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Publication Date

2009

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Gromov–Witten Invariants of Toric Fibrations

by

Jeffrey Steven Brown

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

GRADUATE DIVISION

of the

UNIVERSITY OF CALIFORNIA, BERKELEY

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Spring 2010

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Abstract

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Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Alexander Givental, Chair

We prove a conjecture of Artur Elezi [4] in a generalized form suggested by Givental [5]. Namely, our main result relates genus-0 Gromov–Witten invariants of a bundle space with such invariants of the base, provided that the fiber is a toric manifold. When the base is the point, a new proof of mirror theorems by A. Givental [6] and H. Iritani [10] for toric manifolds is obtained.

To Melody Liao

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Acknowledgments

I attended Givental's lecture at MSRI [5], and immediately began working on the Conjecture he described therein.

I eventually learned Givental's approach to Gromov–Witten theory and solved the problem simultaneously. Before working on the problem, I had tried a variety of problems and accumulated a lot of research experience in symplectic topology.

I learned repeatedly from Sasha to approach a difficult problem by understanding something simple about it, and by having a definite plan from the start.

I thank Melody Liao. Because of her support, I could concentrate well on mathematics, with little time left in graduate school.

I thank my family. I thank my friends from Baltimore, including Aaron, Matt, Megan, April, Jaime and Joe. I thank my friends from the bay area, too numerous to name, including Pavan Segal, Sasha Peterka, Moogig Purevdorj and Nate Brown.

0.1 Formulations

1.1. Genus-0 Gromov–Witten invariants. Given a compact (almost) Kähler manifold M , its *genus-0 descendant potential* is defined as:

$$\mathcal{F}_M := \sum_{n=0}^{\infty} \sum_{D \in MC} \frac{Q^D}{n!} \int_{[M_{0,n,D}]} \prod_{a=1}^n \sum_{k=0}^{\infty} \text{ev}_a^*(t_k) \psi_a^k.$$

Here $M_{0,n,D}$ stands for the moduli space of degree- D stable maps to M of genus-0 holomorphic curves with n marked points, $[M_{0,n,D}]$ — its virtual fundamental class, MC — the Mori cone of M , i.e. the semigroup of classes in the lattice $H_2(M)$ representable by compact holomorphic curves, Q^D — the element in the *Novikov ring* (i.e. a power-series completion of the semigroup algebra of the Mori cone) representing the degree $D \in MC$ of the stable maps, ψ_a — the 1st Chern class of the line bundle over $M_{0,n,D}$ formed by cotangent lines to the holomorphic curves at the a -th marked point, ev_a — the map $M_{0,n,D} \rightarrow M$ defined by the evaluation of stable maps at the a -th marked point, $t_k \in H^*(M, \mathcal{Q})$, $k = 0, 1, 2, \dots$, — arbitrary cohomology classes of the target manifold M with coefficients in a suitable *ground ring* $\mathcal{Q}$ (for the moment let it be the rational Novikov ring $\mathbb{Q}[[MC]]$). The explicit inclusion of Novikov's variables into the definition of the potential turns out to be redundant due to the so-called *divisor equation* (see 5.1).

We view $H^2(X, \mathbb{Q})$ as the space of linear functionals on the vector space $H_2(X, \mathbb{Q})$. To define coordinate functionals on $MC(M)$ we only need the subspace $H^{1,1}(M, \mathbb{C}) \subset H^2(M, \mathbb{C})$, as follows. Use the Hodge decomposition

$$H^2(M, \mathbb{C}) = H^{0,2}(M, \mathbb{C}) \oplus H^{1,1}(M, \mathbb{C}) \oplus H^{2,0}(M, \mathbb{C}),$$

and recall that elements of the first and last summands have the trivial pairing with elements of $MC(M)$. Choose a basis of Kähler classes for the middle summand such that each element lies in $H^2(X, \mathbb{Z})$ [9]. Since Kähler classes have positive integer pairing on curve classes, we may view $MC(M)$ as a subcone of the non-negative integer cone in the integer lattice of rank $\dim_{\mathbb{C}} H^{1,1}(M, \mathbb{C})$. Thus, we may view Novikov's variables as power series. In particular, there is a well-defined *order of vanishing (at the origin)* for any element of $H^*(M, \mathcal{Q})$, in the sense of formal power series.

Following [7, 3], one associates to $\mathcal{F}_M$ a Lagrangian cone $\mathcal{L}_M$ in a *symplectic loop space* $(\mathcal{H}, \Omega)$. Let H denote the cohomology space $H^*(M, \mathcal{Q})$, $(\cdot, \cdot)$ the Poincaré pairing on H , and $1 \in H$ the unit element. Take $\mathcal{H} := H((1/z))$. It consists of Laurent series in one indeterminate $1/z$ with vector coefficients. Equip $\mathcal{H}$ with a $\mathcal{Q}$ -valued non-degenerate symplectic form

$$\Omega(f, g) := \frac{1}{2\pi i} \oint (f(-z), g(z)) dz.$$

The subspaces $\mathcal{H}_+ := H[z]$ and $\mathcal{H}_- := z^{-1}H[[z^{-1}]]$ form a Lagrangian polarization of $(\mathcal{H}, \Omega)$, which identifies $\mathcal{H}$ with $T^*\mathcal{H}_+$. To a point $q = q_0 + q_1z + q_2z^2 + \dots \in \mathcal{H}_+$, associate a sequence $t = (t_0, t_1, t_2, \dots)$ of elements $t_k \in H^*(X, \mathcal{Q})$ according to the *dilaton shift* convention:

$$t_0 + t_1z + t_2z^2 + \dots = 1z + q_0 + q_1z + q_2z^2 + \dots.$$

Define a Lagrangian section $\mathcal{L}_M$ as the graph of the differential of $\mathcal{F}_M$ at the dilaton-shifted point:

$$\mathcal{L}_M := \{(p, q) \in T^*\mathcal{H}_+ | p = d_t\mathcal{F}_M\}.$$

According to general theory of genus-0 Gromov–Witten invariants, the section $\mathcal{L}_M$ considered as a submanifold in $(\mathcal{H}, \Omega)$ is (a germ at a dilaton-shifted point of) an *overruled Lagrangian cone with the vertex at the origin*. Here being *overruled* means that each tangent space T to $\mathcal{L}_M$ is tangent to $\mathcal{L}_M$ exactly along the subspace zT (see [3, 7]). This property is invariant under the action of the *twisted loop group* $L^{(2)}GL(H)$. By definition, it consists of those invertible Laurent series $W(z)$ with values in $\text{End}(H)$ which preserve the symplectic form Ω (i.e. satisfy $W^*(-z)W(z) = 1$, where $*$ means “adjoint” with respect to the Poincare pairing).

An overruled Lagrangian cone $\mathcal{L}_M \subset (\mathcal{H}, \Omega)$ is determined by its intersection (known as the J-function) with the subspace $-1z + z\mathcal{H}_-$. More precisely, the J-function $\tau \mapsto J(z, \tau)$ is defined as a Laurent $1/z$ -series with coefficients in H depending on $\tau \in H$ and characterized by the property:

$$J(-z, \tau) = -1z + \tau + O(1/z) \in \mathcal{L}_M.$$

Explicitly, for any $\phi \in H$,

$$(J(z, \tau), \phi) = (1, \phi)z + (\tau, \phi) + \sum_{n,D} \frac{Q^D}{n!} \int_{[M_{0,n+1,D}]} \text{ev}_1^*(\tau) \cdots \text{ev}_n^*(\tau) \frac{\text{ev}_{n+1}^*(\phi)}{z - \psi_{n+1}}.$$

Barannikov [1], in a mirror context, constructed a function whose values are obtained as the single intersection points of semi-infinite subspaces in a space of Laurent-series.

1.2. Toric fibrations. Kähler toric manifolds can be obtained by symplectic reduction from linear spaces.

Let a torus T^N act by diagonal unitary transformations on the Hermitian space $\mathbb{C}^N$. Denote by $\mu : \mathbb{C}^N \rightarrow \mathbb{R}^N := \text{Lie}^*(T^N)$ the moment map of this action, $\mu(z_1, \dots, z_N) = (|z_1|^2, \dots, |z_N|^2)$. Let a torus T^K be embedded as a subtorus $T^K \subset T^N$. The moment map $\mathbb{C}^N \rightarrow \mathbb{R}^K := \text{Lie}^*(T^K)$ is the composition of μ with the projection $\mathbb{R}^N \rightarrow \mathbb{R}^K$ dual to the embedding $\text{Lie } T^K \subset \text{Lie } T^N$ of the Lie algebras. We denote by $\mathbf{m} = (m_{ij} | i = 1, \dots, K, j = 1, \dots, N)$ the integer $K \times N$ -matrix of this projection. Applying symplectic reduction over a chosen value ω of the moment

map, we obtain a symplectic toric variety $X = \mathbb{C}^N //_{\omega} T^K$. Since the actions of T^N and T^K on $\mathbb{C}^N$ commute, X carries a canonical action of T^N .

We will assume that X is non-singular and compact. In fact any compact Kähler toric manifold X of dimension $N - K$ can be obtained by such reduction, with $\mathbb{R}^K$ canonically identified with $H^2(X, \mathbb{R})$. The T^K -fibration $(\mathbf{m} \circ \mu)^{-1}(\omega) \rightarrow X$ endows X with K tautological T^N -equivariant line bundles, whose 1st Chern classes we denote by $-p_1, \dots, -p_K$. They represent a basis in $\mathbb{R}^K$ of integer lattice vectors, and generate the algebra $H^*(X)$.

Let B be any Kähler manifold, $L_1, \dots, L_N$ line bundles over B , and $\Lambda_j = c_1(L_j^*)$, $j = 1, \dots, N$. In the vector bundle $\oplus L_j$ with the structure group T^N , replace the fiber with the toric T^N -space X . We obtain a toric fibration $\pi : E \rightarrow B$ of Kähler manifolds. It carries a canonical fiberwise action of T^N . The total space E is endowed with K tautological line bundles whose 1st Chern classes we denote by $-P_1, \dots, -P_K$. They restrict to the fibers to $-p_1, \dots, -p_K$, and generate $H^*(E)$ as an algebra over $H^*(B)$.

