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Some rigidity results for coinduced actions and structural results for group von Neumann algebras

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Some rigidity results for coinduced actions and structural results for group von Neumann algebras

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by

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The dissertation of Daniel Drimbe is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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of which the dissertation author was the primary investigator and author.

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ABSTRACT OF THE DISSERTATION

Some rigidity results for coinduced actions and structural results for group von Neumann algebras

by

Daniel Drimbe

Doctor of Philosophy in Mathematics

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The first result which we prove in this dissertation is a cocycle superrigidity theorem for a large class of coinduced actions. In particular, if Σ is an infinite index subgroup of a countable group Γ, we consider a probability measure preserving (pmp) action $\Sigma \curvearrowright X_0$ and let $\Gamma \curvearrowright X$ be the coinduced action. Assume either that Γ has property (T) or that Σ is amenable and Γ is a product of non-amenable groups. Using Popa's deformation/rigidity theory we prove $\Gamma \curvearrowright X$ is $\mathcal{U}_{fin}$-cocycle superrigid, that is any cocycle for this action to a $\mathcal{U}_{fin}$ (e.g. countable) group $\mathcal{V}$ is cohomologous to a homomorphism from Γ to $\mathcal{V}$. This is done in Chapter II.

We then study in Chapter III structural results of group von Neumann algebras arising from certain lattices following the joint work [DHI16] with Daniel Hoff and Adrian Ioana. We describe all tensor product decompositions of $L(\Gamma)$ for icc countable groups Γ that are measure equivalent to a product of non-elementary hyperbolic groups. In particular, we show that $L(\Gamma)$ is prime, unless Γ is a product of infinite groups, in which case we prove a unique prime factorization result for $L(\Gamma)$. As a corollary we obtain that if Γ is an icc irreducible lattice in a product of connected non-compact rank one simple Lie groups with finite center, then the II_1 factor $L(\Gamma)$ is prime. In particular, we deduce that the II_1 factors associated to the arithmetic groups $\text{PSL}_2(\mathbb{Z}[\sqrt{d}])$ and $\text{PSL}_2(\mathbb{Z}[S^{-1}])$ are prime, for any square-free integer $d \geq 2$ with $d \not\equiv 1 \pmod 4$ and any finite non-empty set of primes S. This provides the first examples of prime II_1 factors arising from lattices in higher rank semisimple Lie groups.

Finally, we prove in Chapter IV W*-superrigidity for a large class of coinduced actions. We prove that if Σ is an amenable almost-malnormal subgroup of an icc property (T) countable group Γ, the coinduced action $\Gamma \curvearrowright X$ from an arbitrary pmp action $\Sigma \curvearrowright X_0$ is W*-superrigid. More precisely, if $\Lambda \curvearrowright Y$ is another free ergodic pmp action such that the crossed-product von Neumann algebras are isomorphic $L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$, then the actions are conjugate. We also prove a similar statement if Γ is an icc non-amenable group which is measure equivalent to a product of two infinite groups. In particular, we obtain that any Bernoulli action of such a group Γ is W*-superrigid.

Chapter I

Introduction

I.1 Background

A central theme in the theory of von Neumann algebras is the classification of $L(\Gamma)$ in terms of the group Γ and of $L^\infty(X) \rtimes \Gamma$ in terms of the group action $\Gamma \curvearrowright (X, \mu)$. The most interesting case is when Γ is infinite conjugacy class (icc) and, respectively, when the action $\Gamma \curvearrowright (X, \mu)$ is free, ergodic and probability measure preserving (pmp). These conditions guarantee that the corresponding von Neumann algebras are II_1 *factors*, i.e. indecomposable infinite dimensional von Neumann algebras which admit a trace. Moreover, Singer has proven in [Si55] that the isomorphism class of $L^\infty(X) \rtimes \Gamma$ only depends on the equivalence relation given by the orbits of $\Gamma \curvearrowright (X, \mu)$. This led to the study of group actions up to orbit equivalence [Dy58] and we refer to [Sh04, Fu09, Ga10] for surveys about recent activity in this new branch of ergodic theory.

The strong amenable/non-amenable dichotomy is crucial in the classification of II_1 factors. If the groups are amenable, the classification is complete. More precisely, the greatly celebrated work of Alain Connes [Co76] shows that all icc amenable groups and all the free ergodic pmp actions give rise to the same von Neuman algebra, known as the hyperfinite II_1 factor. In contrast, the non-amenable case is much more challenging and it has led to a challenging, but beautiful rigidity theory. Various aspects of the groups and actions are remembered by their von Neumann algebras in the non-amenable case. One of the major recent achievements has been the discovery of classes of groups and group actions that can be entirely reconstructed from their von Neumann algebras. This progress has been made possible by the success of the deformation/rigidity theory developed by Sorin Popa (see [Po07, Va10a, Io12a, Io17] for surveys).

I.1.1 Organization

At the beginning of this thesis we establish the necessary preliminaries in Section I.2, the motivation for this work in Section I.3 and present the main results in Section I.4. We continue in Chapter II with presenting a cocycle superrigidity theorem for coinduced actions, following [Dr15]. Chapter III follows the joint work [DHI16] with Daniel Hoff and Adrian Ioana in which we describe all tensor product decompositions of von Neumann algebras associated to groups that are measure equivalent to a product of non-elementary hyperbolic groups. Finally, following [Dr17] we present W^*-superrigidity for coinduced actions.

I.2 Preliminaries

I.2.1 Von Neumann algebras

Let H be a separable complex Hilbert space. Denote by $B(H)$ the $*$-algebra of bounded linear operators on H, where for each $T \in B(H)$ we define $T^* \in B(H)$ by $\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$, for all $\xi, \eta \in H$. We denote by $1 \in B(H)$, the identity operator.

We endow $B(H)$ with the following two different topologies. We say that a sequence of operators $T_n \in B(H)$ converges to $T \in B(H)$ in the

- *norm topology* if $\|T_n - T\| \to 0$, as $n \to \infty$.
- *weak operator topology (wot)* if for all $\xi, \eta \in H$, $\langle T_n\xi, \eta\rangle \to \langle T\xi, \eta\rangle$, as $n \to \infty$.

A *von Neumann algebra* M is a unital $*$-subalgebra of $B(H)$ that is closed in the weak operator topology. Given any subset $S \subset B(H)$ which is closed under the adjoint $*$-operation, the *commutant* S' is a von Neumann algebra.
An essential theorem in the theory of von Neumann algebras is *von Neumann's Bicommutant Theorem* which states that all von Neumann algebras arise in this way. More precisely, von Neumann has proven that a unital $*$-subalgebra $M \subset B(H)$ is a von Neumann algebra if and only if $M'' = M$.

In the following we provide some important examples of von Neumann algebras. In particular we describe the construction of group and group measure space von Neumann algebras due to Murray and von Neumann [MvN36, MvN43].

Example I.2.1. Given a standard probability space (X, μ), the abelian algebra $L^\infty(X, \mu) \subset B(L^2(X, \mu))$ is a von Neumann algebra. Here $L^\infty(X, \mu)$ is represented on the Hilbert space $L^2(X, \mu)$ by pointwise multiplication.

Example I.2.2. Let Γ be a countable group and denote by $\{\delta_h\}_{h\in\Gamma}$ the canonical orthonormal basis of $\ell^2(\Gamma)$. The left regular representation $u : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ is given by $u_g(\delta_h) = \delta_{gh}$, for all $g, h \in \Gamma$. The *group von Neumann algebra* $L(\Gamma)$ is the closure of $\mathbb{C}\Gamma$ =span $\{u_g\}_{g\in\Gamma}$ in the weak operator topology.

Example I.2.3. Let $\Gamma \curvearrowright (X, \mu)$ be a probability measure preserving (pmp) action of a countable group Γ on a standard probability space (X, μ). Denote by $(\sigma_g)_{g\in\Gamma}$ the associated action of Γ on $L^\infty(X)$, i.e. $\sigma_g(a)(x) = a(g^{-1} \cdot x)$. Note that both Γ and $L^\infty(X)$ can be represented on the Hilbert space $L^2(X, \mu) \otimes \ell^2(\Gamma)$ through the formulas:

$$u_g(b \otimes \delta_h) = \sigma_g(b) \otimes \delta_{gh} \text{ and } a(b \otimes \delta_h) = ab \otimes \delta_h,$$

for all $g, h \in \Gamma$, $a \in L^\infty(X)$ and $b \in L^2(X,\mu)$.

The *group measure space von Neumann algebra* $L^\infty(x) \rtimes \Gamma$ is $\{u_g, a | g \in \Gamma, a \in L^\infty(X)\}''$, the von Neumann algebra generated by $\{u_g\}_{g\in\Gamma}$ and $L^\infty(X)$.

I.2.2 Tracial von Neumann algebras

The study of tracial von Neumann algebras has attracted a lot of interest. A von Neumann algebra M is called *tracial* if there exists a faithful normal positive linear functional $\tau : M \to \mathbb{C}$ which satisfies $\tau(1) = 1$ and $\tau(xy) = \tau(yx)$, for all $x, y \in M$. The map τ is called a *trace* for M.

Any tracial von Neumann algebra (M,τ) admits a canonical or standard representation on a Hilbert space. Indeed, denote by $L^2(M)$ the Hilbert space obatined by completing M with respect to the 2-norm: $\|x\|_2 = \tau(x^*x)^{1/2}$. Then the left multiplication on M extends to an injective $*$-homomorphism $\pi : M \to B(L^2(M))$.

Example I.2.4. For any countable group Γ and any pmp action $\Gamma \curvearrowright X$ on a standard probabilty space (X,μ), the von Neumann algebras $L(\Gamma)$ and $L^\infty(X) \rtimes \Gamma$ are tracial.

A von Neumann algebra M is called of *type II$_1$* if it is tracial and infinite dimensional. M is called a *factor* if it has trivial center, i.e. $\mathcal{Z}(M) = M' \cap M = \mathbb{C}1$.

Example I.2.5. The following examples of II$_1$ factors are of high interest in the theory of von Neumann algebras.

- $L(\Gamma)$, for any infinite conjugacy class (icc) countable group Γ.
- $L^\infty(X,\mu) \rtimes \Gamma$, for any free ergodic pmp action $\Gamma \curvearrowright (X,\mu)$ of a countable group Γ on a standard probability space (X,μ).

I.2.3 Amenable von Neumann algebras and relative amenability

A tracial von Neumann algebra (M,τ) is called *amenable* if there exists a positive linear functional $\varphi:\mathbb{B}(L^2(M))\to\mathbb{C}$ such that $\varphi_{|M}=\tau$ and φ is *M-central*, in the following sense: $\varphi(xT)=\varphi(Tx)$, for all $x\in M$ and $T\in\mathbb{B}(L^2(M))$.

Amenability plays a big role in the theory of von Neumann algebras. To ilustrate this, we mention the celebrated results of Connes [Co76] which show that M is amenable if and only if M is *hyperfinite*, i.e. M can be written as the wot closure of an increasing sequence of finite dimensional unital $*$-subalgebras.

In this thesis we make extensive use of the notion of relative amenability introduced by Ozawa and Popa. Let $p\in M$ be a projection, and $P\subset pMp, Q\subset M$ be von Neumann subalgebras. Following [OP07, Section 2.2] we say that P is *amenable relative to Q inside M* if there exists a positive linear functional $\varphi:p\langle M,e_Q\rangle p\to\mathbb{C}$ such that $\varphi_{|pMp}=\tau$ and φ is P-central.

I.2.4 Popa's intertwining-by-bimodules

We next recall from [Po03, Theorem 2.1 and Corollary 2.3] the powerful *intertwining-by-bimodules* technique of Popa.

Theorem I.2.6 ([Po03])**.** *Let (M,τ) be a tracial von Neumann algebra and $P\subset pMp, Q\subset qMq$ be von Neumann subalgebras. Let $\mathcal{U}\subset\mathcal{U}(P)$ be a subgroup such that $\mathcal{U}''=P$.*

Then the following are equivalent:

- *There exist projections $p_0\in P, q_0\in Q$, a $*$-homomorphism $\theta:p_0Pp_0\to q_0Qq_0$ and a non-zero partial isometry $v\in q_0Mp_0$ such that $\theta(x)v=vx$, for all $x\in p_0Pp_0$.*

- *There is no sequence $u_n\in\mathcal{U}$ satisfying $\|E_Q(x^*u_ny)\|_2\to 0$, for all $x,y\in pMq$.*

If one of these equivalent conditions holds true, then we write $P \prec_M Q$, and say that a corner of P embeds into Q inside M. If $Pp' \prec_M Q$ for any non-zero projection $p' \in P' \cap pMp$, then we write $P \prec_M^s Q$.

Convention. Whenever the ambient algebra (M, τ) is clear from the context, we will write $P \prec Q$ instead of $P \prec_M Q$. We will also say that P *is amenable relative to* Q instead of P *is amenable relative to* Q *inside* M.

I.3 Classifiction of probability measure preserving actions and of von Neumann algebras

Question. A central problem in operator algebras is to understand how much of the group Γ and of the group action $\Gamma \curvearrowright X$ is "remembered" by their von Neumann algebras $L(\Gamma)$ and $L^\infty(X) \rtimes \Gamma$, respectively.

The work of Connes [Co76] shows that amenable groups manifest a striking lack of rigidity: algebraic properties of the group (e.g. torsion freeness) and properties of the action (e.g. mixing) are lost once we pass to the von Neumann algebraic level. On the other hand, the non-amenable case has led to a complex and interesting rigidity theory. A huge progress has been made in this direction (see [Po07, Va10a, Io12a, Io17] for surveys), nevertheless there are still some famous open problems which show how hard, but interesting is the case when the groups are non-amenable.

Connes' rigidity conjecture. [Co82] If Γ is an icc countable group with property (T) (e.g. $\Gamma = SL_n(\mathbb{Z})$ with $n \geq 3$) and Λ is a countable group such that $L(\Gamma) \simeq L(\Lambda)$, then $\Gamma \simeq \Lambda$.

The Free Group Factor Problem. Is it true that if m and n are positive integers such that $L(\mathbb{F}_m) \simeq L(\mathbb{F}_n)$, then $m = n$?

The intense activity in the area has recently culminated with Popa and Vaes' resolution of the group measure space version of the Free Group Factor Problem:

Theorem I.3.1. *[PV11] If $L^\infty(X) \rtimes \mathbb{F}_m \simeq L^\infty(Y) \rtimes \mathbb{F}_n$, where $\mathbb{F}_m \curvearrowright X$ and $\mathbb{F}_n \curvearrowright Y$ are two free ergodic pmp actions of the free groups $\mathbb{F}_m$ and $\mathbb{F}_n$, respectively, then $m = n$.*

I.3.1 Classification of probability measure preserving actions

Two free ergodic pmp actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are called

- *conjugate* if there exists a measure space isomorphism $\theta : X \to Y$ and a group isomorphism $d : \Gamma \to \Lambda$ such that $\theta(gx) = d(g)\theta(x)$, for all $g \in \Gamma$ and a.e. $x \in X$.
- *orbit equivalent* (OE), if there exists a measure space isomorphism $\theta : X \to Y$ such that $\theta(\Gamma x) = \Lambda\theta(x)$, for a.e. $x \in X$.
- *W*-equivalent* if the associated group measure space von Neumann algebras $L^\infty(X) \rtimes \Gamma$ and $L^\infty(Y) \rtimes \Lambda$ are isomorphic.

Singer proved in [Si55] that two actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are OE if and only if there exists an isomorphism of the group measure space algebras $L^\infty(X) \rtimes \Gamma$ and $L^\infty(Y) \rtimes \Lambda$ which preserves the so-called Cartan algebras $L^\infty(X)$ and $L^\infty(Y)$. In particular, this gives the following implications:

$$\text{Conjugacy} \quad \Rightarrow \quad \text{OE} \quad \Rightarrow \quad \text{W}^*\text{-equivalency.}$$

Rigidity appears whenever an implication in the previous diagram can be reversed for all actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ belonging to two classes of actions. The most extreme form of rigidity happens when this can be achieved without assuming any restrictions on the second class of actions. Therefore, an action $\Gamma \curvearrowright (X,\mu)$ is called *OE-superrigid* *(respectively W*-superrigid)* if whenever $\Lambda \curvearrowright (Y,\nu)$ is a free ergodic pmp action OE (respectively W*-equivalent) to $\Gamma \curvearrowright X$, then the two actions are conjugate.

I.4 The main results of the thesis and the content of the chapters

I.4.1 Chapter II: Cocycle and orbit equivalence superrigidity for coinduced actions

A general principle, going back to Zimmer [Zi84] and made precise by Popa [Po05], asserts that:

$$\text{Cocycle superrigidity} \;\Rightarrow\; \text{OE-superrigidity.} \tag{I.4.1}$$

Here we say that $\Gamma \curvearrowright X$ is *cocycle superrigid*, if every cocycle $w : \Gamma \times X \to \Lambda$, where Λ is an arbitrary countable group, is cohomologous to a group homomorphism. Behind the principle (I.4.1) is the remark that any OE between two free ergodic pmp actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ gives rise to the so-called Zimmer cocycle $w : \Gamma \times X \to \Lambda$. Once this cocycle is untwisted, one essentially obtains that the two actions are conjugate.

Remark I.4.1. This observation implies that the study of cocycles is an important approach in the classification of pmp actions up to OE.

In his breakthrough work [Po05, Po06a], Popa used his deformation/rigidity theory to prove a remarkable cocycle superrigidity theorem for Bernoulli actions of groups with property (T) and of products of non-amenable groups. Popa discovered in [Po01, Po03] that Bernoulli actions satisfy a remarkable deformation property, called *malleability*, i.e. there exists a continuous family of Γ-equivariant automorphisms of $(X \times X, \mu \times \mu)$, which connect the identity to the flip $F : (x,y) \to (y,x)$. Popa's cocycle superrigidity theorem holds actually for this more general class of malleable actions, which includes also Gaussian actions (see[Po05] and [Fu06]).

Our first main result generalizes Popa's cocycle superrigidity theorem to coinduced actions. Since coinduced actions are not necessary malleable in the sense of Popa, we use a different deformation introduced by Adrian Ioana in [Io06a] for Bernoulli actions with the base any tracial von Neumann algebra. In our work, we adapt the deformation of [Io06a] to the context of general coinduced actions.

First we recall that if $\Sigma \subset \Gamma$ is a subgroup of a group Γ and $\Sigma \overset{\sigma_0}{\curvearrowright} (X_0, \mu_0)$ is a pmp action, then there is a canonical way to obtain another pmp action $\Gamma \overset{\sigma}{\curvearrowright} (X_0, \mu_0)^{\Gamma/\Sigma} =: (X, \mu)$, called *the coinduced action of* σ_0. If Σ is the trivial group, then σ is precisely the Bernoulli action $\Gamma \curvearrowright (X_0, \mu_0)^{\Gamma}$ (see Definition II.1.1 for the precise definition).

Theorem I.4.2 (see Theorem A)**.** *Let Σ be an infinite index subgroup of a property (T) countable group Γ. Let $\Sigma \overset{\sigma_0}{\curvearrowright} X_0$ be a pmp action of Σ on a non-trivial standard probability space (X_0, μ_0) and let $\Gamma \curvearrowright X$ be the coinduced action associated to σ_0. Then $\Gamma \curvearrowright X$ is cocycle superrigid.*

Kazhdan's property (T) holds for a broad class of countable groups including higher rank lattices, e.g. $SL_n(\mathbb{Z})$, with $n \geq 3$ [Ka67]. Note that the conclusion of Theorem I.4.2 allows to coinduce from an arbitrary action $\Sigma \curvearrowright X_0$. In particular, any coinduced action of $SL_3(\mathbb{Z})$ from an arbitrary infinite index subgroup is cocycle superrigid. In Theorem B we have extended the class of groups for which the conclusion of Theorem I.4.2 holds to product groups.

I.4.2 Chapter III: Prime II_1 factors arising from lattices in higher rank

The third chapter is the result of a collaboration with Daniel Hoff and Adrian Ioana [DHI16]. The goal is to prove primeness results for certain irreducible lattices Γ in higher rank. We do this by describing all tensor product decompositions of $L(\Gamma)$.

A II_1 factor M is called *prime* if it is not isomorphic to a tensor product of II_1 factors. In [Po83], Popa proved that the free groups on uncountably many generators give rise to prime II_1 factors. By using Dan Voiculescu's free probability theory, Liming Ge showed that the free group factors $L(\mathbb{F}_n)$ are prime [Ge96]. Subsequently, other primeness results were found, but a common feature of these results is that the groups Γ for which $L(\Gamma)$ was proven to be prime, have "rank one" properties such as hyperbolicity .

In spite of the remarkable advances made in the study of II_1 factors in the last 15 years, little is known about the structure of II_1 factors associated to lattices in higher rank semisimple Lie groups. We have provided the first examples of lattices in higher rank semisimple Lie groups which give rise to prime II_1 factors. More precisely, we have proven:

Theorem I.4.3 (see Theorem D)**.** *If Γ is an icc irreducible lattice in a product $G = G_1 \times ... \times G_n$ of $n \geq 1$ connected non-compact rank one simple real Lie groups with finite center, then the II_1 factor $L(\Gamma)$ is prime.*

As a corollary, we obtain that $M = L(\mathrm{PSL}_2(\mathbb{Z}[\sqrt{2}]))$ is prime. As a particular case of [CdSS15, Corollary C], one has that M is not isomorphic to $L(\mathbb{F}_2 \times \mathbb{F}_2)$. Our result considerably strengthens this fact by showing that $L(\mathrm{PSL}_2(\mathbb{Z}[\sqrt{2}]))$ is not isomorphic to $L(\Gamma_1 \times \Gamma_2)$, for any non-trivial countable groups Γ_1, Γ_2.

The following theorem is the main technical result of [DHI16] which in particular proves Theorem I.4.3. We completely classify all tensor product decompositions $L(\Gamma) = P_1 \bar{\otimes} P_2$. For doing this, we use a combination of techniques from Popa's deformation/rigidity theory.

Theorem I.4.4 (see Theorem F)**.** *Let Γ be a countable icc group and denote $M = L(\Gamma)$. Assume that Γ is measure equivalent to a product $\Lambda = \Lambda_1 \times ... \times \Lambda_n$ of $n \geq 1$ non-elementary hyperbolic groups $\Lambda_1, ..., \Lambda_n$. Suppose that $M = P_1 \overline{\otimes} P_2$, for some II_1 factors P_1 and P_2.*

Then there exist a decomposition $\Gamma = \Gamma_1 \times \Gamma_2$, a partition $S_1 \sqcup S_2 = \{1, ..., n\}$ and a unitary $u \in M$ such that:

1. *uP_iu^* is stably isomorphic to $L(\Gamma_i)$, for any $i \in \{1, 2\}$.*

2. *Γ_i is measure equivalent to $\times_{j \in S_i} \Lambda_j$ for any $i \in \{1, 2\}$.*

See Definition III.1.6 for the definition of measure equivalence. We also mention that another application of Theorem I.4.4 gives a prime factorization result for tensor products of II_1 factors arising from irreducible lattices in products of rank one simple Lie groups.

I.4.3 Chapter IV: W*-superrigidity for coinduced actions

Popa has proved in [Po03, Po04] a W*-rigidity result, in which one can deduce conjugacy of two actions out of an isomorphism of their crossed product von Neumann algebras. More precisely, he proved the following: let $\Gamma \curvearrowright X$ be a free ergodic pmp action of an icc countable group Γ with property (T) and let $\Lambda \curvearrowright Y := Y_0^{\Lambda}$ be a Bernoulli action of a countable group Λ. Popa proves that if the two actions have their corresponding von Neumann algebras isomorphic, then the actions are conjugate. After this, Ioana proved in [Io10] that any Bernoulli action of an icc property (T) group is W*-superrigid.

In [Dr17] we provide a large class of W*-superrigid coinduced actions. Before writing the result, we recall some notions. A subgroup Σ of a countable group Γ is called *n-almost malnormal* if for any $g_1, g_2, ..., g_n \in \Gamma$ such that $g_i^{-1} g_j \notin \Sigma$ for all $i \neq j$, the subgroup $\cap_{i=1}^n g_i \Sigma g_i^{-1}$ if finite. The subgroup Σ is called *almost malnormal* if it is n-almost malnormal for some $n \geq 1$.

Theorem I.4.5 (see Theorem H). *Let* Γ *be an icc group which admits an infinite normal subgroup* Γ_0 *with relative property (T) and let* Σ *be an amenable almost malnormal subgroup of* Γ. *Let* σ_0 *be a pmp action of* Σ *on a non-trivial standard probability space* (X_0, μ_0) *and denote by* σ *the coinduced action of* Γ *on* $X := X_0^{\Gamma/\Sigma}$.
Then $\Gamma \overset{\sigma}{\curvearrowright} X$ *is W*-superrigid.*

Ioana, Popa and Vaes have proven W*-superrigidity for Bernoulli actions of product groups in [IPV10]. In Theorem I we extend also the class of groups for which the conclusion of Theorem I.4.5 holds to product groups. We actually prove a more general statement which provides in particular a larger class of groups for which any Bernoulli action is W*-superrigid. More precisely, we obtain:

Corollary I.4.6 (see Corollary J). *Let* Γ *be an icc non-amenable group which is measure equivalent to a product of two infinite groups. Let* (X_0, μ_0) *be a non-trivial standard probability space.*
Then the Bernoulli action $\Gamma \curvearrowright X_0^\Gamma$ *is W*-superrigid.*

In particular, we obtain that if Γ is an icc lattice in a product $G = G_1 \times \dots G_n$ of $n \geq 2$ connected non-compact semisimple Lie groups, then any Bernoulli action of Γ is W*-superrigid. This follows using Armand Borel's theorem which gives us that any G_i contains a lattice (see [Bo63] and [Ra72, Theorem 14.1]).

Terminology

We fix notation regarding tracial von Neumann algebras and countable groups. We denote by $L^2(M)$ the completion of a tracial von Neumann algebra (M,τ) with respect to the norm $\|x\|_2 = \sqrt{\tau(x^*x)}$ and consider the standard representation $M \subset \mathbb{B}(L^2(M))$. Unless stated otherwise, we will always assume that M is separable, i.e. $L^2(M)$ is a separable Hilbert space. For a set $\mathcal{S} \subset \mathbb{B}(L^2(M))$, we denote by $\mathcal{S}'$ its commutant. If $\mathcal{S}$ is closed under adjoint, then by von Neumann's double commutant theorem, $\mathcal{S}'' = (\mathcal{S}')'$ is exactly the von Neumann algebra generated by $\mathcal{S}$. We denote by $\mathcal{U}(M)$ the group of unitary elements of M, by $(M)_1 = \{x \in M \mid \|x\| \leq 1\}$ the unit ball of M, and by $\mathcal{Z}(M) = M \cap M'$ the center of M.

Let $P \subset M$ be a von Neumann subalgebra, which we will always assume to be unital. We denote by $e_P : L^2(M) \to L^2(P)$ the orthogonal projection onto $L^2(P)$, by $E_P : M \to P$ the conditional expectation onto P, and by $\mathcal{N}_M(P) = \{u \in \mathcal{U}(M) \mid uPu^* = P\}$ the *normalizer of* P *in* M. The subalgebra $P \subset M$ is called *regular* if $\mathcal{N}_M(P)'' = M$. *Jones' basic construction* of the inclusion $P \subset M$ is defined as the von Neumann subalgebra of $\mathbb{B}(L^2(M))$ generated by M and e_P, and is denoted by $\langle M, e_P\rangle$. If $J : L^2(M) \to L^2(M)$ denotes the involution given by $J(x) = x^*$, for every $x \in M$, then $\langle M, e_P\rangle = (JPJ)' \cap \mathbb{B}(L^2(M))$.

Let $S, T \subset \Gamma$ be two subsets. We denote by $\langle S\rangle$ the group generated by S, and by $C_S(T) = \{g \in S \,|\, gh = hg,$ for all $h \in T\}$ the *centralizer of* T *in* S.

Chapter II

Cocycle and orbit equivalence superrigidity for coinduced actions

II.1 Introduction and statement of main results

II.1.1 Introduction

The goal of this chapter is to prove a general cocycle superrigidity theorem for *coinduced actions* (see Definition II.1.1) and derive several consequences to orbit equivalence and von Neumann algebras.

The classification of probability measure preserving (pmp) actions of countable groups on standard probability spaces up to orbit equivalence has attracted a lot of interest in the last two decades (see the surveys [Po07, Fu09, Ga10, Va10a, Io12a]).

If the groups are amenable, the classification up to orbit equivalence is done. More precisely, Orstein and Weiss proved in [OW80] (see also [Dy58, CFW81]) that all the free ergodic pmp actions of countable amenable groups are orbit equivalent. In contrast, the non-amenable case is much more challenging and complex. Remarkably, several classes of actions which are *rigid* in the sense that one can deduce conjugacy from

OE, have been discovered. The most extreme form of rigidity for orbit equivalence is OE-superigidity. The first OE-superrigidity result was obtained by Furman in the late 1990s by building on Zimmer's cocycle superrigidity [Zi84]. He showed that many actions of higher rank lattices, including the action $SL_n(\mathbb{Z}) \curvearrowright \mathbb{T}^n$, for $n \geq 3$ is OE-superrigid [Fu98, Fu99]. After this, a number of striking OE-superrigidity results were obtained [MS02, Po05, Po06a, Ki06, Io08, PV08, Ki09, PS09, Io14, TD14, CK15, GITD16].

In particular, in his breakthrough work [Po05, Po06a], Popa used his deformation/rigidity theory to prove a remarkable cocycle superrigidity theorem for Bernoulli actions of groups with property (T) and of products of non-amenable groups. More precisely, if $\Gamma \curvearrowright X$ is such an action, Popa obtained that every cocycle with values in a countable (and more generally, in a $\mathcal{U}_{fin}$) group is cohomologous with a group homomorphism. By applying his cocycle superrigidity theorem to cocycles arising from orbit equivalence, he proved that the action $\Gamma \curvearrowright X$ is OE-superrigid.

II.1.2 Statement of the main results

Our main result provides a generalization of Popa's cocycle superrigidity theorem to coinduced actions. We first review some basic concepts starting with the construction of coinduced actions (see e.g. [Io06b]).

Definition II.1.1. Let Γ be a countable group and let Λ be a subgroup. Let $\phi : \Gamma/\Lambda \to \Gamma$ be a section. Define the cocycle $c : \Gamma \times \Gamma/\Lambda \to \Sigma$ by the formula

$$c(g, i) = \phi^{-1}(gi)g\phi(i),$$

for all $g \in \Gamma$ and $i \in \Gamma/\Lambda$.

Let $\Lambda \overset{\sigma_0}{\curvearrowright} (X_0, \mu_0)$ be a pmp action, where (X_0, μ_0) is a non-trivial standard probability space. We define an action $\Gamma \overset{\sigma}{\curvearrowright} X_0^{\Gamma/\Lambda}$, called the coinduced action of σ_0, as follows:

$$\sigma_g((x_i)_{i\in\Gamma/\Lambda}) = (x'_i)_{i\in\Gamma/\Lambda}, \text{ where } x'_i = c(g^{-1},i)^{-1}x_{g^{-1}i}.$$

Note the following remarks:

- σ is a pmp action of Γ on the standard probability space $X_0^{\Gamma/\Lambda}$.
- if we consider the trivial action of $\Lambda = \{e\}$ on X_0, then the coinduced action of Γ on $X_0^{\Gamma/\{e\}} = X_0^{\Gamma}$ is the Bernoulli action.

Alternatively, it can be seen that the coinduced action $\Gamma \overset{\sigma}{\curvearrowright} X_0^{\Gamma/\Lambda}$ can be identified with the natural action of Γ on $\{f : \Gamma \to X_0 | f(g\lambda) = \sigma_0(\lambda)(f(g)), \forall g \in \Gamma, \forall \lambda \in \Lambda\}$.

We say that the inclusion $\Gamma_0 \subset \Gamma$ of countable groups has the *Kazhdan's relative property* (T) if for every $\epsilon > 0$, there exist $\delta > 0$ and $F \subset \Gamma$ finite such that if $\pi : \Gamma \to \mathcal{U}(K)$ is a unitary representation and $\xi \in K$ is a unit vector satisfying $\|\pi(g)\xi - \xi\| < \delta$, for all $g \in F$, then there exists $\xi_0 \in K$ such that $\|\xi - \xi_0\| < \epsilon$ and $\pi(h)\xi_0 = \xi_0$, for all $h \in \Gamma_0$. The group Γ has the *property* (T) if the inclusion $\Gamma \subset \Gamma$ has the relative property (T). To give some examples, $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ has the relative property (T) and $SL_n(\mathbb{Z})$, $n \geq 3$, has the property (T) [Ka67, Ma82].

An infinite subgroup H of Γ is *w-normal* in Γ if there exist an ordinal β and intermediate subgroups $H = H_0 \subset H_1 \subset \cdots \subset H_\beta = \Gamma$ such that for all $0 < \alpha \leq \beta$, the group $\cup_{\alpha' < \alpha} H_{\alpha'}$ is normal in H_α. Denote by $\mathcal{U}_{fin}$ the class of Polish groups which arise as closed subgroups of the unitary groups of II_1 factors. In particular, all countable discrete groups and all compact Polish groups belong to $\mathcal{U}_{fin}$. These two notions are due to Popa [Po05].

For a Polish group G, a measurable map $w : \Gamma \times X \to G$ is called a *cocycle* if it satisfies the relation $w(\gamma_1\gamma_2, x) = w(\gamma_1, \gamma_2 x)w(\gamma_2, x)$, for all $\gamma_1, \gamma_2 \in \Gamma$ and for almost every $x \in X$. Two cocycles $w, w' : \Gamma \times X \to G$ are *cohomologous* if there exists a measurable map $\phi : X \to G$ such that $w'(\gamma, x) = \phi(\gamma x)w(\gamma, x)\phi(x)^{-1}$, for all $\gamma \in \Gamma$ and for almost every

$x \in X$. An action $\Gamma \curvearrowright (X, \mu)$ is called *$\mathcal{U}_{fin}$-cocycle superrigid* if every cocycle with values in a group from $\mathcal{U}_{fin}$ is cohomologous with a group homomorphism.

The following theorem is our first main result, which generalizes Popa's cocycle superrigidity theorem for Bernoulli actions of property (T) groups to coinduced actions (see [Po05] and also [Fu06, Va06]).

Theorem A (Groups with relative property (T))**.** *Let Λ be a subgroup of a countable group Γ. Let $H \subset \Gamma$ be a subgroup with relative property (T). Assume that there does not exist a finite index subgroup H_0 of H which is contained in a conjugate $g^{-1}\Lambda g$ of Λ, for some $g \in \Gamma$.*

Take $\mathcal{V} \in \mathcal{U}_{fin}$. Let σ_0 be a pmp action of Λ on a standard probability space (X_0, μ_0) and σ the coinduced action of Γ on $X := X_0^{\Gamma/\Lambda}$.

Then, any cocycle $w : \Gamma \times X \to \mathcal{V}$ for the restriction of σ to H is cohomologous to a group homomorphism $d : H \to \mathcal{V}$.
Moreover, if H is w-normal in Γ, then w is cohomologous to a group homomorphism $d : \Gamma \to \mathcal{V}$ and therefore $\Gamma \curvearrowright X$ is $\mathcal{U}_{fin}$-cocycle superrigid.

In particular, Theorem A implies that if Γ has property (T) (e.g. $\Gamma = SL_n(\mathbb{Z}), n \geq 3$) and Λ is an infinite index subgroup of Γ (e.g. Λ is cyclic), then any coinduced action of Γ from Λ is $\mathcal{U}_{fin}$-cocycle superrigid.

In [Po06a, Corollary 1.2], Popa proved a cocycle superrigidity theorem for the Bernoulli action of product groups analogous with [Po05, Corollary 5.4]. The next theorem generalizes this result to coinduced actions.

Theorem B (Product groups)**.** *Let Γ be a countable group and Λ be an amenable subgroup. Let H and H' be infinite commuting subgroups of Γ such that H' is non-amenable. Assume that there does not exist a finite index subgroup H_0 of H which is contained in a conjugate $g^{-1}\Lambda g$ of Λ, for some $g \in \Gamma$.*

Take $\mathcal{V} \in \mathcal{U}_{fin}$. Let σ_0 be a pmp action of Λ on a standard probability space (X_0, μ_0) and σ the coinduced action of Γ on $X := X_0^{\Gamma/\Lambda}$.

Then, any cocycle $w : \Gamma \times X \to \mathcal{V}$ for the restriction of σ to HH' is cohomologous to a group homomorphism $d : HH' \to \mathcal{V}$.
Moreover, if H is w-normal in Γ, then w is cohomologous to a group homomorphism $d : \Gamma \to \mathcal{V}$ and therefore $\Gamma \curvearrowright X$ is $\mathcal{U}_{fin}$-cocycle superrigid.

The proof of Theorem B goes along the same lines as the proof of [Po06a, Theorem 4.1]. First, we untwist the cocycle on H using the rigidity gained from the non-amenability of H' (instead of using property (T) as in Theorem A). Then, using weak mixing properties of coinduced actions and the fact that H is normal in HH', we are able to untwist the cocycle on HH'.

We will prove in this paper a more general version of Theorems A and B dealing with coinduced actions of Γ on $A^{\Gamma/\Lambda}$ that arise from actions of Λ on arbitrary tracial von Neumann algebras A.

As an immediate consequence of Theorems A and B, we deduce the following OE-superrigidity result for coinduced actions.

Corollary C (OE-superrigidity)**.** *Let Γ be a countable subgroup with no non-trivial finite normal subgroups and Λ a subgroup. Let $H \subset \Gamma$ be a w-normal subgroup. Assume that there does not exist a finite index subgroup H_0 of H which is contained in a conjugate $g^{-1}\Lambda g$ of Λ, for some $g \in \Gamma$. Assume either that H has the relative property (T) or that Λ is amenable and there exists a non-amenable subgroup of Γ which commutes with H.*
Let σ_0 be a pmp action of Λ on a standard probability space (X_0, μ_0) and σ the coinduced action of Γ on $X := X_0^{\Gamma/\Lambda}$. If $\Gamma \overset{\sigma}{\curvearrowright} X$ is free, then it is OE-superrigid.

We need in Corollary C the freeness assumption of the coinduced action since the proof uses Proposition II.5.2. See Lemma II.5.3 for a large class of coinduced actions that are free. In particular, if $\cap_{g \in \Gamma} g\Lambda g^{-1} = \{e\}$ and (X_0, μ_0) is non-atomic, then $\Gamma \curvearrowright X$ is free.

Corrolary C proves for example that any coinduced action of $SL_3(\mathbb{Z})$ from a cyclic subgroup is OE-superrigid. We contrast this with the remark that any coinduced action of $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$ from one of the copies of $\mathbb{Z}$ is not OE-superrigid. Bowen proved a stronger result which is OE-flexibility for coinduced actions [B10]. In particular, he proved that any two coinduced actions of $\mathbb{F}_2$ from one of the copies of $\mathbb{Z}$ are OE.

II.1.3 Applications to W*-superrigidity

For every measure preserving action $\Gamma \curvearrowright X$ of a countable group Γ on a standard probability space, we associate *the group measure space von Neumann algebra* $L^\infty(X) \rtimes \Gamma$ [MvN36]. If the action $\Gamma \curvearrowright X$ is free, ergodic and pmp, then $L^\infty(X) \rtimes \Gamma$ is a II_1 factor which contains $L^\infty(X)$ as a *Cartan subalgebra*, i.e. a maximal abelian von Neumann algebra whose normalizer generates $L^\infty(X) \rtimes \Gamma$.

Two pmp actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ on two standard probability spaces (X,μ) and (Y,ν) are said to be W*-*equivalent* if $L^\infty(X) \rtimes \Gamma$ is isomorphic with $L^\infty(Y) \rtimes \Lambda$. It can be seen that orbit equivalence is stronger than W*-equivalence. Moreover, Singer proved in [Si55] that two free ergodic pmp actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are orbit equivalent if and only if they are W*-equivalent via an isomorphism which identifies the Cartan subalgebras $L^\infty(X)$ and $L^\infty(Y)$. The action $\Gamma \curvearrowright (X,\mu)$ is W*-*superrigid* if whenever $\Lambda \curvearrowright (Y,\nu)$ is a free ergodic measure preserving action W*-equivalent with $\Gamma \curvearrowright (X,\mu)$, then the two actions are conjugate. Therefore, W*-superrigidity for an action $\Gamma \curvearrowright X$ integrates two different rigidity aspects, which are hard to obtain: OE-superrigidity and *uniqueness of group measure space Cartan subalgebras*. The latter means that whenever $M = L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda$, then the Cartan subalgebras $L^\infty(X)$ and $L^\infty(Y)$ are unitarily conjugate in M.

A few years ago, the first example of *virtually* W*-superrigid actions (i.e. conjugacy is obtained up to finite index subgroups) was found in [Pe09] building on results of [Io08, OP08]. Soon after, Popa and Vaes discovered the first concrete families of W*-superrigid actions [PV09] and Ioana proved that Bernoulli actions of icc property (T) groups are W*-superrigid

[Io10]. Subsequently, several other classes of W*-superrigid actions have been found in [FV10, CP10, HPV10, Io10, IPV10, Va10b, CS11, CSU11, PV11, PV12, Bo12b, CIK13, CK15, GITD16]. By applying Theorems A and B we will deduce W*-superrigidity for a large class of coinduced actions. To obtain these examples, we will use several results in the literature which prove uniqueness of group measure space Cartan subalgebras for various classes of groups.

We denote by $\mathcal{C}$ the class of all countable groups Γ which satisfy one of the following conditions:

1. [CP10] $\Gamma = \Gamma_1 \times \Gamma_2$, where Γ_i is icc and admits an unbounded cocycle into a mixing representation and a non-amenable icc subgroup with the relative property (T), for $i \in \{1,2\}$;

2. [PV11, PV12] $\Gamma = \Gamma_1 \times \Gamma_2 \times ... \times \Gamma_n$ is a finite product of non-elementary hyperbolic groups with $n \geq 2$;

3. [Io12b] Γ is a finite product of groups of the form $\Gamma_1 *_\Sigma \Gamma_2$, each one of them satisfying:

   - $[\Gamma_1 : \Sigma] \geq 2$, $[\Gamma_2 : \Sigma] \geq 3$;
   - there exist $g_1, g_2, ..., g_n \in \Gamma$ such that $\cap_{i=1}^n g_i \Sigma g_i^{-1}$ is finite.

     In addition, we assume than one of the factors $\Gamma_1^0 *_{\Sigma^0} \Gamma_2^0$ of Γ satisfies the conditions: Γ_1^0 has property (T) and Σ^0 is a normal subgroup of Γ_2^0.

If $\Gamma \in \mathcal{C}$ satisfies condition (i), we say that $\Gamma \in \mathcal{C}_i$, whenever $i \in \{1,2,3\}$. For $\Gamma \in \mathcal{C}$, we fix a subgroup Λ satisfying the following:

1. If $\Gamma \in \mathcal{C}_1$, take Λ an amenable subgroup of Γ_1;

2. If $\Gamma \in \mathcal{C}_2$, take Λ an amenable subgroup of one of the factors which appears in Γ;

3. If $\Gamma \in \mathcal{C}_3$, take Λ an amenable subgroup of $\Gamma_1^0 *_{\Sigma^0} \Gamma_2^0$ such that Σ^0 does not have a finite index subgroup which is contained in a conjugate of Λ (e.g. Λ can be taken to be the commutant of Σ^0 in Γ_2^0).

Theorems A and B combined with [CP10, Corollary 5.3][PV12, Theorem 1.1][Io12b, Theorem 1.1] give us the following W*-superrigidity result.

Corollary II.1.2. *Let $\Gamma \in \mathcal{C}$ a group with no non-trivial finite normal subgroups and Λ a subgroup chosen as before. Let $\Lambda \curvearrowright X_0$ be a pmp action on a standard probability space X_0 and let $\Gamma \curvearrowright X$ be the coinduced action of $\Lambda \curvearrowright X_0$. If $\Gamma \curvearrowright X$ is free, then it is W*-superrigid.*

Example II.1.3. If we take $\Gamma = \Gamma_1 *_\Sigma (\Sigma \times \Lambda) \in \mathcal{C}_3$, Corollary II.1.2 gives another proof of W*-superrigidity for the coinduced action proved in [PV09, Example 6.9].

II.2 Preliminaries and cocycle rigidity

At the beginning of this section we review some basic tools of Popa concerning cocycles and continue by introducing the free malleable deformation for Bernoulli actions. The last point will be a cocycle rigidity result of Popa adapted to the context of the free malleable deformations.

II.2.1 Perturbation of cocycles, property (T) and extensions

Let σ be a trace preserving action of Γ on a tracial von Neumann algebra P. A map $w : \Gamma \to \mathcal{U}(P)$ is called a *cocycle* if $w_{gh} = w_g \sigma_g(w_h)$, for all $g, h \in \Gamma$. Two cocycles $w, w' : \Gamma \to \mathcal{U}(P)$ are called *cohomologous* if there exists a unitary $v \in P$ such that $w_g \sigma_g(v) = v w'_g$, for all $g \in \Gamma$.

Lemma II.2.1. *([Po05, Lemma 2.12]) Let w, w' be cocycles for a trace preserving action σ of a group Γ on a tracial von Neumann algebra Q. The following statements are true:*

1. *If there exists* $\delta > 0$ *such that* $\|w_g - w'_g\|_2 \le \delta$, *for all* $g \in \Gamma$, *then there exists a partial isometry* $v \in Q$ *such that* $\|v - 1\|_2 \le 4\delta^{1/2}$ *and* $w_g \sigma_g(v) = v w'_g$, *for all* $g \in \Gamma$.

2. *If for any* $\epsilon > 0$ *there exists* $u \in \mathcal{U}(Q)$ *such that* $\|w_g \sigma_g(u) - u w'_g\|_2 \le \epsilon$, *for all* $g \in \Gamma$, *then* w *and* w' *are cohomologous.*

3. *If* w *and* w' *are cohomologous and* $v \in Q$ *is a partial isometry satisfying* $w_g \sigma_g(v) = v w'_g$, *for all* $g \in \Gamma$, *then there exists* $u \in \mathcal{U}(Q)$ *such that* $uv^*v = v$ *and* $w_g \sigma_g(u) = u w'_g$, *for all* $g \in \Gamma$.

Let σ be a trace preserving action of a countable group Γ on a tracial von Neumann algebra Q. Take $w : \Gamma \to \mathcal{U}(Q)$ a cocycle. Let δ be a positive real number and a finite subset F of Γ. Denote $\Omega_w(\delta, F) = \{w' : \Gamma \to \mathcal{U}(Q) | w' \text{ is a cocycle such that } \|w_g - w'_g\|_2 \le \delta, \forall g \in F\}$. Assuming this context, we have the following result:

Lemma II.2.2. *([Po05, Lemma 4.2]) Let* $H \subset \Gamma$ *be a subgroup with the relative property (T). Then for every cocycle* $w : \Gamma \to \mathcal{U}(Q)$ *and* $\epsilon > 0$, *there exist* $\delta > 0$ *and* F *a finite subset of* Γ *such that for all* $w' \in \Omega_w(\delta, F)$, *there exists* $v \in Q$ *partial isometry satisfying* $\|v - 1\|_2 \le \epsilon$ *and* $w'_h \sigma_h(v) = v w_h$, *for all* $h \in H$.

Definition II.2.3. Let Γ be a countable group and σ be a trace preserving action on a tracial von Neumann algebra (P, τ). The action σ is *weak mixing* if for every $\epsilon > 0$ and finite subset F of $P \ominus \mathbb{C}$, there exists $g \in \Gamma$ such that $|\tau(y^* \sigma_g(x)| \le \epsilon$, for all $x, y \in F$.

Note that if $P = L^\infty(X)$, for (X, μ) a standard probability space, then the action $\Gamma \overset{\sigma}{\curvearrowright} P$ is weakly mixing if and only if the corresponding action $\Gamma \curvearrowright X$ is weakly mixing.

Proposition II.2.4. *([Po05, Proposition 3.6]) Let* σ *and* σ' *be trace preserving actions of a countable group* Γ *on tracial von Neumann algebras* P *and* N *and let* w *be a cocycle for* $\sigma \otimes \sigma'$. *Let* $H \subset \Gamma$ *be an infinite normal subgroup and assume that* σ *is weak mixing on* H. *If* $w_h \in N$, *for all* $h \in H$, *then* $w_g \in N$, *for all* $g \in \Gamma$.

II.2.2 Coinduced actions for tracial von Neumann algebras and the free product deformation

The coinduced action for tracial von Neumann algebras is defined as in Section II.1.2. More precisely, let Γ be a countable group and let Λ be a subgroup. Let $\phi : \Gamma/\Lambda \to \Gamma$ be a section. Define the cocycle $c : \Gamma \times \Gamma/\Lambda \to \Lambda$ by the formula

$$c(g,x) = \phi^{-1}(gx)g\phi(x),$$

for all $g \in \Gamma$ and $x \in \Gamma/\Lambda$.

Let $\Lambda \overset{\sigma_0}{\curvearrowright} (A, \tau_0)$ be a trace preserving action, where (A, τ_0) is a tracial von Neumann algebra. We define an action $\Gamma \overset{\sigma}{\curvearrowright} A^{\Gamma/\Lambda}$, called the coinduced action of σ_0, as follows:

$$\sigma_g((a_h)_{h\in\Gamma/\Lambda}) = (a'_h)_{h\in\Gamma/\Lambda},$$

where $a'_h = c(g^{-1}, h)^{-1} a_{g^{-1}h}$.

Note that σ is a trace preserving action of Γ on the tracial von Neumann algebra $A^{\Gamma/\Lambda}$.

Remark II.2.5. Let $\Lambda \overset{\sigma_0}{\curvearrowright} (X_0, \mu_0)$ be a pmp action, where (X_0, μ_0) is a standard probability space. We consider the associated action of Λ on $L^\infty(X_0, \mu_0)$. On one hand, we obtain a coinduced action $\Gamma \overset{\sigma}{\curvearrowright} L^\infty(X_0, \mu_0)^{\Gamma/\Lambda}$. We also call σ, the associate action of Γ on $X_0^{\Gamma/\Lambda}$. Note that σ is precisely the usual coinduced action of Γ obtained from the action of Λ on X_0.

In [Io06a], Ioana introduced a malleable deformation for general Bernoulli actions, where the base is any tracial von Neumann algebra. This is a variant of the malleable deformation discovered by Popa [Po03] in the case of Bernoulli actions with abelian or hyperfinite base. Here we adapt the deformation of [Io06a] to the context of general coinduced actions.

Let Γ be a countable group and Λ be a subgroup. Let A be a tracial von Neumann algebra and $\Lambda \overset{\sigma_0}{\curvearrowright} A$ be a trace preserving action. Take $\Gamma \overset{\sigma}{\curvearrowright} A^{\Gamma/\Lambda}$ the corresponding coinduced action. Let σ' be a trace preserving action of Γ on another tracial von Neumann algebra (N, τ').

Denote by $\tilde{A}$ the tracial von Neumann algebra $A * L(\mathbb{Z})$, which is the free product of A and $L(\mathbb{Z})$. Take $u \in L(\mathbb{Z})$ the canonical generating Haar unitary. Let $h = h^* \in L(\mathbb{Z})$ be such that $u = \exp(ih)$ and set $u_t = \exp(ith)$ for all $t \in \mathbb{R}$. Denote by $P = A^{\Gamma/\Lambda}$ and $\tilde{P} = \tilde{A}^{\Gamma/\Lambda}$ the tensor product von Neumann algebras and define $\theta : \mathbb{R} \to \mathrm{Aut}(\tilde{P})$ by

$$\theta_t(\otimes_{h\in\Gamma/\Lambda} a_h) = \otimes_{h\in\Gamma/\Lambda} \mathrm{Ad}(u_t)(a_h),$$

where $\otimes_{h\in\Gamma/\Lambda} a_h \in \tilde{P}$ is an elementary tensor.

We observe that θ_t extends naturally as an automorphism of $\tilde{P}\bar{\otimes}N$. Define also $\beta \in \mathrm{Aut}(\tilde{P}\bar{\otimes}N)$ by $\beta_{|P\bar{\otimes}N} = id_{P\bar{\otimes}N}$ and $\beta(\otimes_{h\in F} u) = \otimes_{h\in F} u^*$, for all finite subsets F of Γ/Λ. Notice that the action σ extends naturally to an action $\tilde{\sigma}$ on $\tilde{P}$ by letting $\tilde{\sigma}_g(\otimes_{h\in F} u) = \otimes_{h\in F} u$, for all finite subsets F of Γ/Λ. We denote by ρ the tensor product action $\sigma \otimes \sigma'$ of Γ on $P\bar{\otimes}N$ and by $\tilde{\rho}$ the tensor product action $\tilde{\sigma} \otimes \sigma'$ of Γ on $\tilde{P}\bar{\otimes}N$.

