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Controlling Interest Rate Risk and Return with Futures

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INTRODUCTION

Dramatic increases in interest rate levels and volatility since the early 1970s have motivated the creation of hedging instruments and immunization models. The creation of interest rate futures markets has enriched the hedging opportunities of participants in the debt instruments markets. The hedger's objective is the optimal control of inherent risk due to adverse interest rates changes. The hedger's problem is straightforward: a methodological choice of the hedging instruments and the optimal hedge ratio. In this paper methods using interest rates futures for controlling risk and return are proposed and demonstrated empirically.

The concept of duration, introduced for the first time by Macaulay (1938) has proven to be a useful measure of bond exposure to interest rate risk and consequently has been widely employed in immunization models. Originally, duration was defined as a weighted average of the payment periods where the weights are related to the present values of the payments of each period. The role of duration as a proxy for interest rate risk was originally proposed by Hicks (1939), and was rediscovered later by other authors. Fisher and Weil (1971) demonstrate that in the absence of default and taxes, an investor can immunize a bond portfolio against parallel interest rates changes by matching duration with holding period. Immunization is defined as obtaining a realized yield for a given period that is at least equal to the promised yield to maturity. Bierwag (1979) and Bierwag and Kaufman (1979) develop portfolio immunization policies for interest rates following multiplicative random shock stochastic processes.

Fong and Vasicek(1984) find sufficient conditions related to the magnitude of term structure change, for which lower bounds of the terminal portfolio value can be defined. Motivated by the inability of traditional specification of duration to provide a perfect proxy for basis risk in a realistic environment, Cox, Ingersoll and Ross(1979) propose a "stochastic duration". While stochastic duration better characterizes bond sensitivity to interest rates governed by mean-reverting elastic diffusion processes, its applicability is complicated by estimation and computation problems.

Traditional immunization techniques attempt to protect bond portfolios against a particular type of interest rate change by matching duration with the planning horizon. The achievement of duration target values is pursued by rebalancing the asset mix of the portfolio by selling bonds of undesirable duration and purchasing those of desired duration. This strategy often appears to be expensive due to relatively large transaction costs. Usually the target duration value changes before it is reached. Also, rebalancing a portfolio to achieve a desired duration is difficult in thin secondary markets.

The presence of interest rate futures contracts reduces the expense of duration rebalancing. Use of futures enables hedgers to substantially reduce or in some cases even eliminate the risk inherent in interest rate shifts. Traded for the first time in 1975 on the Chicago Board of Trade (CBOT) in the form of GNMA futures, interest rate futures contracts have proliferated and currently enjoy wide application as instruments for hedging interest rate risk. The primary problem of the current futures markets is that contract maturities only extend two years into the future.

The literature on hedging interest rate risk with futures is still developing. Ederington (1979) derived a mean-variance model to minimize variability of returns. Kolb and Chiang(1982), Gay and Kolb(1982) and Gay et al.(1983) extend the notion of duration from spot instruments to futures contracts. They define the duration of a interest rate futures contract as the duration of the underlying deliverable instrument. Much of the theory pertaining to the capacity of duration to measure the sensitivity to interest rates changes can be applied with some modifications to futures contracts. The duration approach proposed in the work by Kolb, Gay, and Chiang represents a promising methodology. Using the Kolb model as a starting point, Hillard(1984) constructs minimum variance hedge ratios for a portfolio of futures rather than one single contract. He derives necessary and sufficient conditions for the existence of zero variance hedges in a one-period framework.

The futures hedging models developed so far in the literature assume either deterministic interest rates changes or rate changes dependent on a single state variable. The model developed in the present paper takes into consideration the sensitivity of both spot and futures positions to changes of their implied yield. This allows measurement of the hedging performance of a given instrument with respect to a particular portfolio.

An important question any fixed income hedger faces is the specific contract to be used for risk control. Typically, there are no futures markets for the particular spot instrument to be hedged, so cross hedging is necessary. Often cross hedging is assumed to be the reason for all residual return variance. However, even if futures markets existed for all the instruments to be hedged, variations of futures prices are not perfectly correlated with spot prices. The basis (difference between the futures price and spot price) varies and is a source of uncertainty which also must be controlled. Contract choice is also important to the minimization of this basis risk.

Many futures hedging strategies developed to date are set in a one-period framework. However, for the practitioner, hedging is not a one-period task. The asset mix of the portfolio and the maturity structure change with time, and thus hedge ratios need to be readjusted. Rollover of positions in the futures markets is also necessary because of the inavailability of contracts with settlement dates further than two years into the future. This paper demonstrates continuously adjusted hedging strategies where these sources of risk (and return) are controlled using available instruments.

The plan of the paper is as follows. Section II presents a continuous time, stochastic model which leads to a description of the value of a spot-plus-futures bond portfolio. Section III presents strategies for controlling interest rate trend and volatility, respectively. Section IV presents an empirical performance test of the proposed strategy. In Section V, we give some brief concluding remarks.

II. THE MODEL

Let B denote a bond with a continuous cash flow schedule $c:(0,T) \rightarrow R+$. The implied rate of return at time t equates the net present value of the generated cash flow to its market price P(t). It is the solution r_B to the implicit equation:

$$P(t) = \int_t^M e^{-r_B(t)(s-t)} c_B(s)ds \qquad (1)$$

While the payment rate schedule c is defined at the time the bond is issued, and typically remains unchanged throughout the life of the security, the market price, P(t), varies to accommodate for changes of the implied yield which in equilibrium must equal the market yield for the remaining time to maturity.

The yield to maturity $r_B(t)$, is assumed to follow a diffusion process of the type:

$$dr_B(t) = f_B(r)dt + \sigma_B dz_1 \qquad (2)$$

where the drift term $f_B(r)$ and σ_B represent the instantaneous drift and volatility of the process, and dz represents drawings from a normal density function with mean zero and variance dt.

This is similar to Cox, Ingersoll and Ross (CIR),(1985), who develop a general equilibrium framework for the term structure of interest rates and show the spot rate to follow a mean-reverting diffusion process. The drift term $f_B(r)$ is of the form:

$$f_B(r) = k(\theta - r) \tag{3}$$

and defines an elastic random walk. For $k,\theta > 0$ the interest rates dynamics correspond to a continuous time first order autoregressive process where the randomly moving interest rate is elastically pulled toward a central location θ. The parameter k defines the speed of adjustment.

