Title

A Boundary Element Algorithm Using Compatible Boundary Displacements and Tractions

Authors

Piltner, Reinhard

Taylor, Robert

Publication Date

1988-11-01

REPORT NO.
UCB/SEMM-88/19

STRUCTURAL ENGINEERING
MECHANICS AND MATERIALS

A BOUNDARY ELEMENT ALGORITHM USING COMPATIBLE BOUNDARY DISPLACEMENTS AND TRACTIONS

BY

R. PILTNER and R. L. TAYLOR

NOVEMBER 1988

DEPARTMENT OF CIVIL ENGINEERING
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA

A BOUNDARY ELEMENT ALGORITHM USING COMPATIBLE BOUNDARY DISPLACEMENTS AND TRACTIONS

R. PILTNER AND R. L. TAYLOR

Department of Civil Engineering, University of California, Berkeley, CA 94720, U.S.A.

SUMMARY

With the aid of Muskhelishvili's complex plane elasticity solution representation compatible displacement and stress fields are constructed. The complex functions in these formulas are represented by Cauchy-integrals, which are discretized along the boundary with the aid of complex shape functions for each boundary element. The constructed displacement and stress functions satisfy the Navier-equations and the equilibrium equations, respectively. The use of fifth order complex shape functions with continuous second complex derivatives gives numerical results of high accuracy.

1. INTRODUCTION

In the last years the development, improvement and application of boundary element methods and integral equation methods, respectively were increasing [1-15]. The starting point for most boundary element methods is a weighted residual statement, containing a domain integral over the product of a weighting function and the considered differential equation plus boundary integrals over the product of weighting functions and boundary condition terms. In order to get a boundary integral procedure so called fundamental solutions of the considered differential equation are used for the weighting functions. The substitution of the fundamental solution for a chosen point into the system of differential equations under consideration gives a vector of Dirac-delta-functions. After integration by part of the original weighted residual statement this property of the fundamental solution enables us to replace the remaining domain integral by an integral free term so that a boundary integral statement is obtained.

In the common boundary integral methods for elasticity problems independent shape functions for the boundary displacements and the boundary tractions are assumed. Normaly it is not possible to evaluate the boundary integrals analytically, so that numerical integration is needed. Applying numerical integration one must take into consideration that the integrands have singularities along the boundary curve.

A different strategy to get a boundary element algorithm for plane elasticity problems is described in the present paper. The starting point for this boundary element algorithm is the formulation of displacements and stresses in terms of arbitrary complex functions. This complex formulation which was used by Muskhelishvili in his fundamental work [16] ensures that the Navier-equations and the equilibrium equations, respectively, are automatically satisfied for any choice of complex functions.

It should be noticed that it is possible to represent also 3-dimensional elastic stress and displacement fields with the aid of complex valued functions [17-20] and that this 3-dimensional complex representation of displacements and stresses is equivalent to the real representation of Neuber/Papkovich [21-23].

The complex functions in the plane elasticity solution representation can be assumed to be complex power series in curvilinear coordinates via conformal mapping techniques as it was done in [24-26] to construct special finite elements, which are constructed via boundary integral evaluations. Here we use more general trial functions for the complex functions in Muskhelishvili's formulas in order to obtain a boundary integral algorithm.

For the complex functions we take Cauchy-integrals, which relate harmonic function values inside the solution domain to function values on the boundary [27-33]. The boundary is discretized into a number of straight line boundary elements, where complex shape functions are assumed. After the substitution of the complex shape functions into the Cauchy-integrals the integrals can be evaluated analytically. For the boundary element nodes Cauchy principle values have to be calculated as the integrands in the Cauchy-integrals are singular.

After the evaluation of the Cauchy-integrals, the consideration of all possible limit cases and the substitution of the integral results into the Muskhelishvili formulas we obtain displacement and stress functions which depend on discrete function values on the boundary. The use of the boundary conditions for chosen collocation points gives us a set of linear equations, from which the unknown discrete function values on the boundary can be computed.

2. REPRESENTATION OF DISPLACEMENTS AND STRESSES

Using two complex functions $\Phi(z)$ and $\Psi(z)$ we can represent the displacements and stresses for plane elasticity in the form [16]

$$\begin{aligned} 2\mu u &= \mathrm{Re}[\kappa\Phi(z) - z\overline{\Phi'(z)} - \overline{\Psi(z)}] \\ 2\mu v &= \mathrm{Im}[\kappa\Phi(z) - z\overline{\Phi'(z)} - \overline{\Psi(z)}] \\ \sigma_{xx} &= \mathrm{Re}[2\Phi'(z) - \bar{z}\Phi''(z) - \Psi'(z)] \\ \sigma_{yy} &= \mathrm{Re}[2\Phi'(z) + \bar{z}\Phi''(z) + \Psi'(z)] \\ \tau_{xy} &= \mathrm{Im}[\bar{z}\Phi''(z) + \Psi'(z)] \end{aligned} \tag{1}$$

where $z=x+iy$, $\mu=E/(2(1+\nu))$ and $\kappa=(3\text{-}4\nu)$ for plane strain and $\kappa=(3\text{-}\nu)/(1+\nu)$ for plane stress. $(\ )'$ denotes differentiation with respect to the complex variable z and $\overline{(\ )}$ denotes the complex conjugate. Re and Im mean "real part of" and "imaginary part of", respectively.

Every complex function f(z) (f stands now for Φ and Ψ) in a domain Ω with the boundary Γ can be represented by the Cauchy integral formula [27-33]

$$f(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta-z}d\zeta \tag{2}$$

where $f(\zeta)$ represents the boundary values of f(z) along the boundary curve Γ. Complex derivatives can be obtained with the aid of the formula

$$f^{(n)}(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta \tag{3}$$

For an elasticity problem with given boundary displacements $\bar{u}$, $\bar{v}$ and boundary tractions $\bar{T}_x$, $\bar{T}_y$ the boundary values of the complex functions $\Phi(z)$, $\Psi(z)$ are not known a priori. So we assume complex trial functions for the boundary values of Φ and Ψ after we have discretized the boundary into N boundary elements (Figure 1).

For the sake of simplicity we model the boundary with the aid of straight line boundary elements. The boundary elements have to lie on or enclose the real boundary since

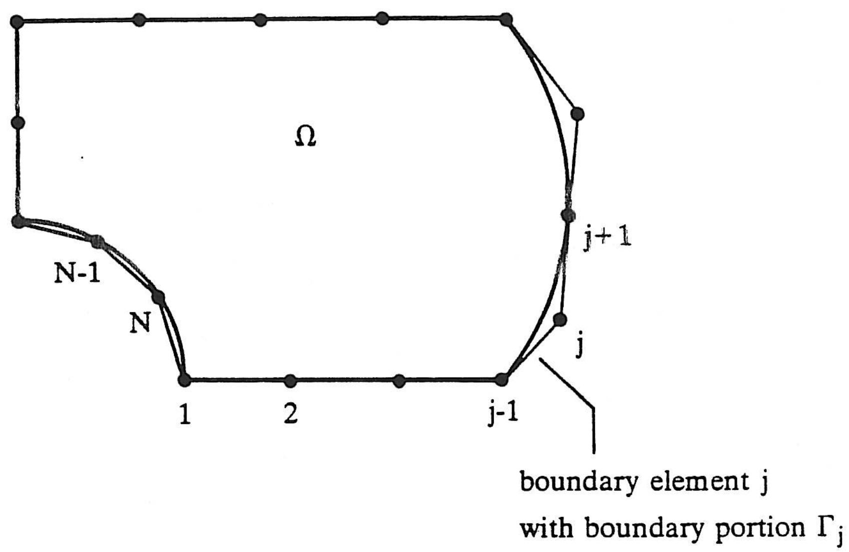


Figure 1: Boundary element discretization

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta-z}d\zeta = 0 \tag{4}$$

if z is in the exterior of Γ .

The modeling with straight line elements is no particular restriction as we can also fulfill boundary conditions on selected points of curved boundary segments (see example 2).
For an approximative solution we represent the boundary values of the complex function f (which stands for Φ and Ψ) with the aid of shape functions $\hat{f}^j$ for the boundary elements in the following way:

$$f(z)=\frac{1}{2\pi i}\sum_j\int_{\Gamma_j}\frac{\hat{f}^j(\zeta)}{\zeta-z}d\zeta \tag{5}$$

where Γ_j is the boundary portion of element j.
As the stress relationships (1) contain complex derivatives of the functions $\Phi(z)$ and $\Psi(z)$ we have to study what kind of shape functions can be used. From equation (3) it seems to be sufficient to require that only function values and not derivatives have to be continuous from element to element. A look at the relationships for the example of a linear complex shape function will show that C^0 continuity is not sufficient.

