it is an extension of* $\mathrm{Gal}(\overline{E_S}/E_S)$ *(resp.* $\mathrm{Gal}(E_S^p/E_S)$*) by $\mathscr{A}(G_S)^\circ$ (resp. $\mathscr{A}(G_{S,p})^\circ$). Moreover, there is a canonical isomorphism*

$$\mathscr{A}(G_S) *_{\mathscr{A}(G_S)^\circ} \mathscr{E}(G_S) \cong \mathscr{A}(G_S) \times \mathrm{Gal}(\overline{E_S}/E_S)$$

and

$$\mathscr{A}(G_{S,p}) *_{\mathscr{A}(G_{S,p})^\circ} \mathscr{E}(G_{S,p}) \cong \mathscr{A}(G_{S,p}) \times \mathrm{Gal}(E_S^p/E_S).$$

There is a right action of $\mathscr{E}(G_S)$ on $\mathscr{A}(G_S) \times S^+$ given by right conjugation via the map $\mathscr{E}(G_S) \to \mathscr{A}(G_S) \times \mathrm{Gal}(\overline{E_S}/E_S) \to \mathscr{A}(G_S)$ on the first factor and right multiplication on the second factor. There is also an action of $\mathscr{A}(G_S)$ on $\mathscr{A}(G_S) \times S^+$ defined by right multiplication on the first factor and ignoring the second factor. Thus, there is an action of $\mathscr{A}(G_S) *_{\mathscr{A}(G_S)^\circ} \mathscr{E}(G_S) \cong \mathscr{A}(G_S) \times \mathrm{Gal}(\overline{E_S}/E_S)$ on $[\mathscr{A}(G_S) \times S^+]/\mathscr{A}(G_S)^\circ$. Similarly, we can define an action of $\mathscr{A}(G_{S,p}) *_{\mathscr{A}(G_{S,p})^\circ} \mathscr{E}(G_{S,p}) \cong \mathscr{A}(G_{S,p}) \times \mathrm{Gal}(E_S^p/E_S)$ on $S^+_{U_{S,p}}$.

Proposition 3.4.7 ([28, Prop 3.3.10])**.** *For $S, S' \in \{X, X', X''\}$, there is an isomorphism of E_S^p schemes*

$$S'_{U_{S',p}} \cong [\mathscr{A}(G_{S',p}) \times S^+_{U_{S,p}}]/\mathscr{A}(G_{S,p})^\circ$$

that respects $\mathrm{Gal}(E_{S'}^p/E_{S'})$ *action where the Galois group acts on the right side via the isomorphism* $\mathscr{A}(G_{S',p}) *_{\mathscr{A}(G_{S',p})^\circ} \mathscr{E}(G_{S',p}) \cong \mathscr{A}(G_{S',p}) \times \mathrm{Gal}(E_{S'}^p/E_{S'})$.

Here, we have that $\mathscr{A}(G_{S,p})^\circ$ acts on $\mathscr{A}(G_{S',p})$ via $\mathscr{A}(G_{S,p})^\circ \cong \mathscr{A}(G_{S',p})^\circ \to \mathscr{A}(G_{S',p})$. If there is an integral model $\mathcal{S}_{U_{S,p}}$ for S such that the action of G^{ad} extends to it, then this isomorphism gives us a way to transfer it to all other S' by simply defining

$$\mathcal{S'}_{U_{S',p}} := [\mathscr{A}(G_{S',p}) \times \mathcal{S}^+_{U_{S,p}}]/\mathscr{A}(G_{S,p})^\circ$$

as $\mathcal{O}_{E^p_{S'}}$-schemes and then using Galois descent to descend an $\mathcal{O}_{E_{S'}}$-scheme.

Theorem 3.4.8. *Let $v \mid p$ be a prime of E_X and $v'' \mid p$ be a prime of $E_{X''}$. If $U^p \subset G(\mathbb{A}_f^p)$ and $U''^p \subset G''(\mathbb{A}_f^p)$ are such that $U_pU^p \subset U(N)$ and $U''_pU''^p \subset U''(N)$ for some $N \geq 3$, then there is a projective system of integral models $\mathcal{X}_{U_pU^p}$ (resp. $\mathcal{X}''_{U''_pU''^p}$) of $X_{U_pU^p}$ (resp. $X''_{U''_pU''^p}$) over $\mathcal{O}_{E_X,v}$ (resp. $\mathcal{O}_{E_{X''},v''}$) such that:*

1. *If p is not ramified in F nor B, the scheme $\mathcal{X}_U$ (resp. $\mathcal{X}''_{U''}$) is smooth over $\mathcal{O}_{E_X,v}$ (resp. $\mathcal{O}_{E_{X''},v''}$);*
2. *The local rings of the integral models are étale locally isomorphic to the local rings of $\mathbb{M}^{\mathrm{loc}}$.*
3. *The p-adic completion of $\mathcal{X}_U$ (resp. $\mathcal{X}''_{U''}$) has a p-adic uniformization by a Rapoport–Zink space.*

Proof. When $E_{X',v'}$ is unramified over $\mathbb{Q}_p$, then the group G' has a hyperspecial local model over $\mathcal{O}_{E_{X'},v'}$. Then by [28, Lem. 3.4.5], the extension property implies that the action of $\mathscr{A}(G')$ extends to the integral model $\mathcal{X}'_{U'_p}$. Let $\mathcal{X'}^+_{U'}$ be the closure of $X'^+_{U'}$ in $\mathcal{X}'_{U'}$ and let

$$\mathcal{X'}^+_{U'_p} := \varprojlim_{U'^p} \mathcal{X'}^+_{U'_pU'^p}.$$

We then define

$$\mathcal{X}_{U_p} = [\mathscr{A}(G_p) \times \mathcal{X'}^+_{U'_p}]/\mathscr{A}(G'_p)^\circ$$

and

$$\mathcal{X''}_{U''_p} = [\mathscr{A}(G''_p) \times \mathcal{X'}^+_{U'_p}]/\mathscr{A}(G'_p)^\circ.$$

The action of $\mathscr{E}(G'_p)$ on $X'^+_{U'_p}$ extends to $\mathcal{X'}^+_{U'_p}$ and hence we get an action of

$$\mathscr{A}(G_p) *_{\mathscr{A}(G'_p)^\circ} \mathscr{E}(G'_p) \cong \mathscr{A}(G_p) \times \mathrm{Gal}(E^p_X/E_X)$$

on $\mathcal{X}_{U_p}$ (resp. $\mathscr{A}(G''_p) \times \mathrm{Gal}(E^p_{X''}/E_{X''})$ on $\mathcal{X}''_{U''_p}$). Thus, we can use the Galois action to descend this scheme to a integral model for X_{U_p} (resp. $X''_{U''_p}$) defined over $\mathcal{O}_{E_X,v}$ (resp. $\mathcal{O}_{E_{X''},v''}$. When p is not ramified in F nor B, then the integral model $\mathcal{X}'_{U'_p}$ corresponds to unramified PEL datum and is smooth.

When p is ramified, then G'_p is no longer hyperspecial and there is no extension property. However, the action of $\mathscr{E}(G'_p)$ still extends to $\mathcal{X'}^+_{U'_p}$ by [27, Cor. 4.6.15]. The statements about the local rings and p-adic uniformization follow from them holding for $\mathcal{X}'_{U'_p}$. $\square$

By gluing these models together, we have an integral model $\mathcal{X}_U$ over $\operatorname{Spec}\mathcal{O}_{E_X}$ for whenever $U = \prod_p U_p \subset U(N)$ for $N \geq 3$ and $U_p = U_p(0)$ is maximal whenever p is ramified in E_X. We now extend the integral model for $\mathcal{X}_U$ when $U = \prod_p U_p$ is maximal at all primes. Take a prime p that is not ramified in E_X such that $U(p) = U_p(1)U^p$ is maximal at all primes away from p. Then define

$$\mathcal{X}_U := \mathcal{X}_{U(p)}/(U/U(p))$$

as the quotient stack. Since the $\mathcal{X}_{U_pU^p}$ form a projective system, the definition of $\mathcal{X}_U$ does not depend on the choice of prime p.

3.5 p-Divisible Groups

When $U' = U'_pU'^p \subset U'(N)$, the functor $\mathcal{F}'_{U'}$ is represented by $\mathcal{X}'_{U'}$ and so we get a universal abelian scheme $\mathcal{A}'_{U'} \to \mathcal{X}'_{U'}$ lying over it. We use the ideas of [28, 27] to transport the p-divisible group $\mathcal{H}'_{U'} := \mathcal{A}'_{U'}[p^\infty]$ to p-divisible groups over $\mathcal{X}_U$ and $\mathcal{X}''_{U''}$. In order to do so, we first give a description of $\mathcal{H}'_{U'}$ over $X'^+_{U'}$ and then describe an action of $\mathscr{E}(G')$ on it.

Recall that $\mathscr{A}(G')^\circ$ depends only on the derived group $G'^{\mathrm{der}} = \operatorname{Res}_{F/\mathbb{Q}} B^1$, elements of norm 1. The center is $Z(G'^{\mathrm{der}}) = \operatorname{Res}_{F/\mathbb{Q}} F^1$, elements of F of norm 1. By Shapiro's lemma, the adjoint map $G'(\mathbb{Q}) \to G'^{\mathrm{ad}}(\mathbb{Q})$ is surjective and so we can write

$$\mathscr{A}(G')^\circ = B^1/F^1$$

where F^1 is the elements of F with norm 1. We can also determine $\mathscr{A}(G_S)$ for $S = X, X', X''$. We have that $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+ = \mathcal{O}^{\times,+}_{B,(p)}/\mathcal{O}^{\times,+}_{F,(p)}$, where the $+$ superscript denotes the elements with norm that is totally positive in F. Also $G''^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+ = (\mathcal{O}^{\times,+}_{B,(p)} \times_{\mathcal{O}^{\times,+}_{F,(p)}} \mathcal{O}^{\times,+}_{E,(p)})/\mathcal{O}^{\times,+}_{E,(p)} \cong \mathcal{O}^{\times,+}_{B,(p)}/\mathcal{O}^{\times,+}_{F,(p)}$ because the norm of all elements of $E^\times$ are totally positive in F. Thus, we get that

$$\mathscr{A}(G) = G(\mathbb{A}_f)/Z(\mathbb{Q})^- *_{G(\mathbb{Q})_+/Z(\mathbb{Q})} G^{\mathrm{ad}}(\mathbb{Q})^+ = (B \otimes_{\mathbb{Q}} \mathbb{A}_f)^\times/\mathcal{O}^{\times,-}_F,$$

and

$$\mathscr{A}(G_p) = G(\mathbb{A}^p_f)/Z(\mathbb{Z}_{(p)})^- *_{G(\mathbb{Z}_{(p)})_+/Z(\mathbb{Z}_{(p)})} G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+ = (B^p)^\times/\mathcal{O}^{\times,-}_{F,(p)},$$

where $B^p := B \otimes_{\mathbb{Q}} \mathbb{A}^p_f$. Similarly, we get that $\mathscr{A}(G'') = G''(\mathbb{A}_F)/\mathcal{O}^{\times,-}_E$ and $\mathscr{A}(G''_p) = G''(\mathbb{A}^p_f)/\mathcal{O}^{\times,-}_{E,(p)}$. Shapiro's lemma doesn't apply to G', but we can write it as

$$\mathscr{A}(G') = G'(\mathbb{A}_f)B^{\times,+}/E^{\times,-}$$

and

$$\mathscr{A}(G'_p) = G'(\mathbb{A}^p_f)G''(\mathbb{Z}_{(p)})_+/\mathcal{O}^{\times,-}_{E,(p)}.$$

Finally, we have that

$$\mathscr{A}(G'_p)^\circ \cong \mathscr{A}(G^{\mathrm{der}}_p)^\circ \cong \mathcal{O}^1_{B,(p)}/\mathcal{O}^1_{F,(p)}$$

Over X'^{+}, we can write the p-divisible group $\mathcal{H}'$ as

$$H'|_{X'^{+}} = B_p/\mathcal{O}_{B,p} \times X'^{+} = B_p/\mathcal{O}_{B,p} \times B^1\backslash(\mathcal{H}^{+})^{\Sigma} \times (B \otimes_{\mathbb{Q}} \mathbb{A}_f)^1.$$

To descend down to $X'^{+}_{U'_p}$, we quotient by the action of $G_p'^{\mathrm{der}}(\mathbb{Z}_p) = \mathcal{O}^1_{B,p}$ to get

$$H'^{+}_{U'_p} := H'|_{X'^{+}_{U'_p}} = [B_p/\mathcal{O}_{B,p} \times X'^{+}]/\mathcal{O}^1_{B,p},$$

where $U'_p = G_p'^{\mathrm{der}}(\mathbb{Z}_p) = \mathcal{O}^1_{B,p}$ acts on $B_p/\mathcal{O}_{B,p}$ by right multiplication.