As the rate r_B changes, so does the price of the security, and the holder of the bond incurs a price risk due to interest rate volatility. This paper describes the nature of the stochastic process followed by the value of a spot bond portfolio to which interest rates futures positions have been added. Hedging is done by controlling the parameters which define the stochastic evolution of the portfolio value.

Denote by $F_{t,T}$ the price of the futures contract at time t, with a settlement date T, and denote by n_t the number of contracts at time t. A positive n_t indicates a long position in the contract, while a negative corresponds to a short position.

The contractual yield implied by the future's price is the solution, $r_F(t)$, to the equation:

$$F_{t,T} = \int_T^M e^{-r_F(t)(s-T)} c_F(s)ds \tag{4}$$

This implied yield $r_F(t)$ of the futures underlying instrument can be viewed as the future yield expected to prevail at time T for the deliverable bond. For a long time it was thought that in the absence of arbitrage opportunities for an equivalent period of time futures rates must equal forward rates implied in the term structure of interest rates. CIR, et al (1983) showed that whenever interest rates are stochastic, and non-parallel rate shifts occur along the term structure, forward and futures price will differ. However, Rendleman and Carabini (1979) and Cornell and Reinganum (1981) conclude that arbitrage opportunities based on empirical differences in forward and futures rates are not readily available, as the two rates are nearly equal.

More important is the relation between the yield to maturity on a bond and on a related future. While r_B is not equal to r_F, variations of the two rates are highly correlated. We assume that r_F follows a diffusion process of the same nature as r_B but with parameters of its own.

$$dr_F(t) = f_F(r_F)dt + \sigma_F dz_2 \tag{5}$$

From (2) it follows that the variances of the instantaneous spot rate and that of the futures implied yield are:

$$Var(dr_B) = Var(f_B dt + \sigma_B dz_1) = \sigma^2{}_B dt \tag{6}$$

$$Var(dr_F) = \sigma^2{}_F dt, \tag{7}$$

respectively.

The correlation coefficient of the bond rate variation and contractual implied rate variation is assumed to be constant in time:

$$\rho_{BF} = \frac{Cov(dr_B, dr_F)}{\sigma_B \sigma_F dt} \tag{8}$$

The coefficient ρ_{BF} describes the extent to which the implied yield to maturity on the bond is correlated with implied yield to maturity on the futures contract. The greater the value of this coefficient the better the capability of the specific futures contract to control risk. The correlation coefficient can be estimated based on historical data. Making the notational substitution :

$$\sigma_{BF} = \rho_{BF}\ \sigma_B \sigma_F \tag{9}$$

it follows that the covariance of implied rate variations can be written as:

$$Cov(dr_B, dr_F) = \sigma_{BF} dt \tag{10}$$

Given the stochastic differential equation (2) governing the evolution of the rate r_B, the partial differential equation satisfied by the value, P, of the bond can be derived. Apply Ito's lemma to the function:

$$P(t) = P(r_B(t), t)$$

and obtain

$$dP(t) = \frac{\partial P}{\partial t} dt + \frac{\partial P}{\partial r_B} dr_B + 1/2 \frac{\partial^2 P}{\partial r_B^2} (dr_B)^2 \tag{11}$$

It follows immediately that

$$\frac{\partial P}{\partial t} = -c_B(t) + r_B(t)P(t) \tag{12}$$

The above expression states that, ceteris paribus, time variation induces a bond price decrease proportional to the cash flow rate and price increase over any time interval proportional to the promised rate of return.

Calculating the other partial derivatives involved in equation (11), we obtain:

$$\frac{\partial P}{\partial r_B} = \frac{\partial}{\partial r_B} \left(\int_t^M e^{-r_B(t)(s-t)} c_B(s)ds\right) =$$
$$= \int_t^M (t-s)\ e^{-r_B(t)(s-t)} c_B(s)ds \tag{13}$$

The continuous time equivalent of Macaulay's duration is defined as:

$$D_B(t) = \frac{\int_t^M (s-t)\ e^{-r_B(t)(s-t)} c_B(s)ds}{P(t)} \tag{14}$$

Substitution of (14) into (13) yields:

$$\frac{\partial P(t)}{\partial r_B} = -D_B(t)\cdot P(t) \tag{15}$$

Equation (18) states that variations of the bond price induced by instantaneous rate changes are proportional to the bond duration and of opposite sign. This result is consistent with the view of duration as a measure of price sensitivity to interest rates changes as described by CIR (1979).

The second order derivative in equation (11) is

$$\frac{\partial^2 P}{\partial r_B^2} = {}_t\int^M (s-t)^2 \, e^{-r_B(t)(s-t)} c_B(s)ds \tag{16}$$

One alternate specification for duration proposed by Cooper(1979) second order effects is:

$$H_B(t) = \left({}_t\int^M (s-t)^2 \, e^{-r_B(s-t)} c_B(s)ds\right)/P(t) \tag{17}$$

Cooper finds duration H to be a meaningful measure of interest rate sensitivity when term structure changes differ substantially from parallel shifts. Duration H describing the second order interest rate effect is related to volatility of rate changes. Substituting (17) into (16),

$$\frac{\partial^2 P}{\partial r_B^2} = H_B(t)P(t) \tag{18}$$

A relationship analogous to equation (11) holds true for the value variation induced by implied yield changes in the futures market. For each contract held long the future's price variation is:

$$dF(t) = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial r_F} dr_F + 1/2 \; \frac{\partial^2 F}{\partial r_F^2} (dr_F)^2 \tag{19}$$

where

$$\frac{\partial F(t)}{\partial t} = \frac{\partial}{\partial t} \; {}_T\int^M e^{-r_F(t)(s-T)} c_F(s)ds = 0 \tag{20}$$

Equation (20) states that, ceteris paribus, the passage of time does not change the quote price of the contract since all factors affecting the contract price pertain to time past the delivery date T and thus remain unchanged.