Remark II.2.6. Notice that $\tilde{\rho}$ commutes with the automorphims β and θ_t for all t. Thus, we can consider β and θ_t as automorphisms of $(P\bar{\otimes}N) \rtimes \Gamma$ and $(\tilde{P}\bar{\otimes}N) \rtimes \Gamma$, by extending them in a natural way. Also note that $\beta\theta_t = \theta_{-t}\beta$ and $\beta^2 = id$.

II.2.3 Finite union of translates of a subgroup and a fixed point lemma

Lemma II.2.7. *Let H be a group and H_i subgroups, for $1 \le i \le n$. Suppose that there exist finite subsets F_i of H such that*

$$H = \cup_{i=1}^n F_i H_i.$$

Then there exists $i \in \{1, 2, ..., n\}$ such that H_i is a subgroup of finite index in H.

Proof. We will proceed by induction over n. For $n = 1$ it is clear. Let us suppose the statement is true for $n-1$ and prove it for n. We consider the case where H_n is a subgroup of infinite index in H, otherwise we are done.

Let us write a partition of H via the infinite index subgroup H_n:

$$H = F_n H_n \cup (\cup_{k=1}^{\infty} h_k H_n),$$

where $h_j^{-1} h_i \notin H_n$, for all $i \neq j$ and $h_k^{-1} h_0 \notin H_n$, for all $k \geq 1$ and $h_0 \in F_n$.

Then, $\cup_{i=1}^{\infty} h_i H_n \subset \cup_{i=1}^{n-1} F_i H_i$. Since H can be written as finite union of translates of $\cup_{i=1}^{\infty} h_i H_n$, we obtain that H can be also written as finite union of translates of $\cup_{i=1}^{n-1} F_i H_i$. Thus,

$$H = \cup_{i=1}^{n-1} F_i' H_i,$$

with F_i' some finite subsets of H. Now we can apply the induction hypothesis and conclude that at least one of the H_i's is a subgroup of finite index in H for an $i \in \{1, 2, ..., n\}$. ∎

Remark II.2.8. The following proposition is a consequence of [PV06, Lemma 2.4], but we include a proof for the reader's convenience.

Proposition II.2.9. *Let Γ be a countable group and Λ a subgroup. Let H be another subgroup of Γ. Then there exists a finite set $F \subset \Gamma/\Lambda$ such that $gF \cap F \neq \varnothing$, for all $g \in H$ if and only if there exists a subgroup H_0 of finite index of H such that H_0 is contained in a conjugate $g^{-1}\Lambda g$ of Λ.*

Proof. Let us suppose that there exists a finite set $F \subset \Gamma/\Lambda$ such that $gF \cap F \neq \varnothing$, for all $g \in H$. Let $F = \{f_1, f_2, \dots f_n\}$. Then for all $h \in H$, there exist $i, j \in \{1, 2, ..., n\}$ such that $hf_j\Lambda = f_i\Lambda$. We obtain that $H \subset \cup_{i,j=1}^{n} f_i \Lambda f_j^{-1}$.

Let $H_{ij} := \{h \in H | hf_j\Lambda = f_i\Lambda\}$ and notice that $H = \cup_{i,j=1}^{n} H_{ij}$. For $i \neq j$, if $H_{ij} \neq \varnothing$, take $g_{ij} \in H_{ij}$ an arbitrary element. Observe that $H_{ij} = g_{ij} H_{jj}$. For $i \neq j$, if $H_{ij} = \varnothing$, choose

g_{ij} to be the neutral element. This allows us to write H in the form $H = \cup_{i,j=1}^{n} g_{ij} H_{jj}$, which is sufficient for applying Lemma III.6.5, where g_{ii} is the neutral element for all $i \in \{1, 2, ..., n\}$.

Notice that $H_{ii} = H \cap f_i \Lambda f_i^{-1}$ and at least one of these subgroups is of finite index in H because of Lemma III.6.5.

The converse is easy. This finishes the proof. ∎

For the following lemma we use the notations from Section II.2.2.

Lemma II.2.10. *Let H be a subgroup of Γ. Assume that there does not exist a subgroup H_0 of finite index in H such that H_0 is contained in a conjugate $g^{-1}\Lambda g$ of Λ. Let w_h and w_h' be arbitrary elements in $P \bar{\otimes} N$, for all $h \in H$, and define the map $\alpha : H \to \mathbb{B}(L^2(\tilde{P} \bar{\otimes} N))$ by $\alpha_h(x) = \gamma(w_h') \tilde{\rho}_h(x) w_h$, where $\gamma \in \{id, \theta_1\}$. Let S be the $\| \cdot \|_2$-closed linear subspace of $\tilde{P}$ generated by $\gamma(P)P$. Then*

$$\{\xi \in \tilde{P} \bar{\otimes} N | \alpha_h(\xi) = \xi, \forall h \in H\} \subset S \otimes L^2(N).$$

Proof. We begin the proof with a claim which will prove the lemma.

Claim. For any $\epsilon > 0$ and $\xi, \eta \in \tilde{P} \bar{\otimes} N$ with $\xi, \eta \perp S \bar{\otimes} N$, there exists $h \in H$ such that

$$|\langle \xi, \alpha_h(\eta) \rangle| \leq \epsilon \|\xi\|_2 \|\eta\|_2.$$

To prove the claim, we can assume $\|\xi\|_2 = \|\eta\|_2 = 1$. Let us take $\xi_0, \eta_0 \in \tilde{P} \bar{\otimes} N$ with $\| \cdot \|_2$ norm smaller than 1 and F a finite subset of Γ/Λ such that

$$\|\xi - \xi_0\|_2 \leq \epsilon/2, \quad \xi_0 = \sum_{i=1}^{n} p_i \otimes n_i, \quad p_i \in \tilde{A}^F \subset \tilde{P}, \quad n_i \in N, \quad p_i \perp S, \forall i \in \{1, 2, ..., n\}.$$

and

$$\|\eta - \eta_0\|_2 \le \epsilon/2, \quad \eta_0 = \sum_{i=1}^{n} q_i \otimes m_i, \quad q_i \in \tilde{A}^F \subset \tilde{P}, \quad m_i \in N, \quad q_i \perp S, \forall i \in \{1, 2, ..., n\}.$$

Proposition II.2.9 allows us to take $h \in H$, such that $hF \cap F = \varnothing$. By the triangle inequality we have

$$\begin{aligned} |\langle \xi, \alpha_h(\eta)\rangle| &\le |\langle \xi - \xi_0, \alpha_h(\eta)\rangle| + |\langle \xi_0, \alpha_h(\eta - \eta_0)\rangle| + |\langle \xi_0, \alpha_h(\eta_0)\rangle| \\ &\le \epsilon/2 + \epsilon/2 + |\langle \xi_0, \alpha_h(\eta_0)\rangle|. \end{aligned}$$

We will prove the claim if we show that

$$\langle \xi_0, \alpha_h(\eta_0)\rangle = \langle \xi_0, \gamma(w_h')\tilde{\rho}_h(\eta_0)w_h\rangle = 0. \tag{II.2.1}$$

By linearity and continuity (weak operator topology) we may suppose that $w_h = \otimes_{F'} a_j \otimes n, w_h' = \otimes_{F'} a_j' \otimes n' \in P \bar{\otimes} N$ are elementary tensors with $F' \subset \Gamma/\Lambda$ a finite subset and $\otimes_{F'} a_j, \otimes_{F'} a_j' \in A^{\Gamma/\Lambda} = P, n, n' \in N$. By the above we may assume that $\xi_0 = p_0 \otimes n_0, \eta_0 = q_0 \otimes m_0 \in \tilde{A}^F \bar{\otimes} N$, p_0 and q_0 orthogonal to S and $n_0, m_0 \in N$.

This scalar product in the formula (II.2.1) will be proven to be 0 by computing it more explicitly. First, notice that the elements from $\tilde{P}$ which appear in the scalar product belong to$\tilde{A}^{F \cup hF \cup F'}$. Denote by $\tilde{\tau}$ the trace on $\tilde{P}$. Then, since $F \cap hF = \varnothing$, we have the decomposition

$$\langle \xi_0, \gamma(w')\rho_h(\eta_0)w\rangle = \tilde{\tau}(b_1)\tilde{\tau}(b_2),$$

where $b_1 = \otimes_{F \cap F'} a_j^* \gamma(a_j'^*) p_0 \in \tilde{A}^F$ and $b_2 \in \tilde{A}^{(hF \cup F') \smallsetminus F} \bar{\otimes} N$.

The first factor is 0 because p_0 is orthogonal to S. This proves the claim. □

Now, we can finish the proof of the lemma. Take $v \in \tilde{P} \bar{\otimes} N$ such that $\alpha_h(v) = v$, for all $h \in H$. Write $v = v_0 + v_1$ with $v_0 \in S \otimes L^2(N)$ and $v_1 \perp S \otimes L^2(N)$. Since $S \otimes L^2(N)$ is

α-invariant, we get that v_0 and v_1 are α-invariant. The claim gives us that $v_1 = 0$, which implies that $v \in S\bar{\otimes}N$. This ends the lemma. ■

II.2.4 Cocycle rigidity

The following proposition is the first part of [Po05, Proposition 3.2]. Before writing the result, let us introduce some terminology.

Let Γ be a countable group and let σ be a trace preserving action of Γ on a tracial von Neumann algebra Q. We recall that a *local cocycle* for the action σ is a map w on Γ with values in the set of partial isometries of Q which satisfies $w_g\sigma_g(w_h) = w_{gh}$, for all $g, h \in \Gamma$.

Let σ' be a trace preserving action of Γ on another tracial von Neumann algebra N and denote by ρ the tensor product action $\sigma \otimes \sigma'$. For a cocycle $w : \Gamma \to \mathcal{U}(Q\bar{\otimes}N)$, we denote by $w^l : \Gamma \to \mathcal{U}(Q\bar{\otimes}Q\bar{\otimes}N)$ the image of w via the canonical isomorphism and inclusion $Q\bar{\otimes}N \simeq Q\bar{\otimes}1\bar{\otimes}N \subset Q\bar{\otimes}Q\bar{\otimes}N$. Similarly, we denote by w^r the image of w via the canonical isomorphism and inclusion $Q\bar{\otimes}N \simeq 1\bar{\otimes}Q\bar{\otimes}N \subset Q\bar{\otimes}Q\bar{\otimes}N$.

Proposition II.2.11. *[Po05, Proposition 3.2] Let σ be a weak mixing trace preserving action of Γ on a tracial von Neumann algebra Q and σ' a trace preserving action of Γ on another tracial von Neumann algebra N. Let $w : \Gamma \to \mathcal{U}(Q\bar{\otimes}N)$ be a cocycle for the action ρ. Let $b \in L^2(Q\bar{\otimes}Q\bar{\otimes}N)$ be a non-zero element and $p \in \mathcal{P}(Q\bar{\otimes}1\bar{\otimes}N)$, a non-zero projection such that $pb = b$ and $w^l_g\bar{\sigma}_g(b)w^{r*}_g = b$, for all $g \in \Gamma$, where $\bar{\sigma} := \sigma \otimes \sigma \otimes \sigma'$. Then, there exist a partial isometry $v \in Q\bar{\otimes}N$ and a local cocycle $w'_g \in \mathcal{U}(v^*vN\sigma'_g(v^*v))$ such that $vv^* \leq p, v^*v \in N$ and $w_g(\sigma_g \otimes \sigma'_g)(v) = vw'_g$, for all $g \in \Gamma$.*

Remark II.2.12. Let us explain why the first part of [Po05, Proposition 3.2] can be written as above.

- In [Po05] the tracial von Neumann algebra (Q, τ) is extended to a larger tracial

von Neumann algebra $(\tilde{Q}, \tilde{\tau})$ satisfying the following properties: it exists a trace preserving action $\tilde{\sigma}$ of Γ on $\tilde{Q}$ which extends σ and an automorphism α_1 of $\tilde{Q}$ which satisfies $\overline{sp}^w\, Q\alpha_1(Q) = \tilde{Q}$ and $\tilde{\tau}(x\alpha_1(y)) = \tau(x)\tau(y)$, for all $x, y \in Q$.
In particular, it can be chosen $\tilde{Q} = Q\bar{\otimes}Q$.

- Notice that b can be chosen in $L^2(\tilde{Q}\bar{\otimes}N)$ in [Po05, Proposition 3.2], not necessary in $\tilde{Q}\bar{\otimes}N$, since the proof uses only this information.

From now on until the end of the section, we assume the following context. Let Λ be a subgroup of a countable group Γ. Let σ_0 be a trace preserving action of Λ on a tracial von Neumann algebra A and σ the coinduced action of Γ on $P := A^{\Gamma/\Lambda}$. Let us consider a trace preserving action σ' of Γ on another tracial von Neumann algebra N.

Denote by ρ the tensor product action $\sigma\bar{\otimes}\sigma'$ of Γ on $P\bar{\otimes}N$, by $\tilde{\rho}$ the tensor product action $\tilde{\sigma}\otimes\sigma'$ of Γ on $\tilde{P}\bar{\otimes}N$ and by $\bar{\sigma}$ the tensor product action $\sigma\otimes\sigma\otimes\sigma'$ of Γ on $P\bar{\otimes}P\bar{\otimes}N$.

Let $w : \Gamma \to \mathcal{U}(P\bar{\otimes}N)$ be a cocycle for ρ. Define the representations $\pi : \Gamma \to \mathcal{U}(L^2(P\bar{\otimes}P\bar{\otimes}N))$ and $\gamma : \Gamma \to \mathcal{U}(\overline{sp}\, P\theta_1(P) \otimes L^2(N))$, by $\pi_g(b) = w_g^l \bar{\sigma}_g(b) w_g^{r*}$ and $\gamma_g(c) = w_g\tilde{\rho}_g(c)\theta_1(w_g)^*$. Here we have denoted by $\overline{sp}\; P\theta_1(P)$ the $\|\cdot\|_2$-closed linear subspace generated by $\{x\theta_1(y) | x, y \in P\}$.

Notice that $L^2(P\bar{\otimes}P\bar{\otimes}N)$ and $\overline{sp}\; P\theta_1(P)\otimes L^2(N)$ may be viewed as left $P\bar{\otimes}N$ Hilbert modules with the actions $(p\otimes n)\cdot(x\otimes y\otimes n') := px\otimes y\otimes nn'$ and, respectively, $(p\otimes n)\cdot x\theta_1(y)\otimes n' := px\theta_1(y)\otimes nn'$, for all $p, x, y \in P$ and $n, n' \in N$. The following lemma makes Proposition II.2.11 useful in our context in which we work with the free product deformation. The proof is a straightforward verification.

Lemma II.2.13. *The map $U : L^2(P\bar{\otimes}P\bar{\otimes}N) \to \overline{sp}\; P\theta_1(P)\otimes L^2(N)$ defined by $U(p_1\otimes p_2\otimes n) = p_1\theta_1(p_2)\otimes n$, with $p_1, p_2 \in P, n \in N$, is an isomorphism of Hilbert spaces which intertwines the representations π and γ. Moreover, U intertwines the left $P\bar{\otimes}N$ - module structures of these Hilbert spaces.*

In order to apply Proposition II.2.11, we need the weak mixing property for the coinduced action.

Lemma II.2.14. *Let H be a subgroup of Γ with the property that there is no finite index subgroup H_0 of H which is contained in a conjugate $g\Lambda g^{-1}$ of Λ. Then the coinduced action σ is weak mixing on H.*

Ioana proved this result for coinduced actions on standard probability spaces in [Io06b, Lemma 2.2], but the proof also works for tracial von Neumann algebras.

Using the same arguments as in the second part of the proof of [Po05, Proposition 3.2], we obtain the following result:

Theorem II.2.15. *Let Γ be a countable group and Λ be a subgroup. Let H be a subgroup of Γ with the property that there is no finite index subgroup H_0 of H such that H_0 is contained in a conjugate $g\Lambda g^{-1}$ of Λ. Let $w : \Gamma \to \mathcal{U}(P\bar{\otimes}N)$ be a cocycle for the action ρ. If $w_{|H}$ and $\theta_1(w)_{|H}$ are cohomologous, then $w_{|H}$ is cohomologous to a cocycle with values in N.*

Proof. We will use Proposition II.2.11 and a maximality argument.

Denote by $\mathcal{W}$ the set of pairs (v, w') with $v \in P\bar{\otimes}N$ partial isometry satisfying $v^*v \in N$ and $w' : \Gamma \to \mathcal{U}(v^*vN\sigma'(v^*v))$ local cocycle for ρ such that $vw'_g = w_g\rho_g(v)$, for all $g \in \Gamma$.

We endow $\mathcal{W}$ with the order: $(v_0, w'_0) \le (v_1, w'_1)$ iff $v_0 = v_1v_0^*v_0, v_0^*v_0w'_1(g) = w'_0(g)$, for all $g \in \Gamma$. $\mathcal{W}$ is an inductive set and let $(v_0, w'_0) \in \mathcal{W}$ be a maximal element.

Claim. v_0 is a unitary.

Proof of the claim. Note that the claim finishes the proof. Let us prove the claim by contradiction. Suppose v_0 is not a unitary. Denote by $v = v_0\theta_1(v_0^*)$. Then $vv^* = v_0v_0^*$ and a direct computation gives us that $w_g\tilde{\rho}_g(v) = v\theta_1(w_g)$. Indeed, since $\rho_g(w^*_{g^{-1}}) = w_g$ and

$\rho_g(w_0'(g^{-1})^*) = w_0'(g)$, we have

$$\begin{aligned} w_g\tilde{\rho}_g(v) &= w_g\tilde{\rho}_g(v_0)\tilde{\rho}_g(\theta_1(v_0^*)) = v_0 w_0'(g)\tilde{\rho}_g(\theta_1(v_0^*)) \\ &= v_0\theta_1(\tilde{\rho}_g(v_0 w_0'(g^{-1}))^*) = v_0\theta_1(\tilde{\rho}_g(w_{g^{-1}}\rho_{g^{-1}}(v_0))^*) \\ &= v_0\theta_1(v_0^*\rho_g(w_{g^{-1}}^*)) = v_0\theta_1(v_0^* w_g) \\ &= v\theta_1(w_g). \end{aligned}$$

Since w and $\theta_1(w)$ are cohomologous, by Lemma II.2.1 we obtain the existence of a partial isometry $v' \in \tilde{P}\bar{\otimes}N$ such that $w_g\tilde{\rho}_g(v') = v'\theta_1(w_g)$ and $v'v'^* = 1 - vv^*, v'^*v' = 1 - v^*v$.

Next, Lemma II.2.10 implies that $v' \in \overline{sp}\, P\theta_1(P)\bar{\otimes}N$, which allows us to use Lemma II.2.13. Since v' is a fixed point for γ, $U^{-1}(v')$ is a fixed point for π. Now we can apply Proposition II.2.11 to obtain the existence of a partial isometry $v_1 \in P\bar{\otimes}N$ with the left support majorized by $l(U^{-1}(v'))$ and right support in N which satisfies $v_1 w_1'(g) = w_g\tilde{\rho}_g(v_1)$ for some local cocycle $w_1' : \Gamma \to \mathcal{U}(v_1v_1^* N\sigma'(v_1v_1^*))$. Here we denote by $l(U^{-1}(v'))$ the left support of $U^{-1}(v')$.

Notice that $l(U^{-1}(v'))$ is majorized by $v'v'^* = 1 - v_0v_0^*$. Indeed, by Lemma II.2.13, U intertwines the $P\bar{\otimes}N$ left module structure. Now, since $v'v'^* = 1 - vv^* = 1 - v_0v_0^* \in P\bar{\otimes}N$, we have $U^{-1}(v') = U^{-1}(v'v'^*v') = v'v'^*U^{-1}(v')$, which proves the claim. □

Thus, in the finite von Neumann algebra $\tilde{P}\bar{\otimes}N$ we have $v_1^*v_1 \sim v_1v_1^* \le 1 - v_0v_0^* \sim 1 - v_0^*v_0$. Since the first and the last projection lies in N, we obtain that $v_1^*v_1 \le 1 - v_0^*v_0$ in N (by working with the central trace).

Now, we conclude as in the proof of [Po05, Proposition 3.2]. By multiplying v_1 to the right with a partial isometry in N and conjugate w_1' appropriately, we may assume $v_1^*v_1 \le 1 - v_0^*v_0$. But then, $(v_0 + v_1, w_0' + w_1') \in \mathcal{W}$ and strictly majorizes (v_0, w_0'), which contradicts the maximality assumption. ■

II.3 Proof of Theorem A

We will prove the following theorem, which is the general version of Theorem A dealing with coinduced actions of Γ on $A^{\Gamma/\Lambda}$ that arise from actions of Λ on arbitrary tracial von Neumann algebras A.

Theorem II.3.1 (Groups with relative property (T))**.** *Let Γ be a countable group and Λ be a subgroup. Let $H \subset \Gamma$ be a subgroup with relative property (T). Assume that there does not exist a subgroup H_0 of finite index in H such that H_0 is contained in a conjugate $g^{-1}\Lambda g$ of Λ.*

Let σ_0 be a trace preserving action of Λ on a tracial von Neumann algebra A and σ the coinduced action on $P := A^{\Gamma/\Lambda}$. Let us consider another action σ' on a tracial von Neumann algebra N. Denote by ρ the tensor product action $\sigma \otimes \sigma'$ of Γ on $P \bar{\otimes} N$.

Then, any cocycle $w : \Gamma \to \mathcal{U}(P \bar{\otimes} N)$ for the restriction of ρ to H is cohomologous to a cocycle of the form $w' : H \to \mathcal{U}(N)$.

Moreover, if H is w-normal in Γ, then w is cohomologous to a cocycle of the form $w' : \Gamma \to \mathcal{U}(N)$.

From now on, in this section we use the same notations as in Section II.2.2. The first step of the proof of Theorem II.3.1 is to prove that $w_{|H}$ and $\theta_1(w)_{|H}$ are cohomologous. This is obtained by the following result which is [Po05, Lemma 4.6] adapted to the free product deformation.

Proposition II.3.2. *[Po05, Lemma 4.6] Let Λ be a subgroup of Γ. Let $H \subset \Gamma$ be a subgroup with relative property (T) such that there does not exist a subgroup H_0 of finite index in H which is contained in a conjugate $g^{-1}\Lambda g$ of Λ.*

Let σ_0 be a trace preserving action of Λ on a tracial von Neumann algebra A and σ the coinduced action on $P = A^{\Gamma/\Lambda}$. Consider a trace preserving action σ' on a tracial von Neumann algebra N.

Let $w : \Gamma \to P\bar{\otimes}N$ be a cocycle for the action ρ on $P\bar{\otimes}N$. Then $w_{|H}$ and $\theta_1(w)_{|H}$ are cohomologous as cocycles for the action $\tilde{\rho}_{|H}$ on $\tilde{P}\bar{\otimes}N$.

The proof of Proposition II.3.2 is almost identical to that of [Po05, Lemma 4.6], but we include it for completeness. At the end of the proof of [Po05, Lemma 4.6], it is used the weak mixing property and therefore is obtained that a certain element belongs to a smaller algebra. The difference is that in the proof of Proposition II.3.2 is used Lemma II.2.10 to obtain the same result.

Proof.[Proof of Proposition II.3.2] It is enough to prove that $\forall \epsilon > 0, \exists v \in \tilde{P}\bar{\otimes}N$ partial isometry such that $\|v^*v - 1\|_2 \leq \epsilon$ and

$$w_h\tilde{\rho}_h(v) = v\theta_1(w_h), \forall h \in H.$$

Indeed, if this holds, take a unitary $u \in \mathcal{U}(\tilde{P}\bar{\otimes}N)$ satisfying $uv^*v = v$. By triangle inequality, we get that

$$\|w_h\tilde{\rho}_h(u) - u\theta_1(w_h)\|_2 \leq 2\|u - v\|_2 = 2\|1 - v^*v\|_2 \leq 2\epsilon, \forall h \in H.$$

Using now Lemma II.2.1, we get that $w_{|H}$ and $\theta_1(w)_{|H}$ are cohomologous.

We now prove the first statement of this proof in two steps.

Step 1. For all $\epsilon > 0$, there exist $v_0 \in \tilde{P}\bar{\otimes}N$ and $n \in \mathbb{N}$ such that $\|v_0^*v_0 - 1\|_2 \leq \epsilon$ and

$$w_h\tilde{\rho}_h(v_0) = v_0\theta_{1/2^n}(w_h), \forall h \in H. \tag{II.3.1}$$

This is just an application of Lemma II.2.2. Indeed, the lemma gives us the existence of a partial isometry $v_0 \in \tilde{P}\bar{\otimes}N$ and $n \in \mathbb{N}$, satisfying $\|v_0 - 1\|_2 \leq \epsilon/2$ such that formula II.3.1 holds. Using the triangle inequality, we get that $\|v_0^*v_0 - 1\|_2 \leq \epsilon$.

Step 2. Assume that there exist a partial isometry $v \in \tilde{P}\bar{\otimes}N$ and $t \in (0,1)$ satisfying

$$w_h\tilde{\rho}_h(v) = v\theta_t(w_h), \forall h \in H. \tag{II.3.2}$$

Then there exists a partial isometry $v' \in \tilde{P}\bar{\otimes}N$ satisfying $\|v\|_2 = \|v'\|_2$ and

$$w_h\tilde{\rho}_h(v') = v'\theta_{2t}(w_h), \forall h \in H.$$

For proving Step 2, we will use the properties of the automorphism β. Since $\beta\theta_t = \theta_{-t}\beta$ and $\beta_{|P\bar{\otimes}N} = id_{P\bar{\otimes}N}$ we get that

$$w_h\tilde{\rho}_h(\beta(v)) = \beta(v)\theta_{-t}(w_h), \forall h \in H.$$

Define $v' = \theta_t(\beta(v)^*v)$. We get

$$\begin{aligned} v'^*w_h &= \theta_t(v^*\beta(v)\theta_{-t}(w_h)) \\ &= \theta_t(v^*w_h\tilde{\rho}_h(\beta(v))) \\ &= \theta_t(\theta_t(w_h)\tilde{\rho}_h(v^*\beta(v))) \\ &= \theta_{2t}(w_h)\tilde{\rho}_h(v'^*), \end{aligned}$$

which implies that

$$w_h\tilde{\rho}_h(v') = v'\theta_{2t}(w_h), \forall h \in H.$$

Let us prove now that $\|v\|_2 = \|v'\|_2$. Since $\|v'\|_2 = \|\beta(v)^*v\|_2$, it's enough to prove that $\beta(vv^*) = vv^*$. By taking the adjoint in II.3.2, we obtain that

$$w_h\tilde{\rho}_h(vv^*)w_h^* = vv^*, \forall h \in H.$$

By Lemma II.2.10, we obtain that $vv^* \in P\bar{\otimes}N$, so $\beta(vv^*) = vv^*$. This ends the proof. ■

The proof of Theorem II.3.1 is now an easy consequence of Proposition II.3.2 and Theorem II.2.15.

Proof or Theorem II.3.1

By Proposition II.3.2, there exists a unitary $v \in \tilde{P}\bar{\otimes}N$ such that

$$w_h\rho_h(v) = v\theta_1(w_h), \forall h \in H.$$

Theorem II.2.15 gives us the existence of a cocycle $w' : H \to \mathcal{U}(N)$ cohomogous to w. More precisely, we have

$$w_h = uw'_h\rho_h(u^*), \quad \forall h \in H,$$

for a unitary $u \in \mathcal{U}(P\bar{\otimes}N)$.

For the moreover part, notice that Lemma II.2.14 implies that the coinduced action is weak mixing on H. Thus, we can apply Proposition II.2.4 and obtain that $u^*w_g\rho_g(u) \in N$, for all $g \in \Gamma$. This allows us to define w' on Γ and obtain that w is cohomologous to 3 a cocycle with values in N on Γ. ∎

Remark II.3.3. Let us see that Theorem II.3.1 implies Theorem A. Denote $P = L^\infty(X)$. Since $\mathcal{V}$ belongs to $\mathcal{U}_{fin}$, we can consider a tracial von Neumann algebra N such that $\mathcal{V} \subset \mathcal{U}(N)$. We consider Γ acts on $P\bar{\otimes}N$ by the tensor product action $\sigma \otimes id$, where the action of Γ on N is the trivial one. Define the cocycle $w : \Gamma \to \mathcal{U}(P\bar{\otimes}N)$, by $w_g(x) = w(g, g^{-1}x)$, for a.e. $x \in X$. Theorem II.3.1 allows us to untwist w to a homomorphism $d_0 : \Gamma \to \mathcal{U}(N)$. We conclude by applying [Po05, Proposition 3.5] so we can untwist w to a homomorphism $d : \Gamma \to \mathcal{V}$.

II.4 Proof of Theorem B

In this section we prove Theorem II.4.1, which is a more general version of Theorem B dealing with coinduced actions of Γ on $A^{\Gamma/\Lambda}$ that arise from actions of Λ on arbitrary tracial von Neumann algebras A. The deduction of Theorem B from Theorem II.4.1 is obtained by using the same arguments as in Remark II.3.3.

Theorem II.4.1 (Product groups)**.** *Let Γ be a countable group and Λ be an amenable subgroup. Let H and H' be infinite commuting subgroups of Γ such that H' is non-amenable. Assume that H does not have a subgroup H_0 of finite index in H such that H_0 is contained in a conjugate $g^{-1}\Lambda g$ of Λ.*

Let σ_0 be a trace preserving action of Λ on a tracial von Neumann algebra A and σ the coinduced action on $P := A^{\Gamma/\Lambda}$. Let us consider another action σ' on a tracial von Neumann algebra N. Denote by ρ the tensor product action $\sigma\bar{\otimes}\sigma'$ of Γ on $P\bar{\otimes}N$.

Then, any cocycle $w : \Gamma \to \mathcal{U}(P\bar{\otimes}N)$ for the restriction of ρ to HH' is cohomologous to a cocycle of the form $w' : HH' \to \mathcal{U}(N)$.
Moreover, if H is w-normal in Γ, then w is cohomologous to a cocycle of the form $w' : \Gamma \to \mathcal{U}(N)$.

We use the same notations as in Section II.2.2. We still consider σ the coinduced action on P, σ' a trace preserving action on a tracial von Neumann algebra N and the free product deformation θ_t.

The following result is known as Popa's *transversality* lemma.

Lemma II.4.2. *([Po06a, Lemma 2.1]) For every $s \in (0, 1/2)$ and $x \in P\bar{\otimes}N$, we have*

$$\|\theta_{2s}(x) - x\|_2 \leq 2\|\theta_s(x) - E_{P\bar{\otimes}N}(\theta_s(x))\|_2.$$

Lemma II.4.3. *Let Γ be a countable group and Λ an amenable subgroup. Let F be a finite* *subset of* Γ/Λ. *Denote* $N_F = \{g \in \Gamma | gF = F\}$, *where* Γ *acts on* Γ/Λ *by left multiplication. Then* N_F *is amenable.*

Proof. The action of N_F on F, by left multiplication, is well defined. Denote by S_F the group of bijections on the finite set F. We obtain a homomorphism $\phi : N_F \to S_F$, defined by $\phi(g)\bar{f} = g\bar{f}$, for all $g \in N_F$ and $\bar{f} \in F$.

Notice that $\ker\phi$, the kernel of ϕ, is amenable. Indeed, if $f\Lambda \in F$, then $ker\phi \subset f\Lambda f^{-1}$. Since Λ is amenable, $\ker\phi$ is amenable. Note that $\phi(N_F)$, the image of ϕ, is amenable, being a finite group.

Since $\ker\phi$ and $\phi(N_F)$ are amenable groups, we conclude that N_F is amenable. ■

Theorem II.4.4. *Let* Γ *be a countable group and* Λ *an amenable subgroup. Let* H *and* H' *be infinite commuting subgroups of* Γ *such that* H' *is non-amenable. Denote* $\tilde{M} = (\tilde{P}\bar{\otimes}N) \rtimes H$ *and* $M = (P\bar{\otimes}N) \rtimes H$.
Let $w : H' \to \mathcal{U}(P\bar{\otimes}N)$ *be a cocycle for* ρ *and define the representation* $\pi : H' \to \mathcal{U}(L^2(\tilde{M}) \ominus L^2(M))$ *by* $\pi_g(x) = w_g\tilde{\rho}_g(x)w_g^*$. *Then* π *has spectral gap.*

Remark II.4.5. In Theorem II.4.4 the action $\tilde{\rho}_{|H'}$ is considered to be extended in a natural way to $(\tilde{P}\bar{\otimes}N) \rtimes H$. This is possible since H and H' commute.

Proof.[Proof of Theorem II.4.4]

Let $\mathcal{B} = \{1 = \eta_0, \eta_1, ..\} \subset A$ be an orthonormal basis of $L^2(A)$. Denote by u the canonical Haar unitary of $L(\mathbb{Z})$. Thus, we obtain an orthonormal basis for $L^2(A * L(\mathbb{Z}))$ given by

$$\tilde{\mathcal{B}} = \{u^{n_1}\eta_{j_1}u^{\eta_2}...\eta_{j_k} | j_1, ..j_{k-1} \geq 1, k \in \mathbb{N}\} = \{1 = \tilde{\eta_0}, \tilde{\eta_1}, ..\},$$

as in [Io06a, Proposition 2.3]. Also, we have that

$$\mathcal{N} = \{\otimes_{f\in\Gamma/\Lambda}\eta_{i_f} | \{f | i_f \neq 0\} \text{ is finite}\}$$

and

$$\tilde{\mathcal{N}} = \{\otimes_{f\in\Gamma/\Lambda}\tilde{\eta_{i_f}} | \{f | i_f \neq 0\} \text{ is finite}\}$$

are orthonormal bases for $L^2(P)$ and, respectively, for $L^2(\tilde{P})$.

Let $x = \otimes_{f\in\Gamma/\Lambda}\tilde{\eta}_{i_f} \in \tilde{\mathcal{N}}$. Denote $F_x = \{f \in \Gamma/\Lambda | \tilde{\eta}_{i_f} \in \tilde{\mathcal{B}} \setminus \mathcal{B}\}$ and $K_F^0 = \overline{sp}\{x \in \tilde{\mathcal{N}} | F_x = F\}$. Notice that $K_F^0 \perp K_{F'}^0$, whenever $F \neq F'$ are finite subsets of Γ/Λ. This implies that

$$L^2(\tilde{P}) \ominus L^2(P) = \overline{sp}\,\tilde{\mathcal{N}} \setminus \mathcal{N} = \oplus K_F^0,$$

where the direct sum runs over all finite non empty subsets $F \subset \Gamma/\Lambda$.

Thus,

$$L^2(\tilde{P}\bar{\otimes}N) \ominus L^2(P\bar{\otimes}N) = \oplus K_F^1,$$

where the direct sum runs over all finite non empty subsets $F \subset \Gamma/\Lambda$ and $K_F^1 = K_F^0 \otimes L^2(N)$.

Finally, we get the decomposition

$$L^2(\tilde{M}) \ominus L^2(M) = \oplus K_F,$$

where the direct sum runs over all finite non empty subsets $F \subset \Gamma/\Lambda$ and $K_F = \overline{sp}\{K_F^1 u_h | h \in H\}$.

Claim 1. We can decompose $L^2(\tilde{M}) \ominus L^2(M) = \oplus_{i\in I}\overline{sp}\,\pi(H')M\xi_i M$, where $\{\xi_i\}_{i\in I}$ is a family of vectors from $L^2(\tilde{P})$ and each $\xi_i \in K_F$ for some non empty finite set $F \subset \Gamma/\Lambda$.

Proof of the claim 1. Let S be the set of elementary tensors $\otimes_{i\in F}\eta_i$, with F finite subset of Γ/Λ such that each η_i is an element of $\tilde{A}$ which starts and ends with a non-trivial power of u. Then Γ acts on S and choose T to be a set of representatives for this action. Then, $L^2(\tilde{M}) \ominus L^2(M) = \oplus_{\xi\in T}\overline{sp}\,\pi(H')M\xi M$. □

Denote by $\lambda_{H'}$ the left regular representation of H' on $l^2(H')$.

Claim 2. $\pi \preceq \lambda_{H'}$, i.e. π is weakly contained in $\lambda_{H'}$.

We suppose the claim holds and we prove it after the end of this theorem. For finishing the proof, note that the non-amenability of H' implies $1_{H'} \npreceq \lambda_{H'}$. Thus, $1_{H'} \npreceq \pi$, which means that π has spectral gap on H. This proves the theorem. ■

We now prove Claim 2 from the proof of Theorem II.4.4 using the same notations.

Lemma II.4.6. $\pi \preceq \lambda_{H'}$.

Proof. For every $F \subset \Gamma/\Lambda$, non empty finite subset, denote $H'_F = \{h' \in H' | h'F = F\}$, where Γ acts on Γ/Λ by left multiplication. Since F is finite and Λ is amenable, Lemma IV.2.1 implies that H'_F is an amenable group.

Let us take a family of vectors $\{\xi_i\}_{i\in I}$ as in the first claim of Theorem II.4.4. Fix $i \in I$ and let F be a finite subset of Γ/Λ such that $\xi_i \in K_F$.

Note that

$$\langle \pi_g(x), x\rangle = 0, \forall x \in K_F, g \notin H'_F. \tag{II.4.1}$$

Let us observe that we can decompose $\overline{sp}\ \pi(H')M\xi_i M = \oplus_{j\in J}\ \overline{sp}\ \pi(H')\eta_{ij}$ in cyclic subspaces, with $\eta_{ij} \in K_F$. Indeed, by taking a maximal family of vectors $\{\eta_{ij}\}_{j\in J}$ with the property that $\overline{sp}\, \pi(H'_F)\eta_{ij}$ are mutually orthogonal, we get the decomposition $\overline{sp}\, \pi(H'_F)M\xi_i M = \oplus_{j\in J}\, \overline{sp}\, \pi(H'_F)\eta_{ij}$. Since $\xi_i \in K_F$ and K_F is a $\pi(H'_F)$ invariant subspace, we obtain that $\eta_{ij} \in K_F$. Since H'_F is a subgroup of H', (II.4.1) implies that the decomposition $\overline{sp}\, \pi(H')M\xi_i M = \oplus_{j\in J}\, \overline{sp}\, \pi(H')\eta_{ij}$ also holds. Indeed, (II.4.1) implies that $\overline{sp}\, \pi(H')\eta_{ij}$ is orthogonal on $\overline{sp}\, \pi(H')\eta_{ij'}$, for all $j, j' \in J$, with $j \neq j'$. This proves the claim.

Fix $j \in J$. Define the cyclic representations $\theta : H' \to \mathcal{U}(\overline{sp}\, \pi(H')\eta_{ij})$ and $\theta_F : H'_F \to \mathcal{U}(\overline{sp}\, \pi(H'_F)\eta_{ij})$ as the restrictions of π, respectively of $\pi_{|H'_F}$ to the coresponding cyclic subspaces.

Let

$$\tilde{\theta} := \mathrm{Ind}_{H'_F}^{H'} \theta_F : H' \to \mathcal{U}(l^2(H'/H'_F) \otimes \overline{sp}\, \pi(H'_F)\eta_{ij})$$

be the induced representation of θ_F defined by

$$\tilde{\theta}_g(\delta_x \otimes \eta) = \delta_{gx} \otimes [\theta_F(c(g,x))\eta], \tag{II.4.2}$$

for all $g \in H', x \in H'/H'_F$ and $\eta \in \overline{sp}\,\pi(H'_F)\eta_{ij}$, where $c : H' \times H'/H'_F \to H'_F$ is the canonical cocycle defined as in section II.1.2. Recall that $c(g,x) = \phi^{-1}(gx)g\phi(x)$, for all $g \in H'$ and $x \in H'/H'_F$, with $\phi : H'/H'_F \to H'$ an arbitrary fixed section. Moreover, ϕ can be chosen such that $\phi(H'_F) = e$, with e the neutral element of H'. This implies $c(g, H'_F) = g$, for all $g \in H'_F$.

Claim. The induced representation $\tilde{\theta}$ contains θ as a subrepresentation.

Proof of the Claim. Define the positive definite function $\varphi : H' \to \mathbb{C}$ by $\varphi(g) =< \theta(g)\eta_{ij}, \eta_{ij} >$, for $g \in H'$. The formula (II.4.1) implies that φ is zero on $H' \smallsetminus H'_F$, since $\eta_{ij} \in K_F$.

Denote $\tilde{\eta} := \delta_{eH'_F} \otimes \eta_{ij} \in l^2(H'/H'_F) \otimes \overline{sp}\,\pi(H'_F)\eta_{ij}$. A direct computation gives us that

$$< \tilde{\theta}(g)\tilde{\eta}, \tilde{\eta} >=< \theta(c(g, eH'_F)\eta_{ij}, \eta_{ij}) >=< \theta(g)\eta_{ij}, \eta_{ij}) >= \varphi(g),$$

for all $g \in H'_F$. For $g \notin H'_F$, the formula (II.4.2) gives us that$< \tilde{\theta}(g)\tilde{\eta}, \tilde{\eta} >= 0$.

Thus, we have obtained that

$$< \tilde{\theta}(g)\tilde{\eta}, \tilde{\eta} >= \varphi(g)$$

for all $g \in H'$. Since θ is a cyclic representation, we get that θ is contained in $\tilde{\theta}$. This ends the claim. □

Now, we can finish the proof of the lemma. Since H'_F is amenable, we have $1_{H'_F} \preceq \lambda_{H'_F}$ (see [BHV08, Theorem G.3.2], for example). By Fell absorbing principle, we get that $\theta_F \preceq \lambda_{H'_F}$. [BHV08, Theorem F.3.5] gives us continuity of weak containment with respect to induction. This implies that $\tilde{\theta} = \mathrm{Ind}_{H'_F}^{H'}\theta_F \preceq \mathrm{Ind}_{H'_F}^{H'}\lambda_{H'_F} = \lambda'_H$. Since θ is contained in $\tilde{\theta}$, we get that $\theta \leq \lambda_{H'}$.

Denote by $\theta_i : H' \to \mathcal{U}(\overline{sp}\ \pi(H')M\xi_i M)$ the restriction of π to the subspace $\overline{sp}\,\pi(H')M\xi_i M$. The decomposition $\overline{sp}\,\pi(H')M\xi_i M = \oplus \overline{sp}\,\pi(H')\eta_{ij}$ gives us that $\theta_i \leq \lambda_{H'}$.

The decomposition given by the first claim implies that $\pi = \oplus_{i\in I}\theta_i$. Thus, $\pi \leq \lambda_{H'}$, which ends the proof of the lemma. ■

Proof of Theorem II.4.1

Define the representation $\pi : \Gamma \to \mathcal{U}(L^2((\tilde{P}\bar{\otimes}N) \rtimes H) \ominus L^2((P\bar{\otimes}N) \rtimes H))$ by $\pi_g(x) = w_g\tilde{\rho}_g(x)w_g^*$ and denote $\tilde{M} = (\tilde{P}\bar{\otimes}N) \rtimes H$ and $M = (P\bar{\otimes}N) \rtimes H$ as in the previous theorem.

Theorem II.4.4 implies that π has spectral gap on H'. Thus, for all $\epsilon > 0$, there exist $\delta > 0$ and $F' \subset H'$ finite, such that if $u \in \mathcal{U}(\tilde{M})$ satisfies $\|\pi_h(u) - u\|_2 \leq \delta, \forall h' \in F'$, then $\|u - E_M(u)\|_2 \leq \epsilon$.

Let us proceed now as in [[Po06a],Theorem 4.1]. Denote $\bar{u}_g := w_g u_g, g \in \Gamma$. Since the map $s \to \theta_s(\bar{u}_{h'})$ is continuous in $\|\cdot\|_2$, for all $h' \in F'$, it follows that for small enough s, we get that

$$\|\theta_{-s/2}(\bar{u}_{h'}) - \bar{u}_{h'}\|_2 \leq \delta/2,$$

for all $h' \in F'$. Because H and H' commute, $\bar{u}_{h'}$ and $\bar{u}_g$ commute for all $h' \in F'$ and $g \in H$. Thus, we get that

$$\|[\theta_{s/2}(\bar{u}_g), \bar{u}_{h'}]\|_2 = \|\bar{u}_g, \theta_{-s/2}(\bar{u}_{h'})\|_2 \leq 2\|\theta_{-s/2}(\bar{u}_{h'}) - \bar{u}_{h'}\| \leq \delta,$$

for all $h' \in F'$ and $g \in H$.

Notice that $\pi_g(x) = \bar{u}_g x \bar{u}_g^*$, for all $g \in \Gamma$. A direct computation gives us that

$$\|\pi_{h'}(\theta_{s/2}(\bar{u}_g)) - \theta_{s/2}(\bar{u}_g)\|_2 = \|[\theta_{s/2}(\bar{u}_g), \bar{u}_{h'}]\|_2 \leq \delta,$$

for all $h' \in F'$ and $g \in H$, which implies that

$$\|\theta_{s/2}(\bar{u}_g) - E_M(\theta_{s/2}(\bar{u}_g))\|_2 \le \epsilon.$$

Using Lemma II.4.2, we get that $\|\theta_s(\bar{u}_g) - \bar{u}_g\|_2 \le 2\epsilon$, for all $g \in H$. The set $K := \overline{co}^w\{\bar{u}_g\theta_s(\bar{u}_g)^*|g \in H\}$ is convex weakly compact, and for all $\xi \in K$ and $g \in H$, we have $\bar{u}_g\xi\theta_s(\bar{u}_g)^* \in K$. Thus, if we denote by $\xi_0 \in K$ the unique element of minimal norm $\|\|_2$, then we get that $\bar{u}_g\xi_0\theta_s(\bar{u}_g)^* = \xi_0$, for all $g \in H$. This is equivalent to

$$w_g\tilde{\rho}_g(\xi_0) = \xi_0\theta_s(w_g),$$

for all $g \in H$. Taking $v \in \tilde{P}\bar{\otimes}N$, to be the partial isometry of ξ_0, we get that

$$w_g\tilde{\rho}_g(v) = v\theta_s(w_g),$$

for all $g \in H$.

Since

$$\|\bar{u}_g\theta_s(\bar{u}_g)^* - 1\|_2 = \|\bar{u}_g - \theta_s(\bar{u}_g)\|_2 \le 2\epsilon,$$

we get that $\|\xi_0 - 1\|_2 \le 2\epsilon$, which implies that $\|v - 1\|_2 \le 4(2\epsilon)^{1/2}$. This proves Step 1 of the proof of Proposition II.3.2. In combination with Step 2 from the proof of Proposition II.3.2, the conclusion follows as in the proof of the same proposition. Meaning, we obtain that $w_{|H}$ and $\theta_1(w)_{|H}$ are cohomologous.

As in the proof of Theorem II.3.1, we use Theorem II.2.15 to deduce the existence of a unitary $u \in \mathcal{U}(P\bar{\otimes}N)$ and of a cocycle $w' : H \to \mathcal{U}(N)$ such that

$$w_h = uw'_h\rho_h(u^*), \quad \forall h \in H.$$

We have $H \triangleleft HH'$, because H and H' commute. Since the restriction of ρ to H is weakly mixing, by using Proposition II.2.4 we obtain a cocycle w' with values in N which is cohomologous to w on HH'.

For the moreover part, we apply again Proposition II.2.4 as in the proof of Theorem II.3.1. ■

II.5 Applications to W*-superrigidity

We record the results [CP10, Corollary 5.3], [PV12, Theorem 1.1], [Io12b, Theorem 1.1] in the following theorem, which give uniqueness of group measure space Cartan subalgebras for groups in $\mathcal{C}$.

Theorem II.5.1. *Let $\Gamma \in \mathcal{C}$ and let $\Gamma \curvearrowright X$ be a free ergodic pmp action on a standard probability space X. Suppose there exists $\Lambda \curvearrowright Y$ a free ergodic pmp action on a standard probability space Y such that $M = L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda$. Then there exists a unitary $u \in M$ such that $uL^\infty(X)u^* = L^\infty(Y)$.*

The following result is a particular case of [Po05, Theorem 5.6] (see also [Fu06, Theorem 1.8]).

Proposition II.5.2. *[Po05, Theorem 5.6] Let Γ be a countable group with no non-trivial finite normal subgroups. Let $\Gamma \curvearrowright (X,\mu)$ be a free pmp action, where (X,μ) is a standard probability space. If $\Gamma \curvearrowright (X,\mu)$ is $\mathcal{U}_{fin}$-cocycle superrigid, then $\Gamma \curvearrowright (X,\mu)$ is OE-superrigid.*

In [Io06b, Lemma 2.1] it is proved that if we coinduce from free actions, we obtain free actions. The following lemma gives another sufficient condition for obtaining free coinduced actions. We include a proof for the reader's convenience.

Lemma II.5.3. *[Io06b, Lemma 2.1] Let Γ be a countable group and Λ a subgroup of infinite index. Let $\Lambda \overset{\sigma_0}{\curvearrowright} (X_0,\mu_0)$ be a pmp action on the standard probability space (X_0,μ_0) which* *has no atoms and let* $\Gamma \overset{\sigma}{\curvearrowright} (X,\mu) := (X_0,\mu_0)^{\Gamma/\Lambda}$ *be the coinduced action. Suppose* $\cap_{g\in\Gamma} g\Lambda g^{-1}$ *is finite and* $\cap_{g\in\Gamma} g\Lambda g^{-1} \cap Fix(\Lambda \curvearrowright X_0) = \{e\}$*, where* $Fix(\Lambda \curvearrowright X_0)$ *consists of those elements* $g \in \Lambda$ *for which* $\{x_0 \in X_0 | gx_0 = x_0\}$ *has measure* 1*. Then* $\Gamma \curvearrowright X$ *is free.*

Proof. Define $A_g = \{(x_h)_{h\in\Gamma/\Lambda} \in X | \sigma_g((x_h)_h) = (x_h)_h\}$ for $g \in \Gamma$. Recall that $\sigma_g((x_h)_h) = (x'_h)_h$, where $x'_h = \phi^{-1}(gh)g\phi(h)x_{g^{-1}h}$ and $\phi : \Gamma/\Lambda \to \Gamma$ is a section.
If $g_0 \notin \cap_{g\in\Gamma} g\Lambda g^{-1}$, there exists $g_1 \in \Gamma$ such that $g_0^{-1}g_1\Lambda \neq g_1\Lambda$. Then

$$\begin{aligned} A_{g_0} &= \{(x_h)_h \in X | x_h = \phi(g_0 h)^{-1} g_0 \phi(h) x_{g_0^{-1}h}, \forall h \in \Gamma/\Lambda\} \\ &\subset \{(x_h)_h \in X | x_{g_1\Lambda} = \phi(g_0 g_1 \Lambda)^{-1} g_0 \phi(g_1\Lambda) x_{g_0^{-1}g_1\Lambda}\} \end{aligned}$$

has measure 0 since X_0 is non-atomic.

Now, if $g_0 \in \Sigma := \cap_{g\in\Gamma} g\Lambda g^{-1} \setminus \{e\}$, we have $g^{-1}g_0 g \in \Sigma \setminus \{e\}$, for all $g \in \Gamma$. The hypothesis implies that $C_\lambda := \{x_0 \in X_0 | \lambda x_0 = x_0\}$ has measure less than 1, for all $\lambda \in \Sigma \setminus \{e\}$. Then,

$$\begin{aligned} A_{g_0} &= \{(x_h)_{h\in\Gamma/\Lambda} | x_h = \phi(h)^{-1} g_0 \phi(h) x_h, \forall h \in \Gamma/\Lambda\} \\ &= \textstyle\prod_{h\in\Gamma/\Lambda} C_{\phi(h)^{-1} g_0 \phi(h)} \end{aligned}$$

has measure 0. Indeed, since Σ is finite, there exists $g_1 \in \Sigma \setminus \{e\}$ such that $\{h \in \Gamma/\Lambda | \phi(h)^{-1} g_0 \phi(h) = g_1\}$ is an infinite set. This implies A_{g_0} has measure 0 since $\mu_0(C_{g_1}) < 1$.
■

The following result proves cocycle superrigidity for coinduced actions of groups from $\mathcal{C}$.

Theorem II.5.4. *Let* $\Gamma \in \mathcal{C}$ *and* Λ *a subgroup defined as in Corollary II.1.2. Let* $\Lambda \curvearrowright X_0$ *be a pmp action on a standard probability space* X_0 *and let* $\Gamma \curvearrowright X$ *be the coinduced action from* $\Lambda \curvearrowright X_0$*. Then* $\Gamma \curvearrowright X$ *is* $\mathcal{U}_{fin}$*-cocycle superrigid.*

Proof. We apply Theorems A and B and let us use the notations from these theorems.