We can now describe H' over all of X'. Under the isomorphism of $\overline{E}$-schemes

$$X' \cong [\mathscr{A}(G') \times X'^{+}]/\mathscr{A}(G')^{\circ} = [\mathscr{A}(G') \times X'^{+}]/B^1,$$

the p-divisible group can be written as

$$H'|_{X'} \cong B_p/\mathcal{O}_{B,p} \times [\mathscr{A}(G') \times X'^{+}]/B^1.$$

After dividing by $G'_p(\mathbb{Z}_p)$, we get

$$H'_{U'_p}|_{X'_{U'_p}} \cong [\mathscr{A}(G'_p) \times [B_p/\mathcal{O}_{B,p} \times X'^{+}]/\mathcal{O}^1_{B,p}]/\mathcal{O}^1_{B,(p)} \cong [\mathscr{A}(G'_p) \times H'^{+}_{U'_p}]/\mathcal{O}^1_{B,(p)},$$

where $\mathscr{A}(G'_p)^{\circ} = \mathcal{O}^1_{B,(p)}/\mathcal{O}^1_{F,(p)} \subset (B \otimes_{\mathbb{Q}} \mathbb{A}^p_f)^{\times}/(F \otimes_{\mathbb{Q}} \mathbb{A}^p_f)^{\times}$ acts trivially on $B_p/\mathcal{O}_{B,p}$. In this way, we can also define p-divisible groups $H_{U_p}, H''_{U''_p}$ over $X_{U_p}, X''_{U''_p}$ respectively as

$$\begin{aligned} H_{U_p}|_{X_{U_p}} &\cong \big(\mathscr{A}(G_p) \times [B_p/\mathcal{O}_{B,p} \times X'^{+}]/\mathcal{O}^1_{B,p}\big)/\mathscr{A}(G'_p)^{\circ} \\ &\cong \Big(B^{p,\times}/\mathcal{O}^{\times}_{F,(p)} \times H'^{+}_{U'_p}\Big)/\mathcal{O}^1_{B,(p)}, \end{aligned}$$

and

$$\begin{aligned} H''_{U''_p}|_{X''_{U''_p}} &\cong \big(\mathscr{A}(G''_p) \times [B_p/\mathcal{O}_{B,p} \times X'^{+}]/\mathcal{O}^1_{B,p}\big)/\mathscr{A}(G'_p)^{\circ} \\ &\cong \Big(G''(\mathbb{A}^p_f)/\mathcal{O}^{\times}_{E,(p)} \times H'^{+}_{U'_p}\Big)/\mathcal{O}^1_{B,(p)}, \end{aligned}$$

Theorem 3.5.1. *Whenever $U \subset U(N)$ (resp. $U'' \subset U''(N)$) for $N \geq 3$, there exists a p-divisible group $\mathcal{H}_U$ over $\underline{\mathcal{X}_U}$ (resp. $\mathcal{H}''_{U''}$ over $\mathcal{X}''_{U''}$) such that the formal completion $\widehat{\mathcal{X}_U}$ along its special fiber over $\overline{k_{E_X,v}}$ is the universal deformation space of $\mathcal{H}_{\overline{k_{E_X,v}}}$.*

Proof. We have already translated the p-divisible group $\mathcal{H}'_{U'_p}$ to p-divisible groups over the generic fiber X_{U_p} and $X''_{U''_p}$. Over $\mathcal{X}'_{U'_p}$, we have an integral model for $\mathcal{H}'_{U'_p}$ by taking the p^{∞} torsion of the universal abelain scheme $\mathcal{A}'_{U'_p} \to \mathcal{X}'_{U'_p}$. We can restrict it to the connected component to get

$$\mathcal{H}'^{+}_{U'_p} := \mathcal{H}'_{U'_p}|_{\mathcal{X}'^{+}_{U'_p}}.$$

Set

$$\mathcal{H}_{U_p} := \left(\mathscr{A}(G_p) \times \mathcal{H}'^{+}_{U'_p}\right)/\mathscr{A}(G'_p)^\circ,$$

and

$$\mathcal{H}''_{U''_p} := \left(\mathscr{A}(G''_p) \times \mathcal{H}'^{+}_{U'_p}\right)/\mathscr{A}(G'_p)^\circ.$$

The action of $\mathscr{E}(G_p^{\mathrm{der}})$ on $\mathcal{X}'^{+}_{U'_p}$ extends to an action on $\mathcal{H}'^{+}_{U'_p}$ by acting trivially on $B_p/\mathcal{O}_{B,p}$ and thus there is an action of $\mathscr{A}(G_p) \times \mathrm{Gal}(E_X^p/E_X)$ on $\mathcal{H}_{U_p}$ that is compatible with the structure morphism $\mathcal{H}_{U_p} \to \mathcal{X}_{U_p}$. Hence, we can descend $\mathcal{H}_{U_p}$ and $\mathcal{H}''_{U''_p}$ down to p-divisible groups defined over $\mathcal{O}_{E_X,v}$ and $\mathcal{O}_{E_{X''},v''}$ respectively whose generic fibers can be identified with H_{U_p} and $H''_{U''_p}$ in a way respecting the structure morphism down to X_{U_p} and $X''_{U''_p}$. For finite level, we can simply take $\mathcal{H}_{U_pU^p} = \mathcal{H}_{U_p}/U^p$ where U^p acts below on $\mathcal{X}_{U_p}$ and acts trivially on the fibers of $\mathcal{H}_{U_p} \to \mathcal{X}_{U_p}$.

The statement about universal deformation spaces follows from $\mathcal{X}'_{U'}$ representing the functor of isomorphism classes of abelian schemes whose p^∞-part is $\mathcal{H}'_{U'}$. □

3.6 Hodge Bundles

In order to calculate the height of a partial CM point, we will take the height of a special point of $\mathcal{X}_U$ with respect to metrized Hodge bundle $\widehat{\mathcal{L}_U}$ on $\mathcal{X}_U$, which we will introduce. Following [76], we define this system $\{\mathcal{L}_U\}_U$ as the canonical bundle

$$\mathcal{L}_U := \omega_{\mathcal{X}_U/\mathcal{O}_{E_X}}.$$

The benefit of using the canonical bundle is that the system $\{\mathcal{L}_U\}_U$ is compatible with pullbacks along the canonical maps $\mathcal{X}_{U_1} \to \mathcal{X}_{U_2}$ with $U_1 \subset U_2$. We call $\mathcal{L}_U$ the *Hodge bundle* of $\mathcal{X}_U$. At the infinite places, the metric is given by

$$\left\| \bigwedge_{\sigma\in\Sigma} dz_\sigma \right\| = \prod_{\sigma\in\Sigma} \mathrm{Im}(2z_i).$$

We now relate this line bundle with our p-divisible group. Let $S = \operatorname{Spec} \mathcal{O}_{E_X^p,p}$ be the ring of integers of the maximal extension of E_X unramified over p. Set $\Omega(\mathcal{H}_S) := \mathrm{Lie}(\mathcal{H}_S)^\vee$ and $\Omega(\mathcal{H}_S^t) := \mathrm{Lie}(\mathcal{H}_S^t)^\vee$. Let $\mathbb{D}(\mathcal{H}_S)$ and $\mathbb{D}(\mathcal{H}_S^t)$ be the covariant Dieudonné crystals attached to the p-divisible groups. Then [41, Chap. IV] gives us a short exact sequence

$$0 \to \mathrm{Lie}(\mathcal{H}_S^t)^\vee \to \mathbb{D}(\mathcal{H}_S) \to \mathrm{Lie}(\mathcal{H}_S) \to 0$$

of $\mathcal{O}_S \otimes \mathcal{O}_E$ modules. Applying the Gauss–Manin connection on $\mathbb{D}(\mathcal{H}_S)$ gives the chain of maps

$$\Omega(\mathcal{H}_S^t) \to \mathbb{D}(\mathcal{H}_S) \xrightarrow{\nabla} \mathbb{D}(\mathcal{H}_S) \otimes \Omega^1_{\mathcal{X}_S/S} \to \Omega(\mathcal{H}_S)^\vee \otimes \Omega^1_{\mathcal{X}_S/S},$$

which gives a map

$$\Omega^{1,\vee}_{\mathcal{X}_S/S} \to \mathrm{Hom}(\Omega(\mathcal{H}_S^t), \Omega(\mathcal{H}_S)^\vee).$$

Both $\Omega(\mathcal{H}_S)^\vee$ and $\Omega(\mathcal{H}_S^t)^\vee$ have an action by $\mathcal{O}_E$ whose determinant is the product of the reflex norms of $\phi \sqcup \phi'$ and $\overline{\phi} \sqcup \phi'$. We can thus decompose the line bundles over S as

$$\Omega(\mathcal{H}_S)^\vee \to \bigoplus_{\tau \in \mathrm{Hom}(E,\overline{\mathbb{Q}})} \Omega(\mathcal{H}_S)^\vee_\tau.$$

Let $\omega(\mathcal{H}_S)_\tau := \det \Omega(\mathcal{H}_S)_\tau$. We define $\Omega(\mathcal{H}_S^t)_\tau$ and $\omega(\mathcal{H}_S^t)_\tau$ similarly. The dimension of $\Omega(\mathcal{H}_S)_\tau$ is 1 if $\tau \in \phi$, 2 if $\tau \in \phi'$, and 0 if $\tau \in \overline{\phi'}$. The dimension of $\Omega(\mathcal{H}_S^t)_\tau$ is $2 - \dim \Omega(\mathcal{H}_S)_\tau$. Thus, we get a map

$$\Omega^{1,\vee}_{\mathcal{X}_S/S} \to \mathrm{Hom}(\Omega(\mathcal{H}_S^t), \Omega(\mathcal{H}_S)^\vee) \to \bigoplus_{\tau\in\phi} \omega(\mathcal{H}_S^t)^\vee_\tau \otimes \omega(\mathcal{H}_S)^\vee_\tau.$$

Taking the determinant of this map and then repeating the map for $\overline{\phi}$ gives the map

$$\omega^{-2}_{\mathcal{X}_S/S} \to \otimes_{\tau\in\phi} \mathcal{N}(\mathcal{H}_S, \tau)^\vee \otimes \mathcal{N}(\mathcal{H}_S, \overline{\tau})^\vee,$$

where $\mathcal{N}(\mathcal{H}_S, \tau) \cong \omega(\mathcal{H}_S)_\tau \otimes \omega(\mathcal{H}_S^t)_{\overline{\tau}}$.

Note that $E_X = E_{\phi\sqcup\overline{\phi}}$ is the reflex field of the set $\phi \sqcup \overline{\phi}$. Let $\mathfrak{d}_{\Sigma,p}$ be the p-part of the relative discriminant of the image of $\mathcal{O}_{E_X,p} \otimes_{\mathbb{Z}} \mathcal{O}_E$ in $\widetilde{E_{\phi\sqcup\overline{\phi},p}}$, over $\mathcal{O}_{E_X,p}$. View $\mathfrak{d}_{\Sigma,p}$ as a divisor of S. Moreover, let $\mathfrak{d}_{B,p}$ be the divisor corresponding to the ramification of B.