$$\frac{\partial F}{\partial r_F} = \frac{\partial}{\partial r_F} \left(\int_T^M e^{-r_F(t)(s-T)} c_F(s) ds \right) \tag{21}$$

In a continuous time framework the futures contract duration (see Kolb and Chiang (1980), Gay and Kolb (1983) and Kolb (1985)), calculated at the contractual implied yield is:

$$D_F(t) = \frac{\left(\int_T^M e^{-r_F(t)(s-T)} c_F(s) ds \right)}{F(t)} \tag{22}$$

so

$$\frac{\partial F(t)}{\partial r_F} = -D_F(t) F(t) \tag{23}$$

The second derivative of the futures contract quote price with respect to the rate r_F is

$$\frac{\partial^2 F}{\partial r_F^2} = H_F(t) F(t) \tag{24}$$

where $H_F(t)$ represents the Cooper duration of the futures deliverable instrument. Now we have all the elements necessary to compute the variation of the hedge portfolio composed of bond B and positions in the futures contract F. While no investment is made in the futures market at time the position is assumed, price variations in that market induce value changes of the spot and futures portfolio. Denote by dV the value variation of the spot-futures portfolio. Then,

$$dV = dP + n_t \; dF \tag{25}$$

where n_t represents the number of contracts bought or sold. After substituting (11) and (19) into (25), the spot-futures portfolio price variation satisfies the following stochastic differential equation:

$$\begin{aligned} dV = {} & [-C_B(t) + r_B(t)P(t) - D_B(t)P(t)f_B(t) - n_t D_F(t)F(t)f_F(t) \\ & + 1/2\; H_B(t)P(t)\sigma_B^2 + 1/2\; n_t H_F(t)F(t)\sigma_F^2]\; dt \\ & - D_B(t)P(t)\sigma_B dz_1 - n_t D_F(t)F(t)\sigma_F dz_2 \end{aligned} \tag{26}$$

The portfolio variation dV has a deterministic and a random component. The deterministic part varies linearly with a drift factor and the length of the time variation dt. The random component, depends on the variations dz_1 and dz_2. Given that the random terms dz_1 and dz_2 are normally distributed with mean zero and variance dt, the instantaneous variance of the random part of equation (26) can be calculated as follows:

$$\begin{aligned} & Var(D_B(t)P(t)\sigma_B dz_1 + n_t D_F(t)F(t)\sigma_F dz_2) = \\ & D_B^2(t)P^2(t)\sigma_B^2 dt + n_t^2 D_F^2(t)F^2(t)\sigma_F^2 dt + \\ & 2D_B(t)D_F(t)P(t)F(t)\sigma_B\sigma_F \; Cov(dz_1,\; dz_2) \end{aligned} \tag{27}$$

It can be easily shown that $Cov(dr_B, dr_F) = \sigma_B \sigma_F \, Cov(dz_1, dz_2)$ and $Cov(dz_1, dz_2) = \rho_{BF} dt$.

It is convenient to write the stochastic differential equation (26) in terms of portfolio instantaneous return, namely:

$$\tilde{R}_p(dt) = \frac{dV + c_B(t)dt}{P(t)} \tag{28}$$

The instantaneous rate of return of the portfolio is equal to the ratio of the total capital gain on the portfolio (spot and futures position) plus interest cash flow to the value of the portfolio. The market value of the spot-futures portfolio is equal to the market value of the bond since no capital outlay is required to take a position in the futures contract. Substituting (28) and (27) into (29) it follows that the instantaneous return $\tilde{R}_p(dt)$ over an infinitesimal time interval dt can be written in the form of a stochastic differential equation as follows:

$$\tilde{R}_p(dt) = \mu_p(t, n_t)\, dt + \sigma_p \, dz_3 \tag{29}$$

where the drift term $\mu_p(t, n_t)$ represents the instantaneous expected rate of return, the term $\sigma_p(t, n_t)$ represents the volatility of the instantaneous rate of return and dz_3 is a random variable distributed $N(0,dt)$. The specification of the expected instantaneous return and volatility are as follows:

$$\mu_p(t, n_t) = r_B(t) - D_B(t)f_B(t) - n_tD_F(t) \cdot \frac{F(t)}{P(t)} f_F(t) \tag{30}$$

$$+ 1/2\ H_B(t)\sigma_B^2 + 1/2\ n_tH_F(t) \frac{F\ (t)}{P\ (t)} \sigma_F^2$$

and

$$\sigma_p^2(t, n_t) = D_B^2(t)\sigma_B^2 + n_t^2D_F^2 \frac{F^2(t)}{P^2(t)}\sigma_F^2 \tag{31}$$

$$+ 2\ D_B(t)D_F(t)\ n_t\frac{F(t)}{P(t)}\sigma_{BF}$$

The following section analyzes the hedger's ability to control the expected instantaneous portfolio return and volatility of instantaneous portfolio return.

III RISK-RETURN CONTROL

Expression (30) portrays the instantaneous expected rate of return of a spot-futures portfolio as a function of the number of contracts assumed. The expected rate of return depends on the implied promised rate of return. The higher this promised rate of return, the higher the instantaneous rate on the portfolio. On the other hand, expected return is also dependent on the interest rate drift. In an increasing rates environment, the position in the spot market will induce a decrease in portfolio return. If interest rates display a tendency to rise, ($f_B(t)>0$), the value of the hedge portfolio tends to decrease and thus reduce its expected rate of return. This explains why the factor multiplying the interest rates trend factor $f_B(t)$ in equation (30) has a negative sign. The contribution of the bond drift to the hedge portfolio's expected return is proportional to the bond sensitivity to rate changes as measured by the Macaulay duration.

A similar remark applies for the position in the interest rates futures market. Here however, the contribution of contractual implied rate trend depends on the futures contract duration $D_F(t)$, on the ratio of contract value to bond market value, and position size in the futures market. A long position coupled with a positive interest rates trends ($f_B(t)>0$) will contribute to a decrease in expected rate of return on the portfolio, while, a short position will decrease the trend effect induced by the long spot position.