3. LINEAR COMPLEX SHAPE FUNCTION

A linear complex shape function for a boundary element with end nodes (j-1) and j (Figure 2) can be written as

$$\hat{f}^j(\zeta) = \frac{z_j - \zeta}{z_j - z_{j-1}} f_{j-1} + \frac{\zeta - z_{j-1}}{z_j - z_{j-1}} f_j \tag{6}$$

The complex value of ζ within the boundary element with nodes (j-1) and j can be expressed with the aid of a real normalized boundary coordinate s by

$$\zeta(s) = z_{j-1} + (z_j - z_{j-1})\, s \quad \cdot \text{ where } 0 \leq s \leq 1 \tag{7}$$

so that we can rewrite (6) as

$$\hat{f}^j(\zeta(s)) = (1 - s)\, f_{j-1} + s\, f_j \tag{8}$$

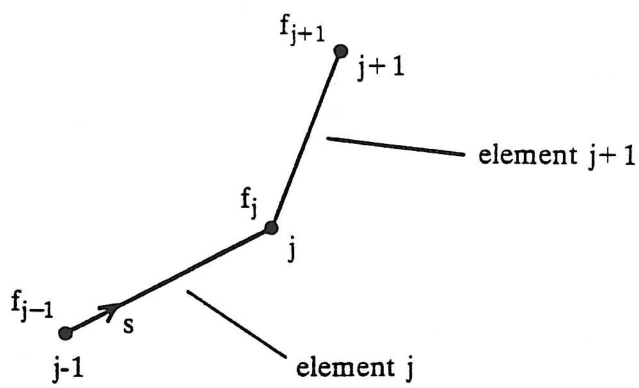


Figure 2: Boundary element notation

With $d\zeta = (z_j - z_{j-1})ds$ we obtain the contribution $f^j(z)$ of the boundary element j in the form

$$f^j(z) = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{\hat{f}^j(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{s=0}^{s=1} \frac{\hat{f}^j}{s - K} ds \tag{9}$$

where $$K = \frac{z - z_{j-1}}{z_j - z_{j-1}} \tag{10}$$

Since the boundary element shape function $\hat{f}^j$ is expressible as a polynomial in s we can use the formulas

$$\int \frac{s^n}{s-K} ds = \sum_{i=0}^{n-1} \frac{1}{n-i} K^i s^{n-i} + K^n \ln(s-K)$$

and (11)

$$\int_0^1 \frac{s^n}{s-K} ds = \sum_{i=0}^{n-1} \frac{1}{n-i} K^i + K^n \ln \frac{z_j - z}{z_{j-1} - z}$$

respectively, to evaluate the integral (9) analytically.

For the example of a linear boundary element shape function we obtain the contribution

$$f^j(z) = \frac{1}{2\pi i} \left\{ f_{j-1} \left[-1 + (1-K) \ln \frac{z_j - z}{z_{j-1} - z} \right] + f_j \left[1 + K \ln \frac{z_j - z}{z_{j-1} - z} \right] \right\} \tag{12}$$

for element j . From the next boundary element we get the contribution

$$f^{j+1}(z) = \frac{1}{2\pi i} \left\{ f_j \left[-1 + (1-L) \ln \frac{z_{j+1} - z}{z_j - z} \right] + f_{j+1} \left[1 + L \ln \frac{z_{j+1} - z}{z_j - z} \right] \right\} \tag{13}$$

where $L = \dfrac{z - z_j}{z_{j+1} - z_j}$ (14)

After the summation of all element contributions we obtain the complex trial function f(z) in the form

$$f(z) = [\, f_1 H_1(z) + f_2 H_2(z) + \cdots + f_j H_j(z) + \cdots + f_N H_N(z) \,] \tag{15}$$

where the f_j are complex function values at the chosen boundary element nodes. Using $f_j = f_j^r + i\, f_j^i$ (15) can be written in real form as

$$\mathrm{Re}[f(z)] = \Big[f_1^r\,\mathrm{Re}[H_1(z)] + f_2^r\,\mathrm{Re}[H_2(z)] + \cdots + f_j^r\,\mathrm{Re}[H_j(z)] + \cdots + f_N^r\,\mathrm{Re}[H_N(z)]$$
$$- f_1^i\,\mathrm{Im}[H_1(z)] - f_2^i\,\mathrm{Im}[H_2(z)] - \cdots - f_j^i\,\mathrm{Im}[H_j(z)] - \cdots - f_N^i\,\mathrm{Im}[H_N(z)] \Big]$$

and (16)

$$\mathrm{Im}[f(z)] = \Big[f_1^r\,\mathrm{Im}[H_1(z)] + f_2^r\,\mathrm{Im}[H_2(z)] + \cdots + f_j^r\,\mathrm{Im}[H_j(z)] + \cdots + f_N^r\,\mathrm{Im}[H_N(z)]$$
$$+ f_1^i\,\mathrm{Re}[H_1(z)] + f_2^i\,\mathrm{Re}[H_2(z)] + \cdots + f_j^i\,\mathrm{Re}[H_j(z)] + \cdots + f_N^i\,\mathrm{Re}[H_N(z)] \Big]$$

respectively. To investigate the behavior of f(z) for the case that z approaches a boundary node we write explicitly the general function term $H_j(z)$ as well as the neighboring terms $H_{j-1}(z)$ and $H_{j+1}(z)$:

$$f(z) = \frac{1}{2\pi i}\Bigg\{ \cdots + f_{j-1}\left[\frac{z - z_{j-2}}{z_{j-1} - z_{j-2}} \ln\frac{z_{j-1} - z}{z_{j-2} - z} + \frac{z_j - z}{z_j - z_{j-1}} \ln\frac{z_j - z}{z_{j-1} - z} \right]$$
$$+ f_j \left[\frac{z - z_{j-1}}{z_j - z_{j-1}} \ln\frac{z_j - z}{z_{j-1} - z} + \frac{z_{j+1} - z}{z_{j+1} - z_j} \ln\frac{z_{j+1} - z}{z_j - z} \right] \qquad (17)$$
$$+ f_{j+1}\left[\frac{z - z_j}{z_{j+1} - z_j} \ln\frac{z_{j+1} - z}{z_j - z} + \frac{z_{j+2} - z}{z_{j+2} - z_{j+1}} \ln\frac{z_{j+2} - z}{z_{j+1} - z} \right] + \cdots \Bigg\}$$

From (17) we can see that f(z) contains singular terms for $z=z_j$, $z=z_{j-1}$ and $z=z_{j+1}$ so that we have to look for the existence of limit values. These limit values exist for f(z). For the example of z approaching the node point z_j we obtain the limit value

$$\lim_{z\to z_j} f(z) = \frac{1}{2\pi i}\Bigg\{ \cdots + f_{j-2}\left[\frac{z - z_{j-3}}{z_{j-2} - z_{j-3}} \ln\frac{z_{j-2} - z}{z_{j-3} - z} + \frac{z_{j-1} - z}{z_{j-1} - z_{j-2}} \ln\frac{z_{j-1} - z}{z_{j-2} - z} \right]$$

$$
\begin{aligned}
&+ f_{j-1}\left[\frac{z_j - z_{j-2}}{z_{j-1} - z_{j-2}} \ln\frac{z_{j-1} - z_j}{z_{j-2} - z_j}\right] \\
&+ f_j \left[\ln\frac{z_{j+1} - z_j}{z_{j-1} - z_j}\right] \\
&+ f_{j+1}\left[\frac{z_{j+2} - z_j}{z_{j+2} - z_{j+1}} \ln\frac{z_{j+2} - z_j}{z_{j+1} - z_j}\right] \\
&+ f_{j+2}\left[\frac{z - z_{j+1}}{z_{j+2} - z_{j+1}} \ln\frac{z_{j+2} - z}{z_{j+1} - z} + \frac{z_{j+3} - z}{z_{j+3} - z_{j+2}} \ln\frac{z_{j+3} - z}{z_{j+2} - z}\right] + \cdots \Bigg\}
\end{aligned}
\tag{18}
$$

Since the stresses contain first and second order complex derivatives we have also to check the existence of limit values for $f'(z)$ and $f''(z)$ for the case that z approaches a node point. From relationship (17) we can get the first derivative of $f(z)$:

$$
\begin{aligned}
f'(z) = \frac{1}{2\pi i}\Bigg\{ \cdots &+ f_{j-1}\left[\frac{1}{z_{j-1} - z_{j-2}} \ln\frac{z_{j-1} - z}{z_{j-2} - z} - \frac{1}{z_j - z_{j-1}} \ln\frac{z_j - z}{z_{j-1} - z}\right] \\
&+ f_j \left[\frac{1}{z_j - z_{j-1}} \ln\frac{z_j - z}{z_{j-1} - z} - \frac{1}{z_{j+1} - z_j} \ln\frac{z_{j+1} - z}{z_j - z}\right] \\
&+ f_{j+1}\left[\frac{1}{z_{j+1} - z_j} \ln\frac{z_{j+1} - z}{z_j - z} - \frac{1}{z_{j+2} - z_{j+1}} \ln\frac{z_{j+2} - z}{z_{j+1} - z}\right] + \cdots \Bigg\}
\end{aligned}
\tag{19}
$$

Unfortunately there are no limit values when z approaches z_j , z_{j-1} or z_{j+1} so that the use of linear complex shape functions is not possible for the plane elasticity solution representation (1). In order to obtain expressions for $f'(z)$ and $f''(z)$ which have limit values for all boundary nodes $z=z_j$ we have to use complex shape functions with continuous values of f , f' , f'' at every node point. This means that the complex shape functions have to be at least fifth order polynomials.

Using relationship (1) we have to require Φ , Φ' , Φ'' and Ψ , Ψ' to be continuous at the element nodes as Φ appears with a second order derivative in (1) whereas Ψ appears with a first order derivative. But for the numerical examples in this paper the same type of functions have been used for Ψ and Φ so that both Φ and Ψ have continuous second order complex derivatives.

