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UNIVERSITY OF CALIFORNIA,
IRVINE

A Morse Theory for the Cohomology of Primitive Forms on Symplectic Manifolds

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

David Clausen

Dissertation Committee:
Associate Professor Li-Sheng Tseng, Chair
Professor Zhiqin Lu
Professor Richard Schoen

2023

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DEDICATION

To my family.

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ABSTRACT OF THE DISSERTATION

A Morse Theory for the Cohomology of Primitive Forms on Symplectic Manifolds

By

David Clausen

Doctor of Philosophy in Mathematics

University of California, Irvine, 2023

Associate Professor Li-Sheng Tseng, Chair

On a symplectic manifold, Tsai, Tseng, and Yau introduced a coholomogy of differential forms that is analogous to the Dolbeault cohomology for symplectic manifolds. Such forms are called primitive forms. We develop a Morse theory for these primitive forms, including a Morse-type Cone complex of pairs of critical points that has isomorphic cohomology to the primitive cohomology. The differential of the complex consists of gradient flows and an integration of the symplectic form over spaces of gradient flow lines. We prove that the complex is independent of the choice of metric and Morse function. We also derive Morse style inequalities for the cohomology of the Cone complex and thus the primitive cohomologies. Also, we develop a Witten deformation of the Cone complex, which provides a Witten deformation of the differential operators associated to the cohomology.

Chapter 1

Introduction

We begin by introducing the background to define the primitive forms.

1.1 Filtered Cohomologies

Let (M^{2n}, ω) be a symplectic manifold.

Definition 1.1. *For a coordinate chart $(U, x_1, ...x_{2n})$ of M, Define the Poisson bivector as the map*

$$\Lambda = \frac{1}{2}(\omega^{-1})^{ij}\iota_{\frac{\partial}{\partial x_i}}\iota_{\frac{\partial}{\partial x_i}}, 1 \leq i, j \leq 2n$$

where $(\omega^{-1})^{ij}$ is the inverse matrix of ω_{ij} and $\iota_{\partial_{x_i}}$ is the interior product. $\Lambda : \Omega^*(M) \rightarrow \Omega^{*-2}(M)$ is globally defined on (M, ω), so it does not depend on coordinate chart.

Proof: To see this, note that if $\omega = \frac{1}{2}\omega_{ij}dx_i \wedge dx_j$ in one chart, and $\omega = \frac{1}{2}\omega'_{k\ell}dy_k \wedge dy_\ell$ in another, then $\omega_{ij} = \frac{\partial y_k}{\partial x_i}\omega'_{k\ell}\frac{\partial y_\ell}{\partial x_j}$. Thus $(\omega^{-1})^{ij} = \frac{\partial x_j}{\partial y_\ell}(\omega'^{k\ell})^{-1}\frac{\partial x_i}{\partial y_k}$, Then

$$2\Lambda = (\omega^{-1})^{ij}\iota_{\frac{\partial}{\partial x_i}}\iota_{\frac{\partial}{\partial x_i}} = \frac{\partial x_j}{\partial y_\ell}(\omega'^{k\ell})^{-1}\frac{\partial x_i}{\partial y_k}\iota_{\frac{\partial}{\partial x_i}}\iota_{\frac{\partial}{\partial x_i}} = (\omega'^{k\ell})^{-1}\iota_{\frac{\partial x_i}{\partial y_k}\frac{\partial}{\partial x_i}}\iota_{\frac{\partial x_j}{\partial y_\ell}\frac{\partial}{\partial x_j}} = (\omega'^{k\ell})^{-1}\iota_{\frac{\partial}{\partial y_k}}\iota_{\frac{\partial}{\partial y_\ell}}$$

which is $2\Lambda'$ so they agree on the overlap of coordinate charts and are thus globally define.

Definition 1.2. *Define the maps*

$$L : \Omega^*(M) \to \Omega^{*+2}(M)$$

by $LA_k = \omega \wedge A_k$ *and* $H : \Omega^*(M) \to \Omega^*(M)$ *by*

$$H = \sum_k (n-k)\Pi_k$$

where $\Pi_k : \Omega^*(M) \to \Omega^k(M)$ *is the projection,* $\Pi_k(A) = A$ *if* $A \in \Omega^k(M)$ *and* $\Pi_k(A) = 0$ *if* $A \notin \Omega^k(M)$.

Lemma 1.1.1. *the three operators* L, Λ, H *satisfy the* $\mathfrak{sl}_2$ *commutivity relations, or*

$$[\Lambda, L] = H$$
$$[H, \Lambda] = 2\Lambda$$
$$[H, L] = -2L$$

Proof: The last two relations follow from degree considerations. For the first relation, we can simply work with Darboux coordinates $\omega = dp_i \wedge dq_i, \Lambda = \iota_{\frac{\partial}{\partial q_j}}\iota_{\frac{\partial}{\partial p_j}}$. Then

$$\begin{aligned}
\Lambda L &= \sum_{i,j} \iota_{\frac{\partial}{\partial q_j}}\iota_{\frac{\partial}{\partial p_j}}(dp_i \wedge dq_i) \\
&= \sum_{i,j} \iota_{\frac{\partial}{\partial q_j}}[\iota_{\frac{\partial}{\partial p_j}}(dp_i \wedge dq_i) + (dp_i \wedge dq_i)\iota_{\frac{\partial}{\partial p_j}}] \\
&= \sum_{i,j}[\iota_{\frac{\partial}{\partial q_j}}(\delta_{ji} \wedge dq_i) - \delta_{ji} \wedge dq_i\iota_{\frac{\partial}{\partial q_j}} + \iota_{\frac{\partial}{\partial q_j}}(dp_i \wedge dq_i)\iota_{\frac{\partial}{\partial p_j}} + (dp_i \wedge dq_i)\iota_{\frac{\partial}{\partial q_j}}\iota_{\frac{\partial}{\partial p_j}}] \\
&= \sum_{i,j}[\iota_{\frac{\partial}{\partial q_j}}(dq_j) - \delta_{ji} \wedge dq_i\iota_{\frac{\partial}{\partial q_j}} - (dp_i \wedge \delta_{ji})\iota_{\frac{\partial}{\partial p_j}}] + L\Lambda \\
&= \sum_{i,j}[1 - dq_j \wedge \iota_{\frac{\partial}{\partial q_j}} - (dp_j\wedge)\iota_{\frac{\partial}{\partial p_j}}] + L\Lambda
\end{aligned}$$

and since we are summing over $1 \leq j \leq n$ and $1 \leq j \leq n$, we have $\Lambda L - L\Lambda = n - dq_j \wedge \iota_{\frac{\partial}{\partial q_j}} - (dp_j\wedge)\iota_{\frac{\partial}{\partial p_j}}$. Now note that if this is applied to a k-form $A = A_{I,J}dp_I dq_J$, then $\sum_j dp_j \wedge \iota_{\frac{\partial}{\partial p_j}} A_{I,J}dp_I dq_J = |I|A_{I,J}dp_I dq_J$ and $\sum_j dq_j \wedge \iota_{\frac{\partial}{\partial q_j}} A_{I,J}dp_I dq_J = |J|A_{I,J}dp_I dq_J$, thus

$$(\Lambda L - L\Lambda)A = nA - (|I| + |J|)A = (n-k)A$$

Corollary 1.1.2. *For any $\mathfrak{sl}_2$ there is a Lefschetz decomposition $\Omega^k = \bigoplus_{k=2r+s} L^r P_s$ where $P_s = \{B_s \in \Omega^s : \Lambda B_s = 0\}$ are called primitive forms.*

Proof: see [7]

Using this, we define filtered forms

Definition 1.1.3. *A form $A_k \in \Omega^k(M)$ is p-filtered if*

$$A_k = B_k + LB_{k-2} + ... + L^p B_{k-2p},$$

or its Lefschetz decomposition does not possess terms with terms L^{p+1} or higher. We denote the filtered k-forms by $F^p\Omega^k = \{A_k \in \bigoplus_{k=2r+s}^{s\leq p} L^r P_s\}$

We are interested in how the differential interacts with the filtration. For that, we first define the following two operators

Definition 1.1.4. *Let $A_k \in \Omega_k$ have a Lefschetz decomposition $A_k = A_k = B_k + LB_{k-2} + ...L^s B_{k-2s}$. Define the projection operator $\Pi^p : \Omega_k \to F^p\Omega^k$ by $\Pi^p A_k = B_k + LB_{k-2} + ...L^p B_{k-2p}$ by projecting to the first p components. Also, define the reflection operator $*_r : Omega_k \to \Omega^{2n-k}$ by $*_r(A_k) = *_r(B_k + LB_{k-2} + ... + L^p B_{k-2p}) = L^{n-k}B_k + L^{n-k+1}B_{k-2} + ... + L^{n-k+p}B_{k-2p}$*

Using these operators one can define an elliptic complex on $F^p\Omega^k$ for a particular p

Theorem 1.1. *Define $d_- := *_r d *_r$ and $d_+ := \Pi^p d$. Then*

$$\begin{array}{ccccccccc} 0 & \xrightarrow{d_+} & F^p\Omega^0 & \xrightarrow{d_+} & F^p\Omega^1 & \xrightarrow{d_+} & \dots & \xrightarrow{d_+} & F^p\Omega^{n+p} \\ & & & & & & & & \downarrow{\scriptstyle \partial_+\partial_-} \\ 0 & \xleftarrow{d_-} & F^p\Omega^0 & \xleftarrow{d_-} & F^p\Omega^1 & \xleftarrow{d_-} & \dots & \xleftarrow{d_-} & F^p\Omega^{n+p} \end{array}$$

is an elliptic complex.

Proof: See [12]. Therefore, we have the following cohomologies.

Definition 1.3 (Filtered cohomologies)**.** *Let $F^pH^* = \{F^pH^0_+, \dots, F^pH^{p+n}_+, F^pH^{p+n}_-, \dots, F^pH^0_-\}$, where*

$$F^pH^k_+ = \frac{ker(d_+) \cap F^p\Omega^k)}{d_+(F^p\Omega^{k-1})} \qquad k = 0, 1, 2, \dots, n+p-1$$

$$F^pH^{n+p}_+ = \frac{ker(\partial_+\partial_-) \cap F^p\Omega^k}{d_+(F^p\Omega^{n+p-1})}$$

$$F^pH^{n+p}_- = \frac{ker(d_-) \cap F^p\Omega^{n+p}}{\partial_+\partial_-(F^p\Omega^{n+p})}$$

$$F^pH^k_- = \frac{ker(d_-) \cap F^p\Omega^k}{d_-(F^p\Omega^{k+1})} \qquad k = 0, 1, 2, \dots, n+p-1$$

note that $F^pH_+^{n+p}$ and $F^pH_-^{n+p}$ these are second order differential operators. The filtered complex has an alternative description in terms of the mapping cone complex, which we now describe

Definition 1.4. *Define* $Cone^k(\omega^{p+1}) = \{\eta + \theta\xi : \eta \in \Omega^k(M), \xi \in \Omega^{k-1}(M)\} = \Omega^k(M) \bigoplus \theta\Omega^{k-1}(M)$ *on the symplectic manifold* M*, where* θ *is a formal object satisfying* $d\theta = \omega^{p+1}$

Applying the exterior derivative gives

$$d(\eta_k + \theta\xi_{k-1}) = d\eta_k + \omega^{p+1} \wedge \xi_{k-1} + \theta(-d\xi_{k-1})$$

Thus we can define d_C on $\begin{bmatrix} \eta_k \\ \xi_{k-1} \end{bmatrix}$ as

$$d_C = \begin{bmatrix} d & \omega^{p+1}\wedge \\ 0 & -d \end{bmatrix}.$$

Thus we have an elliptic complex $(\text{Cone}(\omega^{p+1}), d_C)$.

Theorem 1.2. *The cohomology of* $Cone(\omega^{p+1})$ *is isomorphic to the cohomology of the filtered complex*

$$\begin{aligned} F^pH_+^k &\cong H^k(Cone(\omega^{p+1})) & 0 \le k < n+p \\ F^pH_+^{n+p} &\cong H^{n+p}(Cone(\omega^{p+1})) & \\ F^pH_-^{n+p} &\cong H^{n+p+1}(Cone(\omega^{p+1})) & \\ F^pH_-^k &\cong H^{2n+p+1-k}(Cone(\omega^{p+1})) & 0 \le k < n+p \end{aligned}$$

Proof: see [12] and [11].

We wish to develop a Morse theory for the filtered cohomology, below we will give a brief review of Morse Theory.

1.2 Morse Theory

Definition 1.5. *A function $f : M \to \mathbb{R}$ on a manifold is a Morse function if at critical points where $df = 0$, the Hessian $\left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{ij}$ is non-degenerate. If p is a critical point, then the index of p is defined to be the dimension of the negative definite subspace of T_pM with respect to $\left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{ij}$.*

For a Riemannian metric g, we let $\phi : \mathbb{R} \to M$ be the gradient flow by -f, $\dot{\phi}(t) = -\nabla f$. This allows us to define two submanifolds related to the flow.

Definition 1.6. *Let p be a critical point of a Morse function. Define $U_p = \{y \in M : \lim_{t \to -\infty} \phi_t(y) = p\}$ to be the unstable manifold of p, and $S_p = \{y \in M : \lim_{t \to \infty} \phi_t(y) = p\}$ to be the stable manifold of p. We say a pair (f, g) is Morse-Smale provided for all $p \neq q$, $U_p \pitchfork S_q$.*

One can flow by $-\text{grad}(f)$ from a critical point p to another point q. Being Morse-Smale ensures that if there is a flow from p to q, then $ind(p) > ind(q)$

Definition 1.7. *Let $C_k(M, f, g) =$ the free $\mathbb{Z}$ module generated by the critical points p of f with $ind(p) = k$. We define the boundary operator $\partial_{f,g} : C_k(M, f, g) \to C_{k-1}(M, f, g)$*

$$\partial_{f,g} p_k = \sum_{ind(q)=k-1} \#\tilde{M}(p, q) q_{k-1}$$

Where $\tilde{M}(p, q) = (U_p \cap S_q)/\sim$ is the submanifold of points that flow from p to q modded by the flow, counted with orientation.

The transversality condition ensures that $\tilde{M}(p, q)$ is a zero dimensional manifold. If (f, g) is Morse-Smale, we have the following theorem [19].

Theorem 1.3. *the following is a differential complex*

$$C_{\dim M}(M,f,g) \xrightarrow{\partial_{f,g}} C_{\dim M-1}(M,f,g) \xrightarrow{\partial_{f,g}} \dots \xrightarrow{\partial_{f,g}} C_0(M,f,g)$$

The Morse homology is then $(H_k)_{C(f)}(M,f,g) = \dfrac{Ker(\partial_{f,g})_k}{Im(\partial_{f,g})_{k-1}}$

Note that, from [19], $(H_k)_{C(f)}(M,f,g) \cong H^k_{dR}(M)$. Thus the Morse homology is independent of the Morse function f and the metric g. Witten showed in [18] can also use the Morse function to deform the exterior derivative.

Definition 1.8 (Witten Deformation)**.** *If f is a Morse function, then for $t \in \mathbb{R}$, define the deformed differential $d_t = e^{-tf} d e^{tf} = d + tdf\wedge$*
if we have a metric g, we can also define the deformed adjoint $d_t^* = e^{tf} d^* e^{-tf} = d^* + t\iota_{\nabla f}$ and the deformed Laplacian $\Delta_t = d_t d_t^* + d_t^* d_t = \Delta + t(df \wedge d^* + d\iota_{\nabla f} + \iota_{\nabla f} d + d^* df\wedge) + t^2|df|^2$.

In [18], the factor $t^2|df|^2$ dominates as t goes to infinity and the harmonic solutions localize around critical points, which provides a proof of the Morse inequalities.

Theorem 1.4 (Morse Inequalities)**.** *Let $m_k = \dim C_k$ be the number of Morse points of index k. The Morse polynomial is defined as $M(t) = \sum_{k=0}^{\dim M} m_k t^k$. If $P(t) = \sum_{k=0}^{\dim M} b_k t^k$ is the Poincaré polynomial, then*

$$M(t) = P(t) + (1+t)Q(t)$$

where $Q(t) = \sum_{k=0}^{\dim M} a_k t^k$ is a nonnegative polynomial, $a_k \geq 0$.
Thus we have the weak Morse inequalities

$$m_k \geq b_k \text{ for all } k$$

and the strong Morse inequalities

$$\sum_{k=0}^{i}(-1)^{i-k}m_k \geq \sum_{k=0}^{i}(-1)^{i-k}b_k \textit{ for all } i$$

Our goal is to derive a Morse theory for the filtered cohomologies, and in certain cases generalize from ω^{p+1} it to other closed forms ψ. This would involve a Morse complex, Morse inequalities, and Witten Deformation.

	$\Omega^*(M)$	$C(M,f,g)$	$\mathrm{Cone}(\omega^{p+1})$	Morse $\mathrm{Cone}(\omega^{p+1})$?
elements	differential forms	critical points p_k	$\begin{bmatrix}\eta\\ \xi\end{bmatrix}$	?
differential	exterior derivative d	Morse flow ∂	$d_C = \begin{bmatrix}d & \omega\wedge\\ 0 & -d\end{bmatrix}$	?
Weak Morse Inequalities	$b_k \leq m_k = \#$ of critical points of index k		$b_k^{\omega^{p+1}} = \dim(H^k(\mathrm{Cone}(\omega))) \leq ?$	
Strong Morse Inequalities	$\sum_{k=0}^{i}(-1)^{i-k}b_k \leq \sum_{k=0}^{i}(-1)^{i-k}m_k$		$\sum_{k=0}^{i}(-1)^{i-k}b_k^{\omega^{p+1}} \leq ?$	

Chapter 2

Symplectic Morse Chain Complex

2.1 Preliminaries

The following work is derived from joint work with Tseng and Tang in [4]. Let (M^d, ψ) be a closed manifold of dimension d equipped with a geometric structure given by a closed differential ℓ-form, $\psi \in H^\ell(M)$. For example, we might consider a symplectic manifold (M^{2n}, ω).

Our symplectic Morse complex is motivated by a result of Tanaka-Tseng [11] that relates the cochain complex that underlies the TTY cohomologies with the Cone complex of the wedge product map $\omega^{p+1} : \Omega^\bullet(M) \to \Omega^{\bullet+2p+2}(M)$ on the space of differential forms.

2.2 Morse Cone Complex: Cone(c(ψ))

2.2.1 Preliminaries: Morse Complex and $c(\psi)$

To begin, let f be a Morse function and g a Riemannian metric on M. We will assume that (f, g) satisfy the standard Morse-Smale transversality condition. The elements of the Morse cochain complex $C^\bullet(M, f)$ are $\mathbb{R}$-modules with generators critical points of f, graded by the index of the critical points, with boundary operator ∂ determined by the counting of gradient lines, i.e.

$$\partial q_k = \sum_{ind(r)=k+1} n(r_{k+1}, q_k)\, r_{k+1},$$

where $n(r_{k+1}, q_k) = \#\widetilde{\mathcal{M}}(r_{k+1}, q_k)$ is a count of the moduli space of gradient flow lines with orientation modulo reparametrization.

Note that Morse theory is typically presented as a homology theory, and hence, flowing from index k to index $k-1$ critical points. To match up with the cochain complex of differential forms, we here work with the dual Morse cochain complex. Hence, our ∂ is the adjoint of the usual Morse boundary map under the inner product $\langle q_{k_i}, q_{k_j}\rangle = \delta_{ij}$.

Following Austin-Braam [1] and Viterbo [16], we define

$$c(\psi) q_k = \sum_{ind(r)=k+\ell} \left(\int_{\overline{\mathcal{M}(r_{k+\ell}, q_k)}} \psi \right) r_{k+\ell}$$

where $\psi \in \Omega^\ell(M)$ is an ℓ-form and $\mathcal{M}(r_{k+\ell}, q_k)$ is the submanifold of all points that flow from $r_{k+\ell}$ to q_k, oriented as in [1]. From Appendix A, we have the Leibniz-type product relation

$$\partial c(\psi) + (-1)^{\deg(\psi)+1} c(\psi)\partial = -c(d\psi)$$

specifying a sign convention that is ambiguous in Austin-Braam [1] and Viterbo [16]. Thus, for instance, for $\psi = \omega$, the symplectic structure, we have the relation

$$\partial c(\omega) - c(\omega)\partial = -c(d\omega) = 0.$$

2.2.2 Chain Map Between $\mathrm{Cone}(\psi)$ and $\mathrm{Cone}(c(\psi))$

As explained by Bismut, Zhang and Laudenbach [2, 19], there is a chain map $\mathcal{P} : \Omega^k(M) \to C^k(M, f)$ between differential forms and the Morse cochain complex given by

$$\mathcal{P}\phi = \sum_{q_k \in Crit(f)} \left(\int_{\overline{U_{q_k}}} \phi \right) q_k$$

where $\phi \in \Omega^k(M)$ and U_q is the set of all points on a gradient flow away from q. Being a chain map,

$$\partial \mathcal{P} = \mathcal{P}\, d\,. \tag{2.1}$$

We are interested to find an analogous chain map relating $\mathrm{Cone}(\psi) = (\Omega^\bullet(M) \oplus \theta\, \Omega^{\bullet-\ell+1}(M),\, d_C)$ with $\mathrm{Cone}(\psi) = (C^\bullet(M, f) \oplus C^{\bullet-\ell+1}(M, f),\, \partial_C)$, where as given in Definition of the mapping cone,

$$d_C : \Omega^k(M) \oplus \theta\, \Omega^{k-l+1}(M) \to \Omega^{k+1}(M) \oplus \theta\, \Omega^{k-l+2}(M)$$
$$\partial_C : C^k(M, f) \oplus C^{k-l+1}(M, f) \to C^{k+1}(M, f) \oplus\, C^{k-l+2}(M, f)$$

with

$$d_C = \begin{bmatrix} d & \psi \\ 0 & (-1)^{\ell-1} d \end{bmatrix}, \qquad \partial_C = \begin{bmatrix} \partial & c(\psi) \\ 0 & (-1)^{\ell-1} \partial \end{bmatrix}.$$

The chain map, which we will label by $\mathcal{P}_C$, that links the two cone complexes will need to satisfy $\partial_C \mathcal{P}_C = \mathcal{P}_C d_C$. In fact, such a map exists and can be expressed in an upper-triangular matrix form.