The expected return variation induced by interest rate trend is:

$$-D_B(t)f_B(t) - n_t D_F(t) \cdot \frac{F(t)}{P(t)} f_F(t) \qquad (32)$$

Based on (32) a hedge ratio to reduce the effects of interest rate trend changes on portfolio instantaneous return can be derived.

$$n_t = - \frac{D_B(t)P(t)f_B(t)}{D_F(t)F(t)f_F(t)} \qquad (33)$$

Holding this number of futures contracts controls the hedge portfolio's instantaneous expected return. Assuming that the trend factor in the spot market equals the trend of the futures implied rate of return ($f_B = f_F$), the hedge ratio (33) becomes:

$$n_t^* = - \frac{D_B(t)P(t)}{D_F(t)F(t)} \qquad (34)$$

If applied in a one-period framework, the hedge ratio of (34) is compatible with the duration-based ratio proposed by Kolb(1985). However, we can see this is not the ratio which minimizes portfolio risk.

The sensitivity of expected portfolio return to interest rate volatility is measured by the Cooper duration, H. As interest rate volatility increases, so does the expected return for holding the portfolio to compensate the investor for bearing additional risk. The extent by which the expected return is increased is proportional to the Cooper duration H, and the position in the futures market n_t. The change in expected instantaneous return induced by interest rate volatility is:

$$1/2H_B(t)\sigma_B^2 + 1/2n_t H_F(t)\frac{F(t)}{P(t)}\sigma_F^2 \qquad (35)$$

The volatility of the instantaneous rate of return of the spot-futures portfolio as measured by its variance is indicated in (31). It depends on the volatility of the spot and futures implied yields, namely and σ_B and σ_F, and is a function of the spot and futures duration, and of the position in the futures market. Note that the drift terms, f_B and f_F, defining the stochastic differential equations for the two rates cancels from the expression for the portfolio's instantaneous rate volatility. Thus, portfolio rate volatility depends on interest rate volatility, but does not depend on interest rate drift.

Equation (31) is used to construct a hedge strategy for minimization of instantaneous rate of return volatility.

$$\frac{\partial\ \sigma_p^2(t,n_t)}{\partial n_t} = 2D_B(t)\ \frac{F(t)}{P(t)}\ \sigma_{BF} + 2n_t D_F\ \frac{F^2(t)}{P^2(t)}\ \sigma_F^2 \qquad (36)$$

To minimize the return variance, set $\partial\sigma_p^2/\partial n_t$ equal to zero and solve for n_t. This gives the minimum instantaneous return variance hedge ratio:

$$n_t^{**} = -\frac{D_B(t)P(t)}{D_F(t)F(t)} \cdot \frac{\sigma_{BF}}{\sigma_F^2} \tag{37}$$

The hedge ratios n_t^* and n_t^{**} differ by an adjustment factor which takes into consideration the covariance and volatility of the spot and futures implied rate. Using (8), the minimum variance hedge ratio of (37) becomes:

$$n_t^{**} = n_t^* \rho_{BF} \cdot \frac{\sigma_B}{\sigma_F} \tag{38}$$

which means that the minimum variance hedge ratio can be obtained from the ratio for controlling interest rate trend by multiplying it with an adjustment factor. The adjustment factor is proportional to the correlation between spot and futures rates. If no correlation exists between the two rates, the adjustment factor is zero, and the instrument is not suitable for risk control. On the other hand, the higher the correlation between the two rates, the greater the hedge ratio. The adjusted hedge ratio is also dependent on the variability of the spot rate. The greater the volatility, σ_B, of the spot rate, the greater the hedging needs, and thus the higher the adjustment factor. The volatility of the futures implied rate is also important. If the futures contract has little volatility, then more contracts will be required for the hedge. This can be seen from the adjustment factor, where the parameter σ_F appears as the denominator. Performance of this hedge ratio is tested empirically in the next section of the paper.

To find the minimum return volatility substitute the hedge ratio of (38) into (31) and using (8), it follows that:

$$\min(\sigma_p^2) = \sigma_p^2(n_t^{**}) = D_B^2(t)\sigma_B^2 \ (1-\rho_{BF}) \tag{39}$$

This shows that the correlation between the bond and the future are a measure of hedging adequacy. $\rho_{BF} = 1$ is a well known necessary and sufficient condition for existence of a perfect hedge (ie zero instantaneous portfolio rate variance). The coefficient ρ_{BF} is a measure of the quality of the "cross" hedge provided by a certain futures contract with respect to a particular bond or bond portfolio. The closer to 1 the coefficient ..., the lower the instantaneous return volatility. For instance when a Treasury Bill is hedged with a T-Bill futures contract, this coefficient is expected to be close to 1. However, differences between the spot yield and futures yield do exist which make the coefficient close to, but different from 1. This difference accounts for a portion of the "unhedgeable" risk, the remainder being the basis variation as the futures contracts are rolled over. The next section empirically documents the achievable control over risk and return.

IV EMPIRICAL EVIDENCE

To test the properties of the proposed hedging strategies, an empirical study was conducted. The hedge of a long term government bond with a coupon of 4.5% annual interest payable semiannually has been simulated for the period 1/30/80 thru 4/30/85. The bond used for illustration was issued in June 1960 and expired in August, 1985. The bond data were provided by the Center for Research In Security Prices (CRSP), University of Chicago, while the quotes for interest rates futures contracts were extracted from a Chicago Board of Trade (CBOT) tape.

Four risk-return control strategies are compared: 1) unhedged bond, 2) dynamic control of interest rates trend, 3) dynamic control of interest rate volatility, and 4) fixed hedge ratio strategy.

Because of the long term maturity of the bond it is expected that futures written on long term debt instruments will provide a higher rate correlation. Consequently, T-Bond futures contracts have been selected for the hedging strategies. The T-Bond futures contract is traded on the CBOT exchange. Contracts are available for delivery months March, June, September and December and the quoted price is for a 15 year 8% coupon T-Bond. There are at all times 12 contracts corresponding to settlement dates for the next two years.

No single contract covered the entire period under study. Consequently, three contracts which differ in their settlement dates have been used as follows: contract F 1 deliverable in June of 1982 for hedging from 01/31/80 thru 03/31/80, contract F 2 deliverable in December of 1982 for period 03/31/80 thru 09/30/82 and contract F 3 deliverable in June of 1985 for the period 09/30/82 thru 03/29/85.