4. FIFTH ORDER COMPLEX SHAPE FUNCTIONS

A fifth order complex shape function with the complex node values f_j , f_j' , f_j'' , f_{j-1} , f_{j-1}' , f_{j-1}'' can be written in the following form:

$$\hat{f}^j(\zeta(s)) = \frac{1}{2}\Big\{ [\, 2 - 20s^3 + 30s^4 - 12s^5 \,]\, f_{j-1} \tag{20}$$
$$+ [\, 2s - 12s^3 + 16s^4 - 6s^5 \,]\, f_{j-1}'(z_j - z_{j-1})$$
$$+ [\, s^2 - 3s^3 + 3s^4 - s^5 \,]\, f_{j-1}''(z_j - z_{j-1})^2$$
$$+ [\, 20s^3 - 30s^4 + 12s^5 \,]\, f_j$$
$$+ [\, -8s^3 + 14s^4 - 6s^5 \,]\, f_j'(z_j - z_{j-1})$$
$$+ [\, s^3 - 2s^4 + s^5 \,]\, f_j''(z_j - z_{j-1})^2 \Big\}$$

and

$$\hat{f}^j(\zeta(s)) = N_1(s)\, f_{j-1} + N_2(s)\, f_{j-1}'(z_j - z_{j-1}) + N_3(s)\, f_{j-1}''(z_j - z_{j-1})^2 +$$
$$N_4(s)\, f_j + N_5(s)\, f_j'(z_j - z_{j-1}) + N_6(s)\, f_j''(z_j - z_{j-1})^2 \tag{21}$$

respectively, where

$$s = \frac{\zeta - z_{j-1}}{z_j - z_{j-1}} \tag{22}$$

and

$$N_1(s) = [\, 2 - 20s^3 + 30s^4 - 12s^5 \,]/2$$
$$N_2(s) = [\, 2s - 12s^3 + 16s^4 - 6s^5 \,]/2$$
$$N_3(s) = [\, s^2 - 3s^3 + 3s^4 - s^5 \,]/2 \tag{23}$$
$$N_4(s) = [\, 20s^3 - 30s^4 + 12s^5 \,]/2$$
$$N_5(s) = [\, -8s^3 + 14s^4 - 6s^5 \,]/2$$
$$N_6(s) = [\, s^3 - 2s^4 + s^5 \,]/2$$

After the substitution of (20) into (5) and evaluation of the Cauchy-integral we obtain the approximation function f(z) which depends on the boundary values f_j, f_j', f_j'' (j=1,2,...,N) of the chosen N node points on the boundary. From the evaluated f(z) one can obtain the first and second complex derivatives $f'(z)$ and $f''(z)$, respectively. But in order to get compact expressions for $f'(z)$ and $f''(z)$ it is not convenient to evaluate the derivatives from the expression for f(z). It is easier to use the formulas

$$f'(z)=\frac{1}{2\pi i}\sum_j\int_{\Gamma_j}\frac{\hat{f}^{j\,\prime}(\zeta)}{\zeta-z}d\zeta \tag{24}$$

and

$$f''(z)=\frac{1}{2\pi i}\sum_j\int_{\Gamma_j}\frac{\hat{f}^{j\,\prime\prime}(\zeta)}{\zeta-z}d\zeta \tag{25}$$

Equation (24) is only valid if the shape functions $\hat{f}^j$ are continuous at the node points, whereas (25) is only valid if both $\hat{f}^j$ and $\hat{f}^{j\,\prime}$ are continuous at the node points. The complex derivatives of the fifth order shape functions are given in the appendix.

Substitution of (20) into (9) and integration gives us the element contribution $f^j(z)$ in the form

$$\begin{aligned} f^j(z) = \frac{1}{4\pi i}\Bigg\{ & f_{j-1}\left[-\frac{47}{30} - 3K - 9K^2 + 24K^3 - 12K^4 + 2N_1(K)\ln\frac{z_j - z}{z_{j-1} - z}\right] \\ & + (z_j - z_{j-1})f_{j-1}'\left[\frac{4}{5} - \frac{13}{6}K - 6K^2 + 13K^3 - 6K^4 + 2N_2(K)\ln\frac{z_j - z}{z_{j-1} - z}\right] \\ & + (z_j - z_{j-1})^2 f_{j-1}''\left[\frac{1}{20} + \frac{1}{4}K - \frac{11}{6}K^2 + \frac{5}{2}K^3 - K^4 + 2N_3(K)\ln\frac{z_j - z}{z_{j-1} - z}\right] \\ & + f_j\left[\frac{47}{30} + 3K + 9K^2 - 24K^3 + 12K^4 + 2N_4(K)\ln\frac{z_j - z}{z_{j-1} - z}\right] \\ & + (z_j - z_{j-1})f_j'\left[-\frac{11}{30} - \frac{5}{6}K - 3K^2 + 11K^3 - 6K^4 + 2N_5(K)\ln\frac{z_j - z}{z_{j-1} - z}\right] \\ & + (z_j - z_{j-1})^2 f_j''\left[\frac{1}{30} + \frac{1}{12}K + \frac{1}{3}K^2 - \frac{3}{2}K^3 + K^4 + 2N_6(K)\ln\frac{z_j - z}{z_{j-1} - z}\right]\Bigg\} \end{aligned} \tag{26}$$

Adding neighboring element contributions we obtain for f(z) the representation

$$f(z) = \frac{1}{4\pi i}\Bigg\{ \cdots + f_j \Bigg[3K + 9K^2 - 24K^3 + 12K^4 + 2N_4(K)\ln\frac{z_j - z}{z_{j-1} - z}$$

$$- 3L - 9L^2 + 24L^3 - 12L^4 + 2N_1(L)\ln\frac{z_{j+1} - z}{z_j - z}\Bigg]$$

$$+ f_j' \Bigg[(z_j - z_{j-1})\left\{ -\frac{11}{30} - \frac{5}{6}K - 3K^2 + 11K^3 - 6K^4 + 2N_5(K)\ln\frac{z_j - z}{z_{j-1} - z}\right\}$$

$$+ (z_{j+1} - z_j)\left\{ \frac{4}{5} - \frac{13}{6}L - 6L^2 + 13L^3 - 6L^4 + 2N_2(L)\ln\frac{z_{j+1} - z}{z_j - z}\right\}\Bigg]$$

$$+ f_j'' \Bigg[(z_j - z_{j-1})^2\left\{ \frac{1}{30} + \frac{1}{12}K + \frac{1}{3}K^2 - \frac{3}{2}K^3 + K^4 + 2N_6(K)\ln\frac{z_j - z}{z_{j-1} - z}\right\} +$$

$$+ (z_{j+1} - z_j)^2\left\{ \frac{1}{20} + \frac{1}{4}L - \frac{11}{6}L^2 + \frac{5}{2}L^3 - L^4 + 2N_3(L)\ln\frac{z_{j+1} - z}{z_j - z}\right\}\Bigg] + \cdots \Bigg\} \tag{27}$$

where $$K = \frac{z - z_{j-1}}{z_j - z_{j-1}} \quad \text{and} \quad L = \frac{z - z_j}{z_{j+1} - z_j} \tag{28}$$

The complex derivatives $f'(z)$ and $f''(z)$ are given in the appendix.

A detailed look at the behavior of the functions $f(z)$, $f'(z)$ and $f''(z)$ on the boundary shows that all limit values exist. Within the evaluation of the limit values we have to treat cases of the form $\infty - \infty$ and $0^*\infty$.

We observe that all logarithm terms in (27) are multiplied by the complex shape functions N_i $(i=1,2,...,6)$. Accordingly, the logarithm terms of $f'(z)$ and $f''(z)$ given in the appendix are multiplied.by the shape function derivatives $\dot{N}_i$ and $\ddot{N}_i$, respectively. If we take for example the limit value of the term

$$N_1(L)\ \ln\frac{z_{j+1}-z}{z_j-z} = (2-20L^3+30L^4-12L^5)\ \ln\frac{z_{j+1}-z}{z_j-z} \tag{29}$$

of equation (27) for $z=z_{j+1}$ we get after applying several times l'Hospital's rule the limit expression

$$\lim_{z\to z_{j+1}} N_1(L)\ \ln\frac{z_{j+1}-z}{z_j-z} = \lim_{z\to z_{j+1}} \frac{\ln(z_{j+1}-z)}{\dfrac{1}{N_1(L)}} = \lim_{z\to z_{j+1}} \frac{6N_1'N_1'' + 2N_1N_1'''}{[z_{j+1}-z]\ N_1'''' - 3N_1'''} = 0 \tag{30}$$

where

$$N_1'(L) = \frac{dN_1}{dz} = \frac{dN_1}{dL}\ \frac{dL}{dz} = \dot{N}_1\ \frac{dL}{dz}$$

$$\lim_{z\to z_{j+1}} N_1(L) = 0$$

$$\lim_{z\to z_{j+1}} N_1'(L) = 0$$

$$\lim_{z\to z_{j+1}} N_1''(L) = 0 \tag{31}$$

$$\lim_{z\to z_{j+1}} N_1'''(L) = \frac{-120}{(z_{j+1}-z_j)^2} \neq 0$$

Accordingly we obtain

$$\lim_{z\to z_{j-1}} N_i(K)\ \ln\frac{z_j-z}{z_{j-1}-z} = \lim_{z\to z_{j-1}} \dot{N}_i(K)\ \ln\frac{z_j-z}{z_{j-1}-z} = \lim_{z\to z_{j-1}} \ddot{N}_i(K)\ \ln\frac{z_j-z}{z_{j-1}-z} = 0 \tag{32}$$

(where i = 4, 5, 6)

$$\lim_{z\to z_{j+1}} N_i(L)\ \ln\frac{z_{j+1}-z}{z_j-z} = \lim_{z\to z_{j+1}} \dot{N}_i(L)\ \ln\frac{z_{j+1}-z}{z_j-z} = \lim_{z\to z_{j+1}} \ddot{N}_i(L)\ \ln\frac{z_{j+1}-z}{z_j-z} = 0 \tag{33}$$

(where i = 1, 2, 3)