To a degree $\mathcal{D} \in H_2(E)$ of holomorphic curves in E , we associate the degree $D := \pi_*(\mathcal{D}) \in H_2(B)$ of its projection to the base, and the degrees $d_i := P_i(\mathcal{D})$, $i = 1, \dots, K$, with respect to the classes P_i . In the Novikov ring of E , we will represent $\mathcal{D}$ by the monomial $q^d Q^D$, where $q^d = q_1^{d_1} \cdots q_K^{d_K}$, and Q^D represents D in the Novikov ring of B .

In the formulation below we use the following notation:

$$\begin{aligned} t &= (t_1, \dots, t_K), \quad Pt = \sum_{i=1}^K P_i t_i, \quad dt = \sum_{i=1}^K d_i t_i, \\ U_j &= \sum_{i=1}^K P_i m_{ij} - \Lambda_j, \quad U_j(\mathcal{D}) = \sum_{i=1}^K d_i m_{ij} - \Lambda_j(D). \end{aligned}$$

1.3. Main results.

Theorem 1. *Decompose the J -function of the overruled Lagrangian cone $\mathcal{L}_B$, corresponding to the base B of a toric fibration $E \rightarrow B$, according to the degrees of curves:*

$$J(z, \tau) = \sum_{D \in MC(B)} J_D(z, \tau) Q^D,$$

and introduce the **hypergeometric modification**

$$I_E(z, t, \tau, q, Q) := e^{Pt/z} \sum_{d \in \mathbb{Z}^K, D \in MC(B)} \frac{J_D(z, \tau) Q^D q^d e^{dt}}{\prod_{j=1}^N \prod_{m=1}^{U_j(\mathcal{D})} (U_j + mz)}.$$

Then for all (t, τ) , the series $I_E(-z)$ lies in the overruled Lagrangian cone $\mathcal{L}_E$ corresponding to the total space E .

The products in the denominator are interpreted as ratios of the values of the Gamma-function:

$$\prod_{m=1}^n (U + mz) := \prod_{m=-\infty}^n (U + mz) / \prod_{m=-\infty}^0 (U + mz).$$

Thus, when $n < 0$, we obtain a product in the numerator instead.

In the special case when $E = \text{proj}(\oplus_j L_j)$ is a projective fibration over B , the algebra $H^*(E)$ is generated over $H^*(B)$ by one generator P satisfying the relation $(P - \Lambda_1) \cdots (P - \Lambda_N) = 0$.

Corollary 1. *The overruled Lagrangian cone $\mathcal{L}_E$ of the projective fibration contains $I(-z)$, where*

$$I(z, t, \tau, q, Q) := e^{Pt/z} \sum_{D \in MC(B)} J_D(z, \tau) Q^D \sum_{d \in \mathbb{Z}} \frac{q^d e^{dt}}{\prod_{j=1}^N \prod_{m=1}^{d-\Lambda_j(D)} (P - \Lambda_j + mz)}.$$

Note that the summation range $d \in \mathbb{Z}$ actually reduces to $d \geq \min_j \Lambda_j(D)$ since otherwise the numerator contains $\prod_j (P - \Lambda_j) = 0$. We will see later that a similar phenomenon takes place in the situation of general toric fibrations. As a result, the effective summation range in the series I_E stays within the Mori cone of E .

Corollary 2 (Elezi's conjecture [4]). *Taking $\Lambda_1 = 0$, and assuming that all L_j^* and $c_1(T_B) - \sum_j \Lambda_j$ are nef, we have: $I = z + \tau + Pt + O(z^{-1})$, i.e. the series I represents the J -function of the projective fibration $E \rightarrow B$ at the points $\tau + Pt \in H^*(E, \mathbb{Q})$, where τ is a class of degree ≤ 2 in $H^*(B, \mathbb{Q})$.*

When $B = pt$, we have $J_B = ze^{\tau/z}$, leading to Iritani's mirror theorem for arbitrary toric manifolds.

Corollary 3 (Iritani's theorem [10].) *For all values of (τ, t) , the series $I_X(-z)$, where*

$$I_X(z, t, \tau, q) := z e^{(\tau+Pt)/z} \sum_d \frac{q^d e^{dt}}{\prod_{j=1}^N \prod_{m=1}^{\sum_i d_i m_{ij}} (\sum_i P_i m_{ij} + mz)},$$

lies on the overruled Lagrangian cone $\mathcal{L}_X$ of the toric manifold X .

Corollary 4 (Givental's theorem [6]). *When the toric manifold X is Fano, then the series I_X represents the J -function of X at the points $\tau \oplus Pt \in H^0(X) \oplus H^2(X)$.*

1.4. Remarks.

Although the results are stated above in the setting of Kähler manifolds, they extend without complications to the general setting of symplectic toric fibrations and almost Kähler structures.

Furthermore, in all formulations one may assume that cohomology groups are *equivariant* with respect to the fiberwise action of the torus T^N on the toric fibration $E \rightarrow B$. Respectively, all Gromov–Witten invariants become equivariant, taking values in the coefficient ring $H^*(BT^N, \mathbb{Q}) = \mathbb{Q}[\lambda_1, \dots, \lambda_N]$ of the equivariant cohomology

theory. In this case, the classes P_i and Λ_j are understood as T^N -equivariant 1st Chern classes of the respective line bundles, and are invertible in the field of fractions of the coefficient ring. In fact Theorem 1 follows in the limit $\lambda = 0$ from its equivariant counterpart.

To prove the equivariant version of Theorem 1, we will first show in Section 2 that the equivariant counterpart of the overruled Lagrangian cone $\mathcal{L}_E$ is the solution set of a certain recursion relation. This is an unpublished result of A. Givental, and an easy special case of the general fixed point localization formula for Gromov–Witten invariants in the case of non-isolated fixed points [2].

Cohomology classes entering in the definition of I_E have equivariant counterparts, so we may consider I_E as taking values in equivariant cohomology. The equivariant version of the hypergeometric modification series I_E has essential singularity at $z = 0$ and simple poles at $z \neq 0$. To prove that the series satisfies the recursion relation, one needs to show that: (i) applying a certain linear transformation removes the essential singularity at $z = 0$, and (ii) residues at the simple poles are controlled recursively. In Section 3, we show (ii) by decomposing terms of the series I_E into elementary fractions in a straightforward way. The task (i) relies on properties of oscillating integrals arising in mirror theory. It is accomplished in Section 5 in a way resembling the proof [3] of Quantum Lefschetz Theorem. This is preceded by a general discussion in Section 4 of asymptotics of oscillating integrals.

0.2 Localization

2.1. Fixed sections. Let $X = \mathbb{C}^N //_{\omega} T^K$ be a compact toric manifold as in 1.2. For X to be non-singular, it is necessary that $\omega \in \mathbb{R}^K$ is a regular value of the moment map $\mathbf{m} \circ \mu$. The image of the moment map is a *picture* of the 1st orthant $\mathbb{R}_+^N$ “drawn” (by means of the projection $\mathbf{m}$) in $\mathbb{R}^K$. The fiber $\mathbf{m}^{-1}(\omega) \subset \mathbb{R}_+^N$ is the momentum polyhedron of the torus T^N/T^K action on X . The vertices of the momentum polyhedron represent fixed points of the torus action on X . Each vertex corresponds to a K -dimensional face of the 1st orthant whose picture contains ω . We will label the fixed point by the multi-index $\alpha = \{j_1 < \dots < j_K\}$ specifying the coordinates of the corresponding K -dimensional face.¹

Consider now a toric fibration $\pi : E \rightarrow B$ with the fiber X (as in 1.2). Fixed points of the torus $T := T^N$ acting fiberwise on E form sections $\alpha : B \rightarrow E$, one for each fixed point $\alpha \in X^T$ of the torus action on the fiber. Torus-equivariant intersection theory on the total space E of the fibration can be completely characterized in terms of intersection theory on the base B by the following elegant residue formula describing (via fixed point localization) the push-forward to $H_T^*(B) = H^*(B, H^*(BT^N))$ of a

¹Pictures in $\mathbb{R}^K = H^2(X, \mathbb{R})$ of such K -dimensional faces contain, together with ω , its connected component $\mathcal{K}$ in the regular value locus of the moment map. $\mathcal{K}$ coincides with the Kähler cone of X .

T^N -equivariant cohomology class $f \in H_T^*(B)[P_1, \dots, P_K]$:

$$\pi_* f = \sum_{\alpha \in X^T} \text{Res}_\alpha f(P) \frac{dP_1 \wedge \dots \wedge dP_K}{U_1(P) \dots U_N(P)}.$$

The factors $U_j = \sum_i P_i m_{ij} - \Lambda_j$, $j = 1, \dots, N$, can be interpreted as Poincare-duals of the torus-invariant divisors represented by hyperplane faces of the momentum polyhedron $\mathbf{m}^{-1}(\omega)$. Here Res_α refers to the residue of the K -form at the pole $U_{j_1} = \dots = U_{j_K} = 0$ corresponding to the fixed point $\alpha = \{j_1, \dots, j_K\}$, i.e. at the point $P = P^\alpha$ determined by

$$\sum_i P_i^\alpha m_{ij} = \Lambda_j, \quad \forall j \in \alpha.$$

The formula for the push-forward uses the wedge product symbol in a non-standard way. Namely, write

$$dP_1 \wedge \dots \wedge dP_k = \frac{dP_1 \wedge \dots \wedge dP_k}{dU_{j_1} \wedge \dots \wedge dU_{j_K}} dU_{j_1} \wedge \dots \wedge dU_{j_K}$$

where the ratio of K -forms is equal to $\det^{-1}(m_{i,j_s})$, which is ± 1 for smooth toric fibers. The wedge symbol, as it is used in the formula for the push-forward, is an instruction to compute residue integrals with the dP_i 's reordered according to the exterior algebra so as to offset this sign.