Definition 2.2.1. *Let* $\mathcal{P}_C : \mathrm{Cone}^\bullet(\psi) \to \mathrm{Cone}^\bullet(c(\psi))$ *be the upper-triangular matrix map*

$$\mathcal{P}_C = \begin{bmatrix} \mathcal{P} & K \\ 0 & \mathcal{P} \end{bmatrix}$$

where $K : \Omega^{k-\ell+1}(M) \to C^k(M, f)$ *acting on* $\xi \in \Omega^{k-\ell+1}(M)$ *is defined by*

$$K\xi = (-1)^\ell \left[\mathcal{P}(\psi \wedge d^* G\xi) - c(\psi)\mathcal{P}(d^* G\xi)\right] + \partial_{k,\perp}^{-1}\left(\mathcal{P}(\psi \wedge \mathcal{H}\xi) - c(\psi)\mathcal{P}(\mathcal{H}\xi)\right), \qquad (2.2)$$

in terms of the Hodge decomposition with respect to the de Rham Laplacian $\Delta = dd^* + d^*d$*:*

$$\xi = (\mathcal{H} + \Delta G)\xi = \mathcal{H}\xi + dd^* G\xi + d^* dG\xi\,,$$

where $\mathcal{H}\xi$ *is the harmonic component and* G *is the Green's operator.*

We explain the notation $\partial_{k,\perp}^{-1}$ in the second term for the definition of K in (2.2).

Let γ be a closed $(k - \ell + 1)$-form. Then, it is known that $\mathcal{P}\psi \wedge \gamma$ and $c(\psi)\mathcal{P}\gamma$ are cohomologous. (See, for instance, Austin-Braam [1, Section 3.5] or Viterbo [16, Lemma 4].). Then $\mathcal{P}(\psi \wedge \gamma) - c(\psi)\mathcal{P}\gamma = \partial b$ for some $b \in C^k(M, f)$. Note that $C^k(M, f)$ is an inner product space under $\langle q_{k_i}, q_{k_j}\rangle = \delta_{ij}$, so we have an orthogonal splitting, $C^k(M, f) = \ker \partial_k \oplus (\ker \partial_k)^\perp$, and that ∂_k is an isomorphism between $\mathrm{Im}(\partial_k) \subset C^{k+1}(M, f)$ and $C^k(M, f)/\ker \partial_k$, which is isomorphic to $(\ker \partial_k)^\perp$. Thus, it follows from the finite-dimensional assumption on $C^k(M, f)$ and $C^{k+1}(M, f)$ that we can define a right inverse $\partial_{k,\perp}^{-1} : \mathrm{Im}(\partial_k) \to (\ker \partial_k)^\perp \subset C^k(M, f)$, and $\partial_{k,\perp}^{-1}(\mathcal{P}(\psi \wedge \gamma) - c(\psi)\mathcal{P}\gamma) \in C^k(M, f)$. For the second term of K in (2.2), $\gamma = \mathcal{H}\xi$ is the closed form that is the harmonic component of ξ.

With $\mathcal{P}_C$ defined, we now show that it is a chain map.

Theorem 2.2.2. $\mathcal{P}_C : \mathrm{Cone}^\bullet(\psi) \to \mathrm{Cone}^\bullet(c(\psi))$ *is a chain map. In particular,*

$$\partial_C\, \mathcal{P}_C = \mathcal{P}_C\, d_C\,. \tag{2.3}$$

Proof. The right and the left hand side of (2.3) acting on $\eta + \theta\xi \in \mathrm{Cone}^k(\psi)$ give

$$\mathcal{P}_C\, d_C = \begin{bmatrix} \mathcal{P} & K \\ 0 & \mathcal{P} \end{bmatrix} \begin{bmatrix} d & \psi \\ 0 & (-1)^{\ell-1} d \end{bmatrix} = \begin{bmatrix} \mathcal{P}d & \mathcal{P}\psi + (-1)^{\ell-1} K d \\ 0 & (-1)^{\ell-1}\mathcal{P}d \end{bmatrix},$$
$$\partial_C\, \mathcal{P}_C = \begin{bmatrix} \partial & c(\psi) \\ 0 & (-1)^{\ell-1}\partial \end{bmatrix} \begin{bmatrix} \mathcal{P} & K \\ 0 & \mathcal{P} \end{bmatrix} = \begin{bmatrix} \partial\mathcal{P} & c(\psi)\mathcal{P} + \partial K \\ 0 & (-1)^{\ell-1}\partial\mathcal{P} \end{bmatrix}.$$

Since $\mathcal{P}$ is a chain map (2.1), i.e. $d\,\mathcal{P} = \mathcal{P}\,\partial$, the only entry we need to check comes from the off-diagonal one,

$$\mathcal{P}\psi + (-1)^{\ell-1} K d = c(\psi)\mathcal{P} + \partial K\,,$$

or equivalently, we need to show that

$$\mathcal{P}\psi - c(\psi)\mathcal{P} = \partial K + (-1)^{\ell} K d\,, \tag{2.4}$$

is a graded chain homotopy. To compute $Kd\xi$, note first that $\mathcal{H}d\xi = 0\,,\ \forall \xi \in \Omega^{k-l+1}(M)$. Therefore, we find that

$$Kd\xi = (-1)^{\ell}\left[\mathcal{P}(\psi \wedge d^* G d\xi) - c(\psi)\mathcal{P}d^* G d\xi\right] = (-1)^{\ell}\left[\mathcal{P}(\psi \wedge d^* dG\xi) - c(\psi)\mathcal{P}d^* dG\xi\right]\,,$$

having used (2.2) and the fact that $Gd = dG$. Now, for the $\partial K\xi$ term, we have

$$\begin{aligned}
\partial K\xi &= \partial\left((-1)^{\ell}\left[\mathcal{P}(\psi \wedge d^* G\xi) - c(\psi)\mathcal{P}d^* G\xi\right] + \partial_{k,\perp}^{-1}(\mathcal{P}(\psi \wedge \mathcal{H}\xi) - c(\psi)\mathcal{P}\mathcal{H}\xi)\right) \\
&= (-1)^{\ell}\left[\mathcal{P}d(\psi \wedge d^* G\xi) - \partial c(\psi)\mathcal{P}d^* G\xi\right] + \partial\left(\partial_{k,\perp}^{-1}(\mathcal{P}(\psi \wedge \mathcal{H}\xi) - c(\psi)\mathcal{P}\mathcal{H}\xi)\right) \\
&= (-1)^{\ell}\left[(-1)^{\ell}\mathcal{P}(\psi \wedge dd^* G\xi) - (-1)^{\ell}c(\psi)\partial\mathcal{P}d^* G\xi\right] + \mathcal{P}\psi \wedge \mathcal{H}\xi - c(\psi)\mathcal{P}\mathcal{H}\xi \\
&= \mathcal{P}(\psi \wedge dd^* G\xi) - c(\psi)\mathcal{P}dd^* G\xi + \mathcal{P}(\psi \wedge \mathcal{H}\xi) - c(\psi)\mathcal{P}\mathcal{H}\xi
\end{aligned}$$

Altogether, we find for the right-hand side of (2.4)

$$\begin{aligned}
\partial K\xi + (-1)^{\ell} K d\xi &= \mathcal{P}(\psi \wedge dd^* G\xi) - c(\psi)\mathcal{P}dd^* G\xi + \mathcal{P}(\psi \wedge \mathcal{H}\xi) - c(\psi)\mathcal{P}\mathcal{H}\xi \\
&\qquad + \left[\mathcal{P}(\psi \wedge d^* dG\xi) - c(\psi)\mathcal{P}d^* dG\xi\right] \\
&= \mathcal{P}\left(\psi \wedge (dd^* G\xi + d^* dG\xi + \mathcal{H}\xi)\right) - c(\psi)\mathcal{P}(dd^* G\xi + d^* dG\xi + \mathcal{H}\xi) \\
&= \mathcal{P}(\psi \wedge \xi) - c(\psi)\mathcal{P}\xi.
\end{aligned}$$

Thus, K is a graded chain homotopy of $\mathcal{P}\psi$ and $c(\psi)\mathcal{P}$, and therefore, $\mathcal{P}_C\, d_C = \partial_C\, \mathcal{P}_C$. $\square$

2.2.3 Isomorphism of Cohomologies via Five Lemma

A mapping cone cochain complex can be described by a short exact sequence of chain maps. For the differential forms case, we have

$$0 \longrightarrow (\Omega^k(M), d) \xrightarrow{\iota_{dR}} (\mathrm{Cone}^k(\psi)), d_C) \xrightarrow{\pi_{dR}} (\Omega^{k-\ell+1}(M), (-1)^{\ell-1} d) \longrightarrow 0 \quad (2.5)$$

where ι is the inclusion in the first component $\iota(a) = \begin{bmatrix} a \\ 0 \end{bmatrix}$ and π is the projection to the second component $\pi\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = b$. It is easy to check that these maps are chain maps:

$$\iota_{dR} d\eta = \begin{bmatrix} d\eta \\ 0 \end{bmatrix} = d_C \iota_{dR} \eta$$

and

$$\pi_{dR} d_C \left(\begin{bmatrix} \eta \\ \xi \end{bmatrix}\right) = \pi_{dR} \begin{bmatrix} d\eta + \psi \wedge \xi \\ (-1)^{\ell-1} d\xi \end{bmatrix} = (-1)^{\ell-1} d\xi = (-1)^{\ell-1} d \left(\pi_{dR} \begin{bmatrix} \eta \\ \xi \end{bmatrix}\right) .$$

The short exact sequence (2.5) implies the following long exact sequence for the cohomology of Cone(ψ)

$$\ldots \longrightarrow H^{k-\ell}_{dR}(M) \xrightarrow{[\psi\wedge]} H^k_{dR}(M) \xrightarrow{[\iota_{dR}]} H^k(\text{Cone}(\psi)) \xrightarrow{[\pi_{dR}]} H^{k-\ell+1}_{dR}(M) \longrightarrow \ldots \tag{2.6}$$

Analogously, for Cone($c(\psi)$), we also have the short exact sequence of chain maps

$$0 \longrightarrow (C^k(M, f), \partial) \xrightarrow{\iota_{C(f)}} (\text{Cone}^k(c(\psi)), \partial_C) \xrightarrow{\pi_{C(f)}} (C^{k-\ell+1}(M, f), (-1)^{\ell-1}\partial) \longrightarrow 0 \tag{2.7}$$

and the long exact sequence of cohomology

$$\ldots \longrightarrow H^{k-\ell}_{C(f)}(M) \xrightarrow{[c(\psi)]} H^k_{C(f)}(M) \xrightarrow{[\iota_{C(f)}]} H^k(\text{Cone}(c(\psi))) \xrightarrow{[\pi_{C(f)}]} H^{k-\ell+1}_{C(f)}(M) \longrightarrow \ldots \tag{2.8}$$

The two short exact sequences, (2.5) and (2.7), fit into a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\Omega^k(M), d) & \xrightarrow{\iota_{dR}} & (\mathrm{Cone}^k(\psi)), d_C) & \xrightarrow{\pi_{dR}} & (\Omega^{k-\ell+1}(M), (-1)^{\ell-1}d) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathcal{P}} & & \downarrow{\scriptstyle \mathcal{P}_C} & & \downarrow{\scriptstyle \mathcal{P}} & & \\
0 & \longrightarrow & (C^k(M,f), \partial) & \xrightarrow{\iota_{C(f)}} & (\mathrm{Cone}^k(c(\psi)), \partial_C) & \xrightarrow{\pi_{C(f)}} & (C^{k-\ell+1}(M,f), (-1)^{\ell-1}\partial) & \longrightarrow & 0
\end{array} \tag{2.9}$$

The commutativity can be checked as follows:

$$\iota_{C(f)}(\mathcal{P}(\eta)) = \begin{bmatrix} \mathcal{P}\eta \\ 0 \end{bmatrix} = \begin{bmatrix} \mathcal{P} & K \\ 0 & \mathcal{P} \end{bmatrix} \begin{bmatrix} \eta \\ 0 \end{bmatrix} = \mathcal{P}_C(\iota_{dR}(\eta)) \, ,$$

$$\pi_{C(f)} \left(\mathcal{P}_C \begin{bmatrix} \eta \\ \xi \end{bmatrix} \right) = \pi_{C(f)} \begin{bmatrix} \mathcal{P}\eta + K\xi \\ \mathcal{P}\xi \end{bmatrix} = \mathcal{P}\xi = \mathcal{P} \left(\pi_{dR} \begin{bmatrix} \eta \\ \xi \end{bmatrix} \right) .$$

The short exact commutative diagram (2.9) gives a long commutative diagram of cohomologies:

$$\begin{array}{ccccccccc}
H^{k-\ell}_{dR}(M) & \xrightarrow{[\psi\wedge]} & H^k_{dR}(M) & \xrightarrow{[\iota_{dR}]} & H^k(\mathrm{Cone}(\psi)) & \xrightarrow{[\pi_{dR}]} & H^{k-\ell+1}_{dR}(M) & \xrightarrow{[\psi\wedge]} & H^{k+1}_{dR}(M) \\
\downarrow{\scriptstyle [\mathcal{P}]} & & \downarrow{\scriptstyle [\mathcal{P}]} & & \downarrow{\scriptstyle [\mathcal{P}_C]} & & \downarrow{\scriptstyle [\mathcal{P}]} & & \downarrow{\scriptstyle [\mathcal{P}]} \\
H^{k-\ell}_{C(f)}(M) & \xrightarrow{[c(\psi)]} & H^k_{C(f)}(M) & \xrightarrow{[\iota_{C(f)}]} & H^k(\mathrm{Cone}(c(\psi))) & \xrightarrow{[\pi_{C(f)}]} & H^{k-\ell+1}_{C(f)}(M) & \xrightarrow{[c(\psi)]} & H^{k+1}_{C(f)}(M)
\end{array} \tag{2.10}$$

We can check that each square commutes. The outer squares commute since $\mathcal{P}(\psi \wedge \xi)$ and $c(\psi)\mathcal{P}\xi$ are cohomologous when both ξ and ψ are d-closed, i.e.

$$[\mathcal{P}][\psi\wedge] = [c(\psi)][\mathcal{P}]$$

as was shown by Austin-Braam in [1, Section 3.5]. The middle two squares commute follows from the commutativity of the chain maps in (2.9). Furthermore, the vertical map $[\mathcal{P}]$ is an isomorphism as shown by Bismut-Zhang and Laudenbach [2, Theorem 2.9] (see also [19, Theorem 6.4]).

We can now apply the Five Lemma to (2.10) which implies that the middle vertical map $[\mathcal{P}_C]$ is also an isomorphism on cohomology, and thus we prove Theorem **??**.

Theorem 2.2.3. $\mathcal{P}_C : (\text{Cone}^\bullet(\psi), d_C) \to (\text{Cone}^\bullet(c(\psi)), \partial_C)$ *is a* $\mathbb{Z}$ *graded quasi-isomorphism.*

2.2.4 Example: Cone$(c(\omega))$ on T^4

We now work out a simple example of the Cone$(c(\psi))$ complex on the four-dimensional torus, $M = T^4 = \mathbb{R}^4/\mathbb{Z}^4$. We will describe the torus using Euclidean coordinates, x_i with $i = 1, 2, 3, 4$, with identification $x_i \sim x_i + 1$. We are interested in the symplectic case where $\psi = \omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$.

The complex Cone$(c(\omega))$ is dependent on the choice of the metric and the Morse function. For simplicity, we will work with the flat metric, $g = \sum dx_i^2$ and choose the Morse function to be

$$f = 2 - \frac{1}{2}\sum_{i=1}^{4} \cos(2\pi x_i)\,.$$

This Morse function has several desirable properties that are straightforward to prove:

(i) the non-degenerate critical points are located at $x_i = [0]$ or $x_i = [\frac{1}{2}]$ and have Morse index equal to the number of coordinates which are equal to $[\frac{1}{2}]$;

(ii) the Morse differential ∂ is 0;

(iii) the pair (f, g) is Morse-Smale.

Because of (ii), the ∂_C map in the equation above reduces to the $c(\omega)$ map. Hence, we are interested in pairs of critical points whose indices differ by two, e.g q_{k+1} has two more $[\frac{1}{2}]$ coordinates than q_{k-1}. Also, note that $\mathcal{M}(q_{k+1}, q_{k-1})$ will be a two-dimensional face with two of the coordinates fixed and two coordinates spanning the entire coordinate interval $[0, 1]$ when we take the closure.

In the Table 2.1 below, we give the cohomologies of $H(\mathrm{Cone}(c(\omega)))$ and $H(\mathrm{Cone}(\omega))$ on $(T^4, \omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4)$. We use a multi-index notation of $I = \{i_1 ... i_j\}$ in increasing order such that $dx_I = dx_{i_1} \wedge ... \wedge dx_{i_j}$, q_0 denotes the index 0 point, and q_I denotes the point with $\frac{1}{2}$ in entry $i_1, ... i_j$, i.e. $q_{13} = q_{\left[\frac{1}{2}, 0, \frac{1}{2}, 0\right]}$. The orientation of the submanifolds are chosen such that $\mathcal{P} dx_I = q_I$. Further, since $\partial = 0$ and $G\, dx_I = 0$, we also have $K\, dx_I = 0$.

Notice that $c(\omega) q_I$ only picks out critical points that have two coordinates of q_I changed from $[0]$ to $[\frac{1}{2}]$ in either the 1-2 or 3-4 directions. Thus, we find that

$$
\begin{aligned}
&c(\omega) q_0 = q_{12} + q_{34}\,, \qquad c(\omega) q_{12} = q_{1234}\,, \qquad c(\omega) q_{34} = q_{1234}\,, \\
&c(\omega) q_1 = q_{134}\,, \qquad c(\omega) q_2 = q_{234}\,, \qquad c(\omega) q_3 = q_{123}\,, \qquad c(\omega) q_4 = q_{124}\,,
\end{aligned}
$$

with all other critical points mapped to zero when acted upon by $c(\omega)$.

j	0	1	2
$H^j(\mathrm{Cone}(\omega))$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} dx_1 \\ 0 \end{bmatrix}, \begin{bmatrix} dx_2 \\ 0 \end{bmatrix}, \begin{bmatrix} dx_3 \\ 0 \end{bmatrix}, \begin{bmatrix} dx_4 \\ 0 \end{bmatrix}$	$\begin{bmatrix} dx_{13} \\ 0 \end{bmatrix}, \begin{bmatrix} dx_{14} \\ 0 \end{bmatrix}, \begin{bmatrix} dx_{23} \\ 0 \end{bmatrix}, \begin{bmatrix} dx_{24} \\ 0 \end{bmatrix}, \begin{bmatrix} dx_{12} - dx_{34} \\ 0 \end{bmatrix}$
$H^j(\mathrm{Cone}(c(\omega)))$	$\begin{bmatrix} q_0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} q_1 \\ 0 \end{bmatrix}, \begin{bmatrix} q_2 \\ 0 \end{bmatrix}, \begin{bmatrix} q_3 \\ 0 \end{bmatrix}, \begin{bmatrix} q_4 \\ 0 \end{bmatrix}$	$\begin{bmatrix} q_{13} \\ 0 \end{bmatrix}, \begin{bmatrix} q_{14} \\ 0 \end{bmatrix}, \begin{bmatrix} q_{23} \\ 0 \end{bmatrix}, \begin{bmatrix} q_{24} \\ 0 \end{bmatrix}, \begin{bmatrix} q_{12} - q_{34} \\ 0 \end{bmatrix}$

j	3	4	5
$H^j(\mathrm{Cone}(\omega))$	$\begin{bmatrix} dx_{123} \\ 0 \end{bmatrix}, \begin{bmatrix} dx_{124} \\ 0 \end{bmatrix}, \begin{bmatrix} dx_{234} \\ 0 \end{bmatrix}, \begin{bmatrix} dx_{234} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ dx_{12} - dx_{34} \end{bmatrix}$	$\begin{bmatrix} 0 \\ dx_{123} \end{bmatrix}, \begin{bmatrix} 0 \\ dx_{124} \end{bmatrix}, \begin{bmatrix} 0 \\ dx_{134} \end{bmatrix}, \begin{bmatrix} 0 \\ dx_{234} \end{bmatrix}$	$\begin{bmatrix} 0 \\ dx_{1234} \end{bmatrix}$
$H^j(\mathrm{Cone}(c(\omega)))$	$\begin{bmatrix} q_{123} \\ 0 \end{bmatrix}, \begin{bmatrix} q_{124} \\ 0 \end{bmatrix}, \begin{bmatrix} q_{234} \\ 0 \end{bmatrix}, \begin{bmatrix} q_{234} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ q_{12} - q_{34} \end{bmatrix}$	$\begin{bmatrix} 0 \\ q_{123} \end{bmatrix}, \begin{bmatrix} 0 \\ q_{124} \end{bmatrix}, \begin{bmatrix} 0 \\ q_{134} \end{bmatrix}, \begin{bmatrix} 0 \\ q_{234} \end{bmatrix}$	$\begin{bmatrix} 0 \\ q_{1234} \end{bmatrix}$

Table 2.1: Cohomology of Cone(ω) versus Cone($c(\omega)$) on $(T^4, \omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4)$.

Chapter 3

Symplectic Morse Inequalities

We have a symplectic Morse complex $Cone(c(\omega), \partial_C)$, but it is of interest to ask if we have decomposition of $\partial = \partial_+^m + c(\omega)\partial_-^m$ or primitive Morse cochains and homologies without regard to Cone. We provide partial answers. To begin recall from Weibel [17, page 6] that there are two complexes, the kernel and cokernel complex, associated to a chain map

Definition 3.0.1. *The kernel complex of* $c(\omega)$ *is the complex* $\ker(c(\omega)) \subset C^\bullet(M, f) = \{b \in C^\bullet(M, f) : c(\omega)b = 0\}$*, with differential* $\partial_{\ker}$

Definition 3.0.2. *The cokernel complex of* $c(\omega)$ *is the complex* $\operatorname{coker}(c(\omega)) = \{[\{a\} \in C/Im\ (c(\omega))\}$ *with differential* $\partial_\Pi^m([a]) = [\partial a] \in C/Im\ (c(\omega))$

The analogy to the kernel complex in differential forms are those $\phi \in \Omega^\bullet(M)$ such that $\omega \wedge \psi = 0$, which are precisely the forms $\omega^{n-k}P^k$. These are a chain complex under $(\omega^{n-k}P^k, d) = (\omega^{n-k}P^k, \omega\partial_-)$ that is quasi-isomorphic to (P^k, ∂_-) under the symplectic reflection $*_r : \beta_k \to \omega^{n-k}\beta_k$.