The evaluation of the limit values of the function terms with the coefficients f_j , f_j' , f_j'' depends on the location of z. For the node point $z=z_j$ we obtain

$$f(z_j) = \frac{1}{4\pi i}\left\{ f_j \left[2\ln\frac{z_j - z_{j+1}}{z_j - z_{j-1}} \right] + f_j' \left[\frac{4}{5}\left\{ (z_j - z_{j-1}) + (z_{j+1} - z_j) \right\} \right] + f_j'' \left[\frac{1}{20}\left\{ -(z_j - z_{j-1})^2 + (z_{j+1} - z_j)^2 \right\} \right] + \text{remaining terms} \right\}$$

$$f'(z_j) = \frac{1}{4\pi i}\left\{ f_j \left[-\frac{5}{z_j - z_{j-1}} - \frac{5}{z_{j+1} - z_j} \right] + f_j' \left[2\ln\frac{z_j - z_{j+1}}{z_j - z_{j-1}} \right] + f_j'' \left[\frac{z_j - z_{j-1}}{4} + \frac{z_{j+1} - z_j}{4} \right] + \text{remaining terms} \right\} \tag{34}$$

$$f''(z_j) = \frac{1}{4\pi i}\left\{ f_j \left[\frac{20}{(z_j - z_{j-1})^2} - \frac{20}{(z_{j+1} - z_j)^2} \right] + f_j' \left[-\frac{16}{z_j - z_{j-1}} - \frac{16}{z_{j+1} - z_j} \right] + f_j'' \left[2\ln\frac{z_j - z_{j+1}}{z_j - z_{j-1}} \right] + \text{remaining terms} \right\}$$

The remaining terms can be either singular expressions with limit values or regular expressions. For the node point $z=z_{j-1}$ we get the limit values with the aid of (32) and $K(z=z_{j-1})=0$, $L(z=z_{j-1})=(z_{j-1}-z_j)/(z_{j+1}-z_j)$ whereas for node point $z=z_{j+1}$ we use (33) and

$K(z=z_{j+1})=(z_{j+1}-z_{j-1})/(z_j-z_{j-1})$ as well as $L(z=z_{j+1})=1$.

So the constructed approximation trial function f(z) is continuous and finite everywhere in the domain Ω and on the boundary Γ .

5. THE COMPUTATION OF COMPLEX LOGARITHM TERMS

The complex logarithm

$$g(z) = \ln (z_j - z) = \ln (r_j e^{i\phi_j}) = \ln r_j + i \phi_j \qquad (35)$$

is a multivalued function. In order to define a single valued function in our analysis region we need to select an appropriate branch cut starting from the branch point z_j (Figure 3).

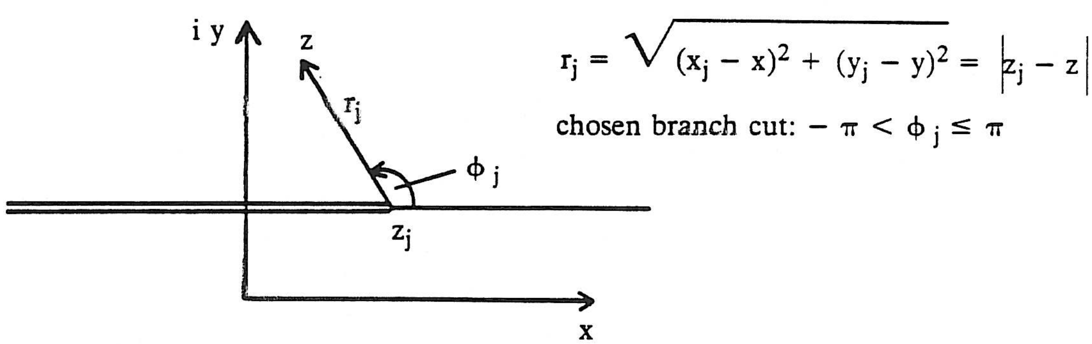


Figure 3: Branch cut definition

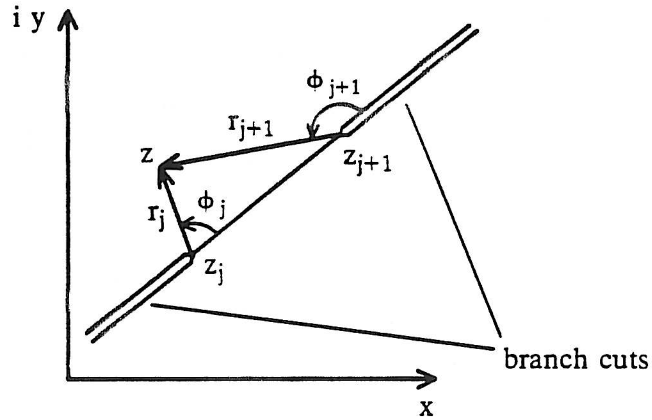
There are an infinite number of possibilities to define a branch cut. Along the chosen branch cut the logarithmic function has a jump.

Although logarithmic functions have jumps we can use these functions for general plane elasticity problems since the logarithm of the fraction $(z_{j+1} - z)/(z_j - z)$ can be written as

$$\ln\frac{z_{j+1} - z}{z_j - z} = \ln (z_{j+1} - z) - \ln (z_j - z) \qquad (36)$$

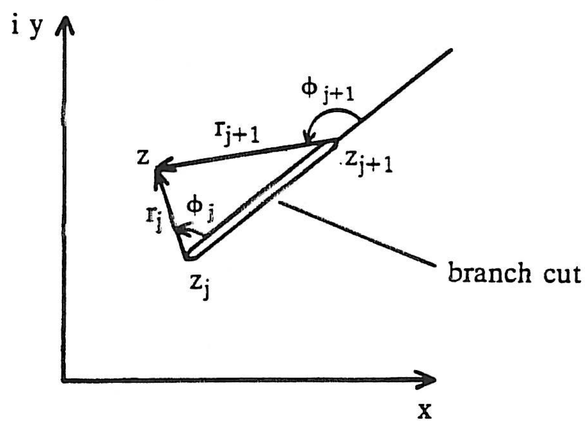
$$= \ln r_{j+1} + i \phi_{j+1} - \ln r_j - i \phi_j$$

Accordingly, the branch cut definition for ϕ_{j+1} and ϕ_j can be chosen in such a way that $\ln[(z_{j+1} - z)/(z_j - z)]$ has no jumps inside the solution domain of an elasticity problem.



a) branch cut definition: $-\pi < \phi_j \leq \pi$

$0 \leq \phi_{j+1} < 2\pi$



b) branch cut definition: $0 \leq \phi_j < 2\pi$

$-\pi < \phi_{j+1} \leq \pi$

Figure 4: Two branch cut definitions for $\ln[(z_{j+1} - z)/(z_j - z)]$

The choice of branch cuts for boundary element calculations is not arbitrary. A branch cut definition as in Figure 4a for a boundary element between node points z_j and z_{j+1} would cause a jump of function values inside the solution domain if a branch cut line crossed the solution domain Ω as shown in Figure 5.

With the definition of the branch cut according to Figure 4b we obtain a branch cut line of

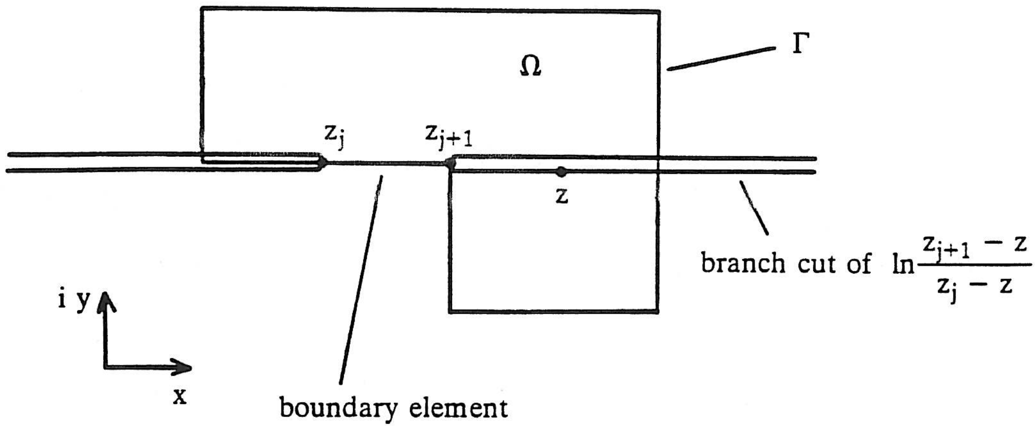


Figure 5: Branch cut crossing the solution domain Ω

finite length. The definition shown in Figure 4b can be used for arbitrary solution domains for which the boundary is discretized into straight line boundary elements. Thus the branch cut of a boundary element must lie exactly on the element. The cut is arranged in such a way that one surface of the cut is identical with the boundary element line and the opposite surface lies outside the solution domain. Using the branch cut definition of Figure 4b for the boundary element algorithm we enclose the solution domain with a series of branch cuts of finite length (Figure 6).

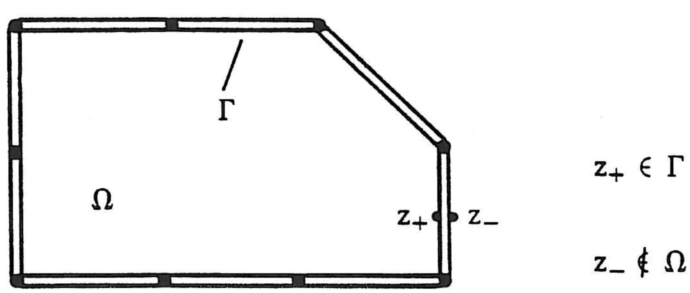


Figure 6: Solution domain Ω enclosed by branch cuts along the boundary elements

As there is a jump of function values from z_+ to z_- (where z_+ lies on the boundary of Γ of the domain Ω and z_- lies a little outside Ω) caused by the branch cut of the logarithmic function one has to take special care in programming the evaluation of the angles of the complex logarithms to avoid wrong values due to roundoff errors.