We note that: (i) the normal bundle to the fixed point section α is the sum of $N - K$ line bundles with the 1st Chern classes

$$\alpha^* U_j = \sum_i P_i^\alpha m_{ij} - \Lambda_j, \quad \text{where } j \notin \alpha,$$

and (ii) a point $D \in MC(B)$, lifted to E by this section, is represented in the Novikov ring of E by the monomial $Q^D q^{P^\alpha(D)} = Q^D q_1^{P_1^\alpha(D)} \dots q_K^{P_K^\alpha(D)}$.

2.2. The cone $\mathcal{L}_E$. The overruled Lagrangian cone $\mathcal{L}_E$ in the torus-equivariant genus-0 Gromov–Witten theory of the total space E of the toric fibration lies in the appropriate symplectic loop space $(\mathcal{H}, \Omega)$. The space is actually a module over the ground ring $\mathcal{Q}$, which we currently take to be the Novikov ring of E tensored with the field of fractions $\mathbb{Q}(\lambda)$ of $H^*(BT^N)$. Pending further completions, $\mathcal{H}$ consists of Laurent series in $1/z$ with coefficients in $H = H^*(E, \mathcal{Q})$. A point in the cone can be written as

$$\mathbf{F}(-z, t) = -1z + t(z) + \sum_{n, D, d} \frac{Q^D q^d}{n!} (\text{ev}_1)_* \left[\frac{1}{-z - \psi_1} \prod_{i=2}^{n+1} (\text{ev}_i^* t)(\psi_i) \right],$$

where $(\text{ev}_1)_*$ denotes the virtual push-forward by the evaluation map $\text{ev}_1 : E_{0, n+1, \mathcal{D}} \rightarrow E$, and $t(z) = \sum_{k=0}^{\infty} t_k z^k$ is a polynomial with arbitrary coefficients $t_k \in H$.

Denote by $\mathbf{F}^\alpha := \alpha^* \mathbf{F}$ restrictions of $\mathbf{F}$ (considered as a cohomology class of E) to the fixed point sections α . The series $\mathbf{F}^\alpha$ lie in the space of Laurent series in $1/z$ with coefficients in $H^*(B, \mathcal{Q})$. In terms of the push-forwards $\alpha_* : H^*(B, \mathcal{Q}) \rightarrow H^*(E, \mathcal{Q})$ by the sections and their normal Euler classes e^α , we have:

$$\mathbf{F} = \sum_{\alpha \in X^T} \alpha_* \left(\frac{\mathbf{F}^\alpha}{e^\alpha} \right), \quad \text{where } e^\alpha = \prod_{j \notin \alpha} U_j(P^\alpha).$$

2.3. Twisted Gromov–Witten invariants. Consider the base B of the toric fibration embedded in the total space E as a fixed section $\alpha : B \rightarrow E$. Torus-equivariant Gromov–Witten invariants of a *neighborhood* of this section can be defined via fixed point localization as certain intersection indices in moduli spaces of stable maps to the fixed locus $\alpha(B)$. They coincide with such invariants of B *twisted* (in the sense of [3]) by the normal bundle N^α of the fixed section in E . More specifically, the genus-0 descendant potential of the twisted theory is defined by the formula:²

$$\mathcal{F}_{B, N^\alpha} := \sum_{n, D} \frac{Q^D q^{P^\alpha(D)}}{n!} \int_{[B_{0,n,D}]} Euler_T^{-1}(N_{0,n,D}^\alpha) \prod_{a=1}^n \sum_{k=0}^\infty ev_a^*(t_k) \psi_a^k.$$

Here $Euler_T^{-1}$ is the inverse T -equivariant Euler class of complex vector bundles³, and $N_{0,n,D}^\alpha := (\text{ft}_{n+1})_* ev_{n+1}^* N^\alpha$ denotes the virtual vector bundle over $B_{0,n,D}$ obtained as the K-theoretic push-forward along the family of curves $\text{ft}_{n+1} : B_{0,n+1,D} \rightarrow B_{0,n,D}$ of the bundle N^α pulled-back from B by $ev_{n+1} : B_{0,n+1,D} \rightarrow B$.

Let $\mathcal{L}^\alpha$ be the overruled Lagrangian cone corresponding to the twisted theory. The cone lies in the symplectic loop space $(\mathcal{H}^\alpha, \Omega^\alpha)$ constructed using the twisted Poincare pairing $(a, b)^\alpha = \int_B Euler_T^{-1}(N^\alpha) ab$ on $H^\alpha := H^*(B, \mathcal{Q})$. Let T denote a tangent space to the cone $\mathcal{L}^\alpha$ at a point $\mathbf{f}$. The same space T is tangent to $\mathcal{L}^\alpha$ everywhere along zT . Let $-z + u^\alpha + O(1/z)$ be the point of the J-function of this cone that lies in zT . Here u^α depends on $\mathbf{f} \in \mathcal{L}^\alpha$, and is an element of H^α .

To each vector $w \in H^\alpha$, associate the vector in T that projects to w along $\mathcal{H}_-^\alpha$. The operator thus defined is an element of the loop group $LGL(H^\alpha)$, and is represented by an operator Laurent series of the form $1 + O(1/z)$. The inverse element, which we denote by $S_{u^\alpha}(-z)$, is characterized as the operator $1/z$ -series which transforms T to $\mathcal{H}_+^\alpha$. From the fact that T is Lagrangian, it follows that $S_{u^\alpha}^*(z) S_{u^\alpha}(-z) = 1$ (i.e. S_{u^α} lies in the twisted loop group). We conclude that for every point $\mathbf{f} \in \mathcal{L}^\alpha$ there exists a unique $u^\alpha \in H^\alpha$ such that $S_{u^\alpha} \mathbf{f} \in z\mathcal{H}_+^\alpha$.

2.4. Fixed stable maps. The general description of genus 0 stable maps $\Sigma \rightarrow M$ whose equivalence class is fixed by the action of a torus T on the target space goes back to Kontsevich's work [11]. According to it, each irreducible component of the curve Σ must be mapped onto an orbit of dimension 0 or 1 of the complexified torus

²Here $Q^D q^{P^\alpha(D)}$ represents degree- D curves of B considered as curves in $\alpha(B) \subset E$.

³In [3], twisting by arbitrary invertible multiplicative characteristic classes is allowed.

${}^{\mathbb{C}}T$. A 1-dimensional orbit is a projective line connecting two 0-dimensional orbits (i.e. fixed points). An irreducible component of Σ mapped onto such orbit with degree k must have ramifications of degree k over the fixed points. We will call such irreducible components *legs* (of *multiplicity* k). Removing all legs from Σ leaves a forest of rational curves mapped to the fixed point locus M^T in the target space. Integration over fixed point components in the moduli spaces of stable maps reduces therefore to evaluation of certain twisted Gromov–Witten invariants of M^T .

In our situation, 1-dimensional orbits of the torus ${}^{\mathbb{C}}T$ in the toric manifold X correspond to edges of the momentum polyhedron. If two vertices of the polyhedron are connected by an edge, then there is exactly one 1-dimensional orbit connecting two corresponding fixed points (say, α and β) in X . Respectively, each fiber of the toric fibration $E \rightarrow B$ contains a copy of this 1-dimensional orbit, connecting the fixed point sections α and β .

Two fixed points α and β are connected by a 1-dimensional orbit exactly when the union $\alpha \cup \beta$ of the multi-indices α and β has cardinality $k + 1$. The orbit itself is a toric $\mathbb{C}P^1 = \mathbb{C}^{k+1}/_{\omega} T^k$ obtained by symplectic reduction from the face of the 1st orthant whose coordinates have indices from $\alpha \cup \beta$. From this, one can easily derive the following relations (see [6]). Denote by $j_{\pm}(\alpha, \beta)$ the indices such that $j_+(\alpha, \beta) \in \beta - \alpha$, and $j_-(\alpha, \beta) \in \alpha - \beta$, by $\chi_{\alpha, \beta}$ the equivariant 1-st Chern class of the line bundle over B formed by the tangent lines to the 1-dimensional orbit at the fixed points α , and by $d_{\alpha, \beta}$ the degree of the 1-dimensional orbit as a rational curve in the fiber X . Then for a given fixed point α , any index $j_+ \notin \alpha$ can play the role of $j_+(\alpha, \beta)$, while β and $j_-(\alpha, \beta)$ are uniquely determined by it. By fixed point localization on $\mathbb{C}P^1$, we find:

$$d_{\alpha, \beta} = \frac{P^{\alpha} - P^{\beta}}{\chi_{\alpha, \beta}}, \quad U_j(d_{\alpha, \beta}) = \frac{\alpha^* U_j - \beta^* U_j}{\chi_{\alpha, \beta}},$$

where $\chi_{\alpha, \beta} = \alpha^* U_{j_+(\alpha, \beta)} = -\beta^* U_{j_-(\alpha, \beta)}$. It follows that

$$U_{j_{\pm}(\alpha, \beta)}(d_{\alpha, \beta}) = 1 \text{ and } \forall j \in \alpha \cap \beta, \alpha^* U_j = \beta^* U_j = 0, U_j(d_{\alpha, \beta}) = 0.$$

2.5. Recursion. In general, the value of a T -equivariant cohomology class f on the invariant fundamental class of a manifold (or orbifold) M is computed as

$$\int_M f = \int_{M^T} \frac{i^* f}{Euler_T(\mathcal{N})},$$

where $i : M^T \rightarrow M$ is the embedding of the fixed point locus, $\mathcal{N}$ is the normal bundle to M^T in M , and $Euler_T$ is the T -equivariant Euler class. The use of fixed point localization in application to integrals over *virtual* fundamental classes of moduli spaces of stable maps has been justified by Graber–Pandharipande [8]. Our nearest goal is to characterize points of the cone $\mathcal{L}_E$ by a recursion relation which comes from fixed point localization in moduli spaces $E_{0, n+2, \mathcal{D}}$.