The analogy to the cokernel in differential forms would be if we pick a representative of $C/\text{Im}\ (c(\omega))$ to be $\phi \in \Omega^\bullet(M)$ which is orthogonal to $\omega \wedge \Omega^\bullet(M)$ under the hodge inner product with compatible metric. Such forms satisfy $0 = \langle\langle \phi, \omega \wedge \alpha \rangle\rangle = \langle\langle \Lambda\phi, \alpha \rangle\rangle$, which are precisely the primitive forms $P^\bullet(M,\omega)$. The differential for this map is $\partial_+ = \Pi^0 d$ on primitive forms.

We might guess from these similarities that these are Morse analogies of the primitive forms. Unfortunately, we need more structure on the Morse function. From Weibel [17, page 24], $\mathrm{coker}(c(\omega))), \mathrm{Cone}(c(\omega)), \ker(c(\omega))$ fit together in a long exact sequence

$$\ldots H^{k-1}(\ker(c(\omega))) \to H^k(\mathrm{Cone}(c(\omega)) \to H^k(\mathrm{coker}(c(\omega))) \to H^k(\ker(c(\omega)))\ldots$$

3.1 Improved Morse Inequalities

We will show using the complex above that we can improve our Cone Morse inequalities. To do this, let $\mathrm{w}_k = \mathrm{rank}[\wedge\omega] : H^k(M) \to H^{k+2}(M)$ and let $v_k = \mathrm{rank}\ c(\omega) : C^k(M,f) \to C^{k+2}(M,f)$. Then we know from [12] that

$$\begin{aligned} b_k^\omega &= (\dim \mathrm{coker}[\wedge\omega] : H^k(M) \to H^{k+2}(M)) + (\dim \ker[\wedge\omega] : H^{k-1}(M) \to H^{k+1}(M)) \\ &= b_k - \mathrm{w}_{k-2} + b_{k-1} - \mathrm{w}_{k-1} \end{aligned}$$

Note that we find the following theorem, that could be considered as an analogy of the Morse inequalities.

Theorem 3.1. *Let Cone$(c(\omega))$ be the Cone complex for the symplectic form ω, then*

$$b_k^\omega \le m_k - v_{k-2} + m_{k-1} - v_{k-1}$$

Proof. To start, note that from Weibel [17] that

$$\ldots H^{k-1}(\ker(c(\omega))) \xrightarrow{f_{ker}^{k-1}} H^k(\mathrm{Cone}(c(\omega))) \xrightarrow{f_{Cone}^k} H^k(\mathrm{coker}(c(\omega))) \xrightarrow{f_{coker}^k} H^k(\ker(c(\omega)))\ldots$$

is a long exact sequence. Note that this being an exact sequence is true if and only if the following are short exact sequences.

$$0 \to Im(f_{ker}^{k-1}) \xrightarrow{f_{ker}^{k-1}} H^k(\mathrm{Cone}(c(\omega)) \xrightarrow{f_{Cone}^{k}} Im(f_{Cone}^{k}) \to 0$$
$$0 \to Im(f_{Cone}^{k}) \xrightarrow{f_{Cone}^{k}} H^k(\mathrm{coker}(c(\omega)) \xrightarrow{f_{coker}^{k}} Im(f_{coker}^{k}) \to 0$$
$$0 \to Im(f_{Coker}^{k}) \xrightarrow{f_{coker}^{k}} H^k(\mathrm{ker}(c(\omega)) \xrightarrow{f_{ker}^{k}} Im(f_{coker}^{k}) \to 0$$

Thus

$$\begin{aligned} b_k^\omega &= \dim Im(f_{ker}^{k-1}) + \dim Im(f_{Cone}^{k}) \\ &\leq \dim H^{k-1}(\mathrm{ker}(c(\omega)) + \dim H^k(\mathrm{coker}(c(\omega)) \\ &\leq \dim(\mathrm{ker}^{k-1}(c(\omega))) + \dim(\mathrm{coker}^k(c(\omega)) \\ &= m_{k-1} - \mathrm{w}_{k-1} + m_k - \mathrm{w}_{k-2} \end{aligned}$$

□

We thus have an analogous form of our weak inequalities. Now for the weak inequalities, we have

Lemma 3.1.1. *If* $v_k = \mathit{rank}\ c(\omega) : C^k(M,f) \to C^{k+2}(M,g)$ *and* $w_k = \mathit{rank}\ [c(\omega)] : H^k_{C(M,f)}(M,\mathbb{R}) \to H^{k+2}_{C(M,f)}(M,\mathbb{R}) = \mathit{rank}\ [\omega\wedge] : H^k_{dR}(M,\mathbb{R}) \to H^{k+2}_{dR}(M,\mathbb{R})$*, then* $w_k \leq v_k$

Using this inequalities above, we have the following immeadiate corollaries:

Corollary 3.1.2.

$$b_k - v_{k-2} + b_{k-1} + v_{k-1} \leq b_k^\omega \leq m_k - w_{k-2} + m_{k-1} - w_{k-1}$$

We now prove the strong Morse inequalities

Theorem 3.2. *If* $b_k^\omega = \dim(H^k(Cone(\omega)) = \dim(H^k(Cone(c(\omega))),$ *then*

$$b_k - v_{k-2} = \sum_{i=0}^{k}(-1)^{i-k}b_i^\omega \leq \sum_{i=0}^{k}(-1)^{i-k}(m_i - v_{i-2} + m_{i-1} - v_{i-1}) = m_k - v_{k-1}$$

Proof. The first and last inequalities follow from the telescoping sum. For the middle inequality, note that

$$\begin{aligned}
\sum_{i=0}^{k}(-1)^{i-k}b_i^\omega &= \sum_{i=0}^{k}(-1)^{i-k}(\dim Im(f_{ker}^{i-1}) + \dim Im(f_{Cone}^{i})) \\
&= \sum_{i=1}^{k}(-1)^{i-k}(\dim Im(f_{ker}^{i-1}) + \dim Im(f_{Cone}^{i})) \\
&+ (\dim Im(f_{Coker}^{i-1}) - \dim Im(f_{Coker}^{i-1})) \\
&= \dim Im(f_{Cone}^{i})) + \sum_{i=0}^{k}(-1)^{i-k}(\dim Im(f_{ker}^{i-1}) \\
&- \dim Im(f_{Cone}^{i-1})) + (\dim Im(f_{Coker}^{i-1}) - \dim Im(f_{Coker}^{i-1})) \\
&= \dim Im(f_{Cone}^{i})) + \sum_{i=0}^{k}(-1)^{i-k}(\dim Im(f_{ker}^{k-1}) + \dim Im(f_{Coker}^{i-1}) \\
&- (\dim Im(f_{Coker}^{i-1}) + \dim Im(f_{Cone}^{i-1}))) \\
&= \dim Im(f_{Cone}^{i})) + \sum_{i=0}^{k}(-1)^{i-k}(\dim H^{i-1}(\ker(c(\omega))) \\
&- \dim(H^{i-1}(\operatorname{coker}(c(\omega))))) \\
&\leq \dim(H^{i}(\operatorname{coker}(c(\omega))) + \sum_{i=0}^{k}(-1)^{i-k}(\dim H^{i-1}(\ker(c(\omega))) \\
&- \dim(H^{i-1}(\operatorname{coker}(c(\omega))))) \\
&= \sum_{i=0}^{k}(-1)^{i-k}(\dim H^{i-1}(\ker(c(\omega))) + \sum_{i=0}^{k}(-1)^{i-k}\dim(H^{i}(\operatorname{coker}(c(\omega))))) \\
&= \sum_{i=0}^{k}(-1)^{i-k}(\dim(\ker^{i}(c(\omega))) + \sum_{i=0}^{k}(-1)^{i-k}\dim((\operatorname{coker}^{i}(c(\omega)))))
\end{aligned}$$

Where in the last line we use the strong Morse inequalities for the complexes $\ker(c(\omega))$ and $\operatorname{coker}(c(\omega))$. Since the dimensions of these complexes are $m_{i-1} - v_{k-1}$ and $m_k - v_{k-2}$, respectively, we have

$$\begin{aligned}\sum_{i=0}^{k}(-1)^{i-k}b_i^{\omega} &\leq \sum_{i=0}^{k}(-1)^{i-k}(m_{i-1}-v_{i-1}) + \sum_{i=0}^{k}(-1)^{i-k}(m_i - v_{i-2}) \\ &= \sum_{i=0}^{k}(-1)^{i-k}(m_{i-1}-v_{i-1}+m_i-v_{i-2}) \\ &= m_k - v_{k-1}\end{aligned}$$

And we have thus proven our Cone complex strong Morse inequality. □

Corollary 3.1.3. *If f is a perfect Morse function, then $m_k = b_k$ and therefore our Morse inequalities become equalities* $\dim H^k(Cone(\omega) = m_{k-1}-v_{k-1}+m_k-v_{k-2}$ *and* $\sum_{i=0}^{k}(-1)^i b_i^{\omega} = m_k - v_{k-1}$

Proof. Note that we have $b_k - \mathrm{w}_{k-2} + b_{k-1} - \mathrm{w}_{k-1} \leq m_k - v_{k-2} + m_{k-1} - v_{k-1}$ from our weak Morse inequalities. Also, note that if we have a perfect Morse function $\dim H^k = \dim C^k$, so our differential is zero $\partial = 0$. In particular, this implies that $[c(\omega)]$ and $c(\omega)$ are the same map (as there is only one element in each cohomology class) thus $v_k = \mathrm{w}_k$. combining $m_k = b_k$ with $v_k = \mathrm{w}_k$ implies $b_k - \mathrm{w}_{k-2} + b_{k-1} - \mathrm{w}_{k-1} = m_k - v_{k-2} + m_{k-1} - v_{k-1}$, so the alternating sums are also equal. Thus the inequalities are equalities for a perfect Morse function. □

3.2 Example

We now provide an example that shows the necessity of having both terms, m_k and $m_{k-\ell+1}$, in the weak Morse-type inequality of Theorem for Cone(ψ).

Example 3.2.1. *Let (X, σ) be the six-dimensional, closed, symplectic manifold constructed by Cho in [3] where the symplectic form $\omega = \sigma$ is not hard Lefschetz type. We will be working with $p = 0$, or the primitive case. Topologically, X can be described as a two-sphere bundle over a projective $K3$ surface and has the following notable properties [3, Theorem 1.3]: (i) X is simply connected; (ii) the odd-degree cohomology, $H^1(X) = H^3(X) = H^5(X) = 0$.*

We will consider the TTY cohomology for (X, σ) in the $p = 0$ case, i.e. $F^{p=0}H(X, \sigma) \cong \mathrm{Cone}(\sigma)(X)$, and $b_k^\omega = \dim(H^k(Cone(\omega))$. From [12], we find

$$
\begin{aligned}
& b_0^\omega = b_7^\omega = 1\,, \\
& b_1^\omega = b_6^\omega = 0\,, \\
& b_2^\omega = b_5^\omega = b_2(X) - 1\,, \\
& b_3^\omega = \dim\left[\ker\left(\sigma : H^2(X) \to H^4(X)\right)\right] > 0\,, \\
& b_4^\omega = \dim\left[\mathrm{coker}\left(\sigma : H^2(X) \to H^4(X)\right)\right] > 0\,.
\end{aligned}
$$

Note that $b_3^0 = b_4^0 > 0$. Since (X, σ) is not hard Lefschetz, which implies that the map, $\sigma : H^2(X) \to H^4(X)$, can not be an isomorphism.

For the inequalities, we can choose to work with a perfect Morse function on X. That such exists is due to a a result of Smale [9, Theorem 6.3] which states that any simply-connected manifold of dimension greater than five that has no homology torsion has a perfect Morse function. (No homology torsion here can be seen from applying the Gysin sequence to X as a two-sphere bundle over $K3$.) Since X has trivial odd-degree cohomology, this implies that

$$
\begin{aligned}
& m_0 = m_6 = 1\,, \\
& m_1 = m_3 = m_5 = 0\,, \\
& m_2 = m_4 = b_2(X)\,.
\end{aligned}
$$

It is straightforward to check that the bounds are satisfied. In particular, for the weak TTY-Morse bound of the morse inequality, the $k = 3, 4$ *case corresponds to*

$$b_3^0 \leq m_3 + m_2 - 1 = m_2 - 1\,,$$
$$b_4^0 \leq m_4 + m_3 - 1 = m_4 - 1\,.$$

The above demonstrate the necessity of having both terms, m_k *and* $m_{k-\ell+1}$*, in our strong Cone Morse inequalities (and in our weak Cone Morse inequalities for the middle case) in order for the inequalities to hold generally.*

Chapter 4

Symplectic Witten Deformation

4.1 Preliminaries

In this chapter we generalize Zhang's [19] work showing the Witten deformation for De Rahm cohomology to the Cone complex. We focus in particular on $F^0\Omega^k$, also called the primitive forms.

4.1.1 Witten Deformation on the Cone Laplacian

To begin, consider a circle bundle over a symplectic manifold M^{2n} given by

$$\begin{array}{ccc} S^1 & \longrightarrow & X \\ & & \Big\downarrow f \\ & & M^{2n} \end{array}$$

Let $\Omega_M(X)$ denote the space of S^1 invariant differential forms on E (which is isomorphic to Cone(ω)),

$$\Omega^k_M(X) = \Omega^k(M) \oplus \theta\, \Omega^{k-1}(M) = \{\eta_k + \theta \wedge \eta_{k-1}, \eta_k \in \Omega^k(M), \xi_{k-1} \in \Omega^{k-1}(M)\},$$

where θ is geometrically the global angular form.

If g is a compatible metric on M with ω, on X choose the metric

$$g_X = \pi^* g + \theta \otimes \theta.$$

If $\{\phi_1, ...\phi_{2n}\}$ is a pointwise orthonormal basis for $T^*_p M$, then $\{\theta, \phi_1, ...\phi_{2n}\}$ is an orthonormal basis for $T^*_z X$, where z is a point in the pre-image, i.e. $\pi(z) = p$. The Hodge star on X is thus

$$*_X(\eta_k + \theta\xi_{k-1}) = *\xi_{k-1} + (-1)^k \theta \wedge *\eta_k.$$

The differential operator is

$$d_X \eta = d_X(\eta_k + \theta \wedge \xi_{k-1}) = d\eta_k + \omega \wedge \xi_{k-1} - \theta \wedge d\xi_{k-1}.$$

The adjoint of d with respect to the metric g_X is

$$d^* = (-1)^k *_X d *_X$$

so

$$d^*_X(\eta_k + \theta\xi_{k-1}) = d^*\eta_k + \theta(\Lambda\eta_k - d^*\xi_{k-1})$$

where $\Lambda = (-1)^k * \omega * = L^*$, assuming the metric is compatible. If $\sigma_k = \begin{bmatrix} \eta_k \\ \xi_{k-1} \end{bmatrix} \in \text{Cone}(\omega)$ is a vector in the mapping Cone of $\omega\wedge : \Omega^*(M) \to \Omega^{*+2}(M)$, then as Cone($\omega$) is quasi-isomorphic to $\Omega^*_M(X)$, [11] we have the corresponding matrix operators

$$d_C = \begin{bmatrix} d & \omega \\ 0 & -d \end{bmatrix} d_C^* = \begin{bmatrix} d^* & 0 \\ \Lambda & -d^* \end{bmatrix}.$$

And if we do a Witten deformation, with f a Morse function, we have

$$d_{C,t} = e^{-tf} d_C e^{tf} = \begin{bmatrix} d + tdf & \omega \\ 0 & -d - tdf \end{bmatrix} = \begin{bmatrix} d_t & \omega \\ 0 & -d_t \end{bmatrix}.$$

$$d_{C,t}^* = e^{tf} d_C^* e^{-tf} = \begin{bmatrix} d^* + t\iota_{\nabla f} & 0 \\ \Lambda & -d^* - t\iota_{\nabla f} \end{bmatrix} = \begin{bmatrix} d_t^* & \Lambda \\ 0 & -d_t^* \end{bmatrix}$$

We can define the deformed Laplacian as

$$\begin{aligned}
\Delta_{C,t} &= d_{C,t} d_{C,t}^* + d_{C,t}^* d_{C,t} \\
&= \begin{bmatrix} dd^* + d^*d + \omega\Lambda & d^*\omega - \omega d^* \\ \Lambda d - d\Lambda & dd^* + d^*d + \Lambda\omega \end{bmatrix} + t \begin{bmatrix} \frac{\partial^2 f}{\partial x^i \partial x^j}[dx_i, \frac{\partial}{\partial x_j}] & \iota_{\nabla f}\omega - \omega\iota_{\nabla f} \\ \Lambda df - df\Lambda & \frac{\partial^2 f}{\partial x^i \partial x^j}[dx_i, \frac{\partial}{\partial x_j}] \end{bmatrix} \\
&+ t^2 \begin{bmatrix} |df|^2 & 0 \\ 0 & |df|^2 \end{bmatrix} \\
&= \begin{bmatrix} \Delta + \omega\Lambda & d^*\omega - \omega d^* \\ \Lambda d - d\Lambda & \Delta + \Lambda\omega \end{bmatrix} + t \begin{bmatrix} \frac{\partial^2 f}{\partial x^i \partial x^j}[dx_i, \frac{\partial}{\partial x_j}] & \iota_{\nabla f}\omega - \omega\iota_{\mathrm{grad}(f)} \\ \Lambda df - df\Lambda & \frac{\partial^2 f}{\partial x^i \partial x^j}[dx_i, \frac{\partial}{\partial x_j}] \end{bmatrix} + t^2 \begin{bmatrix} |df|^2 & 0 \\ 0 & |df|^2 \end{bmatrix} \\
&= \begin{bmatrix} \Delta_t + \omega\Lambda & -d_t^{\Lambda *} \\ -d_t^{\Lambda} & \Delta_t + \Lambda\omega \end{bmatrix}
\end{aligned}$$

where $dx_i, \partial/\partial x_i$ come from local coordinates and

$$\begin{aligned}
d_t^{\Lambda} &= d_t\Lambda - \Lambda d_t = (d + tdf)\Lambda - \Lambda(d + tdf) = d\Lambda - \Lambda d + t(df\Lambda - \Lambda df) \\
d_t^{\Lambda *} &= \omega d_t^* - d_t^*\omega = \omega(d^* + t\iota_{\nabla f}) - (d^* + t\iota_{\nabla f})\omega = \omega d^* - d^*\omega + t(\omega\iota_{\nabla f} - \iota_{\nabla f}\omega)
\end{aligned}$$

Lemma 4.1.1. *If $d_{tf,C}\sigma = 0$ and $d^*_{C,tf}\sigma = 0$, then $d_{C,t(-f)} *_X \sigma = 0$ and $d^*_{C,t(-f)} *_X \sigma = 0$*

Proof: To begin, note that since X is $2n+1$ dimensional, that $*_X *_X = (-1)^{k(2n+1-k)} Id = (-1)^{k(1-k)} Id = Id$

$$\begin{aligned} 0 &= d_{tf,C}\sigma \\ &= e^{-tf} d_C e^{tf}\sigma \\ &= *_X *_X e^{t(-f)} d_C e^{-t(-f)} *_X *_X \sigma \\ &= (-1)^k *_X e^{t(-f)} d^*_C e^{-t(-f)} (*_X \sigma) \\ &= (-1)^k *_X d^*_{t(-f),C} (*_X \sigma) \end{aligned}$$

And taking the $(-1)^k *_X$ implies $d^*_{t(-f),C}(*_X\sigma) = 0$ Likewise

$$\begin{aligned} 0 &= d^*_{tf,C}\sigma \\ &= e^{tf} d^*_C e^{-tf}\sigma \\ &= (-1)^k *_X e^{-t(-f)} d_C e^{t(-f)} *_X \sigma \\ &= (-1)^k *_X d_{t(f),C} (*_X \sigma) \end{aligned}$$

Lemma 4.1.2. *If σ is an eigenvalue of $\Delta_{C,tf}$, then $*_X\sigma$ is an eigenvalue of $\Delta_{C,t(-f)}$*

Proof: if $\Delta_{C,t(f)}\sigma = \lambda\sigma$, implies

$$\begin{aligned}
\lambda\sigma &= d_{t(f),C} d^*_{t(f),C}\sigma + d^*_{t(f),C} d_{t(f),C}\sigma \\
&= (-1)^k e^{-tf} d_C e^{2tf} d^*_C e^{-tf}\sigma + (-1)^{k+1} e^{tf} d^*_C e^{-2tf} d_C e^{tf}\sigma \\
&= (-1)^k e^{-tf} d_C e^{2tf} *_X d_C *_X e^{-tf}\sigma + (-1)^{k+1} e^{tf} *_X d_C *_X e^{-2tf} d_C e^{tf}\sigma \\
&= (-1)^k e^{-tf} *_X *_X d_C e^{2tf} *_X d_C e^{-tf}(*_X\sigma) \\
&\qquad + (-1)^{k+1} e^{tf} *_X d_C *_X e^{-2tf} d_C e^{tf} *_X (*_X\sigma) \\
&= (-1)^{k-1} *_X (-e^{t(-f)} *_X d_C e^{-t(-f)} *_X e^{-t(-f)} d_C e^{-tf}(*_X\sigma) \\
&\qquad + e^{-t(-f)} d_C e^{t(-f)} e^{t(-f)} *_X d_C e^{-t(-f)} *_X (*_X\sigma)) \\
&= (-1)^{k-1} *_X (-(-1)^{2n+1-(k+1)} e^{t(-f)} d^*_C e^{-t(-f)} d_{t(-f),C}(*_X\sigma) \\
&\qquad + d_{t(-f),C}(-1)^{2n+1-k} e^{t(-f)} d^*_C e^{-t(-f)}(*_X\sigma) \\
&= (-1)^{k-1}(-1)^{2n+1-k} *_X (d^*_{t(-f),C} d_{t(-f),C}(*_X\sigma) + d_{t(-f),C} d^*_{t(-f),C}(*_X\sigma)) \\
&= *_X \Delta_{C,t(-f)}(*_X\sigma)
\end{aligned}$$

And taking $*_X$ of both sides gives $\lambda *_X \sigma = \Delta_{t(-f),C}(*_X\sigma)$

Note that Lemma 2.1 or 2.2 immediately implies the following corollary.