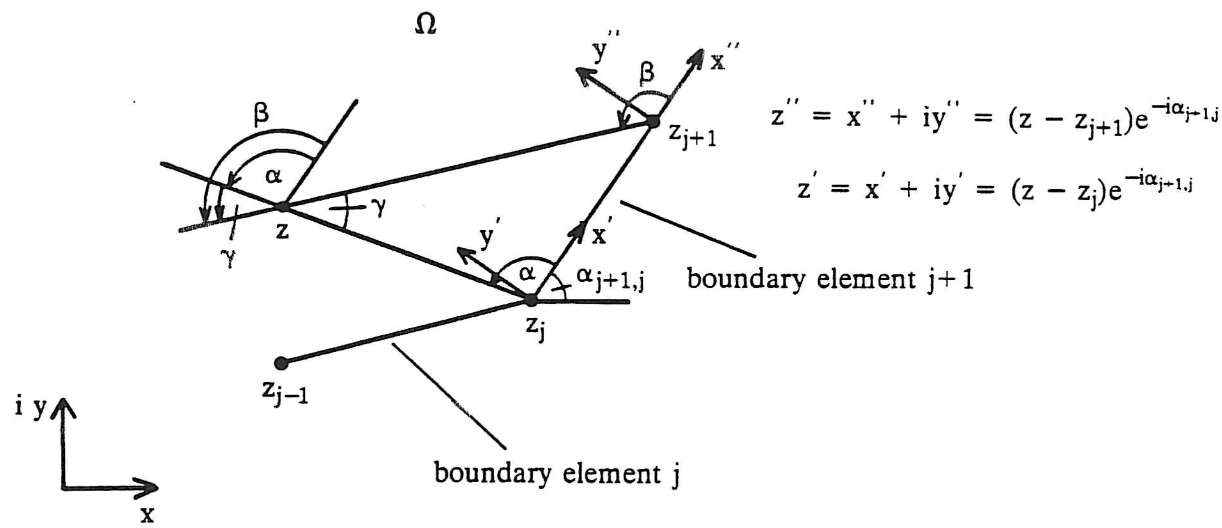


Figure 7: Representation of domain point z in local coordinates x', y' and x'', y'' with origin in the boundary element nodes z_j and z_{j+1}, respectively.

With the notations explained in Figure 7 we can compute the complex value of $\ln[(z_{j+1} - z)/(z_j - z)]$ in the following way:

$$\ln\frac{z_{j+1} - z}{z_j - z} = \ln\frac{(z - z_{j+1})e^{-i\alpha_{j+1,j}}}{(z - z_j)e^{-i\alpha_{j+1,j}}} = \ln\left|\frac{z_{j+1} - z}{z_j - z}\right| + i\gamma \tag{37}$$

where
$$\alpha_{j+1,j} = \arctan\frac{y_{j+1} - y_j}{x_{j+1} - x_j} \tag{38}$$

$$\begin{aligned}\gamma &= \quad \beta \quad - \quad \alpha \\ &= \arg\left([z - z_{j+1}]e^{-i\alpha_{j+1,j}}\right) - \arg\left([z - z_j]e^{-i\alpha_{j+1,j}}\right)\end{aligned} \tag{39}$$

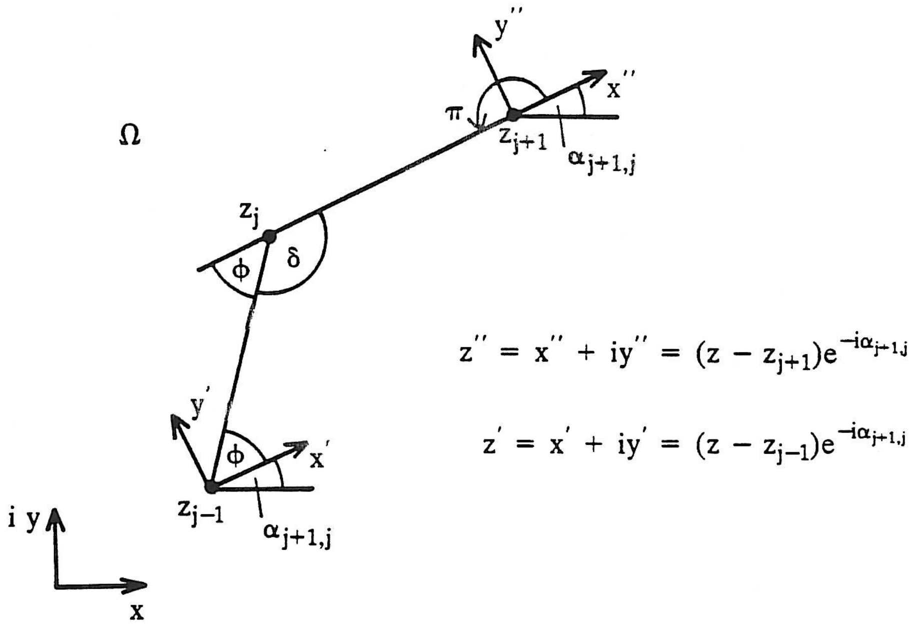


Figure 8: Notations for the evaluation of $\ln[(z_j - z_{j+1})/(z_j - z_{j-1})]$

For the computation of $\ln[(z_j - z_{j+1})/(z_j - z_{j-1})]$ appearing in the expressions (34) for the limit case $z = z_j$ we can use the rotated local coordinates x' , y' and x'' , y'' shown in Figure 8 to obtain

$$\ln\frac{z_j - z_{j+1}}{z_j - z_{j-1}} = \ln\frac{(z_j - z_{j+1})e^{-i\alpha_{j+1,j}}}{(z_j - z_{j-1})e^{-i\alpha_{j+1,j}}} = \ln\left|\frac{z_j - z_{j+1}}{z_j - z_{j-1}}\right| + i\delta \tag{40}$$

where $\delta = \pi - \phi$ (41)

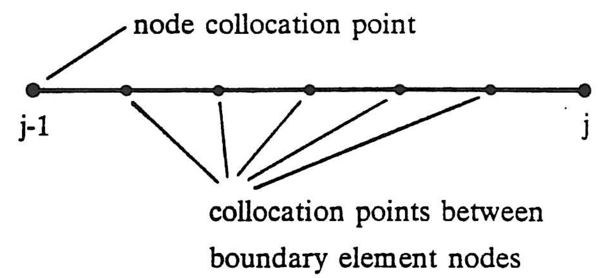
$$= \arg\left\{[z_j - z_{j+1}]e^{-i\alpha_{j+1,j}}\right\} - \arg\left\{[z_j - z_{j-1}]e^{-i\alpha_{j+1,j}}\right\}$$

6. BOUNDARY ELEMENTS AND COLLOCATION POINTS

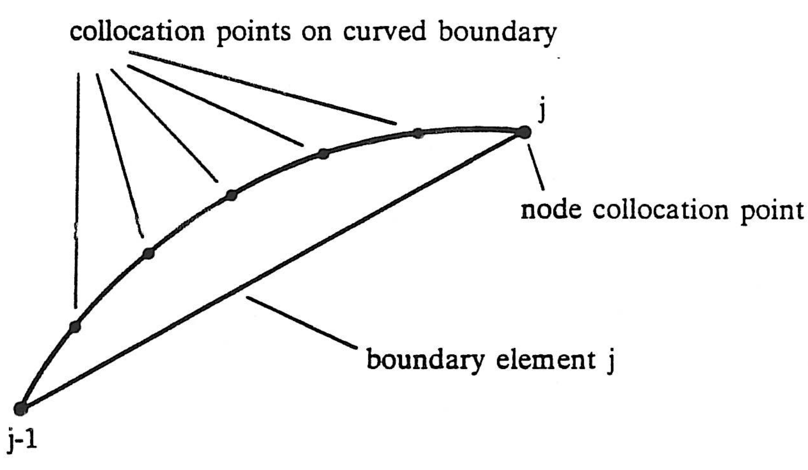
For the complex functions $\Phi(z)$, $\Psi(z)$ in (1) we use the fifth order complex shape functions described above. Since both Φ and Ψ can represent rigid body translations we need to eliminate the linear dependent function terms. This can be done by omitting one complex function term of $\Psi(z)$ using the complex coefficient Ψ_i of node i , which can be easily realized by choosing i=1.

The boundary of a problem domain under consideration is discretized into N boundary elements with N node points. For every node point j we have the six complex unknowns Φ_j , Φ_j' , Φ_j'' , Ψ_j , Ψ_j' , Ψ_j'' except for node 1 where we have only Φ_1 , Φ_1' , Φ_1'' , Ψ_1' , Ψ_1'' . Accordingly, we have $6(N - 1) + 5 = 6N - 1$ complex unknowns and $12N - 2$ real unknowns, respectively.

In order to obtain a system of equations for the evaluation of the unknown parameters we satisfy the given boundary conditions at selected collocation points. The collocation points are chosen to be the node points and equidistant additional points between the node points (Figure 9a).



a)



b)

Figure 9: Choice of collocation points

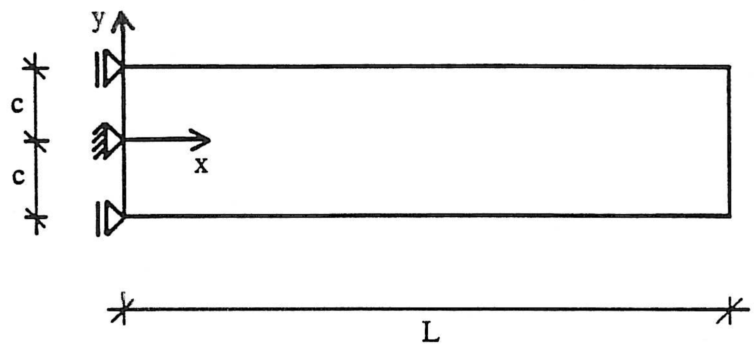
For every boundary element we choose 5 internal collocation points except for the first element, where we have 4 internal collocation points to satisfy the rigid body requirement described above.