Let $\mathbf{F} \in \mathcal{L}_E$ and $\mathbf{F}^\alpha := \alpha^* \mathbf{F}$, i.e.

$$\mathbf{F}^\alpha(-z, t) = -1z + \alpha^* t(z) + \alpha^* \sum_{n, D, d} \frac{Q^D q^d}{n!} (\text{ev}_1)_* \left[\frac{1}{-z - \psi_1} \prod_{i=2}^{n+1} (\text{ev}_i^* t)(\psi_i) \right].$$

We evaluate the sum via fixed point localization, and notice first of all, that a torus-fixed stable map $\Sigma \rightarrow E$ does not contribute to $\mathbf{F}^\alpha$ unless the 1st marked point lands in the fixed section α . When it does, there are two possibilities: the marked point can belong to a *leg* (see 2.4), or to a tree $C \subset \Sigma$ of rational components mapped to the locus $\alpha(B) \subset E$ of the fixed point section $\alpha : B \rightarrow E$.

Examine the first possibility. The leg carrying the 1st marked point is a ramified cover of multiplicity $k > 0$ of a 1-dimensional orbit of the torus ${}^{\mathbb{C}}T$ which lies in a fiber X of the toric fibration $E \rightarrow B$, connects the fixed point α with another fixed point β , and has the degree $d_{\alpha, \beta}$ considered as a curve in X . Contributions of all stable maps of this type to $\mathbf{F}^\alpha$ via fixed point localization can be represented in the form:⁴

$$\sum_{\beta} \sum_{k > 0} \frac{q^{kd_{\alpha, \beta}} \text{Euler}_T^{-1}(\mathcal{N}_{\alpha, \beta}(k))}{k(-z + \chi_{\alpha, \beta}/k)} \mathbf{F}^\beta(-\frac{\chi_{\alpha, \beta}}{k}).$$

Here the factor $-z + \chi_{\alpha, \beta}/k$ is the specialization of $-z - \psi_1$ to the fixed point component (namely, $-\chi_{\alpha, \beta}/k$ is the equivariant 1st Chern class of the line bundle over B formed by the cotangent lines to the leg at the 1st marked point). The symbol $\mathcal{N}_{\alpha, \beta}(k)$ denotes the virtual bundle over B whose fibers describe deformation modes of the 1st leg (in the direction normal to the fixed point locus in the moduli spaces of stable maps to E). The occurrence of the Euler class of this bundle in the denominator is due to the general structure of localization formulas. One can easily compute this Euler class explicitly:

$$\text{Euler}_T(\mathcal{N}_{\alpha, \beta}(k)) = \prod_{m=1}^{k-1} (\alpha^* U_{j_+(\alpha, \beta)} - m \frac{\chi_{\alpha, \beta}}{k}) \prod_{j \notin \beta} \prod_{m=1}^{kU_j(d_{\alpha, \beta})} (\alpha^* U_j - m \frac{\chi_{\alpha, \beta}}{k}).$$

The normal bundle to the fixed point locus in the moduli space contains the smoothing mode of the curve Σ at the node where the leg and the rest of the curve, Σ' , connect. This is a line bundle with the 1st Chern class $-\psi - \chi_{\alpha, \beta}/k$, where ψ is the 1st Chern class of the universal cotangent line bundle to the curves Σ' at the node. Considering the node as the 1st marked point of the curve Σ' , we can therefore represent the sum of all contributions of moduli spaces of curves Σ' as $\mathbf{F}^\beta(-\chi_{\alpha, \beta}/k)$. The extra factor k in the denominator is due to the cyclic symmetry of order k of the leg, which affects the orbifold's fundamental class this way.

⁴Here and later $\beta = \beta(j_+)$ where j_+ runs all indices not in α , as explained in 2.4.

Now put

$$t^\alpha(z) := \alpha^*t(z) + \sum_{\beta} \sum_{k>0} \frac{q^{kd_{\alpha,\beta}} \text{Euler}_T^{-1}(\mathcal{N}_{\alpha,\beta}(k))}{k(-z + \chi_{\alpha,\beta}/k)} \mathbf{F}^\beta(-\frac{\chi_{\alpha,\beta}}{k}),$$

and examine the second possibility, when the 1st marked point of the curve Σ lies on a tree C mapped to $\alpha(B)$. Fixed point components of the moduli spaces of stable maps to E are products of moduli spaces of stable maps to the fixed point sections, and such moduli of the maps $C \rightarrow \alpha(B)$ is one of the factors. Integrating over this factor last, we represent the contribution of each fixed point component as a genus-0 Gromov–Witten invariant of B twisted by the normal bundle N^α . Among the marked points of the curves C , one is the 1st marked point of Σ ; it carries the input $1/(-z - \psi_1)$. Every other marked point could be either a marked point of Σ which happens to lie in C , or a node where a connected component of $\Sigma - C$ is attached to C by a leg. The input at the marked point of this is obtained by adding to $\alpha^*t(\psi_i)$, ($i > 1$) the sum of fixed point localization contributions over all possibilities for the connected component of $\Sigma - C$. The total input coincides with what is denoted above by $t^\alpha(\psi_i)$. Thus we have:

$$\mathbf{F}^\alpha(-z) = -z + t^\alpha(z) + \sum_{n,D} \frac{Q^D q^{P^\alpha(D)}}{n!} (\text{ev}_1)_* \left[\frac{\text{Euler}_T^{-1}(\mathcal{N}_{0,n+1,D})}{-z - \psi_1} \prod_{i=2}^{n+1} t^\alpha(\psi_i) \right],$$

where $(\text{ev}_1)_*$ is the push-forward in the N^α -twisted Gromov–Witten theory B by the map $\text{ev}_1 : B_{0,n+1,D} \rightarrow B$. We conclude that $\mathbf{F}^\alpha(-z)$ lies in the overruled Lagrangian cone $\mathcal{L}^\alpha$ of the twisted Gromov–Witten theory of $\alpha(B) \subset E$.

Let us examine analytical properties of the expression for $\mathbf{F}^\alpha$ as a function of z . Since the class ψ_1 in the sum is nilpotent, each summand with a fixed D and n is polynomial in $1/z$. When D is fixed, but n grows, the degree of the polynomial can grow too. In fact, employing dimensional arguments and the string equation, one can see that for a fixed D the sum over n is a finite linear combination of functions of the form $z^{-k} e^{c/z}$ with positive k and non-zero constant c . Thus the whole sum is a Q -series whose coefficients are meromorphic functions with *essential singularities* at $z = 0$ and no other singularities (including $z = \infty$). The term $t^\alpha(z)$, in the contrary, is a q -series, whose coefficients have simple poles at $z = \chi_{\alpha,\beta}/k$, and (from the summand $\alpha^*t(z)$) a pole at $z = \infty$ of any order ≥ 0 . Thus, $\mathbf{F}^\alpha(z)$ are *power series in the Novikov variables Q, q which have coefficients meromorphic in z , with an essential singularity at $z = 0$, finite order pole at $z = \infty$ and simple poles at $z = -\chi_{\alpha,\beta}/k$, such that the residues at the simple poles satisfy the recursion relation:*

$$\text{Res}_{z=-\frac{\chi_{\alpha,\beta}}{k}} \mathbf{F}^\alpha(z) \, dkz = \frac{q^{kd_{\alpha,\beta}}}{\text{Euler}_T(\mathcal{N}_{\alpha,\beta}(k))} \mathbf{F}^\beta(-\frac{\chi_{\alpha,\beta}}{k}).$$

Note that each elementary fraction $(z + \chi_{\alpha,\beta}/k)^{-1}$ can be expanded into a Laurent series in two ways: inside or outside the circle $|z| = |\chi_{\alpha,\beta}|/k$. Expanding all the elementary fractions as $1/z$ -series renders $\mathbf{F}^\alpha(-z)$ as a Laurent series in $1/z$ in a way it occurs as a component of $\mathbf{F}(-z) \in \mathcal{L}_E$. Expanding all the elementary fractions as z -series renders $\mathbf{F}^\alpha(-z)$ as a Laurent series in $1/z$ in another way, namely the way it occurs as a point on the cone $\mathcal{L}^\alpha$. Note that the polynomial truncation $t^\alpha = [\mathbf{F}^\alpha(z)]_+$ of the latter Laurent series lies not in $H^\alpha[z]$ per se, but in a certain completion of it. Namely, it is a power series in $1/\chi_{\alpha,\beta}$ with coefficients polynomial in z , or equivalently becomes a polynomial in z when reduced modulo any power of $1/\chi_{\alpha,\beta}$. We will subsequently assume that *the ground ring is suitably localized to include inverse powers of $\chi_{\alpha,\beta}$* , and that $\mathcal{L}^\alpha \subset \mathcal{H}^\alpha$ *refers to the cone of the twisted Gromov–Witten theory thus completed*. With these interpretations in mind, we state the result of this section.