Corollary 4.1.3. *let* $\sigma = \begin{bmatrix} \eta \\ \xi \end{bmatrix} \in Cone(\omega)$ *be a harmonic solution of* $\Delta_{C,t}$ *for the Morse function* f*. Then* $*_X\sigma = \begin{bmatrix} *\xi_{k-1} \\ (-1)^k * \eta_k \end{bmatrix}$ *is a harmonic solution of* $\Delta_{C,t}$ *for the Morse function* $-f$

Corollary 4.1.4. *If* σ *is a local harmonic of* $\Delta_{tf,C}$*, then* $*_X\sigma$ *is a local harmonic of* $\Delta_{t(-f),C}$*, and thus a local solution for an index* $2n-k$ *point*

4.1.2 Harmonic Solutions of the Witten Deformed Cone Laplacian

we want to find harmonic $\sigma_k \in Cone(\omega)$, or the forms such that $\Delta_{C,t}\sigma_k = 0$, which is true if and only if $d_{C,t}\sigma_k = 0 = d^*_{tC}\sigma_k$. This amounts to the system of solutions

$$(1) \quad d_t\eta_k + \omega \wedge \xi_{k-1} = 0 \qquad (3) \quad d^*_t\eta_{k-1} = 0$$

$$(2) \quad d_t\xi_{k-1} = 0 \qquad (4) \quad d^*_t\xi_{k-1} - \Lambda\eta_k = 0$$

To prove this, we have the following lemma

Lemma 4.1.5. *Let ϕ_k, ψ_j be differential k and j forms, respectively then $d_t(\phi_k \wedge \psi_j) = d\phi_k \wedge \psi_j + (-1)^k\phi_k \wedge d_t\psi_j$*

Proof: Using $d_t = d + tdf$, we have

$$\begin{aligned} d_t(\phi_k \wedge \psi_j) &= (d + tdf\wedge)(\phi_k \wedge \psi_j \\ &= d\phi_k \wedge \psi_j + (-1)^k\phi_k \wedge d\psi_j + (-1)^k\phi_k \wedge (tdf \wedge \psi_j) \\ &= d\phi_k \wedge \psi_j + (-1)^k\phi_k \wedge d_t\psi_j \end{aligned}$$

Following Witten [18], as t gets large the $t^2|df|^2$ term localizes the solution around critical points. We will show in this chapter that these are in fact the only solutions, but for now we will focus on finding a solution around a critical point p. If the index of p is index $n_f(p)$, we can find Darboux coordinates x_i around and critical points such that $\omega = dx_i \wedge dx_{i+n}$. In [10], Strattman showed there is a Morse function ν such that $\nu = f(p) + \sum_{\ell=1}^{n_f(y)} -x_\ell^2/2 + \sum_{\ell=n_f+1}^{2n} x_\ell^2/2$, where ν is a pullback via symplectomorphism of f (hence we will treat ν as our new f).

define $f_0^k(x) = f^k(x) - f(y)$ and note that locally

$$d_t\psi_j = e^{-tf}d(e^{tf}\psi_j) = e^{-t(f_0^k+f(p))}d(e^{t(f_0^k+f(p))}\psi_j) = e^{-tf_0^k}d(e^{tf_0^k}\psi_j)$$

A similar result shows $d_t^* = e^{tf_0^k} d^* e^{-tf_0^k}$.

We wish to investigate forms in $Cone^k(\omega) = \Omega^k(M) \bigoplus \theta\Omega^{k-1}(M)$, so for $Cone^k(\omega)$ for $k \leq n$, note that for $n_f(p) = k$ (so $f^k(x) = f(p) + \frac{1}{2}(-x_1^2 - x_2^2 - ... - x_k^2 + x_{k+1}^2 + x_{k+2}^2 + ... + x_{2n}^2)$. Then Witten [18] showed that there was a harmonic solution $e^{-t|x|^2/2} dx_1 \wedge ... \wedge dx_k$. Let $\eta_{\mathcal{H},p} = e^{-t|x|^2/2} dx_1 \wedge ... \wedge dx_k$ be this harmonic solution when we set η_k equal to it, and for $n_f(p) = k-1$, let $\xi_{\mathcal{H},p} = e^{-t|x|^2/2} dx_1 \wedge ... \wedge dx_{k-1}$ be the solution when we set ξ_{k-1} equal to it when p is an index $k-1$ critical point. Then two solutions are given below

Theorem 4.1 (Solutions of deformed laplacian for $k \leq n$). *for $k \leq n$ There exist two harmonic solutions for the deformed Laplacian*

One of our solutions is Witten-type solution, namely setting

$$\begin{bmatrix} \eta_k \\ \xi_{k-1} \end{bmatrix} = \begin{bmatrix} \eta_{\mathcal{H},p} \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-t|x|^2/2} dx_1 \wedge ... \wedge dx_k \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-tf_0} \exp\left(\sum_{\ell=k+1}^{2n} x_\ell^2\right) dx_1 \wedge ... \wedge dx_k \\ 0 \end{bmatrix}$$

As Witten showed that $d_t \eta_{\mathcal{H},p} = 0 = d_t^* \eta_{\mathcal{H},p}$, and as $\Lambda = \iota_{\partial_{x_{i+n}}} \iota_{\partial_{x_i}}$, we have $\Lambda \eta_{\mathcal{H},p} = 0$ as k stops before $n+1$, so $d_t \eta_k + \omega \wedge \eta_{k-1} = d_t \eta_k = 0$ and $d_t^* \xi_{k-1} - \Lambda \eta_k = \Lambda \eta_k = 0$ are solutions. Now, suppose $n_f(p) = k-1$, so $f^{k-1}(x) = f(p) + \frac{1}{2}(-x_1^2 - x_2^2 - ... - x_{k-1}^2 + x_k^2 + ... + x_{2n}^2$, $f_0^{k-1} = f^k(x) - f^k(p)$ is an index $k-1$ critical point let $\tau_p = \sum_{i=1}^{k-1} -x_{i+n} dx_i + \frac{1}{2} \sum_{i=k}^{n} x_i dx_{i+n} - x_{i+n} dx_i$ be a local 1-form, and note that $d\tau_p = \omega$. Then another solution is of the form

$$\begin{bmatrix} \eta_k \\ \xi_{k-1} \end{bmatrix} = \begin{bmatrix} -\tau_p \wedge \xi_{\mathcal{H},p} \\ \xi_{\mathcal{H},p} \end{bmatrix} = \begin{bmatrix} e^{-t|x|^2/2} \left(\frac{1}{2} \sum_{i=k}^{n} x_{i+n} dx_i - x_i dx_{i+n}\right) \wedge dx_1 \wedge .... \wedge dx_{k-1} \\ e^{-t|x|^2/2} dx_1 \wedge ... dx_{k-1} \end{bmatrix}$$

Now let $\xi_{k-1} = e^{-tx_\ell^2/2} dx_1 \wedge ... \wedge dx_{k-1} = e^{tf_0^{k-1}} \exp\left(-t \sum_{\ell=k}^{n} x_\ell^2\right) dx_1 \wedge ... dx_{k-1}$ We need to show (η_k, ξ_{k-1}) above localized around $p \in M$ are solutions for (1-4) above, or $d_t \eta_k = -\omega \wedge \xi_{k-1}$,

(2) $d_t\xi = 0$, (3) $d_t^*\eta_k = 0$ and (4) $d_t^*\xi = \Lambda\eta_k$. Since $d_t\xi_{k-1} = 0 = d_t^*\xi_{k-1}$, we satisfy (2) and (4) reduced to $\Lambda\eta_k = 0$.

For (1)

$$\begin{aligned} d_t\eta_k &= d_t(-\tau_p \wedge \xi_{\mathcal{H},p}) \\ &= -d(\tau_p) \wedge \xi_{\mathcal{H},p} + \tau_p \wedge d_t\xi_{\mathcal{H},p} \\ &= -\omega \wedge \xi_{\mathcal{H},p} + \tau_p \wedge 0 = -\omega \wedge \xi_{\mathcal{H},p} \end{aligned}$$

Next since $\Lambda = \iota_{\partial_{x_{i+n}}}\iota_{\partial_{x_i}}$, for the $dx_1 \wedge ...dx_{k-1}$ we need dx_{i+n} for $1 \leq i \leq k-1$, but our sum starts at $k = n$, so this is primitive and $\Lambda\eta_k = 0$.

Computing equation (3) gives

$$\begin{aligned} d_t^*\eta_k &= e^{tf_0^{k-1}} d^*(e^{-tf_0^{k-1}}\eta_k) \\ &= e^{tf_0^{k-1}} d^* \left(e^{-tf_0^{k-1}} e^{-t|x|^2/2} \left(\frac{1}{2} \sum_{i=k}^{n} x_{i+n}dx_i - x_i dx_{i+n} \right) \wedge dx_1 \wedge .... \wedge dx_{k-1} \right) \\ &= e^{tf_0^{k-1}} d^* \left(\exp\left(-t \sum_{\ell=k}^{n} x_\ell^2 \right) \left(\frac{1}{2} \sum_{i=k}^{n} x_{i+n}dx_i - x_i dx_{i+n} \right) \wedge dx_1 \wedge .... \wedge dx_{k-1} \right) \end{aligned}$$

Note that in our codifferential we have no dependence on $x_1, ...x_{k-1}$, so those terms do not contribute to the codifferential and we only have

$$\begin{aligned} d_t^*\eta_k = e^{tf_0^{k-1}} \left(\frac{-1}{2} \right) \sum_{i=k}^{n} \Bigg\{ \partial_{x_i} & \left[x_{i+n} \exp\left(-t \sum_{\ell=k}^{n} x_\ell^2 \right) \right] \\ & - \partial_{x_{i+n}} \left[x_i \exp\left(-t \sum_{\ell=k}^{n} x_\ell^2 \right) \right] \Bigg\} (dx_1 \wedge .... \wedge dx_{k-1}) \\ = e^{tf_0^{k-1}} \left(\frac{-1}{2} \right) \exp\left(-t \sum_{\ell=k}^{n} x_\ell^2 \right) & - 2t \sum_{i=k}^{n} (x_i x_{i+n} - x_{i+n} x_i) (dx_1 \wedge .... \wedge dx_{k-1}) = 0 \end{aligned}$$

Thus we satisfy all three solutions, so

$$\begin{bmatrix} \eta_k \\ \\ \xi_{k-1} \end{bmatrix} = \begin{bmatrix} e^{-tx_\ell^2/2} \left(\frac{1}{2} \sum_{i=k}^{n} x_i dx_{i+n} - x_{i+n} dx_i \right) \wedge dx_1 \wedge .... \wedge dx_{k-1} \\ e^{-tx_\ell^2/2} dx_1 \wedge ... dx_{k-1} \end{bmatrix}$$

Is a non-Witten-type solution for $k \leq n$

Theorem 4.2 (solutions of the deformed Laplacian, $k > n$). *for $k \leq n$ There exist two harmonic solutions for the deformed Laplacian*

Now let us consider solutions to (1)-(4) in $\text{Cone}^k(\omega)$ for $k > n$. If $n_f(p) = k - 1$ have the Witten-type solution

$$\begin{bmatrix} \eta_k \\ \\ \xi_{k-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \xi_{\mathcal{H},p} \end{bmatrix} = \begin{bmatrix} 0 \\ \\ e^{-t|x|^2/2} dx_1 \wedge ... dx_{k-1} \end{bmatrix}$$

Since ξ_{k-1} is harmonic we have $d_t \xi = 0$ (satisfying equation (2)) and so we $d_t^* \xi_k = 0 = \Lambda 0 = \Lambda \eta_k$, so we satisfy equation (4), also $\omega \wedge \xi_{k-1} = 0 = d_t \eta_k$ as it has to have a dx_i term for $1 \leq i \leq n$, so we satisfy equation (1). Finally $d_t^* \eta_k = 0$, so the above is a (Witten-type) solution.

Now around any critical point p so $n_f(p) = k = n + r$, we will show that we have a non-Witten-type solution of the form

$$\begin{bmatrix} \eta_k \\ \\ \xi_{k-1} \end{bmatrix} = \begin{bmatrix} e^{-t|x|^2/2} dx_1 \wedge ... \wedge dx_k \\ e^{-|x|^2/2} (-1)^r \left(\frac{1}{2}\right) \sum_{i=1}^{r} (x_i dx_i + x_{n+i} dx_{n+i}) \wedge dx_1 \wedge ... \widehat{dx_i} ... \widehat{dx_{i+n}} ... \wedge dx_{n+r} \end{bmatrix}$$

To prove this, note that

$$\eta_k = e^{-tx_\ell^2/2} dx_1 \wedge ... \wedge dx_k$$

is Witten harmonic and thus $d_t\eta_k = 0 = d_t^*\eta_k$, so we satisfy equation (3). Also,

$$\eta_k = e^{tf_0}\exp\left(-t\sum_{\ell=k+1}^{n} x_\ell^2\right) dx_1 \wedge ... \wedge dx_k = e^{tf_0}\exp\left(-t\sum_{\ell=k+1}^{n} x_\ell^2\right)\frac{\omega^r}{r!} \wedge dx_{r+1} \wedge ... \wedge dx_n$$

As any $dx_j \in dx_{r+1} \wedge ... \wedge dx_n$ will cancel out with any $dx_j \wedge dx_{j+n}$ terms in ω^r, leaving only the terms that have $dx_1 \wedge dx_{1+n}, ..., dx_r \wedge dx_{r+n} = dx_r \wedge dx_k$, and we get that term $r!$ times. We will also write ξ_{k-1} in a more convenient way as

$$\xi_{k-1} = e^{-|x|^2/2}(-1)^r\left(\frac{1}{2}\right)\sum_{i=1}^{r}(x_i dx_i + x_{i+n}dx_{i+n}) \wedge \frac{\omega^{r-1}}{(r-1)!} \wedge dx_{r+1} ... \wedge dx_n$$

Again, any $dx_j \in dx_{r+1} \wedge ... \wedge dx_n$ will cancel out with any $dx_j \wedge dx_{j+n}$ terms in ω^r, leaving only the terms that have $dx_1 \wedge dx_{1+n}, ..., dx_r \wedge dx_{r+n} = dx_r \wedge dx_k$, and then the $x_i dx_i + x_{i+n}dx_{i+n}$ cancels out the $dx_i dx_{i+n}$ term, removing that, and we have $(r-1)!$ copies

Thus we need to show that ξ_{k-1} is a solution to (1) $\omega \wedge \xi_{k-1} = d_t\eta_k = 0$, (2) $d_t\xi_{k-1} = 0$, and (4) $d_t^*\xi_{k-1} - \Lambda\eta_k = 0$

To check equation number (1), or that $\omega \wedge \xi_{k-1} = 0$, note that each term in ξ_i is of the form

$$e^{-t|x|^2/2}(-1)^r\left(\frac{1}{2}\right)(x_i dx_i + x_{i+n}dx_{i+n}) \wedge dx_1 \wedge ...\widehat{dx_i}...\widehat{dx_{i+n}}... \wedge dx_{n+r}$$

And note that for the terms in $\omega = dx_j \wedge dx_{j+n}$ the terms with $j \neq i$ will cancel out with one of the $dx_1 \wedge ...\widehat{dx_i}...\widehat{dx_{i+n}}... \wedge dx_{n+r}$, while $dx_j \wedge dx_{j+n} = dx_i \wedge dx_{i+n}$ will cancel out with $x_i dx_i + x_{i+n}dx_{i+n}$

Next, for (2), $d_t\xi_{k-1} = 0$.

$$\begin{aligned}
d_t(\xi_{k-1}) &= e^{-tf_0^k} d(e^{tf_0^k}\xi_{k-1}) \\
&= e^{-tf_0^k} d\left(\exp\left(-t\sum_{\ell=1}^{k} x_\ell^2\right)(-1)^r\left(\frac{1}{2}\right)\sum_{i=1}^{r}(x_i dx_i + x_{n+i}dx_{n+i}) \wedge \frac{\omega^{r-1}}{(r-1)!}\bigwedge_{j=r+1}^{n} dx_j\right) \\
&= e^{-tf_0^k} d\left(\exp\left(-t\sum_{\ell=1}^{k} x_\ell^2\right)\right)(-1)^r\left(\frac{1}{2}\right)\sum_{i=1}^{r}(x_i dx_i + x_{i+n}dx_{i+n}) \wedge \frac{\omega^{r-1}}{(r-1)!}\bigwedge_{j=r+1}^{n} dx_j
\end{aligned}$$

Where we have used that $d(\omega) = 0, d(x_i dx_i + x_{i+n}dx_{i+n}) = 0$, and $d(\bigwedge_{j=r+1}^{n} dx_j) = 0$. Focusing on the remaining term gives

$$\begin{aligned}
e^{-tf_0^k} d\left(\exp\left(-t\sum_{\ell=1}^{k} x_\ell^2\right)\right) &= e^{-tf_0^k}\left(\exp\left(-t\sum_{s=1}^{k} x_s^2\right)\sum_{\ell=1}^{k} -2tx_\ell dx_\ell\right) \\
&= e^{-t|x|^2/2}\sum_{\ell=1}^{k} -2tx_\ell dx_\ell
\end{aligned}$$

$$\begin{aligned}
d_t(\xi_{k-1}) = e^{-tf}&\left(\exp\left(-t\sum_{\ell=1}^{k} x_\ell^2\right)(-tx_k dx_k \wedge x_r dx_1 \wedge ... \wedge dx_{k-1}\right. \\
&\left. tx_r dx_r \wedge x_{n+i}(-1)^{n-1} dx_1 \wedge ... \wedge \widehat{dx_r} \wedge ... \wedge dx_k)\right) \\
&= e^{-tx_\ell^2/2}(-tx_k x_r(-1)^{k-1} dx_1 \wedge ...dx_k - tx_r x_k(-1)^{r-1}(-1)^{n-1} dx_1 \wedge ... \wedge dx_k) \\
&= e^{-tx_\ell^2/2}(-tx_k x_r(-1)^{n+r-1} dx_1 \wedge ...dx_k - tx_r x_k(-1)^{n+r-2} dx_1 \wedge ... \wedge dx_k) = 0
\end{aligned}$$

For equation (2), note that (2) is equivalent to

$$\begin{aligned} d_t^*\xi_{k-1} = \Lambda\eta_k &= \frac{\omega^{r-1}}{(r-1)!}e^{tf_0}\exp\left(-t\sum_{\ell=k+1}^{n} x_\ell^2\right) dx_{r+1}\wedge ... \wedge dx_n \\ \Leftrightarrow \qquad e^{tf_0}d^*(e^{-tf_0}\xi_{k-1}) &= \frac{\omega^{r-1}}{(r-1)!}e^{tf_0}\exp\left(-t\sum_{\ell=k+1}^{n} x_\ell^2\right) dx_{r+1}\wedge ... \wedge dx_n \\ \Leftrightarrow \qquad d^*(e^{-tf_0}\xi_{k-1}) &= \frac{\omega^{r-1}}{(r-1)!}\exp\left(-t\sum_{\ell=k+1}^{n} x_\ell^2\right) dx_{r+1}\wedge ... \wedge dx_n \end{aligned}$$

We thus need to anti-codifferentiate this term, which gives us

$$\begin{aligned} e^{-tf_0^k(x)}\xi_{k-1} &= \exp\left(-t\sum_{\ell=k+1}^{n} x_\ell^2\right)\frac{\omega^{r-1}}{(r-1)!}\wedge\left(\left(\frac{1}{2}\right)\sum_{i=1}^{r} x_i dx_i + x_{n+i}dx_{n+i}\right) \\ &\wedge dx_{r+1}\wedge ... \wedge dx_n \\ \Leftrightarrow \xi_{k-1} &= e^{tf_0^k}\exp\left(-t\sum_{\ell=k+1}^{n} x_\ell^2\right)\frac{\omega^{r-1}}{(r-1)!}\wedge\left(\left(\frac{1}{2}\right)\sum_{i=1}^{r} x_i dx_i + x_{n+i}dx_{n+i}\right) \\ &\wedge dx_{r+1}\wedge ... \wedge dx_n \\ \Leftrightarrow \xi_{k-1} &= e^{-t|x|^2/2}\frac{\omega^{r-1}}{(r-1)!}\wedge\left(\left(\frac{1}{2}\right)\sum_{i=1}^{r} x_i dx_i + x_{n+i}dx_{n+i}\right) \\ &\wedge dx_{r+1}\wedge ... \wedge dx_n = \iota_{-\tau_p'^\sharp}\eta_{\mathcal{H},p} \end{aligned}$$

Where we know by Lemma 4.2.1 that the solution is $*_X$ then send $x_k \to x_{2n-k+1}$ of the solutions we found in Theorem 2.4 hence this solution has to be $\eta = *_X(-\tau_p \wedge e^{-tx_\ell^2}dx_{2n}\wedge ... \wedge dx_{2n-k+2} = \iota_{\tau'^\sharp}dx_1 \wedge ...dx_{k-1}$, where $\tau' = \sum_{i=0}^{2n-k+2} -x_{n-i}dx_{2n-i} + \frac{1}{2}\sum_{k=2n-k+2}^{n} x_{2n-i}dx_{n-i} - x_{n-i}dx_{2n-i}$. Also, this is localized, and to see that this antidifferentiates, note that since Next, note that the in the sum $\sum_{i=1}^{r} x_i dx_i + x_{n+i}dx_{n+i}$ it cancels out with every term in the $dx_1\wedge ... \wedge \widehat{dx_r}\wedge ... \wedge dx_{n+r-1}$ except for $i = r$. Also, when we take d_t^*, the e^{tf_0} gets cancelled out, and the only part of the Gaussian is $\exp\left(-t\sum_{\ell=k+1}^{n} x_\ell^2\right)$, which will not be differentiated by the any of the dx_i. If we call $\phi = \exp\left(-t\sum_{\ell=k+1}^{n} x_\ell^2\right)(-1)^{r-1}dx_1\wedge ... \wedge \widehat{dx_r}\wedge ... \wedge dx_{n+r-1}$

Then this is not impacted by the codifferential, and we get

$$\begin{aligned}
d_t^*\xi_{k-1} &= d_t^*\left(\phi \wedge \left(\frac{1}{2}\right)\sum_{i=r} x_i dx_i + x_{n+i}dx_{n+i}\right) \\
&= e^{-tf_0^k}\phi * \left(\frac{1}{2}\right)(1+1) \\
&= e^{-tf_0^k}\exp\left(-t\sum_{\ell=k+1}^{n} x_\ell^2\right)(-1)^{r-1}dx_1 \wedge \ldots \wedge \widehat{dx_r} \wedge \ldots \wedge dx_{n+r-1} \\
&= e^{-tf_0^k}\exp\left(-t\sum_{\ell=k+1}^{n} x_\ell^2\right)(-1)^{r-1}dx_1 \wedge \ldots \wedge \widehat{dx_r} \wedge \ldots \wedge dx_{n+r-1} \\
&= e^{-tf_0^k}\exp\left(-t\sum_{\ell=k+1}^{n} x_\ell^2\right)\frac{\omega^{r-1}}{(r-1)!} = \Lambda\eta_k
\end{aligned}$$

So this satisfies (1).