Theorem 3.6.1. *For any choice of partial CM-type ϕ lying above Σ, we have*

$$\omega^2_{\mathcal{X}_S/S}(\mathfrak{d}_{\Sigma,p}\mathfrak{d}_{B,p})^{-1} \cong \bigotimes_{\tau\in\phi} \mathcal{N}(\mathcal{H}_S, \tau) \otimes \mathcal{N}(\mathcal{H}_S, \overline{\tau}).$$

Proof. We have the short exact sequence

$$0 \to \mathrm{Lie}(\mathcal{H}_S^t)^\vee \to \mathbb{D}(\mathcal{H}_S) \to \mathrm{Lie}(\mathcal{H}_S) \to 0$$

of $\mathcal{O}_S \otimes \mathcal{O}_E$ modules. Since the formal completion of $\mathcal{X}_S/S$ along its special fiber over $\overline{k(S)}$ is the universal deformation space of $\mathcal{H}_S$, [41] gives us that the tangent bundle $\Omega^{1,\vee}_{\mathcal{X}_S/S}$ corresponds to choosing a lift of $\mathrm{Lie}(\mathcal{H}_S^t)^\vee$ and $\mathrm{Lie}(\mathcal{H}_S)$ in $\mathbb{D}(\mathcal{H}_S)_{S'}$, where $S' = \operatorname{Spec} \mathcal{O}_{E_X^p,p}[\varepsilon]/(\varepsilon^2)$, that respects the pairing from ψ and $\mathcal{O}_S \otimes \mathcal{O}_E$ action.

For each $\tau \in \mathrm{Hom}(E, \overline{\mathbb{Q}})$, we can take the τ component of the short exact sequence to get

$$0 \to \Omega(\mathcal{H}_S^t)_\tau \to \mathbb{D}(\mathcal{H}_S)_\tau \to \Omega(\mathcal{H}_S)^\vee_\tau \to 0.$$

For τ lying above Σ^c, either $\Omega(\mathcal{H}_S^t)_\tau$ or $\Omega(\mathcal{H}_S)_\tau$ is 0 meaning that there is only one choice for a lift of $\Omega(\mathcal{H}_S^t)_\tau$ and $\Omega(\mathcal{H}_S)_\tau$. For τ lying above Σ, both are of dimension 1. The pairing

$$\mathbb{D}(\mathcal{H}_S)_{S'} \times \mathbb{D}(\mathcal{H}_S^t)_{S'} \to S'$$

decomposes into an orthogonal sum of pairings

$$\mathbb{D}(\mathcal{H}_S)_{S',\tau} \times \mathbb{D}(\mathcal{H}_S^t)_{S',\overline{\tau}} \to S'.$$

Thus, choosing a lift of $\Omega(\mathcal{H}_S)_\tau$ determines the choice for $\Omega(\mathcal{H}_S)_{\overline{\tau}}$ under the canonical isomorphism $\mathbb{D}(\mathcal{H}_S^t) \cong \mathbb{D}(\mathcal{H}_S)^\vee$. The Hodge filtration gives us that the choice of lift of $\Omega(\mathcal{H}_S^t)_\tau$ is a torsor of $\mathrm{Hom}(\Omega(\mathcal{H}_S^t)_\tau, \Omega(\mathcal{H}_S)_\tau^\vee)$ giving us a map with finite cokernel

$$\Omega^{1,\vee}_{\mathcal{X}_S/S} \subset \mathrm{Hom}(\Omega(\mathcal{H}_S^t), \Omega(\mathcal{H}_S)^\vee) \to \bigoplus_{\tau\in\phi} \mathrm{Hom}(\Omega(\mathcal{H}_S^t)_\tau, \Omega(\mathcal{H}_S)_\tau^\vee).$$

Taking determinants, we get a map

$$\omega^{-1}_{\mathcal{X}_S/S} \to \bigotimes_{\tau\in\phi} \omega(\mathcal{H}_S^t)_\tau^\vee \otimes \omega(\mathcal{H}_S)_\tau^\vee.$$

If we choose $\overline{\phi}$ instead of ϕ, we get the same morphism with $\overline{\tau}$ and tensoring the two gives

$$\omega^{-2}_{\mathcal{X}_S/S} \to \bigotimes_{\tau\in\phi} \omega(\mathcal{H}_S^t)_\tau^\vee \otimes \omega(\mathcal{H}_S)_\tau^\vee \otimes \omega(\mathcal{H}_S^t)_{\overline{\tau}}^\vee \otimes \omega(\mathcal{H}_S)_{\overline{\tau}}^\vee.$$

Let $\pi \in S$ be a generator for S over $\mathcal{O}_{E_X,p}$. For a subset $\Psi \subset \mathrm{Hom}(E, \mathbb{C})$, let

$$f_\Psi(t) = \prod_{\tau\in\Psi}(t - \tau(\pi)).$$

We see that $f_{\phi\cup\overline{\phi}}(t) \in \mathcal{O}_{E_X,p}[t]$ because it is invariant under any automorphism that fixes the underlying places of F under ϕ. Then we see that the image of $\mathcal{O}_{E_X,p} \otimes_{\mathbb{Z}} \mathcal{O}_E$ in $\widetilde{E_{\phi\sqcup\overline{\phi},p}}$ is simply $\mathcal{O}_{E_X,p}[t]/f_{\phi\sqcup\overline{\phi}}(t)$ and so $\mathfrak{d}_{\phi\sqcup\overline{\phi},p} \subset \mathcal{O}_{E_X,p}$ is the ideal generated by the discriminant of $f_{\phi\sqcup\overline{\phi}}$,

By [75, Cor. 2.5], we have that

$$\Omega(\mathcal{H}_S) \cong \frac{\mathcal{O}_S[t]}{f_{\phi\sqcup\phi'}(t)f_{\overline{\phi}\sqcup\phi'}(t)}, \qquad \Omega(\mathcal{H}_S^t) \cong \frac{\mathcal{O}_S[t]}{f_{\overline{\phi}\sqcup\overline{\phi'}}(t)f_{\phi\sqcup\overline{\phi'}}(t)}.$$

Each element of the tangent bundle corresponds to an element of $\Omega(\mathcal{H}_S^t)^\vee \otimes \Omega(\mathcal{H}_S)^\vee$. So, the image of $\omega^{-2}_{\mathcal{X}_S/S}$ is the determinant of the image of $\Omega(\mathcal{H}_S^t) \otimes \Omega(\mathcal{H}_S)$ in

$$\prod_{\tau\in\phi\cup\overline{\phi}} \Omega(\mathcal{H}_S^t)_\tau \otimes \Omega(\mathcal{H}_S)_\tau \cong \prod_{\tau\in\phi\sqcup\overline{\phi}} \mathcal{O}_S[t]/(t-\tau(\pi)) \otimes \mathcal{O}_S[t]/(t-\tau(\pi)).$$

by mapping t to $(t, t, \ldots, t)$. We deal with $\Omega(\mathcal{H}_S)$ and $\Omega(\mathcal{H}_S^t)$ separately. A basis for $\Omega(\mathcal{H}_S)$ is given by $1, t, \ldots, t^{2|\phi|-1}$, so the lattice formed by the image of $\Omega(\mathcal{H}_S)$ in $\prod_{\tau\in\phi\sqcup\overline{\phi}} \Omega(\mathcal{H}_S)_\tau \cong \prod_{\tau\in\phi\sqcup\overline{\phi}} \mathcal{O}_{S,\tau}$ is generated by $\{(\tau(\pi)^i)_{\tau\in\phi\cup\overline{\phi}}\}_{0\le i<2|\phi|}$. To calculate the index of this lattice relative to the maximal lattice, we take the determinant of a $2|\phi| \times 2|\phi|$ matrix whose ij-th element is $\tau_i(\pi)^j$. The ideal generated by the determinant of this Vandermonde matrix is

$$\left(\prod_{1 \le i < j \le 2|\phi|} |\tau_j(\pi) - \tau_i(\pi)| \right) = \mathfrak{d}_{\Sigma,p}^{1/2}.$$

Doing the same for $\Omega(\mathcal{H}_S^t)$ nets an additional factor of $\mathfrak{d}_{\Sigma,p}^{1/2}$.

Finally, when B is ramified, the pairing $\mathbb{D}(\mathcal{H}_S)_{S'} \times \mathbb{D}(\mathcal{H}_S^t)_{S'} \to S'$ is not perfect but rather $\Lambda^\vee = \frac{1}{\omega_\mathfrak{q}}\Lambda$, meaning our choice in $\Omega(\mathcal{H}_S)$ must lie in $\omega_\mathfrak{q}\Omega(\mathcal{H}_S)$, giving an additional factor of $(\omega_\mathfrak{q}) = \mathfrak{d}_{B,p}^{1/2}$ since $\mathfrak{p}$ was specified to be ramified wherever B was. Doing the same for $\mathcal{H}_S^t$ gets us another factor of $\mathfrak{d}_{B,p}^{1/2}$. □

Recall that the p-divisible group $\mathcal{H}_S$ originated from the p-infinity torsion of a universal abelian scheme $\mathcal{A}'_{U'}$ over $\mathcal{X}'_{U'}$ since the connected components of $\mathcal{X}'_{U'}$ and $\mathcal{X}_U$ are isomorphic. Thus for each archimedean place of S, we can use the same norm from the Hermitian pairing

$$\|\cdot\| \colon W(A'_{U'}, \tau) \otimes W(A'^t_{U'}, \overline{\tau}) \to \mathbb{C}$$

to make

$$\widehat{\mathcal{N}(\mathcal{H}_S, \tau)} := \left(\omega(\mathcal{H}_S)_\tau \otimes \omega(\mathcal{H}_S^t)_{\overline{\tau}}, \|\cdot\|\right)$$

into a metrized line bundle.

Theorem 3.6.2. *The Kodaira–Spencer isomorphism in Theorem 3.6.1 respects the norms at infinity and hence extends to an isomorphism of metrized line bundles*

$$\widehat{\mathcal{L}_U}^2 (\mathfrak{d}_{\Sigma,p}\mathfrak{d}_{B,p})^{-1} \cong \bigotimes_{\tau \in \phi} \widehat{\mathcal{N}(\mathcal{H}_S, \tau)} \otimes \widehat{\mathcal{N}(\mathcal{H}_S, \overline{\tau})}.$$

Proof. At the places at infinity, the Dieudonné module $\mathbb{D}(\mathcal{H}_S)$ is naturally isomorphic to the first de Rham homology of $A \to S$. Thus, the Kodaira–Spencer morphism comes from the Hodge filtration

$$0 \to \Omega(A^t/S) \to H_1^{\mathrm{dR}}(A/S) \to \Omega(A/S) \to 0.$$

For each $\tau \colon E \to \mathbb{C}$, we can look at the τ-component of the filtration

$$0 \to \Omega(A^t/S)_\tau \to H_1^{\mathrm{dR}}(A/S)_\tau \to \Omega(A/S)_\tau \to 0$$

For $\tau \colon E \to \mathbb{C}$ lying above Σ^c, there is no contribution from either line bundle so we can restrict ourselves to considering the τ lying above Σ, and specifically for $\tau \in \phi$. For these τ, we have

$$\Omega(A^t/S)_\tau \to H_1^{\mathrm{dR}}(A/S)_\tau \xrightarrow{\nabla} H_1^{\mathrm{dR}}(A/S)_\tau \otimes \Omega^1_{X_S/S} \to \Omega(A/S)^\vee \otimes \Omega^1_{X_S/S}.$$

Explicitly, we have that $H_1^{\mathrm{dR}}(A/S)_\tau \cong V_\tau \cong B \otimes_{E,\tau} \mathbb{C}$. We can choose an isomorphism $B \otimes_{E,\tau} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$ so that $h(i)_\tau \subset M_2(\mathbb{R})$ acts on V_τ via right transpose action $h(i) \cdot (z_1, z_2) = (z_1, z_2)\overline{h(i)}$. Then in terms of the complex uniformization, an element $z = x + iy \in \mathcal{H}_\tau^\pm$ in the complex half planes corresponds to a conjugate of $h(i)$ and $\Omega(A^t/S)_\tau \cong V_\tau^{0,-1}$ is the subset of $\mathbb{C}^2$ for which $h(i)$ acts as $-i$. Computation shows that $\Omega(A^t/S)_\tau \cong \mathbb{C}(z, 1) \subset \mathbb{C}^2$. Moreover, we have that $\mathrm{Lie}(A/S) = V^{-1,0} \cong \mathbb{C}(\overline{z}, 1) \subset \mathbb{C}^2$. Thus, explicitly, the map above gives

$$\begin{array}{ccccccc} \Omega(A^t/S)_\tau & \longrightarrow & H_1^{\mathrm{dR}}(A/S)_\tau & \xrightarrow{\nabla} & H_1^{\mathrm{dR}}(A/S)_\tau \otimes \Omega^1_{X_S/S} & \longrightarrow & \Omega(A/S)^\vee \otimes \Omega^1_{X_S/S} \\ (z,1) & \longmapsto & (z,1) & \longmapsto & (1,0) \otimes dz = \frac{(z,1)-(\overline{z},1)}{2iy} \otimes dz & \longmapsto & \frac{-(\overline{z},1)}{2iy} \otimes dz. \end{array}$$