The studied period has been divided into 63 monthly subperiods at which times the market value has been assessed based on the quoted bond price. Column 2 of Table I lists the quote dates for each period. Bond market value and interest paid are listed in columns 3 and 4 of the same table.

Annualized promised rates of return for the bond held to maturity have been calculated at the beginning of each period based on market price and coupon schedule. The results are listed in column 5 of Table I which indicate the evolution in time of rate r_B, while Figure 1 represents it graphically. As one can easily see the covered period displays a large volatility, rates ranging from 9.48% to 16.33%. The actual rates of return of the unhedged bond (column 6 of Table I) were calculated based on expected market prices and interest paid each month as:

$$R_u^i = \frac{P_{i+1} - P_i + I_i}{P_i}, \quad i=1,\ldots N \tag{40}$$

where P_i, P_{i+1} represent market prices of the bond at times i and i+1 respectively, and I_i represents the interest payment during period i. Actual annualized returns of the unhedged bond are listed in column 6 of Table 1. The deviations of actual returns from promised returns calculated as:

$$R_{uD}^i = R_u^i - r_B^i, \quad i=1,\ldots N \tag{41}$$

are tabulated in column 7 of the same table. The bubbles in Figure 2 represent graphically the actual annualized returns while the continuous line represents the evolution of the promised rate r_B. As one can see, actual annualized returns differ from promised returns and therefore hedging appears to be warranted.

Two continuously balanced hedging strategies are simulated, both using the same contracts but differing in the specification of their hedge ratio. One strategy rebalances the spot-futures portfolio on a monthly basis using control of dynamic portfolio trend as in (34), and a second strategy uses dynamic control of portfolio volatility as in (37). The performance of the two strategies are compared with each other and with the unhedged and fixed ratio strategies.

The parameters σ_B, σ_F, σ_{BF} and ρ_{BF} have been estimated as described below. The implied rate variations:

$$\Delta r_B^i = r_B^{i+1} - r_B^i. \; i=1,\ldots N \tag{42}$$

and

$$\Delta r_F^i = r_F^{i+1} - r_F^i, \; i=1,\ldots N \tag{43}$$

Estimation of the volatilities σ_B and σ_F are based on equations (6) and (7). Classical inference techniques have been used to derive the parameters σ_B, σ_F, ρ_{BF} and Adj with the following properties:

$$\text{Var}(\Delta r_B) = \sigma_B^2 \Delta t \qquad \text{Cov}(\Delta r_B, \Delta r_F) = \sigma_{BF}^2 \Delta t$$

$$\text{Var}(\Delta r_F) = \sigma_F^2 \Delta t \qquad \rho_{BF} = \text{Cov r}(\Delta r_B, \Delta r_F) \tag{44}$$

$$\text{Adj} = \rho_{BF} \cdot \frac{\sigma_B}{\sigma_F}$$

Table 2 indicates the estimated values of the parameters needed for definition of the hedge ratios. Column 6 of Table 2 tabulated the correlation between the bond and the future, ρ_{BF}, which erodes with the passage of time. As the bond maturity becomes shorter, bond rates and futures contract rates become less correlated. The resulting adjustment factors for each of the three contracts are indicated in column 7 of Table 2.

The actual hedge ratios are indicated in columns 2 of Tables 3 and 4. The absolute value of the hedge ratio decreases as time to maturity decreases. This can be explained by the fact that with the elapse of time, bond duration decreases. Consequently actual bond returns become less sensitive to interest rate changes, therefore reducing the number of contracts which need to be sold.

Gains and losses incurred in each period are indicated in Tables 3 and 4. They have been computed based on the position assumed in the futures market and quote price spread between the beginning and end of the period. Actual returns for both strategies have been calculated based on bond market prices, interest paid and gain (loss) in the futures market. Actual returns and deviations from promised returns are tabulated in columns 5 and 6 of Table 3 and columns 4 and 5 of Table 4.

For this sample period, in the unhedged case, on average the actual return was more than 4% below the promised yield to maturity. Both hedging strategies produce an average excess return above the promised rate of return. While the portfolio trend strategy generates an "average" excess return of almost 4%, the portfolio volatility strategy produces an "average" excess return of over 8%. This occurs in a sample period which covers 5 years with interest rate trends in both directions (see Figure 1), but the overall tendency was for a rate decline.

Both hedging strategies reduce the variance of the rate deviation. The volatility strategy 3 reduces the variance of rate deviation by 53.61%, while the trend strategy 2 reduces the variance of rate deviation by 49.92%. The numbers are summarized in Table 5.

To determine how our dynamic hedging strategies compare with fixed hedge ratio strategies we calculated the ex-post variance reduction for a set of hedge ratios in the interval (0,-1.8). The results are represented graphically in Figure 3. The two horizontal lines mark a better level of variance reduction for both strategies 2 and 3 compared to the fixed hedge ratio strategy. The maximum variance reduction for the static strategy was below 43%. Even ex-post are we unable to find a fixed hedge ratio which outperforms the continuously updated strategies.

IV CONCLUDING REMARKS

Assuming nominal spot and contractual futures implied rates to follow elastic mean-reverting continuous time stochastic processes, the stochastic nature of the value of a spot-futures bond portfolio is determined using Ito's lemma. Continuously adjusted hedge strategies for controlling portfolio risk and return are derived and empirically analyzed.

Performance of four strategies are compared: 1) no hedge, 2) continuous control of interest rate trend, 3) continuous control of interest rate volatility and 4) fixed hedge ratio. We find the dynamic strategies (2 and 3) outperform the no-hedge strategy not only with respect to risk measures, but also in terms of average expected return for this sample period. The strategy for control of interest rate volatility (strategy 3) outperformed the strategy for control of interest rate trend (strategy 2) in terms of both risk and average expected return. Both dynamic strategies clearly outperform the fixed hedge ratio approach. No ex-post fixed hedge ratio outperformed any of the proposed ex-ante dynamic strategies in terms of risk reduction.