And note that the $x_i dx_i$ and $x_{n+i}dx_{n+i}$ will cancel out with the terms in the $dx_1 \wedge \ldots \wedge dx_{n+r}$, thus giving 0 for $\omega \wedge \xi_i$.

Thus this combination of η_k, ξ_{k-1} satisfies our equations. Thus for $Cone^k(\omega)$ for $k > n$, we have a non-Witten solution of the form

$$\begin{bmatrix} \eta_k \\ \xi_{k-1} \end{bmatrix} = \begin{bmatrix} \eta_{\mathcal{H},p} \\ \iota_{-\tau_p'^\sharp}\eta_{\mathcal{H},p} \end{bmatrix} = \begin{bmatrix} e^{-tx_\ell^2/2}dx_1 \wedge \ldots dx_k \\ e^{-tx_\ell^2/2}\frac{\omega^{r-1}}{(r-1)!} \wedge \left(\left(\frac{1}{2}\right)\sum_{i=1}^{r} x_i dx_i + x_{n+i}dx_{n+i}\right) \wedge dx_{r+1} \wedge \ldots \wedge dx_n \end{bmatrix}$$

4.1.3 Bounding the Eigenvalues of the Witten Deformed Cone Lapacian Spectrum

Let $\begin{bmatrix} \eta_k \\ \xi_{k-1} \end{bmatrix}$ be a non-harmonic eigenform of Δ_{tC} with eigenvalue λ. We wish to show that λ goes to infinity linearly as t goes to infinity.

Theorem 4.3 (Localized eigenvalues of Δ_{tC} grow like ct)**.** *Suppose* $\begin{bmatrix} \eta_k \\ \xi_{k-1} \end{bmatrix}$ *is orthogonal to* *our solutions found above. then*

$$\left\langle \Delta_{tC} \begin{bmatrix} \eta_k \\ \xi_{k-1} \end{bmatrix}, \begin{bmatrix} \eta_k \\ \xi_{k-1} \end{bmatrix} \right\rangle \geq ct \left|\left| \begin{matrix} \eta_k \\ \xi_{k-1} \end{matrix} \right|\right|^2 + O(||\eta_k||^2, ||\xi_{k-1}||^2)$$

In particular, any non-harmonic eigenforms eigenvalue grows on an order of t

If we compute

$$\begin{aligned}
\left\langle \Delta_{tC} \begin{bmatrix} \eta_k \\ \xi_{k-1} \end{bmatrix}, \begin{bmatrix} \eta_k \\ \xi_{k-1} \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} \Delta_{tM} + \omega\Lambda & -d_t^{\Lambda *} \\ -d_t^{\Lambda} & \Delta_{tM} + \Lambda\omega \end{bmatrix} \begin{bmatrix} \eta_k \\ \xi_{k-1} \end{bmatrix}, \begin{bmatrix} \eta_k \\ \xi_{k-1} \end{bmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} \Delta_{tM}\eta_k + \omega\Lambda\eta_k - d_t^{\Lambda *}\xi_{k-1} \\ -d_t^{\Lambda}\eta_k + \Delta_{tM}\xi_{k-1} + \Lambda\omega\xi_{k-1} \end{bmatrix}, \begin{bmatrix} \eta_k \\ \xi_{k-1} \end{bmatrix} \right\rangle \\
&= \langle \Delta_{tM}\eta_k, \eta_k \rangle + \langle \omega\Lambda\eta_k, \eta_k \rangle - \langle d_t^{\Lambda *}\xi_{k-1}, \eta_k \rangle \\
&+ \langle \Delta_{tM}\xi_{k-1}, \xi_{k-1} \rangle - \langle d_t^{\Lambda}\eta_k, \xi_{k-1} \rangle + \langle \Lambda\omega\xi_{k-1}, \xi_{k-1} \rangle \\
&= \langle \Delta_{tM}\eta_k, \eta_k \rangle - \langle d_t^{\Lambda}\eta_k, \xi_{k-1} \rangle + \langle \Delta_{tM}\xi_{k-1}, \xi_{k-1} \rangle \\
&- \langle d_t^{\Lambda *}\xi_{k-1}, \eta_k \rangle + ||\Lambda\eta_k||^2 + ||\omega\xi_{k-1}||^2
\end{aligned}$$

To start, note that Δ_{tM} is self adjoint and has a basis of orthogonal eigenforms like $H_{m_i}(\sqrt{t}x_i)e^{-t|x|^2/2}dx_I$. Note that this has an eigenvalue for Δ_{tM} of $2t(\ell + \sum m_i)$ where ℓ is the number missing/different in $dx_I = dx_{i_1} \wedge ... \wedge dx_{i_k}$ from harmonic form's $dx_1 \wedge ... \wedge dx_{n_f}$. Therefore, we can write as a linear combination of these $\eta_k = H_{m_i^\eta}(\sqrt{t}x_i)e^{-t|x|^2/2}dx_{I_k^\eta} = \eta^i, \xi_{k-1} = H_{m_i^\xi}(\sqrt{t}x_i)e^{-t|x|^2/2}dx_{I_{k-1}^\xi} = \xi^i$ (call ξ^i, η^i a simple one). Next, as Δ_{tC} is self adjoint, we have that

$$\begin{bmatrix} \eta_k \\ \xi_{k-1} \end{bmatrix} \in (\ker \Delta_{C,t})^\perp, \text{ so } \left\langle \begin{bmatrix} \eta_k \\ \xi_{k-1} \end{bmatrix}, \begin{bmatrix} \eta_{\mathcal{H}} \\ 0 \end{bmatrix} \right\rangle = 0 = \left\langle \begin{bmatrix} \eta_k \\ \xi_{k-1} \end{bmatrix}, \begin{bmatrix} -\tau_p \wedge \xi_{\mathcal{H}} \\ \xi_{\mathcal{H}} \end{bmatrix} \right\rangle$$

where $\eta_{\mathcal{H}}, \xi_{\mathcal{H}}$ are Witten harmonic and thus we cannot have η_k with an orthogonal component, as then it would not be orthogonal to $\begin{bmatrix} \eta_{\mathcal{H}} \\ 0 \end{bmatrix}$ However, we can have a particular $\xi^i = C\xi_{\mathcal{H}}$ harmonic, but if this is the case then

$$0 = \left\langle \begin{bmatrix} \eta^i \\ \xi^i \end{bmatrix}, \begin{bmatrix} -\tau_p \wedge C\xi_{\mathcal{H}} \\ C\xi_{\mathcal{H}} \end{bmatrix} \right\rangle = \langle \eta^i, -\tau_p \wedge C\xi_{\mathcal{H}} \rangle + \langle C\xi_{\mathcal{H},p}, \xi_{\mathcal{H}} \rangle = C(\langle \eta^i, -\tau_p \wedge \xi_{\mathcal{H}} \rangle + ||\xi_{\mathcal{H},p}||^2)$$

Thus we have that η^i has to be some combination of the

$$-\tau_p \wedge \xi_{\mathcal{H}} = \frac{1}{2} \sum_{j=n_f+1}^{n} x_{n+j} e^{-t|x|^2/2} dx_j \wedge dx_1 \wedge ... \wedge dx_{n_f} - x_j e^{-t|x|^2/2} dx_{n+j} dx_1 \wedge ... \wedge dx_{n_f}$$

so in particular, η^i is a linear combination of

$$\eta^i = (\frac{-||\xi_{\mathcal{H},p}||^2}{||x_j dx_{n+j} \wedge \xi_{\mathcal{H},p}||^2}) a_{n+j}(-x_j dx_{n+j}) \wedge \xi_{\mathcal{H},p} + a_j dx_j) \wedge \xi_{\mathcal{H},p}$$

where we know $\sum a_j = 1$. Normalizing so $||\xi_{\mathcal{H},p}||^2 = 1$, then $\xi_{\mathcal{H},p} = \frac{t^{n/2}}{\pi^{n/2}} e^{-t|x|^2} dx_1 \wedge ... \wedge dx_{n_f}$, so

$$\begin{aligned} x_j dx_{n+j} \wedge \xi_{\mathcal{H},p} &= \frac{t^{n/2}}{\pi^{n/2}} x_j e^{-t|x|^2/2} dx_j \wedge dx_i \wedge ... \wedge dx_{n_f} \\ &= \frac{1}{\sqrt{2t}} \left(\frac{t^{n/2}}{\sqrt{2}\pi^{n/2}} H_1(\sqrt{t} x_j) e^{-t|x|^2/2} dx_j \wedge dx_i \wedge ... \wedge dx_{n_f} \right) \end{aligned}$$

where the term in the parentheses has norm 1, so $||x_j dx_{n+j} \wedge \xi_{\mathcal{H},p}||^2 = \frac{1}{2t}$, and thus

$$\eta^i = 2t(a_{n+j} x_j dx_{n+j} - a_j x_{n+j} dx_j \wedge \xi_{\mathcal{H},p})$$

. Noting that $x_j e^{-t|x|^2/2} dx_j \wedge dx_i \wedge ... \wedge dx_{n_f}$ is an eigenform of the Witten laplacian with

$m = 1$ and $\ell = 1$) we have

$$\left\langle \Delta_{C,t} \begin{bmatrix} \eta^i \\ \xi_{\mathcal{H}} \end{bmatrix}, \begin{bmatrix} \eta^i \\ \xi_{\mathcal{H}} \end{bmatrix} \right\rangle = \langle \Delta_{tM} \eta^i, \eta^i \rangle - \langle d_t^\Lambda \eta^i, \xi_{\mathcal{H}} \rangle + \langle \Delta_{tM} \xi_{\mathcal{H}}, \xi_{\mathcal{H}} \rangle$$

$$- \langle d_t^{\Lambda *} \xi_{\mathcal{H}}, \eta^i \rangle + ||\Lambda \eta^i||^2 + ||\omega \xi^i||^2$$

$$= 4t||\eta^i||^2 - 2\langle d_t^\Lambda \eta^i, \xi_{\mathcal{H}} \rangle + 0 + ||\Lambda \eta_k||^2 + ||\omega \xi_{k-1}||^2$$

If we calculate $d_t^\Lambda \eta^i = (d_t \Lambda - \Lambda d_t)\eta^i$, note that for a particular $\eta^j = tx_{n+j} e^{-t|x|^2/2} dx_j \wedge dx_1 \wedge \ldots \wedge dx_{n_f}$ has no $dx_r dx_{r+n}$ (τ_{n_f} was chosen in this way) so $\Lambda \eta_j = \iota_{\partial_{r+n}} \iota_{\partial_r} \iota_{\partial_{i+n}} \eta^i = 0$ Thus

$$\begin{aligned} d_t^\Lambda \eta_j &= (d_t \Lambda - \Lambda d_t)\eta_j \\ &= -\Lambda d_t(tx_n dx_{n+j} \xi_{\mathcal{H},p} \\ &= -t\Lambda(d + tdf \wedge)(x_n dx_{n+j} \xi_{\mathcal{H},p}) \\ &= -t\Lambda(d(x_j dx_{n+j}) \wedge \xi_{\mathcal{H},p} - x_j dx_{n+j} \wedge d\xi_{\mathcal{H},p} - x_j dx_{n+j}(tdf \wedge \xi_{\mathcal{H},p})) \\ &= -t\Lambda(dx_j \wedge dx_{n+j} \wedge \xi_{\mathcal{H},p} - x_j dx_{n+j} \wedge d_t \xi_{\mathcal{H},p}) \\ &= -t\Lambda(dx_j \wedge dx_{n+j} \wedge \xi_{\mathcal{H},p} - x_j dx_{n+j} \wedge 0) \end{aligned}$$

so we have $d_t^\Lambda \eta_j = -t\Lambda(dx_{n+j} \wedge dx_j \xi_{\mathcal{H},p}$, and as $\xi_{\mathcal{H},p}$ is primitive, this just gives $d_t^\Lambda \eta_j = -t\xi_{\mathcal{H},p}$ A similar result for $\eta_{n+j} = -x_j dx_{n+j} \wedge \xi_{\mathcal{H},p}$ also gives $d_t^\Lambda \eta^i = -t\xi_{\mathcal{H},p}$ And adding all these terms together gives

$$d_t^\Lambda \eta^i = \sum -2ta_j \xi_{\mathcal{H},p} = -2t\xi_{\mathcal{H},p}$$

. Plugging these in, we get

$$\left\langle \Delta_{C,t} \begin{bmatrix} \eta^i \\ \xi_{\mathcal{H}} \end{bmatrix}, \begin{bmatrix} \eta^i \\ \xi_{\mathcal{H}} \end{bmatrix} \right\rangle = 4t||\eta^i||^2 - 2\langle d_t^\Lambda \eta^i, \xi_{\mathcal{H}}\rangle + ||\Lambda\eta_k||^2 + ||\omega\xi_{k-1}||^2$$
$$= 4t||\eta^i||^2 - 2\langle -t\xi_{\mathcal{H}}, \xi_{\mathcal{H}}\rangle + ||\Lambda\eta_k||^2 + ||\omega\xi_{k-1}||^2$$
$$= 4t||\eta^i||^2 + 4t||\xi_{\mathcal{H}},||^2 + ||\Lambda\eta_k||^2 + ||\omega\xi_{k-1}||^2$$
$$> 4t(||\eta^i||^2 + ||\xi_{\mathcal{H},p}||^2) + O(||\Lambda\eta^i||^2, ||\omega\xi||^2)$$

Thus in this case, we are also bounded by ct for some $c = 2$. We thus have a bound for the case of a ξ^i harmonic, and also that η^i cannot be not harmonic.

Before we continue, we first need to discuss the Witten symplectic laplacian $\Delta_{tM}^\Lambda = d_t^\Lambda d_t^{\Lambda*} + d_t^{\Lambda*} d_t^\Lambda$. To start, note that $d_t^\Lambda = d_t\Lambda - \Lambda d_t = *_s d_t *_s$, and similarly $d_t^{\Lambda*} = \omega d_t^* - d_t^*\omega = *_s d_t^* *_s = (d_t^\Lambda)^\dagger$ Therefore, using that $*_s^2 = \mathbb{I}$, we have

$$\Delta_{tM}^\Lambda = d_t^\Lambda d_t^{\Lambda*} + d_t^{\Lambda*} d_t^\Lambda$$
$$= *_s d_t *_s *_s d_t^* *_s + *_s d_t^* *_s *_s d_t *_s$$
$$= *_s (d_t d_t^* + d_t^* d_t) *_s$$
$$= *_s \Delta_{tM} *_s$$

Therefore since $*_s = *_s^{-1}$, $\Delta_{tM}^\Lambda = *_s \Delta_{tM} *_s^{-1}$ has the eigenforms $*_s H_{m_i}(\sqrt{t}x_i)e^{-t|x|^2/2}dx_I = H_{m_i}(\sqrt{t}x_i) *_S dx_I$ with eigenvalues $2t(\ell^\Lambda + \sum m_i)$, where ℓ^Λ is the number of forms in $*_s dx_I$ missing/different from $dx_1 \wedge ... \wedge dx_{n_f}$

Next, note that

$$\begin{aligned}||d_t^{\Lambda}\eta_k||^2 &= \langle d_t^{\Lambda}\eta_k, d_t^{\Lambda}\eta_k\rangle \\ &\leq \langle d_t^{\Lambda}\eta_k, d_t^{\Lambda}\eta_k\rangle + \langle d_t^{\Lambda*}\eta_k, d_t^{\Lambda*}\eta_k\rangle \\ &= \langle d_t^{\Lambda*}d_t^{\Lambda}\eta_k, \eta_k\rangle + \langle d_t^{\Lambda}d_t^{\Lambda*}\eta_k, \eta_k\rangle \\ &= \langle \Delta_{tM}^{\Lambda}\eta_k\rangle\end{aligned}$$

a similar argument works for $||d_t^{\Lambda*}\xi_{k-1}||^2 \leq \langle \Delta_{tM}^{\Lambda}\xi_{k-1}, \xi_{k-1}\rangle$

With these inequalities, we are now able to show that

$$\left\langle \Delta_{C,t}\begin{bmatrix}\eta_k \\ \xi_{k-1}\end{bmatrix}, \begin{bmatrix}\eta_k \\ \xi_{k-1}\end{bmatrix}\right\rangle \geq ct\left|\left|\begin{matrix}\eta_k \\ \xi_{k-1}\end{matrix}\right|\right|^2 + O(||\eta_k||^2, ||\xi_{k-1}||^2)$$

We will now focus on the two components $\langle d_t^{\Lambda}\eta_k, \xi_{k-1}\rangle$ and $\langle d_t^{\Lambda*}\xi_{k-1}, \eta_k\rangle$ To start, we begin with $\langle d_t^{\Lambda*}\xi_{k-1}, \eta_k\rangle = \langle d_t^{\Lambda}\eta_k, \xi_{k-1}\rangle$, which we will use later. Next, note that by the Cauchy-Schwarz inequality, we have that $\langle d_t^{\Lambda}\eta_k, \xi_{k-1}\rangle \leq ||\eta_k||||\xi_{k-1}||$. Next, using the arithmetic-geometric mean inequality with $C||d_t^{\Lambda}\eta_k||^2, C||\xi_{k-1}||^2$, for C a constant we have

$$\langle d_t^{\Lambda}\eta_k, \xi_{k-1}\rangle \leq ||\eta_k||||\xi_{k-1}|| \leq \frac{1}{2}(\frac{1}{C}||d_t^{\Lambda}\eta_k||^2 + C||\xi_{k-1}||^2) = \frac{1}{2C}||d_t^{\Lambda}\eta_k||^2 + \frac{C}{2}||\xi_{k-1}||^2$$

If we try to bound

$$\left\langle \Delta_{C,t}\begin{bmatrix}\eta_k \\ \xi_{k-1}\end{bmatrix}, \begin{bmatrix}\eta_k \\ \xi_{k-1}\end{bmatrix}\right\rangle$$

with $\eta_k = \eta^i, \xi_{k-1} = \xi^i$ the sum of basic forms and re-indexing for the second term, we have

$$\left\langle \Delta_{C,t} \begin{bmatrix} \eta^i \\ \xi^i \end{bmatrix}, \begin{bmatrix} \eta^j \\ \xi^j \end{bmatrix} \right\rangle = \langle \Delta_{tM}\eta^i, \eta^j\rangle - \langle d_t^{\Lambda}\eta^i, \xi^j\rangle + \langle \Delta_{tM}\xi^i, \xi^j\rangle -$$

$$\langle d_t^{\Lambda *}\xi^i, \eta^j\rangle + \langle \omega\Lambda\eta^i, \eta^j\rangle + \langle \Lambda\omega\xi^i, \xi^j\rangle$$

$$= \langle \lambda_{\eta^i}\eta^i, \eta^j\rangle - 2\langle d_t^{\Lambda}\eta^i, \xi^j\rangle + \langle \lambda_{\xi^i}\xi^i, \xi^j\rangle + O(||\eta^i||||\eta^j||, ||\xi^i||||\xi^j||)$$

$$= \langle \lambda_{\eta^i}\eta^i, \eta^j\rangle - 2\langle d_t^{\Lambda}\eta^i, \xi^j\rangle + \langle \lambda_{\xi^i}\xi^i, \xi^j\rangle + O(||\eta^i||||\eta^j||, ||\xi^i||||\xi^j||)$$

$$= \lambda_{\eta^i}||\eta^i||^2 - 2\langle d_t^{\Lambda}\eta^i, \xi^j\rangle + \lambda_{\xi^i}||\xi^i||^2 + O(||\eta^i||||\eta^j||, ||\xi^i||||\xi^j||)$$

We now rewrite ξ^i as $A\xi_{\mathcal{H},p} + \xi^i$ where ξ^i are all now nonharmonic. Note that to be orthogonal to $\begin{bmatrix} -\tau_p \wedge \xi_{\mathcal{H}} \\ \xi_{\mathcal{H}} \end{bmatrix}$ there must exist a correstponding η^i of the form $\eta^a = t(a_j x_j dx_{n+j} - a_{n+j} x_j dx_j) \wedge A\xi_{\mathcal{H},p}$. Thus, if we split the term above out term out, we get

$$= \lambda_{\eta^i}||\eta^i||^2 - 2\langle d_t^{\Lambda}\eta^i, \xi^j\rangle + \langle \lambda_{\xi^i}||\xi^i||^2 + O(||\eta^i||||\eta^j||, ||\xi^i||||\xi^j||)$$

$$= \lambda_{\eta^i}||\eta^i||^2 - 2\langle d_t^{\Lambda}\eta^i, \xi^j\rangle - 2\langle d_t^{\Lambda}\eta^i, A\xi_{\mathcal{H},p}\rangle + \lambda_{\eta^a}||\eta^a||^2 - 2\langle d_t^{\Lambda}\eta^a, \xi^j\rangle - 2\langle d_t^{\Lambda}\eta^a, A\xi_{\mathcal{H},p}\rangle$$

$$+ \lambda_{\xi^i}||\xi^i||^2 + \lambda_{\xi_{\mathcal{H},p}}||A\xi_{\mathcal{H},p}||^2 + O(||\eta^i||||\eta^j||, ||\xi^i||||\xi^j||)$$

Next, note that $\eta^i = H_{m_r}(\sqrt{t}x_r)e^{-t|x|^2/2}dx_{I^i}$, then $d_t\eta^i$ (using that the hermite polynomials satisfy the relations $H'_{m_r}(\sqrt{t}x_r = \sqrt{t}(2m_r)H'_{m_r-1}(\sqrt{t}x_r), \sqrt{t}xH_{m_r}(\sqrt{t}x) = H_{m_r+1}(\sqrt{t}x) -$

$\sqrt{t}(m_r-1)H_{m_r-1}(\sqrt{t}x))$ will give

$$\begin{aligned}
d_t\eta^i &= (d+tdf)H_{m_r}(\sqrt{t}x_r)e^{-t|x|^2/2}\wedge dx_{I_i} \\
&= (d(H_{m_r}(\sqrt{t}x_r)) \pm tx_r dx_r H_{m_r}(\sqrt{t}x_r)e^{-t|x|^2/2})\wedge dx_{I^i} \\
&= H_{m_{s\neq r}}(\sqrt{t}x_{m_s})(\sqrt{t}H'_{m_r}(\sqrt{t}x_r)e^{-t|x|^2/2})dx_r \\
&- \sqrt{t}(1\mp 1)\sqrt{t}x_r H_{m_r}(\sqrt{t}x_r)e^{-t|x|^2/2}dx_r)\wedge dx_{I^i} \\
&= H_{m_{s\neq r}}(\sqrt{t}x_{m_s})\Big(2\sqrt{t}m_r)H_{m_r-1}(\sqrt{t}x_r)e^{-t|x|^2/2}) \\
&-\sqrt{t}(1\mp 1)(H_{m_r+1}(\sqrt{t}x)-\sqrt{t}(m_r-1)H_{m_r-1}(\sqrt{t}x))\Big)\,dx_r\wedge dx_{I^i}
\end{aligned}$$

so there are two eigenfunctions $H_{m_{s\neq r}}(H_{m_r+1})$ and $H_{m_{s\neq r}}(H_{m_r-1}$ in the form $dx_r\wedge dx_{I^i}$ and this is summed over the 2n possible r, so we have at most 4n basic eigenforms in $d_t\eta^i$ where η^i is basic. Next note that $\Lambda=\iota_{\partial_{r+n}}\iota_{\partial_r}$ can have at most n terms when applied to a basic η^i. Thus, $d_t^\Lambda\eta^i=d_t\Lambda\eta^i-\Lambda d_t\eta_i$ can have $4n(n)$ basic terms for $d_t\Lambda$ and $n(4n)$ terms for Λd_t so we get at most $8n^2$ terms (some of these will cancel out) Thus, for each of the η^i, the term $\langle d^{t\Lambda}\eta^i,\xi^j\rangle$ is nonzero for at most $8n^2$ ξ^j.