Thus at infinity, the isomorphism $\omega_{X_S/S} \to \bigotimes_{\tau \in \phi} \omega(A^t/S)_\tau \otimes \omega(A/S)_\tau$ gives

$$\bigwedge_{\tau \in \phi} dz \mapsto \bigotimes_{\tau \in \phi} 2iy_\tau \frac{(z_\tau, 1)}{(\overline{z_\tau}, 1)},$$

and taking norms gives $\prod_{\tau \in \phi} 2y_\tau$ on both sides. $\square$

3.7 Special Points

To calculate heights of special points of $\mathcal{X}_U$, we relate the height to heights on $\mathcal{X}'_{U'}$, which represent Faltings heights, through $\mathcal{X}''_{U''}$. Let (E, ϕ) be a partial CM-type with $F \subset E$ the totally real subfield of index 2 and let $\Sigma = \phi|_F \subset \mathrm{Hom}(F, \mathbb{R})$. Let B be a quaternion algebra over F such that B is ramified at infinity at $\Sigma \subset \mathrm{Hom}(F, \mathbb{R})$ and whose finite ramification set is a subset of the primes for which E is ramified. Then we can embed $E \to B$ because $E_\mathfrak{p}$ embeds into $B_\mathfrak{p}$ at every place $\mathfrak{p}$ of F. Let $\{\mathcal{X}_U\}_U$ be the tower of Shimura varieties associated to this particular quaternion algebra B. The embedding $E \to B$ gives us an embedding of $T_E := \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m \to G$ and hence we get a set of CM points of X_U which are parametrized, under the complex uniformization, by points $(z, t) \in (\mathcal{H}^\pm)^\Sigma \times G(\mathbb{A}_f)$ where z is determined by the cocharacter $h_\sigma \colon \mathbb{C} \cong E_\tau \to B_\sigma$ for each $\tau \in \phi$ and $\sigma \colon F \to \mathbb{R}$ lying below it, and $t \in T_E(\mathbb{A}_f)$. Fix one of these CM points $P \in \mathcal{X}_U(\overline{\mathbb{Q}})$.

Pick a complementary partial CM-type ϕ' to ϕ. We can construct the tower $X'_{U'}$ which represents the functor $\mathcal{F}'_{U'}$ as before. For any choice of element $t' \in (T_E \times T_E \cap G')(\mathbb{A}_f)$, the cocharacter formed from $z \in (\mathcal{H}^\pm)^\Sigma$ and h_E is a point $P' = [(z, h_E), t'] \in X'_{U'}(\overline{\mathbb{Q}})$, which represents an abelian variety A' with multiplication by $\mathcal{O}_E$. From the determinant condition of $\mathcal{F}'_{U'}$, we have that A' is isogenous to a product of abelian varieties $A_1 \times A_2$, one with CM by E of type $\phi \sqcup \phi'$ and the other of CM-type $\overline{\phi} \sqcup \phi'$.

We now compare the points P and P' by embedding both X_U and $X_{U'}$ into $X''_{U''}$. Recall that $G'' = \mathrm{Res}_{F/\mathbb{Q}}[(B^\times \times E^\times)/F^\times]$ where $F^\times \subset B^\times \times E^\times$ by $a \mapsto (a, a^{-1})$. This gives rise to the Shimura variety $X''_{U''}$. The embedding $G' \to G''$ gives an embedding $X'_{U'} \to X''_{U''}$. To relate X_U and $X''_{U''}$, we take the quotient map $\mathrm{Res}_{F/\mathbb{Q}}(B^\times \times E^\times) \to G''$ and the former gives rise to a Shimura variety $X_U \times Y_J$ where Y_J is the zero-dimensional Shimura variety associated with datum the torus T_E and morphism h_E as in the definition of G'', and $J \subset T_E(\mathbb{A}_f)$ is an open compact subgroup. This quotient map of Shimura datum gives rise to a surjective morphism

$$X_U \times Y_J \to X''_{U''}$$

of Shimura varieties, where $U'' = U \cdot J \subset G''(\mathbb{A}_f)$. Thus, we have the following morphisms of algebraic groups

$$G \leftarrow G \times T_E \to G'' \leftarrow G'$$

which gives rise to the chain of morphisms of Shimura varieties

$$X_U \leftarrow X_U \times Y_J \to X''_{U''} \leftarrow X'_{U'}.$$

However, given a point $y \in Y_J$, we are able to construct a map $X_U \to X_U \times Y_J \to X''_{U''}$.

Proposition 3.7.1. *We can choose $y \in Y_J$ and $P' \in X'_{U'}$ such that $P \in X_U$ and $P' \in X'_{U'}$ have the same image P'' in $X''_{U''}$.*

Proof. For a fixed $P \in X_U$, we can choose a representative $[z, t] \in (\mathcal{H}^\pm)^\Sigma \times G(\mathbb{A}_f)$ under the complex uniformization. If we let $t' = (t, t^{-1})$, then $t' \in (T_E \times T_E \cap G')(\mathbb{A}_f)$ because $\nu(t') = 1 \in \mathbb{G}_m$. Letting $y \in Y_J$ be the point corresponding to the choice of $t^{-1} \in T_E(\mathbb{A}_f)$ makes it so $(P, y) \in X_U \times Y_J$ and $P' = [z \times h_E, t'] \in X'_{U'}$ have the same image in $X''_{U''}$. □

All the geometric points of Y_J are defined over $E_{X'}$, so the integral model $\mathcal{X}_U$ for X_U gives rise to an integral model $\mathcal{X}_U \times \mathcal{Y}_J$ for $X_U \times Y_J$. We have an p-divisible group $\mathcal{H}''_{U''}$ defined over $\mathcal{X}''_{U''}$. We also define a p-divisible group I over Y_J by defining

$$I_J := (E_p/\mathcal{O}_{E,p} \times Y)/J.$$

Let $K/E_{X'}$ be a finite extension. Suppose that we have points $x \in X_U(K)$ and $y \in Y_J(K)$. These give rise to a point $x'' \in X''_{U''}(K)$.

Proposition 3.7.2 ([75, Prop 5.3])**.** *There are canonical isomorphisms*

$$\mathrm{Lie}(\mathcal{H}''_{x''}) \cong \mathrm{Lie}(\mathcal{H}_x) \otimes_{\mathcal{O}_{E,p} \otimes \mathcal{O}_K} \mathrm{Lie}(\mathcal{I}^t_y)^\vee, \quad \mathrm{Lie}(\mathcal{H}''^t_{x''}) \cong \mathrm{Lie}(\mathcal{H}^t_x) \otimes_{\mathcal{O}_{E,p} \otimes \mathcal{O}_K} \mathrm{Lie}(\mathcal{I}^t_y).$$

Define $\mathcal{N}''(\mathcal{H}''_{x''}, \tau) := \omega(\mathcal{H}''_{x''})_\tau \otimes \omega(\mathcal{H}''_{x''})_{\overline{\tau}}$. Then the previous proposition immediately gets us that

$$\mathcal{N}''(\mathcal{H}''_{x''}, \tau) \cong \mathcal{N}(\mathcal{H}_x, \tau).$$

Theorem 3.7.3. *Let d_B be a positive generator of $N_{F/\mathbb{Q}}\mathfrak{d}_B$ and let $d_\Sigma = d_{\phi \sqcup \overline{\phi}}$. We have that*

$$h_{\widehat{L_U}}(P_U) = \sum_{\tau \in \phi} \big(h(\phi \sqcup \phi', \tau) + h(\overline{\phi} \sqcup \phi', \overline{\tau}) \big) + \frac{1}{2g} \log d_B d_\Sigma.$$

Proof. By Theorem 3.6.1, we get that

$$2h_{\widehat{\mathcal{L}_U}}(P_U) = \sum_{\tau\in\phi} h_{\widehat{\mathcal{N}(\tau)}}(P_U) + h_{\widehat{\mathcal{N}(\overline{\tau})}}(P_U) + \frac{1}{2g}\log d_B d_\Sigma,$$

with the extra factor of g coming from the fact that we defined the height over $\mathbb{Q}$, which is $[F:\mathbb{Q}]$ times larger than the usual height defined over F. By the previous proposition, we have that

$$h_{\widehat{\mathcal{N}(\tau)}}(P_U) = h_{\widehat{\mathcal{N}''(\tau)}}(P''_{U''}).$$

Then by our choice of $y \in Y_J$ and $P' \in X'_{U'}$, the point $P''_{U''}$ is the image of $P' \in X'_{U'}$ which represents an abelian variety that is isogenous to a product of CM abelian varieties, one of CM-type $\phi \sqcup \phi'$ and the other of CM-type $\overline{\phi} \sqcup \phi'$. Thus, we get that

$$h_{\widehat{\mathcal{N}''(\tau)}}(P''_{U''}) = h_{\widehat{\mathcal{N}'(\tau)}}(P'_{U'}) = h(\phi\sqcup\phi',\tau) + h(\overline{\phi}\sqcup\phi',\overline{\tau}).$$

□

This result does not depend on the choice of complementary CM-type ϕ' and so summing over all such complementary CM-types nets us the following.

Theorem 3.7.4. *Suppose that $U = \prod_v U_v$ is a maximal compact subgroup of $G(\mathbb{A}_f)$. Then*

$$\begin{aligned}\frac{1}{2}h_{\widehat{\mathcal{L}_U}}(P_U) =& \frac{1}{2^{|\Sigma^c|}}\sum_{\Phi\supset\phi} h(\Phi) - \frac{|\Sigma^c|}{g2^g}\sum_{\Phi} h(\Phi) \\ &+ \frac{1}{8}\log d_{E/F,\Sigma} d_\Sigma^{-1} + \frac{1}{4}\log d_\phi d_{\overline{\phi}} + \frac{1}{4g}\log d_B d_\Sigma + \frac{|\Sigma|}{4g}\log d_F,\end{aligned}$$

where the first summation is over all full CM-types which contain ϕ, and the second summation over all full CM-types of E.

Proof. We note $h(\Phi) = h(\overline{\Phi})$ and $h(\Phi,\tau) = h(\overline{\Phi},\overline{\tau})$. So we can write

$$\begin{aligned}h_{\widehat{\mathcal{L}_U}}(P_U) - \frac{1}{2g}\log d_B d_\Sigma =& \sum_{\tau\in\phi\sqcup\phi'} h(\phi\sqcup\phi',\tau) + \sum_{\tau\in\phi\sqcup\overline{\phi'}} h(\phi\sqcup\overline{\phi'},\tau) \\ &- \sum_{\tau\in\phi'}\big(h(\phi\sqcup\phi',\tau) + h(\phi\sqcup\overline{\phi'},\overline{\tau})\big).\end{aligned} \qquad (*)$$

Let (Φ_1,Φ_2) be a nearby pair of full CM-types meaning that $|\Phi_1\cap\Phi_2| = g-1$ and let $\tau_i = \Phi_i\backslash(\Phi_1\cap\Phi_2)$. Then Theorem 3.2.6 tells us the quantity $h(\Phi_1,\tau_1)+h(\Phi_2,\tau_2)$ is independent of the choice of nearby pair, so we will denote it by h_{nb}. By [75, Cor. 2.6], we have that

$$\sum_{\Phi} h(\Phi) = g2^{g-1}h_{\mathrm{nb}} - 2^{g-2}\log d_F.$$

Now we sum equation (*) over all complementary types ϕ' to get

$$2^{|\Sigma^{\mathsf{c}}|}h_{\widehat{\mathcal{L}_U}}(P_U) - \frac{2^{|\Sigma^{\mathsf{c}}|}}{2g}\log d_B d_\Sigma = 2\sum_{\Phi\supset\phi}\sum_{\tau\in\Phi} h(\Phi,\tau) - |\Sigma^{\mathsf{c}}|2^{|\Sigma^{\mathsf{c}}|}h_{\mathrm{nb}}.$$