For $\Gamma \in \mathcal{C}_1$, we want to apply Theorem B. If we take $H' = \Gamma_1$ and $H = \Gamma_2$, the conditions of Theorem B are satisfied, so we obtain the claim.

If $\Gamma \in \mathcal{C}_2$, consider $\Gamma = \Gamma_1 \times \Gamma_2 \times ... \times \Gamma_n$, with all the Γ_i's non-elementary hyperbolic groups. Without loss of generality suppose that $\Lambda \subset \Gamma_1$. We apply again Theorem B. By taking $H' = \Gamma_1$ and $H = \Gamma_2 \times ... \times \Gamma_n$ we notice that the conditions of this theorem are again satisfied.

Let $\Gamma \in \mathcal{C}_3$. Since Σ^0 is contained in Γ_1^0, the hypothesis implies that Γ_1^0 does not have finite index subgroups which are contained in a conjugate of Λ. We want to apply Theorem A for $H = \Gamma_1^0$.

Take $\mathcal{V} \in \mathcal{U}_{fin}$ and $w : \Gamma \times X \to \mathcal{V}$ a cocycle for $\Gamma \curvearrowright X$. Theorem A implies that there exist $\phi : X \to \mathcal{V}$ such that $\phi(gx)^{-1} w(g,x)\phi(x)$ is independent of x on Γ_1^0. Since Σ^0 is normal in Γ_2^0 and contained in Γ_1^0, Lemma II.2.14 combined with Proposition II.2.4 implies that $\phi(gx)^{-1} w(g,x)\phi(x)$ is independent of x on Γ_2^0. This proves that w is cohomologous to a group homomorphism on $\Gamma_1^0 *_{\Sigma^0} \Gamma_2^0$.

Now, since $\Gamma_1^0 *_{\Sigma^0} \Gamma_2^0$ is normal in Γ, we apply again Lemma II.2.14 and Proposition II.2.4 to obtain that $\phi(gx)^{-1} w(g,x)\phi(x)$ is independent of x on Γ. This ends the proof. ■

Proof of the Corollary II.1.2. Combining Proposition II.5.2 with Theorem II.5.4, we obtain that $\Gamma \curvearrowright X$ is OE-superrigid. Lemma II.2.14 proves that $\Gamma \curvearrowright X$ is ergodic and we conclude using Theorem II.5.1. ■

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Chapter II is, in part, a reprint of the material as it appears in

[Dr15] D. Drimbe: *Cocycle and orbit equivalence superrigidity for coinduced actions*, to appear in Ergodic Theory Dynam. Systems,

of which the dissertation author was the primary investigator and author.

Chapter III

Prime II_1 factors arising from lattices in higher rank

by

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III.1 Introduction

III.1.1 Background and statement of results

An important theme in operator algebras is the study of tensor product decompositions of II_1 factors. A II_1 factor M is called *prime* if it is not isomorphic to a tensor product of II_1 factors. In [Po83], Popa proved that the free groups on uncountably many generators give rise to prime II_1 factors. By using Voiculescu's free probability theory, Ge showed that the free group factors $L(\mathbb{F}_n)$, $2 \leq n \leq \infty$, are also prime, thus providing

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the first examples of separable prime II_1 factors [Ge96]. Ozawa then used subtle C*-algebraic methods to prove that for any icc hyperbolic group Γ, the II_1 factor $L(\Gamma)$ is *solid*, that is, the relative commutant of any diffuse subalgebra is amenable [Oz03]. Since solid non-amenable II_1 factors are clearly prime, this recovers the primeness of $L(\mathbb{F}_n)$. By developing a new technique based on closable derivations, Peterson proved that $L(\Gamma)$ is prime, for any non-amenable icc group Γ that admits an unbounded 1-cocycle into its left regular representation [Pe06]. Popa then used his powerful deformation/rigidity theory to give a new proof of solidity of $L(\mathbb{F}_n)$ [Po06c]. For additional primeness results, see [Oz04, Po06a, CI08, CH08, Va10b, Bo12a, HV12, DI12, CKP14, Ho15].

A common feature of these results is that the groups Γ for which $L(\Gamma)$ was proven to be prime have "rank one" properties, such as hyperbolicity or the existence of certain unbounded (quasi) 1-cocycles. On the other hand, the primeness question for the "higher rank" arithmetic groups $\mathrm{PSL}_n(\mathbb{Z})$, $n \geq 3$, is notoriously hard and remains open. Moreover, in spite of the remarkable advances made in the study of II_1 factors in the last 15 years (see the surveys [Po07, Va10a, Io12a]), little is known about the structure of II_1 factors associated to lattices in higher rank semisimple Lie groups. In fact, while II_1 factors arising from lattices in connected rank one simple Lie groups have already been shown to be prime in [Oz03], not a single example of a lattice, whose II_1 factor is prime, in either a higher rank simple or semisimple Lie group is known.

Our first main result provides the first examples of lattices in higher rank semisimple Lie groups which give rise to prime II_1 factors. More precisely, we prove:

Theorem D. *If Γ is an icc irreducible lattice in a product $G = G_1 \times ... \times G_n$ of $n \geq 1$ connected non-compact rank one simple real Lie groups with finite center, then the II_1 factor $L(\Gamma)$ is prime.*

More generally, if $\Gamma \in \mathcal{L}$ is an icc group, then the II_1 factor $L(\Gamma)$ is prime.

Before stating a consequence of Theorem D, we will first explain the terminology used, give several examples of groups to which Theorem D applies, and compare it with a result in the literature.

Definition III.1.1. We denote by $\mathcal{L}$ the family of countable groups Γ which can be realized as an irreducible lattice in a product $G = G_1 \times \ldots \times G_n$ of $n \geq 1$ locally compact second countable groups such that (i) G_j admits a lattice that is a non-elementary hyperbolic group, for every $1 \leq j \leq n$, (ii) G_j does not admit an open normal compact subgroup, for some $1 \leq j \leq n$, and (iii) Γ does not contain a non-trivial element which commutes with an open subgroup of G.

A subgroup $\Gamma < G$ is called a **lattice** if it is discrete and the homogeneous space G/Γ carries a G-invariant Borel probability measure. A lattice $\Gamma < G$ in a product group $G = G_1 \times \ldots \times G_n$ is called **irreducible** if its projection onto $\times_{i \neq j} G_i$ is dense, for every $1 \leq j \leq n$.

Remark III.1.2. Assume that G_j is a connected non-compact simple Lie group (or algebraic group), for every $1 \leq j \leq n$. Then condition (ii) of Definition III.1.2 is satisfied. Since any element of $G_j, 1 \leq j \leq n$, which commutes with an open subgroup is necessarily central, condition (iii) is satisfied by any icc lattice $\Gamma < G$. Here, we point out that if $G_1, \ldots, G_n$ are of rank one then condition (i) is also satisfied, and provide several examples of countable groups belonging to $\mathcal{L}$.

1. If G is a connected non-compact rank one simple real Lie group with finite center, then any co-compact lattice $\Lambda < G$ is non-elementary hyperbolic. This in particular applies to $G = \mathrm{SL}_2(\mathbb{R})$. Moreover, in this case, $\mathrm{SL}_2(\mathbb{Z})$ and the free group $\mathbb{F}_2$ arise as lattices of G.

2. Let G be a rank one simple algebraic group over a locally compact non-archimedean field. Then any such group admits a lattice $\Lambda < G$ which is a finitely generated free group (see [BK90, Corollaries 4.8 and 4.14] and [Lu91, Theorem 2.1]). In particular,

this applies to $G = \mathrm{SL}_2(\mathbb{Q}_p)$, where $\mathbb{Q}_p$ denotes the field of p-adic numbers for a prime p.

3. Let $d \geq 2$ be a square-free integer and S be a finite non-empty set of primes. Denote by $\mathcal{O}_d$ the ring $\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$, if $d \equiv 1 \pmod 4$, and the ring $\mathbb{Z}[\sqrt{d}]$, otherwise. Denote by $\mathbb{Z}[S^{-1}]$ the ring of rational numbers whose denominators have all prime factors from S. Then $\mathrm{SL}_2(\mathcal{O}_d)$ and $\mathrm{SL}_2(\mathbb{Z}[S^{-1}])$ can be realized as irreducible lattices in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{R}) \times (\prod_{p \in S} \mathrm{SL}_2(\mathbb{Q}_p))$, respectively. Since the same holds if SL_2 is replaced with PSL_2, it follows that $\mathrm{PSL}_2(\mathcal{O}_d)$ and $\mathrm{PSL}_2(\mathbb{Z}[S^{-1}])$ belong to $\mathcal{L}$.

Remark III.1.3. Let $G = G_1 \times \ldots \times G_n$ be as in the first part of Theorem D and $\Gamma < G$ be an irreducible, but not necessarily icc, lattice. Then the center $Z(\Gamma)$ of Γ is contained in $Z(G)$ and $\Gamma/Z(\Gamma)$ is icc. Thus, $\Gamma/Z(\Gamma)$ is an irreducible icc lattice in $G/Z(G)$. By Remark III.1.2(1) it follows that $\Gamma/Z(\Gamma) \in \mathcal{L}$ and Theorem D implies that the II_1 factor $L(\Gamma/Z(\Gamma))$ is prime.

Remark III.1.4. Let $G = G_1 \times \ldots \times G_n$ be as in the first part of Theorem D. Popa and Vaes proved that any lattice $\Gamma < G$ is Cartan-rigid: any II_1 factor $L^\infty(X) \rtimes \Gamma$ arising from a free ergodic pmp action of Γ has a unique Cartan subalgebra up to unitary conjugacy (see [PV12, Theorem 1.3]). Moreover, their proof shows that $L(\Gamma)$ does not have a Cartan subalgebra (see [PV12, Section 5]). Both the approach of [PV12] and our proof of Theorem D use the fact that the lattices in G are measure equivalent to a product of non-elementary hyperbolic groups. However, unlike the results from [PV12], the conclusion of Theorem D does not hold for arbitrary lattices $\Gamma < G$, as it obviously fails for product lattices $\Gamma = \Gamma_1 \times \ldots \times \Gamma_n$, whenever $n \geq 2$ and $\Gamma_i < G_i$ is a lattice for all $1 \leq i \leq n$. To prove Theorem D we will perform a detailed analysis which shows that if $\Gamma < G$ is any icc lattice such that $L(\Gamma)$ is not prime, then Γ is a product group and thus cannot be an irreducible lattice.

The following corollary in an immediate consequence of Theorem D.

Corollary E. *Let $d \geq 2$ be a square-free integer and S be a finite non-empty set of primes. Then $L(PSL_2(\mathcal{O}_d))$ and $L(PSL_2(\mathbb{Z}[S^{-1}]))$ are prime II_1 factors.*

Remark III.1.5. [CdSS15, Corollary] implies that $L(\mathrm{PSL}_2(\mathbb{Z}[\sqrt{2}]))$ is not isomorphic to $L(\Gamma_1 \times \Gamma_2)$, for any non-amenable groups Γ_1 and Γ_2 in Ozawa's class $\mathcal{S}$ [Oz04, BO08]. Corollary E strengthens this fact by showing that $L(\mathrm{PSL}_2(\mathbb{Z}[\sqrt{2}]))$ is prime and hence not isomorphic to $L(\Gamma_1 \times \Gamma_2)$, for any non-trivial countable groups Γ_1 and Γ_2.

Theorem D will be deduced from a general result which describes all tensor product decompositions of II_1 factors associated to groups that are measure equivalent to products of hyperbolic groups. Before stating this result, let us recall the notion of measure equivalence due to Gromov [Gr91], and the construction of amplifications of II_1 factors.

Definition III.1.6. Two countable groups Γ and Λ are called **measure equivalent** if there exist commuting free measure preserving actions of Γ and Λ on a standard measure space (Ω, m), such that the actions of Γ and Λ each admit a finite measure fundamental domain.

Natural examples of measure equivalent groups are provided by pairs of lattices Γ, Λ in an unimodular locally compact second countable group G. Indeed, endowing G with a Haar measure m and the left and right translation actions of Γ and Λ shows that Γ and Λ are measure equivalent.

If M is a II_1 factor and $t > 0$, then the **amplification** M^t is defined as the isomorphism class of $p(M\overline{\otimes}\mathbb{B}(\ell^2(\mathbb{N}))p$, where $p \in M\overline{\otimes}\mathbb{B}(\ell^2(\mathbb{N}))$ is a projection satisfying $(\tau \otimes \mathrm{Tr})(p) = t$. Here, τ and Tr denote the canonical traces of M and $\mathbb{B}(\ell^2(\mathbb{N}))$, respectively. Finally, recall that if $M = P_1 \overline{\otimes} P_2$, for some II_1 factors P_1 and P_2, then for every $t > 0$ we have a natural identification $M = P_1^t \overline{\otimes} P_2^{1/t}$.

The following theorem is the main technical result of this paper:

Theorem F. *Let Γ be a countable icc group and denote $M = L(\Gamma)$. Assume that Γ is measure equivalent to a product $\Lambda = \Lambda_1 \times ... \times \Lambda_n$ of $n \geq 1$ non-elementary hyperbolic groups $\Lambda_1, ..., \Lambda_n$. Suppose that $M = P_1 \overline{\otimes} P_2$, for some II_1 factors P_1 and P_2.*

Then there exist a decomposition $\Gamma = \Gamma_1 \times \Gamma_2$, a partition $S_1 \sqcup S_2 = \{1, ..., n\}$, a decomposition $M = P_1^t \overline{\otimes} P_2^{1/t}$, for some $t > 0$, and a unitary $u \in M$ such that

1. *$P_1^t = uL(\Gamma_1)u^*$ and $P_2^{1/t} = uL(\Gamma_2)u^*$.*
2. *Γ_1 is measure equivalent to $\times_{j \in S_1} \Lambda_j$ and Γ_2 is measure equivalent to $\times_{j \in S_2} \Lambda_j$.*

In order to put Theorem F in a better perspective, we first emphasize a new rigidity phenomenon that Theorem F leads to, and then discuss several applications of it.

Connes' classification of injective factors implies that no algebraic information regarding an icc amenable group Γ can be recovered from $L(\Gamma)$ [Co76]. In sharp contrast, Theorem F implies that for a natural and wide class of groups Γ, any tensor product decomposition of $L(\Gamma)$ must arise from a direct product decomposition of Γ. This adds to the few known instances where algebraic properties of the von Neumann algebra $L(\Gamma)$ can be promoted to algebraic properties of the group Γ. We highlight here two recent developments in this direction: Ioana, Popa and Vaes' discovery of the first classes of "W*-superrigid" groups [IPV10] (see Berbec and Vaes [BV12] for the only other known examples), and Chifan, de Santiago and Sinclair's "product rigidity" theorem [CdSS15].

There are three main applications of Theorem F. First, we use Theorem F to deduce Theorem D. To briefly indicate how this works, let $\Gamma \in \mathcal{L}$ be an icc group. Then Γ can be realized as an irreducible lattice in a locally compact group $G = G_1 \times ... \times G_n$ which also admits a product of non-elementary hyperbolic groups as a lattice. Assuming that $L(\Gamma)$ is not prime, we apply Theorem F to conclude that Γ decomposes as a product of infinite groups. In the case $G_1, \ldots, G_n$ are non-compact simple Lie groups with finite center, such a decomposition can be ruled out by appealing to Margulis' normal subgroup theorem (see

[Zi84, Theorem 8.1.1]). In the general case, we will show that such product decompositions do not exist by using a stronger version of Theorem F (see Theorem III.7.1).

Secondly, Theorem F allows us to prove a unique prime factorization result for tensor products of II_1 factors arising from irreducible lattices in products of rank one simple Lie groups.

Corollary G. *Let Γ be a countable icc group which is measure equivalent to a product of $n \geq 1$ non-elementary hyperbolic groups. Denote $M = L(\Gamma)$.*

Then there exists a unique (up to permutation of factors) decomposition $\Gamma = \Gamma_1 \times \ldots \times \Gamma_k$, for some $1 \leq k \leq n$, such that $L(\Gamma_i)$ is a prime II_1 factor, for every $1 \leq i \leq k$. Moreover, the following hold:

1. *If $M = P_1 \overline{\otimes} P_2$, for some II_1 factors P_1, P_2, then there exist a partition $I_1 \sqcup I_2 = \{1, ..., k\}$ and a decomposition $M = P_1^t \overline{\otimes} P_2^{1/t}$, for some $t > 0$, such that $P_1^t = \overline{\otimes}_{i \in I_1} L(\Gamma_i)$ and $P_2^{1/t} = \overline{\otimes}_{i \in I_2} L(\Gamma_i)$, modulo unitary conjugacy in M.*

2. *If $M = P_1 \overline{\otimes} \ldots \overline{\otimes} P_m$, for some $m \geq k$ and II_1 factors $P_1, ..., P_m$, then $m = k$ and there exists a decomposition $M = P_1^{t_1} \overline{\otimes} ... \overline{\otimes} P_k^{t_k}$ for some $t_1, ..., t_k > 0$ with $t_1 t_2 \ldots t_k = 1$ such that after permutation of indices and unitary conjugacy we have $L(\Gamma_i) = P_i^{t_i}$, for all $1 \leq i \leq k$.*

3. *In (2), the assumption $m \geq k$ can be omitted if each P_i is assumed to be prime.*

Corollary G in particular applies if $\Gamma = \Gamma_1 \times ... \times \Gamma_k$, where $\Gamma_i \in \mathcal{L}, 1 \leq i \leq k$, are icc groups.

The first unique prime factorization results for II_1 factors were obtained by Ozawa and Popa in their pioneering work [OP03]. More precisely, [OP03] established conclusions (1)-(3) of Corollary G for $M = L(\Gamma_1) \overline{\otimes} ... \overline{\otimes} L(\Gamma_k)$ whenever Γ_i, $1 \leq i \leq k$, are icc non-amenable groups which are either hyperbolic or discrete subgroups (in particular, lattices) of connected simple Lie groups of rank one. In the meantime, several other unique prime factorization results have been obtained in [Pe06, CS11, SW11, Is14, CKP14, HI15, Ho15, Is16]. Corollary G is the first unique prime factorization result that applies to II_1 factors coming from irreducible lattices in certain higher rank semisimple Lie groups. It implies in particular that if Γ is any irreducible lattice in a product of $n \geq 1$ connected non-compact rank one simple Lie groups with finite center and $\Gamma_0 = \Gamma/Z(\Gamma)$, then the II_1 factors

$$L(\Gamma_0)^{\overline{\otimes}k} = \underbrace{L(\Gamma_0)\overline{\otimes}...\overline{\otimes}L(\Gamma_0)}_{k \text{ times}}, \quad k \geq 1,$$

are pairwise non-isomorphic. This generalizes the case $n = 1$ obtained in [CH88], for lattices in the simplectic groups $\mathrm{Sp}(m,1)$, and in [OP03], for lattices in arbitrary connected non-compact rank one simple Lie groups with finite center.

Our last application of Theorem F relates to prime factorization for measure equivalence. In [MS02], Monod and Shalom proved a series of striking rigidity results for orbit and measure equivalence. In particular, they also studied groups Γ which are measure equivalent to a product $\Lambda = \Lambda_1 \times ... \times \Lambda_n$ of non-elementary hyperbolic groups (more generally, of groups in the class $\mathcal{C}_{\text{reg}}$). In this context, they proved a prime factorization result: if $\Gamma = \Gamma_1 \times ... \times \Gamma_m$ is itself a product group and all the groups involved are torsion-free, then $m \leq n$, and if $m = n$ then, after permutation of the indices, Γ_i is measure equivalent to Λ_i, for $1 \leq i \leq n$ (see [MS02, Theorem 1.16] and [Sa09, Theorem 3]). Theorem F recovers and strengthens this result in the case Γ is icc and Λ_i are hyperbolic. More precisely, it implies that if instead of assuming that Γ is a product of m infinite groups, one merely requires that $L(\Gamma)$ is a tensor product of m II_1 factors, then $m \leq n$, and if $m = n$, then there exists a unique product decomposition $\Gamma = \Gamma_1 \times ... \times \Gamma_m$ such that the above conclusion holds.

III.1.2 Comments on the proof of Theorem F

Since all of our main results are deduced from Theorem F, we outline briefly and informally its method of proof. Let Γ be an icc group which is measure equivalent to a product $\Lambda = \Lambda_1 \times ... \times \Lambda_n$ of non-elementary hyperbolic groups. By [Fu98], Γ and Λ must have stably orbit equivalent actions. To simplify notation, assume that Γ and Λ admit in fact orbit equivalent actions, i.e. there exist free ergodic pmp actions of Γ and Λ on a probability space (X, μ) whose orbits are equal, almost everywhere. Denote $M = L^\infty(X) \rtimes \Gamma = L^\infty(X) \rtimes \Lambda$.

Our goal is to classify all tensor product decompositions $L(\Gamma) = P_1 \overline{\otimes} P_2$. To achieve this goal, we use a combination of techniques from Popa's deformation/rigidity theory.

First, we use repeatedly the relative strong solidity property of hyperbolic groups (see Section III.2.4) established in the breakthrough work [PV11, PV12], to conclude the existence of a partition $S_1 \sqcup S_2 = \{1, \dots, n\}$ such that letting $\Lambda_{S_i} = \times_{j \in S_i} \Lambda_j$ for $i \in \{1, 2\}$, we have

$$P_1 \prec L^\infty(X) \rtimes \Lambda_{S_1}, \quad \text{and} \quad P_2 \prec L^\infty(X) \rtimes \Lambda_{S_2}, \tag{III.1.1}$$

where $P \prec Q$ denotes that a corner of P embeds into Q inside the ambient algebra, in the sense of Popa [Po03]. For simplicity below, we will write $P \sim Q$ to indicate that $Pp' \prec Q$ and $Qq' \prec P$, for all non-zero projections p' and q' in the relative commutants of P and Q.

To see the importance of (III.1.1), note that for each i, we have $P_i \subset L(\Gamma) \subset L^\infty(X) \rtimes \Gamma$, and in this sense P_i is "far away" from $L^\infty(X)$. This remains true after passing through the intertwining in (III.1.1), and hence one thinks of the image of P_i as being not far from $L(\Lambda_{S_i})$ in $L^\infty(X) \rtimes \Lambda_{S_i}$. The critical consequence of (III.1.1) is then that it allows one to show that each P_i inherits a weaker form of the relative strong solidity present in $L(\Lambda_{S_i})$.

In particular, if we follow [PV09] and consider the comultiplication $*$-homomorphism $\Delta : M \to M\overline{\otimes}L(\Gamma)$ given by $\Delta(au_g) = au_g \otimes u_g$, for all $a \in L^\infty(X), g \in \Gamma$, it allows us to conclude that

$$\Delta(L^\infty(X) \rtimes \Lambda_{S_1}) \prec M\overline{\otimes}P_1, \quad \text{and} \quad \Delta(L^\infty(X) \rtimes \Lambda_{S_2}) \prec M\overline{\otimes}P_2. \tag{III.1.2}$$

This is achieved in the first part of Section 5. The conclusion (III.1.2) enables us to then make crucial use of an ultrapower technique from [Io11] (see Section 4) in combination with the transfer of rigidity principle from [PV09] to find subgroups $\Sigma_1, \Sigma_2 < \Gamma$ such that

$$L^\infty(X) \rtimes \Sigma_1 \sim L^\infty(X) \rtimes \Lambda_{S_1}, \quad \text{and} \quad L^\infty(X) \rtimes \Sigma_2 \sim L^\infty(X) \rtimes \Lambda_{S_2}; \tag{III.1.3}$$

$$P_1 \sim L(\Sigma_1), \quad \text{and} \quad P_2 \sim L(\Sigma_2). \tag{III.1.4}$$

This is achieved in the second part of Section 5.

We then use (III.1.3) to deduce that Σ_i is measure equivalent to Λ_{S_i}, for all $i \in \{1,2\}$ (see Section 3).

Finally, inspired by results in [CdSS15], we show that (III.1.4) implies that, after replacing Σ_i with a commensurable subgroup $\Gamma_i < \Gamma$ we have $\Gamma = \Gamma_1 \times \Gamma_2$ with $P_i = L(\Gamma_i)$ for all $i \in \{1,2\}$, modulo unitary conjugacy and amplification (see Section 6). This altogether proves Theorem F.

III.1.3 Organization of the paper

Besides the introduction and a section of preliminaries, this paper has five other sections: Sections 3-6 are devoted to the different ingredients of the proof of Theorem F, as explained above. In Section 7, we finalize the proof of Theorem F and derive the rest of our main results.

III.2 Preliminaries

III.2.1 Equivalence relations

In this subsection we follow the work of Feldman and Moore [FM77] and fix some notations regarding countable equivalence relations. Let (X,μ) be a standard probability space. A *countable pmp equivalence relation* R on (X,μ) is an equivalence relation that satisfies the following:

- $[x]_{\mathcal{R}}$, the $\mathcal{R}$-equivalence class of x, is countable or finite, for almost everywhere (a.e.) $x \in X$,
- $\mathcal{R}$ is a Borel subset of $(X \times X, \mu \times \mu)$,
- any Borel automorphism θ of X satisfying $(\theta(x), x) \in \mathcal{R}$, for a.e. $x \in X$, preserves the measure μ.

For a pmp action $\Gamma \curvearrowright (X,\mu)$ of a countable group Γ on a standard probability space (X,μ), we denote by $\mathcal{R}(\Gamma \curvearrowright X) = \{(x,y) \in X \times X \,|\, \Gamma \cdot x = \Gamma \cdot y\}$ the associated orbit equivalence relation. For a countable pmp equivalence relation $\mathcal{R}$ on (X,μ) and a measurable subset $Y \subset X$, we denote by $\mathcal{R}|_Y = \mathcal{R} \cap (Y \times Y)$ the restriction of $\mathcal{R}$ to Y. For every $x \in X$, $[x]_{\mathcal{R}}$ denotes its equivalence class. We denote by $[[\mathcal{R}]]$ the set of partially defined measurable isomorphisms $\theta : Y = \mathrm{dom}(\theta) \to Z = \mathrm{ran}(\theta)$ between measurable subsets $Y, Z \subset X$ which satisfy $(\theta(x), x) \in \mathcal{R}$, for almost every $x \in Y$. The group of measurable isomorphisms $\theta : X \to X$ which satisfy $(\theta(x), x) \in \mathcal{R}$, for almost every $x \in X$, is called the *full group* of $\mathcal{R}$ and denoted by $[\mathcal{R}]$.

Finally, two pmp actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are called *stably orbit equivalent (SOE)* if there exist non-negligible measurable subsets $X_0 \subset X$ and $Y_0 \subset Y$, and a measure preserving isomorphism $\theta : (X_0, \mu(X_0)^{-1}\mu|_{X_0}) \to (Y_0, \nu(Y_0)^{-1}\nu|_{Y_0})$ such that $(\theta \times \theta)(\mathcal{R}(\Gamma \curvearrowright$

$X)|_{X_0}) = \mathcal{R}(\Lambda \curvearrowright Y)|_{Y_0}$. If this holds for $X_0 = X$ and $Y_0 = Y$, the actions are called *orbit equivalent (OE)*.

III.2.2 Intertwining-by-bimodules

We continue with two remarks and two lemmas, in which we collect several elementary facts concerning Popa's intertwining-by-bimodules technique (Theorem I.2.6).

Remark III.2.1. In the context of Theorem I.2.6, let $(\tilde{M}, \tilde{\tau})$ be a tracial von Neumann algebra such that $M \subset \tilde{M}$ and $\tilde{\tau}_{|M} = \tau$. If $P \prec_M Q$, then clearly $P \prec_{\tilde{M}} Q$. But the following fact, which we will use throughout the paper, also holds: if $P \prec^s_M Q$, then $P \prec^s_{\tilde{M}} Q$. To see this, assume that $P \prec^s_M Q$ and let $p' \in P' \cap \tilde{M}$ be a non-zero projection. Let $p'' \in P' \cap M$ be the support projection of $E_M(p')$. Since $Pp'' \prec_M Q$, we can find projections $p \in P, q \in Q$, a $*$-homomorphism $\theta : pPpp'' \to qQq$, and a non-zero partial isometry $v \in qMpp''$ such that $\theta(x)v = vx$, for all $x \in pPpp''$. Let $\tilde{\theta} : pPpp' \to qQq$ be the $*$-homomorphism given by $\tilde{\theta}(xp') = \theta(xp'')$, for all $x \in pPp$, and put $\tilde{v} = vp'$. Then $\tilde{\theta}$ is well-defined and $\tilde{\theta}(y)\tilde{v} = \tilde{v}y$, for all $y \in pPpp'$. Since $\tilde{v} \neq 0$, we get that $Pp' \prec_{\tilde{M}} Q$.

Remark III.2.2. Let P and $Q_i \subset M_i$, $1 \leq i \leq m$, be tracial von Neumann algebras. Let $\mathcal{U} \subset \mathcal{U}(P)$ a subgroup such that $\mathcal{U}'' = P$, and $\pi_i : P \to M_i$, $1 \leq i \leq m$, be trace preserving $*$-homomorphisms. Assume that there exist $\delta > 0$ and $a_i, b_i \in M_i$, $1 \leq i \leq m$, such that $\sum_{i=1}^m \|E_{Q_i}(a_i \pi_i(u) b_i)\|_2^2 \geq \delta$, for all $u \in \mathcal{U}$. Then $\pi_i(P) \prec_{M_i} Q_i$, for some $1 \leq i \leq m$. Indeed, the above inequality implies that a corner of the von Neumann algebra generated by $\{\oplus_{i=1}^m \pi_i(u) \,|\, u \in \mathcal{U}\}$ embeds into $\oplus_{i=1}^m Q_i$ inside $\oplus_{i=1}^m M_i$, which implies the desired conclusion (see [IPP05, proof of Theorem 4.3] for details).

Lemma III.2.3. *Let (M, τ) be a tracial von Neumann algebra and $P \subset pMp, Q \subset qMq, R \subset rMr$ be von Neumann subalgebras. Then the following hold:*

1. [Va08] *Assume that $P \prec_M Q$ and $Q \prec^s_M R$. Then $P \prec_M R$.*

2. *Assume that* $Pz \prec_M Q$*, for any non-zero projection* $z \in \mathcal{N}_{pMp}(P)' \cap pMp \subset \mathcal{Z}(P' \cap pMp)$*. Then* $P \prec^s_M Q$*.*

3. *Assume that* $P \prec_M Q$*. Then there is a non-zero projection* $z \in \mathcal{N}_{pMp}(P)' \cap pMp$ *such that* $Pz \prec^s_M Q$*.*

4. *Assume that* $P \prec_M Q$*. Then there is a non-zero projection* $z \in \mathcal{N}_{qMq}(Q)' \cap qMq$ *such that* $P \prec_M Qq'$*, for every non-zero projection* $q' \in Q' \cap M$ *with* $q' \leq z$*.*

Proof. (1) This part is precisely [Va08, Lemma 3.7].

(2) & (3) Using Zorn's lemma and a maximality argument, we can find a projection $z \in P' \cap pMp$ such that $Pz \prec^s_M Q$ and $P(p-z) \nprec_M Q$. We claim that $z \in \mathcal{N}_{pMp}(P)' \cap pMp$. This claim clearly implies both (2) and (3).

Let us first show that $z \in \mathcal{Z}(P' \cap pMp)$. Otherwise, we can find non-zero equivalent projections $p_1, p_2 \in P' \cap pMp$ satisfying $p_1 \leq z, p_2 \leq p - z$. Let $u \in \mathcal{U}(P' \cap pMp)$ such that $up_1u^* = p_2$. Then $uPp_1u^* = Pp_2$, which contradicts that $Pp_1 \prec_M Q$, while $Pp_2 \nprec_M Q$. This shows that indeed $z \in \mathcal{Z}(P' \cap pMp)$. Now, if $u \in \mathcal{N}_{pMp}(P)$, then $uzu^* \in \mathcal{Z}(P' \cap pMp)$ and $Puzu^* = uPzu^* \prec^s_M Q$. The maximality property of z forces $uzu^* \leq z$, hence $uzu^* = z$. This proves the claim.

(4) Let $p_0 \in P, q_0 \in Q$ be projections, $\theta : p_0Pp_0 \to q_0Qq_0$ a $*$-homomorphism, and $v \in q_0Mp_0$ a non-zero partial isometry such that $\theta(x)v = vx$, for all $x \in p_0Pp_0$. Let $r \in Q' \cap qMq$ be the support projection of $E_{Q' \cap qMq}(vv^*)$. Let $r' \in Q' \cap qMq$ be a non-zero projection with $r' \leq r$. Let $\psi : p_0Pp_0 \to q_0r'(Qr')q_0r'$ be given by $\psi(x) = \theta(x)r'$ and $w = r'v \in q_0r'Mp_0$. Then $\psi(x)w = wx$, for all $x \in p_0Pp_0$. Since $E_{Q' \cap qMq}(wv^*) = r'E_{Q' \cap qMq}(vv^*) \neq 0$, we get that $w \neq 0$, hence $P \prec_M Qr'$.

Let $z' \in \mathcal{Z}(Q' \cap qMq)$ be the central support of r, and put $z = \vee_{u \in \mathcal{N}_{qMq}(Q)} uz'u^* \in \mathcal{N}_{qMq}(Q)' \cap qMq$. If $q' \in Q' \cap qMq$ is a non-zero projection with $q' \leq z$, we can find $u \in \mathcal{N}_{qMq}(Q)$ such that $q'uz'u^* \neq 0$. This implies the existence of non-zero equivalent projections $q'', r' \in Q' \cap qMq$ such that $q'' \le q'$ and $r' \le uru^*$. As $u^*r'u \le r$, we get that $P \prec_M Qu^*r'u = u^*(Qr')u$, hence $P \prec_M Qr'$. This implies that $P \prec_M Qq''$ and since $q'' \le q'$, we derive that $P \prec_M Qq'$. This finishes the proof. ∎

Lemma III.2.4. *Let $\Lambda < \Gamma$ be a countable groups.*

1. *If $L(\Gamma) \prec_{L(\Gamma)} L(\Lambda)$, then Λ has finite index in Γ.*
2. *If Λ has finite index in Γ, then $L(\Gamma) \prec^s_{L(\Gamma)} L(\Lambda)$.*

Proof. (1) Assume that $L(\Gamma) \prec_{L(\Gamma)} L(\Lambda)$. This implies that the $L(\Gamma)$-$L(\Gamma)$-bimodule $L^2(\langle L(\Gamma), e_{L(\Lambda)}\rangle)$ contains a non-zero $L(\Gamma)$-central vector (see [Po03, Theorem 2.1]). Therefore, the unitary representation $\pi : \Gamma \to \mathcal{U}(L^2(\langle L(\Gamma), e_{L(\Lambda)}\rangle))$ given by $\pi(g)(\xi) = u_g \xi u_g^*$, has a non-zero invariant vector. As π is isomorphic to a subrepresentation of the representation $\Gamma \curvearrowright \oplus_{k\in\Gamma} \ell^2(\Gamma/k\Lambda k^{-1})$, we deduce that $\ell^2(\Gamma/k\Lambda k^{-1})$ contains a non-zero Γ-invariant vector, for some $k \in \Gamma$. This implies that $k\Lambda k^{-1}$ and hence Λ has finite index in Γ.

(2) Assume that $[\Gamma : \Lambda] < \infty$. Let $g_1, ..., g_m \in \Gamma$ such that Γ is the disjoint union of $\{g_i\Lambda\}_{i=1}^m$. Fix any non-zero projection $p \in L(\Gamma)' \cap L(\Gamma) = \mathcal{Z}(L(\Gamma))$. Then $0 < \|p\|_2^2 = \|up\|_2^2 = \sum_{i=1}^m \|E_{L(\Lambda)}(u_{g_i}^* up)\|_2^2$, for every $u \in \mathcal{U}(L(\Gamma))$. This shows that $L(\Gamma)p \prec_{L(\Gamma)} L(\Lambda)$, and the conclusion follows. ∎

III.2.3 Relative amenability

We continue with two lemmas containing several elementary facts regarding relative amenability.

Lemma III.2.5. *Let (M, τ) be a tracial von Neumann algebra, and $P \subset pMp, Q \subset M$ be von Neumann subalgebras. Then the following hold:*

1. *Assume that P is amenable relative to Q. Then Pp' is amenable relative to Q, for every projection $p' \in P' \cap pMp$.*

2. Assume that p_0Pp_0p' is amenable relative to Q, for some projections $p_0 \in P, p' \in P' \cap pMp$. Let z be the smallest projection belonging to $\mathcal{N}_{pMp}(P)' \cap pMp$ such that $p_0p' \leq z$. Then Pz is amenable relative to Q.

3. Assume that $P \prec^s_M Q$. Then P is amenable relative to Q.

Proof. (1) Let $\varphi : p\langle M, e_Q\rangle p \to \mathbb{C}$ be a P-central positive linear functional such that $\varphi_{|pMp} = \tau$. Then the restriction of φ to $p'\langle M, e_Q\rangle p'$ witnesses that Pp' is amenable relative to Q.

(2) Let $p'' \in \mathcal{Z}(P' \cap pMp)$ be the smallest projection such that $p_0p' \leq p''$. By [Io12b, Remark 2.2], Pp'' is amenable relative to Q. Since $z = \vee_{u \in \mathcal{N}_{pMp}(P)} up''u^*$, we can find $p_n \in \mathcal{Z}(P' \cap pMp)p''$ and $u_n \in \mathcal{N}_{pMp}(P)$ such that $z = \sum_n u_n p_n u_n^*$. Since $Pz \subset \bigoplus_n u_n P p_n u_n^*$ and Pp_n is amenable relative to Q for every n by part (1), it follows that Pz is amenable relative to Q.

(3) If P is not amenable relative to Q, then there is a non-zero projection $z \in \mathcal{Z}(P' \cap pMp)$ such that Pz' is not amenable relative to Q, for any non-zero projection $z' \in \mathcal{Z}(P' \cap pMp)z$. Since $Pz \prec_M Q$, we can find projections $p_0 \in P$, $q_0 \in Q$, a $*$-homomorphism $\theta : p_0Pp_0z \to q_0Qq_0$, and a non-zero partial isometry $v \in q_0Mp_0z$ such that $\theta(x)v = vx$, for all $x \in p_0Pp_0z$. Then $v^*v = p_0p'$, for a projection $p' \in (P' \cap pMp)z$, and $vv^* \in \theta(p_0Pp_0z)' \cap q_0Qq_0$. Since $\theta(p_0Pp_0z) \subset q_0Qq_0$, by part (1), $\theta(p_0Pp_0)vv^*$ is amenable relative to Q. Since $\theta(p_0Pp_0)vv^*$ is unitarily conjugate to p_0Pp_0p', the latter algebra is also amenable relative to Q. By part (2), we can find a projection $z' \in \mathcal{Z}(P' \cap pMp)$ such that $p_0p' \leq z'$ and Pz' is amenable relative to Q. This contradicts the definition of z. ■

Lemma III.2.6. *Let (M, τ) be a tracial von Neumann algebra and $Q \subset M$ a von Neumann subalgebra. Let $(P_i)_{i \in I} \subset pMp$ be an increasing net of von Neumann subalgebras, and denote $P = (\cup_{i \in I} P_i)''$.*

If P_i is amenable relative to Q, for every $i \in I$, then P is amenable relative to Q.

Proof. Let $\lim_i$ denote a state on $\ell^\infty(I)$ which extends the usual limit. For every $i \in I$, let $\varphi_i : p\langle M, e_Q\rangle p \to \mathbb{C}$ be a P_i-central positive linear functional such that $\varphi_{i|pMp} = \tau$. We define $\varphi : p\langle M, e_Q\rangle p \to \mathbb{C}$ by letting $\varphi(T) = \lim_i \varphi_i(T)$, for every $T \in p\langle M, e_Q\rangle p$.

Then φ is a positive linear functional and $\varphi_{|pMp} = \tau$. Moreover, φ is P_i-central, for every $i \in I$. To see this, let $x \in P_i$, for some $i \in I$, and $T \in p\langle M, e_Q\rangle p$. If $j \in I$ satisfies $j \geq i$, then $P_i \subset P_j$ and thus $x \in P_j$. Hence, we have $\varphi_j(xT) = \varphi_j(Tx)$, for every $j \geq i$, which implies that $\varphi(xT) = \varphi(Tx)$.

Let $A \subset P$ be the set of all $x \in P$ such that $\varphi(xT) = \varphi(Tx)$, for every $T \in p\langle M, e_Q\rangle p$. By the above, A contains $\cup_{i\in I} P_i$. On the other hand, the Cauchy-Schwarz inequality implies that $|\varphi(xT)| \leq \sqrt{\varphi(x^*x)\varphi(TT^*)} \leq \|x\|_2\|T\|$ and similarly that $|\varphi(Tx)| \leq \|x\|_2\|T\|$, for all $x \in pMp$ and $T \in p\langle M, e_Q\rangle p$. This implies that A is closed in $\|.\|_2$. Hence, $A = P$ and thus φ is P-central. ■

We next record the following useful result:

Lemma III.2.7. *Let (M, τ) be a tracial von Neumann algebra and $Q_1, Q_2 \subset M$ von Neumann subalgebras which form a commuting square, i.e. $E_{Q_1} \circ E_{Q_2} = E_{Q_2} \circ E_{Q_1}$. Assume that Q_1 is regular in M. Let $P \subset pMp$ be a von Neumann subalgebra. Then the following hold:*

1. [PV11] *If P is amenable relative to Q_1 and Q_2, then P is amenable relative to $Q_1 \cap Q_2$.*
2. *If $P \prec^s_M Q_1$ and $P \prec^s_M Q_2$, then $P \prec^s_M Q_1 \cap Q_2$.*

Proof. Part (1) is precisely [PV11, Proposition 2.7]. Part (2) follows easily by adapting the proof of [PV11, Proposition 2.7]. For completeness we include a proof, using the notation therein.

Assume that $P \prec^s_M Q_1$ and $P \prec^s_M Q_2$. Let $p' \in P' \cap pMp$ be a non-zero projection. We will prove the conclusion of part (2) by showing that $Pp' \prec_M Q_1 \cap Q_2$. To this end, for $i \in \{1, 2\}$, we let $\text{Tr}_i : \langle M, e_{Q_i}\rangle \to \mathbb{C}$ be the canonical semifinite trace given by $\text{Tr}_i(xe_{Q_i}y) =$

$\tau(xy)$, for all $x, y \in M$. Let $\mathcal{T}_i : L^1(\langle M, e_{Q_i}\rangle) \to L^1(M)$ given by $\tau(\mathcal{T}_i(T)x) = \mathrm{Tr}_i(Tx)$, for all $T \in L^1(\langle M, e_{Q_i}\rangle)$ and $x \in M$.

Since $P \prec^s_M Q_1$, we get that $Pp' \prec_M Q_1$. By [Po03, Theorem 2.1] we can find a non-zero projection $e_1 \in (Pp')' \cap p'\langle M, e_{Q_1}\rangle p'$ such that $\mathrm{Tr}_1(e_1) < \infty$. Let $p'' \in M$ be the support projection of $\mathcal{T}_1(e_1)$. Then $p'' \in (Pp')' \cap p'Mp' = p'(P' \cap pMp)p'$. Since $P \prec^s_M Q_2$, we get that $Pp'' \prec_M Q_2$. By [Po03, Theorem 2.1] we can find a non-zero projection $e_2 \in (Pp'')' \cap p''\langle M, e_{Q_2}\rangle p''$ with $\mathrm{Tr}_2(e_2) < \infty$.

Next, consider the M-M-bimodule $\mathcal{H} = L^2(\langle M, e_{Q_1}\rangle) \otimes_M L^2(\langle M, e_{Q_2}\rangle)$ and put $\xi = e_1 \otimes_M e_2 \in \mathcal{H}$. Then $x\xi = \xi x$, for all $x \in Pp''$. Moreover, since $p''e_2 = e_2 \neq 0$ and p'' is the support projection of $\mathcal{T}_1(e_1)$ we have $\mathcal{T}_1(e_1)^{1/2}e_2 \neq 0$. Since $\|\xi\|^2 = \langle e_1 \otimes_M e_2, e_1 \otimes_M e_2\rangle = \langle \mathcal{T}_1(e_1)e_2, e_2\rangle = \|\mathcal{T}_1(e_1)^{1/2}e_2\|^2$, we deduce that $\xi \neq 0$.

Now, by the last part of the proof of [PV11, Proposition 2.7], the M-M-bimodule $\mathcal{H}$ is contained in a multiple of ${}_ML^2(\langle M, e_Q\rangle)_M$, where $Q = Q_1 \cap Q_2$. Since $0 \neq \xi = p''\xi p''$, we derive the existence of a non-zero vector $\eta \in p''L^2(\langle M, e_Q\rangle)p''$ such that $x\eta = \eta x$, for all $x \in Pp''$. Then $\zeta = \eta^*\eta \in L^1(\langle M, e_Q\rangle)_+$ satisfies $0 \neq \zeta = p''\zeta p''$ and $x\zeta = \zeta x$, for all $x \in Pp''$. Let $t > 0$ such that the spectral projection $f = \mathbf{1}_{[t,\infty)}(\zeta)$ is non-zero. Then $f \in (Pp'')' \cap p''\langle M, e_Q\rangle p''$ and since $tf \leq \zeta$, we get that $\mathrm{Tr}(f) \leq \mathrm{Tr}(\zeta)/t < \infty$, where $\mathrm{Tr} : \langle M, e_Q\rangle \to \mathbb{C}$ denotes the canonical semifinite trace. By [Po03, Theorem 2.1] we conclude that $Pp'' \prec_M Q$ and hence that $Pp' \prec_M Q$, as desired. ■

For the last result of this subsection, assume the following context: let $\Gamma \curvearrowright (X, \mu)$, $\Lambda \curvearrowright (Y, \nu)$ be stably orbit equivalent free ergodic pmp actions. Thus, there exists $\ell \geq 1$ such that we can view X as a subset of $Y \times \mathbb{Z}/\ell\mathbb{Z}$ satisfying $\mathcal{R}(\Gamma \curvearrowright X) = \mathcal{R}(\Lambda \times \mathbb{Z}/\ell\mathbb{Z} \curvearrowright Y \times \mathbb{Z}/\ell\mathbb{Z})|_X$, where $\mathbb{Z}/\ell\mathbb{Z}$ acts on itself by addition. Hence, $L^\infty(X) \rtimes \Gamma = pMp$, where $M = L^\infty(Y \times \mathbb{Z}/\ell\mathbb{Z}) \rtimes (\Lambda \times \mathbb{Z}/\ell\mathbb{Z})$ and $p = 1_X$. If $B = L^\infty(Y) \otimes \mathbb{M}_\ell(\mathbb{C})$, then we identify $M = B \rtimes \Lambda$, where Λ acts trivially on $\mathbb{M}_\ell(\mathbb{C})$.

Lemma III.2.8. *Let Σ be a subgroup of Λ such that $L(\Gamma)$ is amenable relative to $B \rtimes \Sigma$*

inside M.

Then Σ *is co-amenable in* Λ, *i.e.* $\ell^\infty(\Lambda/\Sigma)$ *admits a left* Λ*-invariant state.*

Proof. Assume first that $(\nu \times c)(X) \geq 1$, where c denotes the counting measure on $\mathbb{Z}/\ell\mathbb{Z}$. Then by using the ergodicity of the actions, we may assume that the inclusion $X \subset Y \times \mathbb{Z}/\ell\mathbb{Z}$ satisfies $Y \subset X$, where Y denotes its copy $Y \times \{0\} \subset Y \times \mathbb{Z}/\ell\mathbb{Z}$. Thus, we have $q = 1_Y \leq p = 1_X$ and $qBq = L^\infty(Y)$. Put $A = L^\infty(X)$ and note that $A \rtimes \Gamma = pMp$ and $L^\infty(Y) \rtimes \Lambda = q(A \rtimes \Gamma)q$. We also denote by $\{u_g\}_{g\in\Gamma} \subset A \rtimes \Gamma$ and $\{v_h\}_{h\in\Lambda} \subset B \rtimes \Lambda$ the canonical unitaries implementing the actions of Γ and Λ on A and B, respectively. We end this paragraph by observing that $\{v_h q\}_{h\in\Lambda} \subset L^\infty(Y) \rtimes \Lambda$ are precisely the canonical unitaries which implement the action of Λ on $L^\infty(Y)$.

Next, since $L(\Gamma)$ is amenable relative to $B \rtimes \Sigma$ inside M, there exists a positive linear functional $\varphi : p\langle M, e_{B\rtimes\Sigma}\rangle p \to \mathbb{C}$ which is $L(\Gamma)$-central and satisfies $\varphi_{|pMp} = \tau$.

Let $\mathcal{D} \subset q\langle M, e_{B\rtimes\Sigma}\rangle q$ be the von Neumann algebra generated by $\{v_h q e_{B\rtimes\Sigma}\, q v_h^*\}_{h\in\Lambda}$. If $h \in \Lambda \setminus \Sigma$, then $e_{B\rtimes\Sigma} v_h q e_{B\rtimes\Sigma} = E_{B\rtimes\Sigma}(v_h) q e_{B\rtimes\Sigma} = 0$. On the other hand, if $h \in \Sigma$, then $v_h q \in B \rtimes \Sigma$, thus $v_h q e_{B\rtimes\Sigma} = e_{B\rtimes\Sigma} v_h q$. Let $S \subset \Lambda$ be a complete set of representatives for Λ/Σ. The above observations imply that the formula $\pi(f) = \sum_{h\in S} f(h\Sigma) v_h q\, e_{B\rtimes\Sigma}\, q v_h^*$ defines a $*$-isomorphism $\pi : \ell^\infty(\Lambda/\Sigma) \to \mathcal{D}$. Moreover, we have $\pi(k \cdot f) = v_k q \pi(f) q v_k^*$, for every $k \in \Lambda$ and $f \in \ell^\infty(\Lambda/\Sigma)$.

Now, we claim that $\varphi(v_k q T q v_k^*) = \varphi(T)$, for all $k \in \Lambda$ and $T \in \mathcal{D}$. Since $v_k q \in \mathcal{N}_{q(A\rtimes\Gamma)q}(Aq)$, we can find mutually orthogonal projections $a_g \in Aq$, $g \in \Gamma$, such that $v_k q = \sum_{g\in\Gamma} u_g a_g$ and $\sum_{g\in\Gamma} a_g = q$, where both series converge in $\|.\|_2$. Note that $Aq = L^\infty(Y)$ commutes with $\mathcal{D}$, hence a_g commutes with $\mathcal{D}$, for every $g \in \Gamma$. Moreover, $a_g q v_k^* = a_g u_g^*$, for every $g \in \Gamma$. Also, the Cauchy-Schwarz inequality implies that $|\varphi(xV))|, |\varphi(Vx)| \leq \|x\|_2 \|V\|$, for every $x \in pMp$ and $V \in p\langle M, e_{B\rtimes\Sigma}\rangle p$. By combining these facts with the fact that φ is

$L(\Gamma)$-central we obtain that

$$\begin{aligned} \varphi(v_k q T q v_k^*) &= \textstyle\sum_{g\in\Gamma} \varphi(u_g a_g T q v_k^*) \\ &= \textstyle\sum_{g\in\Gamma} \varphi(u_g T a_g q v_k^*) \\ &= \textstyle\sum_{g\in\Gamma} \varphi(u_g T a_g u_g^*) \\ &= \textstyle\sum_{g\in\Gamma} \varphi(T a_g) \\ &= \varphi(T). \end{aligned}$$

It is now clear that the positive linear functional $\varphi \circ \pi : \ell^\infty(\Lambda/\Sigma) \to \mathbb{C}$ is Λ-left invariant, which implies that Σ is co-amenable in Λ. This finishes the proof in the case $(\nu \times c)(X) \geq 1$.