Theorem 2. *Points $\{\mathbf{F}^\alpha(-z)\}$ of the overruled Lagrangian cone $\mathcal{L}_E$ are characterized by the following conditions:*

- (i) $\mathbf{F}^\alpha(-z) \in \mathcal{L}^\alpha$,
- (ii) $\mathbf{F}^\alpha$ are power series in the Novikov variables Q, q whose coefficients are analytic functions of z with essential singularities at $z = 0$, finite order poles at $z = \infty$, simple poles at $z = \chi_{\alpha,\beta}/k$, $k = 1, 2, 3, \dots$, and such that the residues at the simple poles satisfy the recursion relations:

$$\text{Res}_{z=-\frac{\chi_{\alpha,\beta}}{k}} \mathbf{F}^\alpha(z) \, dkz = \frac{q^{kd_{\alpha,\beta}}}{\text{Euler}_T(\mathcal{N}_{\alpha,\beta}(k))} \mathbf{F}^\beta(-\frac{\chi_{\alpha,\beta}}{k}),$$

where

$$\text{Euler}_T(\mathcal{N}_{\alpha,\beta}(k)) = \prod_{m=1}^{k-1} (\alpha^* U_{j+(\alpha,\beta)} - m \frac{\chi_{\alpha,\beta}}{k}) \prod_{j \notin \beta} \prod_{m=1}^{kU_j(d_{\alpha,\beta})} (\alpha^* U_j - m \frac{\chi_{\alpha,\beta}}{k}).$$

We have established that every point on $\mathcal{L}_E$ satisfies (i) and (ii). Let us prove now that if series $\{\mathbf{F}^\alpha\}$ satisfy these conditions, then they represent a point in $\mathcal{L}_E$. Indeed, since $\mathbf{F}^\alpha(-z) \in \mathcal{L}^\alpha$, there exists a unique $u^\alpha \in H^\alpha$ such that $\mathbf{G}^\alpha(z) := S_{u^\alpha}(-z)\mathbf{F}^\alpha(z) \in z\mathcal{H}_+^\alpha$ (see 2.3). Combining this with the property (ii), we conclude that as a function of z , $\mathbf{G}^\alpha$ satisfies the following recursion relation:

$$(*) \quad \mathbf{G}^\alpha(z) = q^\alpha(z) + \sum_{\beta, k} S_{u^\alpha}(\frac{\chi_{\alpha,\beta}}{k}) \frac{q^{kd_{\alpha,\beta}} \text{Euler}_T^{-1}(\mathcal{N}_{\alpha,\beta}(k))}{kz + \chi_{\alpha,\beta}} S_{u^\beta}(-\frac{\chi_{\alpha,\beta}}{k}) \mathbf{G}^\beta(-\frac{\chi_{\alpha,\beta}}{k}),$$

where q^α are (Q, q) -series with coefficients polynomial in z . Indeed, for $m > 0$,

$$\frac{z^{-m} - (-\chi_{\alpha,\beta}/k)^{-m}}{kz + \chi_{\alpha,\beta}} = O(\frac{1}{z}).$$

Since $S_{u^\alpha}(-z) = 1 + O(1/z)$ is a $1/z$ -series, we see that the difference between $\mathbf{G}^\alpha(z)$ and the R.H.S. of the recursion relation is $O(1/z)$, and hence vanishes, because both sides lie in $\mathcal{H}_+^\alpha$.

Given arbitrary $\{q^\alpha\}$, and $\{u^\alpha\}$, a solution $\{\mathbf{G}^\alpha\}$ to the system of recursion relations (*) is computed by successive q -adic approximations, and is therefore unique. The set of corresponding $\mathbf{F}^\alpha \in \mathcal{L}^\alpha$ is reconstructed by the application of $S_{u^\alpha}^{-1}(-z)$. In particular, $\alpha^*t(z)$ are related to $q^\alpha(z)$ by

$$z + \alpha^*t(-z) = [S_{u^\alpha}^{-1}(-z)q^\alpha(z)]_+.$$

It remains to show that the values $\{u^\alpha\}$ are unambiguously determined by $\{\alpha^*t\}$ in view of the additional constraint that $\mathbf{G}^\alpha \in z\mathcal{H}_+^\alpha$ (rather than $\mathcal{H}_+^\alpha$). Indeed, if such uniqueness is established, we conclude that $\{\mathbf{F}^\alpha\}$ coincide with the components of the point $\mathbf{F}$ on the cone $\mathcal{L}_E$ which corresponds to the Gromov–Witten invariants of E with the inputs t at the marked points.

To verify the required uniqueness, consider first the same problem *classically*, i.e. modulo Novikov variables Q and q . Then recursion relation (*) degenerates into $\mathbf{G}^\alpha = q^\alpha$, the S-matrices turn into $S_{u^\alpha}(-z) = e^{-u^\alpha/z}$ (where $u^\alpha \in H$), so that we have:

$$q^\alpha(-z) = [e^{u^\alpha/z}(-z + \alpha^*t(z))]_+.$$

The additional constraints $q^\alpha(0) = 0$ assume the same form of the universal fixed point equation

$$u = \sum_{m=0}^{\infty} t_m \frac{u^m}{m!},$$

where t_m are coefficients of $\alpha^*t = \sum t_m z^m$. The fixed point equation has a unique formal solution $u = u(t_0, t_1, t_2, \dots)$ on the space of polynomials. Using the formal Inverse Function Theorem, we conclude that the values of u^α can be uniquely found by successive (Q, q) -adic approximations from the relations between $\{\alpha^*t\}$ and $\{q^\alpha\}$, the recursion relations (*), and the additional constraints $\mathbf{G}^\alpha \in z\mathcal{H}_+^\alpha$.

0.3 Recursion

To prove the equivariant version of Theorem 1, it suffices to show that $\mathbf{F} = I_E$ satisfies conditions (i) and (ii) of Theorem 2. The hypergeometric modification I_E is a (q, Q) -series whose coefficients have simple poles at $z = -\alpha^*U_j/k$, finite order poles at $z = \infty$, and essential singularities at $z = 0$. Thus we need to show that: (i) $\alpha^*I_E \in \mathcal{L}^\alpha$, and (ii) residues at the simple poles satisfy the recursion relation of Theorem 2. We postpone (i) until Section 5, and deal with (ii) here by computing the residues explicitly.

We have:

$$\mathbf{F}^\alpha(z) := \alpha^* I_E(z) = e^{P^\alpha t/z} \sum_D \sum_{d' \in \mathbb{Z}^K} \frac{J_D(z, \tau) Q^D q^{d'} e^{d' t}}{\prod_{j=1}^N \prod_{m=1}^{\sum_i d'_i m_{ij} - \Lambda_j(D)} (\alpha^* U_j + mz)}.$$

It will be convenient to put $d_i := d'_i - P_i^\alpha(D)$, and use that

$$\sum_i P_i^\alpha(D) m_{ij} - \Lambda_j(D) = \alpha^* U_j(D) (= 0 \ \forall j \in \alpha),$$

to obtain:

$$\mathbf{F}^\alpha(z) = e^{P^\alpha t/z} \sum_D \sum_{d \in \mathbb{Z}^K} \frac{J_D(z, \tau) (Q^D q^{P^\alpha(D)}) q^d e^{dt} e^{P^\alpha(D)t}}{\prod_{j \in \alpha} \prod_{m=1}^{U_j(d)} (mz) \prod_{j \notin \alpha} \prod_{m=1}^{U_j(d) + \alpha^* U_j(D)} (\alpha^* U_j + mz)}.$$

We see that if $U_j(d) < 0$ for some $j \in \alpha$, then the term contains a factor $(0z)$ in the numerator. Thus, the effective summation range is over those d for which $U_j(d) \geq 0$ for all $j \in \alpha$. For each fixed point α , this range lies in the Mori cone of the fiber X of our toric fibration (because the Kähler cone of X lies in the simplicial cone spanned by $\{U_j | j \in \alpha\}$). Since the monomials $Q^D q^{P^\alpha(D)}$ represent degrees of holomorphic curves in the fixed section $\alpha : B \rightarrow E$, we conclude that all series $\alpha^* I_E$ are supported in the Mori cone of E . The same remains true for the non-equivariant limit of I_E (as we promised in 1.3).

Non-zero poles of $\mathbf{F}^\alpha$ correspond to the choice of a factor $\alpha^* U_j + kz$ with $j \notin \alpha$ and $k > 0$. Given a choice, we put $\alpha \cup \beta = \alpha \cup \{j\}$. As we mentioned in 2.4, this determines β , $d_{\alpha, \beta}$, and $j_\pm(\alpha, \beta)$ such that $j_+(\alpha, \beta) = j$. To single out the contribution of the elementary fraction $(\alpha^* U_{j_+(\alpha, \beta)} + kz)^{-1}$, we need to evaluate all other factors of the product at $z = -\alpha^* U_{j_+(\alpha, \beta)}/k = -\chi_{\alpha, \beta}/k$. Recalling from 2.4 that

$$\alpha^* U_j - k U_j(d_{\alpha, \beta}) \frac{\chi_{\alpha, \beta}}{k} = \beta^* U_j,$$

we find: ⁵

⁵Let us remind ourselves that we are using the analytic continuation convention $\prod_{m=1}^n := \prod_{m=-\infty}^n / \prod_{m=-\infty}^0$.

$$\begin{aligned}
& \text{For } j \notin \alpha \cup \beta, \quad \prod_{m=1}^{U_j(d)+\alpha^*U_j(D)} (\alpha^*U_j - m\frac{\chi_{\alpha,\beta}}{k}) = \\
& \quad \prod_{m=1}^{kU_j(d_{\alpha,\beta})} (\alpha^*U_j - m\frac{\chi_{\alpha,\beta}}{k}) \times \prod_{m=1}^{U_j(d-kd_{\alpha,\beta})+\alpha^*U_j(D)} (\beta^*U_j - m\frac{\chi_{\alpha,\beta}}{k}), \\
& \text{for } j = j_-(\alpha, \beta), \quad \prod_{m=1}^{U_j(d)} (-m\frac{\chi_{\alpha,\beta}}{k}) = \\
& \quad \prod_{m=1}^{U_j(kd_{\alpha,\beta})} (\alpha^*U_j - m\frac{\chi_{\alpha,\beta}}{k}) \times \prod_{m=1}^{U_j(d-kd_{\alpha,\beta})+\alpha^*U_j(D)} (\beta^*U_j - m\frac{\chi_{\alpha,\beta}}{k}) \\
& \text{for } j = j_+(\alpha, \beta), \quad \prod_{m=1, \neq k}^{U_j(d)+\alpha^*U_j(D)} (\alpha^*U_j - m\frac{\chi_{\alpha,\beta}}{k}) = \\
& \quad \prod_{m=1}^{k-1} (\alpha^*U_{j_+(\alpha,\beta)} - m\frac{\chi_{\alpha,\beta}}{k}) \times \prod_{m=1}^{U_j(d-kd_{\alpha,\beta})+\alpha^*U_j(D)-\beta^*U_j(D)} (-m\frac{\chi_{\alpha,\beta}}{k}), \\
& \text{for } j \in \alpha \cap \beta, \quad \prod_{m=1}^{U_j(d)} (-m\frac{\chi_{\alpha,\beta}}{k}) = 1 \times \prod_{m=1}^{U_j(d-kd_{\alpha,\beta})+\alpha^*U_j(D)-\beta^*U_j(D)} (-m\frac{\chi_{\alpha,\beta}}{k}), \\
& \quad (Q^D q^{P^\alpha(D)}) q^d = q^{kd_{\alpha,\beta}} \times (Q^D q^{P^\beta(D)}) q^{d-kd_{\alpha,\beta}+P^\alpha(D)-P^\beta(D)}, \\
& \exp\left(-\frac{P^\alpha t k}{\chi_{\alpha,\beta}}\right) \exp(dt) \exp(P^\alpha(D)t) = \\
& \quad 1 \times \exp\left(-\frac{P^\beta t k}{\chi_{\alpha,\beta}}\right) \exp[(d - kd_{\alpha,\beta} + P^\alpha(D) - P^\beta(D))t] \exp(P^\beta(D)t).
\end{aligned}$$

In the last equality we use $(P^\alpha - P^\beta)/\chi_{\alpha,\beta} = d_{\alpha,\beta}$ from 2.4.