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FIGURE 1

BOND PROMISED YIELD TO MATURITY

As Implied by Coupon and Market Price

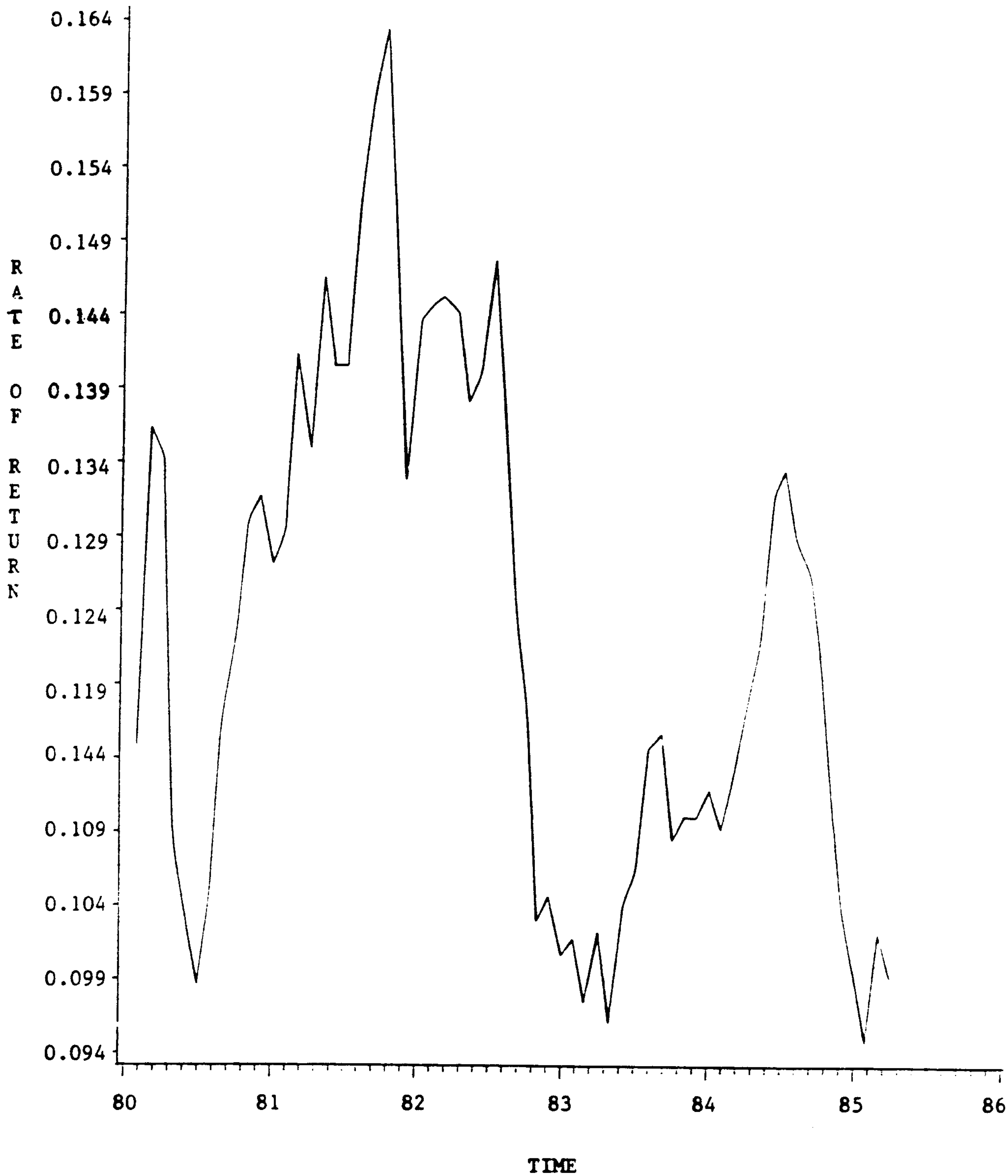

FIGURE 2

EX-POST MONTHLY (ANNUALIZED) RETURNS

Of The Unhedged Portfolio

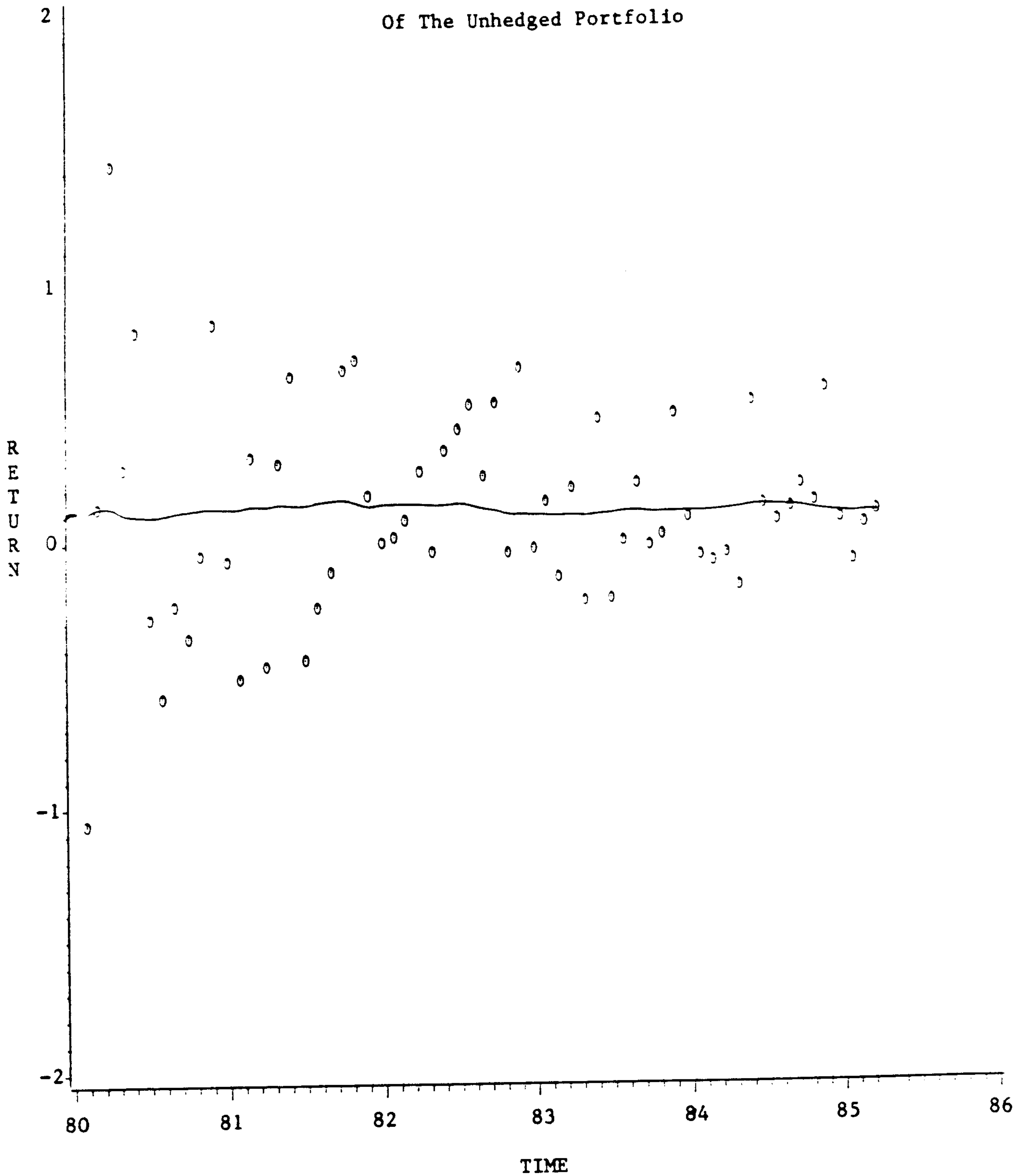

FIGURE 3

EX-POST DEVIATION VARIANCE REDUCTION

As a Function of Fixed Hedge Ratio

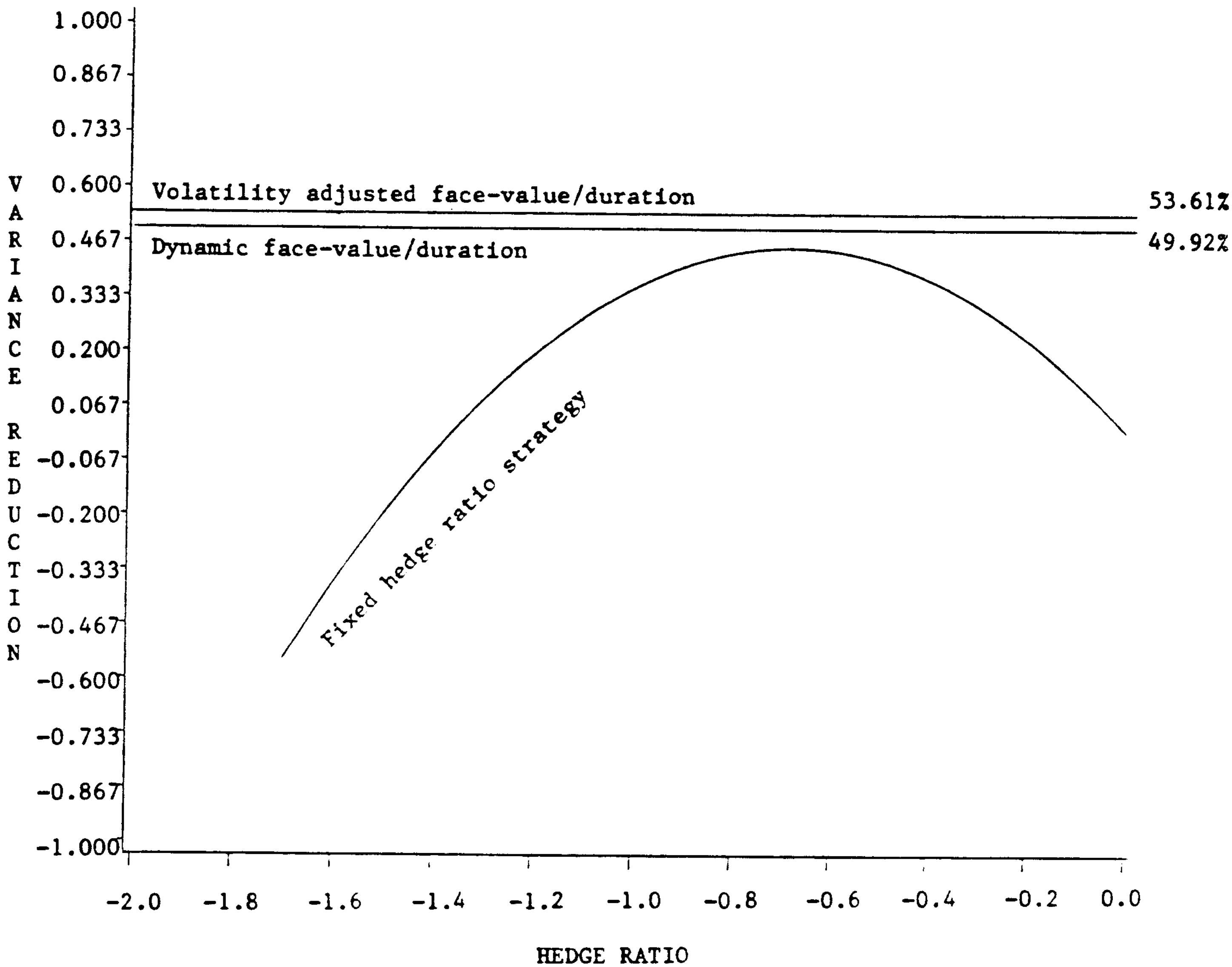

TABLE 1

PROMISED RETURN FOR UNHEDGED PORTFOLIO
Strategy 1

Period No.	Quote Date	Bond Value	Interest	Implied Rate r_B	*Ex Post* Rate R_u	Rate Deviation $R_u - r_B$
1	800131	85437.50	0.00	0.1146	-0.0622	-1.1768
2	800229	77875.00	0.00	0.1363	0.1348	-0.0015
3	800331	78750.00	0.00	0.1343	1.4286	1.2943
4	800430	88125.00	0.00	0.1086	0.2809	0.1723
5	800530	90187.50	3937.50	0.1035	0.7983	0.6948
6	800630	92250.00	0.00	0.0986	-0.2846	-0.3831
7	800731	90062.50	0.00	0.1044	-0.5829	-0.6873
8	800829	85687.50	0.00	0.1166	-0.2363	-0.3529
9	800930	84000.00	0.00	0.1220	-0.3571	-0.4791
10	801030	81500.00	0.00	0.1301	-0.0460	-0.1761
11	801128	81187.50	3937.50	0.1317	0.8268	0.6951
12	801231	82843.75	0.00	0.1271	-0.0679	-0.1950
13	810130	82375.00	0.00	0.1292	-0.5099	-0.6390
14	810227	78875.00	0.00	0.1413	0.3233	0.1820
15	810331	81000.00	0.00	0.1349	-0.4630	-0.5979
16	810430	77875.00	0.00	0.1465	0.2986	0.1520
17	810529	79812.50	3937.50	0.1405	0.6296	0.4891
18	810630	80062.50	0.00	0.1405	-0.4403	-0.5808
19	810731	77125.00	0.00	0.1521	-0.2431	-0.3952
20	810831	75562.50	0.00	0.1593	-0.1092	-0.2685