We now use Theorem 3.2.5 to represent the inner summation as $\sum_{\tau\in\Phi} h(\Phi,\tau)$. Doing so gives

$$\begin{aligned}\frac{1}{2}h_{\widehat{\mathcal{L}_U}}(P_U) =& \frac{1}{2^{|\Sigma^{\mathsf{c}}|}}\sum_{\Phi\supset\phi} h(\Phi) - \frac{|\Sigma^{\mathsf{c}}|}{g2^g}\sum_{\Phi} h(\Phi) \\ &+ \frac{1}{2^{|\Sigma^{\mathsf{c}}|}}\sum_{\Phi\supset\phi}\frac{1}{4[E_\Phi:\mathbb{Q}]}\log(d_\Phi d_{\overline{\Phi}}) + \frac{1}{4g}\log d_B d_\Sigma - \frac{|\Sigma^{\mathsf{c}}|}{4g}\log d_F.\end{aligned}$$

Base changing up to $E' = E^{\mathrm{Gal}}$, we can simplify the first sum of logarithms as

$$\frac{1}{2^{|\Sigma^{\mathsf{c}}|}\cdot 4[E':\mathbb{Q}]}\sum_{\Phi\supset\phi}\log(d_\Phi d_{\overline{\Phi}}) = \sum_{p<\infty}\sum_{\sigma\colon E'\to\overline{\mathbb{Q}_p}}\frac{1}{2^{|\Sigma^{\mathsf{c}}|}\cdot 4[E':\mathbb{Q}]}\sum_{\Phi\supset\phi}\log|d_{\Phi,p}d_{\overline{\Phi},p}|_\sigma.$$

For each $\tau\colon E'\to\overline{\mathbb{Q}_p}$, let π be a generator of $\mathcal{O}_{E,p}$ over $\mathbb{Z}_p$. Let

$$f_\Phi(t) = \prod_{\tau\in\Phi}(t-\tau(\pi)) \in \mathcal{O}_{E_\Phi,p}[t].$$

The image of $\mathcal{O}_{E_\Phi}\times_{\mathbb{Z}}\mathcal{O}_{E,p}$ in $\widetilde{E_{\Phi,p}}$ is $\mathcal{O}_{E_\Phi,p}[t]/f_\Phi(t)$. This means $d_{\Phi,p}$ is the discriminant $f_\Phi(t)$, or

$$d_{\Phi,p} = \prod_{(\tau,\tau')}(\tau(\pi)-\tau'(\pi))^2,$$

where the product is taken over all unordered pairs of distinct $\tau\neq\tau'\in\Phi$. We can write the summation over all $\Phi\supset\phi$ as the sum of $\log|\tau(\pi)-\tau'(\pi)|_\sigma$ over all pairs (τ,τ'), and then subtract the pairs when $\tau=\overline{\tau'}$ and when $\tau\in\phi$ and $\tau'\in\overline{\phi}$, or vice versa. Thus, we can simplify the sum as

$$\sum_{\Phi\supset\phi}\log|d_{\Phi,p}d_{\overline{\Phi},p}|_\sigma = \log\left|\frac{\prod_{(\tau,\tau')\in\mathrm{Hom}(E,\overline{\mathbb{Q}_p})}(\tau(\pi)-\tau'(\pi))^2}{\prod_{\tau\in\Phi}(\tau(\pi)-\overline{\tau}(\pi))^2}\right|_\sigma^{2^{|\Sigma^c|-1}}$$
$$+\log\left|\frac{\prod_{\tau\in\phi}(\tau(\pi)-\overline{\tau}(\pi))^2}{\prod_{(\tau,\tau')\in\phi\sqcup\overline{\phi}}(\tau(\pi)-\tau'(\pi))^2}\right|_\sigma^{2^{|\Sigma^c|-1}}$$
$$+\log\left|\prod_{(\tau,\tau')\in\phi}(\tau(\pi)-\tau'(\pi))^2(\overline{\tau}(\pi)-\overline{\tau'}(\pi))^2\right|_\sigma^{2^{|\Sigma^c|}}.$$

We can simplify the first term as $\log\left|\frac{d_{E,p}}{d_{E/F,p}}\right|_\sigma = \log|d_{F,p}|_\sigma^2$. The second term can be written as $\log\left|\frac{N_{F/E_X}d_{E/F}}{d_\Sigma}\right|_\sigma = \log\left|\frac{d_{E/F,\Sigma}}{\Sigma}\right|_\sigma$. Finally, the last term is $\log|d_{\phi,p}d_{\overline{\phi},p}|_\sigma$.

Plugging this back in gives

$$\frac{1}{2}h_{\widehat{\mathcal{L}_U}}(P_U) = \frac{1}{2^{|\Sigma^c|}}\sum_{\Phi\supset\phi}h(\Phi) - \frac{|\Sigma^c|}{g2^g}\sum_{\Phi}h(\Phi)$$
$$+\frac{1}{8}\log d_{E/F,\Sigma}d_\Sigma^{-1} + \frac{1}{4}\log d_\phi d_{\overline{\phi}} + \frac{1}{4g}\log d_B + \frac{|\Sigma|}{4g}\log d_F.$$

□

3.8 André–Oort for Shimura Varieties

A definition of the height of a partial CM-type was given by [58]. We show that their definition of the partial CM-type is compatible with our quaternionic height. We first recall their definition of the modified height on special points, specialized to the case of a partial CM-type. Let E/F be a CM extension, with $[F:\mathbb{Q}]=g$, and set $R_E := \mathrm{Res}_{F/\mathbb{Q}}E^\times/F^\times$. Let $\phi\subset\mathrm{Hom}(E,\mathbb{C})$ be a partial CM-type and ϕ' be a complementary partial CM-type. Use them to identify $E\otimes_{\mathbb{Q}}\mathbb{R}\overset{\phi\sqcup\phi'}{\cong}\mathbb{C}^g$. Then we have that

$$R_{E,\mathbb{R}} \cong \prod_{\sigma\in\phi\sqcup\phi'}\mathbb{C}_\sigma/\mathbb{R},$$

and we take our homomorphism $h_\phi \colon \mathbb{C}^\times \to R_{E,\mathbb{R}}$ as

$$h_\phi(z) := \left(\prod_{\sigma \in \phi} z_\sigma, \prod_{\sigma \in \phi'} 1_\sigma \right).$$

The Shimura datum (R_E, h_ϕ) and compact open subgroup $K \subset R_E(\mathbb{A}_f)$ give rise to a 0-dimensional Shimura variety T_K whose complex points are

$$T_K(\mathbb{C}) \cong R_E(\mathbb{Q}) \backslash R_E(\mathbb{A}_f)/K.$$

It has a canonical model over a number field E_T. We identify

$$R_E(\mathbb{C}) \cong \prod_{\sigma \in \phi \sqcup \phi'} \mathbb{C}_\sigma.$$

Let $\chi \colon R_E(\mathbb{C}) \to \mathbb{C}$ be the character given by

$$\chi\left(\prod_{\sigma \in \Phi} z_\sigma \right) := \prod_{\sigma \in \phi} \frac{z}{\overline{z}}.$$

Let V be the smallest $\mathbb{Q}$-representation of R_E whose complexification contains χ. Let $\operatorname{Fil}^a V$ be the smallest piece of the Hodge filtration and assume that it is one-dimensional and R_E acts on it via χ. Let $\Lambda \subset V$ be a maximal lattice and now take $K = \prod_p K_p \subset R_E(\mathbb{A}_f)$ to be the stabilizer of Λ. Let ψ be a polarization on V that takes integral values on Λ.

The representation V of R_E gives rise to a vector bundle $\mathcal{V}_K$ over T_K and filtration on it. By [17], over every non-archimedean place v of $\mathcal{O}_{E_T}$ lying over a prime $p \in \mathbb{Z}$, our data $(\Lambda, V, \operatorname{Fil})$ can be functorially identified with data $({}_p\Lambda, {}_pV, {}_p\operatorname{Fil})$ of a filtered vector bundle over $T_{E_{T,v},K}$, the Shimura variety extended over local fields.

To each place v of $\mathcal{O}_{E_T}$, we define a norm on $\operatorname{Fil}^a \Lambda$ as:

- If $v \mid \infty$, then the norm is the Hodge norm given by the polarization q;
- If $v \mid p$ is a non-archimedean place such that
  - T is unramified at p,
  - K_v is maximal, and
  - $p \geq a \dim V + 2$,

  then use the crystalline norm on ${}_pV$;
- For all other places v, use the intrinsic norm on ${}_p\Lambda$.

Now the height of ϕ can be defined as

$$h(\phi) := \sum_v -log\|s\|_v,$$

where s is any element of $\mathcal{V}_K$ and the sum is over all places of $\mathcal{O}_{E_T}$.

The height depends on the choice of lattice Λ and polarization q, but only up to d_E, the discriminant of E.

Theorem 3.8.1 ([58, Lem. 9.4, Thm. 9.5, 9.6])**.** *The height $h(\phi)$ is defined up to $O(\log d_E)$.*

Theorem 3.8.2.

$$2h(\phi) = h_{\widehat{\mathcal{L}_U}}(P_U) + O(\log d_E d_B).$$

Proof. Consider the representation of E on $V = B$ through left multiplication. Our point P_U corresponds to an action whose trace is $\mathrm{Tr}_{\phi \sqcup \phi'} + \mathrm{Tr}_{\overline{\phi} \sqcup \phi'}$. When we take the Shimura variety associated with the adjoint group G^{ad}, then this representation gives a representation of R_E on V/F whose trace is given by the trace of $\frac{z_\tau}{z_{\overline{\tau}}}$, meaning that we get 2χ. Thus, we get a representation of R_E whose complexification contains 2χ. Thus, we are reduced to showing that the choice of lattices at each finite place are the same. However, since our equality is only up to $O(d_E d_B)$, it suffices to consider primes where B, E are unramified and the local norm used in the definition of $h(\phi)$ is given by the crystalline norm.

Let $S = \mathcal{O}_{E'^p_X, p}$ again be the maximal unramified extension of $\mathcal{O}_{E'_X, p}$. To show that the lattices coincide, it suffices to check the two lattices at each S point of X_U. Under the mapping $X_U \to X_U \times Y_J \to X''_{U''}$, the point P_U corresponds to an abelian variety $\mathcal{A}$ with complex multiplication of type $\phi \sqcup \phi' + \overline{\phi} \sqcup \phi'$. By [58, Sec. 9.3], the lattice given by the crystalline norm is the same as the lattice from integral de Rham cohomology. So the lattice at that point is

$$\Omega(\mathcal{A}) \otimes S \cong \Omega(\mathcal{A}[p^\infty]) \otimes S \cong \Omega(\mathcal{H}''_S).$$

However, by Proposition 3.7.2, this is the same as $\Omega(\mathcal{H}_S)$. Moreover, the pairing is perfect here meaning that we get the same lattice on $\Omega(\mathcal{A}^t) \otimes S \cong \Omega(\mathcal{H}''^t_S)$. Thus, twice $h(\phi)$ corresponds to taking the height relative to the lattice $\Omega(\mathcal{A}) \otimes \Omega(\mathcal{A}^t)$ which is $\Omega(\mathcal{H}_S) \otimes \Omega(\mathcal{H}^t_S)$ which by Theorem 3.6.1 is just $\mathcal{L}_U$, as required. $\square$

Chapter 4

Existential Closedness for Shimura Varieties

4.1 Shimura Varieties and Hermitian Symmetric Domains

Shimura Varieties

We briefly review the theory of Shimura varieties. More detailed information can be found in [43].

A Shimura datum is a pair (G, X) where G is a reductive algebraic group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$ conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ from the Deligne torus $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ such that one (and hence all) $h \in X$ satisfy the following three conditions:

(1) The Hodge structure on the Lie algebra of $G_{\mathbb{R}}$ given by $\mathrm{Ad} \circ h$ is of type

$$\{(-1, 1), (0, 0), (1, -1)\},$$

(2) the adjoint action $\mathrm{Ad}h(i)$ is a Cartan involution on the adjoint group $G^{\mathrm{ad}}_{\mathbb{R}}$,

(3) the adjoint group G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

When these three conditions are satisfied, then X has the structure of a disjoint union of Hermitian symmetric domains. Let $\mathbb{A}_f$ denote the ring of finite adeles of $\mathbb{Q}$ and K be a compact open subgroup of $G(\mathbb{A}_f)$. The Shimura variety attached to the triple (G, X, K) is the double quotient

$$\mathrm{Sh}_K(G, X) := G(\mathbb{Q})\backslash X \times [G(\mathbb{A}_f)/K],$$

where $G(\mathbb{Q})$ acts on both X by conjugation and $G(\mathbb{A}_f)/K$ by left multiplication. When K is small, this double coset space has the structure of a smooth quasi-projective variety over $\mathbb{C}$, and moreover has a canonical model over a number field.