In general, let $r \geq 1$ such that $r(\nu \times c)(X) \geq 1$, and put $X_1 = X \times \mathbb{Z}/r\mathbb{Z}$, $\Gamma_1 = \Gamma \times \mathbb{Z}/r\mathbb{Z}$. Then $X_1 \subset Y \times \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z}$ and if we consider a bijection $\mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z} \equiv \mathbb{Z}/\ell r\mathbb{Z}$, we have that $\mathcal{R}(\Gamma_1 \curvearrowright X_1) = \mathcal{R}(\Lambda \times \mathbb{Z}/\ell r\mathbb{Z} \curvearrowright Y \times \mathbb{Z}/\ell r\mathbb{Z})|_{X_1}$. Denote $M_1 = L^\infty(Y \times \mathbb{Z}/\ell r\mathbb{Z}) \rtimes (\Lambda \times \mathbb{Z}/\ell r\mathbb{Z})$, $B_1 = L^\infty(Y) \rtimes \mathbb{M}_{\ell r}(\mathbb{C})$, and identify $M_1 = B_1 \rtimes \Lambda$. Since the inclusion $B_1 \rtimes \Sigma \subset M_1$ is naturally identified to the inclusion $(B \rtimes \Sigma) \otimes \mathbb{M}_r(\mathbb{C}) \subset M \otimes \mathbb{M}_r(\mathbb{C})$, we get that $L^\infty(\mathbb{Z}/r\mathbb{Z}) \rtimes \Gamma_1 \equiv L(\Gamma) \otimes \mathbb{M}_r(\mathbb{C})$ is amenable relative to $B_1 \rtimes \Sigma$ inside M_1. Thus, $L(\Gamma_1)$ is amenable relative to $B_1 \rtimes \Sigma$ inside M_1. Since $(\nu \times c)(X_1) \geq 1$, we can apply the above and derive that Σ is co-amenable in Λ. ■

III.2.4 Relatively strongly solid groups

In his breakthrough work [Oz03], Ozawa proved that II_1 factors arising from non-elementary hyperbolic groups Γ (e.g. $\Gamma = \mathbb{F}_n, 2 \leq n \leq \infty$) are *solid*: if $P_1, P_2 \subset L(\Gamma)$ are commuting von Neumann subalgebras, then either P_1 is not diffuse or P_2 is amenable. In the last ten years, this result has been generalized and strengthened in many ways. Remarkably, Ozawa and Popa proved that if $\Gamma = \mathbb{F}_n, 2 \leq n \leq \infty$, then $L(\Gamma)$ *strongly solid*: the normalizer $\mathcal{N}_{L(\Gamma)}(P)''$ is amenable, for any diffuse amenable von Neumann subalgebra

$P \subset L(\Gamma)$. Chifan and Sinclair extended this to cover all non-elementary hyperbolic groups Γ [CS11].

Most recently, a breakthrough was made by Popa and Vaes who proved that non-abelian free groups and, more generally, non-elementary hyperbolic groups Γ are relatively strong solid [PV11, PV12]. Following [CIK13, Definition 2.7], we say that a countable non-amenable group Γ is *relatively strongly solid* and write $\Gamma \in \mathcal{C}_{\text{rss}}$ if for any trace preserving action $\Gamma \curvearrowright Q$ on a tracial von Neumann algebra (Q, τ) the following holds: if $M = Q \rtimes \Gamma$ and $P \subset pMp$ is any von Neumann subalgebra which is amenable relative to Q, then either $P \prec_M Q$ or the normalizer $\mathcal{N}_{pMp}(P)''$ is amenable relative to Q. Note that $\mathcal{C}_{\text{rss}}$ more generally contains all weakly amenable groups that either admit a proper 1-cocycle into an orthogonal representation weakly contained in the left regular representation [PV11, Theorem 1.6], or are bi-exact [PV12, Theorem 1.4].

We will use repeatedly the following consequence of belonging to $\mathcal{C}_{\text{rss}}$ (see [KV15, Lemma 5.2]).

Lemma III.2.9 ([KV15]). *Let Γ be a group in $\mathcal{C}_{rss}$, and $M = Q \rtimes \Gamma$, where $\Gamma \curvearrowright Q$ is a trace preserving action on a tracial von Neumann algebra. Let $P_1, P_2 \subset M$ be commuting von Neumann subalgebras.*

Then either $P_1 \prec_M Q$ or P_2 is amenable relative to Q.

III.3 From intertwining to measure equivalence

The main goal of this section is to establish the following proposition, which provides the tool used to deduce the measure equivalence in part (2) of Theorem F:

Proposition III.3.1. *Let $\mathcal{R}$ be a countable pmp equivalence relation on (X, μ), and $Y, Z \subset X$ be positive measure subsets. Suppose that $\mathcal{R}|_Y = \mathcal{R}(\Gamma_1 \times \Gamma_2 \curvearrowright Y)$ and $\mathcal{R}|_Z \geq \mathcal{R}(\Lambda \curvearrowright Z)$ for free measure preserving actions of countable groups Γ_1, Γ_2, and Λ. Assume that*

(i) $L^\infty(Y) \rtimes \Gamma_1 \prec_{L(\mathcal{R})} L^\infty(Z) \rtimes \Lambda$, *and*

(ii) $L^\infty(Z) \rtimes \Lambda \prec^s_{L(\mathcal{R})} L^\infty(Y) \rtimes \Gamma_1$.

Then Γ_1 *and* Λ *are measure equivalent.*

Throughout this section, all subsets of probability spaces that we consider are assumed measurable.

In order to prove Proposition III.3.1, we first establish a series of lemmas in subsections III.3.1-III.3.4. The proof of Proposition III.3.1 is then given in Subsection III.3.5.

III.3.1 Essentially finite index subequivalence relations

Consider an inclusion of countable pmp equivalence relations $\mathcal{T} \leq \mathcal{R}$ on (X, μ). Decompose $X = \bigsqcup_{N \in \{1,2,\dots,\aleph_0\}} X_N$, where the X_N are the $\mathcal{R}$-invariant sets defined by

$$X_N = \{x \in X \mid [x]_{\mathcal{R}} \text{ is the union of } N \ \mathcal{T}\text{-classes}\} \quad \text{for} \quad N = 1, 2, \dots, \aleph_0. \qquad \text{(III.3.1)}$$

If $\mu(X_\infty) = 0$, we say that the inclusion $\mathcal{T} \leq \mathcal{R}$ has *essentially finite index.* If in fact there exists $k \geq 1$ such that $\mu(X_N) = 0$ for all $N > k$, the inclusion is said to have *bounded index.*

We will use the following basic fact, whose proof we include for the sake of completeness.

Lemma III.3.2. *Let* $\mathcal{S}, \mathcal{T} \leq \mathcal{R}$ *be inclusions of pmp countable equivalence relations and suppose that* $\mathcal{S} \leq \mathcal{R}$ *has essentially finite (respectively, bounded) index.*

Then $\mathcal{S} \cap \mathcal{T} \leq \mathcal{T}$ *has essentially finite (respectively, bounded) index.*

Proof. Note that if $C_{\mathcal{S}}$ is an $\mathcal{S}$-class and $C_{\mathcal{T}}$ is a $\mathcal{T}$-class, then $C_{\mathcal{S}} \cap C_{\mathcal{T}}$ is either empty or equal to the $(\mathcal{S} \cap \mathcal{T})$-class of any of its elements. Hence for $x \in X$, if $\mathcal{C}_x$ denotes the set of $\mathcal{S}$-classes in $[x]_{\mathcal{R}}$, we have that

$$[x]_{\mathcal{T}} = [x]_{\mathcal{T}} \cap [x]_{\mathcal{R}} = \bigsqcup_{C \in \mathcal{C}_x} ([x]_{\mathcal{T}} \cap C)$$

is the union of at most $|\mathcal{C}_x|$ $(\mathcal{S} \cap \mathcal{T})$-classes. If $\mathcal{S} \leq \mathcal{R}$ is essentially finite (resp. bounded) index, then $|\mathcal{C}_x| < \infty$ (resp. there is $k \geq 1$ such that $|\mathcal{C}_x| < k$) for a.e. $x \in X$, and hence $\mathcal{S} \cap \mathcal{T} \leq \mathcal{T}$ has essentially finite (resp. bounded) index. ■

The product structure $\Gamma_1 \times \Gamma_2$ assumed in Proposition III.3.1 will be exploited via the following lemma:

Lemma III.3.3. *Let $\mathcal{R} = \mathcal{R}(\Gamma_1 \times \Gamma_2 \curvearrowright X)$ for a pmp action of the product of countable groups Γ_1, Γ_2 on (X, μ). Let $\mathcal{T} = \mathcal{R}(\Gamma_1 \curvearrowright X)$, $Y \subset X$ a positive measure subset, $\mathcal{T}_0 \leq \mathcal{T}|_Y$ a subequivalence relation, and $\theta \in [[\mathcal{R}]]$ with $Y = \mathrm{dom}(\theta)$ such that $(\theta \times \theta)(\mathcal{T}_0) \leq \mathcal{T}|_{\theta(Y)}$. Assume that $\mathcal{T}_0 \leq \mathcal{T}|_Y$ has essentially finite (respectively, bounded) index.*

Then there is a sequence of $\mathcal{T}_0$-invariant positive measure $Y_n \subset Y$ with $Y = \bigcup_{n=1}^{\infty} Y_n$ such that $(\theta \times \theta)(\mathcal{T}_0|_{Y_n}) \leq \mathcal{T}|_{\theta(Y_n)}$ has essentially finite (respectively, bounded) index for each $n \geq 1$.

Proof. Enumerate $\Gamma_2 = \{s_n\}_{n=1}^{\infty}$ and let

$$Y_n = \{x \in Y \mid \text{there exists } h_1(x) \in \Gamma_1 \text{ such that } \theta(x) = h_1(x) s_n x\}$$

Then $Y = \bigcup_{n=1}^{\infty} Y_n$, and each Y_n is $\mathcal{T}_0$-invariant, for if $x \in Y_n$ and $(x, x') \in \mathcal{T}_0$, then $(\theta(x), \theta(x')) \in \mathcal{T}$, so there is $k_1 \in \Gamma_1$ such that $\theta(x') = k_1 \theta(x) = k_1 h_1(x) s_n x$.

Now for any $x, x' \in Y_n$ such that $(\theta(x), \theta(x')) \in \mathcal{T}$, there is $k_1 \in \Gamma_1$ such that $\theta(x') = k_1 \theta(x) = k_1 h_1(x) s_n x$, and on the other hand, $\theta(x') = h_1(x') s_n x'$ and so we conclude that $x' = h_1(x')^{-1}k_1h_1(x)x$, giving $(x,x') \in \mathcal{T}$. Thus we have

$$\mathcal{T}_0|_{Y_n} \leq (\theta^{-1} \times \theta^{-1})(\mathcal{T}|_{\theta(Y_n)}) \leq \mathcal{T}|_{Y_n},$$

and since $\mathcal{T}_0|_{Y_n} \leq \mathcal{T}|_{Y_n}$ has bounded index, so too does $\mathcal{T}_0|_{Y_n} \leq (\theta^{-1} \times \theta^{-1})(\mathcal{T}|_{\theta(Y_n)})$ and its image under θ, as desired. ∎

III.3.2 Realizing subequivalence relations as restrictions

We recall in this section a useful construction appearing in [IKT08]. Consider as above an inclusion of countable pmp equivalence relations $\mathcal{S} \leq \mathcal{R}$ on (X,μ) and the decomposition $X = \bigsqcup_{N \in \{1,2,\ldots,\aleph_0\}} X_N$ defined by (III.3.1). For each N, let $\{C_n^{(N)}\}_{0 \leq n < N}$ be a sequence of choice functions, i.e. a sequence of Borel functions $C_n^{(N)} : X_N \to X_N$ such that for each $x \in X_N$ the sequence $\{C_n^{(N)}(x)\}_{n=0}^{N-1}$ contains exactly one element of each $\mathcal{S}$-class contained in $[x]_{\mathcal{R}}$. We take $C_0^{(N)} = \mathrm{Id}_X$.

Each $(x,y) \in \mathcal{R}|_{X_N}$ gives rise to a permutation $\pi_N(x,y) \in S_N$ defined by

$$m = \pi_N(x,y)(n) \iff (C_m^{(N)}(x), C_n^{(N)}(y)) \in \mathcal{S},$$

and the map $\pi_N : \mathcal{R}|_{X_N} \to S_N$ is called the *index cocycle* associated to these choice functions (see [FSZ89]).

Let

$$(\tilde{X}, \lambda) = \bigsqcup_{N \in \{1,2,\ldots,\aleph_0\}} (X_N \times \{0,\ldots,N-1\},\ \mu \otimes c) \tag{III.3.2}$$

where c denotes the counting measure. In the case of an essentially finite index inclusion $\mathcal{S} \leq \mathcal{R}$, we may instead endow the space $\tilde{X}$ with an $\tilde{\mathcal{R}}$-invariant probability measure $\tilde{\mu}$ by normalizing the counting measure:

$$(\tilde{X},\tilde{\mu}) = \bigsqcup_{N\in\{1,2,\dots,\aleph_0\}} \left(X_N \times \{0,\dots,N-1\},\ \mu\otimes\frac{c}{N}\right) \tag{III.3.3}$$

We define a measurable equivalence relation $\tilde{\mathcal{R}}$ on $\tilde{X}$ by

$$((x,n),(y,m)) \in \tilde{\mathcal{R}} \iff (x,y)\in\mathcal{R} \text{ and } n = \pi_N(x,y)(m), \tag{III.3.4}$$

for $x,y\in X_N$, $n,m\in\{0,\dots,N-1\}$. For $x,y\in X_N$ we have $((x,0),(y,0))\in\tilde{\mathcal{R}}$ if and only if $(x,y)\in\mathcal{R}$ and $\pi_N(x,y)(0)=0$ which occurs exactly when $(x,y) = (C_0^{(N)}(x), C_0^{(N)}(y))\in\mathcal{S}$. Thus

$$\mathcal{S} = \tilde{\mathcal{R}}|_{X\times\{0\}} \tag{III.3.5}$$

Now let $p:\tilde{X}\to X$ be the projection map $p(x,n)=x$. Any element $\phi\in[[\mathcal{R}]]$ gives rise to $\tilde{\phi}\in[[\tilde{\mathcal{R}}]]$ defined by

$$\begin{aligned}\tilde{\phi}: p^{-1}(\operatorname{dom}\phi) &\to p^{-1}(\operatorname{ran}\phi)\\ (x,n) &\mapsto (\phi(x), \pi_N(\phi(x),x)(n)) \quad \text{for} \quad x\in X_N,\ n\in\{0,\dots,N-1\},\end{aligned} \tag{III.3.6}$$

such that $\tilde{\phi}\tilde{\psi} = \widetilde{\phi\psi}$ for $\phi,\psi\in[[\mathcal{R}]]$. In particular, if $\mathcal{R}=\mathcal{R}(\Gamma\curvearrowright X)$ is given by the free pmp action of a countable group Γ, then $\tilde{\mathcal{R}}$ is given by the free measure preserving action $\Gamma\curvearrowright\tilde{X}$ defined by

$$g\cdot(x,n) = (gx, \pi_N(gx,x)(n)) \quad \text{for} \quad x\in X_N,\ n\in\{0,\dots,N-1\}. \tag{III.3.7}$$

III.3.3 A stable orbit equivalence-type characterization of measure equivalence

The main purpose of this subsection is to prove Lemma III.3.6, which allows one to deduce that countable groups Γ and Λ admit SOE free pmp actions (and hence are ME) from a seemingly weaker condition. We begin with the following general ergodic-theoretic lemma, whose proof we include for the sake of completeness.

Lemma III.3.4. *Let $\mathcal{R}$ be a countable pmp equivalence relation on (X,μ) and $E \subset X$ a positive measure subset.*

Then there exist a positive measure subset $E_0 \subset E$ and $\phi_0 = \mathrm{id}_{E_0}, \phi_1, \ldots, \phi_k \in [[\mathcal{R}]]$, such that

1. $\mathrm{dom}(\phi_i) = E_0$ *for* $i = 0, \ldots, k$,
2. $\mathrm{ran}(\phi_i) \cap \mathrm{ran}(\phi_j) = \varnothing$ *for* $i \neq j$, *and*
3. $Y = \bigsqcup_{i=0}^{k} \mathrm{ran}(\phi_i)$ *is* $\mathcal{R}$*-invariant.*

Proof. Let Z be the set of $\mathcal{R}$-ergodic invariant probability measures on X, and let $\pi : X \to Z$ denote the ergodic decomposition of μ with respect to $\mathcal{R}$ (see [KM04, Theorem 18.5]). Thus, if we denote $\nu = \pi_*\mu$, then $\mu = \int_Z m \, d\nu(m)$. Consider the natural embedding $L^2(Z) \ni f \mapsto f \circ \pi \in L^2(X)$ and denote by $e : L^2(X) \to L^2(Z)$ the orthogonal projection, noting that for $f \in L^2(X)$, we have that $e(f)$ is given by $e(f)(x) = \int_X f(y) d(\pi(x))(y)$ for almost every $x \in X$.

Since $\mu(E) = \int_Z m(E) \, d\nu$ is positive, the set $Z_1 = \{m \in Z | m(E) > 0\}$ has positive measure. Since each $m \in Z_1$ is $\mathcal{R}$-ergodic, there is a positive measure subset $Z_0 \subset Z_1$ such that either (X, m) is non-atomic for all $m \in Z_0$ or such that there is an integer $k \geq 0$ with m supported on $k+1$ atoms each of measure $\frac{1}{k+1}$ for all $m \in Z_0$. In any case, we can find an integer $k \geq 0$ and a measurable subset $E_0 \subset E \cap \pi^{-1}(Z_0)$ with $m(E_0) = \frac{1}{k+1}$

for all $m \in Z_0$. Moreover, we can then find measurable subsets $E_1, ..., E_k \subset \pi^{-1}(Z_0)$ with $\pi^{-1}(Z_0) = E_0 \cup E_1 \cup ... \cup E_k$ such that $E_i \cap E_j = \emptyset$ for all $0 \leq i < j \leq k$, and $m(E_i) = \frac{1}{k+1}$ for all $0 \leq i \leq k$ and $m \in Z_0$.

Since $\pi^{-1}(Z_0)$ is $\mathcal{R}$-invariant, in order to get the conclusion, it suffices to prove the following claim:

Claim. Let $A, B \subset X$ be measurable sets satisfying $m(A) = m(B)$, for almost every $m \in Z$. Then there is $\theta \in [[\mathcal{R}]]$ such that $\text{dom}(\theta) = A$ and $\text{ran}(\theta) = B$.

To this end, let $\{A_j\}_{j\in J}$ and $\{B_j\}_{j\in J}$ be maximal families of disjoint non-negligible measurable subsets of A and B such that for every $j \in J$ we can find $\theta_j \in [[\mathcal{R}]]$ with $\text{dom}(\theta_j) = A_j$ and $\text{ran}(\theta_j) = B_j$. Since $\sum_{j\in J} \mu(A_j) \leq \mu(A) \leq 1$, we deduce that J is countable. In particular, the sets $A' = \cup_{j\in J} A_j$, $B' = \cup_{j\in J} B_j$, $A'' = A \setminus A'$, and $B'' = B \setminus B'$ are measurable.

Our goal is to show that $\mu(A'') = \mu(B'') = 0$. Granting this, $\theta \in [[\mathcal{R}]]$ given by $\theta(x) = \theta_j(x)$ for all $x \in A_j$ and $j \in J$ satisfies $\theta(A') = B'$, and since $\mu(A \setminus A') = \mu(B \setminus B') = 0$, the claim follows.

Assume by contradiction that $\mu(A'') = \mu(B'') > 0$. For any $m \in Z$ and $j \in J$, since m is $\mathcal{R}$-invariant and $B_j = \theta_j(A_j)$, we have $m(B_j) = m(A_j)$. Together with the assumption made on A and B, this implies that $m(A'') = m(B'')$, for almost every $m \in Z$.

Let us show that there is $\rho \in [\mathcal{R}]$ such that $\mu(\rho(A'') \cap B'') > 0$. Otherwise, we would have that $\int_{B''} \mathbf{1}_{A''} \circ \rho \, \text{d}\mu = 0$, for all $\rho \in [\mathcal{R}]$. Thus, if $\mathcal{K} \subset L^2(X, \mu)$ denotes the $\|.\|_2$-closure of the convex hull of $\{\mathbf{1}_{A''} \circ \rho | \rho \in [\mathcal{R}]\}$, then $\int_{B''} f \, \text{d}\mu = 0$, for every $f \in \mathcal{K}$. If $f \in \mathcal{K}$ denotes the element of minimal $\|.\|_2$, then f is $\mathcal{R}$-invariant, hence $f = e(f)$. Moreover, since $e(\mathbf{1}_{A''} \circ \rho) = e(\mathbf{1}_{A''})$, for all $\rho \in [\mathcal{R}]$, we conclude that $f = e(\mathbf{1}_{A''}) \geq 0$. This and the condition $\int_{B''} f \, \text{d}\mu = 0$ imply that $\pi(x)(A'') = f(x) = 0$, for almost every $x \in B''$. Thus, $\pi(x)(B'') = 0$, for almost every $x \in B''$, contradicting our assumption that $\mu(B'') > 0$.

Finally, let $\tilde{A} = A'' \cap \rho^{-1}(B''), \tilde{B} = \rho(A'') \cap B''$, and $\tilde{\theta} \in [[\mathcal{R}]]$ be the restriction of

ρ to $\tilde{A}$. Since $\mu(\tilde{A}) = \mu(\tilde{B}) > 0$, $\tilde{\theta}(\tilde{A}) = \tilde{B}$, and $\tilde{A} \cap A' = \tilde{B} \cap B' = \varnothing$, this contradicts the maximality of the families $\{A_j\}_{j \in J}$ and $\{B_j\}_{j \in J}$, and finishes the proof of the claim. ∎

Lemma III.3.5. *Let $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X)$ for a free pmp action of a countable group Γ and let $E \subset X$ be a positive measure subset.*

Then there exists a positive measure subset $E_0 \subset E$ with the following property: for any essentially finite index subequivalence relation $\mathcal{T} \leq \mathcal{R}|_{E_0}$ there is a free pmp action $\Gamma \curvearrowright (\tilde{X}, \tilde{\mu})$ such that $\mathcal{T} \cong \mathcal{R}(\Gamma \curvearrowright \tilde{X})|_{\tilde{E}_0}$ for some measurable subset $\tilde{E}_0 \subset \tilde{X}$.

Proof. Let $E_0 \subset E$, $Y \subset X$ and $\phi_0, \ldots, \phi_k \in [[\mathcal{R}]]$ be as in the conclusion of Lemma III.3.4. Let $\mathcal{S} = \mathcal{R}(\Gamma \curvearrowright X)|_Y$ and note that since Y is Γ-invariant we have $\mathcal{S} = \mathcal{R}(\Gamma \curvearrowright Y)$ with Γ acting freely. Define a subequivalence relation $\mathcal{S}_0 \leq \mathcal{S}$ by $\mathcal{S}_0 = \bigsqcup_{j=0}^{k} (\phi_j \times \phi_j)(\mathcal{T})$.

Then for $x \in Y$,

$$[x]_{\mathcal{S}} = \bigsqcup_{j=0}^{k} [x]_{\mathcal{S}} \cap \phi_j(E_0) = \bigsqcup_{j=0}^{k} \phi_j([x]_{\mathcal{S}} \cap E_0)$$

and as $[x]_{\mathcal{S}} \cap E_0$ is the union of finitely many $\mathcal{T}$-classes, we see that $[x]_{\mathcal{S}}$ is the union of finitely many $\mathcal{S}_0$-classes. Thus, $\mathcal{S}_0 \leq \mathcal{S}$ is an essentially finite index inclusion. Let $\Gamma \curvearrowright (\tilde{X}, \tilde{\mu})$ be the free pmp action arising from this inclusion as in (III.3.3) and (III.3.7). Then by (III.3.5) we have $\mathcal{S}_0 \cong \mathcal{R}(\Gamma \curvearrowright \tilde{X})|_{Y \times \{0\}}$ and so $\mathcal{T} = \mathcal{S}_0|_{E_0} \cong \mathcal{R}(\Gamma \curvearrowright \tilde{X})|_{E_0 \times \{0\}}$ as desired. ∎

Lemma III.3.6. *Let $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ be free pmp actions of countable groups. Suppose there are positive measure subsets $E \subset X$, $F \subset Y$ and essentially finite index subequivalence relations $\mathcal{T} \leq \mathcal{R}(\Gamma \curvearrowright X)|_E$ and $\mathcal{S} \leq \mathcal{R}(\Lambda \curvearrowright Y)|_F$ with $\mathcal{T} \cong \mathcal{S}$.*

Then Γ and Λ admit SOE free pmp actions (and hence are measure equivalent).

Proof. Applying Lemma III.3.5 we find a free pmp action $\Gamma \curvearrowright (\tilde{X}, \tilde{\mu})$ and positive measure subsets $E_0 \subset E$, $\tilde{E}_0 \subset \tilde{X}$ such that $\mathcal{T}|_{E_0} \cong \mathcal{R}(\Gamma \curvearrowright \tilde{X})|_{\tilde{E}_0}$. Let $\Psi : \tilde{E}_0 \to E_0$ denote the measure space isomorphism implementing this identification.

Since $\mathcal{T} \cong \mathcal{S}$, let $\theta : E \to F$ be a measure space isomorphism such that $(\theta \times \theta)(\mathcal{T}) = \mathcal{S}$. Let $F_0 = \theta(E_0)$ and again apply Lemma III.3.5 to find a positive measure subset $F_1 \subset F_0$ and a free pmp action $\Lambda \curvearrowright (\tilde{Y}, \tilde{\nu})$ such that $\mathcal{S}|_{F_1} \cong \mathcal{R}(\Lambda \curvearrowright \tilde{Y})|_{\tilde{F}_1}$ for some measurable $\tilde{F}_1 \subset \tilde{Y}$.

Letting $E_1 = \theta^{-1}(F_1)$ and $\tilde{E}_1 = \Psi^{-1}(E_1)$ we see that

$$\mathcal{R}(\Gamma \curvearrowright \tilde{X})|_{\tilde{E}_1} \cong \mathcal{T}|_{E_1} \cong \mathcal{S}|_{F_1} \cong \mathcal{R}(\Lambda \curvearrowright \tilde{Y})|_{\tilde{F}_1},$$

giving the desired stable orbit equivalence. ∎

III.3.4 Intertwining subequivalence relations

We will need the techniques of [Io11] which give the analogue of Popa's intertwining in the setting of countable pmp equivalence relations. Consider an inclusion of countable pmp equivalence relations $\mathcal{S} \leq \mathcal{R}$ on (X, μ) such that each $\mathcal{R}$-class contains infinitely many $\mathcal{S}$-classes. For a positive measure subset $E \subset X$, the formula (III.3.6) gives rise to a unitary representation $\rho : [\mathcal{R}|_E] \to \mathcal{U}(L^2(E \times \mathbb{Z}_{\geq 0}))$ defined by

$$[\rho(\theta)\xi](x, n) = \xi(\tilde{\theta}^{-1}(x, n)) \quad \text{for} \quad \xi \in L^2(E \times \mathbb{Z}_{\geq 0}).$$

For $\xi \in L^2(E \times \mathbb{Z}_{\geq 0})$, denote $S(\xi) = \{x \in E \mid \sup_i |\xi(x, i)| \neq 0\}$. For further reference, we note that if ξ is $\rho([\mathcal{T}])$-invariant, for some subequivalence relation $\mathcal{T} \leq \mathcal{R}|_E$, then $S(\xi)$ is $\mathcal{T}$-invariant.

Following [IKT08] we define a function $\varphi_{\mathcal{S}} : [[\mathcal{R}]] \to [0, 1]$ by

$$\varphi_{\mathcal{S}}(\theta) = \mu(\{x \in \mathrm{dom}(\theta) \mid (\theta(x), x) \in \mathcal{S}\}).$$

The following result established in [Io11] shows the connection between $\varphi_{\mathcal{S}}$, Popa's intertwining, and intertwining of subequivalence relations:

Lemma III.3.7 ([Io11, Lemmas 1.7 and 1.8])**.** *Let $E \subset X$ be a positive measure subset and $\mathcal{T} \leq \mathcal{R}|_E$ a subequivalence relation. Then the following are equivalent:*

1. *$L(\mathcal{T}) \prec_{L(\mathcal{R})} L(\mathcal{S})$.*

2. *There is no sequence $\{\theta_n\}_{n=1}^{\infty} \subset [\mathcal{T}]$ such that $\varphi_{\mathcal{S}}(\psi\theta_n\psi') \to 0$ for all $\psi, \psi' \in [\mathcal{R}]$.*

3. *There is a non-zero $\rho([\mathcal{T}])$-invariant vector $\eta \in L^2(E \times \mathbb{Z}_{\geq 0})$. Moreover, in this case, there is a subequivalence relation $\mathcal{T}_0 \leq \mathcal{T}$ such that for any positive measure subset $E_0 \subset S(\eta)$ there is a positive measure subset $Y \subset E_0$ and $\theta \in [[\mathcal{R}]]$, $\theta : Y \to Z$, satisfying*

   (a) $\mathcal{T}_0|_Y \leq \mathcal{T}|_Y$ has bounded index, and

   (b) $(\theta \times \theta)(\mathcal{T}_0|_Y) \leq \mathcal{S}|_Z$.

In order to exploit strong intertwining $L(\mathcal{T}) \prec^s_{L(\mathcal{R})} L(\mathcal{S})$, we will use the following lemma:

Lemma III.3.8. *Let $\mathcal{T} \leq \mathcal{R}$ be a subequivalence relation such that for all $\mathcal{T}$-invariant subsets $E \subset X$ of positive measure there is no sequence $\{\theta_n\}_{n=1}^{\infty} \subset [\mathcal{T}|_E]$ such that $\varphi_{\mathcal{S}}(\psi\theta_n\psi') \to 0$ for all $\psi, \psi' \in [\mathcal{R}]$.*

Then there is a non-zero $\rho([\mathcal{T}])$-invariant vector $\eta \in L^2(X \times \mathbb{Z}_{\geq 0})$ such that $\mu(S(\eta)) = 1$.

Proof. Let $\mathcal{F}$ be the set of families $\{\eta_i\}_{i \in I} \subset L^2(X \times \mathbb{Z}_{\geq 0})$ of $\rho([\mathcal{T}])$-invariant vectors which satisfy $S(\eta_i) \cap S(\eta_j) = \varnothing$, for all $i \neq j$, and $\|\eta_i\|_2 = \sqrt{\mu(S(\eta_i))} > 0$, for all i. By Zorn's lemma, we can find a family $\{\eta_i\}_{i \in I} \in \mathcal{F}$ that is maximal with respect to inclusion.

We claim that $\sum_{i \in I} \mu(S(\eta_i)) = 1$. Indeed, otherwise $E = X \setminus (\cup_{i \in I} S(\eta_i))$ would be a $\mathcal{T}$-invariant set of positive measure. By applying Lemma III.3.7 (2) $\Rightarrow$ (3) we find a non-zero $\rho(\mathcal{T})$-invariant vector $\xi \in L^2(X \times \mathbb{Z}_{\geq 0})$ with $S(\xi) \subset E$. But then the family $\{\eta_i\}_{i \in I} \cup \{\frac{\sqrt{\mu(S(\xi))}}{\|\xi\|_2}\xi\}$ also belongs to $\mathcal{F}$, which contradicts the maximality of $\{\eta_i\}_{i\in I}$, and thus proves the claim.

It is now clear $\eta = \sum_{i \in I} \eta_i \in L^2(X \times \mathbb{Z}_{\geq 0})$ is a $\rho([\mathcal{T}])$-invariant unit vector with $\mu(S(\eta)) = 1$. ∎

We can now prove the intertwining lemma to be used in the proof of Proposition III.3.1. A countable pmp equivalence relation $\mathcal{T}$ on (Y, ν) is called *aperiodic* if $[y]_{\mathcal{T}}$ is infinite for almost every $y \in Y$.

Lemma III.3.9. *Let $\mathcal{R}$ be a countable pmp equivalence relation on (X, μ), $Y, Z \subset X$ subsets of positive measure, and $\mathcal{T} \leq \mathcal{R}|_Y$, $\mathcal{S} \leq \mathcal{R}|_Z$ subequivalence relations with $\mathcal{T}$ aperiodic.*

If $L(\mathcal{T}) \prec_{L(\mathcal{R})} L(\mathcal{S})$, then there is a subequivalence relation $\mathcal{T}_0 \leq \mathcal{T}$, subsets of positive measure $Y_1 \subset Y$, $Z_1 \subset Z$, and $\theta \in [[\mathcal{R}]]$, $\theta : Y_1 \to Z_1$, such that

1. *$\mathcal{T}_0|_{Y_1} \leq \mathcal{T}|_{Y_1}$ has bounded index, and*
2. *$(\theta \times \theta)(\mathcal{T}_0|_{Y_1}) \leq \mathcal{S}|_{Z_1}$.*

If we assume moreover that $L(\mathcal{T}) \prec^s_{L(\mathcal{R})} L(\mathcal{S})$, then for any positive measure $Y_0 \subset Y$, the subset Y_1 above can be taken with $Y_1 \subset Y_0$.

Proof. Let $\mathcal{S}' = \mathcal{S} \sqcup \{(x,x) \mid x \in X \setminus Z\}$ and note that $L(\mathcal{T}) \prec_{L(\mathcal{R})} L(\mathcal{S})$ implies $L(\mathcal{T}) \prec_{L(\mathcal{R})} L(\mathcal{S}')$. Then by Lemma III.3.7, we can find $\mathcal{T}_0 \leq \mathcal{T}$, positive measure subsets $Y_1 \subset Y$, $Z_1 \subset X$ and $\theta \in [[\mathcal{R}]]$, $\theta : Y_1 \to Z_1$, such that conclusions (1) and (2) hold. Since $\mathcal{T}$ is aperiodic, conclusion (1) forces the $\mathcal{T}_0|_{Y_1}$-class of almost every $x \in Y_1$ to be infinite, and so conclusion (2) forces $\mu(Z_1 \cap Z) = \mu(Z_1)$, and so we may indeed take $Z_1 \subset Z$.

The moreover conclusion follows because Lemma III.3.8 allows us to apply the moreover assertion of Lemma III.3.7 with E_0 a positive measure subset of Y_0. ∎

III.3.5 Proof of Proposition III.3.1

Let $\mathcal{T} = \mathcal{R}(\Gamma_1 \curvearrowright Y)$ and $\mathcal{S} = \mathcal{R}(\Lambda \curvearrowright Z)$. By assumption *(i)* and Lemma III.3.9, there is a subequivalence relation $\mathcal{T}_0 \leq \mathcal{T}$, positive measure subsets $Y_1 \subset Y$, $Z_1 \subset Z$, and $\theta \in [[\mathcal{R}]]$, $\theta : Y_1 \to Z_1$, such that $\mathcal{T}_0|_{Y_1} \leq \mathcal{T}|_{Y_1}$ has bounded index and $(\theta \times \theta)(\mathcal{T}_0|_{Y_1}) \leq \mathcal{S}|_{Z_1}$.

Similarly, by assumption *(ii)* and Lemma III.3.9, there is a subequivalence relation $\mathcal{S}_0 \leq \mathcal{S}$, positive measure subsets $Z_2 \subset Z$, $Y_2 \subset Y$, and $\phi \in [[\mathcal{R}]]$, $\phi : Z_2 \to Y_2$, such that $\mathcal{S}_0|_{Z_2} \leq \mathcal{S}|_{Z_2}$ has bounded index and $(\phi \times \phi)(\mathcal{S}_0|_{Z_2}) \leq \mathcal{T}|_{Y_2}$. Moreover, by Lemma III.3.9 we can take $Z_2 \subset Z_1$, since *(ii)* assumes strong intertwining.

Define $Y_2' \subset Y_1$ by $Y_2' = \theta^{-1}(Z_2)$ and let $\psi = \phi \circ \theta : Y_2' \to Y_2$. Since $\mathcal{S}_0|_{Z_2} \leq \mathcal{S}|_{Z_2}$ has bounded index and $(\theta \times \theta)(\mathcal{T}_0|_{Y_2'}) \leq \mathcal{S}|_{Z_2}$, Lemma III.3.2 gives that

$$\mathcal{S}_0|_{Z_2} \cap (\theta \times \theta)(\mathcal{T}_0|_{Y_2'}) \leq (\theta \times \theta)(\mathcal{T}_0|_{Y_2'})$$

has bounded index. Letting

$$\mathcal{T}_{00} = (\theta^{-1} \times \theta^{-1})(\mathcal{S}_0|_{Z_2} \cap (\theta \times \theta)(\mathcal{T}_0|_{Y_2'})) = (\theta^{-1} \times \theta^{-1})(\mathcal{S}_0|_{Z_2}) \cap \mathcal{T}_0|_{Y_2'} \tag{III.3.8}$$

we see that $\mathcal{T}_{00} \leq \mathcal{T}_0|_{Y_2'}$ has bounded index and therefore so to does $\mathcal{T}_{00} \leq \mathcal{T}|_{Y_2'}$. Moreover, $(\theta \times \theta)(\mathcal{T}_{00}) \leq \mathcal{S}_0|_{Z_2}$ and so

$$(\psi \times \psi)(\mathcal{T}_{00}) \leq \mathcal{T}|_{Y_2}. \tag{III.3.9}$$

As $Y_2', Y_2 \subset Y$, we may regard ψ in $[[\mathcal{R}|_Y]] = [[\mathcal{R}(\Gamma_1 \times \Gamma_2 \curvearrowright Y)]]$ and apply Lemma III.3.3 to find positive measure $Y_3' \subset Y_2'$ such that for $Y_3 = \psi(Y_3')$, the inclusion (III.3.9) has bounded index when restricted to Y_3, i.e. $(\psi \times \psi)(\mathcal{T}_{00}|_{Y_3'}) \leq \mathcal{T}|_{Y_3}$ has bounded index. Let $Z_3 = \theta(Y_3')$.

Then because

$$(\psi \times \psi)(\mathcal{T}_{00}|_{Y_3'}) \leq (\phi \times \phi)(\mathcal{S}_0|_{Z_3}) \leq \mathcal{T}|_{Y_3},$$

we conclude that $(\phi \times \phi)(\mathcal{S}_0|_{Z_3}) \leq \mathcal{T}|_{Y_3}$ has bounded index.

Thus, $\mathcal{S}_0|_{Z_3}$ is a subequivalence relation of $\mathcal{S}|_{Z_3} = \mathcal{R}(\Lambda \curvearrowright Z)|_{Z_3}$ with bounded index whose isomorphic image $(\phi \times \phi)(\mathcal{S}_0|_{Z_3})$ has bounded index in $\mathcal{T}|_{Y_3} = \mathcal{R}(\Gamma_1 \curvearrowright Y)|_{Y_3}$. An application of Lemma III.3.6 finishes the proof. ■

III.4 Transfer of commutation from subalgebras to subgroups

In this section we prove the following result which will be crucial in the proof of Theorem F. This result is an immediate consequence of the "ultrapower technique" developed in [Io11], being essentially contained in the proof of [Io11, Theorem 3.1] (see also [CdSS15, Theorem 3.3] and [KV15, Lemma 5.6]). Nevertheless, for completeness, we include a detailed proof.

Theorem III.4.1 ([Io11])**.** *Let M be a II_1 factor and $p \in M$ a projection such that $pMp = A \rtimes \Gamma$, where $\Gamma \curvearrowright A$ is a trace preserving action on a tracial von Neumann algebra. Let $\Delta : M \to M \overline{\otimes} L(\Gamma)$ be a $*$-homomorphism which satisfies $\Delta(a) = a \otimes 1$ and $\Delta(u_g) = u_g \otimes u_g$, for all $a \in A$ and $g \in \Gamma$. Assume that $P \subset L(\Gamma)$ and $Q \subset qMq$ are von Neumann subalgebras such that $\Delta(Q) \prec_{M \overline{\otimes} L(\Gamma)} M \overline{\otimes} P$.*

Then there exists a decreasing sequence of subgroups $\Omega_k < \Gamma$ such that

1. *$Q \prec_M A \rtimes \Omega_k$, for all $k \geq 1$, and*

2. *$P' \cap L(\Gamma) \prec_{L(\Gamma)} L(\cup_{k \geq 1} C_\Gamma(\Omega_k))$.*

Throughout this section, we assume the setting of Theorem III.4.1. Since M is a II_1 factor, after replacing Q with a unitary conjugate of one its corners, we may clearly assume that $q \leq p$.

In preparation for the proof of Theorem III.4.1, let us introduce some notation. We denote by $\mathcal{G}$ the family of all subgroups $\Sigma < \Gamma$ such that $Q \nprec_M A \rtimes \Sigma$. We may assume that $\mathcal{G}$ is non-empty. Indeed, if $\mathcal{G} = \emptyset$, then $Q \prec_M A$, and thus the conclusion holds with $\Omega_k = \{e\}$, for every $k \geq 1$.

We say that a set $S \subset \Gamma$ is *small relative to* $\mathcal{G}$ if $S \subset \cup_{i=1}^m b_i \Sigma_i c_i$, for some $b_i, c_i \in \Gamma$ and $\Sigma_i \in \mathcal{G}$. We denote by I the family of subsets of Γ that are small relative to $\mathcal{G}$. We order I by inclusion and consider a cofinal ultrafilter $\mathcal{V}$ on I. Thus, $\{S' \in I | S' \supset S\}$ belongs to $\mathcal{V}$, for every $S \in I$.

Lemma III.4.2. *We can find a finite set $F \subset L(\Gamma)$ and $\delta > 0$ such that the following holds: for any $S \subset \Gamma$ which is small relative to $\mathcal{G}$, there exists $g \in \Gamma \setminus S$ such that $\sum_{\alpha,\beta \in F} \|E_P(\alpha u_g \beta)\|_2^2 \geq \delta$.*

Proof. The proof uses the "transfer of rigidity" principle from [PV09] (see the proof of [PV09, Lemma 3.2]). Since $\Delta(Q) \prec_{M \overline{\otimes} L(\Gamma)} M \overline{\otimes} P$, we can find $F \subset (L(\Gamma))_1$ finite and $\kappa > 0$ such that

$$\sum_{\alpha,\beta \in F} \|E_{M \overline{\otimes} P}((1 \otimes \alpha)\Delta(u)(1 \otimes \beta))\|_2^2 \geq \kappa, \quad \text{for every } u \in \mathcal{U}(Q). \tag{III.4.1}$$

Put $\delta = \frac{\kappa}{2\|q\|_2^2}$. Let $S \subset \Gamma$ be small relative to $\mathcal{G}$. Thus, $S \subset \cup_{i=1}^m b_i \Sigma_i c_i$, for some $b_i, c_i \in \Gamma$ and $\Sigma_i \in \mathcal{G}$. For $g \in \Gamma$, we denote $\varphi(g) = \sum_{\alpha,\beta \in F} \|E_P(\alpha u_g \beta)\|_2^2$. Since $F \subset (L(\Gamma))_1$, we have that $\varphi(g) \leq |F|^2$, for every $g \in \Gamma$. Our goal is to show the existence of $g \in \Gamma \setminus S$ such that $\varphi(g) \geq \delta$.

Since $Q \nprec_M A \rtimes \Sigma_i$, for every $i \in \{1, ..., m\}$, by Remark III.2.2 we can find $u \in \mathcal{U}(Q)$

such that

$$\|E_{A\rtimes\Sigma_i}(u_{b_i}^* u u_{c_i}^*)\|_2^2 \le \frac{\kappa}{2m|F|^2}, \quad \text{for every } 1 \le i \le m. \tag{III.4.2}$$

Since $u \in Q \subset qMq \subset q(A \rtimes \Gamma)q$, we can write $u = \sum_{g\in\Gamma} a_g u_g$, where $a_g \in A$. By using I we get that

$$\sum_{g\in S} \varphi(g)\|a_g\|_2^2 \le |F|^2 \sum_{g\in S} \|a_g\|_2^2 \le |F|^2 \sum_{i=1}^{m} \|E_{A\rtimes\Sigma_i}(u_{b_i}^* u u_{c_i}^*)\|_2^2 \le \frac{\kappa}{2}. \tag{III.4.3}$$

On the other hand, since $\Delta(u) = \sum_{g\in\Gamma} a_g u_g \otimes u_g$, equation H rewrites as $\sum_{g\in\Gamma} \varphi(g)\|a_g\|_2^2 \ge \kappa$. In combination with J this gives that $\sum_{g\in\Gamma\smallsetminus S} \varphi(g)\|a_g\|_2^2 \ge \frac{\kappa}{2}$. Since $\sum_{g\in\Gamma\smallsetminus S} \|a_g\|_2^2 \le \|u\|_2^2 = \|q\|_2^2$, it follows that we can find $g \in \Gamma \smallsetminus S$ such that $\varphi(g) \ge \delta$, as claimed. ■

Proof of Theorem III.4.1. Denote $N = L(\Gamma)$. By Lemma III.4.2, for every $S \in I$ we can find $g_S \in \Gamma \smallsetminus S$ such that $\sum_{\alpha,\beta\in F} \|E_P(\alpha u_{g_S}\beta)\|_2^2 \ge \delta$. We put $g = (g_S)_{S\in I} \in \Gamma^{\mathcal{V}}$ and consider the canonical inclusions $\Gamma \subset \Gamma^{\mathcal{V}} \subset \mathcal{U}(N^{\mathcal{V}})$. We define $\Sigma = \Gamma \cap g\Gamma g^{-1}$ and claim that $P' \cap N \prec_N L(\Sigma)$.

Assume by contradiction that this is false. By Theorem I.2.6, we can find a sequence $u_n \in \mathcal{U}(P' \cap N)$ such that $\|E_{L(\Sigma)}(x u_n y)\|_2 \to 0$, for every $x, y \in N$. We denote by $\mathcal{K} \subset L^2(N^{\mathcal{V}})$ the closed linear span of $N u_g N$, and by e the orthogonal projection from $L^2(N^{\mathcal{V}})$ onto $\mathcal{K}$.

Let us show that $\langle u_n \xi u_n^*, \eta\rangle \to 0$, for every $\xi, \eta \in \mathcal{K}$. To prove this, it suffices to show that $\langle u_n x u_g y u_n^*, x' u_g y'\rangle \to 0$, for every $x, y \in N$. But this is clear since $\|E_{L(\Sigma)}(x'^* u_n x)\|_2 \to 0$ and

$$\begin{aligned}\langle u_n x u_g y u_n^*, x' u_g y'\rangle &= \tau(u_g^*(x'^* u_n x) u_g (y u_n^* y'^*)) = \tau(E_N(u_g^*(x'^* u_n x) u_g) y u_n^* y'^*) \\ &= \tau(E_N(u_g^* E_{L(\Sigma)}(x'^* u_n x) u_g) y u_n^* y'^*).\end{aligned}$$

Next, since $\sum_{\alpha,\beta\in F} \|E_{P^{\mathcal{V}}}(\alpha u_g \beta)\|_2^2 = \lim_{S\to\mathcal{V}} (\sum_{\alpha,\beta\in F} \|E_P(\alpha u_{g_S}\beta)\|_2^2) \ge \delta$, we can find $\alpha, \beta \in F$ such that $E_{P^{\mathcal{V}}}(\alpha u_g \beta) \ne 0$. Thus, $\|E_{P^{\mathcal{V}}}(\alpha u_g \beta) - \alpha u_g \beta\|_2 < \|\alpha u_g \beta\|_2$. Since $\alpha u_g \beta \in \mathcal{K}$, we get that $\|e(E_{P^\vee}(\alpha u_g\beta)) - \alpha u_g\beta\|_2 < \|\alpha u_g\beta\|_2$. This implies that $\xi = e(E_{P^\vee}(\alpha u_g\beta)) \in \mathcal{K}$ is non-zero. On the other hand, as e is N-N-bimodular and $u_n \in P' \cap N$, we get that $u_n\xi u_n^* = e(u_n E_{P^\vee}(\alpha u_g\beta)u_n^*) = \xi$ and therefore $\langle u_n\xi u_n^*, \xi\rangle = \|\xi\|_2^2 > 0$, for every n. This contradicts the previous paragraph and thus proves the claim.

Now, enumerate $\Sigma = \{\sigma_j\}_{j\geq 1}$. If $\sigma \in \Gamma$, then σ belongs to Σ if and only if σ commutes with $\{g_S g_{S'}^{-1} | S, S' \in T\}$, for some $T \in \mathcal{V}$. In particular, for every $j \geq 1$, we can find $T_j \in \mathcal{V}$ such that σ_j commutes with $\{g_S g_{S'}^{-1} | S, S' \in T_j\}$. For $k \geq 1$, define $W_k = \cap_{j=1}^k T_j$ and $\Omega_k = \langle g_S g_{S'}^{-1} | S, S' \in W_k\rangle$. Then $W_k \in \mathcal{V}$ and $\Omega_k \supset \Omega_{k+1}$. Since $\sigma_1, ..., \sigma_k \in C_\Gamma(\Omega_k)$, we deduce that $\Sigma = \cup_{k\geq 1} C_\Gamma(\Omega_k)$.

To finish the proof, it suffices to show that if $W \in \mathcal{V}$, then $\Omega = \langle g_S g_{S'}^{-1} | S, S' \in W\rangle$ does not belong to $\mathcal{G}$. Indeed, this implies that $\Omega_k \notin \mathcal{G}$ and hence that $Q \prec_M A \rtimes \Omega_k$, for every $k \geq 1$. Assume by contradiction that $\Omega \in \mathcal{G}$. Fix $S' \in W$. Then $g_S \in \Omega g_{S'}$, for every $S \in W$. Since $\Omega \in \mathcal{G}$, the set $\Omega g_{S'} \subset \Gamma$ is small relative to $\mathcal{G}$. Since $\mathcal{V}$ is cofinal and $W \in \mathcal{V}$, we get that $W \cap \{S'' \in I | S'' \supset \Omega g_{S'}\}$ belongs to $\mathcal{V}$, and hence is non-empty. Let $S'' \in W$ such that $S'' \supset \Omega g_{S'}$. But then we get that $g_S \in S''$, for every $S \in W$. Taking $S = S''$, this contradicts the fact that $g_{S''} \in \Gamma \smallsetminus S''$. ■

III.5 Groups measure equivalent to products of hyperbolic groups and tensor decompositions

The proof of Theorem F is divided between this and the next section. Before stating the main result of this section, we need to introduce some notation.

Notation III.5.1. Let Γ be an icc group which is measure equivalent to a product $\Lambda = \Lambda_1 \times ... \times \Lambda_n$ of $n \geq 1$ groups belonging to $\mathcal{C}_{\text{rss}}$. By [Fu98, Lemma 3.2], Γ and Λ admit stably orbit equivalent free ergodic pmp actions. We may thus find a free ergodic pmp action $\Lambda \curvearrowright (Y,\nu)$ and $\ell \geq 1$, such that the following holds: consider the product action $\Lambda \times \mathbb{Z}/\ell\mathbb{Z} \curvearrowright (Y \times \mathbb{Z}/\ell\mathbb{Z}, \nu \times c)$, where $\mathbb{Z}/\ell\mathbb{Z}$ acts on itself by addition and c denotes the counting measure on $\mathbb{Z}/\ell\mathbb{Z}$. Then there is a non-negligible measurable set $X \subset Y \times \mathbb{Z}/\ell\mathbb{Z}$ and a free ergodic measure preserving action $\Gamma \curvearrowright X$ such that

$$\mathcal{R}(\Gamma \curvearrowright X) = \mathcal{R}(\Lambda \times \mathbb{Z}/\ell\mathbb{Z} \curvearrowright Y \times \mathbb{Z}/\ell\mathbb{Z})|_X.$$

We put $A = L^\infty(X), M = L^\infty(Y \times \mathbb{Z}/\ell\mathbb{Z}) \rtimes (\Lambda \times \mathbb{Z}/\ell\mathbb{Z})$, $p = 1_X$, and note that $A \rtimes \Gamma = pMp$. We identify $L^\infty(\mathbb{Z}/\ell\mathbb{Z}) \rtimes \mathbb{Z}/\ell\mathbb{Z} = \mathbb{M}_\ell(\mathbb{C})$, and use this identification to write $M = B \rtimes \Lambda$, where $B = L^\infty(Y) \otimes \mathbb{M}_\ell(\mathbb{C})$ and Λ acts trivially on $\mathbb{M}_\ell(\mathbb{C})$. We let $\{u_g\}_{g\in\Gamma} \subset A \rtimes \Gamma$ and $\{v_h\}_{h\in\Lambda} \subset B \rtimes \Lambda$ denote the canonical unitaries implementing the actions of Γ and Λ on A and B, respectively.

For a set $T \subset \{1,2,...,n\}$, we denote $\Lambda_T = \times_{j\in T} \Lambda_j$ and let $\widehat{T} = \{1,2,...,n\} \setminus T$.