If we look at the first three terms, $\lambda_{\eta^i}||\eta^i||^2-2\langle d_t^\Lambda\eta^i,\xi^j\rangle-2\langle d_t^\Lambda\eta^i,A\xi_{\mathcal{H},p}\rangle$, and removing the nonzero $\langle d_t^\Lambda \eta^i, \xi^j\rangle$ we get

$$\begin{aligned}
\lambda_{\eta^i}||\eta^i||^2 - 2\langle d_t^\Lambda \eta^i, \xi^j\rangle - 2\langle d_t^\Lambda \eta^i, A\xi_{\mathcal{H},p}\rangle &\geq 2t(\ell^{\eta^i} + \sum m^{\eta^i})||\eta^i||^2 - 2(\frac{1}{2C}||\Delta_{C,t}^\Lambda \eta^i||^2 \\
&+ \frac{C}{2}||\xi^j||^2) - 2(\frac{1}{2C}||\Delta_{C,t}^\Lambda \eta^i||^2 + \frac{C}{2}||A\xi_{\mathcal{H},p}||^2) \\
&= 2t(\ell^{\eta^i} + \sum m^{\eta^i})||\eta^i||^2 - \frac{2t}{C^{ij}}(\ell^{\eta^i\Lambda} + \\
&\sum m^{\eta^i})||\eta^i||^2) - C^{ij}||\xi^j||^2 \\
&- \frac{2t}{C^i_{\mathcal{H},p}}(\ell^{\eta^i\Lambda} + \sum m^{\eta^i})||\eta^i||^2) - C^i_{\mathcal{H},p}||A\xi_{\mathcal{H},p}||^2) \\
&= 2t\left(\ell^{\eta^i} - \left(\frac{1}{C^i_{\mathcal{H},p}} + \sum_j \frac{1}{C^{ij}}\right)\ell^{\eta^i\Lambda}\right. \\
&+ \left.\left(1 - \left(\frac{1}{C^i_{\mathcal{H},p}} + \sum_j \frac{1}{C^{ij}}\right)\right)\sum m^{\eta^i}\right)||\eta^i||^2 \\
&- C^{ij}||\xi^j||^2 - C^i_{\mathcal{H},p}||A\xi_{\mathcal{H},p}||^2)
\end{aligned}$$

Now, recall that η_k is not harmonic, so either $\sum m_{\eta^i} > 1$ or $\ell^\eta \geq 1$. Also, $\ell^{\Lambda\eta} \leq 2n$. Therefore, if we choose $C^{ij}, C^i_{\mathcal{H},p}$ so

$$\left(\frac{1}{C^i_{\mathcal{H},p}} + \sum_j \frac{1}{C^{ij}}\right) \leq \frac{1}{4n} \leq \frac{1}{4}$$

(for instance, by setting $C^{ij} = 4n(8n^2) = C^i_{\mathcal{H},p}$ (as $8n$ is the number of ξ^j we have after removing the zero terms), then in the case of $\ell > 1$ (with $\sum m^{\eta^i} > 0$, we have

$$\begin{aligned}
2t\left(\ell^{\eta^i} - \left(\frac{1}{C^i_{\mathcal{H},p}} + \sum_j \frac{1}{C^{ij}}\right)\ell^{\eta^i\Lambda} + \left(1 - \left(\frac{1}{C^i_{\mathcal{H},p}} + \sum_j \frac{1}{C^{ij}}\right)\right)\sum m^{\eta^i}\right)||\eta^i||^2 \\
> 2t\left(\ell^{\eta^i} - \frac{\ell^{\eta^i\Lambda}}{4n} + \left(1 - \left(\frac{1}{C^i_{\mathcal{H},p}} + \sum_j \frac{1}{C^{ij}}\right)\right)0\right) \\
\geq 2t(1 - \frac{2n}{4n}) \geq t
\end{aligned}$$

Thus the coefficient $c^{\eta^i} = 2\left(\ell^{\eta^i} - \left(\frac{1}{C^i_{\mathcal{H},p}} + \sum_j \frac{1}{C^{ij}}\right)\ell^{\eta^i\Lambda} + \left(1 - \left(\frac{1}{C^i_{\mathcal{H},p}} + \sum_j \frac{1}{C^{ij}}\right)\right)\sum m^{\eta^i}\right) >$

0 gives us a bound of $c^{\eta^i}t||\eta^i||^2$ in this case.

In the case of $\ell^{\eta^i} > 0$, so $\sum m^{\eta^i} > 1$, we have

$$2t\left(\ell^{\eta^i} - \left(\frac{1}{C^i_{\mathcal{H},p}} + \sum_j \frac{1}{C^{ij}}\right)\ell^{\eta^i\Lambda} + \left(1 - \left(\frac{1}{C^i_{\mathcal{H},p}} + \sum_j \frac{1}{C^{ij}}\right)\right)\sum m^{\eta^i}\right)||\eta^i||^2$$
$$> 2t\left(\ell^{\eta^i} - \frac{\ell^{\eta^i\Lambda}}{4n} + \left(1 - \frac{1}{4n}\right)\sum m^{\eta^i}\right)$$
$$\geq 2t\left(0 + \frac{3}{4}\sum m^{\eta^i}\right) \geq 1.5t$$

Thus c^{η^i} above also gives a bound $c^{\eta^i}t||\eta^i||^2$ inn either case, so this positive bound holds in all cases. so we have

$$\lambda_{\eta^i}||\eta^i||^2 - 2\langle d_t^\Lambda \eta^i, \xi^j\rangle - 2\langle d_t^\Lambda \eta^i, A\xi_{\mathcal{H},p}\rangle \geq c^{\eta^i}t||\eta^i||^2 - (4n)(8n)^2||\xi_j|| - 4n(8n)^2|\xi_{\mathcal{H},p}||^2$$

And note that we picked only the ξ^j that had a component with $d^{t\Lambda}\eta^i$, and each ξ^j can only have this happen with a certain number of η^i from the formula for $d^{t\Lambda}\eta^i$, so each $|\xi^j|$ is multiplied by $(4n)(8n)^2$ for the terms where H_{m_r} of η^i are equal or off by 1 and the forms are off by dx_r, which is only 3*(2n) at most. Thus we are not summing ξ^j an infinite number of times, but at most $C^{\eta^j}_{max} < \infty$ times.

If we now look at the next three terms $\lambda_{\eta^a}||\eta^a||^2 - 2\langle d_t^\Lambda \eta^a, \xi^j\rangle - 2\langle d_t^\Lambda \eta^a, A\xi_{\mathcal{H},p}\rangle$, note that our calculation from the $\eta_{\mathcal{H},p}$ harmonic case we have

$$\lambda_{\eta^a}||\eta^a||^2 - 2\langle d_t^\Lambda \eta^a, \xi^j\rangle - 2\langle d_t^\Lambda \eta^a, A\xi_{\mathcal{H},p}\rangle \geq 4t||\eta^a||^2 - 2\langle d_t^\Lambda \eta^a, \xi^j\rangle + 2t||A\xi_{\mathcal{H},p}||^2$$

And if we use our inequality on $\langle d_t^\Lambda \eta^a, A\xi_{\mathcal{H},p}\rangle$, we have

$$\begin{aligned}
\lambda_{\eta^a}||\eta^a||^2 - 2\langle d_t^\Lambda \eta^a, \xi^j\rangle - 2\langle d_t^\Lambda \eta^a, A\xi_{\mathcal{H},p}\rangle &\geq 4t||\eta^a||^2 - 2\langle d_t^\Lambda \eta^a, \xi^j\rangle + 2t||\xi_{\mathcal{H},p}||^2 \\
&\geq 4t||\eta^a||^2 - 2(\frac{1}{2C^{a_{\mathcal{H},p}}}||\Delta^\Lambda \eta^a|| + \frac{C^{aj}}{2}||\xi^j||^2) \\
&+ 2t||\xi_{\mathcal{H},p}||^2 \\
&= \left(4t - \frac{2t}{C^{aj}}\left(\ell^{\eta^a \Lambda} + \sum m^{\eta^a}\right)\right)||\eta^a||^2 \\
&+ 2t||\xi_{\mathcal{H},p}||^2 - C^{aj}||\xi^j||^2
\end{aligned}$$

with η^a having $\sum m^{\eta^a} = 1$. and again choosing the appropriate $C^{aj} = 4n(8n^2)$ where again $8n^2$ is the max number of nonzero $\langle d_t^\Lambda \eta^a, \xi^j\rangle$ (note that we removed the ξ^j associated to $\xi_{\mathcal{H},p}$, so $\sum \frac{1}{C^{aj}} \leq \frac{1}{4n}$), we then have

$$\begin{aligned}
\lambda_{\eta^a}||\eta^a||^2 - 2\langle d_t^\Lambda \eta^a, \xi^j\rangle - 2\langle d_t^\Lambda \eta^a, A\xi_{\mathcal{H},p}\rangle &\geq \left(4t - \frac{2t}{C^{aj}}\left(\ell^{\eta^a \Lambda} + \sum m^{\eta^a}\right)\right)||\eta^a||^2 \\
&+ 2t||\xi_{\mathcal{H},p}||^2 - C^{aj}||\xi^j||^2 \\
&\geq \left(4t - 2t\left(\frac{\ell^{\eta^a \Lambda}}{4n} + \frac{1}{4n}\right)\right)||\eta^a||^2 + 2t||\xi_{\mathcal{H},p}||^2 \\
&- C^{aj}||\xi^j||^2 \\
&\geq \left(4t - 2t\left(\frac{2n}{4n} + \frac{1}{4}\right)\right)||\eta^a||^2 + 2t||\xi_{\mathcal{H},p}||^2 \\
&- C^{aj}||\xi^j||^2 \\
&\geq (4t - 1.5t)\,||\eta^a||^2 + 2t||\xi_{\mathcal{H},p}||^2 - C^{aj}||\xi^j||^2 \\
&\geq 2.5t||\eta^a||^2 + 2t||A\xi_{\mathcal{H},p}||^2 - C^{aj}||\xi^j||^2
\end{aligned}$$

Thus if we define $c^{\eta^a} = 2.5$ and $c^{\xi_{\mathcal{H},p}} = 2$, we have positive bounds for these forms.

Combining all of this together (and recalling $C^{\eta^j}_{max}, C^{i_{\mathcal{H},p}} C^{aj} < 3n(2n)(8n^2)$), we have

$$\left\langle \Delta_{C,t} \begin{bmatrix} \eta^i \\ \xi^i \end{bmatrix}, \begin{bmatrix} \eta^j \\ \xi^j \end{bmatrix} \right\rangle = \lambda_{\eta^i} ||\eta^i||^2 - 2\langle d_t^\Lambda \eta^i, \xi^j \rangle + \lambda_{\xi^i} ||\xi^i||^2 + O(||\eta^i||||\eta^j||, ||\xi^i||||\xi^j||)$$

$$\geq c^{\eta^i} t ||\eta^i||^2 + c^{\eta^a} t ||\eta^a||^2 + c^{\xi_{\mathcal{H},p}} t ||A\xi_{\mathcal{H},p}||^2 + c^{\xi^i} t ||\xi^i||^2$$

$$- C^{\eta^j}_{max} ||\xi^j||^2 - C^{aj} ||\xi^j||^2 - C^{i_{\mathcal{H},p}} ||A\xi_{\mathcal{H},p}||^2 - O(||\eta^i||||\eta^j||, ||\xi^i||||\xi^j||)$$

Thus if we let c_{min} be the minimum of $c^{\eta^i}, c^{\eta^a}, c^{\xi_{\mathcal{H},p}}, c^{\xi^i}$ that are part of the sum (so no $c^{\eta^a}, c^{\xi_{\mathcal{H},p}}$ if they are not part of η_k, ξ_{k-1}) then this inequality shows

$$\left\langle \Delta_{C,t} \begin{bmatrix} \eta^i \\ \xi^i \end{bmatrix}, \begin{bmatrix} \eta^j \\ \xi^j \end{bmatrix} \right\rangle = \lambda_{\eta^i} ||\eta^i||^2 - 2\langle d_t^\Lambda \eta^i, \xi^j \rangle + \lambda_{\xi^i} ||\xi^i||^2 + O(||\eta^i||||\eta^j||, ||\xi^i||||\xi^j||)$$

$$\geq c_{min} t (||\eta^i||^2 + ||\eta^a||^2 + ||A\xi_{\mathcal{H},p}||^2 + ||\xi^i||^2) -$$

$$O(||\eta^i||||\eta^j||, ||\xi^i||^2, ||\xi^i||||\xi^j||)$$

$$= c_{min} t \left|\left| \begin{matrix} \eta^i \\ \xi^i \end{matrix} \right|\right|^2 - O(||\eta^i||||\eta^j||, ||\xi^i||||\xi^j||)$$

we thus have

$$\lambda \left|\left| \begin{matrix} \eta^i \\ \xi^i \end{matrix} \right|\right|^2 = \left\langle \Delta_{C,t} \begin{bmatrix} \eta^i \\ \xi^i \end{bmatrix}, \begin{bmatrix} \eta^j \\ \xi^j \end{bmatrix} \right\rangle \geq ct \left|\left| \begin{matrix} \eta^i \\ \xi^i \end{matrix} \right|\right|^2 - O(||\eta^i||||\eta^j||, ||\xi^i||||\xi^j||)$$

so by driving t large enough it will dominate this inequality (as $O(||\eta^i||||\eta^j||, ||\xi^i||||\xi^j||$ has no factor of t) and as eigenforms are non-zero $\lambda > ct$ for sufficiently large t As another result of our inequality above, we now show that the kernel of Δ^{tx} is generated by only our two harmonic solutions. For if it were a third independent solution, we could project it to the orthogonal complement of our two solutions and get a nonzero harmonic solution in the othorgonal complement of $\begin{bmatrix} \eta_{\mathcal{H},p} \\ 0 \end{bmatrix}, \begin{bmatrix} -\tau_p \wedge \xi_{\mathcal{H},p} \\ \xi_{\mathcal{H},p} \end{bmatrix}$, but then our inequality above would apply to our new solution, but choosing t large enough would give a positive eigenvalue, contradicting that our projection was harmonic. Thus the kernel of $\Delta_{C,t}$ is two dimensional and is generated by $\begin{bmatrix} \eta_{\mathcal{H},p} \\ 0 \end{bmatrix}, \begin{bmatrix} -\tau_p \wedge \xi_{\mathcal{H},p} \\ \xi_{\mathcal{H},p} \end{bmatrix}$, The proofs above work for the case of $n_f \leq n$, for $n_f \geq n$, use that $[\Delta_X, *] = 0$ to change f to $-f$, which now has index less than n, and the proof follows.

4.2 Showing that the Only L^2 Harmonic Solutions Are Localized

Without loss of generality assume that each V_p where $p \in \text{Zero}(f)$ is an open ball of radius $4a$, and assume $n_f \leq n, t > 0$ let γ_p be a smooth bump function such that $\gamma_p(z) = 0$ if $|z| > 2a$ $\gamma_p(z) = 1$ if $|z| < a$. Define

$$\alpha_{p,1}(t) = \sqrt{\int \gamma_p^2 \left\| \begin{matrix} \eta_{\mathcal{H},p} \\ \iota_{-\tau_p'^\sharp} \eta_{\mathcal{H},p} \end{matrix} \right\|^2 dVol}, \alpha_{p,2}(t) = \sqrt{\int \gamma_p^2 \left\| \begin{matrix} -\tau_p \wedge \xi_{\mathcal{H},p} \\ \xi_{\mathcal{H},p} \end{matrix} \right\|^2 dVol}$$

Now define the forms $\rho_{p,1}(t), \rho_{p,2}(t)$ by

$$\rho_{p,1}(t) = \frac{\gamma_p}{\alpha_{p,1}(t)} \begin{bmatrix} \eta_{\mathcal{H},p} \\ \iota_{-\tau_p'^\sharp} \eta_{\mathcal{H},p} \end{bmatrix}, \rho_{p,2}(t) = \frac{\gamma_p}{\alpha_{p,2}(t)} \begin{bmatrix} -\tau_p \wedge \xi_{\mathcal{H},p} \\ \xi_{\mathcal{H},p} \end{bmatrix}$$

Then $\rho_{p,1}(t), \rho_{p,2}(t)$ is a section of unit length with compact support contained in V_p. Also, note that $\rho_{p,1}(t), \rho_{p,2}(t)$ encompass all our local solutions found in section 2 as if $n_f(p) \leq n$, then $\iota_{-\tau_p'^\sharp}\eta_{\mathcal{H},p} = 0$ and if $n_f(p) \geq n$, then $-\tau_p'^\sharp \wedge \xi_{\mathcal{H},p} = 0$. Let $E_{C,t}$ denote the direct sum of the vector spaces generated by the $\rho_{,p,1}(t), \rho_{p,2}(t)$, and let $E_{C,t}^\perp$ be the orthogonal complement to $E_{C,t}$ in $\mathbf{H}^1(Cone(\omega))$, so $\mathbf{H}^1(Cone(\omega)) = E_{C,t} \oplus E_{C,t}^\perp$. Let $p_{C,t}, p_{C,t}^\perp$ denote the orthogonal projections from $\mathbf{H}^1(Cone(\omega)$ to $E_{C,t}$ and $E_{C,t}^\perp$, respectively and decompose the operator $D_{C,t} = d_{C,t} + d_{C,t}^* = d_C + d_C^* + c(\nabla(f))$, where $c(e) = e^* \wedge -\iota_e$ via

$$D_{C,t,1} = p_{C,t}D_{C,t}p_{C,t}, D_{C,t,2} = p_{C,t}D_{C,t}p_{C,t}^\perp, D_{C,t,3} = p_{C,t}^\perp D_{C,t}p_{C,t}, D_{C,t,4} = p_{C,t}^\perp D_{C,t}p_{C,t}^\perp$$

we now have the following results:

Theorem 4.4.

for some $t_0 > 0$, and any $t \geq t_0$, and $0 \leq u \leq 1$, the operator is fredholm

$$D_{C,t,u} = D_{C,t,1} + D_{C,t,4} + u(D_{C,t,2} + D_{C,t,3}) = D_{C,t} + (u-1)(D_{C,t,2} + D_{C,t,3})$$

(ii) the operator$D_{C,t,4} : E_{C,t}^\perp \cap H^1(Cone(\omega) \to E_{C,t}^\perp$ is invertible

To prove these, we need the following inequalities:

Lemma 4.2.1. *there exists a constant $t_0 > 0$ such that for $\sigma \in E_{C,t}^\perp \cap \boldsymbol{H}^1(Cone(\omega)), \sigma' \in E_{C,t}$ and $t \geq t_0$ one has*

$$||D_{t,2}\sigma||_0 \leq \frac{||\sigma||_0}{t},$$
$$||D_{t,3}\sigma'||_0 \leq \frac{||\sigma'||_0}{t}$$

Proof: note that $D_{t,3}$ is the adjoint. of $D_{t,2}$ so if we prove for the first we get the second.

Since $\rho_{p,1}(t), \rho_{p,2}(t)$ has support in V_p, we have

$$\begin{aligned}
D_{t,2}s &= p_{C,t} D_t p^{\perp}_{C,t}\sigma \\
&= p_{C,t} D_t \sigma \\
&= \sum_{p \in Zero(V)} \rho_{p,i}(t) \int_{V_p} \langle \rho_{p,i}(t), D_t\sigma \rangle dv_{V_p} \\
&= \sum_{p \in Zero(V)} \rho_{p,1,t} \int_{V_p} \langle D_t \frac{\gamma_p}{\alpha_{p,1}(t)} \begin{bmatrix} \eta_{\mathcal{H},p} \\ \iota_{-\tau_p'^{\sharp}} \eta_{\mathcal{H},p} \end{bmatrix}, \sigma \rangle \\
&+ \rho_{p,2,t} \int_{V_p} \langle D_t \frac{\gamma_p}{\alpha_{p,2}(t)} \begin{bmatrix} -\tau_p \wedge \xi_{\mathcal{H},p} \\ \xi_{\mathcal{H},p} \end{bmatrix}, \sigma \rangle dv_{V_p}
\end{aligned}$$

And note that γ is constants on $|\mathbf{y}| < a, |\mathbf{y}| > 2a$, so $D_t\rho_{p,1}(t) = 0 = D_t\rho_{p,2}(t)$ (as these are harmonic solutions multiplied by a constant) on $|\mathbf{y}| < a, |\mathbf{y}| > 2a$.