Let $G(\mathbb{R})_+ \subset G(\mathbb{R})$ denote the subgroup that maps into the identity connected component $G^{\mathrm{ad}}(\mathbb{R})^+$ of the adjoint group of G, and let $G(\mathbb{Q})_+ := G(\mathbb{Q}) \cap G(\mathbb{R})_+$ denote the rational points of this subgroup. Let Ω denote a connected component of X and let $\mathcal{C}$ be a set of representatives for the double coset $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$. Then our Shimura variety can be described as a disjoint union of quotients of Hermitian symmetric domains

$$\mathrm{Sh}_K(G, X) \simeq \sqcup_{g \in \mathcal{C}} \Gamma_g \backslash \Omega,$$

where $\Gamma_g = gKg^{-1} \cap G(\mathbb{Q})_+$ is an arithmetic subgroup of $G(\mathbb{R})$.

We limit ourselves to looking at the quotient map associated to exactly one of these connected components. Fix a connected component Ω of X and fix $\Gamma \subset G(\mathbb{Q})_+$ an arithmetic subgroup of $G(\mathbb{R})$ associated with a $g \in \mathcal{C}$ and compact open subgroup $K \subset G(\mathbb{A}_f)$, and let $S := \Gamma \backslash \Omega$ denote the connected component of $\mathrm{Sh}_K(G, X)$ corresponding to Ω and Γ, and $q : \Omega \to S$ the quotient map from modding out by the left action of Γ.

Borel and Harish-Chandra embeddings

The quotient S is a connected component of the full Shimura variety and thus has the structure of a smooth algebraic variety. In order to discuss varieties of $\Omega \times S$, we need to give the $G(\mathbb{R})$ conjugacy class of homomorphisms Ω algebraic structure.

Every point $h \in \Omega$ is a homomorphism $h : \mathbb{S} \to G_{\mathbb{R}}$, and to each we can attach a cocharacter of $G_{\mathbb{C}}$ by taking

$$\mu_h : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_{\mathbb{C}} \to G_{\mathbb{C}}$$

where the first map takes $z \mapsto (z, 1) \in \mathbb{S}_{\mathbb{C}} = \mathbb{C}^\times \times \mathbb{C}^\times$, and the second map is induced by h. This cocharacter gives rise to a filtration $\mathrm{Filt}(\mu)$ of $\mathbf{Rep}_{\mathbb{C}}(G_{\mathbb{C}})$. This functor gives a decreasing filtration $F^\bullet$ on V for each representation (V, ξ) of $G_{\mathbb{C}}$ where $F^p V = \oplus_{i \geq p} V^i$ and $V^i = \{v \in V : \mu(z)v = z^{-i}v\}$. The compact dual Hermitian space $\check{X}$ to X is the $G_{\mathbb{C}}$ conjugacy class of filtrations of $\mathbf{Rep}_{\mathbb{C}}(G_{\mathbb{C}})$ that contains $\mathrm{Filt}(\mu)$.

Let $P_\mu \subset G_{\mathbb{C}}$ be the subgroup fixing the filtration $\mathrm{Filt}(\mu)$ of $\mathbf{Rep}_{\mathbb{C}}(G_{\mathbb{C}})$. It is a parabolic subgroup of $G_{\mathbb{C}}$ and so the bijection $G(\mathbb{C})/P_\mu(\mathbb{C}) \to \check{X}$ endows $\check{X}$ with the structure of a smooth projective complex variety. There is a natural embedding $X \hookrightarrow \check{X}$ given by sending a homomorphism $h \in X$ to the filtration associated to it $\mathrm{Filt}(\mu_h)$. Fix a base point $o \in X$ and let $K_o \subset G_{\mathbb{R}}$ be the subgroup fixing the homomorphism o and let $P_o := P_{\mu_o}$ be the parabolic subgroup associated with μ_o. Then we can identify X and $\check{X}$ with the coset spaces $G(\mathbb{R})/K_o$ and $G(\mathbb{C})/P_o(\mathbb{C})$ respectively. Additionally $K_o = G(\mathbb{R}) \cap P_o(\mathbb{C})$, and the Borel embedding can be seen as the natural map

$$X = G(\mathbb{R})/K_o(\mathbb{R}) \hookrightarrow G(\mathbb{C})/P_o(\mathbb{C}) = \check{X}.$$

In our situation, we are more interested in the Harish-Chandra embedding. Let $\mathfrak{p}^+$ be the holomorphic tangent bundle of $\check{X}$ at o. It is proven in [9] that Ω can be realized in $\mathfrak{p}^+ \simeq \mathbb{C}^N$ as a bounded symmetric domain. Moreover, this tangent bundle can be embedded $\mathfrak{p}^+ \to \check{X}$

as a dense open subset in the compact dual of X. In this way, we will view Ω as a bounded symmetric domain inside $\mathbb{C}^N$ for $N = \dim \mathfrak{p}^+ = \dim \Omega$, and the algebraic structure on Ω will be induced from $\mathbb{C}^N$. In fact, this gives a bijection between Hermitian symmetric domains and bounded symmetric domains. A more detailed discussion can be found in [24, Chapter VIII]

Example 4.1.1. If $(G, X) = (\mathrm{SL}_2, \mathbb{H})$, then for any choice of base point $o \in X$, the stabilizer P_o is a conjugate subgroup of $B := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{C}^\times, b \in \mathbb{C} \right\}$ and so $\check{X} \simeq \mathbb{P}^1(\mathbb{C})$ and $\mathfrak{p}^+ \simeq \mathbb{C}$. The Harish-Chandra embedding realizes X as the unit disk $\mathbb{D} \subset \mathbb{C}$.

We define an algebraic subvariety of $\Omega \subset \mathbb{C}^N$, which is an analytic but not algebraic space, as the restriction to Ω of an algebraic variety of $\mathbb{C}^N$.

Definition 4.1.2. A subset $Y \subset \Omega$ is an *(irreducible) algebraic variety of* X if there exists an (irreducible) algebraic variety $Z \subset \mathbb{C}^N$ such that $Y = \Omega \cap Z$.

Metrics on domains

We will be interested in a comparison between different notions of distance on Ω and so we briefly recall different metrics that can be defined on a general open domain $D \subset \mathbb{C}^n$. In addition to the Euclidean metric, there also exists the Bergman and Carathéodory metrics on a domain $D \subset \mathbb{C}^n$. We first give a description of the Bergman metric, which can also be found in more detail in [24, Chapter VIII].

For a domain $D \subset \mathbb{C}^n$, let $H(D)$ be the space of holomorphic functions $f : D \to \mathbb{C}$ and let

$$H_2(D) := \left\{ f \in H(D) : \int_D |f(z)|^2 dz < \infty \right\}$$

be the subspace of holomorphic functions with bounded L^2 norm. The space $H_2(D)$ is a Hilbert space for the L^2 norm $\langle f, f \rangle = \int_D |f(z)|^2 dz$. For each fixed $w \in D$, the Riesz representation theorem implies the linear functional $f \mapsto f(w)$ is representable by some function $K(\cdot, w) \in H_2(D)$, so that $\langle f, K(\cdot, w) \rangle = f(w)$ for all $f \in H_2(D)$. The function $K(z, w)$ is defined on $D \times D$, it is holomorphic in z and anti-holomorphic in w, satisfies $K(z, w) = \overline{K(w, z)}$, and is called the Bergman kernel of D.

Definition 4.1.3. For each $z \in D$ and tangent vector $\mathbf{v} = \sum_{i=1}^n a_i \frac{\partial}{\partial z_i} \in T_z(D)$, the *Bergman metric* is

$$B_D(z; \mathbf{v})^2 := \sum_{i,j=1}^n a_i \overline{a_j} \frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log K(z, z).$$

From this metric, comes the notion of Bergman distance.

Definition 4.1.4. For each pair of points $z, w \in D$, the *Bergman distance* is

$$b_D(z, w) := \inf_{\gamma} \int_0^1 B_D(\gamma(t); \gamma'(t))dt,$$

where the infimum is taken over all piecewise C^1 curves $\gamma : [0, 1] \to D$ with $\gamma(0) = z$ and $\gamma(1) = w$.

Example 4.1.5. If $D = \mathbb{H} \subset \mathbb{C}$ is the upper half plane, then the Bergman metric corresponds to the Poincaré metric $B_D(z)^2 = \frac{dzd\bar{z}}{y^2}$.

If $D = \mathbb{D} \subset \mathbb{C}$ is the unit disk, the Bergman metric is given by $B_D(z) = \frac{ds}{1-|z|^2}$, where ds denotes the Euclidean metric.

An important property of the Bergman distance is that it is invariant under holomorphic automorphisms. For Shimura varieties, the Hermitian symmetric domain Ω, each $g \in G(\mathbb{R})_+$ acts as a holomorphic automorphism of Ω [43]. Thus, for each $g \in G(\mathbb{R})_+$ and $z, w \in \Omega$, the Bergman distance $b_\Omega(z, w) = b_\Omega(gz, gw)$ is invariant under the action of g.

The Carathéodory–Reiffen metric introduced by Reiffen [62] is another biholomorphic metric that can be defined on bounded domains. While we are primarily interested in the Bergman metric, the Carathéodory metric will help bound the Bergman metric and is easier to worth with.

Let

$$H_\infty(D) := \left\{ f \in H(D) : \sup_{z \in D} |f(z)| < \infty \right\}$$

denote the subspace of holomorphic functions with bounded L^∞ norm.

Definition 4.1.6. For each $z \in D$ and tangent vector $\mathbf{v} = \sum_{i=1}^n a_i \frac{\partial}{\partial z_i} \in T_z(D)$, the *Carathéodory metric* is

$$C_D(z; \mathbf{v}) := \sup \left\{ \mathbf{v}f : f \in H_\infty(D), f(z) = 0, \|f\|_\infty \le 1 \right\}.$$

As with the Bergman metric, we can define the Carathéodory distance between two points.

Definition 4.1.7. For each pair of points $z, w \in D$, the *Carathéodory distance* is

$$c_D(z, w) := \inf_{\gamma} \int_0^1 C_D(\gamma(t); \gamma'(t))dt,$$

where the infimum is taken over all piecewise C^1 curves $\gamma : [0, 1] \to D$ with $\gamma(0) = z$ and $\gamma(1) = w$.

Example 4.1.8. If $D = \mathbb{D}$, then the Carathéodory metric coincides with the Bergman metric and Poincaré metric.

There is a simple relation between them in that the Bergman metric is at least the Carathéodory metric.

Theorem 4.1.9 ([23])**.** *In any domain $D \subset \mathbb{C}^n$, point $z \in D$ and tangent vector $\boldsymbol{v} \in T_z(D)$, we have $C_D(z;\boldsymbol{v}) \leq B_D(z;\boldsymbol{v})$.*

4.2 Boundaries of Hermitian Symmetric Domains

Shilov boundary

A fundamental result from complex analysis says that a bounded domain $D \subset \mathbb{C}$ satisfies the maximum modulus principle; a holomorphic function on $\overline{D}$ achieves its maximum modulus on the boundary of D. In higher dimensions though, we can sometimes say a stronger result in that the maximum modulus must occur on a proper closed subset of the boundary of D.

Definition 4.2.1. Let $D \subset \mathbb{C}^n$ be a bounded domain. The *Shilov boundary* of D is the smallest closed subset $\sigma(D) \subset \partial D$ that satisfies the maximum modulus principle. That is, for any holomorphic function defined in an open neighborhood of the closure $\overline{D}$, the function $|f(z)|$ achieves a maximum at some point $z \in \sigma(D)$.