We define a $*$-homomorphism $\Delta : M \to M \overline{\otimes} L(\Gamma)$ as follows [PV09]. Let $k \geq \tau(p)^{-1}$ be an integer, where τ denotes the trace of M. Let $\widetilde{\Delta} : \mathbb{M}_k(pMp) \to \mathbb{M}_k(pMp) \overline{\otimes} L(\Gamma)$ be the $*$-homomorphism given by $\widetilde{\Delta}(a) = a \otimes 1$ and $\widetilde{\Delta}(u_g) = u_g \otimes u_g$, for all $a \in \mathbb{M}_k(A)$ and $g \in \Gamma$. Let $q \in \mathbb{M}_k(A)$ be a projection satisfying $(\mathrm{Tr} \otimes \tau)(q) = 1$ and $e_{11} \otimes p \leq q$, where Tr is the non-normalized trace of $\mathbb{M}_k$. We fix an identification $\alpha : M \to q\mathbb{M}_k(pMp)q$ which satisfies $\alpha(x) = e_{11} \otimes x$, for all $x \in pMp$. Since $\widetilde{\Delta}(q) = q \otimes 1$, we have that $\widetilde{\Delta}(q\mathbb{M}_k(pMp)q) \subset q\mathbb{M}_k(pMp)q \overline{\otimes} L(\Gamma)$.

Finally, we put $\Delta = (\alpha^{-1} \otimes \mathrm{id}) \circ \widetilde{\Delta} \circ \alpha : M \to M \overline{\otimes} L(\Gamma)$. Then one checks that

$$\Delta(a) = a \otimes 1 \text{ and } \Delta(u_g) = u_g \otimes u_g, \text{ for every } a \in A \text{ and } g \in \Gamma.$$

For further reference, we also record two facts. Firstly, if Γ is icc, then $\Delta(M)' \cap M \overline{\otimes} L(\Gamma) = \mathbb{C}$. Indeed, if Γ is icc, it is easy to see that $\widetilde{\Delta}(\mathbb{M}_k(pMp))' \cap \mathbb{M}_k(pMp) \overline{\otimes} L(\Gamma) = \mathbb{C}$, which gives the fact. The second fact goes back to [IPV10, Proposition 7.2.4]. In the more general context needed below, it is due to [KV15, Proposition 2.4]).

Lemma III.5.2 ([KV15]). *If $N \subset M$ has no amenable direct summand, then $\Delta(N)p'$ is non-amenable relative to $M\overline{\otimes}\mathbb{C}$ inside $M\overline{\otimes}L(\Gamma)$ for any non-zero projection $p' \in \Delta(N)' \cap M\overline{\otimes}L(\Gamma)$.*

The following is the main result of this section:

Theorem III.5.3. *Assume that $L(\Gamma) = P_1\overline{\otimes}P_2$, where $P_1, P_2 \subset L(\Gamma)$ are II_1 factors.*

Then there are subgroups $\Sigma_1, \Sigma_2 < \Gamma$ and a partition $S_1 \sqcup S_2 = \{1, ..., n\}$ such that for all $i \in \{1,2\}$,

1. *$P_i \prec^s_{L(\Gamma)} L(\Sigma_i)$, $L(\Sigma_i) \prec^s_{L(\Gamma)} P_i$,*
2. *$A \rtimes \Sigma_i \prec^s_M B \rtimes \Lambda_{S_i}$, $B \rtimes \Lambda_{S_i} \prec^s_M A \rtimes \Sigma_i$, and*
3. *Σ_i is measure equivalent to Λ_{S_i}.*

The rest of the section is devoted to the proof of Theorem III.5.3. We assume throughout the notation from III.5.1 and that $L(\Gamma) = P_1\overline{\otimes}P_2$.

III.5.1 Outline of proof of Theorem III.5.3

The proof of Theorem III.5.3 is divided between five steps, which we now briefly outline in order to facilitate reading.

Step 1. *There is a partition $T_1 \sqcup T_2 = \{1, ..., n\}$ such that $P_i \prec^s_M B \rtimes \Lambda_{T_i}$, for all $i \in \{1,2\}$.* This conclusion will be obtained in Proposition III.5.5 by using that $\Lambda_j \in \mathcal{C}_{\text{rss}}$, for all $1 \leq j \leq n$.

Step 2. *There is a partition $S_1 \sqcup S_2 = \{1, ..., n\}$ such that $\Delta(B \rtimes \Lambda_{S_i}) \prec_{M\overline{\otimes}L(\Gamma)} M\overline{\otimes}P_i$, for all $i \in \{1,2\}$.* This conclusion will be obtained in Proposition III.5.7 by using that

$\Lambda_j \in \mathcal{C}_{\text{rss}}$, for all j, and the embeddings $\varphi_i : P_i \to \mathbb{M}_{m_i}(B \rtimes \Lambda_{T_i})$ (for some $m_i \geq 1$) provided by **Step 1**.

Step 3. *There is a decreasing sequence of subgroups $\Omega_k < \Gamma$ such that $B \rtimes \Lambda_{S_1} \prec_M A \rtimes \Omega_k$, for all $k \geq 1$, and $P_2 \prec_{L(\Gamma)} L(\cup_{k\geq 1} C_\Gamma(\Omega_k))$.* This is an immediate consequence of **Step 2** and Theorem III.4.1; see Lemma III.5.8.

Step 4. *There is a subgroup $\Sigma_1 < \Gamma$ such that $B \rtimes \Lambda_{S_1} \prec^s_M A \rtimes \Sigma_1$, $A \rtimes \Sigma_1 \prec^s_M B \rtimes \Lambda_{S_1}$, $P_1 \prec^s_{L(\Gamma)} L(\Sigma_1)$, and $L(\Sigma_1) \prec^s_{L(\Gamma)} P_1$.* Specifically, Lemma III.5.10 will show that $\Sigma_1 = \Omega_k$ works, for k large. A key part is showing that $L(\Omega_k) \prec_{L(\Gamma)} P_1$, for large k; see Lemma III.5.9. This uses again that $\Lambda_j \in \mathcal{C}_{\text{rss}}$ for all j and the embeddings $\varphi_i : P_i \to \mathbb{M}_{m_i}(B \rtimes \Lambda_{T_i})$ for $i \in \{1, 2\}$. *Similarly, there is a subgroup $\Sigma_2 < \Gamma$ with analogous properties.*

Step 5. *Σ_i is measure equivalent to Λ_{S_i}, for every $i \in \{1, 2\}$.* This will follow readily by combining the result of **Step 4** with Proposition III.3.1.

Remark III.5.4. Since **Steps 1-3** suffice in order to deduce Corollary E, we include its proof right after **Step 3**.

III.5.2 Step 1

Proposition III.5.5. *There is a partition $T_1 \sqcup T_2 = \{1, ..., n\}$ such that $P_i \prec^s_M B \rtimes \Lambda_{T_i}$, for all $i \in \{1, 2\}$. Moreover, if P_i is amenable relative to $B \rtimes \Lambda_T$, for some $i \in \{1, 2\}$ and $T \subset \{1, .., n\}$, then $T \supset T_i$.*

Proof. For $t \in \{1, ..., n\}$, denote by $\hat{t}$ the set $\{1, ..., n\} \setminus \{t\}$. For $i \in \{1, 2\}$, let $T_i \subset \{1, ..., n\}$ be a minimal set with respect to inclusion such that P_i is amenable relative to $B \rtimes \Lambda_{T_i}$.

We claim that $P_2 \prec^s_M B \rtimes \Lambda_{\{1,...,n\}\setminus T_1}$. This is immediate if $T_1 = \varnothing$.[4] Otherwise consider

[4] In fact, since each P_i is type II_1 and B is type I, after the proposition is proved, the conclusion that $P_i \prec^s_M B \rtimes \Lambda_{T_i}$ for all $i \in \{1, 2\}$ will imply that T_1 and T_2 are nonempty.

any $t \in T_1$. Since $\Lambda_t \in \mathcal{C}_{rss}$, Lemma III.2.9 implies that P_1 is amenable relative to $B \rtimes \Lambda_{\hat{t}}$ or $P_2 \prec_M B \rtimes \Lambda_{\hat{t}}$. Using Lemma III.2.7(1) and the minimality of T_1, it follows that $P_2 \prec_M B \rtimes \Lambda_{\hat{t}}$. Since Γ is icc and the action $\Gamma \curvearrowright X$ is ergodic, we have that $(\mathcal{N}_{pMp}(P_2))' \cap pMp \subset L(\Gamma)' \cap pMp = \mathbb{C}p$. Lemma III.2.3(3) implies that $P_2 \prec^s_M B \rtimes \Lambda_{\hat{t}}$. Since this holds for all $t \in T_1$, Lemma III.2.7(2) implies that $P_2 \prec^s_M B \rtimes \Lambda_{\{1,\dots,n\} \setminus T_1}$ as claimed. Using the minimality of T_2 and Lemma III.2.5(3), we get that $T_1 \cap T_2 = \emptyset$. In a similar way we obtain that $P_1 \prec^s_M B \rtimes \Lambda_{\{1,\dots,n\} \setminus T_2}$.

The remaining part of the proof is to prove that $T_1 \cup T_2 = \{1, \dots, n\}$. We claim that $L(\Gamma)$ is not amenable relative to $B \rtimes \Lambda_T$ inside M, for any proper set $T \subsetneq \{1, \dots, n\}$. Otherwise, Lemma III.2.8 would imply that $\Lambda_T < \Lambda$ is co-amenable, for some $T \subsetneq \{1, \dots, n\}$. This would further give that $\Lambda_{\{1,\dots,n\} \setminus T}$ is amenable, which contradicts the fact that Λ_j is non-amenable, for every $1 \leq j \leq n$.

Next, fixing any $i \in \{1, 2\}$, we claim that $P_i \prec^s_M B \rtimes \Lambda_{T_i}$. This is immediate if $T_i = \{1, \dots, n\}$; otherwise consider any $t \notin T_i$. Then P_i is amenable relative to $B \rtimes \Lambda_{\hat{t}}$ and since $\Lambda_t \in \mathcal{C}_{rss}$, we must have either $P_i \prec_M B \rtimes \Lambda_{\hat{t}}$ or $\mathcal{N}_{pMp}(P_i)''$ amenable relative to $B \rtimes \Lambda_{\hat{t}}$. Since $L(\Gamma) \subset \mathcal{N}_{pMp}(P_i)''$, the previous paragraph implies that $P_i \prec_M B \rtimes \Lambda_{\hat{t}}$, for all $t \notin T_i$. As above, we get that $P_i \prec^s_M B \rtimes \Lambda_{\hat{t}}$, for all $t \notin T_i$. Lemma III.2.7(2) implies now that $P_i \prec^s_M B \rtimes \Lambda_{T_i}$, as claimed.

Thus, in particular $P_i \prec^s_M B \rtimes \Lambda_{T_1 \cup T_2}$, for all $i \in \{1, 2\}$. Applying [BV12, Lemma 2.3] implies that $L(\Gamma) \prec_M B \rtimes \Lambda_{T_1 \cup T_2}$. As above, Lemma III.2.3(3) implies that $L(\Gamma) \prec^s_M B \rtimes \Lambda_{T_1 \cup T_2}$. Applying [BV12, Lemma 2.3] once again gives that $A \rtimes \Gamma \prec_M B \rtimes \Lambda_{T_1 \cup T_2}$. If there exists $t \in \{1, \dots, n\} \setminus (T_1 \cup T_2)$, then we would get that $L(\Lambda_t) \prec_M B \rtimes \Lambda_{\hat{t}}$, which contradicts that Λ_t is infinite. Thus, $T_1 \cup T_2 = \{1, \dots, n\}$. The moreover assertion follows from the minimality of T_1 and T_2 using again Lemma III.2.7(1). ∎

III.5.3 Step 2

Towards the second step of the proof of Theorem III.5.3, we now prove that the each intertwining $P_i \prec^s_M B \rtimes \Lambda_{T_i}$ from Proposition III.5.5 allows us to deduce that P_i itself has a weaker form of relative solidity present in $B \rtimes \Lambda_{T_i}$. More precisely:

Lemma III.5.6. *Let $P = P_i$ and $k = |T_i|$ for some $i \in \{1,2\}$. Then for any tracial von Neumann algebra M_0, any projection $q \in \tilde{M} = M_0 \overline{\otimes} P$, and any commuting subalgebras $Q_0, \dots, Q_k \subset q\tilde{M}q$ we have either*

1. *$Q_0 \prec^s_{\tilde{M}} M_0$, or*
2. *$Q_j q'$ is amenable relative to M_0 inside $\tilde{M}$, for some $j \in \{1, \dots, k\}$ and some non-zero projection $q' \in Q_j' \cap q\tilde{M}q$.*

Proof. Assume that $Q_j q'$ is not amenable relative to M_0 inside $\tilde{M}$ for any $j \in \{1, \dots, k\}$ and non-zero projection $q' \in Q_j' \cap q\tilde{M}q$. We first note that in order to prove the lemma, it suffices to show the conclusion $Q_0 \prec_{\tilde{M}} M_0$. Indeed, if this is known, then for any $z \in \mathcal{N}_{q\tilde{M}q}(Q_0)' \cap q\tilde{M}q \subset (\bigcup_{j=0}^k Q_j)' \cap q\tilde{M}q$, applying the result to the commuting subalgebras $\{Q_j z\}_{j=0}^k \subset z\tilde{M}z$ (noting that $Q_j q'$ is not amenable relative to M_0, for all $j \in \{1, \dots, k\}$ and any non-zero projection $q' \in (Q_j z)' \cap z\tilde{M}z$), we conclude that $Q_0 z \prec_{\tilde{M}} M_0$ and so by Lemma III.2.3(2), $Q_0 \prec^s_{\tilde{M}} M_0$ as desired.

For an integer $m \geq 1$, let $e_{11} \in \mathbb{M}_m(\mathbb{C})$ denote the matrix unit corresponding the $(1,1)$ entry and view M as a non-unital subalgebra of $\mathbb{M}_m(M)$ via the embedding $x \mapsto x \otimes e_{11}$. By Proposition III.5.5 we have that $P \prec^s_M B \rtimes \Lambda_T$ for some $T \subset \{1, \dots, n\}$ with $|T| = k$. Hence we have for some $m \geq 1$ a not necessarily unital $*$-homomorphism $\varphi : P \to \mathbb{M}_m(B \rtimes \Lambda_T)$ and a non-zero partial isometry $v \in \mathbb{M}_{m,1}(M)p$ such that $\varphi(x)v = vx$, for every $x \in P$. We define $e = \varphi(p)$, $\mathcal{B} = \mathbb{M}_m(B)$, and $\mathcal{M} = \mathbb{M}_m(B \rtimes \Lambda_T) \subset \mathbb{M}_m(M)$ and write canonically $\mathcal{M} = \mathcal{B} \rtimes \Lambda_T$. Moreover, we may assume that $E_{\mathcal{M}}(vv^*) \geq ce$, for some $c > 0$.

Replacing φ by $\mathrm{id}\otimes\varphi$ we extend to $\varphi: M_0\overline{\otimes}P \to M_0\overline{\otimes}\mathbb{M}_m(M)$. Note that $\varphi(M_0\overline{\otimes}P) \subset M_0\overline{\otimes}\mathcal{M}$ and that $\varphi(x)v = vx$, for every $x \in M_0\overline{\otimes}P$. Let $f = \varphi(q)$ and $Q = (\bigcup_{j=0}^k \varphi(Q_j))'' \subset f(M_0\overline{\otimes}\mathcal{M})f$.

Claim 1. To prove that $Q_0 \prec_{\tilde{M}} M_0$, it is enough to show that $\varphi(Q_0) \prec^s_{M_0\overline{\otimes}\mathcal{M}} M_0\overline{\otimes}\mathcal{B}$.

Proof of Claim 1. Assume by contradiction that $\varphi(Q_0) \prec^s_{M_0\overline{\otimes}\mathcal{M}} M_0\overline{\otimes}\mathcal{B}$ and $Q_0 \nprec_{\tilde{M}} M_0$. Since $Q_0 \prec_{M_0\overline{\otimes}\mathbb{M}_m(M)} \varphi(Q_0)$, Lemma III.2.3(1) implies that $Q_0 \prec_{M_0\overline{\otimes}\mathbb{M}_m(M)} M_0\overline{\otimes}\mathcal{B}$. From this we get that $Q_0 \prec_{M_0\overline{\otimes}pMp} M_0\overline{\otimes}A$. On the other hand, since $Q_0 \nprec_{\tilde{M}} M_0$, by Theorem I.2.6 we can find a sequence $u_n \in \mathcal{U}(Q_0)$ satisfying $\|E_{M_0}(xu_ny)\|_2 \to 0$, for all $x, y \in \tilde{M}$. Let us show that $\|E_{M_0\overline{\otimes}A}(xu_ny)\|_2 \to 0$, for all $x, y \in M_0\overline{\otimes}pMp$. This assertion will give a contradiction, and thus prove the claim.

To prove the assertion, recalling that $pMp = A \rtimes \Gamma$, it suffices to treat the case $x = 1$ and $y \in L(\Gamma)$. But then since $u_n \in Q_0$ and $Q_0 \subset \tilde{M} \subset M_0\overline{\otimes}L(\Gamma)$ we get that $u_ny \in M_0\overline{\otimes}L(\Gamma)$ and thus $E_{M_0\overline{\otimes}A}(u_ny) = E_{M_0}(u_ny) = E_{M_0}(u_nE_{\tilde{M}}(y))$. As $\|E_{M_0}(u_nE_{\tilde{M}}(y))\|_2 \to 0$, the claim is proven. ◻

Claim 2. $\varphi(Q_j)q'$ is not amenable relative to $M_0\overline{\otimes}\mathcal{B}$ inside $M_0\overline{\otimes}\mathcal{M}$ for any $j \in \{1,\dots,k\}$ and any non-zero projection $q' \in \varphi(Q_j)' \cap f(M_0\overline{\otimes}\mathcal{M})f$.

Proof of Claim 2. Suppose the claim is false. Since $\mathcal{B}$ is amenable, by [OP07, Proposition 2.4(3)], we would conclude that there is $j \in \{1,\dots,k\}$ such that $\varphi(Q_j)q'$ is amenable relative to M_0 inside $M_0\overline{\otimes}\mathcal{M}$ for some non-zero projection $q' \in \varphi(Q_j)' \cap f(M_0\overline{\otimes}\mathcal{M})f$. Thus, by Lemma III.2.5(2), there is a projection $z \in \mathcal{Z}(\varphi(Q_j)' \cap f(M_0\overline{\otimes}\mathbb{M}_m(M))f)$ such that $q' \leq z$ and $\varphi(Q_j)z$ is amenable relative to M_0 inside $M_0\overline{\otimes}\mathbb{M}_m(M)$. Since $E_{M_0\overline{\otimes}\mathcal{M}}(vv^*) \geq ce$, we get that $v^*q'v \neq 0$. Hence we deduce that $z' = v^*zv \in Q_j' \cap q(M_0\overline{\otimes}M)q$ is a non-zero projection such that Q_jz' is amenable relative to M_0 inside $M_0\overline{\otimes}\mathbb{M}_m(M)$, and hence inside $M_0\overline{\otimes}pMp$.

Thus, we can find a Q_jz'-central positive linear functional $\psi: z'\langle M_0\overline{\otimes}pMp, e_{M_0}\rangle z' \to$

$\mathbb{C}$ such that $\psi_{|z'(M_0\overline{\otimes}pMp)z'} = \tau$. The formula $\Psi(T) = \psi(z'Tz')$ defines a Q_j-central positive linear functional $\Psi : \langle M_0\overline{\otimes}pMp, e_{M_0}\rangle \to \mathbb{C}$ such that $\Psi(x) = \tau(xz')$, for any $x \in M_0\overline{\otimes}pMp$.

Note that $L^2(pMp) \cong L^2(P) \otimes \ell^2$, as left P-modules. Thus, we can find a unitary operator $U : L^2(pMp) \to L^2(P) \otimes \ell^2$ such that $U(x\xi) = xU(\xi)$, for any $x \in P$ and $\xi \in L^2(pMp)$. Let $V = \mathrm{id}_{L^2(M_0)} \otimes U : L^2(M_0\overline{\otimes}pMp) \to L^2(M_0\overline{\otimes}P) \otimes \ell^2$ and $\theta : \mathbb{B}(L^2(M_0\overline{\otimes}P)) \to \mathbb{B}(L^2(M_0\overline{\otimes}pMp))$ be the $*$-homomorphism given by $\theta(T) = V^*(T\otimes\mathrm{id}_{\ell^2})V$. Then $\theta(\langle M_0\overline{\otimes}P, e_{M_0}\rangle) \subset \langle M_0\overline{\otimes}pMp, e_{M_0}\rangle$ and $\theta(x) = x$, for every $x \in M_0\overline{\otimes}P$. Thus, if $\tilde{\Psi} : \langle M_0\overline{\otimes}P, e_{M_0}\rangle \to \mathbb{C}$ is given by $\tilde{\Psi}(T) = \Psi(\theta(T))$, then $\tilde{\Psi}$ is Q_j-central and satisfies $\tilde{\Psi}(x) = \tau(xz')$, for every $x \in M_0\overline{\otimes}P$. If we let $z'' \in Q_j' \cap q(M_0\overline{\otimes}P)q$ be the support projection of $E_{M_0\overline{\otimes}P}(z')$, then [OP07, Theorem 2.1] implies that Q_jz'' is amenable relative to M_0 inside $\tilde{M} = M_0\overline{\otimes}P$, which is a contradiction. □

For $j \in \{1,\dots,k\}$ and $S \subset T$, let $q_{j,S}$ be the maximal projection in $\mathcal{Z}(Q' \cap f(M_0\overline{\otimes}\mathcal{M})f)$ such that $\varphi(Q_j)q_{j,S}$ is amenable relative to $M_0\overline{\otimes}(\mathcal{B} \rtimes \Lambda_S)$. Noting that $S' \subset S$ implies $q_{j,S'} \leq q_{j,S}$, set

$$z_{j,S} = q_{j,S} - \bigvee_{S'\subsetneq S} q_{j,S'}, \tag{III.5.1}$$

so that $z_{j,S}z_{j,S'} = 0$ whenever $S \neq S'$ by Lemma III.2.7(1). Since $q_{j,T} = f$ it follows that if we let $\mathcal{F}_j = \{S \subset T | z_{j,S} \neq 0\}$, then $\sum_{S\in\mathcal{F}_j} z_{j,S} = f$ with the summands being mutually orthogonal.

Claim 3. If $j \neq j'$ and $S \in \mathcal{F}_j$, $S' \in \mathcal{F}_{j'}$ with $z_{j,S}z_{j',S'} \neq 0$, then $S \cap S' = \emptyset$.

Proof of Claim 3. For any $\ell \in S$ and any nonzero projection $z \leq z_{j,S}$, $z \in \mathcal{Z}(Q' \cap f(M_0\overline{\otimes}\mathcal{M})f)$, we must have $\varphi(Q_j)z$ non-amenable relative to $M_0\overline{\otimes}(\mathcal{B}\rtimes\Lambda_{T\smallsetminus\{\ell\}})$. Otherwise, using Lemma III.2.7(1) would give $\varphi(Q_j)z$ is amenable relative to $M_0\overline{\otimes}(\mathcal{B}\rtimes\Lambda_{S\smallsetminus\{\ell\}})$ implying $z \leq q_{j,S\smallsetminus\{\ell\}} \leq 1 - z_{j,S}$ (this last inequality coming from equation (III.5.1)). Thus, decomposing $M_0\overline{\otimes}\mathcal{M} = (M_0\overline{\otimes}(\mathcal{B} \rtimes \Lambda_{T\smallsetminus\{\ell\}})) \rtimes \Lambda_\ell$ and using that $\Lambda_\ell \in \mathcal{C}_{\mathrm{rss}}$ and Lemma III.2.9 we conclude that

$$\varphi(Q_{j'})z \prec_{M_0\overline{\otimes}\mathcal{M}} M_0\overline{\otimes}(\mathcal{B} \rtimes \Lambda_{T\smallsetminus\{\ell\}}).$$

Since

$$\mathcal{N}_{z_{j,S}(M_0\overline{\otimes}\mathcal{M})z_{j,S}}(\varphi(Q_{j'})z_{j,S})' \cap z_{j,S}(M_0\overline{\otimes}\mathcal{M})z_{j,S} \subset \mathcal{Z}((Qz_{j,S})' \cap z_{j,S}(M_0\overline{\otimes}\mathcal{M})z_{j,S}),$$

it follows by Lemma III.2.3(2) that $\varphi(Q_{j'})z_{j,S} \prec^s_{M_0\overline{\otimes}\mathcal{M}} M_0\overline{\otimes}(\mathcal{B} \rtimes \Lambda_{T\smallsetminus\{\ell\}})$. Applying Lemma III.2.7(2) to intersect over $\ell \in S$, we find that $\varphi(Q_{j'})z_{j,S} \prec^s_{M_0\overline{\otimes}\mathcal{M}} M_0\overline{\otimes}(\mathcal{B} \rtimes \Lambda_{T\smallsetminus S})$. Lemma III.2.5(3) then implies that $\varphi(Q_{j'})z_{j,S}$ is amenable relative to $M_0\overline{\otimes}(\mathcal{B} \rtimes \Lambda_{T\smallsetminus S})$. Hence $z_{j,S} \leq q_{j',T\smallsetminus S}$, and so

$$0 < z_{j',S'}z_{j,S} \leq z_{j',S'}q_{j',T\smallsetminus S} \leq z_{j',S'}q_{j',S'\cap(T\smallsetminus S)}$$

which forces $S' \cap (T \smallsetminus S) = S'$ (that is, $S \cap S' = \varnothing$), since otherwise $q_{j',S'\cap(T\smallsetminus S)} \leq 1 - z_{j',S'}$ by equation (III.5.1). □

Claim 4. For each $\ell \in T$ we have $\bigvee\{z_{j,S} | \ell \in S, 1 \leq j \leq k, S \in \mathcal{F}_j\} = f$.

Proof of Claim 4. To prove the claim it suffices to show that for any non-zero projection $q' \in \mathcal{Z}(Q' \cap f(M_0\overline{\otimes}\mathcal{M})f)$, we have $\bigcup\{S \in \mathcal{F}_j | 1 \leq j \leq k, z_{j,S}q' \neq 0\} = T$. Indeed, assuming this condition, let $\ell \in T$ and put $f' = \bigvee\{z_{j,S} | \ell \in S, 1 \leq j \leq k, S \in \mathcal{F}_j\}$. Then $q' = f - f'$ satisfies $z_{j,S}q' = 0$, for every $1 \leq j \leq k$ and $S \in \mathcal{F}_j$ such that $\ell \in S$. The assumed condition forces $q' = 0$ and hence $f' = f$.

For each $1 \leq j \leq k$, using the fact that $\sum_{S\in\mathcal{F}_j} z_{j,S} = f$, pick (recursively) some $S_j \in \mathcal{F}_j$ such that $z_{j,S_j}q' \neq 0$ and $z_{j,S_j}z_{j',S_{j'}} \neq 0$ for all $j' \leq j$. Then using Claim 3 we have

$$|\bigcup\{S \in \mathcal{F}_j | 1 \leq j \leq k, z_{j,S}q' \neq 0\}| \geq \sum_{j=1}^{k} |S_j|.$$

By Claim 2 we have $|S|>0$ for all $S\in\mathcal{F}_j$, $j\in\{1,\dots,k\}$, so each of the $k=|T|$ terms in the above sum is positive. Thus $|\bigcup\{S\in\mathcal{F}_j|1\le j\le k, z_{j,S}q'\ne 0\}|=|T|$ and the claim follows.[5] □

Claim 5. $\varphi(Q_0)\prec^s_{M_0\overline{\otimes}\mathcal{M}} M_0\overline{\otimes}(\mathcal{B}\rtimes\Lambda_{T\setminus\{\ell\}})$ for each $\ell\in T$.

Proof of Claim 5. Fix $\ell\in T$. By Lemma III.2.3(2) it is enough to show that

$$\varphi(Q_0)z\prec_{M_0\overline{\otimes}\mathcal{M}} M_0\overline{\otimes}(\mathcal{B}\rtimes\Lambda_{T\setminus\{\ell\}})$$

for any $z\in\mathcal{N}_{f(M_0\overline{\otimes}\mathcal{M})f}(\varphi(Q_0))'\cap f(M_0\overline{\otimes}\mathcal{M})f\subset\mathcal{Z}(Q'\cap f(M_0\overline{\otimes}\mathcal{M})f)$. Fix any such z and note that by Claim 4 we can find $j\in\{1,\dots,k\}$ and $S\in\mathcal{F}_j$ such that $\ell\in S$ and $zz_{j,S}\ne 0$. It follows that $\varphi(Q_j)z$ is not amenable relative to $M_0\overline{\otimes}(\mathcal{B}\rtimes\Lambda_{T\setminus\{\ell\}})$, otherwise Lemma III.2.7(1) would give $\varphi(Q_j)zz_{j,S}$ amenable relative to $M_0\overline{\otimes}(\mathcal{B}\rtimes\Lambda_{S\setminus\{\ell\}})$ implying $zz_{j,S}\le q_{j,S\setminus\{\ell\}}\le 1-z_{j,S}$ (this last inequality coming from equation (III.5.1)). Decomposing $M_0\overline{\otimes}\mathcal{M}=(M_0\overline{\otimes}(\mathcal{B}\rtimes\Lambda_{T\setminus\{\ell\}}))\rtimes\Lambda_\ell$ and using that $\Lambda_\ell\in\mathcal{C}_{\text{rss}}$ and Lemma III.2.9 we conclude that $\varphi(Q_0)z\prec_{M_0\overline{\otimes}\mathcal{M}} M_0\overline{\otimes}(\mathcal{B}\rtimes\Lambda_{T\setminus\{\ell\}})$, as desired. □

Note that the subalgebras $\{M_0\overline{\otimes}(\mathcal{B}\rtimes\Lambda_{T\setminus\{\ell\}})\}_{\ell\in T}$ pairwise form commuting squares, are each regular in $M_0\overline{\otimes}\mathcal{M}$, and have $\bigcap_{\ell\in T}M_0\overline{\otimes}(\mathcal{B}\rtimes\Lambda_{T\setminus\{\ell\}})=M_0\overline{\otimes}\mathcal{B}$. Hence Claim 5 together with Lemma III.2.7(2) implies that $\varphi(Q_0)\prec^s_{M_0\overline{\otimes}\mathcal{M}} M_0\overline{\otimes}\mathcal{B}$. By Claim 1, this concludes the proof of the lemma. ■

Proposition III.5.7. *There is a partition $S_1\sqcup S_2=\{1,...,n\}$ such that $\Delta(B\rtimes\Lambda_{S_i})\prec_{M\overline{\otimes}L(\Gamma)} M\overline{\otimes}P_i$, for all $i\in\{1,2\}$.*

Proof. Set $\tilde{M}=M\overline{\otimes}L(\Gamma)=M\overline{\otimes}P_1\overline{\otimes}P_2$, for $T\subset\{1,\dots,n\}$ let $Q_T=\Delta(L(\Lambda_T))$, and define $Q=(\bigcup_{j=1}^n Q_j)''=\Delta(L(\Lambda))$. For $i\in\{1,2\}$, let $\hat{i}$ denote the element in $\{1,2\}\setminus\{i\}$.

Claim 1. There are $i\in\{1,2\}$, $S_i\subset\{1,\dots,n\}$ with $|S_i|=|T_i|$, and a non-zero projection $q\in\mathcal{Z}(Q'\cap\tilde{M})$ such that Q_jq' is not amenable relative to $M\overline{\otimes}P_{\hat{i}}$ for all $j\in S_i$ and any non-zero projection $q'\in\mathcal{Z}((Qq)'\cap q\tilde{M}q)$.

[5] This type of reasoning also implies that $|S|=1$ for any $S\in\mathcal{F}_j, j\in\{1,\dots,k\}$, but we will not need this.

Proof of Claim 1. For $j \in \{1,\ldots,n\}$, $i \in \{1,2\}$, let $q_{j,i}$ be the maximal projection in $\mathcal{Z}(Q' \cap \tilde{M})$ such that $Q_j q_{j,i}$ is amenable relative to $M\overline{\otimes}P_{\hat{i}}$ inside $\tilde{M}$. Then $Q_j q'$ is non-amenable relative to $M\overline{\otimes}P_{\hat{i}}$ for all projections $q' \in \mathcal{Z}(Q' \cap \tilde{M})$ with $q' \le 1 - q_{j,i}$, so it suffices to find $S_i \subset \{1,\ldots,n\}$ with $|S_i| \ge |T_i|$ and $\bigwedge_{j\in S_i}(1-q_{j,i}) \neq 0$. Note that for each j we have $Q_j q_{j,1} q_{j,2} = \Delta(L(\Lambda_j)) q_{j,1} q_{j,2}$ amenable relative to M by Lemma III.2.7(1) and hence Lemma III.5.2 forces $q_{j,1}q_{j,2} = 0$.

Let $S_1 \subset \{1,\ldots,n\}$ be a maximal subset satisfying $q_1 = \bigwedge_{j\in S_1}(1-q_{j,1}) \neq 0$. If $|S_1| \ge |T_1|$ the claim holds with $i = 1$ and we are done. Otherwise, $S_2 = \widehat{S_1}$ will have $|S_2| \ge |T_2|$ and by the maximality of S_1, for any $j \in S_2$ we have $q_1 \le q_{j,1} \le 1 - q_{j,2}$ and hence $\bigwedge_{j\in S_2}(1-q_{j,2}) \ge q_1 \neq 0$ so that the claim holds with $i = 2$. □

For ease of notation, we assume without loss of generality that Claim 1 holds for $i = 1$. Set $S_2 = \hat{S}_1$.

Claim 2. $\Delta(L(\Lambda_{S_i})) = Q_{S_i} \prec_{\tilde{M}} M\overline{\otimes}P_i$ for all $i \in \{1,2\}$.

Proof of Claim 2. We apply Lemma III.5.6 with $M_0 = M\overline{\otimes}P_2$ to the commuting subalgebras $Q_{S_2}q, \{Q_j q\}_{j\in S_1} \subset q\tilde{M}q$. Alternative (2) of Lemma III.5.6 cannot hold, for if there were $j \in S_1$ and a non-zero projection $q' \in (Q_j q)' \cap q\tilde{M}q$ with $Q_j q'$ amenable relative to $M\overline{\otimes}P_2$, Lemma III.2.5(2) would give a projection $q'' \in \mathcal{N}_{q\tilde{M}q}(Q_j q)' \cap q\tilde{M}q \subset \mathcal{Z}((Qq)' \cap q\tilde{M}q)$ with $q' \le q''$ (so $q'' \neq 0$) and $Q_j q''$ amenable relative to $M\overline{\otimes}P_2$, contradicting Claim 1. Thus Lemma III.5.6 gives that $Q_{S_2}q \prec^s_{\tilde{M}} M\overline{\otimes}P_2$. This implies that $Q_{S_2} \prec_{\tilde{M}} M\overline{\otimes}P_2$, and that $Q_{S_2}q$ is amenable relative to $M\overline{\otimes}P_2$ by Lemma III.2.5(3).

Hence for all $j \in S_2$ we have $Q_j q$ amenable relative to $M\overline{\otimes}P_2$. It follows that $Q_j q'$ is not amenable relative to $M\overline{\otimes}P_1$ for any $j \in S_2$ and non-zero projection $q' \in \mathcal{Z}((Qq)' \cap q\tilde{M}q)$. Otherwise, Lemma III.2.7(1) would give $Q_j q' = \Delta(L(\Lambda_j))$ amenable relative to M, contradicting Lemma III.5.2. We then apply Lemma III.5.6 with $M_0 = M\overline{\otimes}P_1$ to the commuting subalgebras $Q_{S_1}q, \{Q_j q\}_{j\in S_2} \subset q\tilde{M}q$, and as before we conclude that

$Q_{S_1} q \prec^s_{\tilde{M}} M \overline{\otimes} P_1$ and hence $Q_{S_1} \prec_{\tilde{M}} M \overline{\otimes} P_1$, establishing Claim 2. □

We now finish the proof of the proposition. For any $i \in \{1,2\}$, since $\mathcal{U}(\Delta(B \rtimes \Lambda_{S_i}))$ is generated by $\{\Delta(bu) : b \in \mathcal{U}(B), u \in \mathcal{U}(L(\Lambda_{S_i}))\}$ if we did not have $\Delta(B \rtimes \Lambda_{S_i}) \prec_{\tilde{M}} M \overline{\otimes} P_i$ there would be sequences $\{b_n\} \subset \mathcal{U}(B), \{u_n\} \subset \mathcal{U}(L(\Lambda_{S_i}))$ such that $\|E_{M \overline{\otimes} P_i}(x \Delta(b_n u_n) y)\|_2 \to 0$ for all $x, y \in \tilde{M}$. But then for any $x, y \in P_{\hat{i}}$, using the fact that $\Delta(B) \subset M \overline{\otimes} P_i$ we would have

$$\|E_{M \overline{\otimes} P_i}(x \Delta(u_n) y)\|_2 = \|\Delta(b_n) E_{M \overline{\otimes} P_i}(x \Delta(u_n) y)\|_2 = \|E_{M \overline{\otimes} P_i}(x \Delta(b_n u_n) y)\|_2 \to 0.$$

Since $\tilde{M} = M \overline{\otimes} P_i \overline{\otimes} P_{\hat{i}}$ it would further follow that $\|E_{M \overline{\otimes} P_i}(x \Delta(u_n) y)\|_2 \to 0$ for all $x, y \in \tilde{M}$, which would contradict Claim 2. Hence we must have $\Delta(B \rtimes \Lambda_{S_i}) \prec_{\tilde{M}} M \overline{\otimes} P_i$ as desired. ■

III.5.4 Step 3

Next, by combining **Step 2** and Theorem III.4.1, we obtain:

Lemma III.5.8. *We can find a decreasing sequence of subgroups $\Omega_k < \Gamma$ such that*

- $B \rtimes \Lambda_{S_1} \prec_M A \rtimes \Omega_k$, *for all $k \geq 1$, and*
- $P_2 \prec_{L(\Gamma)} L(\cup_{k \geq 1} C_\Gamma(\Omega_k))$.

Proof. By Proposition III.5.7 we have that $\Delta(B \rtimes \Lambda_{S_1}) \prec_{M \overline{\otimes} L(\Gamma)} M \overline{\otimes} P_1$. Since $P_2 \subset P_1' \cap L(\Gamma)$, the conclusion follows from Theorem III.4.1. ■

III.5.5 Proof of Corollary E

Let $\Gamma = \mathrm{PSL}_2(R)$, where either $R = \mathcal{O}_d$, for a square-free integer $d \geq 2$, or $R = \mathbb{Z}[S^{-1}]$, for a non-empty set of primes S. Then the centralizer $C_\Gamma(g)$ of any non-trivial element $g \in \Gamma \setminus \{e\}$ is solvable, hence amenable. This follows from the following fact which can be derived by using for instance the Jordan normal form of matrices: if $A \in \mathrm{SL}_2(\mathbb{R}) \setminus \{\pm I\}$, then the group $\{B \in \mathrm{SL}_2(\mathbb{R}) | AB = \pm BA\}$ is solvable. In particular, we deduce that Γ is icc and does not contain two commuting non-amenable subgroups.

Assume by contradiction that $L(\Gamma)$ is not prime and write $L(\Gamma) = P_1 \overline{\otimes} P_2$. Since Γ is non-amenable, we may assume without loss of generality that P_2 is non-amenable. Since $\Gamma \in \mathcal{L}$ by Remark III.1.2, Γ is measure equivalent to a product $\Lambda = \Lambda_1 \times ... \times \Lambda_n$ of $n \geq 1$ non-elementary hyperbolic groups (where $n = 2$, if $R = \mathcal{O}_d$, and $n = |S| + 1$, if $R = \mathbb{Z}[S^{-1}]$). Since non-elementary hyperbolic groups are in class $\mathcal{C}_{\mathrm{rss}}$ by [PV12], we are in the setting of III.5.1. Thus, we may find a decreasing sequence of subgroups $\Omega_k < \Gamma$ satisfying Lemma III.5.8. Since Λ_i is non-amenable, for every $1 \leq i \leq n$, and P_2 is non-amenable, it follows that for large enough k we have that both Ω_k and $C_\Gamma(\Omega_k)$ are non-amenable. This contradicts the previous paragraph. ∎

III.5.6 Step 4

This step is divided between two lemmas. We start with the following:

Lemma III.5.9. *Let Ω_k be the decreasing sequence of subgroups of Γ provided by Lemma III.5.8.*

Then for any large enough $k \geq 1$ we have that $L(\Omega_k) \prec_{L(\Gamma)} P_1$.

Proof. Let $i \in \{1, 2\}$. By Proposition III.5.5, $P_i \prec_M B \rtimes \Lambda_{T_i}$. We can thus find a not necessarily unital $*$-homomorphism $\varphi_i : P_i \to \mathbb{M}_{m_i}(M)$ and a non-zero partial isometry $v_i \in M_{m_i,1}(M)p$ such that $\varphi_i(x) v_i = v_i x$, for every $x \in P_i$, and $\varphi(P_i) \subset \mathcal{M}_i$, where $\mathcal{M}_i = \mathbb{M}_{m_i}(B \rtimes \Lambda_{T_i})$, for some $m_i \geq 1$. Here, we view $P_i \subset M$ as non-unital subalgebras of $\mathbb{M}_{m_i}(M)$ via the embedding $x \mapsto x \otimes e_{11}$, where $e_{11} \in \mathbb{M}_{m_i}(\mathbb{C})$ is the matrix unit corresponding the $(1,1)$ entry. Moreover, we may assume that $E_{\mathcal{M}_i}(v_i v_i^*) \geq c_i \varphi_i(1)$, for some $c_i > 0$. We define $\mathcal{B}_i = \mathbb{M}_{m_i}(B)$ and write canonically $\mathcal{M}_i = \mathcal{B}_i \rtimes \Lambda_{T_i}$.

We claim that $\varphi_i(P_i)p'$ is not amenable relative to $\mathcal{B}_i \rtimes \Lambda_{T_i \setminus \{j\}}$ inside $\mathbb{M}_{m_i}(M)$, for any $j \in T_i$ and any non-zero projection $p' \in \varphi_i(P_i)' \cap \varphi_i(1)\mathbb{M}_{m_i}(M)\varphi_i(1)$ with $p' \le v_i v_i^*$. Otherwise, it would follow that $P_i v_i^* p' v_i$ is amenable relative to $\mathcal{B}_i \rtimes \Lambda_{T_i \setminus \{j\}}$ inside $\mathbb{M}_{m_i}(M)$. Note that $v_i^* p' v_i$ is a non-zero projection in $P_i' \cap (p \otimes e_{11})\mathbb{M}_{m_i}(M)(p \otimes e_{11}) = P_i' \cap pMp$. But, recalling that $pMp = A \rtimes \Gamma$, $P_1 \overline{\otimes} P_2 = L(\Gamma)$, Γ is icc, and the action $\Gamma \curvearrowright A$ is ergodic, we get that $\mathcal{N}_{pMp}(P_i)' \cap pMp \subset L(\Gamma)' \cap A \rtimes \Gamma = \mathbb{C}p$. Thus, by Lemma III.2.5(2), we would get that P_i amenable relative to $\mathcal{B}_i \rtimes \Lambda_{T_i \setminus \{j\}}$ inside $\mathbb{M}_{m_i}(M)$. This contradicts the moreover assertion of Proposition III.5.5.

Next, we define $\varphi = \varphi_1 \otimes \varphi_2 : L(\Gamma) = P_1 \overline{\otimes} P_2 \to \mathbb{M}_{m_1}(M) \overline{\otimes} \mathbb{M}_{m_2}(M)$. Then $\varphi(L(\Gamma)) \subset \mathcal{M}$, where $\mathcal{M} = \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$. We let $v = v_1 \otimes v_2$ and note that $\varphi(x)v = vx$, for every $x \in L(\Gamma)$. We denote $e = \varphi(1) \in \mathcal{M}$ and $\mathcal{B} = \mathcal{B}_1 \overline{\otimes} \mathcal{B}_2$. Then $\mathcal{M} = \mathcal{B} \rtimes \Lambda$, where we consider the product action of $\Lambda = \Lambda_{T_1} \times \Lambda_{T_2}$ on $\mathcal{B}$. The rest of the proof is split between three claims.

Claim 1. If a von Neumann subalgebra $Q \subset L(\Gamma)$ satisfies $\varphi(Q) \prec_{\mathcal{M}} \mathcal{B} \rtimes \Lambda_{T_1}$, then $Q \prec_{L(\Gamma)} P_1$.

Proof of Claim 1. Assuming that $Q \nprec_{L(\Gamma)} P_1$, we will prove that $\varphi(Q) \nprec_{\mathcal{M}} \mathcal{B} \rtimes \Lambda_{T_1}$. By applying Theorem I.2.6, we can find a sequence $u_n \in \mathcal{U}(Q)$ such that $\|E_{P_1}(u_n a)\|_2 \to 0$, for all $a \in L(\Gamma)$.

For every $i \in \{1,2\}$, let $\psi_i : M \to \mathbb{M}_{m_i}(M)$ be the embedding given by $\psi_i(x) = x \otimes e_{11}$. Let $\psi = \psi_1 \otimes \psi_2 : L(\Gamma) = P_1 \overline{\otimes} P_2 \to \mathbb{M}_{m_1}(M) \overline{\otimes} \mathbb{M}_{m_2}(M)$. We claim that

$$\|E_{\mathbb{M}_{m_1}(M)\overline{\otimes}\mathcal{B}_2}(a\psi(u_n)b)\|_2 \to 0, \quad \text{for all } a, b \in \mathbb{M}_{m_1}(M)\overline{\otimes}\mathbb{M}_{m_2}(M). \tag{III.5.2}$$

By using that $\mathcal{B}_2 = \mathbb{M}_{m_2}(B)$ and the position of $A \subset B$, we find $\alpha_1, ..., \alpha_D, \beta_1, ..., \beta_D \in \mathbb{M}_{m_2}(M)$ such that $E_{\mathcal{B}_2}(x) = \sum_{d=1}^D \alpha_d E_{\psi_2(A)}(\alpha_d^* x \beta_d)\beta_d^*$, for every $x \in \mathbb{M}_{m_2}(M)$. This allows us to reduce III.5.2 to showing that $\|E_{\mathbb{M}_{m_1}(M)\overline{\otimes}\psi_2(A)}(a\psi(u_n)b)\|_2 \to 0$, for all $a, b \in \mathbb{M}_{m_1}(M)\overline{\otimes}\psi_2(p)\mathbb{M}_{m_2}(M)\psi_2(p)$. Since $\psi_2(p)\mathbb{M}_{m_2}(M)\psi_2(p) = \psi_2(pMp) = \psi_2(A \rtimes \Gamma)$

and $E_{\mathbb{M}_{m_1}(M)\overline{\otimes}\psi_2(A)}$ is $\mathbb{M}_{m_1}(M)\overline{\otimes}\psi_2(A)$-bimodular, it is enough to treat the case when $a = 1\otimes\psi_2(\xi)$, $b = 1\otimes\psi_2(\zeta)$, for some $\xi,\zeta \in L(\Gamma)$.

In this case we have $a\psi(u_n)b = (\psi_1\otimes\psi_2)((1\otimes\xi)u_n(1\otimes\zeta)) \in P_1\overline{\otimes}\psi_2(L(\Gamma))$. Since $E_{\psi_2(A)}(\psi_2(x)) = \tau(x)\psi_2(p)$, for every $x \in P_2$, we get that $E_{\mathbb{M}_{m_1}(M)\overline{\otimes}\psi_2(A)}((\psi_1\otimes\psi_2)(x)) = (\psi_1\otimes\tau)(x)\psi_2(p)$, for all $x \in L(\Gamma)$. Also, note that $(\psi_1\otimes\tau)(x) = (\psi_1\otimes\tau)(1\otimes E_{P_2})(E_{P_1}\otimes 1)(x) = ((\psi_1\otimes\tau)\circ E_{P_1\overline{\otimes}P_2})(x)$, for every $x \in P_1\overline{\otimes}L(\Gamma)$, and that $(1\otimes\xi)u_n(1\otimes\zeta) \in P_1\overline{\otimes}L(\Gamma)$. By combining these fact we get that

$$\begin{aligned}E_{\mathbb{M}_{m_1}(M)\overline{\otimes}\psi_2(A)}(a\psi(u_n)b) &= (\psi_1\otimes\tau)((1\otimes\xi)u_n(1\otimes\zeta))\psi_2(p)\\ &= ((\psi_1\otimes\tau)\circ E_{P_1\overline{\otimes}P_2})((1\otimes\xi)u_n(1\otimes\zeta))\psi_2(p)\\ &= \psi_1(E_{P_1}(u_nE_{P_2}(\zeta\xi)))\psi_2(p),\end{aligned}$$

where in order to get that the last equality we used the fact that for all $\alpha\in P_1,\beta\in P_2$ we have

$$\begin{aligned}(1\otimes\tau)(E_{P_1}\otimes E_{P_2})((1\otimes\xi)(\alpha\otimes\beta)(1\otimes\zeta)) &= E_{P_1}(\alpha)\tau(\beta\zeta\xi)\\ &= E_{P_1}(\alpha)\tau(\beta E_{P_2}(\zeta\xi))\\ &= E_{P_1}((\alpha\otimes\beta)E_{P_2}(\zeta\xi)).\end{aligned}$$

Since $\|E_{P_1}(u_nE_{P_2}(\zeta\xi))\|_2\to 0$, equation III.5.2 follows.

Let $c = c_1c_2 > 0$. Since $E_{\mathcal{M}}(vv^*) \geq c\varphi(1) = ce$, if $w = v^*E_{\mathcal{M}}(vv^*)^{-1}$, then $\varphi(x) = E_{\mathcal{M}}(vxw)$, for any $x\in L(\Gamma)$. Let $a,b\in\mathcal{M}$. Since $\mathcal{B}\rtimes\Lambda_{T_1}\subset\mathcal{M}$, we have $E_{\mathcal{B}\rtimes\Lambda_{T_1}}(a\varphi(u_n)b) = E_{\mathcal{B}\rtimes\Lambda_{T_1}}(avu_nwb)$. Using that $\mathcal{B}\rtimes\Lambda_{T_1} = \mathcal{M}_1\overline{\otimes}\mathcal{B}_2\subset\mathbb{M}_{m_1}(M)\overline{\otimes}\mathcal{B}_2$ in combination with III.5.2, the claim follows. □

To finish the proof, it suffices to show that Claim 1 applies to $Q = L(\Omega_k)$, for k large enough. This will be achieved by combining Claims 2 and 3 below. We fix $j\in T_2$ and denote

$T = \{1, ..., n\} \setminus \{j\}$. For $k \geq 1$, we put $N_k = \varphi(L(C_\Gamma(\Omega_k)))$, and let $f_k \in \mathcal{Z}(N_k' \cap e\mathcal{M}e)$ be the maximal projection such that $N_k f_k$ is amenable relative to $\mathcal{B} \rtimes \Lambda_T$ inside $\mathcal{M}$.

Claim 2. $\varphi(L(\Omega_k))(e - f_k) \prec^s_{\mathcal{M}} \mathcal{B} \rtimes \Lambda_T$, for any $k \geq 1$.

Proof of Claim 2. Since $\varphi(L(\Omega_k)) \subset N_k' \cap e\mathcal{M}e$, by parts (1) and (2) of Lemma III.2.3, it suffices to show that $(N_k' \cap e\mathcal{M}e)z \prec_{\mathcal{M}} \mathcal{B} \rtimes \Lambda_T$, whenever $z \in \mathcal{Z}((N_k' \cap e\mathcal{M}e)' \cap e\mathcal{M}e)(e - f_k)$ is a non-zero projection. Since $\mathcal{Z}((N_k' \cap e\mathcal{M}e)' \cap e\mathcal{M}e) \subset \mathcal{Z}(N_k' \cap e\mathcal{M}e)$, we get $z \in \mathcal{Z}(N_k' \cap e\mathcal{M}e)$. Since $z \leq e - f_k$, the maximality of f_k implies that $N_k z$ is not amenable relative to $\mathcal{B} \rtimes \Lambda_T$. Since $(N_k' \cap e\mathcal{M}e)z$ and $N_k z$ commute, and we can decompose $\mathcal{M} = (\mathcal{B} \rtimes \Lambda_T) \rtimes \Lambda_j$, where $\Lambda_j \in \mathcal{C}_{\text{rss}}$, Lemma III.2.9 implies that $(N_k' \cap e\mathcal{M}e)z \prec_{\mathcal{M}} \mathcal{B} \rtimes \Lambda_T$. This proves the claim. □

Next, put $N = \varphi(L(\cup_{k\geq 1} C_\Gamma(\Omega_k)))$. Since $P_2 \prec_{L(\Gamma)} L(\cup_{k\geq 1} C_\Gamma(\Omega_k))$ by Proposition III.5.8, and P_2 is regular in $L(\Gamma)$, Lemma III.2.3(3) implies that $P_2 \prec^s_{L(\Gamma)} L(\cup_{k\geq 1} C_\Gamma(\Omega_k))$. Thus $\varphi(P_2) \prec^s_{\varphi(L(\Gamma))} N$, hence $\varphi(P_2) \prec^s_{\mathbb{M}_{m_1}(M) \overline{\otimes} \mathbb{M}_{m_2}(M)} N$, so in particular

$$\varphi(P_2)vv^* \prec_{\mathbb{M}_{m_1}(M) \overline{\otimes} \mathbb{M}_{m_2}(M)} N.$$

Using Lemma III.2.3(4) we find a non-zero projection $e' \in \mathcal{Z}(N' \cap e(\mathbb{M}_{m_1}(M) \overline{\otimes} \mathbb{M}_{m_2}(M))e)$ such that

$$\varphi(P_2)vv^* \prec_{\mathbb{M}_{m_1}(M) \overline{\otimes} \mathbb{M}_{m_2}(M)} Nf$$

, for any non-zero projection $f \in N' \cap e(\mathbb{M}_{m_1}(M) \overline{\otimes} \mathbb{M}_{m_2}(M))e$ with $f \leq e'$.