Factors on the R.H.S. which come before the multiplication sign “ $\times$ ” form the recursion coefficients $q^{kd_{\alpha,\beta}} Euler_T^{-1}(\mathcal{N}_{\alpha,\beta}(k))$. Factors which come after the multiplication sign form the term of the series $\mathbf{F}^\beta$ evaluated at $z = -\chi_{\alpha,\beta}/k$ and with the summation index d replaced with $d - kd_{\alpha,\beta} + P^\alpha(D) - P^\beta(D)$. Reversing this change in the summation index, we conclude that

$$\text{Res}_{z=-\frac{\chi_{\alpha,\beta}}{k}} \mathbf{F}^\alpha(z) dkz = \frac{q^{kd_{\alpha,\beta}}}{Euler_T(\mathcal{N}_{\alpha,\beta}(k))} \mathbf{F}^\beta(-\frac{\chi_{\alpha,\beta}}{k}),$$

as required.

0.4 Asymptotics

4.1. Stationary phase asymptotics. We discuss here basic properties of

complex oscillating integrals

$$\int e^{f(x)/z} a(x) dx$$

and their asymptotics as $z \rightarrow 0$. For simplicity of notation we assume all integrals one-dimensional. Generalizations to higher dimensions are straightforward and are left to the reader.

Let $x = 0$ be a non-degenerate critical point of the *phase function* f , i.e. $f(x) = f(0) - x^2/2\sigma^2 + \alpha x^3 + \beta x^4 + \dots$, and let $a(x) = a_0 + a_1 x + a^2 x^2 + \dots$. The *stationary phase asymptotics* of the oscillating integral assumes the form:

$$\int e^{f(x)/z} a(x) dx \sim \sqrt{2\pi z} \sigma e^{f(0)/z} \sum_{k \geq 0} A_k z^k,$$

where the coefficients A_k are obtained by the following procedure. Make the change $x = \sqrt{z} y$, and replace a fixed integration interval $-m \leq x \leq m$ with the infinite interval $-\infty < y < \infty$ (to which $[-\frac{m}{\sqrt{z}}, \frac{m}{\sqrt{z}}]$ tends as $z \rightarrow 0$):

$$\sqrt{z} e^{f(0)/z} \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2} \exp(\alpha \sqrt{z} y^3 + \beta z y^4 + \dots) (a_0 + a_1 \sqrt{z} y + a_2 z y^2 + \dots) dy.$$

Expanding the integrand as a power series in $\sqrt{z}$ and evaluating momenta of the Gaussian distribution

$$\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2} y^n dy = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \sqrt{2\pi} \sigma^{n+1} (n-1)!! & \text{if } n \text{ is even,} \end{cases}$$

we obtain the required asymptotical expansion. Note that the sign of $\sigma := 1/\sqrt{-f''(0)}$ depends on the choice of a branch of the square root, but the values of the *asymptotical coefficients* A_k do not. In this construction, the *amplitude* a may depend formally on z . Also, the phase function f and/or the amplitude a may depend on additional parameters, in which case the *critical value* $f(0)$, *Hessian* $-\sigma^{-2}$, and asymptotical coefficients do too. The following (rather obvious) proposition also allows for such parametric dependence.

Proposition 1. *Suppose that the 1-form in the integrand of an oscillating integral is the total Lie derivative along a vector field $v(x)\partial/\partial x$. Then the stationary phase asymptotics of this integral is trivial:*

$$\int d(e^{f(x)/z} a(x) v(x)) \sim 0.$$

Proof. After the change $x = \sqrt{z} y$ and series expansion, we arrive at the sequence of integrals

$$\int_{-\infty}^{\infty} d(e^{-y^2/2\sigma^2} y^n) = \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2} (n y^{n-1} - y^{n+1} \sigma^{-2}) dy = 0.$$

The proposition follows from the (obvious) fact that the zero answer is obtained by substituting respective values of momenta of the Gaussian distribution (in lieu of actual integration). Taking $v = 1$ we obtain:

Corollary 1 (integration by parts). *The following oscillating integrals have the same stationary phase asymptotics:*

$$\int e^{f/z} f' g \, dx \quad \text{and} \quad -z \int e^{f/z} g' \, dx.$$

The following two corollaries follow from their infinitesimal version established by Proposition 1.

Corollary 2. *If a one-parameter family of oscillating integrals is obtained from each other by the flow of a vector field, then the stationary phase asymptotics does not depend on the parameter.*

Corollary 3. *Stationary phase asymptotics of an oscillating integral does not change under (formal or analytic) change of variables in the integral in a neighborhood of the non-degenerate critical point.*

Proposition 2. *The asymptotics of the derivative of an oscillating integral with respect to a parameter is obtained by differentiating the asymptotics of the integral.*

Proof. Let

$$\int e^{f(x,\epsilon)/z} a(x,\epsilon) dx \sim \sqrt{2\pi z} \, \sigma(\epsilon) \, e^{f(x_{cr}(\epsilon))/z} \sum_k A_k(\epsilon) z^k.$$

The RHS is obtained by evaluating momenta of Gaussian distributions after the change

$$x = x_{cr}(\epsilon) + \sqrt{z} \, \sigma(\epsilon) y,$$

where $x_{cr}(\epsilon)$ is the non-degenerate critical point of the phase function depending on the parameter, and $1/\sigma^2 = f''(x_{cr}(\epsilon), \epsilon)$. Applying $z\partial/\partial\epsilon$ to the RHS is equivalent to differentiating the integrand termwise *after* the change of variables. On the other hand, we have:

$$z \frac{\partial}{\partial \epsilon} \int e^{f/z} a dx = \int e^{f/z} \left(\frac{\partial f}{\partial \epsilon} a + z \frac{\partial a}{\partial \epsilon} \right) dx.$$

Since the RHS has the same phase function as the original integral, the asymptotics of the derivative integral is obtained by applying the same operations: the change of variables, expansion of the integrand into a series, and evaluation of momenta, *preceded* however by the differentiation of the initial integrand. The change $(y, \epsilon) \mapsto (x(y, \epsilon), \epsilon)$ transforms $\frac{\partial}{\partial \epsilon}$ into

$$\frac{\partial}{\partial \epsilon} + \frac{\partial x}{\partial \epsilon} \frac{\partial}{\partial x}.$$

The difference with $\partial/\partial\epsilon$ is a vector field $(\partial x/\partial\epsilon)\partial/\partial x$ (depending on the parameter ϵ). Thus the required independence of the asymptotics of the order of the operations follows from Proposition 1.

Corollary. *Given an oscillating integral*

$$\int e^{f(x,\epsilon)/z} a(x, \epsilon) dx \sim \sqrt{2\pi z} \sigma(\epsilon) e^{f(x_{cr}(\epsilon), \epsilon)/z} \sum_{k \geq 0} A_k(\epsilon) z^k,$$

depending formally on ϵ , and such that $f(x, 0)$ has a non-degenerate critical point $x_{cr}(0)$, consider it as an oscillating integral with the phase function $f(x, 0)$ and the amplitude depending formally on z and $\mu = \epsilon/z$:

$$\int e^{f(x,0)/z} \exp\left(\frac{f(x, z\mu) - f(x, 0)}{z}\right) a(x, z\mu) dx.$$

Then the asymptotics of the latter oscillating integral coincides with the asymptotics of the former one at $\epsilon = z\mu$:

$$\sqrt{2\pi z} \sigma(z\mu) e^{f(x_{cr}(z\mu), z\mu)/z} \sum_{k \geq 0} A_k(z\mu) z^k.$$

Indeed, it suffices to check that for each $n \geq 0$, the n th derivatives $(\partial/\partial\mu)^n$ of the two asymptotics coincide at $\mu = 0$. This follows by iterative application of Proposition 2 to the derivation $z\partial/\partial\epsilon = \partial/\partial\mu$.

4.2. D-modules generated by J-functions. Let $\mathcal{L} \subset \mathcal{H}$ be an overruled Lagrangian cone in a symplectic loop space $(\mathcal{H}, \Omega)$, and let J be its J-function. Tangent spaces to $\mathcal{L}$ vary in a family $H \ni \tau \rightarrow T_\tau$, where $J(-z, \tau) \in \mathcal{L}$ is taken for the application point of T_τ . The J-function satisfies a system of 2nd order PDE:

$$z \frac{\partial^2 J(z, \tau)}{\partial \tau^\alpha \partial \tau^\beta} = \sum_{\gamma} F_{\alpha\beta}^\gamma(\tau) \frac{\partial J(z, \tau)}{\partial \tau^\gamma},$$

where $\tau = \sum \tau^\alpha \phi_\alpha$ is a coordinate system on H .