Example 4.2.2. If $D = \mathbb{D} \times \mathbb{D} \subset \mathbb{C}^2$, then the Shilov boundary is $S^1 \times S^1$. It satisfies the maximum modulus principle by applying the single variable maximum modulus principle in the first coordinate and then the second coordinate. It is a proper closed subset of $\partial D = (\mathbb{D} \times S^1) \sqcup (S^1 \times \mathbb{D}) \sqcup (S^1 \times S^1)$.

Proposition 4.2.3. *Let $D \subset \mathbb{C}^n$ be a bounded domain. Then $\sigma(D)$ is Zariski dense in $\mathbb{C}^n$.*

Proof. Suppose for contradiction that $\sigma(D) \subset Z(f)$ was in the zero set of some non-zero algebraic function. Then $f(z) = 0$ for all $z \in \sigma(D)$. By definition, the function $|f(z)|$ restricted to $\overline{D}$ achieves a maximum at some $z \in \sigma(D)$, so function must vanish on all of $\overline{D}$. The function $f : \mathbb{C}^n \to \mathbb{C}$ vanishes on an open set in $\mathbb{C}^n$ and hence on all of $\mathbb{C}^n$, so $f = 0$, contradicting our assumption on f. Therefore $\sigma(D)$ does not lie in any proper variety of $\mathbb{C}^n$ and is Zariski dense. □

In the case of D being a Hermitian symmetric domain, the boundary has a well-studied decomposition. The boundary inherits an action of $\mathrm{Hol}(D)$. After partitioning the boundary of D into disjoint $\mathrm{Hol}(D)$ orbits, each boundary component is either a Hermitian symmetric domain of smaller dimension, or the unique closed orbit [71, Part I.5]. This unique closed orbit is precisely the Shilov boundary of D.

Example 4.2.4. If $D = \mathbb{D} \times \mathbb{D} \subset \mathbb{C}^2$, then the boundary of D can be split up as $\partial D = (\mathbb{D} \times S^1) \sqcup (S^1 \times \mathbb{D}) \sqcup (S^1 \times S^1)$. The first two components are Hermitian symmetric domains of smaller dimension, and the unique closed orbit is the Shilov boundary $S^1 \times S^1$.

Behavior near Shilov boundary

In the case of the upper half-plane, as the imaginary part of a point $z \in \mathbb{H}$ tends to zero, a hyperbolic ball of fixed radius centered at z converges uniformly in the Euclidean metric to a real point. In the general case of a Hermitian symmetric domain of higher dimension though, this is not necessarily true. While it still holds that as we take the center $z \in D$ of a hyperbolic ball of fixed radius to a point on the boundary $\bar{z} \in \partial D$ of, the Euclidean volume of the ball will go to 0 and all the points in the ball will converge to the boundary [7], the points in the ball may not all converge to the same point on the boundary. For instance, taking the center $(z, w) \in D = \mathbb{D} \times \mathbb{D}$ of a hyperbolic ball to a point on the boundary $(\bar{z}, w) \in S^1 \times \mathbb{D} \subset \partial D$ by keeping the second coordinate fixed and only changing the first coordinate makes the points in the ball converge to a hyperbolic ball $\{\bar{z}\} \times B \subset S^1 \times \mathbb{D}$ in a smaller Hermitian domain. However, if the center converges to a point on the Shilov boundary, the hyperbolic ball centered there will converge in the Euclidean metric to the same point on the Shilov boundary.

Proposition 4.2.5. *Let $\Omega \subset \mathbb{C}^N$ be the bounded realization of a Hermitian symmetric domain. For any fixed real number $r \geq 0$ and $z' \in \sigma(D)$,*

$$\lim_{z \to z', z \in \Omega} \sup\{\|w - z\| : w \in \Omega, b_\Omega(z, w) \leq r\} = 0$$

Proof. The Shilov boundary corresponds to the set of points in $\overline{\Omega}$ of maximal Euclidean distance from the origin [35, Theorem 6.5]. Let the radius of Ω to be $R = \sup_{s \in \Omega} \|s\|$. Let $B_R \subset \mathbb{C}^N$ denote the ball of radius R centered at the origin. From the definition of the Carathéodory metric, we must have $C_{B_R}(z; \mathbf{v}) \leq C_\Omega(z; \mathbf{v})$ for any $z \in \Omega \subset B_R$ and $\mathbf{v} \in T_z(\Omega) = T_z(B_R)$. Moreover, the Carathéodory metric is less than the Bergman metric and hence

$$C_{B_R}(z; \mathbf{v}) \leq C_\Omega(z; \mathbf{v}) \leq B_\Omega(z; \mathbf{v}).$$

Integrating along paths gives $c_{B_R}(z, w) \leq b_\Omega(z, w) \leq r$. The point $z' \in \sigma(\Omega)$ is on the Shilov boundary with $\|z'\| = R$ and so as $z \to z'$ we have $z \to \partial B_R$ as well. The inequality $c_{B_R}(z, w) \leq r$ means that w is in a ball of fixed Carathéodory radius centered at z and as $z \to \partial B_R$, and hence $\|z - w\| \to 0$ by [32, Prop. 17]. $\square$

Limit sets of arithmetic subgroups

The complex points of a connected component of a Shimura variety can be described as the quotient of a Hermitian symmetric domain Ω by the action of an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})_+$. We would like to know the behavior of $\overline{\Gamma x}$, the Euclidean closure of the Γ-orbit of some point $x \in \Omega$. The group Γ acts discretely on Ω and hence any limit points of Γx must lie on the boundary $\partial\Omega$. For the upper half-plane $\Omega = \mathbb{H}$, Γ is a Fuchsian group of the first type and the limit set of the orbit of any point $\overline{\Gamma x} \supset \partial\mathbb{H}$ consists of the real line and the point at infinity. Again, by looking at the case of $\mathbb{D} \times \mathbb{D}$, this result does not hold in higher dimensions but as before, the Shilov boundary provides a suitable substitute.

Proposition 4.2.6 ([6, Sec. 3.6])**.** *Let Ω be a Hermitian symmetric domain acted on by a Zariski dense arithmetic subgroup $\Gamma \subset G(\mathbb{Q})_+$. There exists a subset $\Lambda_\Gamma \subset \sigma(\Omega)$, Zariski dense inside $\mathbb{C}^N$, such that for any point $x \in \Omega$, the limit set of the Γ-orbit of x contains $\Lambda_\Gamma \subset \overline{\Gamma x}$.*

[6] only proves that Λ_Γ is Zariski dense inside of $\sigma(\Omega)$, but then using Proposition 4.2.3, we get that Λ_Γ is in fact Zariski dense in all of $\mathbb{C}^N$.

4.3 Proof of Theorem 1.3.1

Now that we have Propositions 4.2.5 and 4.2.6, the proof of Theorem 1.3.1 proceeds, *mutatis mutandis*, like the j-function case found in [20].

Theorem 4.3.1 (Rouche's Theorem [1])**.** *Let $D \subset \mathbb{C}^n$ a bounded domain with continuous boundary ∂D and let $f, g : \bar{D} \to \mathbb{C}^n$ be two continuous functions, whose restriction to D are holomorphic and whose zeroes are isolated. If at each point $\boldsymbol{x}$ in ∂D, $\|f(\boldsymbol{x})\| > \|g(\boldsymbol{x})\|$, then f and $f+g$ have the same number of zeroes in D, counting by multiplicity.*

Proposition 4.3.2. *Let $\Gamma \subset G(\mathbb{Q})_+$ be an arithmetic subgroup and Λ_Γ be as in Proposition 4.2.6, let $q : \Omega \to S$ be the quotient map for Γ, let $U \subset \mathbb{C}^N$ be a Euclidean open set such that $U \cap \Lambda_\Gamma \neq \varnothing$, and let $p : U \to S$ be a holomorphic function. Then, the equation $q(Z) = p(Z)$ has infinitely many solutions with $Z \in U \cap \Omega$. Moreover, the closure of the set of solutions contains $U \cap \Lambda_\Gamma$.*

Proof. Fix some $Z_0 \in U \cap \Lambda_\Gamma$. The quotient map q is surjective so choose $Z_1 \in \Omega$ such that $q(Z_1) = p(Z_0)$. There exists a small Euclidean closed ball $B \subset \Omega$ around Z_1 such that $q(Z) \neq q(Z_1)$ for all $Z \in B \backslash \{Z_1\}$ because q is locally a diffeomorphism. Moreover, since Γ is a discrete subgroup, we may shrink B so that gB is disjoint from B for all $g \in \Gamma$ unless $gZ_1 = Z_1$. By Proposition 4.2.6, there exists a sequence $\{\gamma_k\}_k \in \Gamma$ such that $\|\gamma_k Z_1 - Z_0\| \to 0$. By taking a subsequence, we may assume that each $\gamma_k Z_1$, and hence $\gamma_k B$, is disjoint.

Define $\delta := \min_{Z \in \partial B} \|q(Z) - q(Z_1)\| > 0$. Using the continuity of p, choose a Euclidean open neighborhood $W \subset U$ of Z_0 satisfying

$$Z \in W \implies \|p(Z) - p(Z_0)\| < \delta/2.$$

The supremum $\sup_{Z \in \partial B} b_D(Z, Z_1)$ is finite because ∂B is a compact set and hence B lies within a ball of finite Bergman radius. Since the centers $\gamma_k Z_1$ tend towards Z_0 on the boundary of Ω, Proposition 4.2.5 implies the translates $\gamma_k B$ also tend uniformly to Z_0, and so there exists some N such that $\gamma_k B \subset W$ for all $k > N$. The function q is invariant under Γ action and thus for all $Z \in \partial(\gamma_k B) \subset W$

$$\|q(Z) - p(Z_0)\| = \|q(\gamma_k^{-1} Z) - f(Z_1)\| \geq \delta > \|p(Z_0) - p(Z)\|.$$

The function $q(Z) - p(Z_0)$ has an isolated zero in $\Gamma_k B$ at $Z = \gamma_k Z_1$ since $q(\gamma_k Z_1) - p(Z_0) = q(Z_1) - q(Z_1) = 0$. The functions q and p are holomorphic on $\gamma_k B$ so Rouche's Theorem applied to $q(Z) - p(Z_0)$ and $p(Z_0) - p(Z)$ says their sum $q(Z) - p(Z)$ also has a zero in $\gamma_k B$. This holds for every $k > N$, giving infinitely many solutions to the system of equations $f(Z) = p(Z)$, one in each $\gamma_k B$, converging to Z_0. Moreover, the γ_k were refined so that the $\gamma_k B$ are disjoint, meaning that the solutions are all distinct. Our initial choice of point $Z_0 \in U \cap \Lambda_\Gamma$ was arbitrary and hence all of $U \cap \Lambda_\Gamma$ must lie in the closure of the set of solutions. $\square$

Theorem 4.3.3 (Inverse Function Theorem)**.** *Let V be a complex manifold of dimension n and $F : V \to \mathbb{C}^n$ a holomorphic function. If $x \in V$ is a point such that the Jacobian at x has rank n, then there is an open neighborhood W of x and a holomorphic inverse $G : F(W) \to W$ such that $F \circ G = \mathrm{id}_{F(W)}$ and $G \circ F = \mathrm{id}_W$.*

We will use the Inverse Function Theorem in conjunction with Proposition 4.3.2 to prove Theorem 1.3.1.

Proof of Theorem 1.3.1. First we restrict ourselves to the case when $\dim V = N$, in which $\pi : V \to \mathbb{C}^N$ is a finite map with a Zariski dense image. The projection π is a regular map and Chevalley's theorem implies the image of π is a constructible set: a finite union of intersections of Zariski open and closed sets. Since the image is Zariski dense, we may make a further reduction to the case when the image of π is a Zariski open set of $\mathbb{C}^N$.

Let U be the image under π of the smooth locus of V. Then $U \cap \Lambda_\Gamma \neq \varnothing$ because the latter is Zariski dense by Proposition 4.2.6. For every point $z \in U \cap \Lambda_\Gamma$, choose a preimage $x \in V$ such that $\pi(x) = z$. The point x lies in the smooth locus of so the Jacobian at x is of maximal rank N. Hence, there is a Euclidean open neighborhood W of x and a holomorphic map $\pi(W) \to W$. Combining this with the projection down to S gives a holomorphic map $\pi(W) \to S$. Proposition 4.3.2 gives infinitely many solutions which are intersection points of $E_q \cap V \cap W$. We can find an open neighborhood for all $z \in U \cap \Lambda_\Gamma$ so the closure of $\pi(E_q \cap V)$ contains $U \cap \Lambda_\Gamma$, and hence is Zariski dense in $\mathbb{C}^N$.