21	810930	74875.00	0.00	0.1633	0.6511	0.4878
22	811030	78937.50	0.00	0.1486	0.6888	0.5403
23	811130	83468.75	3937.50	0.1328	0.1752	0.0424
24	811231	80750.00	0.00	0.1437	0.0000	-0.1437
25	820129	80750.00	0.00	0.1447	0.0186	-0.1261
26	820226	80875.00	0.00	0.1452	0.0835	-0.0617
27	820331	81437.50	0.00	0.1442	0.2671	0.1229
28	820430	83250.00	0.00	0.1381	-0.0360	-0.1741
29	820528	83000.00	3937.50	0.1400	0.3434	0.2033
30	820630	81437.50	0.00	0.1477	0.4236	0.2760
31	820730	84312.50	0.00	0.1369	0.5159	0.3790
32	820831	87937.50	0.00	0.1235	0.2473	0.1238
33	820930	89750.00	0.00	0.1172	0.5223	0.4050
34	821029	93656.25	0.00	0.1030	-0.0400	-0.1430
35	821130	93343.75	3937.50	0.1047	0.6548	0.5501
36	821230	94500.00	0.00	0.1007	-0.0238	-0.1245
37	830131	94312.50	0.00	0.1018	0.1511	0.0493
38	830228	95500.00	0.00	0.0975	-0.1335	-0.2310
39	830331	94437.50	0.00	0.1023	0.2025	0.1002
40	830429	96031.25	0.00	0.0961	-0.2226	-0.3187
41	830531	94250.00	3937.50	0.1042	0.4615	0.3573
42	830630	93937.50	0.00	0.1062	-0.2156	-0.3218
43	830729	92250.00	0.00	0.1147	0.0016	-0.1131
44	830831	92262.44	0.00	0.1157	0.2179	0.1021
45	830930	93937.50	0.00	0.1085	-0.0160	-0.1245
46	831031	93812.50	0.00	0.1101	0.0240	-0.0861
47	831130	94000.00	3937.50	0.1100	0.4787	0.3687
48	831230	93812.50	0.00	0.1119	0.0879	-0.0240
49	840131	94500.00	0.00	0.1092	-0.0556	-0.1648
50	840229	94062.50	0.00	0.1126	-0.0797	-0.1924
51	840330	93437.50	0.00	0.1176	-0.0482	-0.1657
52	840430	93062.50	0.00	0.1214	-0.1733	-0.2947
53	840531	91718.75	3937.50	0.1320	0.5233	0.3914