Now note that we chose

$$\alpha_{p,1}(t) = \sqrt{\int \gamma_p^2 \left\| \begin{matrix} \eta_{\mathcal{H},p} \\ \iota_{-\tau_p'^{\sharp}} \eta_{\mathcal{H},p} \end{matrix} \right\|^2 dVol}, \alpha_{p,2}(t) = \sqrt{\int \gamma_p^2 \left\| \begin{matrix} -\tau_p \wedge \xi_{\mathcal{H},p} \\ \xi_{\mathcal{H},p} \end{matrix} \right\|^2 dVol}$$

so $\rho_{p,1}(t) = \frac{\gamma_p}{\alpha_{p,1}(t)} \begin{bmatrix} \eta_{\mathcal{H},p} \\ \iota_{-\tau_p'^{\sharp}} \eta_{\mathcal{H},p} \end{bmatrix}, \rho_{p,2}(t) = \frac{\gamma_p}{\alpha_{p,1}(t)} \begin{bmatrix} -\tau_p \wedge \xi_{\mathcal{H},p} \\ \xi_{\mathcal{H},p} \end{bmatrix}$ have norm 1. Thus note that, as $\eta_{\mathcal{H},p}, \xi_{\mathcal{H},p}, -\tau_p \wedge \xi_{\mathcal{H},p}, \iota_{-\tau_p'^{\sharp}} \eta_{\mathcal{H},p}$ are either $e^{-tx_i^2} dx_I$ or $x_j e^{-tx_i^2} dx_J$ where $dx_J = \iota_{n+j} dx_I$ or $dx_{n+j} \wedge dx_I$. Thus, on the interval $a < |y| < 2a$, these are bounded above by $\int_{\mathbb{R}^n} (1 + 2a^n) e^{-t|x|^2/2} dVol = \max(1, (2a)^n) \left(\frac{2t}{\pi}\right)^{2n/2}$, and bounded below by $\int_{B_a} \max(1, 2a^n) e^{-t|x|^2/2} dVol = C_a \left(\frac{2t}{\pi}\right)^{2n/2}$, and thus $C_a' t^n \leq \frac{1}{\alpha_{p,i}(t)} \leq C_a'' t^n$. Thus when we look at the integrals

$$\int_{V_p} \langle D_t(\frac{\gamma_p}{\alpha_{p,1}(t)} \begin{bmatrix} \eta_{\mathcal{H},p} \\ \iota_{-\tau_p'^{\sharp}} \eta_{\mathcal{H},p} \end{bmatrix}), s \rangle dv_{V_p}, \int_{V_p} \langle D_t(\frac{\gamma_p}{\alpha_{p,2}(t)} \begin{bmatrix} -\tau_p \wedge \xi_{\mathcal{H},p} \\ \xi_{\mathcal{H},p} \end{bmatrix}), s \rangle dv_{V_p}.$$

Mote that we can restrict these to $a < |x| < 2a$, where $\gamma\eta_{\mathcal{H},p}, \Lambda\eta_{\mathcal{H},p}, \xi_{\mathcal{H},p}, -\omega \wedge \xi_{\mathcal{H},p}$ are bounded by $C_a''' e^{-ta^2}$, also note that

$$\begin{aligned} d_t + d_t^*(\gamma\psi) &= (d\gamma) \wedge \psi + \gamma d_t \psi + (d^* + t\iota_\nabla f)\gamma\psi \\ &= (d\gamma) \wedge \psi + \gamma d_t \psi + (\partial_i \gamma \psi_I)\iota_{\partial i} dx_I + t\iota_{\nabla f}\gamma\psi_I dx_I \\ &= (d\gamma) \wedge \psi + \gamma d_t \psi + (\partial_i \gamma)\psi_I \iota_{\partial_i} dx_I + \gamma\partial_i \psi_I \iota_{\partial i} dx_I + \gamma t\iota_{\nabla f}\psi_I dx_I \\ &= (d\gamma) \wedge \psi + \gamma d_t \psi + (\partial_i \gamma)\psi_I + \gamma d^* \psi_I + t\gamma\iota_{\nabla f}\psi \\ &= (d\gamma) \wedge \psi + \gamma d_t \psi + (\partial_i \gamma)\psi_I \iota_{\partial_i} dx_I + \gamma d_t^* \psi \end{aligned}$$

Therefore for $\psi = \eta_{\mathcal{H},p}, \iota_{-\tau_p'^\sharp}\eta_{\mathcal{H}}, -\tau_p \wedge \xi_{\mathcal{H},p}, \xi_{\mathcal{H},p}$ the terms we can get in $D_{C,t}$ are $\Lambda\psi, \omega \wedge \psi$ (which are bounded by $\max(1, 2a)e^{-ta^2/2}$) and $(d\gamma) \wedge \psi, \gamma d_t \psi + (\partial_i \gamma)\psi_I \iota_{\partial_i} dx_I + \gamma d_t^* \psi$ (which are bounded by $(\max \gamma) \max(1, 2a)e^{-ta^2/2}$. therefore, we have

$$\begin{aligned} \int_{V_p} \langle D_T(\frac{\gamma}{\alpha_{p,i,t}} \begin{bmatrix} \eta_{\mathcal{H},p} \\ \iota_{-\tau_p'^\sharp}\eta_{\mathcal{H},p} \end{bmatrix}, \sigma\rangle dv_{V_p} &\leq ||\sigma||_0 \sqrt{\int_{a<|x|<2a} \frac{C_\gamma}{\alpha_{p,i,t}} e^{-ta^2/2} dV_{V_p}} \\ &\leq C_1 t^n e^{-ta^2/2} ||\sigma||_0 \leq \frac{C_2 ||\sigma||_0}{t} \end{aligned}$$

Therefore, the norm of $D_{t,2}s$ is

$$||D_{t,2}\sigma|| \leq \left|\left|\rho_{p,i}(t)\frac{C||\sigma||_0}{t}\right|\right| = \frac{C||\sigma||_0}{t}$$

and we thus have our first inequality.

Next, note that this implies that $D_{C,t,2}$ and $D_{C,t,3}$ are compact operators, and thus $D_{C,t,u} = D_{C,t} + (u-1)(D_{C,t,2} + D_{C,t,3})$ is a fredholm operator plus a compact operator, hence fredholm.

To show that the operator$D_{C,t,4} : E_{C,t}^\perp \cap \mathbf{H}_1(Cone(\omega)) \to E_{C,t}^\perp$ is invertible , we follow Bismut and Zhang [2] will show that for $t \geq t_2$ and $\sigma \in E_{C,t}^\perp$, we have $||D_{C,t,4}\sigma||_0 \geq C_3\sqrt{t}||\sigma||_0$.

To do this, we break into cases:

Lemma 4.2.2. *Case 1*

if Supp(s) $\in V_p(4a)$ (the ball of radius 4a) Let E_p be a euclidean space containing V_p. Define $\rho'_{p,1,t} = \left(\frac{t}{\pi}\right)^{n/2} e^{-tx_i^2/2}\rho_{p,1,t}$ and $\rho'_{p,2,t} = \left(\frac{t}{\pi}\right)^{n/2} e^{-tx_i^2}\rho_{p,t,2}$, and define $p'_{C,t}$ to be the projection onto the subspace of $\mathbf{H}_0(E_p)$ spanned by the $\rho'_{p,i,t}$. Since $p_{C,t}s = 0$, we have that

$$\begin{aligned} p'_{C,t}\sigma &= p'_{C,t}\sigma - p_{C,t}\sigma \\ &= \sum_{p\in \text{zero(f)}} \rho'_{p,i,t} \int_{E_p} (1-\gamma(|x|)) \left(\frac{t}{\pi}\right)^{n/2} \langle e^{-t|x|^2/2}\rho_{p,i,t}, \sigma\rangle dVol_{E_p} \end{aligned}$$

As $\gamma = 1$ near p, this is 0 in the ball of radius $|a|$ and since σ has support in the ball of radius $4a$ A similar result to lemma 3.2

$$||p'_{C,t}(\sigma)|| \leq 2 \leq C'_4\sqrt{t}||\sigma||^2$$

Next, note that $D_{C,t}\rho' = 0$, so $D_{C,t}p'_{C,t}\sigma = 0$, and as $\sigma - p'_{C,t}\sigma \in (E'_{C,t})^{\perp}$ we can apply our inequalities above and get

$$||D_{C,t}\sigma||^2 = ||D_{C,t}(\sigma - p'\sigma)||^2 \geq C_6t||s - p'\sigma||^2 \geq C_6t||\sigma||_0 - C_7\sqrt{t}||\sigma||^2$$

Thus $||D_{C,t}\sigma||_0 \geq \frac{C_8\sqrt{t}}{2}||\sigma||_0$,

Lemma 4.2.3. *Case 2*

$\text{Supp } (\sigma) \subset M \setminus \bigcup_{p \in \text{zero}(f)} V_p(2a)$ (and still $\sigma \in E_{C,t}^{\perp} \cap H^1(M)$) to prove this, recall that

$$
\begin{aligned}
D_{C,t}^2 &= D^2 + \begin{bmatrix} 0 & -d_t^{\Lambda *} \\ -d_t^{\Lambda} & 0 \end{bmatrix} + t^2 |df|^2 I \\
&= D^2 + \begin{bmatrix} 0 & -d^{\Lambda *} \\ -d^{\Lambda} & 0 \end{bmatrix} + t \begin{bmatrix} 0 & d^*\omega - \omega d^* \\ \Lambda d - d\Lambda & 0 \end{bmatrix} + t^2 \begin{bmatrix} |df|^2 & 0 \\ 0 & |df|^2 \end{bmatrix},
\end{aligned}
$$

where since we are away from the zeroes of f, $|df|^2 \geq C_9$, thus

$$
\begin{aligned}
||D_t \sigma||^2 &= \langle \langle D_{C,t}^2 \sigma, \sigma \rangle \rangle \\
&\geq (C_9 t^2 - C_{10} t - C_{11}) ||\sigma||^2
\end{aligned}
$$

From which we can conclude $||D_t \sigma|| \geq C_{12} \sqrt{t} ||\sigma||$

Case 3: Let $\tilde{\gamma} \in C^{\infty}(M)$ be defined such that on $V_p, p \in \text{zero}(f)$ that $\tilde{\gamma}(y) = \gamma_p(|y|/2)$ and $\tilde{\gamma}|_{M \setminus \bigcup V_p(4a)} = 0$ Now for $\sigma \in E_{C,t}^{\perp} \cap \mathbf{H}^1(M)$ one can see that $\tilde{\gamma}\sigma \in E_{C,t}^{\perp} \cap \mathbf{H}^1(M)$, as $ga\tilde{m}ma = 1$ in the regions of V_p so it is still 0 there. also $||D_T \sigma|| \geq ||D_T s + (-\tilde{\gamma} D_T s|| + || - \tilde{\gamma}$

Lemma 4.2.4. *Case 3*

Thus by Case 1 and 2, one deduces there is a C_{13} such that for $t \geq t_1 + t_2$,

$$
\begin{aligned}
||D_t \sigma||_0 &\geq \frac{1}{2} (||(1 - \tilde{\gamma} D_t \sigma||_0 + ||\tilde{\gamma} D_t \sigma||_0) \\
&\geq \frac{1}{2} \left(||D_t (1 - \tilde{\gamma})\sigma + [D, \gamma]\sigma||_0 + ||D_{C,t} \tilde{\gamma} + [\gamma, D]\sigma||_0 \right) \\
&\geq \frac{\sqrt{t} C_8}{2} ||(1 - \gamma)\sigma||_0 + \sqrt{C}_6 ||\tilde{\gamma}\sigma||_0) - C_9 ||\sigma||_0
\end{aligned}
$$

Where $C_{10} = \min\{\sqrt{C_6}/2, C_8/2\}$. Thus we have that the operator $D_{C,t,4} : E_T^{\perp} \cap \mathbf{H}^1(M) \to E_T^{\perp}$ is invertible.

Theorem 4.5. *There exists a T_5 so that for $T > T_5, \sigma \in H^1(Cone(\omega))$ $||D_{C,t,1}\sigma|| \leq \frac{C_{14}||\sigma||}{T}$*

To show this, note that for the operator $D_{C,t,1}$ that

$$||D_{C,t,1}\sigma||_0 \leq \frac{C_{14}||\sigma||_0}{t}$$

to prove this, we proceed in a similar method to our first inequality, using

$$p_{C,t}\sigma = \sum_{p \in Zero(V)} \rho_{p,i}(t) \int_{V_p} \langle \rho_{p,i}(t), \sigma \rangle dv_{V_p}$$

And note that if we take $D_t\rho_{p,i}(t)$, then this is 0 in the region $|x-p| < a$ and $|x-p| > a$ and from a similar argument to lemma 3.2 $||D_t\rho_{p,i}(t)|| \leq \frac{C_3}{t}$. Then using that $\rho_{p,i}(t)$ has norm 1

$$\begin{aligned}
||p_{C,t}D_tp_{C,t}\sigma||_0 &= \left|\left| \sum_{p \in Zero(V)} \rho_{p,i}(t) \int_{V_p} D_T\rho_{p,i}(t) \langle \rho_{p,i}(t), \sigma \rangle dv_{V_p} \right|\right| \\
&\leq \sum_{p \in Zero(V)} \left|\left| (\rho_{p,i}(t)) \langle \rho_{p,i}(t), \sigma \rangle \int_{V_p} D_T\rho_{p,i}(t) dv_{V_p} \right|\right| \\
&\leq \sum_{p \in Zero(V)} \left|\left| \langle \rho_{p,i}(t), \sigma \rangle \frac{C_3}{t} \right|\right| \\
&\leq \sum_{p \in Zero(V)} \frac{C_3}{t} ||\rho_{p,i}(t)||_0 ||\sigma||_0 \\
&\leq \frac{C_{12}||\sigma||_0}{t}
\end{aligned}$$

Definition 4.1. *Let for $c > 0$, let $E_{C,t}(c)$ denote the direct sum of eigenspaces of $D^2_{C,t}$ with eigenvalues in $[-c, c]$. since $D^2_{C,t}$ is a self adjoint linear operator $E(c)$ is finite dimensional subspace of $\boldsymbol{H}^0(Cone(\omega))$*

Let $P_{C,t}(c)$ denote the projection operator from $\mathbf{H}^0(Cone(\omega))$ to $E_{C,t}(c)$

Lemma 4.2.5. *There exists a* $C_1 > 0$ *such that for* $t \geq t_3$ *and* $\sigma \in E_T$

$$||P_{C,t}(c)\sigma - \sigma||_0 \leq \frac{C_1}{T}||\sigma||_0$$

Proof: Let $\delta = \{\lambda \in \mathbb{C} : |\lambda| = C_\lambda\}$ By Lemma 3.2 and Theorem 3.6, we have that for any $\lambda \in \delta, t \geq t_1 + t_2, \sigma' \in \mathbf{H}^1(Cone(\omega)$,

$$\begin{aligned}||(\lambda - D_{C,t})\sigma'||_0 \geq & \frac{1}{2}\left(||\lambda p_{C,t}\sigma' - D_{t,1}p_{C,t}\sigma - D_{t,2}p_{C,t}^{\perp}\sigma'||_0 + ||\lambda p_T^{\perp} - D_{t,3}p_{C,t}\sigma' - \right. \\ & \left. D_{t,4}p_{C,t}^{\perp}\sigma'||_0\right) \\ & \geq \frac{1}{2}\left(\left(C_\lambda - \frac{C_3}{t} - \frac{C_{12}}{t}\right)||p_{C,t}\sigma'||_0 + \left(C_6\sqrt{t} - C_\lambda - \frac{C_3}{T}\right)||p_{C,t}^{\perp}\sigma'||_0\right)\end{aligned}$$

By the inequality, for $t_4 \geq t_1 + t_2$ and $C_{13} > 0$ such that for any $t > t_4$ and $\sigma' \in \mathbf{H}^1(Cone(\omega)$

$$||(\lambda - D_{C,t})\sigma'||_0 \geq C_{13}||\sigma'||_0$$

Thus for any $\lambda \in \delta$, $\lambda - D_{C,t} : \mathbf{H}^1(Cone(\omega)) \to \mathbf{H}^0(Cone(\omega))$ is invertible, so the resolvent $(\lambda - D_{C,t})^{-1}$ is well defined. By the basic spectral theory for operators, for $\sigma \in E_T$ one has

$$P_{C,t}(c)\sigma - \sigma = \frac{1}{2\pi\sqrt{-1}}\int_\delta ((\lambda - D_{C,t})^{-1} - \lambda^{-1})\sigma d\lambda.$$

Since with $p_{C,t}$ the projection to E_T, we have $p_{C,t}^{\perp}\sigma = 0$ Thus using the inequality above, we have

$$\begin{aligned}(\lambda - D_{C,t})^{-1} - \lambda^{-1})\sigma &= \lambda^{-1}(\lambda - D_{C,t})^{-1}D_{C,t}\sigma \\ &= \lambda^{-1}(\lambda - D_{C,t})^{-1}(D_{C,t,1}\sigma + D_{C,t,3}\sigma)\end{aligned}$$

One deduces by Lemma 3.2 and above we have

$$||\lambda - D_{C,t})^{-1}(D_{C,t,1}\sigma + D_{C,t,3}\sigma)||_0 \leq C_{13}^{-1}||D_{t,1}\sigma + D_{t,3}\sigma||_0$$
$$\leq C_{13}^{-1}(\frac{C_{14}+C_2}{T}||\sigma||_0$$

and plugging this into the integral gives

$$||P_{C,t}(c)\sigma - \sigma||_0 = \left|\left|\frac{1}{2\pi\sqrt{-1}}\int_\delta ((\lambda - D_{C,t})^{-1} - \lambda^{-1})\sigma d\lambda\right|\right|_0$$
$$\leq \frac{1}{2\pi}\int_\delta ||\lambda^{-1}(\lambda - D_{C,t})^{-1}(D_{C,t,1}\sigma + D_{C,t,3}\sigma)||_0 d\lambda$$
$$\leq \frac{C_\lambda}{2\pi}\int_\delta ||C_{13}^{-1}(\frac{C_{14}+C_2}{T}||\sigma||_0 d\lambda$$
$$\leq \frac{C_{15}}{T}||\sigma||_0$$

Theorem 4.6. *Let $F_{C,t}^{[0,c]}$ be the space of all eigenforms of Δ_{tf} with eigenvalues in $[0,c]$. Then for t large enough, $(F_{C,t}^{[0,1]}, d_{tC})$ is a chain complex with* $\dim(F_{tC}^{[0,1]}) = m_k + m_{k-1}$

Proof: by lemma 3.7 applied to the $\rho_{p,i,t}$ when t is large enough, $P_{C,t}(c)\rho_{p,i,t}$ will be linearly independent (as if they are not linearly dependent, then $\sigma = \rho_{p,i,t}$ and $\sigma' = a\rho'_{p',i',t}$ would have $P_{C,t}(c)\sigma = P_{C,t}(c)\sigma'$, but then by (3.7) we would have

$$\frac{C_1}{T}||\sigma - \sigma'||_0 \geq ||P_{C,t}(c)\sigma - P_{C,t}(c)\sigma' - (\sigma - \sigma')||_0 = ||\sigma - \sigma'||_0$$

Thus for $t \geq t_5$, we have $\dim(E_{C,t}(c)) \geq \dim(E_{C,t})$. Now assume for the purposes of contradiction that $\dim(E_{C,t}(c)) \geq \dim(E_{C,t})$. Then there is a nonzero $\sigma \in E_{C,t}(c)$ that is orthogonal to $P_{C,t}(c)E_{C,t}$, or $\langle\sigma, P_{C,t}\rho_{p,i}(t)\rangle_{\mathbf{H}^0(Cone(\omega)} = 0$ for any $\rho_{p,i}$. Then from lemma 4.2 and 4.7, we have that

$$\begin{aligned}
p_t\sigma &= \sum_{p\in zero(df)} \langle \sigma, \rho_{p,i}(t)\rangle \rho_{p,i}(t) \\
&= \sum_{p\in zero(df)} \langle \sigma, \rho_{p,i}(t)\rangle \rho_{p,i}(t) - \sum_{p\in zero(df)} \langle \sigma, P_{C,t}\rho_{p,i}(t)\rangle P_{C,t}(c)\rho_{p,i}(t) \\
&= \sum_{p\in zero(df)} \langle \sigma, \rho_{p,i}(t)\rangle (\rho_{p,i}(t) - P_{C,t}(c)\rho_{p,i}(t)) + \\
&\sum_{p\in zero(df)} \langle \sigma, \rho_{p,i}(t) - P_{C,t}\rho_{p,i}(t)\rangle P_{C,t}(c)\rho_{p,i}(t)
\end{aligned}$$

By lemma 3.2, there exists a $C_3 > 0$ so when $t \geq t_5$ $||p_{C,t}\sigma||_0 \leq \frac{C_3}{T}||\sigma||_0$, and thus

$$||p_{C,t}^{\perp}\sigma||_0 \geq ||\sigma||_0 - ||p_{C,t}\sigma||_0 \geq ||\sigma||_0 \geq C_{15}||\sigma||_0$$

using this and lemma 3.7, when $T > 0$ is large enough

$$\begin{aligned}
C_{15}C_4\sqrt{t}||\sigma||_0 &\geq ||D_{C,t}p_{C,t}^{\perp}\sigma||_0 \\
&= ||D_{C,t}\sigma - D_{C,t}p_{C,t}\sigma||_0 \\
&= ||D_{C,t}\sigma - D_{C,t,1}\sigma - D_{C,t,3}\sigma||_0 \\
&\leq ||D_{C,t}\sigma||_0 + ||D_{C,t,1}\sigma||_0 + ||D_{C,t,3}\sigma||_0 \\
&\leq ||D_{C,t}\sigma||_0 + \frac{C_{12} + C_3}{T}||\sigma||_0
\end{aligned}$$

From which one gets $|D_{C,t}\sigma||_0 \geq C_{15}C_4\sqrt{T}||\sigma||_0 - \frac{C_{12}+C_3}{T}||\sigma||_0$ which contradicts that $\sigma \in E_{C,t}(c)$ is an eigenspace for t large enough.

Thus one has

$$\dim(E_{C,t}(c)) = \dim E_{C,t} = \sum_k m_k + m_{k-1} = 2\sum_k m_k$$

Morevoer $E_{C,t}$ is generated by the $P_{C,t}(c)\rho_{p,i}(t)$

Now to prove Theorem 3.8, for any integer k such that $0 \leq k \leq 2n+1$ let Q_i denote the projection from $\mathbf{H}^0(Cone(\omega))$ onto the L^2 completion of $Cone^k(\omega)$. Since $\Delta_{C,t}$ preserves the $\mathbb{Z}$ grading of $\Omega^*(M)$, for any eigenvector σ of $D_{C,t}$ associated with an eigenvalue $\mu \in [-c, c]$

$$\Delta_{C,t} Q_k \sigma = Q_k \Delta_{C,t} \sigma = Q_k \mu^2 \sigma = \mu^2 Q_k \sigma$$

That is $Q_k \sigma$ is an eigenform of $\Delta_{C,t}$ with eigenvalue μ^2

We thus need to show that $\dim Q_k E_{C,t}(c) = m_k + m_{k-1}$ To prove this, note that by lemma 3.7

$$||Q_{n_f(p)} P_{C,t}(c) \rho_{p,i}(t) - \rho_{p,i}(t)||_0 \leq \frac{C_3}{T}$$

so for t large enough the $Q_{n_f(p)} P_{C,t}(c) \rho_{p,i}(t)$ are linearly independent. Thus for each k

$$\dim Q_k E_{C,t}(c) \geq m_k + m_{k-1}$$

however, we also have (as every element in $\mathbf{H}^0(Cone(\omega))$ is a linear combination of $2n+1$ form)

$$\sum_{k=0}^{2n+1} \dim Q_k E_{C,t}(c) = \sum_{k=0}^{2n+1} \dim E_{C,t}(c) = \sum_k m_k + m_{k-1} = 2 \sum_k m_k$$

from this and $\dim Q_k E_{C,t}(c) \geq m_k + m_{k-1}$ we have

$$\dim Q_k E_{C,t}(c) = m_k + m_{k-1}$$

Remark: as Zhang [19] also proves, since $c > 0$ is arbitrary, as $T \to \infty$ the eigenvalues of $[0, c]$ converge to 0.