Indeed, for any family $\tau \rightarrow I(-z, \tau) \in \mathcal{L}$ transverse to the ruling subspaces $zT_\tau \subset \mathcal{L}$, the derivatives $\partial I(-z, \tau)/\partial \tau^\beta$ form a basis of T_τ as a $\mathcal{Q}[z]$ -module, while $z\partial I(-z, \tau)/\partial \tau^\beta$ lie in $zT_\tau \subset \mathcal{L}$. Therefore the 2nd derivatives $z\partial^2 I(-z, \tau)/\partial \tau^\alpha \partial \tau^\beta$ lie in T_τ and are expressible as linear combinations of the basis, i.e.

$$z \frac{\partial^2 I(z, \tau)}{\partial \tau^\alpha \partial \tau^\beta} = \sum_{\gamma} A_{\alpha\beta}^\gamma(z, \tau) \frac{\partial I(z, \tau)}{\partial \tau^\gamma}$$

where $A_{\alpha\beta}^\gamma$ are suitable coefficients polynomial in z . When I is the J-function, the LHS lies in $z\mathcal{H}_-$, while $\partial I(-z, \tau)/\partial \tau^\gamma$ form a basis of the quotient space $z\mathcal{H}_-/\mathcal{H}_-$. Thus the RHS lies in $z\mathcal{H}_-$ only if the coefficients $A_{\alpha\beta}^\gamma$ do not depend on z .⁶

⁶This line of reasoning is due to S. Barannikov [1].

In fact the coefficients $F_{\alpha\beta}^\gamma(\tau)$ (and more generally, the values $A_{\alpha\beta}^\gamma(0, \tau)$) are structure constants of the *quantum cup-product* $\bullet$:

$$\phi_\alpha \bullet \phi_\beta = \sum_{\gamma} F_{\alpha\beta}^\gamma(\tau) \phi_\gamma.$$

Together with the pairing $(\cdot, \cdot)$, it provides the tangent spaces $T_\tau H$ with the structure of a Frobenius algebra. In Gromov–Witten theory,

$$F_{\alpha\beta}^\gamma = \sum_{n,d} \frac{Q^d}{n!} \int_{[M_{0,n+3,d}]} \prod_{i=1}^n \text{ev}_i^*(\tau) \text{ev}_{n+1}^*(\phi_\alpha) \text{ev}_{n+2}^*(\phi_\beta) \text{ev}_{n+3}^*(\phi^\gamma),$$

where $\{\phi^\gamma\}$ is the basis of H Poincare-dual to $\{\phi_\gamma\}$.

Using the quantum cup-product, we can rewrite the PDE system for the J-function in a more invariant form:

$$\forall v, w \in H, \quad z \partial_v \partial_w J = \partial_{v \bullet w} J.$$

These equations can be considered as defining relations of the D-module generated by the J-function. They allow one to represent 2nd derivatives of J as linear combinations of first derivatives. Note that $v \bullet w$ depends on τ . As a result, further differentiations of these equations contain terms involving derivatives of $v \bullet w$. However such terms come with an extra z . Arguing inductively, we conclude:

Proposition 3 ([3]). *For any $v_1, \dots, v_m \in H$, the higher directional derivatives $(z \partial_{v_1}) \cdots (z \partial_{v_m}) J(z, \tau)$ can be expressed as linear combinations of 1st derivatives $z \partial_{\phi^\gamma} J(z, \tau)$ with coefficients which are functions of (z, τ) polynomial in z . Modulo (z) , the direction vector of this linear combination coincides with the quantum cup-product $v_1 \bullet \cdots \bullet v_m$, i.e.*

$$(z \partial_{v_1}) \cdots (z \partial_{v_m}) J = z \partial_{v_1 \bullet \cdots \bullet v_m} J + o(z).$$

4.3. Action of pseudo-differential symbols on J-functions. We describe here in a general form a key argument from the proof of Quantum Lefschetz Theorem found in [3].

Let $(p_1, \dots, p_n)$ be coordinates on the space H^* corresponding to the basis $(\phi_1, \dots, \phi_n)$ of H . Let $\Phi(p_1, \dots, p_n)$ be a polynomial. We consider it as the symbol of a differential operator $\Phi(z \partial_{\phi_1}, \dots, z \partial_{\phi_n})$ with constant coefficients.

Lemma. *Adjoin a formal parameter ν to the ground ring $\mathcal{Q}$, and consider the overruled Lagrangian cone $\mathcal{L}$ completed in the ν -adic topology of $\mathcal{Q}[[\nu]]$. Then*

$$e^{-\nu \Phi(-z \partial)/z} J(-z, \tau) \in \mathcal{L}.$$

Proof. Let us assume first that $\Phi(0) = 0$. According to Proposition 3, the action of the high order differential operator $\Phi(z\partial)$ on the J-function can be, in the quasi-classical approximation, replaced with the action of the 1st order operator $z\partial_{\Phi(p\bullet)}$ where $\Phi(p\bullet) = \Phi(\phi_1\bullet, \dots, \phi_n\bullet)1$ is the value of Φ computed in the quantum cohomology algebra. Therefore $e^{\nu\Phi(z\partial)/z}J$ can be written as

$$\left[1 + \nu \sum_{\gamma=1}^n a_{\gamma}(z, \tau; \nu) z \partial_{\phi_{\gamma}} \right] e^{\nu\partial_{\Phi(p\bullet)}} J(z, \tau),$$

where the coefficients a_{γ} reduced modulo any power of ν are polynomial in z . By Taylor's formula,

$$I := e^{\nu\partial_{\Phi(p\bullet)}} J(z, \tau) = J(z, \tau + \nu\Phi(p\bullet)).$$

The point $I(-z, \tau)$ lies in $\mathcal{L}$ since it is the value of the J-function, only at a shifted point. Thus the whole expression is obtained by adding to this value a linear combination of the derivatives $z\partial_{\phi_{\alpha}}I$ which lie in $zT_I(-z, \tau) \subset \mathcal{L}$, so that the whole sum lies in $\mathcal{L}$.

In the case when $\Phi(0) \neq 0$, it suffices to add that

$$e^{\nu\Phi(0)/z} J(z, \tau) = J(z, \tau + \nu\Phi(0))$$

due to the *string equation* $z\partial_1 J = J$.

Corollary. *The conclusion of the Lemma remains true even if the differential symbol $\Phi(p, z, \nu)$ is allowed to depend on z and ν (provided that modulo any power of ν it is polynomial in z).*

Remark. In Proposition 3, and hence in the results of 4.3, one can replace the J-function by any function $\tau \mapsto I(-z, \tau) \in \mathcal{L}$ transverse to the ruling spaces zT_{τ} .

0.5 Mirrors

5.1. Mirrors of toric manifolds. In equivariant Gromov–Witten theory of toric manifolds, the *mirror* of the toric manifold X (see 1.2 for notations) is defined as the following oscillating integral

$$\mathcal{I}(z, qe^t, \lambda) := \int e^{\sum_{j=1}^N (x_j + \lambda_j \ln x_j)/z} \frac{d \ln x_1 \wedge \dots \wedge d \ln x_N}{d \ln q_1 e^{t_1} \wedge \dots \wedge d \ln q_K e^{t_K}}$$

over suitable cycles in subvarieties of $\mathbb{C}^N$ given by the equations:

$$\prod_{j=1}^N x_j^{m_{ij}} = q_i e^{t_i}, \quad i = 1, \dots, K.$$

To a fixed point $\alpha = (j_1, \dots, j_K) \in X^T$, one associates a cycle C_α of integration which is $\mathbb{R}_+^{N-K}$ in the chart $\{x_j | j \notin \alpha\}$. On this cycle, the variables x_j with $j \in \alpha$ can be expressed via the above relations in terms of the coordinates $x_j, j \notin \alpha$. Put

$$\mathcal{I}_\alpha(z, t, q, \lambda) := \int_{C_\alpha} e^{\sum_{j=1}^N (x_j + \lambda_j \ln x_j)/z} \bigwedge_{j \notin \alpha} d \ln x_j.$$

Theorem 3. *Let $J(z, \tau, Q)$ be the J-function of the base B of the toric fibration $E \rightarrow B$ with the fiber X . Then*

$$q^{-P_\alpha/z} \mathcal{I}_\alpha(z, t, q, z\partial_\Lambda) J(z, \tau, Q) = \alpha^* I_E(z, t, \tau, q, Q) \prod_{j \notin \alpha} \int_0^\infty e^{(x - \alpha^* U_j \ln x)/z} d \ln x.$$

Proof. As it was mentioned in 1.1, the dependence of the genus-0 descendant potential on Novikov's variables is governed by *divisor equations*. They can be stated in terms of the cone $\mathcal{L}_B$ as follows [3, 7]. Novikov's variables Q_i represent degrees (of holomorphic curves) which form a basis in $H_2(B, \mathbb{Q})$. Let $\{\rho_i\}$ denote the dual basis in $H^2(B, \mathbb{Q})$. The linear operator $f \mapsto \rho_i f/z$ lies in the Lie algebra of the twisted loop group and thus defines in the symplectic loop space $(\mathcal{H}, \Omega)$ a linear Hamiltonian vector field which we denote ρ_i/z . Then $\mathcal{L}_B$, considered as a family of Lagrangian cones depending on Q , is invariant under the flows of the vector fields $Q_i \partial/\partial Q_i - \rho_i/z$.

The divisor equations give rise to the following symmetries of the J-function. Let $J(z, \tau, Q) = \sum_D Q^D J_D(z, \tau)$, and $\rho \in H^2(B, \mathbb{Z})$. Then

$$J(z, \tau + t\rho, Q) = e^{\rho t/z} \sum_D Q^D e^{\rho(D)t} J_D(z, \tau),$$

where $\rho(D)$ is the value of the cohomology class ρ on the homology class D .