If the real boundary is curved (as in example 2) we can even choose collocation points on the given curved boundary (Figure 9b).

7. NUMERICAL EXAMPLES

In order to illustrate the accuracy of the proposed boundary element algorithm examples have been chosen for which exact solutions are known.

Example 1 (Figure 10)



Boundary conditions: $u(0,0) = v(0,0) = 0$

$$u(0,\pm c) = 0$$

$$T_x(x,\pm c) = T_y(x,\pm c) = 0$$

$$T_x(L,y) = 0$$

$$T_y(L,y) = \frac{P}{2I}(c^2 - y^2)$$

$$T_x(0,y) = \frac{PLy}{I}$$

$$T_y(0,y) = -\frac{P}{2I}(c^2 - y^2)$$

where $I = 2c^3/3$.

data for calculations: $P = -1$, $L = 16$, $c = 2$, $E = 1$

$$\nu = \begin{cases} 0 \\ 0.3 \\ 0.5 \end{cases}$$

Figure 10: Plane strain elasticity problem

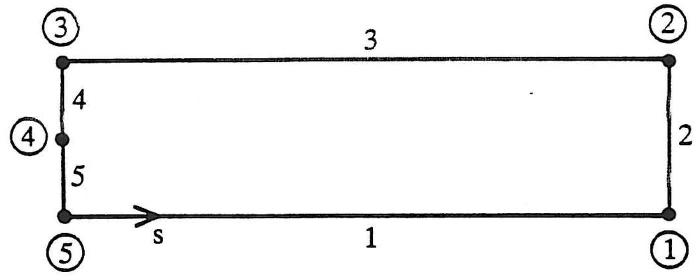


Figure 11: Boundary element discretization for example 1

The plane strain elasticity problem defined in Figure 10 is taken from the book of Hughes ([34], page 220) in which also the exact solution is given on page 255. With 5 boundary elements according to Figure 11 we obtain the exact solution: for the maximum displacement v(16,0) we obtain the values v(16,0) = 264.0 for $\nu = 0$, v(16,0) = 244.14 for $\nu = 0.3$, and v(16,0) = 205.5 for $\nu = 0.5$. The nonvanishing stresses σ_{xx} and τ_{xy} are plotted along the boundary of the domain (Figure 12). The four edges 5 - 1, 1 - 2 , 2 - 3, 3 - 5 appear with the unit length "1" in these plots.

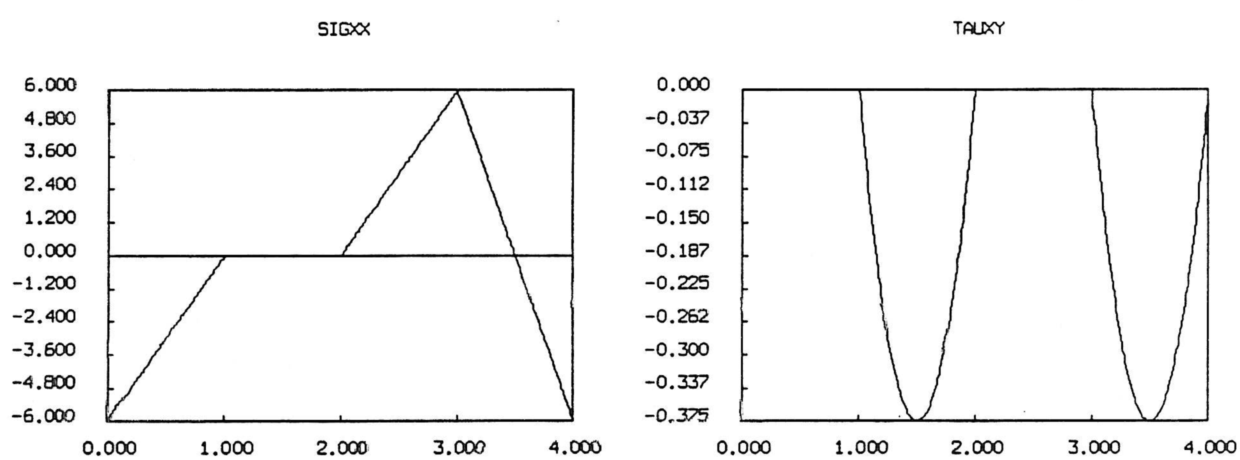


Figure 12: Stresses of example 1 along the domain boundary (starting from node 5 in direction of node 1)

Example 2 (Figure 13)

For a plate strip with a circular hole under normal tension several numerical computations have been made using different ratios between hole radius r_0 and width 2b of the strip (Figure 13). The comparison solution is taken from a paper by Howland [35].

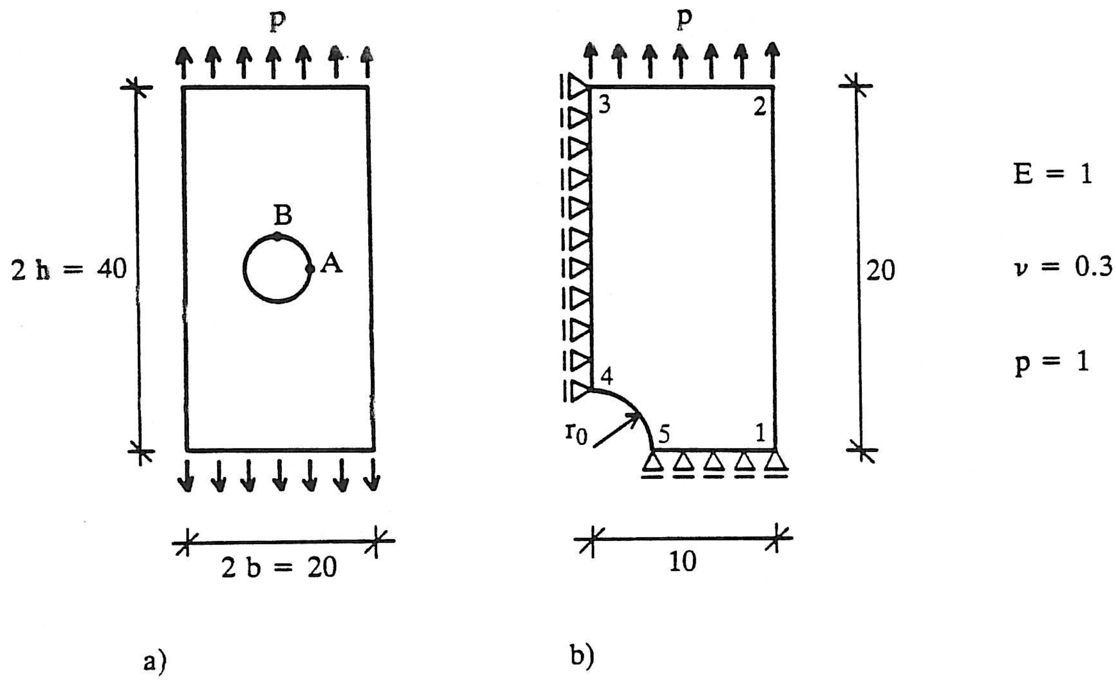


Figure 13: a) Plate strip with circular hole. b) Quarter system

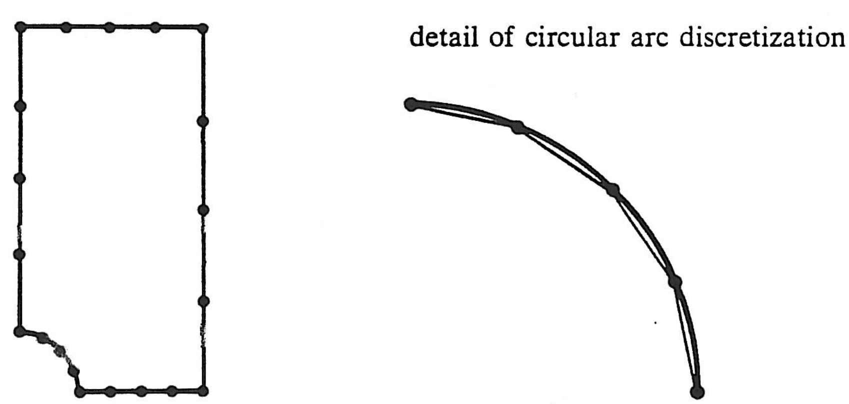


Figure 14: Boundary element discretization with 20 elements

Table I: Stress tip results for a Howland strip for different ratios of r_0/b

case		10 boundary elements	20 boundary elements	Howland [35]
$r_0 = 2$	$\sigma_{\phi\phi}$ at A	3.562	3.141	3.14
$r_0/b = 0.2$	$\sigma_{\phi\phi}$ at B	-1.405	-1.115	-1.11
$r_0 = 3$	$\sigma_{\phi\phi}$ at A	3.524	3.365	3.36
$r_0/b = 0.3$	$\sigma_{\phi\phi}$ at B	-1.429	-1.257	-1.26
$r_0 = 4$	$\sigma_{\phi\phi}$ at A	3.825	3.749	3.74
$r_0/b = 0.4$	$\sigma_{\phi\phi}$ at B	-1.566	-1.449	-1.44

It should be mentioned that although we used straight line boundary elements for the hole surface (Figure 14) all collocation points for the boundary conditions were chosen on the real boundary.