We continue with the following:

Claim 3. $\tau(f_k e') \to 0$, as $k \to \infty$.

Proof of Claim 3. Assume that the claim is false. Since $N_k \subset N_{k+1}$, we have $f_{k+1} \leq f_k$, for any $k \geq 1$. If $f = \bigwedge_k f_k$, then $f \in \mathcal{Z}(N' \cap e\mathcal{M}e)$. Since $f \leq f_k$, we get that $N_k f$ is amenable relative to $\mathcal{B} \rtimes \Lambda_T$ inside $\mathcal{M}$, for all $k \geq 1$. By Lemma III.2.6 we get that $Nf = (\cup_{k\geq 1} N_k f)''$ is amenable relative to $\mathcal{B} \rtimes \Lambda_T$ inside $\mathcal{M}$. Lemma III.2.5(1) then gives that Nfe' is amenable relative to $\mathcal{B} \rtimes \Lambda_T$ inside $\mathcal{M}$.

Since $\tau(fe') = \lim_k \tau(f_k e')$ and the claim is assumed false, $fe' \neq 0$. Since $fe' \leq e'$ belongs to $N' \cap e(\mathbb{M}_{m_1}(M)\overline{\otimes}\mathbb{M}_{m_2}(M))e$, the discussion before the claim gives

$$\varphi(P_2)vv^* \prec_{\mathbb{M}_{m_1}(M)\overline{\otimes}\mathbb{M}_{m_2}(M)} Nfe'.$$

By Lemma III.2.3(2) there is a non-zero projection

$$p' \in \mathcal{Z}((\varphi(P_2)vv^*)' \cap vv^*(\mathbb{M}_{m_1}(M)\overline{\otimes}\mathbb{M}_{m_2}(M))vv^*)$$

with $\varphi(P_2)p' \prec^s_{\mathbb{M}_{m_1}(M)\overline{\otimes}\mathbb{M}_{m_2}(M)} Nfe'$. Lemma III.2.5(3) further gives that $\varphi(P_2)p'$ is amenable relative to Nfe' inside $\mathbb{M}_{m_1}(M)\overline{\otimes}\mathbb{M}_{m_2}(M)$. Since $\varphi(P_2)vv^* = v_1v_1^* \otimes \varphi_2(P_2)v_2v_2^*$ and M is a factor, we get that $p' = v_1v_1^* \otimes p''$, for some projection $p'' \in \varphi_2(P_2)' \cap \varphi_2(1)\mathbb{M}_{m_2}(M)\varphi_2(1)$ with $p'' \leq v_2v_2^*$. It follows that $\varphi_2(P_2)p''$ is amenable relative to Nfe' inside $\mathbb{M}_{m_1}(M)\overline{\otimes}\mathbb{M}_{m_2}(M)$.

By combining the conclusions of the last two paragraphs with [OP07, Proposition 2.4(3)], we deduce that $\varphi_2(P_2)p'' \subset \mathbb{M}_{m_2}(M)$ is amenable relative to $\mathcal{B} \rtimes \Lambda_T$ inside $\mathbb{M}_{m_1}(M)\overline{\otimes}\mathbb{M}_{m_2}(M)$. Since $\mathcal{B} \rtimes \Lambda_T$ and $\mathbb{M}_{m_2}(M)$ are in a commuting square position and regular, by Lemma III.2.7(2), $\varphi_2(P_2)p''$ is amenable relative to their intersection, $\mathcal{B}_2 \rtimes \Lambda_{T_2 \setminus \{j\}}$, inside $\mathbb{M}_{m_1}(M)\overline{\otimes}\mathbb{M}_{m_2}(M)$. As $\varphi_2(P_2)p''$ and $\mathcal{B}_2 \rtimes \Lambda_{T_2 \setminus \{j\}}$ are subalgebras of $\mathbb{M}_{m_2}(M)$, it follows that $\varphi_2(P_2)p''$ is amenable relative to $\mathcal{B}_2 \rtimes \Lambda_{T_2 \setminus \{j\}}$ inside $\mathbb{M}_{m_2}(M)$. This contradicts the second paragraph of the proof of the lemma. □

Next, by combining claims 2 and 3, for every $j \in T_2$, we can find projections $f_{k,j} \in \mathcal{Z}(N_k' \cap e\mathcal{M}e)$ such that $\varphi(L(\Omega_k))(e - f_{k,j}) \prec^s_{\mathcal{M}} \mathcal{B} \rtimes \Lambda_{\{1,\dots,n\} \setminus \{j\}}$, for any $k \geq 1$, and $\tau(f_{k,j}e') \to 0$, as $k \to \infty$.

For $k \geq 1$, let $r_k = \bigvee_{j \in T_2} f_{k,j}$. Then $r_k \in \mathcal{Z}(N_k' \cap e\mathcal{M}e)$ and since $\tau(r_k e') \leq \sum_{j \in T_2} \tau(f_{k,j}e')$, we get that $\tau(r_k e') \to 0$, as $k \to \infty$. In particular, since $0 \neq e' \leq e$, we get that

$e - r_k \neq 0$, for k large enough. On the other hand, since $\varphi(L(\Omega_k))(e-r_k) \prec^s_{\mathcal{M}} \mathcal{B} \rtimes \Lambda_{\{1,\dots,n\}\smallsetminus\{j\}}$, for every $j \in T_2$, and the algebras $\mathcal{B} \rtimes \Lambda_{\{1,\dots,n\}\smallsetminus\{j\}}$, with $j \in T_2$, are in a commuting square position and regular in $\mathcal{M}$, Lemma III.2.7(2) implies that $\varphi(L(\Omega_k))(e-r_k) \prec^s_{\mathcal{M}} \mathcal{B} \rtimes \Lambda_{\{1,\dots,n\}\smallsetminus T_2} = \mathcal{B} \rtimes \Lambda_{T_1}$, for any $k \geq 1$.

Thus, if k is large enough then $\varphi(L(\Omega_k)) \prec_{\mathcal{M}} \mathcal{B} \rtimes \Lambda_{T_1}$, hence $L(\Omega_k) \prec_{L(\Gamma)} P_1$, by Claim 1. ■

We are now ready to complete the proof of **Step 4**.

Lemma III.5.10. *For every $i \in \{1,2\}$ we can find a subgroup $\Sigma_i < \Gamma$ such that*

1. $B \rtimes \Lambda_{S_i} \prec^s_M A \rtimes \Sigma_i$.
2. $A \rtimes \Sigma_i \prec^s_M B \rtimes \Lambda_{S_i}$.
3. $P_i \prec^s_{L(\Gamma)} L(\Sigma_i)$.
4. $L(\Sigma_i) \prec^s_{L(\Gamma)} P_i$.

Proof. Assume for simplicity $i = 1$. By Lemma III.5.8 we can find a decreasing sequence of subgroups $\Omega_k < \Gamma$ such that $B \rtimes \Lambda_{S_1} \prec_M A \rtimes \Omega_k$, for all $k \geq 1$, and $P_2 \prec_{L(\Gamma)} L(\cup_{k\geq 1} C_\Gamma(\Omega_k))$. By Lemma III.5.9, for any $k \geq 1$ large enough $\Sigma_1 := \Omega_k$ satisfies $L(\Sigma_1) \prec_{L(\Gamma)} P_1$ in addition to $B \rtimes \Lambda_{S_1} \prec_M A \rtimes \Sigma_1$. Since $B \rtimes \Lambda_{S_1}$ is regular in the II$_1$ factor M, by Lemma III.2.3(3) we get that $B \rtimes \Lambda_{S_1} \prec^s_M A \rtimes \Sigma_1$. This proves (1).

By Lemma III.2.3(3), we can find a non-zero projection $e \in L(\Sigma_1)' \cap L(\Gamma)$ with $L(\Sigma_1)e \prec^s_{L(\Gamma)} P_1$. By Proposition III.5.5 we have that $P_1 \prec^s_M B \rtimes \Lambda_{T_1}$. By combining these facts with Lemma III.2.3(1) we derive that $L(\Sigma_1)e \prec^s_M B \rtimes \Lambda_{T_1}$. Our next goal is to upgrade this to the following conclusion:

Claim 1. $A \rtimes \Sigma_1 \prec^s_M B \rtimes \Lambda_{T_1}$.

Proof of Claim 1. For $F \subset \Lambda$, let $\mathcal{K}_F \subset L^2(M)$ be the closed linear span of $\{(B \rtimes \Lambda_{T_1})v_g | g \in F\}$. We denote by P_F be the orthogonal projection onto $\mathcal{K}_F$. The proof relies on the following fact: let $R \subset rMr$ be a von Neumann subalgebra and $\mathcal{U} \subset \mathcal{U}(R)$ a subgroup with $\mathcal{U}'' = R$. Then $R \prec^s_M B \rtimes \Lambda_{T_1}$ iff for any $\varepsilon > 0$, there is $F \subset \Lambda$ finite such that $\|u - P_F(u)\|_2 \leq \varepsilon$, for all $u \in \mathcal{U}$. This fact follows from [Va10b, Lemma 2.5] by using that $\Lambda_{T_1} < \Lambda$ is a normal subgroup.

Let $\varepsilon > 0$. Since $A \subset pMp$ is maximal abelian and $e \in L(\Gamma)$, we have $E_{A' \cap pMp}(e) = E_A(e) = \frac{\tau(e)}{\tau(p)} p$. On the other hand, $E_{A' \cap pMp}(e)$ belongs to the closed convex hull of $\{vev^* | v \in \mathcal{U}(A)\}$ (being precisely its element of minimal $\|.\|_2$). We can therefore find $v_1, ..., v_D, w_1, ..., w_D \in \mathcal{U}(A)$ such that $\|p - \sum_{d=1}^D v_d e w_d\|_2 \leq \frac{\varepsilon}{2}$. Since $L(\Sigma_1)e \prec^s_M B \rtimes \Lambda_{T_1}$, by using the above fact, we can find $F \subset \Lambda$ finite such that $\|u_g e - P_F(u_g e)\|_2 \leq \frac{\varepsilon}{2D}$, for any $g \in \Sigma_1$.

By combining the last two inequalities, for every $a \in \mathcal{U}(A)$ and $g \in \Sigma_1$ we have

$$\|au_g - \sum_{d=1}^{D} a(u_g v_d u_g^*) P_F(u_g e) w_d\|_2 \leq \|p - \sum_{d=1}^{D} v_d e w_d\|_2 + D\ \|u_g e - P_F(u_g e)\|_2 \leq \varepsilon.$$

Since $\mathcal{K}_F$ is an A-A-bimodule, we derive that $\sum_{d=1}^D a(u_g v_d u_g^*) P_F(u_g e) w_d \in \mathcal{K}_F$. Hence, we have $\|au_g - P_F(au_g)\|_2 \leq \varepsilon$, for every $a \in \mathcal{U}(A)$ and $g \in \Sigma_1$. Since $\varepsilon > 0$ is arbitrary and the group $\mathcal{U} = \{au_g | a \in \mathcal{U}(A), g \in \Sigma_1\}$ generates $A \rtimes \Sigma_1$, the above fact gives the claim. □

By combining the claim with $B \rtimes \Lambda_{S_1} \prec_M A \rtimes \Sigma_1$ and with Lemma III.2.3(1) we conclude that $B \rtimes \Lambda_{S_1} \prec_M B \rtimes \Lambda_{T_1}$. This readily implies that $S_1 \subset T_1$. By symmetry, we also get that $S_2 \subset T_2$. Since $\{S_1, S_2\}$ and $\{T_1, T_2\}$ are partitions of $\{1, ..., n\}$ we must have that $S_1 = T_1$ and $S_2 = T_2$. Thus, Claim 2 reads $A \rtimes \Sigma_1 \prec^s_M B \rtimes \Lambda_{S_1}$, which proves (2).

We are left with proving (3) and (4), which is done in the following two claims.

Claim 2. $P_1 \prec^s_{L(\Gamma)} L(\Sigma_1)$.

Proof of Claim 2. Since P_1 is regular in $L(\Gamma)$ and $L(\Gamma)$ is a II$_1$ factor, by Lemma III.2.3(3) it suffices to show that $P_1 \prec_{L(\Gamma)} L(\Sigma_1)$. By Proposition III.5.5, $P_1 \prec_M B \rtimes \Lambda_{T_1} =$

$B \rtimes \Lambda_{S_1}$. By combining this with (1) and Lemma III.2.3(1), it follows that $P_1 \prec_M A \rtimes \Sigma_1$.

Assume by contradiction that $P_1 \nprec_{L(\Gamma)} L(\Sigma_1)$. By Theorem I.2.6 we can find $u_n \in \mathcal{U}(P_1)$ such that $\|E_{L(\Sigma_1)}(au_nb)\|_2 \to 0$, for every $a,b \in L(\Gamma)$. We claim that $\|E_{A\rtimes\Sigma_1}(au_nb)\|_2 \to 0$, for every $a,b \in pMp = A\rtimes\Gamma$. Since $E_{A\rtimes\Sigma_1}$ is A-A-bimodular, it suffices to verify this for every $a,b \in L(\Gamma)$. But, since $au_nb \in L(\Gamma)$, we have that $\|E_{A\rtimes\Sigma_1}(au_nb)\|_2 = \|E_{L(\Sigma_1)}(au_nb)\|_2 \to 0$. Since the claim implies that $P_1 \nprec_M A \rtimes \Sigma_1$, we get the desired contradiction. □

Claim 3. $L(\Sigma_1) \prec^s_{L(\Gamma)} P_1$.

Proof of Claim 3. By Proposition III.5.7 we have $\Delta(B \rtimes \Lambda_{S_1}) \prec_{M\overline{\otimes}L(\Gamma)} M\overline{\otimes}P_1$. Since Γ is icc, we get that $\Delta(M)' \cap M\overline{\otimes}L(\Gamma) = \mathbb{C}1$. Therefore, by applying Lemma III.2.3(3) we conclude that $\Delta(B \rtimes \Lambda_{S_1}) \prec^s_{M\overline{\otimes}L(\Gamma)} M\overline{\otimes}P_1$. On the other hand, since $L(\Sigma_1) \subset A \rtimes \Sigma_1$, Claim 1 implies that $L(\Sigma_1) \prec^s_M B \rtimes \Lambda_{S_1}$, and therefore $\Delta(L(\Sigma_1)) \prec^s_{M\overline{\otimes}L(\Gamma)} \Delta(B \rtimes \Lambda_{S_1})$. By combining these facts with Lemma III.2.3(1), we derive that $\Delta(L(\Sigma_1)) \prec^s_{M\overline{\otimes}L(\Gamma)} M\overline{\otimes}P_1$.

Let $p' \in L(\Sigma_1)' \cap L(\Gamma)$ be a non-zero projection. Assuming that $L(\Sigma_1)p' \nprec_{L(\Gamma)} P_1$, we will reach a contradiction, which will prove the claim. By Theorem I.2.6 we can find a sequence $g_n \in \Sigma_1$ such that $\|E_{P_1}(au_{g_n}p'b)\|_2 \to 0$, for every $a,b \in L(\Gamma)$. We claim that $\|E_{M\overline{\otimes}P_1}(a\Delta(u_{g_n})(1\otimes p')b)\|_2 \to 0$, for every $a,b \in M\overline{\otimes}L(\Gamma)$. Since $\Delta(u_{g_n}) \in \mathcal{U}(\Delta(L(\Sigma_1)))$ and $1 \otimes p' \in \Delta(L(\Sigma_1))' \cap M\overline{\otimes}L(\Gamma)$ is non-zero projection (recall that $\Delta(u_g) = u_g \otimes u_g$, for all $g \in \Gamma$), we get that $\Delta(L(\Sigma_1))(1\otimes p') \nprec_{M\overline{\otimes}L(\Gamma)} M\overline{\otimes}P_1$, which contradicts the conclusion of the previous paragraph. Thus, it remains to prove the claim.

Since $E_{M\overline{\otimes}P_1}$ is $M\otimes 1$-$M\otimes 1$-bimodular, we may assume that $a,b \in 1\otimes L(\Gamma)$. But in this case we have $\|E_{M\overline{\otimes}P_1}(a\Delta(u_{g_n})(1\otimes p')b)\|_2 = \|E_{P_1}(au_{g_n}p'b)\|_2 \to 0$, which finishes the proof. ■

III.5.7 Step 5: completion of the proof of Theorem III.5.3

Let $i \in \{1,2\}$. By Lemma III.5.10 we have that $B \rtimes \Lambda_{S_i} \prec^s_M A \rtimes \Sigma_i$ and $A \rtimes \Sigma_i \prec^s_M B \rtimes \Lambda_{S_i}$. Recalling that $A = L^\infty(X)$ and $B \rtimes \Lambda_{S_i} = (L^\infty(Y) \rtimes \Lambda_{S_i}) \otimes \mathbb{M}_\ell(\mathbb{C})$, we get that $L^\infty(Y) \rtimes \Lambda_{S_i} \prec_M L^\infty(X) \rtimes \Sigma_i$ and also that $L^\infty(X) \rtimes \Sigma_i \prec^s_M L^\infty(Y) \rtimes \Lambda_{S_i}$. Since $\Lambda = \Lambda_{S_1} \times \Lambda_{S_2}$, Proposition III.3.1 and implies that Σ_i is measure equivalent to Λ_{S_i}. Together with Lemma III.5.10, this finishes the proof of Theorem III.5.3. ■

III.6 From tensor decompositions to product decompositions

The goal of this section is prove the following result that we will need in the proof of Theorem F. We say that two subgroups Σ, Ω of a countable group Γ are called *commensurable* if we have that $[\Sigma : \Sigma \cap \Omega] < \infty$ and $[\Omega : \Sigma \cap \Omega] < \infty$.

Theorem III.6.1. *Let Γ be a countable icc group, denote $M = L(\Gamma)$, and assume that $M = P_1 \overline{\otimes} P_2$. For every $i \in \{1,2\}$, let $\Sigma_i < \Gamma$ be a subgroup such that $P_i \prec^s_M L(\Sigma_i)$ and $L(\Sigma_i) \prec^s_M P_i$.*

Then we can find a decomposition $\Gamma = \Gamma_1 \times \Gamma_2$, a decomposition $M = P_1^s \overline{\otimes} P_2^{1/s}$, for some $s > 0$, and a unitary $u \in \mathcal{U}(M)$ such that

- *Γ_1 is commensurable to $k\Sigma_1 k^{-1}$, for some $k \in \Gamma$, Γ_2 is commensurable to Σ_2,*
- *$P_1^s = uL(\Gamma_1)u^*$ and $P_2^{1/s} = uL(\Gamma_2)u^*$.*

The proof of Theorem III.6.1 relies on several results. Before continuing, we introduce some terminology. Let Γ be a countable group and $\Sigma < \Gamma$ be a subgroup. Following [CdSS15], we denote by $\mathcal{O}_\Sigma(g) = \{hgh^{-1} | h \in \Sigma\}$ the orbit of $g \in \Gamma$ under the conjugation action of Σ. Note that $\mathcal{O}_\Sigma(g_1 g_2) \subset \mathcal{O}_\Sigma(g_1)\mathcal{O}_\Sigma(g_2)$, thus $|\mathcal{O}_\Sigma(g_1 g_2)| \leq |\mathcal{O}_\Sigma(g_1)||\mathcal{O}_\Sigma(g_2)|$, for all $g_1, g_2 \in \Gamma$.

Therefore, the set $\Delta = \{g \in \Gamma \mid \mathcal{O}_\Sigma(g) \text{ is finite}\}$ is a subgroup of Γ. Moreover, we note that $L(\Sigma)' \cap L(\Gamma) \subset L(\Delta)$.

III.6.1 From commuting subalgebras to almost commuting subgroups

The first step towards proving Theorem III.6.1 is to show the existence of conjugates of finite index subgroups of Σ_1, Σ_2 that "almost" commute, in the sense that they have finite commutator.

Theorem III.6.2. *Let Γ be a countable group and $\Sigma_1, \Sigma_2 < \Gamma$ be two subgroups. Assume that we have $L(\Sigma_1) \prec_{L(\Gamma)} L(\Sigma_2)' \cap L(\Gamma)$.*

Then we can find finite index subgroups $\Omega_1 < k\Sigma_1 k^{-1}$ and $\Omega_2 < \Sigma_2$, for some $k \in \Gamma$, such that the group $[\Omega_1, \Omega_2]$ generated by all commutators $[g,h] = g^{-1}h^{-1}gh$ with $g \in \Omega_1, h \in \Omega_2$, is finite and satisfies $[\Omega_1, \Omega_2] \subset C_\Gamma(\Omega_1) \cap C_\Gamma(\Omega_2)$.

Remark III.6.3. We do not know whether the following more natural, stronger conclusion holds: there exist finite index commuting subgroups $\Omega_1 < k\Sigma_1 k^{-1}$ and $\Omega_2 < \Sigma_2$, for some $k \in \Gamma$. Note, however, that Lemma III.6.4 below implies that this is the case if Σ_1 is finitely generated.

The proof of Theorem III.6.2 relies on the following lemma inspired by [CdSS15, Claims 4.9-4.11].

Lemma III.6.4. *Assume the setting of Theorem III.6.2. Let $\Delta = \{g \in \Gamma \mid \mathcal{O}_{\Sigma_2}(g) \text{ is finite}\}$.*

Then we can find a finite index subgroup $\Omega_1 < \Sigma_1$ and $k \in \Gamma$ such that $k\Omega_1 k^{-1} \subset \Delta$ and $L(k\Omega_1 k^{-1}) \prec_{L(\Delta)} L(\Sigma_2)' \cap L(\Gamma)$.

Proof. Since $L(\Sigma_1) \prec_{L(\Gamma)} L(\Sigma_2)' \cap L(\Gamma)$, by Theorem I.2.6 we can find $k_1, ..., k_n$,

$l_1,...,l_n \in \Gamma$ and a constant $\delta > 0$ such that

$$\sum_{i=1}^{n} \|E_{L(\Sigma_2)'\cap L(\Gamma)}(u_{k_i}u_g u_{l_i})\|_2^2 \geq \delta, \quad \text{for every } g \in \Sigma_1. \tag{III.6.1}$$

If $g \in \Gamma$, then $E_{L(\Sigma_2)'\cap L(\Gamma)}(u_g)$ is equal to $\frac{1}{|\mathcal{O}_{\Sigma_2}(g)|}\sum_{h\in\mathcal{O}_{\Sigma_2}(g)} u_h$, if $g \in \Delta$, and to 0, otherwise. Thus, we have $\|E_{L(\Sigma_2)'\cap L(\Gamma)}(u_g)\|_2^2 = \frac{1}{|\mathcal{O}_{\Sigma_2}(g)|}$, where we make the convention that $\frac{1}{\infty} = 0$.

Let $c = \frac{n}{\delta}$ and define $S = \{g \in \Gamma | \, |\mathcal{O}_{\Sigma_2}(g)| \leq c\}$. By using III.6.1 we get that for any $g \in \Sigma_1$, there is $i \in \{1,...,n\}$ such that $|\mathcal{O}_{\Sigma_2}(k_i g l_i)| \leq c$. Hence, we have $\Sigma_1 \subset \cup_{i=1}^n k_i^{-1} S l_i^{-1}$. For $i \in \{1,...,n\}$, let $a_i \in \Sigma_1 \cap k_i^{-1} S l_i^{-1}$, if $\Sigma_1 \cap k_i^{-1} S l_i^{-1}$ is non-empty, and let $a_i = e$, otherwise.

Since $S \subset \Delta$, we get that $\Sigma_1 \subset \cup_{i=1}^n (k_i^{-1}\Delta k_i)a_i$. This implies that at least one of the groups $\Sigma_1 \cap k_i^{-1}\Delta k_i$, with $1 \leq i \leq n$, has finite index in Σ_1. After renumbering, we find $m \in \{1,...,n\}$ such that the index $[\Sigma_1 : \Sigma_1 \cap k_i^{-1}\Delta k_i]$ is finite, for all $1 \leq i \leq m$, and infinite, for all $m+1 \leq i \leq n$.

Define $\Omega_1 = \cap_{i=1}^m (\Sigma_1 \cap k_i^{-1}\Delta k_i)$. Then Ω_1 has finite index in Σ_1, and $\Omega_1 = \cup_{i=1}^n (\Omega_1 \cap k_i^{-1} S l_i^{-1})$. For $1 \leq i \leq n$, let $b_i \in \Omega_1 \cap k_i^{-1} S l_i^{-1}$, if $\Omega_1 \cap k_i^{-1} S l_i^{-1}$ is non-empty, and $b_i = e$, otherwise. If $i \leq m$, then since $b_i \in \Omega_1 \subset k_i^{-1}\Delta k_i$, we get $k_i b_i k_i^{-1} \in \Delta$, or equivalently $|\mathcal{O}_{\Sigma_2}(k_i b_i k_i^{-1})| < \infty$. Let $0 < d \leq 1$ be a constant such that $d \leq \frac{1}{c^2|\mathcal{O}_{\Sigma_2}(k_i b_i k_i^{-1})|}$, for every $1 \leq i \leq m$.

Next, fix $g \in \Omega_1$. Then $g \in \Omega_1 \cap k_i^{-1} S l_i^{-1}$, for some $1 \leq i \leq n$. Thus, $g b_i^{-1} \in k_i^{-1} S S^{-1} k_i$ and hence $g b_i^{-1} \in \Omega_1 \cap k_i^{-1}\Delta k_i$. Moreover, since $k_i g b_i^{-1} k_i^{-1} \in SS^{-1}$, we get that $k_i g b_i^{-1} k_i^{-1} \in \Delta$ and that $|\mathcal{O}_{\Sigma_2}(k_i g b_i^{-1} k_i^{-1})| \leq c^2$. Now, if $i \leq m$, then $|\mathcal{O}_{\Sigma_2}(k_i g k_i^{-1})| \leq c^2 |\mathcal{O}_{\Sigma_2}(k_i b_i k_i^{-1})| \leq \frac{1}{d}$, hence $\|E_{L(\Sigma_2)'\cap L(\Gamma)}(u_{k_i g k_i^{-1}})\|_2^2 \geq d$. Altogether, since $d \leq 1$, we conclude that

$$\sum_{i=1}^{m} \|E_{L(\Sigma_2)'\cap L(\Gamma)}(u_{k_i g k_i^{-1}})\|_2^2 + \sum_{i=m+1}^{n} \|E_{L(\Omega_1 \cap k_i^{-1}\Delta k_i)}(u_g u_{b_i}^*)\|_2^2 \geq d, \quad \text{for every } g \in \Omega_1. \tag{III.6.2}$$

Since $k_i \Omega_1 k_i^{-1} \subset \Delta$ and $b_i \in \Omega_1$, Remark III.2.2 implies that either $L(k_i \Omega_1 k_i^{-1}) \prec_{L(\Delta)} L(\Sigma_2)'\cap L(\Gamma)$, for some $1 \leq i \leq m$, or that $L(\Omega_1) \prec_{L(\Omega_1)} L(\Omega_1 \cap k_i^{-1}\Delta k_i)$, for some $m+1 \leq i \leq n$.

The latter is however impossible by Lemma III.2.4(1) since the inclusion $\Omega_1 < \Sigma_1$ has finite index and thus the inclusion $\Omega_1 \cap k_i^{-1}\Delta k_i < \Omega_1$ has infinite index, for every $m+1 \le i \le n$. This proves the lemma. ■

Proof of Theorem III.6.2. Let $\Delta = \{g \in \Gamma \mid \mathcal{O}_{\Sigma_2}(g) \text{ is finite}\}$. By Lemma III.6.4, we can find a finite index subgroup $\Omega_1 < k\Sigma_1 k^{-1}$, for some $k \in \Gamma$, such that $\Omega_1 \subset \Delta$ and $L(\Omega_1) \prec_{L(\Delta)} L(\Sigma_2)' \cap L(\Gamma)$. We continue with the following claim. If $A \subset pL(\Gamma)p$ and $B \subset L(\Gamma)$ are von Neumann subalgebras, then we write $A \subset_\varepsilon B$ if $\|a - E_B(a)\|_2 \le \varepsilon$, for every $a \in A$ with $\|a\| \le 1$.

Claim. There exists a non-zero projection $z \in L(\Omega_1)' \cap L(\Delta)$ with the following property: for every $\varepsilon > 0$ we can find a finite index subgroup $\Omega_2 < \Sigma_2$ such that $L(\Omega_1)z \subset_\varepsilon L(\Omega_2)' \cap L(\Gamma)$.

Proof of the claim. By Theorem I.2.6 we can find projections $p \in L(\Omega_1), q \in L(\Sigma_2)' \cap L(\Gamma)$, a non-zero partial isometry $v \in qL(\Delta)p$, and a $*$-homomorphism $\theta : pL(\Omega_1)p \to q(L(\Sigma_2)' \cap L(\Gamma))q$ such that $vx = \theta(x)v$, for every $x \in pL(\Omega_1)p$. Since $v^*v \in (pL(\Omega_1)p)' \cap pL(\Delta)p$, we can find a projection $p' \in L(\Omega_1)' \cap L(\Delta)$ such that $v^*v = pp'$. Let $p'' \in \mathcal{Z}(L(\Omega_1))$ be the central support of p.

We will prove that $z = p''p'$ satisfies the claim. To this end, fix $\varepsilon > 0$ and $x \in L(\Omega_1)$ with $\|x\| \le 1$. Let $v_i \in L(\Omega_1)$ be partial isometries such that $p'' = \sum_{i\ge1} v_i v_i^*$ and $v_i^* v_i \le p$, for every $i \ge 1$. Let $n \ge 1$ such that $\|p'' - \sum_{i=1}^n v_i v_i^*\|_2 \le \frac{\varepsilon}{4}$. Then

$$\|xp''p' - \sum_{i,j=1}^n v_i v_i^* x v_j v_j^* p'\|_2 \le \|xp'' - \sum_{i,j=1}^n v_i v_i^* x v_j v_j^*\|_2 \le \frac{\varepsilon}{2}. \tag{III.6.3}$$

On the other hand, using that v_j and p' commute, for every j, that $v_i^* x v_j \in pL(\Omega_1)p$, for every i, j, and that $yp' = v^*\theta(y)v$, for every $y \in pL(\Omega_1)p$, we derive that

$$\sum_{i,j=1}^n v_i v_i^* x v_j v_j^* p' = \sum_{i,j=1}^n v_i v_i^* x v_j p' v_j^* = \sum_{i,j=1}^n v_i v^* \theta(v_i^* x v_j) v v_j^*.$$

Now, if $g \in \Delta$, then $\mathcal{O}_{\Sigma_2}(g)$ is finite, hence g commutes with a finite index subgroup of Σ_2. Therefore, any finite subset of Δ commutes with some finite index subgroup of Σ_2. This implies that for every $y \in L(\Delta)$ and $\delta > 0$, we can find a finite index subgroup $\Omega_2 < \Sigma_2$ such that $\|y - E_{L(\Omega_2)' \cap L(\Delta)}(y)\|_2 \le \delta$.

Thus, there is a finite index subgroup $\Omega_2 < \Sigma_2$ such that $\|v_i v^* - E_{L(\Omega_2)' \cap L(\Delta)}(v_i v^*)\|_2 \le \frac{\varepsilon}{4n^2}$, for all $1 \le i \le n$. Using these inequalities and the last displayed formula, it follows that

$$\|\sum_{i,j=1}^{n} v_i v_i^* x v_j v_j^* p' - \sum_{i,j=1}^{n} E_{L(\Omega_2)' \cap L(\Delta)}(v_i v^*) \theta(v_i^* x v_j) E_{L(\Omega_2)' \cap L(\Delta)}(v_j v^*)^* \|_2 \le \frac{\varepsilon}{2}. \qquad \text{(III.6.4)}$$

Since $\sum_{i,j=1}^{n} E_{L(\Omega_2)' \cap L(\Delta)}(v_i v^*) \theta(v_i^* x v_j) E_{L(\Omega_2)' \cap L(\Delta)}(v_j v^*)^*$ belongs to $L(\Omega_2)' \cap L(\Delta)$, by combining III.6.3 and III.6.4 we deduce that $\|x p'' p' - E_{L(\Omega_2)' \cap L(\Delta)}(x p'' p')\|_2 \le \varepsilon$. Since $x \in L(\Omega_1)$ with $\|x\| \le 1$ is arbitrary, the claim follows. $\square$

Now, write $z = \sum_{g \in \Delta} c_g u_g$, where $c_g \in \mathbb{C}$. Let $\alpha = \max_{g \in \Delta} |c_g|$ and put $F = \{g \in \Delta | \, |c_g| = \alpha\}$. Then F is a finite set, and there is $\varepsilon > 0$ such that if $k \in \Gamma$ satisfies $\|u_k z - z\|_2 < \varepsilon$, then $kF = F$. Indeed, one can check that $\varepsilon = \alpha - \beta$, where $\beta = \max_{g \in \Delta \setminus F} |c_g|$, works.

The claim gives a finite index subgroup $\Omega_2 < \Sigma_2$ such that $L(\Omega_1) z \subset_{\frac{\varepsilon}{4}} L(\Omega_2)' \cap L(\Gamma)$. As $z \in L(\Delta)$, after replacing Ω_2 with a finite index subgroup, we may assume that $\|z - E_{L(\Omega_2)' \cap L(\Delta)}(z)\|_2 < \frac{\varepsilon}{4}$. Let $g \in \Omega_1$ and $h \in \Omega_2$. Since $\|u_g z - E_{L(\Omega_2)' \cap L(\Delta)}(u_g z)\|_2 \le \frac{\varepsilon}{4}$, we get that $\|u_g z - u_h (u_g z) u_h^*\|_2 \le \frac{\varepsilon}{2}$. Since $\|z - E_{L(\Omega_2)' \cap L(\Delta)}(z)\|_2 < \frac{\varepsilon}{4}$, we also have that $\|z u_h^* - u_h^* z\|_2 < \frac{\varepsilon}{2}$. Altogether, we deduce that $\|u_g z - u_h u_g u_h^* z\|_2 < \varepsilon$, hence $\|z - u_{g^{-1} h g h^{-1}} z\|_2 < \varepsilon$. By the previous paragraph, this implies that $g^{-1} h g h^{-1} F = F$, for every $g \in \Omega_1$ and $h \in \Omega_2$.

Therefore, $[\Omega_1, \Omega_2]$ is finite and contained in the group $\langle F \rangle$ generated by F. Since $z \in L(\Omega_1)' \cap L(\Delta)$ and $F \subset \Delta$, after replacing Ω_1, Ω_2 with finite index subgroups, we may assume that they commute with F. Thus, $[\Omega_1, \Omega_2]$ is finite and $[\Omega_1, \Omega_2] \subset \langle F \rangle \subset C_\Gamma(\Omega_1) \cap C_\Gamma(\Omega_2)$. This finishes the proof. ■

III.6.2 Finite index commensurator

The next step towards proving Theorem III.6.1 is to show that Σ_i is commensurated by a finite index subgroup of Γ, for every $i \in \{1, 2\}$.

Lemma III.6.5. *Let Γ be a countable icc group, denote $M = L(\Gamma)$, and assume that $M = P_1 \overline{\otimes} P_2$. Let $\Sigma < \Gamma$ be a subgroup such that $P_1 \prec^s_M L(\Sigma)$ and $L(\Sigma) \prec^s_M P_1$. Let $\Gamma_0 < \Gamma$ be the subgroup of $g \in \Gamma$ such that Σ and $g\Sigma g^{-1}$ are commensurable.*

Then $[\Gamma : \Gamma_0] < \infty$.

Proof. The proof is inspired by [CdSS15, Claims 4.5 and 4.6]. Let $\Delta = \{g \in \Gamma | \ \mathcal{O}_\Sigma(g) \text{ is finite}\}$. Then $\Delta \subset \Gamma_0$, hence $\Sigma\Delta \subset \Gamma_0$. Indeed, if $k \in \Delta$, then k commutes with a finite index subgroup of Σ, hence $k \in \Gamma_0$. Since $L(\Sigma)' \cap M \subset L(\Delta)$, we have $L(\Sigma) \vee (L(\Sigma)' \cap M) \subset L(\Sigma\Delta) \subset L(\Gamma_0)$. Lemma III.2.4(1) implies that in order to reach the conclusion it is sufficient to prove that

$$M \prec L(\Sigma) \vee (L(\Sigma)' \cap M). \tag{III.6.5}$$

Towards proving III.6.5, we denote $Q_1 = L(\Sigma)$. Then the hypothesis gives that $P_1 \prec^s Q_1$ and $Q_1 \prec^s P_1$. By Lemma III.2.3(4), there is a non-zero projection $z \in \mathcal{Z}(Q_1' \cap M)$ such that $P_1 \prec Q_1 q'$ for every non-zero projection $q' \in (Q_1' \cap M)z$. We claim that $(Q_1' \cap M)z \prec^s P_2$. By Lemma III.2.3(2), it suffices to show that $(Q_1' \cap M)z'z \prec P_2$, for any projection $z' \in \mathcal{Z}((Q_1' \cap M)' \cap M)$ such that $z'z \neq 0$. But $z'z \in (Q_1' \cap M)z$, and thus by the above $P_1 \prec Q_1 z'z$. By [Va08, Lemma 3.5] we derive that $(Q_1' \cap M)z'z \prec P_2 = P_1' \cap M$, which proves our claim.

Next, we denote $Q_2 = Q_1' \cap M$. Then $z \in \mathcal{Z}(Q_2)$ is a non-zero projection such that $Q_2 z \prec^s P_2$ and $P_1 \prec Q_1 q'$, for every non-zero projection $q' \in Q_2 z$. Since we also have that $Q_1 z \prec^s P_1$, we get that $\mathcal{Z}(Q_1)z = Q_1 z \cap Q_2 z$ satisfies $\mathcal{Z}(Q_1)z \prec^s P_1$ and $\mathcal{Z}(Q_1)z \prec^s P_2$. By

Lemma III.2.7(2) we deduce that $\mathcal{Z}(Q_1)z \prec^s P_1 \cap P_2 = \mathbb{C}1$, hence $\mathcal{Z}(Q_1)z$ is completely atomic.

Further, since $Q_1 z \prec P_1$, by [Va08, Lemma 3.5] we get that $P_2 = P_1' \cap M \prec Q_2 z$. By arguing as in the second paragraph, we can find a non-zero projection $z' \in \mathcal{Z}((Q_2 z)' \cap zMz) = \mathcal{Z}(Q_2' \cap M)z$ such that $(Q_2' \cap M)z' \prec^s P_1$. Since $\mathcal{Z}(Q_2' \cap M) \subset \mathcal{Z}(Q_2)$, we have that $z' \in \mathcal{Z}(Q_2)z$. Since $Q_2 z' \prec^s P_2$, by arguing as in the previous paragraph, we get that $\mathcal{Z}(Q_2)z'$ is completely atomic.

Thus, $z' \in \mathcal{Z}(Q_2)$ is a non-zero projection such that $\mathcal{Z}(Q_1)z'$ and $\mathcal{Z}(Q_2)z'$ are completely atomic. By shrinking z' we may assume that in fact $\mathcal{Z}(Q_2)z' = \mathbb{C}z'$. Since $\mathcal{Z}(Q_1)z'$ is completely atomic we can find a non-zero projection $f \in \mathcal{Z}(Q_1)z'$ such that $\mathcal{Z}(Q_1)f = \mathbb{C}f$. But then also $\mathcal{Z}(Q_2)f = \mathbb{C}f$. Therefore, $f \in Q_2 = Q_1' \cap M$ is a projection such that both $Q_1 f$ and $fQ_2 f = (Q_1 f)' \cap fMf$ are II_1 factors. Since $Q_1 f \prec P_1$, by [OP03, Proposition 12], we can find a decomposition $fMf = P_1^{t_1} \overline{\otimes} P_2^{t_2}$, for some $t_1, t_2 > 0$ satisfying $t_1 t_2 = \tau(f)$, and a unitary $u \in fMf$ such that $uQ_1 fu^* \subset P_1^{t_1}$.

Since $f \in Q_2 z$ is a non-zero projection, we have that $P_1 \prec Q_1 f$, hence $P_1^{t_1} \prec_{fMf} uQ_1 fu^*$. We claim that $P_1^{t_1} \prec_{P_1^{t_1}} uQ_1 fu^*$. Otherwise we can find a sequence $u_n \in \mathcal{U}(P_1^{t_1})$ such that $\|E_{uQ_1 fu^*}(au_n b)\|_2 \to 0$, for every $a, b \in P_1^{t_1}$. We will show that $\|E_{uQ_1 fu^*}(a_0 u_n b_0)\|_2 \to 0$, for every $a_0, b_0 \in fMf$, contradicting the fact that $P_1^{t_1} \prec_{fMf} uQ_1 fu^*$. Since $fMf = P_1^{t_1} \overline{\otimes} P_2^{t_2}$, we may assume that $a_0 = a_1 \otimes a_2$ and $b_0 = b_1 \otimes b_2$, for $a_1, a_2 \in P_1^{t_1}$ and $b_1, b_2 \in P_2^{t_2}$. Using that $uQ_1 fu^* \subset P_1^{t_1}$, we get that $\|E_{uQ_1 fu^*}(a_0 u_n b_0)\|_2 = \|E_{uQ_1 fu^*}(a_1 u_n b_1 \otimes a_2 b_2)\|_2 = \|E_{uQ_1 fu^*}(a_1 u_n b_1)\|_2 |\tau(a_2 b_2)| \to 0$. This altogether proves that $P_1^{t_1} \prec_{P_1^{t_1}} uQ_1 fu^*$.

This implies that $fMf = P_1^{t_1} \overline{\otimes} P_2^{t_2} \prec_{fMf} uQ_1 fu^* \overline{\otimes} P_2^{t_2}$. Since $P_2^{t_2} \subset (uQ_1 fu^*)' \cap fMf$, we get that $fMf \prec_{fMf} Q_1 f \vee ((Q_1 f)' \cap fMf) = f(Q_1 \vee (Q_1' \cap M))f$, which proves III.6.5 and the lemma. ■

III.6.3 Proof of Theorem III.6.1

The proof Theorem III.6.1 has two main parts.

In the first part of the proof, we construct two commuting icc subgroups $\Omega_1, \Omega_2 < \Gamma$ which are conjugates of finite index subgroups of Σ_1, Σ_2, and satisfy $[\Gamma : \Omega_1\Omega_2] < \infty$ (compare with [CdSS15, Theorem 4.3]).

Since $L(\Sigma_2) \prec P_2$, [Va08, Lemma 3.5] implies that $P_1 \prec L(\Sigma_2)' \cap M$. Since P_1 is regular in M and M is a II$_1$ factor, Lemma III.2.3(3) implies that $P_1 \prec^s L(\Sigma_2)' \cap M$. Since $L(\Sigma_1) \prec P_1$, by combining this with Lemma III.2.3(1) we deduce that $L(\Sigma_1) \prec L(\Sigma_2)' \cap M$.

By applying Theorem III.6.2, we find finite index subgroups $\Omega_1 < k\Sigma_1 k^{-1}$, $\Omega_2 < \Sigma_2$, for some $k \in \Gamma$, such that $[\Omega_1, \Omega_2]$ is finite and contained in $C_\Gamma(\Omega_1) \cap C_\Gamma(\Omega_2)$. If $i \in \{1, 2\}$, then Lemma III.2.4(2) implies that $L(\Omega_i) \prec^s L(\Sigma_i)$ and $L(\Sigma_i) \prec^s L(\Omega_i)$. Since $L(\Sigma_i) \prec^s P_i$ and $P_i \prec^s L(\Sigma_i)$, we conclude that $L(\Omega_i) \prec^s P_i$ and $P_i \prec^s L(\Omega_i)$.

By applying Lemma III.6.5 to Ω_1 we deduce that $[\Gamma : \Gamma_0] < \infty$, where $\Gamma_0 < \Gamma$ is the subgroup of $g \in \Gamma$ such that Ω_1 and $g\Omega_1 g^{-1}$ are commensurable. Since $[\Gamma : \Gamma_0] < \infty$ and Γ is icc, it follows that $\mathcal{O}_{\Gamma_0}(g)$ is infinite, for every $g \in \Gamma \setminus \{e\}$. From this we deduce that $L(\Gamma_0)' \cap M = \mathbb{C}1$. Using that $P_1 \prec L(\Omega_1)$ and $P_2 \prec L(\Omega_2)$, we find non-zero elements $v, v_1, ..., v_m, w, w_1, ..., w_m \in M$ such that

$$(P_1)_1 v \subset \sum_{i=1}^{m} v_i (L(\Omega_1))_1 \quad \text{and} \quad w(P_2)_1 \subset \sum_{i=1}^{m} (L(\Omega_2))_1 w_i. \tag{III.6.6}$$

We claim that we can find $g \in \Gamma_0$ such that $vu_g w \neq 0$. Indeed, otherwise we would get that $u_g^* v^* v u_g w w^* = 0$, for every $g \in \Gamma_0$. Thus, if K denotes the $\|.\|_2$-closure of the convex hull of $\{u_g^* v^* v u_g | g \in \Gamma_0\}$, then $\xi w w^* = 0$, for all $\xi \in K$. Let $\eta \in K$ be the unique element of minimal $\|.\|_2$. Since the map $K \ni \xi \mapsto u_h^* \xi u_h \in K$ preserves $\|.\|_2$, we get that $u_h^* \eta u_h = \eta$, for all $h \in \Gamma_0$. Thus, $\eta \in L(\Gamma_0)' \cap M = \mathbb{C}1$ and since $\tau(\eta) = \tau(v^* v)$, we deduce that $\eta = \tau(v^* v)1$. But this implies that $0 = \eta w w^* = \tau(v^* v) w w^*$, contradicting that both v and w are non-zero. This proves the claim.

Next, since $[\Omega_1 : \Omega_1 \cap g\Omega_1 g^{-1}] < \infty$, we can find $g_1, ..., g_n \in \Gamma$ such that $\Omega_1 g \subset \cup_{j=1}^n g_j \Omega_1$ and thus $(L(\Omega_1))_1 u_g \subset \sum_{j=1}^n u_{g_j}(L(\Omega_1))_1$. By combining this inclusion with equation III.6.6 we get that

$$(P_1)_1(vu_g w)(P_2)_1 \subset \sum_{i=1}^m \sum_{j=1}^n v_i u_{g_j}(L(\Omega_1))_1 (L(\Omega_2))_1 w_i. \qquad \text{(III.6.7)}$$

Thus, if we denote by $\Omega < \Gamma$ the subgroup generated by Ω_1 and Ω_2, then III.6.7 implies that

$$\mathcal{U}(P_1)\ (vu_g w)\ \mathcal{U}(P_2) \subset \sum_{i=1}^m \sum_{j=1}^n v_i u_{g_j}(L(\Omega))_1 w_i. \qquad \text{(III.6.8)}$$

Let us show that $[\Gamma : \Omega] < \infty$. Otherwise, if $[\Gamma : \Omega] = \infty$, Lemma III.2.4(1) implies that $M \nprec L(\Omega)$. Since the group of unitaries $\{u_1 \otimes u_2 | u_1 \in \mathcal{U}(P_1), u_2 \in \mathcal{U}(P_2)\}$ generates M, by Theorem I.2.6 we can find a sequence $u_n = u_{n,1} \otimes u_{n,2}$, with $u_{n,1} \in \mathcal{U}(P_1)$ and $u_{n,2} \in \mathcal{U}(P_2)$, such that $\|E_{L(\Omega)}(au_n b)\|_2 \to 0$, for every $a, b \in M$. We claim that $\|E_{L(\Omega)}(au_{n,1} bu_{n,2} c)\|_2 \to 0$, for every $a, b, c \in M$. Since this claim contradicts equation III.6.8, we conclude that the assumption $[\Gamma : \Omega] = \infty$ is false. To prove this claim, we may assume that $a = a_1 \otimes a_2, b = b_1 \otimes b_2, c = c_1 \otimes c_2$, where $a_1, b_1, c_1 \in P_1$ and $a_2, b_2, c_2 \in P_2$. But in this case $au_{n,1} bu_{n,2} c = a_1 u_{n,1} b_1 c_1 \otimes a_2 b_2 u_{n,2} c_2 = (a_1 \otimes a_2 b_2) u_n (b_1 c_1 \otimes c_2)$, and therefore $\|E_{L(\Omega)}(au_{n,1} bu_{n,2} c)\|_2 \to 0$ by the above.

Since Γ is icc and $[\Gamma : \Omega] < \infty$, we get that Ω is icc. On the other hand, $[\Omega_1, \Omega_2]$ is a finite central subgroup of Ω. Thus, we must have $[\Omega_1, \Omega_2] = \{e\}$, or, in other words, Ω_1 and Ω_2 commute. Moreover, since Γ is icc, it follows that both Ω_1 and Ω_2 are icc.

In the second part of the proof, we derive the conclusion by repeating almost verbatim part of the proof of [CdSS15, Theorem 4.14]. Nevertheless, we include details for the reader's convenience.

Since $L(\Omega_1)$ is a II_1 factor and $L(\Omega_1) \prec P_1$, by applying [OP03, Proposition 12], we can find a decomposition $M = P_1^t \overline{\otimes} P_2^{1/t}$, for some $t > 0$, and a non-zero partial isometry

$v \in M$ such that $vv^* \in P_2^{1/t}, v^*v \in L(\Omega_1)' \cap M$, and

$$vL(\Omega_1)v^* \subset P_1^t vv^*. \tag{III.6.9}$$

Next, let $H_2 \subset \Gamma$ be the subgroup of $g \in \Gamma$ for which $\mathcal{O}_{\Omega_1}(g)$ is finite. Then $H_2 \supset \Omega_2$ and since Ω_1 is icc, we get that $H_2 \cap \Omega_1\Omega_2 = \Omega_2$. Using that $[\Gamma : \Omega_1\Omega_2] < \infty$, we deduce that $[H_2 : \Omega_2] < \infty$. Let $g_1, ..., g_n \in H_2$ such that $H_2 = \cup_{i=1}^n \Omega_2 g_i$. Since $C_{\Omega_1}(g_i) < \Omega_1$ is a finite index subgroup, for every $i \in \{1, ..., n\}$, we derive that $H_1 := C_{\Omega_1}(H_2) = \cap_{i=1}^n C_{\Omega_1}(g_i)$ is a finite index subgroup of Ω_1. Since $[\Omega_1\Omega_2 : H_1\Omega_2] \leq [\Omega_1 : H_1] < \infty$ and $H_1\Omega_2 \subset H_1H_2$, we get that $[\Gamma : H_1H_2] < \infty$. In particular, it follows that the commuting subgroups $H_1, H_2 < \Gamma$ are icc.

Since $H_1 \subset \Omega_1$, by equation III.6.9 we get that $vL(H_1)v^* \subset P_1^t vv^*$. Since $L(\Omega_1)' \cap M \subset L(H_2)$, we also get that $v^*v \in L(H_2)$. Note that $L(H_2)$ is a II_1 factor and $L(H_2) \subset L(H_1)' \cap M$. By combining these facts and proceeding as in the last paragraph of the proof of [OP03, Proposition 12] (see also the proof of [CdSS15, Theorem 4.14]), we find a unitary $u \in M$ such that

$$uL(H_1)u^* \subset P_1^t. \tag{III.6.10}$$

Let $\Gamma_2 < \Gamma$ be the subgroup of $g \in \Gamma$ for which $\mathcal{O}_{H_1}(g)$ is finite. By repeating the argument from above it follows that Γ_2 is icc, $[\Gamma_2 : H_2] < \infty$, $[H_1 : C_{H_1}(\Gamma_2)] < \infty$, and $[\Gamma : C_{H_1}(\Gamma_2)\Gamma_2] < \infty$. Since $L(H_1)' \cap M \subset L(\Gamma_2)$, equation III.6.10 implies that

$$uL(\Gamma_2)u^* \supset P_2^{1/t}. \tag{III.6.11}$$

Since $L(\Gamma_2)$ is a II_1 factor, by using III.6.11 and applying [Ge95, Theorem A], we find a factor $A \subset P_1^t$ such that $uL(\Gamma_2)u^* = A\overline{\otimes}P_2^{1/t}$. Since $[\Gamma_2 : H_2] < \infty$ and $[H_2 : \Omega_2] < \infty$, we have that $[\Gamma_2 : \Omega_2] < \infty$. In particular, we conclude that Γ_2 and Σ_2 are commensurable.