Thus, we have:

$$J(z, \tau + \sum_j \Lambda_j \ln x_j, Q) = e^{\sum_j \Lambda_j \ln x_j/z} \sum_D J_D(z, \tau) Q^D \prod_j x_j^{\Lambda_j(D)}.$$

Therefore

$$\begin{aligned} \mathcal{I}_\alpha(z, t, q, z\partial_\Lambda) J(z, \tau, Q) &= \int_{C_\alpha} e^{\sum_{j=1}^N (x_j + z\partial_\Lambda_j \ln x_j)/z} J(z, \tau) \bigwedge_{j \notin \alpha} d \ln x_j = \\ &= \int_{C_\alpha} e^{\sum_{j=1}^N x_j/z} J(z, \tau + \sum_{j=1}^N \Lambda_j \ln x_j) \bigwedge_{j \notin \alpha} d \ln x_j = \\ &= \sum_D J_D(z, \tau) Q^D \int_{C_\alpha} e^{\sum_{j=1}^N (x_j + \Lambda_j \ln x_j)/z} \prod_{j=1}^N x_j^{\Lambda_j(D)} \bigwedge_{j \notin \alpha} d \ln x_j. \end{aligned}$$

On the other hand, relations between x_j can be written in a more general form:

$$\forall d \in \mathbb{Z}^K, \quad (qe^t)^d = \prod_j x_j^{U_j(d)},$$

since $U_j(d) = \sum_i d_i m_{ij}$. The map $\mathbb{Z}^K \ni d \mapsto \{U_j(d), j \in \alpha\} \in \mathbb{Z}^K$ is an isomorphism of lattices (this is a necessary condition for the toric variety X to be non-singular at the fixed point α). Therefore

$$e^{\sum_{j \in \alpha} x_j/z} = \sum_{\{d \mid U_j(d) \geq 0 \ \forall j \in \alpha\}} \frac{(qe^t)^d}{\prod_{j \in \alpha} z^{U_j(d)} U_j(d)! \prod_{j \notin \alpha} x_j^{U_j(d)}}.$$

Furthermore, from $\Lambda_j = \sum_i P_i^\alpha m_{ij} - \alpha^* U_j$ (see 2.1), we find:

$$\begin{aligned} e^{\sum_{j=1}^N \Lambda_j \ln x_j/z} &= \prod_{j=1}^N x_j^{-\alpha^* U_j/z} \prod_{i=1}^K \left[\prod_{j=1}^N x_j^{m_{ij}} \right]^{P_i^\alpha/z} = \prod_{j \notin \alpha} x_j^{-\alpha^* U_j/z} \prod_{i=1}^K (q_i e^{t_i})^{P_i^\alpha/z}, \\ \prod_{j=1}^N x_j^{\Lambda_j(D)} &= \prod_{j=1}^N x_j^{-\alpha^* U_j(D)} \prod_{i=1}^K \left[\prod_{j=1}^N x_j^{m_{ij}} \right]^{P_i^\alpha(D)} = \prod_{j \notin \alpha} x_j^{-\alpha^* U_j(D)} \prod_{i=1}^K (q_i e^{t_i})^{P_i^\alpha(D)}. \end{aligned}$$

Using this we rearrange the integrand to obtain:

$$\begin{aligned} q^{-P^\alpha/z} \mathcal{I}_\alpha(z, t, q, z \partial_\Lambda) J(z, \tau, Q) &= e^{P^\alpha t/z} \sum_D J_D(z, \tau) Q^D (qe^t)^{P^\alpha(D)} \times \\ &\sum_{d \in \mathbb{Z}^K} \frac{(qe^t)^d}{\prod_{j \in \alpha} \prod_{m=1}^{U_j(d)} (mz)} \prod_{j \notin \alpha} \int_0^\infty e^{x/z} x^{-\alpha^* U_j/z - U_j(d) - \alpha^* U_j(D) - 1} dx. \end{aligned}$$

Integrating by parts $U_j(d) + \alpha^* U_j(D)$ times, and making assumptions about the values of z and $\alpha^* U_j$ which would guarantee that the integrand vanishes at $x = 0$ and $x = \infty$, we find:

$$\int_0^\infty e^{x/z} x^{-\alpha^* U_j/z - U_j(d) - \alpha^* U_j(D) - 1} dx = \frac{\int_0^\infty e^{(x - \alpha^* U_j \ln x)/z} d \ln x}{\prod_{m=1}^{U_j(d) + \alpha^* U_j(D)} (\alpha^* U_j + mz)}.$$

To complete the proof, substitute this into the previous formula, and compare the result with the expression for $\alpha^* I_E$ from Section 3:

$$\alpha^* I_E = e^{P^\alpha t/z} \sum_D \sum_{d \in \mathbb{Z}^K} \frac{J_D(z, \tau) (Q^D q^{P^\alpha(D)}) q^d e^{dt} e^{P^\alpha(D)t}}{\prod_{j \in \alpha} \prod_{m=1}^{U_j(d)} (mz) \prod_{j \notin \alpha} \prod_{m=1}^{U_j(d) + \alpha^* U_j(D)} (\alpha^* U_j + mz)}.$$

Remark. The assumptions about $\operatorname{Re} z$ and $\operatorname{Re} \Lambda_j$, which guarantee convergence of the integrals and vanishing of the finite terms that come out of integration by parts, may differ for different terms of the series. The theorem should be understood therefore as the identity between coefficients of (Q, q) -series. In the next corollary about *asymptotics* of the integrals, convergence of the integrals is not required, and according to Corollary 1 of Proposition 1, integrations by parts does not generate finite terms.

Corollary. *Let $\hat{\mathcal{I}}_\alpha(z, t, q, \lambda)$ and $\hat{\Gamma}(z, \nu)$ denote stationary phase asymptotics of the oscillating integrals $\mathcal{I}_\alpha$ and $\int_0^\infty e^{(-x+\nu \ln x)/z} d \ln x$ respectively. Then*

$$q^{-P_\alpha/z} \hat{\mathcal{I}}_\alpha(z, t, q, z\partial_\Lambda) J(z, \tau, Q) = \alpha^* I_E(z, t, \tau, q, Q) \prod_{j \notin \alpha} \hat{\Gamma}(-z, \alpha^* U_j).$$

Proof. On the LHS, we have

$$\begin{aligned} \hat{\mathcal{I}}_\alpha(z, t, q, z\partial_\Lambda) J(z, \tau, Q) &= \sum_D Q^D \hat{\mathcal{I}}_\alpha(z, t, q, z\partial_\Lambda) J_D(z, \tau) \\ &= \sum_D Q^D J_D(z, \tau) \hat{\mathcal{I}}_\alpha(z, t, q, \Lambda + z\Lambda(D)), \end{aligned}$$

since $z\partial_{\Lambda_j} J_D = (\Lambda_j + z\Lambda_j(D)) J_D$ due to the divisor equation. The factor $\hat{\mathcal{I}}_\alpha(z, t, q, \Lambda + z\Lambda(D))$ is the stationary phase asymptotics of the integral

$$\int_{C_\alpha} e^{\sum_{j=1}^N (x_j + \Lambda_j \ln x_j)/z} \prod_{j=1}^N x_j^{\Lambda_j(D)} \bigwedge_{j \notin \alpha} d \ln x_j,$$

which depends on the parameters q (as well as t and Λ). According to Corollary of Proposition 2, such asymptotics of a single oscillating integral depending on parameters can be replaced with a suitable q -series of asymptotics of oscillating integrals. Following the steps in the proof of Theorem 3 and applying Corollary 1 of Proposition 1 to justify integration by parts, we arrive at the expression on the RHS.

5.2. The Quantum Riemann–Roch theorem. We have:

$$\hat{\Gamma}(z, \nu) = \sqrt{\frac{2\pi z}{\nu}} \exp \left\{ \frac{\nu \ln \nu - \nu}{z} + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)} \left(\frac{z}{\nu} \right)^{2m-1} \right\}.$$

Here B_{2m} are Bernoulli numbers, and the equality follows from the well-known asymptotics of the logarithm of the Gamma-function $\Gamma(\nu/z)$.

Let $\mathcal{L}$ and $\mathcal{L}^{tw}$ be the overruled Lagrangian cones respectively: of genus 0 Gromov–Witten theory of a target manifold M , and of such a theory *twisted* (in the sense of 2.3) by a line bundle over M with the equivariant 1st Chern class ν . The cone $\mathcal{L}$ lies in

the symplectic loop space $(\mathcal{H}, \Omega)$ based on the Poincare pairing $(a, b) = \int_M ab$, while $\mathcal{L}^{tw}$ lies in $(\mathcal{H}, \Omega^{tw})$ based on $(a, b)^{tw} = \int_M ab/\nu$. The linear map $(\mathcal{H}, \Omega^{tw}) \rightarrow (\mathcal{H}, \Omega)$ defined by $f \mapsto f/\sqrt{\nu}$ is a symplectomorphism.

Theorem ([3]).

$$\mathcal{L} = \frac{\hat{\Gamma}(z, \nu)}{\sqrt{2\pi z}} \mathcal{L}^{tw}.$$

5.3. Completing the proof of Theorem 1. We need to show that $\alpha^* I_E(-z)$ lies in the overruled Lagrangian cone $\mathcal{L}^\alpha$ corresponding to the genus-0 Gromov–Witten theory of B twisted by the normal bundle N^α of $\alpha(B) \subset E$. Due to the Quantum Riemann–Roch Theorem, it suffices to prove that, equivalently,

$$\prod_{j \notin \alpha} \frac{\hat{\Gamma}(z, \alpha^* U_j)}{\sqrt{2\pi z}} \alpha^* I_E(-z)$$

lies in the overruled Lagrangian cone of the “untwisted” theory. Note that in this formula, Novikov’s variables still occur in the form $Q^D q^{P^\alpha(D)}$ to account correctly for degrees of curves in $\alpha(B) \subset E$ considered as curves in E . According to Corollary of Theorem 3, the above expression coincides with

$$(2\pi z)^{-\dim N^\alpha/2} q^{P^\alpha/z} \hat{\mathcal{I}}_\alpha(-z, t, q, -z\partial_\Lambda) \sum_D Q^D J_D(-z, \tau).$$

We make the change $Q^D \mapsto Q^D q^{-P^\alpha(D)}$ to restore the absolute meaning of Novikov’s variables, and apply the divisor equation:

$$q^{P^\alpha/z} \sum_D Q^D q^{-P^\alpha(D)} J_D(-z, \tau) = J(-z, \tau - P^\alpha \ln q, Q).$$

Then it remains to show that

$$(2\pi z)^{-\dim N^\alpha/2} \hat{\mathcal{I}}_\alpha(-z, t, q, -z\partial_\Lambda) J(-z, \tau - P^\alpha \ln q, Q) \in \mathcal{L}_B.$$

Since $J(z, \tau, Q)$ is the J-function of $\mathcal{L}_B$, this follows from the results of 4.3 about actions of pseudo-differential symbols on J-functions.

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