The stress tip results for a coarse mesh with 10 elements and another mesh with 20 elements are given in Table I . For the case $r_0 = 3$ the displacements and tractions along the boundary of the system obtained with a discretization according to Figure 14 are plotted in the Figures 15 and 16, respectively. Instead of real distances between the nodes 5 - 1, 1 - 2, 2 - 3, 3 - 4, 4 - 5 unit lengths have been used in the plots.

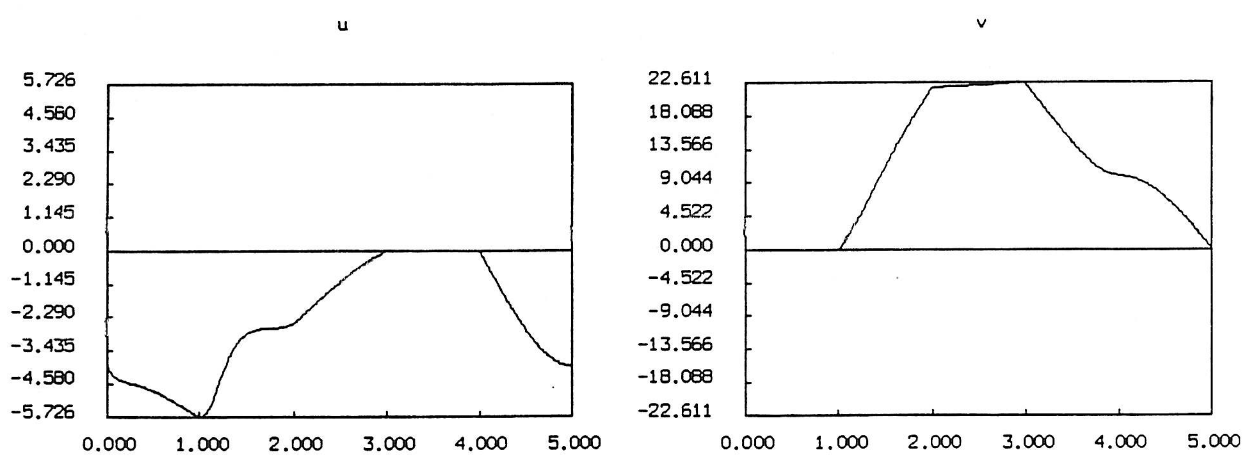


Figure 15: Displacements along the boundary between nodes 5-1-2-3-4-5

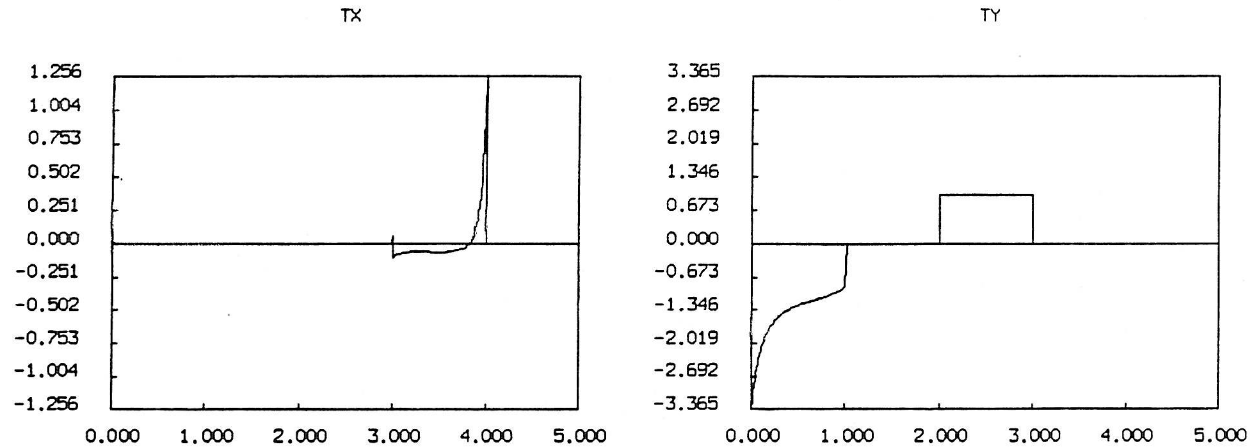
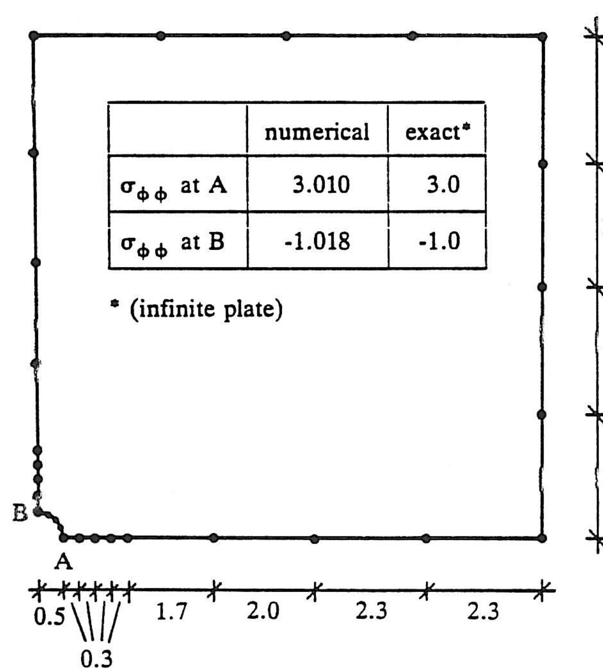


Figure 16: Tractions along the boundary between nodes 5-1-2-3-4-5

Example 3: Model for an infinite plate with hole under tension

As in the finite element method we need small boundary elements in areas, were great changes in stresses are expected, whereas in other areas larger elements are sufficient. As an example for this the boundary element discretization for a plate with a relative small hole (r_0 = 0.5, b = 10 , according to Figure 13) is given in Figure 17 as well as the stress tip results.



	numerical	exact*
$\sigma_{\phi\phi}$ at A	3.010	3.0
$\sigma_{\phi\phi}$ at B	-1.018	-1.0

* (infinite plate)

Figure 17: Boundary element discretization for plate with r_0/b = 0.5/10 and stress tip results

8. CONCLUSIONS

A plane elasticity boundary element concept with compatible displacements and stresses has been derived. The boundary integrals could be integrated analytically. The case of straight line boundary elements is no particular restriction as it is possible within the presented concept to choose collocation points also on curved boundaries. High accuracy could be achieved with simple discretizations.
In a following paper we will discuss the possibility to obtain a symmetric coefficient matrix in the presented boundary element algorithm as well as the evaluation of finite element stiffness matrices via boundary integral techniques.

APPENDIX

The fifth order complex shape functions $\hat{f}^j$ and their complex derivatives can be written in the following form:

$$\hat{f}^j(\zeta(s)) = N_1(s)\, f_{j-1} + N_2(s)\, f'_{j-1}(z_j - z_{j-1}) + N_3(s)\, f''_{j-1}(z_j - z_{j-1})^2 + N_4(s)\, f_j + N_5(s)\, f'_j(z_j - z_{j-1}) + N_6(s)\, f''_j(z_j - z_{j-1})^2 \tag{A1}$$

$$\hat{f}^{j\prime}(\zeta(s)) = \dot{N}_1(s)\, f_{j-1}/(z_j - z_{j-1}) + \dot{N}_2(s)\, f'_{j-1} + \dot{N}_3(s)\, f''_{j-1}(z_j - z_{j-1}) + \dot{N}_4(s)\, f_j/(z_j - z_{j-1}) + \dot{N}_5(s)\, f'_j + \dot{N}_6(s)\, f''_j(z_j - z_{j-1}) \tag{A2}$$

$$\hat{f}^{j\prime\prime}(\zeta(s)) = \ddot{N}_1(s)\, f_{j-1}/(z_j - z_{j-1})^2 + \ddot{N}_2(s)\, f'_{j-1}/(z_j - z_{j-1}) + \ddot{N}_3(s)\, f''_{j-1} + \ddot{N}_4(s)\, f_j/(z_j - z_{j-1})^2 + \ddot{N}_5(s)\, f'_j/(z_j - z_{j-1}) + \ddot{N}_6(s)\, f''_j \tag{A3}$$

where

$$\begin{aligned}
N_1(s) &= [\, 2 - 20s^3 + 30s^4 - 12s^5 \,]/2 \\
N_2(s) &= [\, 2s - 12s^3 + 16s^4 - 6s^5 \,]/2 \\
N_3(s) &= [\, s^2 - 3s^3 + 3s^4 - s^5 \,]/2 \\
N_4(s) &= [\, 20s^3 - 30s^4 + 12s^5 \,]/2 \\
N_5(s) &= [\, -8s^3 + 14s^4 - 6s^5 \,]/2 \\
N_6(s) &= [\, s^3 - 2s^4 + s^5 \,]/2
\end{aligned} \tag{A4}$$

and

$$\dot{N}_1(s) = [\quad - 60s^2 + 120s^3 - 60s^4]/2$$
$$\dot{N}_2(s) = [\ 2 \ - 36s^2 + 64s^3 - 30s^4]/2$$
$$\dot{N}_3(s) = [\ 2s \ - \ 9s^2 + \ 12s^3 - \ 5s^4]/2 \qquad (A5)$$
$$\dot{N}_4(s) = [\quad 60s^2 - 120s^3 + 60s^4]/2$$
$$\dot{N}_5(s) = [\quad -24s^2 + 56s^3 - 30s^4]/2$$
$$\dot{N}_6(s) = [\quad 3s^2 \ - \ 8s^3 \ + \ 5s^4]/2$$

and

$$\ddot{N}_1(s) = [\ - 120s + 360s^2 - 240s^3]/2$$
$$\ddot{N}_2(s) = [\quad - 72s + 192s^2 - 120s^3]/2$$
$$\ddot{N}_3(s) = [\ 2 \ - \ 18s + \ 36s^2 - \ 20s^3]/2 \qquad (A6)$$
$$\ddot{N}_4(s) = [\quad 120s - 360s^2 + 240s^3]/2$$
$$\ddot{N}_5(s) = [\quad -48s + 168s^2 - 120s^3]/2$$
$$\ddot{N}_6(s) = [\quad 6s \ - \ 24s^2 \ + 20s^3]/2$$