Using that $L(\Omega_2) \prec P_2$, we get that $L(\Gamma_2) \prec P_2$, hence $A \prec P_2$. In combination with $A \subset P_1^t$, this implies that A is not diffuse. Since A is a factor, it must be finite dimensional, hence $A = \mathbb{M}_k(\mathbb{C})$, for some $k \geq 1$. Denoting $s = t/k$, we obtain a decomposition $M = P_1^s \overline{\otimes} P_2^{1/s}$ such that

$$uL(\Gamma_2)u^* = P_2^{1/s}. \tag{III.6.12}$$

Finally, let $\Gamma_1 < \Gamma$ be the subgroup of $g \in \Gamma$ for which $\mathcal{O}_{\Gamma_2}(g)$ is finite. Then $C_{H_1}(\Gamma_2) \subset \Gamma_1$, and since Γ_2 is icc we have that $\Gamma_1 \cap C_{H_1}(\Gamma_2)\Gamma_2 \subset C_{H_1}(\Gamma_2)$. Using that $[\Gamma : C_{H_1}(\Gamma_2)\Gamma_2] < \infty$, we get that $[\Gamma_1 : C_{H_1}(\Gamma_2)] < \infty$. In combination with $[k\Sigma_1 k^{-1} : \Omega_1] < \infty$, $[\Omega_1 : H_1] < \infty$ and $[H_1 : C_{H_1}(\Gamma_2)] < \infty$, this implies that Γ_1 and $k\Sigma_1 k^{-1}$ are commensurable.

Using III.6.12, we get that $P_1^s = u(L(\Gamma_2)' \cap M)u^* \subset uL(\Gamma_1)u^*$. By applying [Ge95, Theorem A] again, we find a von Neumann subalgebra $B \subset P_2^{1/s}$ such that $uL(\Gamma_1)u^* = P_1^s \overline{\otimes} B$. Since Γ_2 is icc, we get that $B = uL(\Gamma_1)u^* \cap uL(\Gamma_2)u^* = uL(\Gamma_1 \cap \Gamma_2)u^* = \mathbb{C}1$. Therefore, we have that

$$uL(\Gamma_1)u^* = P_1^s. \tag{III.6.13}$$

It is now clear that III.6.12 and III.6.13 imply that $\Gamma = \Gamma_1 \times \Gamma_2$, which finishes the proof. ∎

III.7 Proofs of main results

In this section we prove Theorems D and F, and Corollary G.

III.7.1 A strengthening of Theorem F

We establish the following strengthening of Theorem F. This result will also be used to derive Theorem D.

Theorem III.7.1. *Let Γ be a countable icc group and assume that Γ is measure equivalent to a product $\Lambda = \Lambda_1 \times \dots \times \Lambda_n$ of $n \geq 1$ groups $\Lambda_1, \dots, \Lambda_n$ which belong to $\mathcal{C}_{rss}$. Assume the* *notation from III.5.1. Suppose that* $L(\Gamma) = P_1 \overline{\otimes} P_2$*, for some* II_1 *factors* P_1 *and* P_2*.*

Then there exist a decomposition $\Gamma = \Gamma_1 \times \Gamma_2$*, a partition* $S_1 \sqcup S_2 = \{1, ..., n\}$*, a decomposition* $L(\Gamma) = P_1^t \overline{\otimes} P_2^{1/t}$*, for some* $t > 0$*, and a unitary* $u \in L(\Gamma)$ *such that*

1. $P_1^t = uL(\Gamma_1)u^*$ *and* $P_2^{1/t} = uL(\Gamma_2)u^*$*,*

2. $A \rtimes \Gamma_i \prec_M^s B \rtimes \Lambda_{S_i}$*,* $B \rtimes \Lambda_{S_i} \prec_M^s A \rtimes \Gamma_i$ *for every* $i \in \{1, 2\}$*, and*

3. Γ_i *is measure equivalent to* Λ_{S_i}*, for every* $i \in \{1, 2\}$*.*

Proof. By applying Theorem III.5.3 we find subgroups $\Sigma_1, \Sigma_2 < \Gamma$ and a partition $S_1 \sqcup S_2 = \{1, ..., n\}$ such that the following conditions hold for all $i \in \{1, 2\}$:

(a) $P_i \prec_{L(\Gamma)}^s L(\Sigma_i)$, $L(\Sigma_i) \prec_{L(\Gamma)}^s P_i$,

(b) $A \rtimes \Sigma_i \prec_M^s B \rtimes \Lambda_{S_i}, B \rtimes \Lambda_{S_i} \prec_M^s A \rtimes \Sigma_i$, and

(c) Σ_i is measure equivalent to Λ_{S_i}.

Further, by using (a), Theorem III.6.1 provides decompositions $\Gamma = \Gamma_1 \times \Gamma_2$ and $L(\Gamma) = P_1^s \overline{\otimes} P_2^{1/s}$, for some $s > 0$, and a unitary $u \in L(\Gamma)$ such that Γ_1 is commensurable to $k\Sigma_1 k^{-1}$, for some $k \in \Gamma$, Γ_2 is commensurable to Σ_2, and condition (1) is satisfied. It is clear that (b) implies (2). Finally, since commensurable groups are measure equivalent, we deduce that Γ_i is measure equivalent to Σ_i hence to Λ_{S_i}, for all $i \in \{1, 2\}$. This shows that condition (3) also holds and finishes the proof. ∎

III.7.2 Proof of Theorem F

Since non-elementary hyperbolic groups belong to $\mathcal{C}_{\text{rss}}$ by [PV12], Theorem F follows from Theorem III.7.1. ∎

III.7.3 Proof of Theorem D

By Remark III.1.2(1), any irreducible lattice in a product of connected non-compact rank one simple Lie groups with finite center belongs to $\mathcal{L}$. Thus, it suffices to prove the second assertion of Theorem D.

Let $\Gamma \in \mathcal{L}$ be an icc group and assume by contradiction that the II_1 factor $L(\Gamma)$ is not prime. Then Γ is an irreducible lattice in a product $G = G_1 \times ... \times G_n$ of $n \geq 1$ locally compact groups, each admitting a non-elementary hyperbolic lattice $\Lambda_j < G_j$, and not all admitting an open normal compact subgroup. Moreover, Γ does not contain a non-trivial element which commutes with an open subgroup of G. Denote $\Lambda = \Lambda_1 \times ... \times \Lambda_n$. Then $\Lambda < G$ is also a lattice, and hence Γ and Λ are measure equivalent. Since non-elementary hyperbolic groups belong to $\mathcal{C}_{\text{rss}}$ by [PV12], we deduce that Γ satisfies the hypothesis of Theorem III.7.1.

To get a contradiction we will apply Theorem III.7.1. We begin by defining a concrete stable orbit equivalence between certain actions of Γ and Λ. Let m be a fixed Haar measure of G, consider the left-right translation action $\Gamma \times \Lambda \curvearrowright (G, m)$ given by $(g,h) \cdot x = gxh^{-1}$, and put $\mathcal{R} = \mathcal{R}(\Gamma \times \Lambda \curvearrowright G)$.

Let $X = G/\Lambda$ and $Y = \Gamma\backslash G$, endowed with left and right translation actions of G, and the unique G-invariant probability measures m_X and m_Y. Let $p : X \to G$ and $q : Y \to G$ be Borel maps such that $p(x) \in x\Lambda$ and $q(y) \in \Gamma y$, for all $x \in X, y \in Y$. Let $\ell \geq 1$ such that $\ell\, m(q(Y)) \geq m(p(X))$. Let $\{X_j\}_{1\leq j\leq \ell}$ be a measurable partition of X such that $m(p(X_j)) \leq m(q(Y))$, for every $1 \leq j \leq \ell$. Since $\mathcal{R}$ is ergodic, we can find $\{\theta_j\}_{1\leq j\leq \ell} \subset [\mathcal{R}]$ such that $\theta_j(p(X_j)) \subset q(Y)$, for every $1 \leq j \leq \ell$. Let $\alpha_j : G \to \Gamma, \beta_j : G \to \Lambda$ be Borel maps such that $\theta_j(x) = \alpha_j(x) x \beta_j(x)$, for almost every $x \in G$.

We define $\iota : X \to Y \times \mathbb{Z}/\ell\mathbb{Z}$ by letting

$$\iota(x) = (\Gamma\theta_j(p(x)), j + \ell\mathbb{Z}), \text{ if } x \in X_j, \text{ for some } 1 \leq j \leq \ell.$$

We view X as a subset of $Y \times \mathbb{Z}/\ell\mathbb{Z}$ by identifying it with $\iota(X)$. Fix $x_1, x_2 \in X$ and let $1 \le j_1, j_2 \le \ell$ such that $x_1 \in X_{j_1}, x_2 \in X_{j_2}$. Then $x_1 \in \Gamma x_2$ iff $p(x_1) \in \Gamma p(x_2)\Lambda$ iff $\theta_{j_1}(p(x_1)) \in \Gamma\theta_{j_2}(p(x_2))\Lambda$ iff $\Gamma\theta_{j_1}(p(x_1)) \in (\Gamma\theta_{j_2}(p(x_2)))\Lambda$. Thus, if $\mathbb{Z}/\ell\mathbb{Z}$ acts on itself by addition, then

$$\mathcal{R}(\Gamma \curvearrowright X) = \mathcal{R}(\Lambda \times \mathbb{Z}/\ell\mathbb{Z} \curvearrowright Y \times \mathbb{Z}/\ell\mathbb{Z})|_X.$$

Since Γ does not contain a non-trivial element which commutes with an open subgroup of G, it is easy to see that the actions $\Gamma \curvearrowright (X, \mu_X)$ and $\Lambda \curvearrowright (Y, \mu_Y)$ are free.

We are therefore in the situation from III.5.1, so we may assume the notation introduced therein: $A = L^\infty(X), B = L^\infty(Y) \otimes \mathbb{M}_\ell(\mathbb{C})$, $M = L^\infty(Y \times \mathbb{Z}/\ell\mathbb{Z}) \rtimes (\Lambda \times \mathbb{Z}/\ell\mathbb{Z}) = B \rtimes \Lambda$. We denote by $\{u_g\}_{g\in\Gamma} \subset A \rtimes \Gamma$ and $\{v_h\}_{h\in\Lambda} \subset M$ the canonical unitaries. Additionally, we let $\Lambda_S = \times_{i\in S}\Lambda_i$, $G_S = \times_{i\in S} G_i$, and $\pi_S : G \to G_S$ denote the canonical projection, for every subset $S \subset \{1, ..., n\}$.

Since $L(\Gamma)$ is not prime, Theorem III.7.1 implies that we can find a decomposition $\Gamma = \Gamma_1 \times \Gamma_2$, with Γ_1 and Γ_2 icc, and a partition $S_1 \sqcup S_2 = \{1, ..., n\}$ such that $A \rtimes \Gamma_i \prec^s_M B \rtimes \Lambda_{S_i}$, for all $i \in \{1, 2\}$. The rest of the proof relies on the following:

Claim. The subgroups $\overline{\pi_{S_1}(\Gamma_2)} \subset G_{S_1}$ and $\overline{\pi_{S_2}(\Gamma_1)} \subset G_{S_2}$ are compact.

Proof of the claim. By symmetry, it suffices to prove the first assertion. Assume by contradiction that $\overline{\pi_{S_1}(\Gamma_2)}$ is not compact. Then we can find a sequence $g_n \in \Gamma_2$ such that $\pi_{S_1}(g_n) \to \infty$, as $n \to \infty$, in G_{S_1}. We claim that

$$\|E_{B\rtimes\Lambda_{S_2}}(u_{g_n} v_k^*)\|_2 \to 0, \text{ for every } k \in \Lambda_{S_1}. \tag{III.7.1}$$

Since $E_{B\rtimes\Lambda_{S_2}}$ is $B \rtimes \Lambda_{S_2}$-bimodular and M is generated by $B \rtimes \Lambda_{S_2}$ together with the unitaries $\{v_k \mid k \in \Lambda_{S_1}\}$ that normalize it, claim III.7.1 readily implies that $\|E_{B\rtimes\Lambda_{S_2}}(a u_{g_n} b)\|_2 \to 0$, for every $a, b \in M$, which contradicts that $A \rtimes \Gamma_2 \prec_M B \rtimes \Lambda_{S_2}$.

For $1 \le j \le \ell$, let $e_j \in L^\infty(X)$ denote the characteristic function of X_j. Since

$\sum_{1\le j\le \ell} e_j = 1_X$, claim III.7.1 reduces to proving

$$\|E_{B\rtimes\Lambda_{S_2}}(e_{j_1}u_{g_n}e_{j_2}v_k^*)\|_2 \to 0, \text{ for every } k\in\Lambda_{S_1} \text{ and } 1\le j_1,j_2\le \ell. \tag{III.7.2}$$

To prove III.7.2, fix $k\in\Lambda_{S_1}$ and $1\le j_1,j_2\le\ell$. For $g\in\Gamma$, the Fourier expansion of $e_{j_1}u_g e_{j_2}$ in $M = B\rtimes\Lambda$ is given by $e_{j_1}u_g e_{j_2} = \sum_{h\in\Lambda\times\mathbb{Z}/\ell\mathbb{Z}} 1_{\{x\in X_{j_1}\cap gX_{j_2}|g^{-1}x=h^{-1}x\}}v_h$. If $x\in X_{j_1}\cap gX_{j_2}$, then $\iota(x) = (\Gamma g^{-1}p(x)\beta_{j_1}(p(x)), j_1+\ell\mathbb{Z})$ and $\iota(g^{-1}x) = (\Gamma p(g^{-1}x)\beta_{j_2}(p(g^{-1}x)), j_2+\ell\mathbb{Z})$. Thus, denoting

$$w(x) = \beta_{j_1}(p(x))^{-1}p(x)^{-1}gp(g^{-1}x)\beta_{j_2}(p(g^{-1}x))\in\Lambda,$$

and recalling that the action $\Lambda \curvearrowright (Y,\mu_Y)$ is free, we get that

$$e_{j_1}u_g e_{j_2} = \sum_{h\in\Lambda} 1_{\{x\in X_{j_1}\cap gX_{j_2}|w(x)=h\}}\, v_{(h,j_1-j_2+\ell\mathbb{Z})}. \tag{III.7.3}$$

From this it follows that

$$\|E_{B\rtimes\Lambda_{S_2}}(e_{j_1}u_g e_{j_2}v_k^*)\|_2^2 \le m_X(\{x\in X|w(x)\in\Lambda_{S_2}k\}), \text{ for every } g\in\Gamma. \tag{III.7.4}$$

Now, let $\varepsilon>0$. Then we can find a compact set $C\subset G_{S_1}$ such that we have

- $\mu_X(\{x\in X|\pi_{S_1}(p(x))\notin C\}\le\frac{\varepsilon}{4}$, and
- $\mu_X(\{x\in X|\pi_{S_1}(\beta_j(p(x))\notin C\}\le\frac{\varepsilon}{4}$, for $j\in\{j_1,j_2\}$.

If $x\in X$ satisfies $w(x)\in\Lambda_{S_2}k$, then $\pi_{S_1}(w(x)) = k$. By using the definition of $w(x)$, the fact that the action of Γ on X is measure preserving, and the last two inequalities one obtains that

$$\mu_X(\{x\in X|w(x)\in\Lambda_{S_2}k\}) \le \varepsilon + \mu_X(\{x\in X|k\in(C^{-1})^2\pi_{S_1}(g)C^2\}). \tag{III.7.5}$$

By combining III.7.4 and III.7.5 we derive that

$$\|E_{B\rtimes\Lambda_{S_2}}(e_{j_1}u_g e_{j_2}v_k^*)\|_2^2 \le \varepsilon + \mu_X(\{x\in X | k \in (C^{-1})^2\pi_{S_1}(g)C^2\}), \text{ for every } g\in\Gamma.$$

Since $\pi_{S_1}(g_n)\to\infty$, we have that $k\notin (C^{-1})^2\pi_{S_1}(g_n)C^2$, for large enough n. Therefore, the last inequality implies that $\limsup_{n\to\infty}\|E_{B\rtimes\Lambda_{S_2}}(e_{j_1}u_{g_n}e_{j_2}v_k^*)\|_2^2 \le \varepsilon$. Since $\varepsilon>0$ is arbitrary, this proves III.7.2 and thus the claim. □

To finish the proof, let $i\in\{1,2\}$. Since Γ_i is infinite and $A\rtimes\Gamma_i \prec_M^s B\rtimes\Lambda_{S_i}$, we get that Λ_{S_i} is infinite, hence S_i is nonempty. Thus, S_i is a proper subset of $\{1,...,n\}$. Therefore, since Γ is an irreducible lattice in G, we derive that $\pi_{S_i}(\Gamma)<G_{S_i}$ is dense. In combination with the claim, this implies that $K_1=\overline{\pi_{S_1}(\Gamma_2)}$ and $K_2=\overline{\pi_{S_2}(\Gamma_1)}$ are normal compact subgroups of G_{S_1} and G_{S_2}, respectively. Thus, $K=K_1\times K_2$ is a normal compact subgroup of $G=G_{S_1}\times G_{S_2}$.

Let $\rho_i : G_{S_i}\to G_{S_i}/K_i$, for $i\in\{1,2\}$, and $\rho=(\rho_1,\rho_2): G\to G/K$ be the canonical projections. If $g_1\in\Gamma_1$ and $g_2\in\Gamma_2$, then $\rho_1(\pi_{S_1}(g_2))=\text{id}$ and $\rho_2(\pi_{S_2}(g_1))=\text{id}$. Thus, we derive that $\rho(g_1g_2)=(\rho_1(\pi_{S_1}(g_1g_2)),\rho_2(\pi_{S_2}(g_1g_2)))=(\rho_1(\pi_{S_1}(g_1)),\rho_2(\pi_{S_2}(g_2)))$, which implies that

$$\rho(\Gamma)=\rho_1(\pi_{S_1}(\Gamma_1))\times\rho_2(\pi_{S_2}(\Gamma_2)). \tag{III.7.6}$$

If $i\in\{1,2\}$, then $\pi_{S_i}(\Gamma)<G_{S_i}$ is dense, hence $\rho_i(\pi_{S_i}(\Gamma))=\rho_i(\pi_{S_i}(\Gamma_i))$ is dense in G_{S_i}/K_i. In combination with III.7.6, we conclude that $\rho(\Gamma)<G/K$ is dense. On the other hand, since $\Gamma<G$ is discrete and $K<G$ is compact, we get that $\rho(\Gamma)<G/K$ is discrete hence closed. Altogether, we deduce that $\rho(\Gamma)=G/K$ and thus $K<G$ is an open normal compact subgroup. This implies that $\pi_{\{j\}}(K)<G_j$ is an open normal compact subgroup, for every $1\le j\le n$, a contradiction. ■

III.7.4 Proof of Corollary G

Let $k \geq 1$ be the largest integer for which there are a decomposition $\Gamma = \Gamma_1 \times ... \times \Gamma_k$ and a partition $T_1 \sqcup ... \sqcup T_k = \{1, ..., n\}$ such that T_i is non-empty and Γ_i is measure equivalent to $\times_{j \in T_i} \Lambda_j$, for all $1 \leq i \leq k$. Theorem F implies that $L(\Gamma_i)$ is a prime II_1 factor, for all $1 \leq i \leq k$. This proves the existence of a decomposition with the desired property.

In order to prove the uniqueness of the decomposition, we establish the following fact: if $\Gamma = \Sigma_1 \times \Sigma_2$, then there is a partition $I_1 \sqcup I_2 = \{1, ..., k\}$ such that $\Sigma_1 = \times_{i \in I_1} \Gamma_i$ and $\Sigma_2 = \times_{i \in I_2} \Gamma_i$. To see this, for $1 \leq i \leq k$, let $\pi_i : \Gamma \to \Gamma_i$ be the canonical projection. Then Γ_i is generated by the commuting subgroups $\pi_i(\Sigma_1)$ and $\pi_i(\Sigma_2)$. Since Γ_i has trivial center, we have that $\pi_i(\Sigma_1) \cap \pi_i(\Sigma_2) = \{e\}$, which implies that $\Gamma_i = \pi_i(\Sigma_1) \times \pi_i(\Sigma_2)$. Since $L(\Gamma_i)$ is prime, we deduce that either $\pi_i(\Sigma_1) = \{e\}$ or $\pi_i(\Sigma_2) = \{e\}$. Since this holds for every $1 \leq i \leq k$, the fact follows.

Now, if $\Gamma = \Sigma_1 \times ... \times \Sigma_l$ is another decomposition such that $L(\Sigma_j)$ is a prime II_1 factor, for every $1 \leq j \leq l$, then the fact implies that $l = k$ and that, after a permutation of indices, we have $\Sigma_i = \Gamma_i$, for every $1 \leq i \leq k$.

(1) Assume that $M = P_1 \overline{\otimes} P_2$, for some II_1 factors P_1 and P_2. By applying Theorem F we find a decomposition $\Gamma = \Sigma_1 \times \Sigma_2$, a decomposition $M = P_1^t \overline{\otimes} P_2^{1/t}$, for some $t > 0$, and a unitary $u \in M$, such that $P_1^t = uL(\Sigma_1)u^*$ and $P_2^{1/t} = uL(\Sigma_2)u^*$. The above fact now clearly implies the conclusion.

(2) & (3) Assume that $M = P_1 \overline{\otimes} \ldots \overline{\otimes} P_m$, where $P_1, \ldots, P_m$ are II_1 factors. Then by induction, part (1) implies that $m \leq k$ and there are a partition $I_1 \sqcup \cdots \sqcup I_m = \{1, \ldots, k\}$, a decomposition $M = P_1^{t_1} \overline{\otimes} \ldots \overline{\otimes} P_m^{t_m}$, for some $t_1, \ldots, t_m > 0$ with $t_1 \ldots t_m = 1$, and a unitary $u \in M$ such that $P_j^{t_j} = u(\overline{\otimes}_{i \in I_j} L(\Gamma_i))u^*$, for every $1 \leq j \leq m$.

If $m \geq k$, then we get that $m = k$. Since I_j is nonempty, it follows that I_j consists of one element, for every $1 \leq j \leq m$. This implies part (2). If P_j is prime, for every $1 \leq j \leq m$, then again it follows that I_j consists of one element, for every $1 \leq j \leq m$. This implies part

(3). ■

III.7.5 Acknowledgment

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Chapter III is, in part, a reprint of the material as it appears in

[DHI16] D. Drimbe, D. Hoff, A. Ioana:*Prime* II_1 *factors arising from irreducible lattices in products of rank one simple Lie groups* to apear in J. Reine. Angew. Math. of which the dissertation author was one of the primary investigators and authors.

Chapter IV

W*-superrigidity for coinduced actions

IV.1 Introduction and statement of the main results

IV.1.1 Introduction.

To every measure preserving action $\Gamma \curvearrowright (X, \mu)$ of a countable group Γ on a standard probability space (X, μ), one associates *the group measure space von Neumann algebra* $L^\infty(X) \rtimes \Gamma$ [MvN36]. If the action $\Gamma \curvearrowright X$ is free, ergodic and probability measure preserving (pmp), then $L^\infty(X) \rtimes \Gamma$ is a II_1 factor which contains $L^\infty(X)$ as a *Cartan subalgebra*, i.e. a maximal abelian von Neumann algebra whose normalizer generates $L^\infty(X) \rtimes \Gamma$. The classification of group measure space II_1 factors $L^\infty(X) \rtimes \Gamma$ is a central problem in the theory of von Neumann algebras.

If the groups are amenable, the classification up to W*-equivalency has been completed in the 1970s. More precisely, the celebrated theorem of Connes [Co76] asserts that all II_1 factors arising from free ergodic pmp actions of countable amenable groups are isomorphic to the hyperfinite II_1 factor. In contrast, the non-amenable case is much more challenging and it has led to a beautiful *rigidity theory* in the sense that one can deduce conjugacy from W*-equivalence. A major breakthrough in the classification of II_1 factors was made by Popa between 2001-2004 through the invention of deformation/rigidity theory (see [Po07, Va10a, Io12a] for surveys). In particular, he obtained the following W*-rigidity result: let $\Gamma \curvearrowright X$ be a free ergodic pmp action of an infinite conjugacy class (icc) countable group Γ which has an infinite normal subgroup with the relative property (T) and let $\Lambda \curvearrowright Y := Y_0^{\Lambda}$ be a Bernoulli action of a countable group Λ. Popa proved that if the two actions are W*-equivalent, then the actions are *conjugate* [Po03, Po04], i.e. there exist a group isomoprhism $d : \Gamma \to \Lambda$ and a measure space isomorphism $\theta : X \to Y$ such that $\theta(gx) = d(g)\theta(x)$ for all $g \in \Gamma$ and almost everywhere (a.e.) $x \in X$.

The most extreme form of rigidity for an action $\Gamma \curvearrowright (X,\mu)$ is W*-superrigidity, i.e. whenever $\Lambda \curvearrowright (Y,\nu)$ is a free ergodic pmp action W*-equivalent to $\Gamma \curvearrowright (X,\mu)$, then the two actions are conjugate. A few years ago, Peterson was able to show the existence of virtually W*-superrigid actions [Pe09]. Soon after, Popa and Vaes discovered the first concrete families of W*-superrigid actions [PV09]. Ioana then proved in [Io10] a general W*-superrigidity result for Bernoulli actions.

Theorem (Ioana, [Io10]). If Γ is an icc property (T) group and (X_0,μ_0) is a non-trivial standard probability space, then the Bernoulli action $\Gamma \curvearrowright (X_0,\mu_0)^{\Gamma}$ is W*-superrigid.

The main ingredient of his proof was the discovery of a beautiful dichotomy result for abelian subalgebras of II_1 factors coming from Bernoulli actions.

Using a similar method, Ioana, Popa and Vaes were able to prove later that any Bernoulli action of an icc non-amenable group which is a product of two infinite groups is also W*-superrigid [IPV10]. A few years ago Boutonnet extended these results to Gaussian actions in [Bo12b]. Several other classes of W*-superrigid actions have been found in [FV10, CP10, HPV10, Va10b, CS11, CSU11, PV11, PV12, CIK13, CK15, Dr15, GITD16].

IV.1.2 Statement of the main results.

Our first theorem is a generalization of Ioana's W*-superrigidity result [Io10, Theorem A] to coinduced actions. Before stating the theorem, we explain first the terminology that we use.

Recall that an inclusion $\Gamma_0 \subset \Gamma$ of countable groups has the *relative property* (T) if for every $\epsilon > 0$, there exist $\delta > 0$ and a finite subset $F \subset \Gamma$ such that if $\pi : \Gamma \to \mathcal{U}(K)$ is a unitary representation and $\xi \in K$ is a unit vector satisfying $\|\pi(g)\xi - \xi\| < \delta$, for all $g \in F$, then there exists $\xi_0 \in K$ such that $\|\xi - \xi_0\| < \epsilon$ and $\pi(h)\xi_0 = \xi_0$, for all $h \in \Gamma_0$. The group Γ has the *property* (T) if the inclusion $\Gamma \subset \Gamma$ has the relative property (T). To give some examples, note that $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ has the relative property (T) and $SL_n(\mathbb{Z})$, $n \geq 3$, has the property (T) [Ka67, Ma82].

We say that a subgroup Σ of a countable group Γ is called *n-almost malnormal* if for any $g_1, g_2, ..., g_n \in \Gamma$ such that $g_i^{-1} g_j \notin \Sigma$ for all $i \neq j$, the group $\cap_{i=1}^n g_i \Sigma g_i^{-1}$ if finite. The subgroup Σ is called *almost malnormal* if it is n-almost malnormal for some $n \geq 1$. Finally, see Definition II.1.1 for recalling the definition of coinduced actions.

Theorem H. *Let Γ be an icc group which admits an infinite normal subgroup Γ_0 with relative property (T) and let Σ be an amenable almost malnormal subgroup of Γ. Let σ_0 be a pmp action of Σ on a non-trivial standard probability space (X_0, μ_0) and denote by σ the coinduced action of Γ on $X := X_0^{\Gamma/\Sigma}$. Then $\Gamma \overset{\sigma}{\curvearrowright} X$ is W^*-superrigid.*

Example IV.1.1. In particular, Theorem H can be applied for $\Gamma = SL_3(\mathbb{Z})$ and $\Sigma = \langle A \rangle$, where $A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ [PV06, Section 7]. See [PV06] for more concrete examples of amenable almost malnormal subgroups of $PSL_n(\mathbb{Z})$, $n \geq 3$. See also [RS10, Theorem 1.1], a result which proves the existence of amenable almost malnormal subgroups of torsion-free uniform lattices in connected semisimple real algebraic groups with no compact factors.

We now generalize Ioana-Popa-Vaes' result [IPV10, Theorem 10.1] to coinduced actions. First, recall that two countable groups Γ and Λ are called *measure equivalent* in sense of Gromov if there exist two commuting free measure preserving actions of Γ and Λ on a standard measure space (Ω, m), such that the actions of Γ and Λ each admit a finite measure fundamental domain [Gr91]. Natural examples of measure equivalent groups are provided by pairs of lattices Γ, Λ in an unimodular locally compact second countable group.

Theorem I. *Let Γ be an icc non-amenable group which is measure equivalent to a product of two infinite groups. Let Σ be an amenable almost malnormal subgroup and let σ_0 be a pmp action of Σ on a non-trivial standard probability space (X_0, μ_0) and denote by σ the coinduced action of Γ on $X := X_0^{\Gamma/\Sigma}$.*
Then $\Gamma \overset{\sigma}{\curvearrowright} X$ is W^*-superrigid.

See Theorem IV.6.3 for a more general statement in which it is assumed instead that Γ is measure equivalent to a group Λ_0 whose group von Neumann algebra $L(\Lambda_0)$ is not prime. Note that Theorems H and I provide a complementary class of W*-superrigid coinduced actions from the one found in [Dr15, Corollary 1.4].

Example IV.1.2. A more general statement of Theorem I can be appplied for $\Sigma \subset \Gamma = \Delta \wr \Sigma$ with Δ non-amenable and Σ amenable (see Remark IV.6.4).

The following remark shows that if Σ is not almost malnormal, the action $\Gamma \curvearrowright X$ is not necessary W*-superrigid. To put this in context, we recall first the notion of OE-superrigidity and Singer's result [Si55]. Two actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are *orbit equivalent* (OE) if there exists a measure space isomorphism $\theta : X \to Y$ such that $\theta(\Gamma x) = \Lambda\theta(x)$, for a.e. $x \in X$. A pmp action $\Gamma \curvearrowright X$ is *OE-superrigid* if whenever $\Lambda \curvearrowright Y$ is a free ergodic pmp action which is OE to $\Gamma \curvearrowright X$, then the two actions are conjugate.

Singer proved in [Si55] that two free ergodic pmp actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are OE if and only if there exists an isomoprhism of the group measure space algebras $L^\infty(X) \rtimes \Gamma$

and $L^\infty(Y) \rtimes \Lambda$ which preserves the Cartan algebras $L^\infty(X)$ and $L^\infty(Y)$. In particular, W*-superrigidity implies OE-superrigidity.

Remark IV.1.3. If Σ is not almost malnormal, the action $\Gamma \curvearrowright X$ may fail to be W*-superrigid. Indeed, suppose Γ is an icc group which splits as a direct product $\Gamma = \Sigma \times \Delta$, with Σ amenable and Δ a non-amenable group. Connes and Jones have found in [CJ82] a class of groups Σ and a class of free ergodic pmp actions $\Sigma \overset{\sigma_0}{\curvearrowright} X_0$ for which the coinduced action $\Gamma \curvearrowright X$ of σ_0 is not W*-superrigid. Precisely, they have proven that $M := L^\infty(X) \rtimes \Gamma$ is *McDuff*, i.e. $M \simeq M \bar{\otimes} R$, where R is the hyperfinite II_1 factor. However, [Dr15, Corollary 1.3] implies that $\Gamma \curvearrowright X$ is OE-superrigid.

Note that Theorem I extends the class of groups whose Bernoulli actions are W*-superrigid. Therefore we record the following result.

Corollary J. *Let Γ be an icc non-amenable group which is measure equivalent to a product of two infinite groups. Let (X_0, μ_0) be a non-trivial standard probability space. Then the Bernoulli action $\Gamma \curvearrowright X_0^\Gamma$ is W*-superrigid.*

We recall the well known theorem due to Borel which asserts that every connected non-compact semisimple Lie group contains a lattice (see [Bo63] and [Ra72, Theorem 14.1]). Using this, we obtain an immediate consequence of Corollary J.

Corollary K. *Let Γ be an icc lattice in a product $G = G_1 \times \cdots \times G_n$ of $n \geq 2$ connected non-compact semisimple Lie groups and let (X_0, μ_0) be a non-trivial standard probability space. Then the Bernoulli action $\Gamma \curvearrowright X_0^\Gamma$ is W*-superrigid.*

Note that a combination of Popa's cocycle superrigidity theorem for product groups [Po06a] and the results on uniqueness of Cartan subalgebras from [PV12] already proves Corollary K, but only in the case when each factor $G_1, \dots, G_n$ is of rank one.

IV.1.3 Comments on the proof of Theorem I

For obtaining the proofs of Theorem H and Theorem I, we adapt the proofs used by Ioana [Io10] and Ioana-Popa-Vaes [IPV10] to the context of coinduced actions. We outline briefly and informally the proof of Theorem I since it has as a consequence Corollary J.

To this end, let Γ be an icc group and let Σ be an almost malnormal subgroup. Assume Γ is measure equivalent to a product $\Lambda_0 = \Lambda_1 \times \Lambda_2$ of two countable groups. By [Fu98], Γ and Λ_0 must have stably orbit equivalent actions. To simplify notation, assume there exist free ergodic pmp actions of Γ and Λ_0 on a probability space (Y_0, μ) whose orbits are equal, almost everywhere. Thus, $L^\infty(Y_0) \rtimes \Gamma = L^\infty(Y_0) \rtimes \Lambda_0$.
Suppose $\Sigma \curvearrowright X_0$ is a pmp action on a non-trivial standard probability space and let $\Gamma \overset{\sigma}{\curvearrowright} X := X_0^{\Gamma/\Sigma}$ be the corresponding coinduced action. Our goal is to show that $\Gamma \overset{\sigma}{\curvearrowright} X$ is W*-superrigid. Assume that $\Lambda \curvearrowright Y$ is an arbitrary free ergodic pmp action such that

$$M := L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda.$$

First, we reduce the problem to showing that the Cartan subalgebras $L^\infty(X)$ and $L^\infty(Y)$ are unitarily conjugated. We do this by proving in Section IV.4 a cocycle superrigidity theorem for $\Gamma \curvearrowright X$. Combined with [Po05, Theorem 5.6], we obtain that $\Gamma \curvearrowright X$ is OE-superrigid. Therefore, by a result of Singer [Si55] it is enough to show that $L^\infty(Y)$ is unitarily cojugate to $L^\infty(X)$ in M. We note that this is actually equivalent to $L^\infty(Y) \prec_M L^\infty(X)$, by [Po06b, Theorem A.1]. See Section I.2.4 for the definition of Popa's intertwining symbol "$\prec$".

As is [Io10], we make use of the decomposition $M = L^\infty(Y) \rtimes \Lambda$ via the comultiplication $\Delta : M \to M \bar{\otimes} M$ defined by $\Delta(bv_\lambda) = bv_\lambda \otimes \lambda$, for all $b \in L^\infty(Y)$ and $\lambda \in \Lambda$, introduced in [PV09]. Here we denote by $\{v_\lambda\}_{\lambda \in \Lambda}$ the canonical unitaries implementing the action of Λ

on $L^\infty(Y)$. The next step is to prove that there exists a unitary $u \in M\bar{\otimes}M$ such that

$$u\Delta(L(\Gamma))u^* \subset L(\Gamma)\bar{\otimes}L(\Gamma). \qquad \text{(IV.1.1)}$$

This is obtained in two steps. A main technical contribution of our paper is to use the rigidity of Γ inherited from the product structure of Λ_0 through measure equivalence. We do this in Section IV.4 by introducing an "amplified" version of the comultiplication map Δ which is defined on the larger von Neumann algebra $(L^\infty(Y_0)\bar{\otimes}L^\infty(X)) \rtimes \Gamma$. Combined with the spectral gap rigidity theorem for coinduced actions (Theorem IV.3.1) proved in Section IV.3, we obtain the conclusion (IV.1.1).

In Section IV.5, following Ioana's idea [Io10], we obtain a dichotomy theorem for certain abelian algebras. The result is a straightforward adaptation of [IPV10, Theorem 5.1] to coinduced actions and has two consequences. First, we obtain

$$\Delta(L^\infty(X))' \cap (M\bar{\otimes}M) \prec L^\infty(X)\bar{\otimes}L^\infty(X).$$

Second, it implies a weaker version of Popa's conjugacy criterion adapted to coinduced actions. This will altogether prove Theorem I.

IV.2 Preliminaries

IV.2.1 Bimodules and weak containment.

Let M, N be tracial von Neumann algebras. An M-N-*bimodule* ${}_M\mathcal{H}_N$ is a Hilbert space $\mathcal{H}$ equipped with two commuting normal unital $*$-homomorphisms $M \to B(\mathcal{H})$ and $N^{\text{op}} \to B(\mathcal{H})$. An $M-N$-bimodule ${}_M\mathcal{H}_N$ is *weakly contained* in a M-N-bimodule ${}_M\mathcal{K}_N$ and we write ${}_M\mathcal{H}_N \underset{weak}{\subset} {}_M\mathcal{K}_N$ if for any $\epsilon > 0$, finite subsets $F \subset M, G \subset N$ and $\xi \in \mathcal{H}$, there exist $\eta_1, \ldots, \eta_n \in \mathcal{K}$ such that

$$|\langle x\xi y, \xi\rangle - \sum_{i=1}^{n} \langle x\eta_i y, \eta_i\rangle| \leq \epsilon, \text{ for all } x \in F, y \in G.$$

Given two bimodules ${}_M\mathcal{H}_N$ and ${}_N\mathcal{K}_P$, one can define the *Connes tensor product* $\mathcal{H} \otimes_N \mathcal{K}$ which is an M-P bimodule (see [Co94, V.Appendix B]). If ${}_M\mathcal{H}_N \underset{weak}{\subset} {}_M\mathcal{K}_N$, then ${}_M\mathcal{H} \otimes_N \mathcal{L}_P \underset{weak}{\subset} {}_M\mathcal{K} \otimes_N \mathcal{L}_P$, for any N-P bimodule $\mathcal{L}$.

IV.2.2 Relative amenability

Let (M, τ) be a tracial von Neumann algebra. Let $p \in M$ be a projection and $P \subset pMp, Q \subset M$ be von Neumann subalgebras. By [OP07, Section 2.2], P is amenable relative to Q inside M if and only if ${}_M L^2(Mp)_P$ is weakly contained in ${}_M L^2(\langle M, e_Q\rangle p)_P$.

A von Neumann subalgebra $P \subset pMp$ is *strongly non-amenable relative to* Q if for all non-zero projections $p_1 \in P' \cap pMp$, the von Neumann algebra $p_1 P$ is non-amenable relative to Q.

For $B \subset M$ a von Neumann subalgebra, we have $L^2(M) \otimes_B L^2(M) \cong L^2(\langle M, e_B\rangle)$ as M-M-bimodules. Note that B is amenable if and only if ${}_M L_2(M) \otimes_B L^2(M)_M \underset{weak}{\subset} {}_M L^2(M) \otimes L^2(M)_M$.

Recall that a countable group Γ is amenable if and only if every unitary representation of Γ is weakly contained in the left regular representation ([BHV08, Theorem G.3.2]). The next lemma is the analogous statement for amenable von Neumann algebras. The result is likely well-known, but for a lack of reference, we include a proof.

Lemma IV.2.1. *Let A be a tracial von Neumann algebra. Then A is amenable if and only if every A-A-bimodule $\mathcal{K}$ is weakly contained in the coarse bimodule $L^2(A) \otimes L^2(A)$.*

Proof. Suppose A is amenable and let $\mathcal{K}$ be an A-A-bimodule. Then the trivial bimodule ${}_A L^2(A)_A$ is weakly contained in the coarse bimodule ${}_A L^2(A) \otimes L^2(A)_A$. Since

$L^2(A) \otimes_A \mathcal{K}$ identifies with $\mathcal{K}$ as A-A bimodules, we obtain that

$$ {}_A\mathcal{K}_A \underset{weak}{\subset} {}_A L^2(A) \otimes \mathcal{K}_A. \tag{IV.2.1} $$

Now, since any right module of A is contained in $\bigoplus_{\mathbb{N}} L^2(A)$ as a right A-submodule, we have that

$$ {}_{\mathbb{C}}\mathcal{K}_A \underset{weak}{\subset} {}_{\mathbb{C}}L^2(A)_A. \tag{IV.2.2} $$

Thus, (IV.2.1) and (IV.2.2) implies that ${}_A\mathcal{K}_A$ is weakly contained in the coarse A-A-bimodule. The converse is clear by taking $\mathcal{K} = L^2(A)$, the trivial A-A-bimodule. ■

We end this subsection by recording an immediate corollary of [DHI16, Lemma 2.6]. We provide a proof for the reader's convenience.

Lemma IV.2.2. *[DHI16, Lemma 2.6] Let P and Q be two von Neumann subalgebras of a tracial von Neumann algebra (M, τ). If P is non-amenable relative to Q, then there exists a non-zero projection $z \in \mathcal{N}_M(P)' \cap M$ such that Pz is strongly non-amenable relative to Q.*

Proof. Using Zorn's lemma and a maximality argument, we can find a projection $z \in P' \cap M$ such that Pz is strongly non-amenable relative to Q and $P(1-z)$ is amenable relative to Q. Using [DHI16, Lemma 2.6] there exists $z_1 \in \mathcal{N}_M(P)' \cap M$ such that $1 - z \le z_1$ and Pz_1 is amenable relative to Q. Therefore, $P(z_1 - (1-z))$ is amenable relative to Q, which implies that $1 - z = z_1 \in \mathcal{N}_M(P)' \cap M$.

■

IV.3 Intertwining of rigid algebras

IV.3.1 The free product deformation for coinduced actions

We recall now the free product deformation introduced by Ioana [Io06a] for general Bernoulli actions defined in Section II.2.2.

Coinduced actions for tracial von Neumann algebras are defined in Section II.2.2. Let Γ be a countable group and let Σ be a subgroup. Let $\Sigma \overset{\sigma_0}{\curvearrowright} (A_0, \tau_0)$ be a trace preserving action, where (A_0, τ_0) is a tracial von Neumann algebra. Let $\Gamma \overset{\sigma}{\curvearrowright} A_0^{\Gamma/\Sigma}$ be the coinduced action of σ_0.

Consider the free product $A_0 * L(\mathbb{Z})$ with respect to the natural traces. Extend canonically σ_0 to an action on $A_0 * L(\mathbb{Z})$. Denote by $\tilde{M} = (A_0 * L(\mathbb{Z}))^{\Gamma/\Sigma} \rtimes_\sigma \Gamma$ the corresponding crossed product of the coinduced action $\Gamma \overset{\sigma}{\curvearrowright} (A_0 * L(\mathbb{Z}))^{\Gamma/\Sigma}$ of σ_0.

Take $u \in L(\mathbb{Z})$ the canonical generating Haar unitary. Let $h = h^* \in L(\mathbb{Z})$ be such that $u = \exp(ih)$ and set $u_t = \exp(ith)$ for all $t \in \mathbb{R}$. Define the deformation $(\alpha_t)_{t\in\mathbb{R}}$ by automorphisms of $\tilde{M}$ by

$$\alpha_t(u_g) = u_g \quad \text{and} \quad \alpha_t(\otimes_{h\in\Gamma/\Sigma} a_h) = \otimes_{h\in\Gamma/\Sigma} \operatorname{Ad}(u_t)(a_h),$$

for all $g \in \Gamma, t \in \mathbb{R}$ and $\otimes_{h\in\Gamma/\Sigma} a_h \in (A_0 * L(\mathbb{Z}))^{\Gamma/\Sigma}$ an elementary tensor.

IV.3.2 Spectral gap rigidity for coinduced actions

Theorem IV.3.1. *Let Γ be an icc countable group and let Σ be an almost malnormal subgroup. Let σ_0 be a pmp action of Σ on a non-trivial standard probability space (X_0, μ_0). Denote by $M = L^\infty(X) \rtimes \Gamma$ the crossed-product von Neumann algebra of the coinduced action $\Gamma \overset{\sigma}{\curvearrowright} (X_0, \mu)^{\Gamma/\Sigma}$ associated to $\Sigma \overset{\sigma_0}{\curvearrowright} (X_0, \mu)$. Let N be an arbitrary tracial von Neumann algebra and suppose $Q \subset p(M \bar{\otimes} N)p$ is a von Neumann subalgebra such that $Q' \cap p(M \bar{\otimes} N)p$ is strongly non-amenable relative to $1 \otimes N$.*

Then,

$$\sup_{b \in \mathcal{U}(Q)} \|(\alpha_t \otimes id)(b) - b\|_2 \text{ converges to 0 as } t \to 0.$$

Theorem IV.3.1 and its proof are similar with other results from the literature [Po06a, Lemma 5.1], [IPV10, Corollary 4.3] and especially with [BV12, Theorem 3.1] (where the generalized Bernoulli action might have non amenable stabilizers) and with [KV15, Theorem 2.6] (which is another version of this result for coinduced actions).

Proof of Theorem IV.3.1.

Put $\mathcal{M} := M \bar{\otimes} N$ and $\tilde{\mathcal{M}} := \tilde{M} \bar{\otimes} N$. The proof of this theorem goes along the same lines as the proof of [BV12, Theorem 3.1]. Therefore, instead of working with the bimodule ${}_{\mathcal{M}}L^2(\tilde{\mathcal{M}} \ominus \mathcal{M})_{\mathcal{M}}$, we use the following $\mathcal{M}$-$\mathcal{M}$-submodule

$$\mathcal{K} := \overline{\text{sp}} \left\{ (\otimes_{i \in \mathcal{F}} a_i) u_g \otimes n \;\middle|\; \begin{array}{c} \mathcal{F} \subset \Gamma/\Sigma \text{ with } k \leq |\mathcal{F}| < \infty, \, n \in N \text{ and } g \in \Gamma \\ a_i \in A_0 * L(\mathbb{Z}) \text{ for all } i \in \mathcal{F} \\ a_i \in (A_0 * L(\mathbb{Z})) \ominus A_0 \text{ for at least } k \text{ elements } i \in \mathcal{F} \end{array} \right\}.$$

Claim 1. The $\mathcal{M}$-$\mathcal{M}$-bimodule $\mathcal{K}$ is weakly contained in the bimodule $L^2(\mathcal{M}) \otimes_{1 \otimes N} L^2(\mathcal{M})$.

Proof of Claim 1. Let $\mathcal{A} \subset A_0 \ominus \mathbb{C}1$ be an orthonormal basis of $L^2(A_0) \ominus \mathbb{C}1$ and denote by u the canonical Haar unitary of $L(\mathbb{Z})$. Define the orthonormal set $\tilde{A} \subset L^2(A_0 * L(\mathbb{Z})) \ominus L^2(A_0)$ by

$$\tilde{\mathcal{A}} := \{u^{n_1} a_1 u^{n_2} a_2 \dots u^{n_{k-1}} a_{k-1} u^{n_k} | k \geq 1, n_j \in \mathbb{Z} \setminus \{0\}, a_j \in \mathcal{A} \text{ for all } j\}$$

This gives us the following orthogonal decomposition of $L^2(A_0 * L(\mathbb{Z}))$ into A_0-A_0 submodules:

$$L^2(A_0 * L(\mathbb{Z})) = L^2(A_0) \oplus \bigoplus_{a \in \tilde{\mathcal{A}}} \overline{A_0 a A_0}. \tag{IV.3.1}$$

If we denote

$$\mathcal{C} := \{(\otimes_{i \in \mathcal{F}} c_i) \otimes 1 | \mathcal{F} \text{ finite }, k \leq |\mathcal{F}| < \infty, c_i \in \tilde{\mathcal{A}}, \text{ for all } i \in \mathcal{F}\},$$

then the decomposition (IV.3.1) implies that the bimodule $\mathcal{K}$ can be written as the linear span $\mathcal{K} = \overline{\text{sp}}_{c \in \mathcal{C}} \mathcal{M} c \mathcal{M}$. To finish the proof of this claim, note that it is enough to consider an element $c \in \mathcal{C}$ and prove that the M-M-bimodule $\overline{\text{sp}}\, McM$ is weakly contained in the coarse bimodule $L^2(M) \otimes L^2(M)$.

Let $c = (\otimes_{i \in \mathcal{F}} c_i) \otimes 1 \in \mathcal{C}$. We denote by $\Gamma_0 := \{g \in \Gamma | gf = f, \text{ for all } f \in \mathcal{F}\}$, the stabilizer of $\mathcal{F}$ for the action $\Gamma \curvearrowright \Gamma/\Sigma$ and by $\Gamma_1 := \{g \in \Gamma | g \cdot \mathcal{F} = \mathcal{F}\}$, the normalizer of $\mathcal{F}$ for the same action. Since Σ is k-almost malnormal and Γ_0 is a finite index subgroup of Γ_1, we obtain that Γ_1 is a finite group.

Denote $P = A \rtimes \Gamma_1$. Since P is amenable, Lemma IV.2.1 implies that the P-P-bimodule $\overline{\text{sp}}\, McM$ is weakly contained in the coarse bimodule $L^2(P) \otimes L^2(P)$. Thus, for each $\epsilon > 0, F \subset \Gamma_1$ and $E \subset A$ finite subsets, there exist $\eta_1, \eta_2, ..., \eta_n \in L^2(P) \otimes L^2(P)$ such that

$$|\langle a u_g c (b u_h)^*, c\rangle - \sum_{i=1}^{n} \langle a u_g \eta_i (b u_h)^*, \eta_i\rangle| \leq \epsilon, \tag{IV.3.2}$$

for all $g, h \in F$ and $a, b \in E$.

Using the canonical inclusion $L^2(P) \subset L^2(M)$, we obtain that $\langle a u_g \eta_i (b u_h)^*, \eta_i\rangle = 0$, for all $(g, h) \in (\Gamma \times \Gamma) \setminus (\Gamma_1 \times \Gamma_1)$ and $a, b \in A$. Note that also $\langle a u_g c (b u_h)^*, c\rangle = 0$, for all $(g, h) \in (\Gamma \times \Gamma) \setminus (\Gamma_1 \times \Gamma_1)$ and $a, b \in A$. Using these observations together with (IV.3.2), we obtain that the M-M-bimodule $\overline{\text{sp}}\, McM$ is weakly contained in the coarse bimodule $L^2(M) \otimes L^2(M)$. This finishes the proof of the claim. □

Denote by $P_{\mathcal{K}}$ the orthogonal projection of $L^2(\tilde{\mathcal{M}})$ onto the closed subspace $\mathcal{K}$.

Claim 2. $\sup_{b\in\mathcal{U}(Q)}\|P_{\mathcal{K}}((\alpha_t\otimes\mathrm{id})(b))\|_2$ converges to 0 as $t\to 0$.

Proof of Claim 2. Suppose the claim is false. Then there exist $\delta>0$, a sequence of positive numbers $t_n\to 0$, as $n\to\infty$, and a sequence of unitaries $b_n\in\mathcal{U}(Q)$ such that $\|P_{\mathcal{K}}((\alpha_{t_n}\otimes\mathrm{id})(b_n))\|_2\geq\delta$, for all $n\geq 1$.