Finally, if $E_q \cap V$ were not Zariski dense in V, then its Zariski closure would have dimension smaller than $\dim V = N$ and so the Zariski closure of $\pi(E_q \cap V)$ would have dimension smaller than N, a contradiction.

For the general case when $\dim V > N$, suppose for the sake of contradiction that $\overline{V \cap E_q}^{\mathrm{Zar}} = W \subsetneq V$ is not Zariski dense in V. Let $V' = V \cap \mathbb{C}^N \times H$, where $H \subset S$ is an intersection of $\dim V - N$ hyperplanes chosen generically so that V' is irreducible, broad, Hodge-generic, and $V' \not\subset W$. Then $\dim V' = N$ with dominant projection onto $\mathbb{C}^N$ and the above proof shows that $V' \cap E_q$ is Zariski dense in V'. But since $V' \cap W$ is a proper subvariety of V', there are elements of $V' \cap E_q$, and hence $V \cap E_q$ not lying in W. This contradicts the definition of W and hence $V \cap E_q$ is Zariski dense in V. $\square$

4.4 Products of Varieties

Totally geodesic subvarieties

In this section, we will define totally geodesic subvarieties following [68] and then state some results from [69] on these subvarieties of Ω and their image in S.

Definition 4.4.1. Let (G, X) be Shimura datum and let Ω be a connected component of X. Let (H, X_H) by a sub-Shimura datum of (G, X). This gives a finite map $S_H \to S$ of Shimura varieties. The Hecke orbits of S_H are called *special* subvarieties of S. For each decomposition $(H^{\mathrm{ad}}, X_H^{\mathrm{ad}}) = (H_1, X_1) \times (H_2, X_2)$ and point $y_2 \in X_2$, the image of $X_1 \times \{y_2\}$ in S is a *weakly special* subvariety of S. For a connected component X_1^+ of X_1 lying in Ω, the subvariety $X_1^+ \times \{y_2\}$ is called a *weakly special* subvariety of Ω.

There is a more general form of weakly special subvarieties called *totally geodesic subvarieties* of Ω, which we will be using. Here we diverge from the terminology used by Ullmo and Yafaev in [69]. What they refer to as weakly special subvarieties, we follow the terminology of [45] and call them totally geodesic. We reserve the notion of weakly special subvarieties for totally geodesic ones that are bi-algebraic.

Definition 4.4.2. The *Mumford–Tate group* of a real algebraic subgroup $F \subset G_{\mathbb{R}}$ is the smallest $\mathbb{Q}$-subgroup $H = \mathrm{MT}(F)$ of G such that $F \subset H_{\mathbb{R}}$. We call F *Hodge-generic* if $\mathrm{MT}(F) = G$.

A totally geodesic subvariety is the $F(\mathbb{R})^+$-orbit of a point $x \in \Omega$. We say that a totally geodesic subvariety $Z = F(\mathbb{R})^+x$ is Hodge-generic if F is Hodge-generic. This is equivalent to saying Z does not lie in a proper weakly special subvariety.

Proposition 4.4.3 ([69])**.** *A subvariety $Z \subset \Omega$ is totally geodesic if and only if there exists a semi-simple real algebraic subgroup $F \subset G_{\mathbb{R}}$ without compact factors and some $x \in \Omega$ such that $h_x : \mathbb{C}^\times \to G(\mathbb{R})$ factors through $FZ_G(F)^\circ$ such that $Z = F(\mathbb{R})^+x$.*

Example 4.4.4. Totally geodesic subvarieties $Z \subset \mathbb{H}^n$ are also called Möbius subvarieties and are cut out by equations of the form $x_i = gx_j$ for some $g \in \mathrm{SL}_2(\mathbb{R})$ or $x_i = c$ for some $c \in \mathbb{H}$. The totally geodesic subvariety $Z = \{(\tau, g\tau, c) : \tau \in \mathbb{H}\} \subset \mathbb{H}^3$ for fixed $g \in \mathrm{SL}_2(\mathbb{R})$ and $c \in \mathbb{H}$ corresponds to the real algebraic subgroup $F(\mathbb{R}) := \{(h, ghg^{-1}, 1) : h \in \mathrm{SL}_2(\mathbb{R})\}$ and $Z = F(\mathbb{R}) \cdot (i, gi, c)$.

These Möbius subvarieties are weakly special subvarieties precisely when all the equations of the form $x_i = gx_j$ satisfy $g \in \mathrm{GL}_2(\mathbb{Q})$.

We recall a result from Ratner theory on the image of these totally geodesic subvarieties of Ω.

Theorem 4.4.5 ([69])**.** *Let $F = F(\mathbb{R})^+$ be a semi-simple subgroup of $G(\mathbb{R})^+$ without compact factors. Let $H = MT(F)$ be the Mumford–Tate group of F. The closure of $\Gamma\backslash\Gamma F$ in $\Gamma\backslash G(\mathbb{R})^+$ is $\Gamma\backslash\Gamma H(\mathbb{R})^+$.*

From this and the fact that for a given $x \in \Omega$, the map $\pi_x : \Gamma\backslash G(\mathbb{R})^+ \to \Gamma\backslash\Omega$ by $\pi_x(g) = g \cdot x$ is closed, we can now describe the image of a totally geodesic subvariety of X^+ in S.

Corollary 4.4.6. *Let $Z \subset \Omega$ be totally geodesic so that $Z = F(\mathbb{R})^+ x$ and let $H = \mathrm{MT}(F)$. Then the Euclidean closure of $q(Z)$ in S is $\Gamma\backslash\Gamma H(\mathbb{R})^+ \cdot x$.*

In general, this closure is a real analytic subset that need not be an algebraic variety. However, if we take its Zariski closure, we will get a weakly special subvariety [69, Theorem 1.2].

Example 4.4.7. If $g \in \mathrm{SL}_2(\mathbb{R})\setminus \mathrm{SL}_2(\mathbb{Q})$, then the real algebraic subgroup given by $F(\mathbb{R}) := \{(h, ghg^{-1}) : h \in \mathrm{SL}_2(\mathbb{R})\}$ is Hodge-generic and since $\Gamma\backslash \mathrm{SL}_2(\mathbb{R})^2 \to \mathbb{C}^2$ by $(h_1, h_2) \mapsto (j(h_1), j(h_2))$ is closed, the set of $(j(\tau), j(g\tau))$ is Euclidean dense in $\mathbb{C}^2$.

Proof of Theorem 1.3.3

First we prove some results on what the intersection of $q(L)$ and W can look like.

Theorem 4.4.8 (Ax–Schanuel [44])**.** *Let $V \subset \mathbb{C}^N \times S$ be an algebraic subvariety and let U be an irreducible component of the intersection $V \cap E_q$ with the graph of the quotient map q. If $\dim U > \dim V - N$, then the projection of U to S is contained in a proper weakly special subvariety of S.*

Lemma 4.4.9. *Let $W \subset S$ be a Hodge generic proper subvariety. There exists a Euclidean dense subset $Z \subset W$ such that for any totally geodesic subvariety $L \subset \Omega$ such that $L \times W$ is broad, any intersection component of $q(L) \cap W$ that intersects Z has the expected dimension $\dim L + \dim W - \dim S$.*

Proof. For each splitting $G^{\mathrm{ad}} = G_1 \times G_2$ and $S = S_1 \times S_2$, we have projection maps $p_i : S \to S_i$. The fiber-dimension theorem says that there exist Zariski open sets $U_i \subset S_i$ such that for any $s \in U_i$, the dimension of the fiber is as expected $\dim W = \dim W_s + \dim p_i(W)$. Define

$$Z := \bigcap_{S=S_1\times S_2} (W \cap p_1^{-1}(U_1) \cap p_2^{-1}(U_2)) \cap \bigcap_{S_M} (W \cap S_M^c),$$

where the first intersection is over all such splittings and the second intersection is over all proper special subvarieties of S. Since there are only countably many ways to split $G^{\mathrm{ad}} = G_1 \times G_2$ into a product of $\mathbb{Q}$-subgroups and countably many special subvarieties of S, none of which contain W, the intersection Z is dense in W.

Now let U be an intersection component of $q(L) \cap W$ so that $U \cap Z \neq \varnothing$ and suppose that $\dim U > \dim L + \dim W - \dim S$. Then Ax–Schanuel says that U is contained in a proper weakly special subvariety of X. Since U contains a point not in any proper special subvariety, it must be contained in a weakly special subvariety obtained from a splitting of

$G^{\mathrm{ad}} = G_1 \times G_2$ and $S = S_1 \times S_2$ so that there exists $x \in X_2$ such that $U \subset S_1 \times \{q(x)\}$. We can choose this splitting so that U is Hodge-generic inside S_1.

Since U is constant on S_2, we know that U is actually in the intersection of the fibers $q(L_x) \cap W_{q(x)} \subset S_1$. But U is not contained in a proper weakly special subvariety of S_1 and hence $\dim U = \dim L_x + \dim W_{q(x)} - \dim S_1$. By the fiber-dimension theorem, we can write the dimension of the fiber of W as $\dim W_{q(x)} = \dim W - \dim p_2(W)$ and $\dim p_2(L) + \dim p_2(W) \geq \dim S_2$ by the broadness condition. Combining this with $\dim U > \dim L + \dim W - \dim S$ gives

$$\dim L < \dim L_x + \dim p_2(L).$$

But this is impossible since L is totally geodesic and equals the orbit of a point under the group action, meaning that all of its fibers are of the same dimension. Thus U must have proper dimension. □

We are now ready to prove the theorem.

Proof of Theorem 1.3.3. There are only countably many ways to split $G^{\mathrm{ad}} = G_1 \times G_2$ and hence $S = S_1 \times S_2$. So we can cut W down with hyperplanes to maintain that $L \times W$ is Hoge-generic and broad, but now $\dim L = \mathrm{codim} W = d$.

Choose a real semi-simple subgroup $F = F(\mathbb{R})^+ \subset G(\mathbb{R})^+$ and point $x \in X$ such that $L = Fx$. For each smooth point $w \in Z \subset W$, where Z is as in Lemma 4.4.9, choose some $g \in G(\mathbb{R})^+$ such that $w = q(gx)$. If $g \in F$ then $w \in q(L)$ and we are done so assume that $g \notin F$. The group F is Hodge-generic meaning $\mathrm{MT}(F) = G(\mathbb{R})^+$ so by Theorem 4.4.5, there exists a sequence $f_i \in F(\mathbb{R})^+$ and $\gamma_i \in \Gamma$ such that $\gamma_i f_i \to g$. Since w is a smooth point of W, in a small neighborhood V of w, there exist functions $p_1, \ldots, p_d : S \to \mathbb{C}$ such that W is cut out by the p_i. Define $P : L \to \mathbb{C}^d$ as the function $P(z) = (p_1(q(gz)), \ldots, p_k(q(gz)))$ and $P_i : L \to \mathbb{C}^k$ by $P_i(z) = (p_1(q(\gamma_i f_i z)), \ldots, p_k(q(\gamma_i f_i z)))$. Hence we have $P_i \to P$ and $P(x) = 0$.

Since $\dim L = \mathrm{codim} W$, and $x \in Z$, their intersection is typical by Lemma 4.4.9 and so x is an isolated zero of P and there exists a open neighborhood $U \subset L$ of x such that $P(y) \neq 0$ for all $y \in U \backslash \{x\}$. Let $\varepsilon = \inf_{y \in \partial U} \|P(y)\| > 0$. Then there exists N such that for all $i \geq N$ the supremum $\sup_{y \in \partial U} \|P_i(y) - P(y)\| < \varepsilon$ and hence Rouche's theorem tells us there exists $x_N \in U$ such that $P_N(x_N) = 0$. But this means that $q(\gamma_N f_N x_N) = q(f_N x_N) \in W$. And since $x_N \in L$ and $f_N \in F(\mathbb{R})^+$ so $f_N x_N \in L$ as well. By shrinking U as necessary, we get a sequence $f_i x_i \in L$ such that $q(f_i x_i) \to w$ and $q(f_i x_i) \in W$. Since this is true for any smooth point of W lying in Z, we get a Euclidean dense intersection $W \cap q(L)$ inside of W. □

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