The complex derivatives of the approximation function f(z) given by equation (27) can be obtained with (A2) and (A3) from equations (24) and (25) as follows:

$$f'(z) = \frac{1}{4\pi i}\left\{ \cdots + f_j \left[\frac{1}{z_j - z_{j-1}} \left\{ 5 + 20K - 90K^2 + 60K^3 + 2\dot{N}_4(K) \ln\frac{z_j - z}{z_{j-1} - z} \right\} \right.\right.$$
$$\left. + \frac{1}{z_{j+1} - z_j} \left\{ -5 - 20L + 90L^2 - 60L^3 + 2\dot{N}_1(L) \ln\frac{z_{j+1} - z}{z_j - z} \right\} \right]$$
$$+ f_j' \left[\quad - 5 - 6K - 41K^2 - 30K^3 + 2\dot{N}_5(K) \ln\frac{z_j - z}{z_{j-1} - z} \right.$$
$$\left. - 14L + 49L^2 + 30L^3 + 2\dot{N}_2(L) \ln\frac{z_{j+1} - z}{z_j - z} \right] \qquad (A7)$$
$$+ f_j'' \left[(z_j - z_{j-1}) \left\{ \frac{1}{12} + \frac{2}{3}K - \frac{11}{2}K^2 + 5K^3 + 2\dot{N}_6(K) \ln\frac{z_j - z}{z_{j-1} - z} \right\} + \right.$$

$$+ \; (z_{j+1} - z_j)\left\{ \frac{1}{4} - \frac{14}{3}L + \frac{19}{2}L^2 - 5L^3 + 2\dot{N}_3(L)\ln\frac{z_{j+1} - z}{z_j - z} \right\}\Bigg] + \cdots \Bigg\}$$

$$f''(z) = \frac{1}{4\pi i}\Bigg\{ \cdots + f_j\Bigg[\frac{1}{(z_j - z_{j-1})^2}\left\{ 20 - 240K + 240K^2 + 2\ddot{N}_4(K)\ln\frac{z_j - z}{z_{j-1} - z} \right\}$$

$$+ \frac{1}{(z_{j+1} - z_j)^2}\left\{ -20 + 240L - 240L^2 + 2\ddot{N}_1(L)\ln\frac{z_{j+1} - z}{z_j - z} \right\}\Bigg]$$

$$+ f_j'\Bigg[\frac{1}{z_j - z_{j-1}}\left\{ -4 + 108K - 120K^2 + 2\ddot{N}_5(K)\ln\frac{z_j - z}{z_{j-1} - z} \right\}$$

$$+ \frac{1}{z_{j+1} - z_j}\left\{ -16 + 132L - 120L^2 + 2\ddot{N}_2(L)\ln\frac{z_{j+1} - z}{z_j - z} \right\}\Bigg] \qquad \text{(A8)}$$

$$+ f_j''\Bigg[-6 - 14K + 20K^2 + 2\ddot{N}_6(K)\ln\frac{z_j - z}{z_{j-1} - z}$$

$$+ 26L - 20L^2 + 2\ddot{N}_3(L)\ln\frac{z_{j+1} - z}{z_j - z}\Bigg] + \cdots \Bigg\}$$

ACKNOWLEDGEMENT

The first author gratefully acknowledges his support from the DFG (Deutsche Forschungsgemeinschaft).

REFERENCES

[1] F.J. Rizzo, "An integral equation approach to boundary value problems of classical elastostatics", Quart. Appl. Math., 25 (1), 83 - 95, (1967)

[2] T.A. Cruse, "Numerical solutions in three-dimensional elastostatics", Int. J. Solids Struct., 5, 1259 - 1274, (1969)

[3] J.C. Lachat, J.O. Watson, "Effective numerical treatment of boundary integral equations: a formulation for three-dimensional elastostatics", Int. J. Num. Methods Eng., 10, 991 - 1005, (1976)

[4] M.A. Jaswon, G.T. Symm, "Integral Equation Methods in Potential Theory and Elastostatics", Academic Press, London 1977

[5] O.C. Zienkiewicz, D.W. Kelly, P. Bettess, "The coupling of finite element and boundary solution procedures", Int. J. Num. Meth. Eng., 11, 355 - 376, (1977)

[6] D.W. Kelly, G.G.W. Mustoe, O.C. Zienkiewicz, "Coupling Boundary Element Methods with other Numerical Methods", pp. 251 - 285 , in: "Developments in Boundary Element Methods - 1", Edited by P.K. Banerjee/ R. Butterfield, Applied Science Publishers, London, 1979

[7] C.A. Brebbia, "The Boundary Element Method for Engineers", Pentech Press, London, 1978

[8] G. Kuhn, W. Möhrmann, "Boundary element method in elastostatics: theory and applications", Appl. Math. Modelling, 7, 97 - 105, (1983)

[9] T.V. Hromadka II, G.C. Pardoen, "Application of the CVBEM to Non-Uniform St. Vernant Torsion", Computer Meth. in Appl. Mech. and Eng., 53, 149 - 161 (1985)

[10] B. Hunt, L. Isaacs, "Integral equation formulation for ground-water flow", J. Hyd. Div., 1197 - 1209, (1981)

[11} C.A. Brebbia (Ed.), "Topics in Boundary Element Research, Vol. 1: Basic Principles and Applications", Springer, Berlin/ Heidelberg/ New York/ Tokyo 1984

[12] C.A. Brebbia, J.C.F. Telles, L.C. Wrobel, "Boundary Element Techniques", Springer, Berlin/ Heidelberg/ New York/ Tokyo 1984

[13] P.K. Banerjee, J.O. Watson, "Developments in Boundary Element Methods - 4", Elsevier, London/ New York 1986

[14] W.L. Wendland, "Asymptotic Accuracy and Convergence for Point Collocation Methods", pp. 230 - 257, Chapt. 9 in: "Topics in Boundary Element Research, Vol. 2: Time-dependent and Vibration Problems", Edited by C.A. Brebbia, Springer, Berlin/ Heidelberg/ New York/ Tokyo 1985

[15] T.A. Cruse (Ed.), "Advanced Boundary Element Methods", Proceedings of the IUTAM Symposium San Antonio, Texas, April 1987, Springer, Berlin / New York/ Tokyo 1988

[16] N.I. Muskhelishvili, "Some Basic Problems of the Mathematical Theory of Elasticity", Noordhoff, Groningen, Holland , 1953

[17] R. Piltner, "The use of complex valued functions for the solution of three-dimensional elasticity problems", Journal of Elasticity, 18, 191 - 225 (1987)

[18] R. Piltner, "The representation of three-dimensional elastic displacement fields with the aid of complex valued functions for several curvilinear coordinates", Mechanics Research Communications, 15 (2), 79 - 85, (1988)

[19] R. Piltner, "The application of a complex 3-dimensional elasticity solution representation for the analysis of a thick rectangular plate", Acta Mechanica, to appear 1988

[20] R. Piltner, "On the representation of three-dimensional elasticity solutions with the aid of complex valued functions", Journal of Elasticity, accepted for publication

[21] H. Neuber, "Kerbspannungslehre", Springer, Berlin, 1958

[22] H. Neuber, "Ein neuer Ansatz zur Lösung räumlicher Probleme der Elastizitätstheorie. Der Hohlkegel unter Einzellast als Beispiel", ZAMM 14, 203 - 212, (1934)

[23] P.F. Papkovich, "Solution generale des equations differentielles fondamentales d' elasticite, exprimee par trois fonctions harmonique", C. R. Acad. Sci. Paris 195, 513 - 515, (1932)

[24] R. Piltner, "Special finite elements with holes and internal cracks", Int. J. Num. Methods. Eng., 21, 1471 - 1485, (1985)

[25] R. Piltner, "Finite Elemente mit Ansätzen im Trefftz'schen Sinne", In: "Finite Elemente - Anwendung in der Baupraxis", (Eds. H. Grundmann, E. Stein, W. Wunderlich), Verlag Ernst + Sohn, Berlin/ München/ Düsseldorf , 1985

[26] R. Piltner, "Special finite elements for an appropriate treatment of local effects", pp. 299 - 314, in: "Local Effects in the Analysis of Structures", (Ed. P. Ladeveze), Elsevier Science Publishers, 1985

[27] P. Henrici, "Applied and Computational Complex Analysis", Wiley, New York/ London, 1986

[28] M.R. Spiegel, "Complex Variables", Mc Graw-Hill, New York, 1964

[29] J.E. Marsden, M.J. Hoffman, "Basic Complex Analysis", Freeman and Company, New York, 1987

[30] R.A. Silverman, "Complex Analysis with Applications", Dover, New York, 1984

[31] J.W. Dettman, "Applied Complex Variables", Dover, New York, 1984

[32] R.V. Churchill, "Complex Variables and Applications", Mc Graw-Hill, New York/ London, 1960

[33] L.V. Ahlfors, "Complex Analysis", Mc Graw-Hill, New York, 1966

[34] T.J.R. Hughes, "The Finite Element Method", Prentice-Hall, New Jersey, 1987

[35] R.C.J. Howland, "On the stresses in the neighbourhood of a circular hole in a strip under tension", Phil. Trans. Roy. Soc., London, A 229, 49 - 86, (1930)