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Appendix A

Morse Stokes' Theorem

We describe here the conventions used to define the differential map ∂ in the Morse cochain complex and also the orientations of the submanifolds which are integrated over in the $c(\psi)$ map of (**??**). A main aim is to prove the following:

Lemma A.0.1 (Leibniz Rule on forms in Morse cohomology)**.** *Let* $\psi \in \Omega^\ell(M)$ *then*

$$\partial\, c(\psi) + (-1)^{\ell+1} c(\psi)\, \partial = -c(d\psi)\,. \tag{A.1}$$

This formula appeared in Austin-Braam [1] and Viterbo [16] though with ambiguous signs. To set our conventions and prove the Lemma, we start with a brief background.

Let ϕ_t be the flow of the vector field $-\nabla f$. For a critical point $r \in Crit(f)$, the stable S_r and unstable U_r submanifolds are defined to be

$$S_r = \{x \in M : \lim_{t\to\infty} \phi_t(x) = r\}\,, \qquad U_r = \{x \in M : \lim_{t\to-\infty} \phi_t(x) = r\}\,,$$

and the moduli spaces of gradient lines between two critical points, $q, r \in Crit(f)$,

$$\mathcal{M}(r,q) = S_q \cap U_r\,, \qquad \widetilde{\mathcal{M}}(r,q) = \frac{S_q \cap U_r}{\{x \sim y : \phi_t(x) = y \text{ for some } t \in \mathbb{R}\}}.$$

We define the orientation of the moduli spaces similar to that in Austin-Braam [1, Section 2.2]. For an oriented manifold M, we first specify an orientation for either the stable submanifolds, or equivalently, the unstable ones. The orientation of one type determines the other by the relation

$$[S_r][U_r] = [M]\,. \tag{A.2}$$

The orientation of the moduli space is then just the orientation of the transversal intersection which can be expressed as

$$[\mathcal{M}(r,q)] = [U_r][M]^{-1}[S_q] = [U_r][U_q]^{-1}\,. \tag{A.3}$$

We will also take as convention

$$[\mathcal{M}(r,q)] = [\widetilde{\mathcal{M}}(r,q)][\nabla f]\,. \tag{A.4}$$

In the special case when $ind(r) = ind(q) + 1$, $\mathcal{M}(r,q)$ is an oriented one-dimensional submanifold of gradient flow lines and $\widetilde{\mathcal{M}}(r,q)$ is an oriented collection of points. Also, recall that the Morse differential is defined by $\partial q = \sum_r n(r,q)\, r$ where

$$n(r,q) = \#\widetilde{\mathcal{M}}(r,q)\,. \tag{A.5}$$

It follows from (A.4) that $n(r,q)$ is equal to the number of gradient lines flowing in the direction of ∇f minus the number flowing in the direction of $-\nabla f$.

As an example of why (A.1) has the correct signs, we first prove the zero-form case with $\psi = h$, a function.

Corollary A.0.2. *If $h \in C^\infty(M)$, then $-c(dh) = \partial c(h) - c(h)\partial$.*

Proof. Evaluating $c(dh)$ by integrating over the gradient curves with orientation, we have

$$\begin{aligned} c(dh)q_k &= \sum_{r_{k+1}} \left(\int_{\overline{\mathcal{M}(r_{k+1}, q_k)}} dh \right) r_{k+1} \\ &= \sum_{r_{k+1}} \left(n(r_{k+1}, q_k)(h(r_{k+1}) - h(q_k)) \right) r_{k+1} \\ &= \sum_{r_{k+1}} h(r_{r+1}) n(r_{k+1}, q_k) r_{k+1} - \sum_{r_{k+1}} n(r_{k+1}, q_k) h(q_k) r_{k+1} \\ &= c(h)\partial q_k - \partial c(h) q_k = (c(h)\partial - \partial c(h)) q_k \end{aligned}$$

where $c(h)q_k = (\int_{\overline{\mathcal{M}(q_k,q_k)}} h)q_k = h(q_k)q_k$. Thus, having taken into account our orientation convention, we find that $-c(dh) = \partial c(h) - c(h)\partial$. ∎

To prove (A.1) in general, we re-express the right-hand side by Stokes' theorem

$$c(d\psi)q_k = \sum_{r_{k+\ell+1}} \left(\int_{\overline{\mathcal{M}(r_{k+\ell+1}, q_k)}} d\psi \right) r_{k+\ell+1} = \sum_{r_{k+\ell+1}} \left(\int_{\partial \mathcal{M}(r_{k+\ell+1}, q_k)} \psi \right) r_{k+\ell+1} .$$

The relevant components of $\partial \mathcal{M}(r_{k+\ell+1}, q_k)$ for integrating ψ consists of

$$\left\{ \bigcup_{p_{k+\ell}} \mathcal{M}(p_{k+\ell}, q_k) \times \widetilde{\mathcal{M}}(r_{k+\ell+1}, p_{k+\ell}) \right\} \bigcup \left\{ \bigcup_{p_{k+1}} \mathcal{M}(r_{k+\ell+1}, p_{k+1}) \times \widetilde{\mathcal{M}}(p_{k+1}, q_k) \right\} .$$

This implies up to signs

$$
\begin{aligned}
&c(d\psi)q_k\\
&=\sum_{r_{k+\ell+1}}\left[\pm\sum_{p_{k+\ell}}\int_{\overline{\mathcal{M}(p_{k+\ell},q_k)}\times\widetilde{\mathcal{M}}(r_{k+\ell+1},p_{k+\ell})}\psi \;\pm\; \sum_{p_{k+1}}\int_{\overline{\mathcal{M}(r_{k+\ell+1},p_{k+1})}\times\widetilde{\mathcal{M}}(p_{k+1},q_k)}\psi\right]r_{k+\ell+1}\\
&=\sum_{r_{k+\ell+1}}\left[\pm\sum_{p_{k+\ell}}\left(\int_{\overline{\mathcal{M}(p_{k+\ell},q_k)}}\psi\right)n(r_{k+\ell+1},p_{k+\ell})\pm\sum_{p_{k+1}}n(p_{k+1},q_k)\left(\int_{\overline{\mathcal{M}(r_{k+\ell+1},p_{k+1})}}\psi\right)\right]r_{k+\ell+1}\\
&=\pm\partial c(\psi)q_k\pm c(\psi)\partial q_k
\end{aligned}
\tag{A.6}
$$

To fix the signs, we will proceed in two steps. First, we make a choice of the orientation of the stable and unstable manifolds at the critical points $\{q_k, p_{k+1}, p_{k+l}, r_{k+l+1}\}$. By (A.3), this determines the orientation of the various moduli spaces that arise in the Stokes' theorem calculation above. Then in step two, we compare the orientation of the relevant boundary components, $\mathcal{M}(p_{k+\ell}, q_k) \times \widetilde{\mathcal{M}}(r_{k+l+1}, p_{k+l})$ and $\mathcal{M}(r_{k+\ell+1}, p_{k+\ell}) \times \widetilde{\mathcal{M}}(p_{k+1}, q_k)$, with the orientation needed to satisfy Stokes' theorem. The relative difference in the orientations will determine the signs in (A.6).

Step 1: Computing the orientation of the moduli spaces.

By (A.3), the orientation of a moduli space $\mathcal{M}(r, q)$ can be determined by the orientation of the unstable submanifolds U_r and U_q. Hence, we will write below our choice for the orientation for the relevant unstable submanifolds explicitly. (The orientation of the stable submanifolds of a critical point are then fixed by (A.2).) Similar to [1, Section 2.2], we will express the orientations in terms of orthonormal frame vectors grouped together by Clifford multiplication.

Let $e_1, \ldots, e_k$ be an orthonormal set of frame vectors that are shared by both U_{q_k} and $U_{r_{k+\ell+1}}$.

Let $e_{k+1}, \ldots, e_{k+\ell+1}$ be the additional frame vectors in $U_{r_{k+\ell+1}}$ defined such that they point in the direction away from q_k towards r_{k+l+1}, i.e. in the direction of ∇f. Then, for p_{k+1}, there is a vector $e_{i_{p_{k+1}}}$ that points along the gradient curve $\mathcal{M}(p_{k+1}, q_k)$ from q_k to p_{k+1}, and for $p_{k+\ell}$, there is a vector $e_{i_{p_{k+\ell}}}$ that points along the gradient curve $\mathcal{M}(r_{k+l+1}, p_{k+l})$ from $p_{k+\ell}$ to $r_{k+\ell+1}$. Note both $e_{i_{p_{k+1}}}$ and $e_{i_{p_{k+\ell}}}$ are defined to point in the direction of ∇f. See Figure A.1 below.

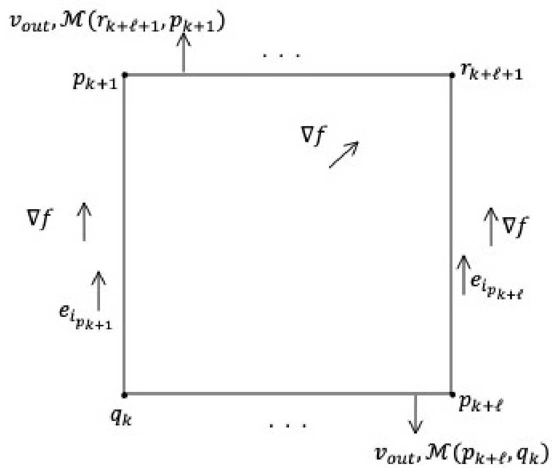

Figure A.1: $\mathcal{M}(r_{k+\ell+1}, q_k)$ with orientations.

Our choice for the orientation of the relevant unstable submanifolds are

$$[U_{q_k}] = e_k \ldots e_1\,, \qquad [U_{p_{k+\ell}}] = e_{k+\ell+1} \ldots \widehat{e_{i_{p_{k+\ell}}}} \ldots e_k \ldots e_1\,,$$
$$[U_{p_{k+1}}] = e_{i_{p_{k+1}}} e_k \ldots e_1\,, \qquad [U_{r_{k+\ell+1}}] = e_{k+\ell+1} \ldots e_k \ldots e_1\,.$$

Then by (A.3), $[\mathcal{M}(r,q)] = [U_r][U_q]^{-1}$, we find the orientations of the moduli spaces:

$$\begin{aligned} [\mathcal{M}(r_{k+\ell+1}, q_k)] &= (e_{k+\ell+1} \ldots e_k \ldots e_1)(e_1 \ldots e_k) = e_{k+\ell+1} \ldots e_{k+1}\,, \\ [\mathcal{M}(p_{k+\ell}, q_k)] &= (e_{k+\ell+1} \ldots \widehat{e_{i_{p_{k+\ell}}}} \ldots e_k \ldots e_1)(e_1 \ldots e_k) = e_{k+\ell+1} \ldots \widehat{e_{i_{p_{k+\ell}}}} \ldots e_{k+1}\,, \\ [\mathcal{M}(r_{k+\ell+1}, p_{k+1})] &= (e_{k+\ell+1} \ldots e_k \ldots e_1)(e_1 \ldots e_k e_{i_{p_{k+1}}}) \\ &= (-1)^{i_{p_{k+1}} - k - 1}\, e_{k+\ell+1} \ldots \widehat{e_{i_{p_{k+1}}}} \ldots e_{k+1}\,. \end{aligned} \tag{A.7}$$

And by (A.4), we also have

$$
\begin{aligned}
[\widetilde{\mathcal{M}}(r_{k+\ell+1}, p_{k+\ell})] &= [\mathcal{M}(r_{k+\ell+1}, p_{k+\ell})][\nabla f]^{-1} \\
&= (e_{k+\ell+1} \dots e_k \dots e_1)(e_1 \dots e_k \dots \widehat{e_{i_{p_{k+\ell}}}} \dots e_{k+\ell+1})(e_{i_{p_{k+\ell}}}) \\
&= (-1)^{k+\ell+1-i_{p_{k+\ell}}} \, , \\
[\widetilde{\mathcal{M}}(p_{k+1}, q_k)] &= [\mathcal{M}(p_{k+1}, q_k)][\nabla f]^{-1} \\
&= (e_{i_{p_{k+1}}} e_k \dots e_1)(e_1 \dots e_k)(e_{i_{p_{k+1}}}) = 1 \, .
\end{aligned}
$$

Hence, we find

$$[\mathcal{M}(p_{k+\ell}, q_k) \times \widetilde{\mathcal{M}}(r_{k+\ell+1}, p_{k+\ell})] = (-1)^{k+\ell+1-i_{p_{k+\ell}}} \, e_{k+\ell+1} \dots \widehat{e_{i_{p_{k+\ell}}}} \dots e_{k+1} \, , \tag{A.8}$$

$$[\mathcal{M}(r_{k+\ell+1}, p_{k+1}) \times \widetilde{\mathcal{M}}(p_{k+1}, q_k)] = (-1)^{i_{p_{k+1}}-k-1} \, e_{k+\ell+1} \dots \widehat{e_{i_{p_{k+1}}}} \dots e_{k+1} \, . \tag{A.9}$$

Step 2: Orientation of the boundary components, $\mathcal{M}(p_{k+\ell}, q_k) \times \widetilde{\mathcal{M}}(r_{k+\ell+1}, p_{k+\ell})$ and $\mathcal{M}(r_{k+\ell+1}, p_{k+1}) \times \widetilde{\mathcal{M}}(p_{k+1}, q_k)$, as specified by Stokes' theorem.

For a manifold N with boundary ∂N, Stokes' theorem holds only if the orientation of the boundary ∂N is chosen such that

$$[v_{out}][\partial N] = [N] \tag{A.10}$$

where v_{out} is the outward pointing normal on the boundary.

For the boundary component $\mathcal{M}(p_{k+\ell}, q_k) \times \widetilde{\mathcal{M}}(r_{k+\ell+1}, p_{k+\ell})$, the outward pointing normal at for instance $p_{k+\ell}$ can be expressed as (see Figure A.1)

$$v_{out,\mathcal{M}(p_{k+\ell},q_k)} = -e_{i_{p_{k+\ell}}} + \sum_{k+j\neq i_{p_{k+\ell}}} a_j e_{k+j} \,.$$

Therefore, the specified orientation from Stokes' theorem (denoted with a subscript 'S') is

$$\begin{aligned}
[\mathcal{M}(p_{k+\ell}, q_k) \times \widetilde{\mathcal{M}}(r_{k+\ell+1}, p_{k+\ell})]_S &= [v_{out,\mathcal{M}(p_{k+\ell},q_k)}]^{-1}[\mathcal{M}(r_{k+\ell+1}, q_k)] \\
&= (-e_{i_{p_{k+\ell}}})(e_{k+\ell+1} \dots e_{k+1}) \\
&= (-1)^{k+\ell+i_{p_{k+\ell}}}\, e_{k+\ell+1} \dots \widehat{e_{i_{p_{k+\ell}}}} \dots e_{k+1} \\
&= -[\mathcal{M}(p_{k+\ell}, q_k) \times \widetilde{\mathcal{M}}(r_{k+\ell+1}, p_{k+\ell})] \qquad \text{(A.11)}
\end{aligned}$$

having used (A.7) in the first line and (A.8) in the last line.

Similarly, for the boundary component $\mathcal{M}(r_{k+\ell+1}, p_{k+1}) \times \widetilde{\mathcal{M}}(p_{k+1}, q_k)$, the outward pointing normal at for instance p_{k+1} can be expressed as (see Figure A.1)

$$v_{out,\mathcal{M}(r_{k+\ell+1},p_{k+1})} = e_{i_{p_{k+1}}} + \sum_{k+j\neq i_{p_{k+1}}} a_j e_{k+j} \,.$$

This gives for the specified orientation from Stokes' theorem

$$\begin{aligned}
[\mathcal{M}(r_{k+\ell+1}, p_{k+1}) \times \widetilde{\mathcal{M}}(p_{k+1}, q_k)]_S &= [v_{out,\mathcal{M}(r_{k+\ell+1},p_{k+1})}]^{-1}[\mathcal{M}(r_{k+\ell+1}, q_k)] \\
&= (e_{i_{p_{k+1}}})(e_{k+\ell+1} \dots e_{k+1}) \\
&= (-1)^{k+\ell+1-i_{p_{k+1}}}\, e_{k+\ell+1} \dots \widehat{e_{i_{p_{k+1}}}} \dots e_{k+1} \\
&= (-1)^{\ell}[\mathcal{M}(r_{k+\ell+1}, p_{k+1}) \times \widetilde{\mathcal{M}}(p_{k+1}, q_k)] \qquad \text{(A.12)}
\end{aligned}$$

having used (A.7) in the first line and (A.9) in the last line.

Finally, with (A.11)-(A.12) and matching up with the corresponding terms in (A.6), we have

$$\begin{aligned} c(d\psi)q_k &= \sum_{r_{k+\ell+1}} \Bigg[-\sum_{p_{k+\ell}} \left(\int_{\overline{\mathcal{M}(p_{k+\ell},q_k)}} \psi \right) n(r_{k+\ell+1}, p_{k+\ell}) \\ &\qquad + \sum_{p_{k+1}} (-1)^{\ell} n(p_{k+1}, q_k) \left(\int_{\overline{\mathcal{M}(r_{k+\ell+1},p_{k+1})}} \psi \right) \Bigg] r_{k+\ell+1} \\ &= -\partial c(\psi) q_k + (-1)^{\ell} c(\psi) \partial q_k \end{aligned}$$

or equivalently, $-c(d\psi) = \partial c(\psi) + (-1)^{\ell+1} c(\psi)\partial$.

Appendix B

Proof of Long Exact Sequence Between Cone, Kernel, and Cokernel

To show we have the isomorphism, we start with the short exact sequence

$$0 \to (\ker(c(\omega)), -\partial) \xrightarrow{F} (\text{Cone}(c(\omega)), d_C) \xrightarrow{G} (\text{Cone}(\iota), d_\iota) \to 0$$

where $(Cone(\iota), d_\iota)$ is the cone chain complex arising from the inclusion $\iota : \text{Im } (c(\omega)) \to C^k$, i.e $\iota(c(\omega)b) = c(\omega)b$, $Cone(\iota) = C^k \oplus c(\omega)C^{k-1}$, and $d_\iota = \begin{pmatrix} d & \iota \\ 0 & -d \end{pmatrix}$. The map F is given by $F(s) = \begin{pmatrix} 0 \\ s \end{pmatrix}$ while $G \begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} t \\ c(\omega)b \end{pmatrix}$ it is clear that F is injective, G is surjective, and $\ker G = \text{Im } F$. It is also straightforward to check that $d_C(F(s)) = \begin{pmatrix} 0 \\ -\partial s \end{pmatrix} = F(-\partial s)$ and $Gd_C \begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} \partial t + c(\omega)b \\ -c(\omega)\partial b \end{pmatrix} = \partial_\iota G \begin{pmatrix} t \\ b \end{pmatrix}$ Thus these are chain maps, and thus we get the long sequence in cohomology. It remains to show that $H^k(\text{Cone}(\iota)) \cong H^k(\text{coker}(c(\omega)))$. To do this, define the map $H : Cone(\iota) \to \text{coker}(c(\omega))$ by $H \begin{pmatrix} t \\ b \end{pmatrix} = \{t\} \in \text{coker}(c(\omega))$. Thus $Hd_\iota \begin{pmatrix} t \\ c(\omega)b \end{pmatrix} = \{\partial t\} = \partial^m_\Pi\{t\} = \partial^m_\Pi H \begin{pmatrix} t \\ c(\omega)b \end{pmatrix}$). Hence we have a chain map.

We will show that $[H]$ is injective. To do this, note that if $\left[H \begin{pmatrix} t \\ c(\omega)b \end{pmatrix} \right] = [\{0\}] \in H^k(\text{coker}(c(\omega)))$, then this implies that $\{t\} \in [\{0\}]$, or $\{t\} = \partial^m_\Pi\{a\} = \{\partial a\}$, which implies $t - \partial a = c(\omega)r$. Note also that $\left[\begin{pmatrix} t \\ c(\omega)b \end{pmatrix} \right]$ being closed implies $-c(\omega)\partial b = 0$ and $\partial t + c(\omega)b = 0$. Note that $-c(\omega)b = \partial t = c(\omega)\partial r$ combining these equations, we find

$$\begin{pmatrix} \partial & \iota \\ 0 & -\partial \end{pmatrix} \begin{pmatrix} a \\ c(\omega)r \end{pmatrix} = \begin{pmatrix} \partial a + c(\omega)r \\ -c(\omega)\partial r \end{pmatrix} = \begin{pmatrix} t \\ -(-c(\omega)\partial b) \end{pmatrix} = \begin{pmatrix} t \\ c(\omega)b \end{pmatrix}$$

Thus $\begin{pmatrix} t \\ c(\omega)b \end{pmatrix}$ is exact, so $[H]$ is injective.

To show $[H]$ is surjective, note that if $[\{a\}] \in H^k(\text{coker}(c(\omega)))$, then $\partial a \in \{0\}$, i.e $\partial a = c(\omega)b$. Thus $c(\omega)\partial b = 0$, so if we consider $\begin{pmatrix} a \\ -c(\omega)b \end{pmatrix}$, then it is clear that $\partial_\iota \begin{pmatrix} a \\ -c(\omega)b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Hence $\left[\begin{pmatrix} a \\ -c(\omega)b \end{pmatrix} \right] \in H^k(\text{coker}(c(\omega)))$, and $[H] \left[\begin{pmatrix} a \\ -c(\omega)b \end{pmatrix} \right] = [\{a\}]$, so $[H]$ is surjective. Hence $[H]$ is a bijection between $H^k(Cone(\iota))$ and $H^k(\text{coker} c(\omega))$. Using $[H]$ and the Zig-Zag lemma, we thus have the following long exact seqeunce on cohomology

$$\dots H^{k-1}(\ker(c(\omega))) \xrightarrow{[F]} H^k(\text{Cone}(c(\omega)) \xrightarrow{[H][G]} H^k(\text{coker}(c(\omega))) \xrightarrow{[\delta][H]^{-1}} H^k(\ker(c(\omega))) \dots$$