Define $\xi_n=P_{\mathcal{K}}((\alpha_{t_n}\otimes\mathrm{id})(b_n))$. For all $x\in Q'\cap p\mathcal{M}p$, we have $\|\xi_n x-x\xi_n\|\to 0$, as $n\to\infty$. Note also that $\liminf_{n\to\infty}\|\xi_n\|_2\geq\delta$ and $\|x\xi_n\|_2\leq\|x\|_2$, for all $x\in\mathcal{M}$. Then, [Ho15, Lemma 2.3] implies that there exists a projection $q\in\mathcal{Z}(Q'\cap p\mathcal{M}p)$ such that the $\mathcal{M}$-$(Q'\cap p\mathcal{M}p)q$ bimodule $L^2(\mathcal{M}q)$ is weakly contained in $\mathcal{K}$. Claim 1 implies now that the $\mathcal{M}$-$(Q'\cap p\mathcal{M}p)q$ bimodule $L^2(\mathcal{M}q)$ is weakly contained in the bimodule $L^2(\mathcal{M})\otimes_{1\otimes N}L^2(\mathcal{M})$. This implies that $(Q'\cap p\mathcal{M}p)q$ is amenable relative to $1\otimes N$ inside $\mathcal{M}$, which contradicts the hypothesis. This proves the claim. □

In order to finish the proof of the theorem we need a variant of Popa's transversality property. In the proof of [BV12, Theorem 3.1] it is proven the following fact for generalized Bernoulli actions: if $\sup_{b\in\mathcal{U}(Q)}\|P_{\mathcal{K}}((\alpha_t\otimes\mathrm{id})(b))\|_2$ converges to 0 as $t\to 0$, then $\sup_{b\in\mathcal{U}(Q)}\|(\alpha_t\otimes\mathrm{id})(b)-b\|_2$ converges to 0 as $t\to 0$. With the same proof we obtain the same result for coinduced actions. Claim 2 completes now the proof of the theorem. ■

For $Q\subset M$ a von Neumann subalgebra, we define $\mathrm{QN}_M(Q)\subset M$ to be the set of all elements $x\in M$ for which there exist $x_1,\dots,x_n,y_1,\dots,y_n$ satisfying $xQ\subset\sum_{i=1}^n Qx_i$ and $Qx\subset\sum_{i=1}^n y_iQ$. The weak closure of $\mathrm{QN}_M(Q)$ is called the *quasi-normalizer of Q inside M* and note that it is a von Neumann subalgebra of M which contains both Q and $Q'\cap M$.

The proof of [IPV10, Theorem 4.2] carries over verbatim and gives us the following result.

Theorem IV.3.2. *Let Γ be an icc countable group and let Σ be an almost malnormal subgroup. Let σ_0 be a pmp action of Σ on a non-trivial standard probability space (X_0,μ_0). Denote by $M=L^\infty(X)\rtimes\Gamma$ the crossed-product von Neumann algebra of the coinduced action*

$\Gamma \overset{\sigma}{\curvearrowright} (X_0, \mu)^{\Gamma/\Sigma}$ *associated to* $\Sigma \overset{\sigma_0}{\curvearrowright} (X_0, \mu)$. *Let* N *be a* II_1 *factor and suppose* $Q \subset p(M \bar{\otimes} N)p$ *is a von Neumann subalgebra. Denote by* P *the quasi-normalizer of* Q *in* $p(M \bar{\otimes} N)p$.

If there exist $0 < t < 1$ *and* $\delta > 0$ *such that*

$$\tau(b^*(\alpha_t \otimes id)(b)) \geq \delta, \text{ for all } b \in \mathcal{U}(Q),$$

then one of the following statements is true:

- $Q \prec 1 \otimes N$,
- $P \prec (A \rtimes \Sigma) \bar{\otimes} N$,
- *there exists a unitary* $u \in M \bar{\otimes} N$ *such that* $uPu^* \subset L(\Gamma) \bar{\otimes} N$.

IV.3.3 Controlling intertwiners and relative commutants

In the Appendix of his PhD thesis [Bo14], Boutonnet has presented a unified approach to the notion of mixing for von Neumann algebras. As a consequence, we obtain results which give us good control over intertwiners between certain subalgebras of von Neumann algebras arising from coinduced actions.

Definition IV.3.3. Let $A \subset N \subset M$ be an inclusion of finite von Neumann algebras. We say that the inclusion $N \subset M$ is *mixing relative* to A if for any sequence of unitaries $\{x_n\} \subset \mathcal{U}(N)$ with $\|E_A(yx_nz)\|_2 \to 0$ for all $y, z \in N$, we have

$$\|E_N(m_1 x_n m_2)\|_2 \to 0 \text{ for all } m_1, m_2 \in M \ominus N.$$

Proposition IV.3.4. *[Bo14, Appendix A] Let* $A \subset N \subset M$ *be an inclusion of finite von Neumann algebras such that* $N \subset M$ *is mixing relative to* A. *Let* $Q \subset pMp$ *be a subalgebra such that* $Q \nprec_M A$. *Denote by* P *the quasi-normalizer of* Q *in* pMp.

1. *If $Q \subset N$, then $P \subset N$.*

2. *If $Q \prec N$, then there exists a non-zero partial isometry $v \in pM$ such that $vv^* \in P$ and $v^*Pv \subset N$.*

3. *If N is a factor and if $Q \prec^s_M N$, then there exists a unitary $u \in \mathcal{U}(M)$ such that $uPu^* \subset N$.*

Lemma IV.3.5. *Let Σ be a subgroup of a countable group Γ. Let $\Sigma \overset{\sigma_0}{\curvearrowright} A_0$ be a tracial action on a non-trivial von Neumann algebra A_0 and let $\Gamma \overset{\sigma}{\curvearrowright} A := A_0^{\Gamma/\Sigma}$ be the coinduced action of σ_0. Let $\Gamma \curvearrowright C$ be another tracial action. Then $C \rtimes \Gamma \subset (C\bar{\otimes}A) \rtimes \Gamma$ is mixing relative to $C \rtimes \Sigma$.*

Proof. Denote $\mathcal{M} := (C\bar{\otimes}A) \rtimes \Gamma$ and $I := \Gamma/\Sigma$. Let $\{x_n\} \subset \mathcal{U}(C \rtimes \Gamma)$ be a sequence of unitaries such that $\|E_{C\rtimes\Sigma}(yx_nz)\|_2 \to 0$, for all $y, z \in C \rtimes \Gamma$. Let $a, b \in \mathcal{M} \ominus (C \rtimes \Gamma)$. We have to show that $\|E_{C\rtimes\Gamma}(ax_nb)\|_2 \to 0$. Since $E_{C\rtimes\Gamma}$ is $C \rtimes \Gamma$-bimodular, we can assume $a, b \in A$. Moreover, we can suppose that there exist a finite subset $J \subset I$ and $j_0 \in J$ such that $a, b = \otimes_{j\in J} b_j \in A_0^J$ with $b_{j_0} \in A_0 \ominus \mathbb{C}$. If $j_0 = g_0\Sigma$ and $J = \{g_1\Sigma, \dots, g_n\Sigma\}$ note that $\Sigma_0 := \{g \in \Gamma | gj_0 \in J\} = \cup_{i=1}^n g_i\Sigma g_0^{-1}$. Now, since $ax_nb = \sum_{g\in\Gamma} aE_C(x_nu_g^*)\sigma_g(b)u_g$, we have

$$\|E_{C\rtimes\Gamma}(ax_nb)\|_2^2 = \sum_{g\in\Sigma_0} |\tau(a\sigma_g(b))|^2 \|E_C(x_nu_g^*)\|_2^2 \le \|a\|_2^2\|b\|_2^2 \sum_{i=1}^n \|E_{C\rtimes\Sigma}(u_{g_i}^* x_n u_{g_0})\|_2^2,$$

which goes to zero because of the assumption. This proves the lemma. ■

Proposition IV.3.4 together with Lemma IV.3.5 give the following result.

Corollary IV.3.6. *Let Σ be a subgroup of a countable group Γ. Let $\Sigma \overset{\sigma_0}{\curvearrowright} A_0$ be a tracial action on a non-trivial tracial von Neumann algebra A_0 and let $\Gamma \overset{\sigma}{\curvearrowright} A := A_0^{\Gamma/\Sigma}$ be the coinduced action of σ_0. Let $\Gamma \curvearrowright C$ be another tracial action and let N be an arbitrary factor. Define $\mathcal{M} := (C\bar{\otimes}A) \rtimes \Gamma$. Suppose $Q \subset p(\mathcal{M}\bar{\otimes}N)p$ is a von Neumann subalgebra such that $Q \nprec (C \rtimes \Sigma)\bar{\otimes}N$. Denote by P the quasi-normalizer of Q inside $p(\mathcal{M}\bar{\otimes}N)p$.*

1. *If* $Q \subset p((C \rtimes \Gamma)\bar{\otimes}N)p$*, then* $P \subset p((C \rtimes \Gamma)\bar{\otimes}N)p$*.*

2. *If* $Q \prec (C \rtimes \Gamma)\bar{\otimes}N$*, then there exists a non-zero partial isometry* $v \in p(\mathcal{M}\bar{\otimes}N)$ *such that* $vv^* \in P$ *and* $v^*Pv \subset (C \rtimes \Gamma)\bar{\otimes}N$*.*

3. *If* $Q \prec^s_{\mathcal{M}\bar{\otimes}N} (C \rtimes \Gamma)\bar{\otimes}N$*, then there exists a unitary* $u \in \mathcal{U}(\mathcal{M}\bar{\otimes}N)$ *such that* $uPu^* \subset (C \rtimes \Gamma)\bar{\otimes}N$*.*

The proof of the following proposition is similar to [Bo12a, Corollary 3.7] and we leave it to the reader.

Proposition IV.3.7. *Let* $\Gamma \curvearrowright C$ *be a tracial action and denote* $M_0 = C \rtimes \Gamma$*. Let* Σ *be an almost malnormal subgroup of* Γ*. Suppose* $Q \subset pM_0p$ *is a von Neumann subalgebra such that* $Q \prec C \rtimes \Sigma$ *and* $Q \nprec C$*. Denote by* P *the quasi-normalizer of* Q *inside* pM_0p*.*
Then $P \prec C \rtimes \Sigma$*.*

IV.4 Rigidity coming from measure equivalence

In this section we establish some results needed in the proof of Theorem I. Throughout the section, we will work with coinduced actions satisfying the following:

Assumption IV.4.1. Let Σ be a subgroup of a countable icc group Γ. Let σ_0 be a pmp action of Σ on a non-trivial standard probability space (X_0, μ_0) and denote by σ the coinduced action of Γ on $X := X_0^{\Gamma/\Sigma}$. Suppose:

- Γ is a non-amenable icc group which is measure equivalent to a group Λ_0 for which the group von Neumann algebra $L(\Lambda_0)$ is not prime.

- Σ is almost malnormal.

Note that since Σ is almost malnormal in Γ, we have that $[\Gamma : \Sigma] = \infty$. Before stating the results of this section, we need to introduce some notation.

Notation IV.4.2. The group von Neumann algebra $L(\Lambda_0)$ is not prime, therefore there exist von Neumann algebras R_1 and R_2, both not of type I, such that $L(\Lambda_0) = R_1\bar{\otimes}R_2$. Since $L(\Lambda_0)$ is diffuse and non-amenable, there exists $z_0 \in \mathcal{Z}(L(\Lambda_0))$ such that R_1z_0 and R_2z_0 are diffuse and $L(\Lambda_0)z_0$ is non-amenable.

The group Γ is measure equivalent to Λ_0. By [Fu98, Lemma 3.2], Γ and Λ_0 admit stably orbit equivalent free ergodic pmp actions. Thus, we may find a free ergodic pmp action $\Gamma \curvearrowright (Z_0, \nu)$ and $\ell \geq 1$, such that the following holds: consider the product action $\Gamma \times \mathbb{Z}/\ell\mathbb{Z} \curvearrowright (Z_0 \times \mathbb{Z}/\ell\mathbb{Z}, \nu \times c)$, where $\mathbb{Z}/\ell\mathbb{Z}$ acts on itself by addition and c denotes the counting measure on $\mathbb{Z}/\ell\mathbb{Z}$. Then there exist a non-negligible measurable set $Y_0 \subset Z_0 \times \mathbb{Z}/\ell\mathbb{Z}$ and a free ergodic measure preserving action $\Lambda_0 \curvearrowright Y_0$ such that

$$\mathcal{R}(\Lambda_0 \curvearrowright Y_0) = \mathcal{R}(\Gamma \times \mathbb{Z}/\ell\mathbb{Z} \curvearrowright Z_0 \times \mathbb{Z}/\ell\mathbb{Z}) \cap (Y_0 \times Y_0).$$

We put $C_0 = L^\infty(Y_0), M_0 = L^\infty(Z_0 \times \mathbb{Z}/\ell\mathbb{Z}) \rtimes (\Gamma \times \mathbb{Z}/\ell\mathbb{Z})$, $p = 1_{Y_0}$, and note that $C_0 \rtimes \Lambda_0 = pM_0p$. We identify $L^\infty(\mathbb{Z}/\ell\mathbb{Z}) \rtimes \mathbb{Z}/\ell\mathbb{Z} = \mathbb{M}_\ell(\mathbb{C})$, and use this identification to write $M_0 = C \rtimes \Gamma$, where $C = L^\infty(Z_0) \otimes \mathbb{M}_\ell(\mathbb{C})$ and Γ acts trivially on $\mathbb{M}_\ell(\mathbb{C})$.

Denote $A = L^\infty(X)$ and let $\{u_g\}_{g\in\Gamma} \subset (C\bar{\otimes}A) \rtimes \Gamma$ denote the canonical unitaries implementing the diagonal action of Γ on $C\bar{\otimes}A$.

Remark IV.4.3. Throughout this section we will use many times the following easy observation (see [Va08, Lemma 3.4]). Let $P \subset pMp$ and $Q \subset qMq$ be von Neumann subalgebras of a tracial von Neumann algebra (M, τ). Then:

- if $p_0Pp_0 \prec Q$ for a non-zero projection $p_0 \in P$, then $P \prec Q$.
- if $Pp' \prec Q$ for a non-zero projection $p' \in P' \cap pMp$, then $P \prec Q$.

Lemma IV.4.4. *Let $w : \Gamma \to \mathcal{U}(A\bar{\otimes}N)$ be a cocycle for the action $\sigma\otimes id$, where N is II_1 factor. Define the $*$-homomorphism $d : C \rtimes \Gamma \to (A \rtimes \Gamma)\bar{\otimes}N\bar{\otimes}(C \rtimes \Gamma)$ by $d(cu_g) =$*

$w_g u_g \otimes c u_g, g \in \Gamma, c \in C$. *Let* $Q \subset pM_0p$ *be a subalgebra and let* $\Sigma_0 \subset \Gamma$ *be a subgroup. The following hold:*

1. *If* $Q \nprec C$ *, then* $d(Q) \nprec 1 \otimes N \bar{\otimes} (C \rtimes \Gamma)$.

2. *If* $[\Gamma : \Sigma_0] = \infty$*, then* $d(L(\Lambda_0)) \nprec (A \rtimes \Sigma_0) \bar{\otimes} N \bar{\otimes} (C \rtimes \Gamma)$.

3. *If* Q *is non-amenable, then* $d(Q)$ *is non-amenable relative to* $1 \otimes N \bar{\otimes} (C \rtimes \Gamma)$.

Proof. Denote $\mathcal{M} = (A \rtimes \Gamma) \bar{\otimes} N \bar{\otimes} M_0$ and $\mathcal{N} = 1 \otimes N \bar{\otimes} M_0$.

(1) Let $\{u_n\}_{n \geq 1} \subset \mathcal{U}(Q)$ be a sequence of unitaries such that $\|E_C(u_n u_g)\|_2 \to 0$, for all $g \in \Gamma$. We claim that

$$\|E_{1 \otimes N \bar{\otimes} M_0}(x d(u_n) y)\|_2 \to 0, \text{ for all } x, y \in \mathcal{M}.$$

Since $E_{\mathcal{N}}$ is $\mathcal{N}$-bimodular, by Kaplansky's density theorem we may assume $x = a u_g \otimes 1 \otimes 1$, $y = b u_h \otimes 1 \otimes 1$ for some $a, b \in A$ and $g, h \in \Gamma$. Then for all $n \geq 1$, we have

$$x d(u_n) y = \sum_{k \in \Gamma} a \sigma_g(w_k) \sigma_{gk}(b) u_{gkh} \otimes E_C(u_n u_k^*) u_k.$$

Therefore, $\|E_{\mathcal{N}}(x d(u_n) y)\|_2 \leq \|a\| \|b\| \|E_C(u_n u^*_{g^{-1}h^{-1}})\|_2 \to 0$.

(2) Assume $d(L(\Lambda_0)) \prec (A \rtimes \Sigma_0) \bar{\otimes} N \bar{\otimes} M_0$. Since $d(C_0) \subset 1 \otimes 1 \otimes C_0$, we obtain $d(C_0 \rtimes \Lambda_0) \prec (A \rtimes \Sigma_0) \bar{\otimes} N \bar{\otimes} M_0$. Therefore $d(L(\Gamma)) \prec (A \rtimes \Sigma_0) \bar{\otimes} N \bar{\otimes} M_0$, which implies $L(\Gamma) \prec L(\Sigma_0)$. Indeed, suppose by contrary that $L(\Gamma) \nprec L(\Sigma_0)$. Then there exists a sequence $u_n \in \mathcal{U}(L(\Gamma))$ such that $\|E_{L(\Sigma_0)}(x v_n y)\|_2 \to 0$, for all $x, y \in L(\Gamma)$. We would like to prove that

$$\|E_{(A \rtimes \Sigma_0) \bar{\otimes} N \bar{\otimes} M_0}(x d(u_n) y)\|_2 \to 0, \tag{IV.4.1}$$

for all $x, y \in (A \rtimes \Gamma) \bar{\otimes} N \bar{\otimes} M_0$. For proving (IV.4.1), it is enough to consider $x = u_g \otimes 1 \otimes 1$ and $y = u_h \otimes 1 \otimes 1$, with $g, h \in \Gamma$. In this case one can check that

$$\|E_{(A \rtimes \Sigma_0) \bar{\otimes} N \bar{\otimes} M_0}(x d(u_n) y)\|_2 = \|E_{L(\Sigma_0)}(u_g u_n u_h)\|_2,$$

which goes to 0. Therefore (IV.4.1) is proven and we obtain that $d(L(\Gamma)) \nprec (A\rtimes\Sigma_0)\bar{\otimes}N\bar{\otimes}M_0$, contradiction.

Thus $L(\Gamma) \prec L(\Sigma_0)$, which implies that Σ_0 has finite index in Γ by [DHI16, Lemma 2.5].

(3) Suppose by contrary that $d(Q)$ is amenable relative to $\mathcal{N}$. Then there exists a positive linear functional $\phi : d(p)\langle\mathcal{M}, e_{\mathcal{N}}\rangle d(p) \to \mathbb{C}$ such that $\phi_{|d(p)\mathcal{M}d(p)} = \tau$ and ϕ is $d(Q)$-central. Define now $\varphi : p\langle M_0, e_{\mathbb{C}}\rangle p \to \mathbb{C}$ by

$$\varphi(\sum_{i=1}^{N} m_i e_{\mathbb{C}} n_i) = \phi(\sum_{i=1}^{N} d(m_i) e_{\mathcal{N}} d(n_i)),$$

where $N \geq 1$, $m_i, n_i \in M_0, i \in \{1,\dots,N\}$. Note that φ is a well defined positive linear functional. Indeed, suppose $\sum_{i=1}^{N} m_i e_{\mathbb{C}} n_i = 0$, with $m_i, n_i \in M_0$, for all $1 \leq i \leq N$. This implies $\sum_{i=1}^{N} d(m_i)\tau(n_i) = 0$. Since $E_{\mathcal{N}}(d(m)) = \tau(m)$, for all $m \in M_0$, we obtain $\sum_{i=1}^{N} d(m_i) E_{\mathcal{N}}(d(n_i)) = 0$, which implies $\sum_{i=1}^{N} d(m_i) e_{\mathcal{N}} d(n_i) = 0$. Therefore, φ is a positive linear functional which is Q-central and $\varphi_{|pM_0p} = \tau$. We obtain Q is amenable, contradiction.
■

Denote by $\mathcal{U}_{fin}$ the class of Polish groups which arise as closed subgroups of the unitary groups of II_1 factors [Po05]. In particular, all countable discrete groups and all compact Polish groups belong to $\mathcal{U}_{fin}$.

Theorem IV.4.5. *(Cocycle superrigidity.) Let $\Gamma \curvearrowright X$ be as in Assumption IV.4.1. Then any cocycle $w : \Gamma \times X \to \Lambda$ valued in a group $\Lambda \in \mathcal{U}_{fin}$ untwists, i.e. there exists a measurable map $\varphi : X \to \Lambda$ and a group homomorphism $d : \Gamma \to \Lambda$ such that $w(g,x) = \varphi(gx)d(g)\varphi(x)^{-1}$ for all $g \in \Gamma$ and a.e. $x \in X$.*

This result was proven in [PS09] for Bernoulli actions using deformations obtained from closable derivations. In our case, we will provide a direct proof for Theorem IV.4.5 which uses only the free product deformation α_t

Proof. Define $A \coloneqq L^\infty(X)$ and let N be a II$_1$ factor such that $\Lambda \subset \mathcal{U}(N)$. We associate to $w : \Gamma \times X \to \mathcal{U}(N)$ the cocycle $w : \Gamma \to \mathcal{U}(A\bar{\otimes}N)$, given by $w_g(x) = w(g, g^{-1}x)$. Define $Q = \{w_g u_g\}''_{g\in\Gamma}$.

Claim. We have

$$\sup_{b\in\mathcal{U}(Q)} \|(\alpha_t \otimes \mathrm{id})(b) - b\|_2 \text{ converges to } 0 \text{ as } t \to 0.$$

Proof of the Claim. As in Lemma IV.4.4 we define the $*$-homomorphism $d : C \rtimes \Gamma \to (A \rtimes \Gamma)\bar{\otimes}N\bar{\otimes}(C \rtimes \Gamma)$ by $d(cu_g) = w_g u_g \otimes cu_g, g \in \Gamma, c \in C$. Denote $\mathcal{M} = (A \rtimes \Gamma)\bar{\otimes}N\bar{\otimes}M_0$. Without loss of generality assume that $R_1 z_0$ is non-amenable. Lemma IV.4.4 implies that $d(R_1 z_0)$ is non-amenable relative to $1 \otimes N\bar{\otimes}(C \rtimes \Gamma)$. By Lemma IV.2.2 there exists a non-zero projection $z \in \mathcal{N}_{d(z_0)\mathcal{M}d(z_0)} d(R_1 z_0)' \cap d(z_0)\mathcal{M}d(z_0)$ such that $d(R_1)z$ is strongly non-amenable relative to $1 \otimes N\bar{\otimes}(C \rtimes \Gamma)$. Using Theorem IV.3.1 we obtain that

$$\sup_{b\in\mathcal{U}(d(R_2)z)} \|(\alpha_t \otimes \mathrm{id} \otimes \mathrm{id})(b) - b\|_2 \text{ converges to } 0 \text{ as } t \to 0.$$

and therefore by Theorem IV.3.2 we obtain that one of the following hold:

1. $d(R_2)z \prec 1 \otimes N\bar{\otimes}(C \rtimes \Gamma)$,
2. $d(L(\Lambda_0))z \prec (A \rtimes \Sigma)\bar{\otimes}N\bar{\otimes}(C \rtimes \Gamma)$,
3. $d(L(\Lambda_0))z \prec L(\Gamma)\bar{\otimes}N\bar{\otimes}(C \rtimes \Gamma)$.

Note that (1) and (2) are not possible by Lemma IV.4.4 since $R_2 z_0$ is diffuse and $[\Gamma : \Sigma] = \infty$. Therefore (3) is true.

Now, together with the remark that $d(C) \subset 1 \otimes 1 \otimes C$ we obtain that $d(C \rtimes \Gamma) \prec L(\Gamma)\bar{\otimes}N\bar{\otimes}(C \rtimes \Gamma)$. One can check directly this fact or use [BV12, Lemma 2.3]. Proceeding in the same way, we obtain actually $d(C \rtimes \Gamma) \prec^s_{\mathcal{M}} L(\Gamma)\bar{\otimes}N\bar{\otimes}(C \rtimes \Gamma)$. Lemma IV.4.4 implies that $d(C \rtimes \Gamma) \nprec L(\Sigma)\bar{\otimes}N\bar{\otimes}(C \rtimes \Gamma)$, so by Corollary IV.3.6 we obtain

$$\sup_{b \in \mathcal{U}(Q)} \|(\alpha_t \otimes \mathrm{id})(b) - b\|_2 \text{ converges to } 0 \text{ as } t \to 0.$$

□

Using a result which goes back to Popa [Po05], the claim implies that the cocycle w untwists (see [Dr15, Theorem 2.15], the proof of [Dr15, Proposition 3.2] and [Dr15, Remark 3.3]). ■

Theorem IV.4.6. *Let $\Gamma \curvearrowright X$ be as in Assumption IV.4.1 and supppose that Σ is amenable. Let $\Lambda \curvearrowright B$ be a tracial action on a non-trivial abelian von Neumann algebra B such that $A \rtimes \Gamma = B \rtimes \Lambda$. Denote by $\Delta : B \rtimes \Lambda \to (B \rtimes \Lambda)\bar{\otimes}L(\Lambda)$ the comultiplication $\Delta(bv_\lambda) = bv_\lambda \otimes v_\lambda$ for all $b \in B$ and $\lambda \in \Lambda$ (we let $\{v_\lambda\}_{\lambda \in \Lambda} \subset B \rtimes \Lambda$ denote the canonical unitaries implementing the action of Λ on B).*

Then there exists a unitary $u \in \mathcal{U}((A \rtimes \Gamma)\bar{\otimes}(A \rtimes \Gamma))$ such that

$$u\Delta(L(\Gamma))u^* \subset L(\Gamma \times \Gamma).$$

Define $M := (C\bar{\otimes}A) \rtimes \Gamma$ and $\theta : M \to M\bar{\otimes}M\bar{\otimes}M$ by $\theta(cau_g) = cu_g \otimes \Delta(au_g)$, for all $c \in C, a \in A$ and $g \in \Gamma$. In the following lemma we record some properties of the unital $*$-homomorphism θ which are similar to the ones of [Io10, Lemma 10.2].

Lemma IV.4.7. *Let $Q \subset qMq$. The following hold:*

1. *If Q is diffuse, then $\theta(Q) \nprec M\bar{\otimes}1\bar{\otimes}M$.*
2. *If $Q \nprec B$, then $\theta(Q) \nprec M\bar{\otimes}M \otimes 1$.*
3. *If Q has no amenable direct summand, then $\theta(Q)$ is strongly non-amenable relative to $M\bar{\otimes}M \otimes 1$ and $M\bar{\otimes}1\bar{\otimes}M$.*

We continue now with the proof of Theorem IV.4.6 and we will give the proof of Lemma IV.4.7 at the end of this section.

Proof of Theorem IV.4.6. Without loss of generality we can assume that $R_1 z_0$ is non-amenable. Take $z \in \mathcal{Z}(R_1 z_0)$ such that $R_1 z$ has no amenable direct summand.

Claim 1. We have $\sup_{b \in \mathcal{U}(\Delta(L(\Gamma)))} \|(\mathrm{id} \otimes \alpha_t)(b) - b\|_2$ converges to 0 as $t \to 0$.

Proof of Claim 1. Note that $\theta(R_1 z)$ is strongly non-amenable relative to $M \bar{\otimes} M \otimes 1$ by Lemma IV.4.7. Therefore by Theorem IV.3.1, we obtain

$$\sup_{b \in \mathcal{U}(\theta(R_2 z))} \|(\mathrm{id} \otimes \mathrm{id} \otimes \alpha_t)(b) - b\|_2 \text{ converges to } 0 \text{ as } t \to 0. \tag{IV.4.2}$$

Using Theorem IV.3.2 we obtain that one of the following three conditions holds:

1. $\theta(R_2 z) \prec M \bar{\otimes} M \otimes 1$,
2. $\theta(L(\Lambda_0) z) \prec M \bar{\otimes} M \bar{\otimes} (A \rtimes \Sigma)$,
3. there exists a unitary $u \in \mathcal{M}$ such that $u \theta(L(\Lambda_0) z) u^* \subset M \bar{\otimes} M \bar{\otimes} L(\Gamma)$.

If (1) holds, Lemma IV.4.7 implies $R_2 z \prec_M B$. By applying [Va08, Lemma 3.5], we obtain $B \prec_M zMz \cap (R_2 z)'$. Note that if $R_2 z \prec_M C \rtimes \Sigma$, Proposition IV.3.7 implies that $L(\Lambda_0) \prec C \rtimes \Sigma$. Using [BV12, Lemma 2.3] we deduce that $C \rtimes \Gamma \prec C \rtimes \Sigma$. This is a contradiction since $[\Gamma : \Sigma] = \infty$.
Therefore $R_2 z \nprec_M C \rtimes \Sigma$ and Corollary IV.3.6 implies that $zMz \cap (R_2 z)' \subset C \rtimes \Gamma$, so $B \prec_M C \rtimes \Gamma$. On the other hand, since $B \subset A \rtimes \Gamma$, we obtain $B \prec_{A \rtimes \Gamma} L(\Gamma)$. Proposition IV.3.7 implies that $B \nprec_{A \rtimes \Gamma} L(\Sigma)$. Finally, using Corollary IV.3.6 we obtain that $A \rtimes \Gamma \prec_{A \rtimes \Gamma} L(\Gamma)$, which is a contradiction.

Now, if (2) holds, we obtain $\theta(L(\Lambda_0)) \prec M \bar{\otimes} M \bar{\otimes} (A \rtimes \Sigma)$. Together with $\theta(C) \subset C \otimes 1 \otimes 1$, we obtain $\theta(M_0) \prec M \bar{\otimes} M \bar{\otimes} (A \rtimes \Sigma)$. Since Σ is amenable, it implies that $\theta(M_0)$

is not strongly non-amenable relative to $M\bar{\otimes}M \otimes 1$. Now, M_0 is a factor, so Lemma IV.4.7 gives that M_0 is amenable, which is a contradiction.

Thus, (3) holds. Since $\theta(C_0) \subset C_0 \otimes 1 \otimes 1$, we obtain

$$\theta(M_0) \prec M\bar{\otimes}M\bar{\otimes}L(\Gamma)$$

With the same computation, we obtain $\theta(M_0) \prec^s_{M\bar{\otimes}M\bar{\otimes}M} M\bar{\otimes}M\bar{\otimes}L(\Gamma)$.

Lemma IV.4.7 implies that $\theta(M_0) \nprec M\bar{\otimes}M\bar{\otimes}L(\Sigma)$ since Σ is amenable and M_0 is a factor. By Corollary IV.3.6 we obtain that $\sup_{b \in \mathcal{U}(\Delta(L(\Gamma))} \|(\mathrm{id}\otimes\alpha_t)(b) - b\|_2$ converges to 0 as $t \to 0$. □

Claim 2. We have $\sup_{b \in \mathcal{U}(\Delta(L(\Gamma)))} \|(\alpha_t \otimes \mathrm{id})(b) - b\|_2$ converges to 0 as $t \to 0$.

Proof of Claim 2. As in Claim 1, by applying Lemma IV.4.7, Theorem IV.3.1 and Theorem IV.3.2 we obtain that one of the following conditions hold:

1. $\theta(R_2 z) \prec M\bar{\otimes}1\bar{\otimes}M$,
2. $\theta(L(\Lambda_0)z) \prec M\bar{\otimes}(A \rtimes \Sigma)\bar{\otimes}M$,
3. there exists a unitary $u \in \mathcal{M}$ such that $u\theta(L(\Lambda_0)z)u^* \subset M\bar{\otimes}L(\Gamma)\bar{\otimes}M$.

Note that by Lemma IV.4.7, (1) is not possible since $R_2 z$ is diffuse. As before, (2) is not possible, which implies (3) holds true and by reasoning as before we obtain the claim. □

Notice that $\Delta(L(\Gamma))$ is a factor since Γ is icc. Using Claim 1 and 2 and by applying twice Theorem IV.3.2 and [IPV10, Lemma 10.2] we obtain the conclusion. ■

Proof of Lemma IV.4.7. The proofs of (1) and (2) are similar to the proof of Lemma IV.4.4.1 (see also the proof of [IPV10, Lemma 10.2]). For proving (3), denote $\mathcal{M} := M\bar{\otimes}M\bar{\otimes}(A \rtimes \Gamma)$ and $\psi : M \to M\bar{\otimes}M$, by $\psi(cau_g) = cu_g \otimes au_g$ for all $c \in C, a \in A$ and $g \in \Gamma$.

Claim 1. We have ${}_{\mathcal{M}}L^2(\mathcal{M}) \otimes_{M\bar{\otimes}M\otimes 1} L^2(\mathcal{M})_{\theta(M)} \underset{weak}{\subset} {}_{\mathcal{M}}L^2(\mathcal{M}) \otimes L^2(A \rtimes \Gamma)_{\psi(M)_{1,4}}$. (here we consider that $\psi(M) \subset M\bar{\otimes}M$ acts to the right on $L^2(M)\otimes L^2(M)\otimes L^2(M)\otimes L^2(M)$ on the first and fourth positions.)

Proof of the Claim 1. Note that we have the identification

$$ {}_{\mathcal{M}}L^2(\mathcal{M}) \otimes_{M\bar{\otimes}M\otimes 1} L^2(\mathcal{M})_{\theta(M)} \simeq_{\mathcal{M}_{1,2,3}} L^2(M\bar{\otimes}M\bar{\otimes}(A \rtimes \Gamma)\bar{\otimes}(A \rtimes \Gamma))_{\theta(M)_{1,2,4}} $$

as $\mathcal{M}$-M-bimodules. Therefore, it is enough to show that

$$ {}_{M\bar{\otimes}M\otimes 1}L^2(M\bar{\otimes}M\bar{\otimes}(A \rtimes \Gamma))_{\theta(M)} \underset{weak}{\subset} {}_{M\bar{\otimes}M\otimes 1}L^2(M\bar{\otimes}M\bar{\otimes}(A \rtimes \Gamma))_{\psi(M)_{1,3}}. $$

Let $\mathcal{B}$ be an orthonormal basis for $L^2(B)$ and note that we have the following orthogonal decomposition into $(M\bar{\otimes}M)$-M-bimodules:

$$ L^2(M\bar{\otimes}M\bar{\otimes}(A \rtimes \Gamma)) = \bigoplus_{b\in\mathcal{B}} \overline{\mathrm{sp}}\,(M\bar{\otimes}M \otimes 1)(1 \otimes 1 \otimes b)\,\theta(M) $$

First, notice that for a fixed $b \in \mathcal{B}$ we have

$$ \overline{\mathrm{sp}}\,(M\bar{\otimes}M \otimes 1)(1 \otimes 1 \otimes b)\,\theta(M) \simeq_{(M\bar{\otimes}M)_{1,2}} L^2(M) \otimes L^2(M) \otimes_B L^2(A \rtimes \Gamma)_{\psi(M)_{1,3}} $$

as $(M\bar{\otimes}M)$-M-bimodules. Indeed, let $m_1, m_2, m_3 \in M$ and let us prove that

$$ \langle (m_1\otimes m_2\otimes 1)(1\otimes 1\otimes b)\theta(m_3), 1\otimes 1\otimes b\rangle = \langle (m_1\otimes m_2\otimes 1)(1\otimes 1\otimes_B 1)\psi(m_3), 1\otimes 1\otimes_B 1\rangle \quad \text{(IV.4.3)} $$

We may assume $m_3 = cau_g$ for some $c \in C, a \in A$ and $g \in \Gamma$. Write $au_g = \sum_{l\in\Lambda} b_l v_l \in B \rtimes \Lambda$, with $b_l \in B$ for all $l \in \Lambda$. Therefore, the LHS of (IV.4.3) equals to

$$ \tau((m_1 \otimes m_2 \otimes b^*b)\theta(m_3)) = \tau(m_1 cu_g \otimes ((m_2 \otimes b^*b)\Delta(au_g))) = \tau(m_1 cu_g)\tau(m_2 b_e). $$

On the other hand, the RHS of (IV.4.3) equals to

$$\tau((m_1 \otimes E_B(m_2))\psi(m_3)) = \tau(m_1 c u_g \otimes E_B(m_2) a u_g) = \tau(m_1 c u_g)\tau(m_2 b_e),$$

which proves (IV.4.3).

Now since B is amenable, we obtain that

$${}_{(M \bar{\otimes} M)_{1,2}} L^2(M) \otimes L^2(M) \otimes_B L^2(A \rtimes \Gamma)_{\psi(M)_{1,3}}$$

is weakly contained in

$${}_{(M \bar{\otimes} M)_{1,2}} L^2(M) \otimes L^2(M) \otimes L^2(A \rtimes \Gamma)_{\psi(M)_{1,3}}.$$

This finishes the proof of the claim. □

Claim 2. We have ${}_{\mathcal{M}} L^2(\mathcal{M}) \otimes L^2(M)_{\psi(M)_{1,4}} \underset{weak}{\subset} {}_{\mathcal{M}} L^2(\mathcal{M}) \otimes L^2(M)_M$.

Proof of the Claim 2. First, note that it is enough to prove

$${}_M L^2(M) \otimes L^2(M)_{\psi(M)} \underset{weak}{\subset} {}_M L^2(M) \otimes L^2(M)_M.$$

Let $\mathcal{C}$ be an orthonormal basis for $L^2(C)$ and note that we have the following orthogonal decomposition into M-M-bimodules:

$$L^2(M) \otimes L^2(M) = \bigoplus_{c \in \mathcal{C}} \overline{\text{sp}}\, M\,(1 \otimes c)\, d(M).$$

Note that $\overline{\text{sp}}\, M\,(1 \otimes c)\, d(M) \cong L^2(M) \otimes_C L^2(M)$ as M-M-bimodules. Indeed, let us take $m_1 = c_1 a_1 u_{g_1}$, $m_2 = c_2 a_2 u_{g_2}$, and note that

$$\langle m_1(1 \otimes c)\psi(m_2), 1 \otimes c\rangle = \langle c_1 a_1 u_{g_1} c_2 u_{g_2} \otimes c a_2 u_{g_2}, 1 \otimes c\rangle = \delta_{g_1,e}\delta_{g_2,e}\tau(c_1 c_2)\tau(a_1)\tau(a_2)$$

and

$$\langle m_1 e_C m_2, e_C \rangle = \tau(E_C(c_1 a_1 u_{g_1}) c_2 a_2 u_{g_2}) = \delta_{g_1,e}\delta_{g_2,e}\tau(c_1 c_2)\tau(a_1)\tau(a_2).$$

This implies that $\overline{\text{sp}}\, M\,(1\otimes c)\,\psi(M) \cong L^2(M)\otimes_C L^2(M)$ as M-M-bimodules. Since C is amenable, the claim is proven. □

Now, assume that $\theta(Q)$ is not strongly non-amenable relative to $M\bar{\otimes}M\otimes 1$. Then there exists a non-zero projection $p\in\theta(Q)'\cap\theta(q)\mathcal{M}\theta(q)$ such that

$$_{\mathcal{M}}L^2(\mathcal{M}p)_{\theta(Q)} \underset{weak}{\subset} {}_{\mathcal{M}}L^2(\mathcal{M})\otimes_{M\bar{\otimes}M\otimes 1} L^2(\mathcal{M})_{\theta(Q)}.$$

Using Claim 1 and 2, we obtain now that $_{\mathcal{M}}L^2(\mathcal{M}p)_{\theta(Q)} \underset{weak}{\subset} {}_{\mathcal{M}}L^2(\mathcal{M})\otimes L^2(Q)_Q$.

Take $z\in Q$ such that $\theta(z)$ is the support projection of $E_{\theta(Q)}(p)$. Note that z is a non-zero central projection in Q and that θ embeds the trivial Qz-Qz-bimodule into $_{\theta(Qz)}L^2(\theta(Qz))_{\theta(Qz)}$. Therefore, $_{Qz}L^2(Qz)_{Qz} \underset{weak}{\subset} {}_{\theta(Qz)}L^2(\mathcal{M})\otimes L^2(Qz)_{Qz}$. Finally, we obtain $_{Qz}L^2(Qz)_{Qz} \underset{weak}{\subset} {}_{Qz}L^2(Qz)\otimes L^2(Qz)_{Qz}$, which means that Qz is amenable, contradiction.

In a similar way, one can prove that $\theta(Q)$ is strongly non-amenable relative to $M\bar{\otimes}1\bar{\otimes}M$. This ends the proof. ■

IV.5 Intertwining of abelian subalgebras

Throughout this section we will use the following notation. Let Γ be a countable group. Let Σ be an almost malnormal subgroup and let σ_0 be a tracial action of Σ on a non-trivial abelian von Neumann algebra A_0. Denote by σ the coinduced action of Γ on $A := A_0^{\Gamma/\Sigma}$. Finally, denote $M = A\rtimes\Gamma$.

The next result is a localization theorem for coinduced actions which goes back to[Io10, Theorem 6.1]. The form presented in this paper is very similar to [IPV10, Theorem

5.1], but written with coinduced actions instead of generalized Bernoulli ones.

Theorem IV.5.1. *Assume that $D \subset M\bar{\otimes}M$ is an abelian von Neumann subalgebra which is normalized by a group of unitaries $(\gamma(s))_{s\in\Lambda}$ that belong to $L(\Gamma)\bar{\otimes}L(\Gamma)$. Denote by P the quasi-normalizer of D inside $M\bar{\otimes}M$. We make the following assumptions:*

1. $D \nprec M \otimes 1$ *and* $D \nprec 1 \otimes M$,
2. $P \nprec M\bar{\otimes}(A \rtimes \Sigma)$ *and* $P \nprec (A \rtimes \Sigma)\bar{\otimes}M$,
3. $P \nprec M\bar{\otimes}L(\Gamma)$ *and* $P \nprec L(\Gamma)\bar{\otimes}M$,
4. $\gamma(\Lambda)'' \nprec L(\Gamma)\bar{\otimes}L(\Sigma)$ *and* $\gamma(\Lambda)'' \nprec L(\Sigma)\bar{\otimes}L(\Gamma)$.

Define $C := D' \cap (M\bar{\otimes}M)$. Then for every non-zero projection $q \in \mathcal{Z}(C)$ we have $Cq \prec A\bar{\otimes}A$.

The proof is identically with the one of [IPV10, Theorem 5.1], since essentially the same computations still hold once we replace generalized Bernoulli actions by coinduced ones.

Next, we obtain a similar statement if one considers an abelian von Neumann algebra in M and not in $M\bar{\otimes}M$.

Theorem IV.5.2. *Assume that $D \subset M$ is an abelian von Neumann subalgebra which is normalized by a group of unitaries $(\gamma(s))_{s\in\Lambda}$ that belong to $L(\Gamma)$. Denote by P the quasi-normalizer of D inside M. We make the following assumptions:*

1. *D is diffuse,*
2. $P \nprec A \rtimes \Sigma$,
3. $P \nprec L(\Gamma)$,

4. $\gamma(\Lambda)'' \nprec L(\Sigma)$.

Define $C := D' \cap M$. *Then for every non-zero projection* $q \in \mathcal{Z}(C)$ *we have* $Cq \prec A$.

As noticed in [Io10], we obtain as a corollary a weaker version of Popa's conjugacy criterion adapted in this case to coinduced actions.

Theorem IV.5.3. *Suppose* Γ *is icc and* Σ *is amenable. Let* $\Lambda \curvearrowright B$ *be another tracial action of a countable group* Λ *on a non-trivial abelian von Neumann algebra* B *such that* $M = A \rtimes \Gamma = B \rtimes \Lambda$ *and* $L(\Lambda) \subset L(\Gamma)$.

Then $B \prec A$.

Proof. The proof is a direct application of Theorem IV.5.2. Note that the quasi-normalizer of the abelian algebra B is M. Now, notice that if $M \prec A \rtimes \Sigma$, by [DHI16, Lemma 2.5.1] we obtain that $[\Gamma : \Sigma] < \infty$. This is not possible since Σ is almost malnormal in Γ. Also $L(\Lambda) \nprec L(\Sigma)$ since Σ is amenable and therefore we obtain $B \prec A$. ∎

IV.6 Proof of the main results

In [Io10], Ioana has proven that any Bernoulli action of an arbitrary icc property (T) group is W*-superrigid. The strategy of his proof was successfully applied also in [IPV10] and [Bo12b].

IV.6.1 A general method for obtaining W*-superrigidity.

Using Ioana's proof, we identify a couple of steps for proving that a certain free ergodic pmp action $\Gamma \curvearrowright X$ is W*- superrigid (see also the introduction of [Bo12b]). Consider an arbitrary free ergodic pmp action $\Lambda \curvearrowright Y$ such that $M := A \rtimes \Gamma = B \rtimes \Lambda$, where $A = L^\infty(X)$ and $B = L^\infty(Y)$. Define the comultiplication $\Delta : M \to M \bar{\otimes} L(\Lambda)$ by $\Delta(bv_\lambda) = bv_\lambda \otimes v_\lambda$, for all $b \in B, \lambda \in \Lambda$, where we denote by $v_\lambda, \lambda \in \Lambda$, the canonical unitaries corresponding to the action of Λ.

Step 1. One has to show that $\Gamma \curvearrowright X$ is OE superrigid. From now on, using Singer's result [Si55], it is enough to assume that B is not unitarily conjugated to A in M, which is equivalent to $B \nprec_M A$ [Po06b, Theorem A.1].

Step 2. One can also assume that there exists a non-zero projection $s_0 \in L(\Lambda)' \cap M$ such that $L(\Lambda)s_0 \nprec L(\Gamma)$.

Step 3. One shows that there exists a unitary $u \in \mathcal{U}(M \bar{\otimes} M)$ such that

$$u\Delta(L(\Gamma))u^* \subset L(\Gamma \times \Gamma).$$

Step 4. Next, one proves that the algebra $C := \Delta(A)' \cap (M \bar{\otimes} M)$ satisfies

$$Cq \prec_{M \bar{\otimes} M} A \bar{\otimes} A \text{ for all } q \in \mathcal{Z}(C).$$

Step 5. Using the previous steps together with a generalization of [Po04, Theorem 5.2], one essentially obtains that there exist a unitary $v \in \mathcal{U}(M \bar{\otimes} M)$, a group homomorphism $\delta : \Gamma \to \Gamma \times \Gamma$ and a character $\omega : \Gamma \to \mathbb{C}$ such that $vCv^* = A \bar{\otimes} A$ and $v\Delta(u_g)v^* = \omega(g)u_{\delta(g)}$, for all $g \in \Gamma$ (the precise statement is the Step 3 of the proof [IPV10, Theorem 10.1]).

Step 6. Using Step 5, one proves that for every sequence $(x_n)_n$ in M for which the Fourier coefficient (w.r.t. the decomposition $M = A \rtimes \Gamma$) converges to 0 pointwise in $\|\cdot\|_2$, then the Fourrier coefficient of $\Delta(x_n)$ (w.r.t. the decomposition $M \bar{\otimes} M = (M \bar{\otimes} A) \rtimes \Gamma$) also converges to 0 pointwise in $\|\cdot\|_2$. This shows $B \prec A$ and Step 1 implies that $\Gamma \curvearrowright X$ is W*-superrigid.

IV.6.2 Proof of Theorem H

We record first the following observation.

Remark IV.6.1. Since Σ is almost malnormal in Γ, using [Dr15, Lemma 5.3], the action $\Gamma \curvearrowright X$ is free (see also [Io06b, Lemma 2.1]).

Proof of Theorem H. Assume that $\Lambda \curvearrowright (Y, \nu)$ is an arbitrary free ergodic pmp action such that

$$M := L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda.$$

We put $A = L^\infty(X), B = L^\infty(Y)$. Define $\Delta : M \to M \bar{\otimes} M$ by $\Delta(bv_s) = bv_s \otimes v_s$, for all $b \in B$ and $s \in \Lambda$, where we denote by $v_s, s \in \Lambda$, the canonical unitaries corresponding to the action of Λ.

Since the action $\Gamma \curvearrowright X$ is OE superrigid (using [Dr15, Theorem A] and [Po05, Theorem 5.6]), Step 1 is completed. To prove Step (2), suppose $L(\Lambda)q \prec L(\Gamma)$ for all $q \in L(\Lambda)' \cap M$. Since Σ is amenable, $L(\Lambda) \nprec L(\Sigma)$, so by Corollary IV.3.6, there exists a unitary $u \in \mathcal{U}(M)$ such that $uL(\Lambda)u^* \subset L(\Gamma)$. Based on Step 1, Theorem IV.5.3 proves that $\Gamma \curvearrowright X$ is W*-superrigid. This completes Step 2. Therefore, we take a non-zero projection $q_0 \in L(\Lambda)' \cap M$ such that $L(\Lambda)q_0 \nprec L(\Gamma)$. Step (3) is obtained by combining Theorem IV.3.2 and [IPV10, Lemma 10.2.5].

Proof of Step (4). Note that Theorem IV.5.1 proves this step by considering the abelian subalgebra $D_0 := \Delta(A)(1 \otimes q_0)$. For showing this, denote $C_0 = D_0' \cap (M \bar{\otimes} q_0 M q_0)$, $C = \Delta(A)' \cap (M \bar{\otimes} M)$ and note that $C_0 = C(1 \otimes q_0)$. Since $L(\Lambda)q_0 \nprec L(\Gamma)$, [Io10, Lemma 9.2.4] implies that $\Delta(M)(1 \otimes q_0) \nprec M \bar{\otimes} L(\Gamma)$. Using [IPV10, Lemma 10.2], we see that all the conditions of Theorem IV.5.1 are satisfied. Therefore, we obtain that $C_0 q \prec A \bar{\otimes} A$, for all $q \in \mathcal{Z}(C_0) = \mathcal{Z}(C)(1 \otimes q_0)$. $\square$

Proof of Step (5). For proving Step (3) of the proof of [IPV10, Theorem 10.1], one only needs to show:

- If H is a subgroup of $\Gamma \times \Gamma$ such that H acts non-ergodically on $A \bar{\otimes} A$, then $\Delta(L(\Gamma)) \nprec (A \bar{\otimes} A) \rtimes H$.

Suppose by contrary that $\Delta(L(\Gamma)) \nprec (A \bar{\otimes} A) \rtimes H$. It is easy to prove that there exists a finite set $T \subset \Gamma$ such that $H \subset (\cup_{t \in T} t\Sigma) \times \Gamma$ or $H \subset \Gamma \times (\cup_{t \in T} t\Sigma)$. This implies that $\Delta(L(\Gamma)) \prec (A \rtimes \Sigma) \bar{\otimes} M$ or $\Delta(L(\Gamma)) \prec M \bar{\otimes} (A \rtimes \Sigma)$. By applying [IPV10, Lemma 10.2.5], we obtain a contradiction. □

Step (6) works in general once the other steps are proven. This finishes the proof of the theorem. ■

IV.6.3 Proof of Theorem I

In this subsection we will prove a more general statement of Theorem I.

Assumption IV.6.2. Let Σ be a subgroup of a countable icc group Γ. Let σ_0 be a pmp action of Σ on a non-trivial standard probability space (X_0, μ_0) and denote by σ the coinduced action of Γ on $X := X_0^{\Gamma/\Sigma}$. Suppose:

- Γ is a non-amenable icc group which is measure equivalent to a group Λ_0 for which the group von Neumann algebra $L(\Lambda_0)$ is not prime.
- Σ is amenable and almost malnormal.

Theorem IV.6.3. *Let $\Gamma \curvearrowright X$ be as in Assumption IV.6.2. Then $\Gamma \curvearrowright X$ is W*-superrigid.*

Proof. The proof of this theorem goes along the same lines as the proof of Theorem H. We point out only the differences. The action $\Gamma \curvearrowright X$ is OE superrigid using Theorem IV.4.5 and [Po05, Theorem 5.6]. Step (3) follows by Theorem IV.4.6. All the other steps follow as in the proof of Theorem H, which finishes the proof. ■

Remark IV.6.4. A careful handling of Thorem IV.4.6 shows that Assumption IV.6.2 can be improved by supposing the weaker assumption that $L(\Lambda_0)$ contains a commuting pair of diffuse subalgebras P_1 and P_2 such that P_2 is non-amenable and $\mathcal{N}_{L(\Lambda_0)}(P_1 \vee P_2)'' = L(\Lambda_0)$ (see also Step 1 of the proof of [IPV10, Theorem 8.2]).

Corollary IV.6.5. *Let* Γ *be an icc non-amenable group which is measure equivalent to a group* Λ_0 *for which* $L(\Lambda_0)$ *is not prime. Then the Bernoulli action* $\Gamma \curvearrowright (X,\mu)^\Gamma$ *is* W^**-superrigid, where* (X,μ) *is a non-trivial standard probability space.*

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