54	840629	91781.25	0.00	0.1335	0.1348	0.0013
55	840731	92812.50	0.00	0.1286	0.0727	-0.0558
56	840831	93375.00	0.00	0.1267	0.1205	-0.0062
57	840928	94312.50	0.00	0.1217	0.2068	0.0851
58	841031	95937.50	0.00	0.1112	0.1407	0.0295
59	841130	97062.50	3937.50	0.1037	0.5679	0.4642
60	841231	97718.75	0.00	0.0993	0.0768	-0.0226
61	850131	98343.75	0.00	0.0948	-0.0839	-0.1787
62	850228	97656.25	0.00	0.1022	0.0538	-0.0485
63	850329	98093.75	0.00	0.0992	0.1032	0.0040

TABLE 2

VOLATILITY PARAMETERS OF FUTURES CONTRACTS

Contract Deliverable	Estimation Period	σ_B	σ_F	σ_{BF}	ρ_{BF}	Adjust Factor
9/82	1/31/80-08/31/82	.03425	.020087	.000611	.88805	1.514489
12/82	03/31/80-11/30/82	.03227	.019651	.000556	.87692	1.44004
6/85	09/30/82-04/30/85	.02001	.017564	.000306	.87142	.9928

$\sigma_B^2 = (Var(\Delta r_B))/\Delta t$ = Instantaneous volatility parameter for bond interest rate.

$\sigma_F^2 = (Var(\Delta r_F))/\Delta t$ = Instantaneous volatility parameter for contractual futures implied rate.

$\sigma_{BF} = (Cov(\Delta r_F, \Delta r_B)/\Delta t$

ρ_{BF} = Correlation coefficients of bond and futures contract implied rate.

Adjust factor = $\rho_{BF} \cdot \frac{\sigma_B}{\sigma_F}$ (Volatility adjustment factor).

TABLE 3

ANNUALIZED ACTUAL RETURNS
Strategy 2

Period No.	Hedge Ratio N_t^*	Futures Price	Futures Gain (loss)	Annualized Return (R_1)	Return Deviation R_1-r_B
1	-1.04	76687.50	5925.12	-0.2300	-0.3446
2	-1.03	71000.00	-706.73	0.0259	-0.1104
3	-0.95	71781.25	-5219.39	0.6332	0.4990
4	-0.96	77281.25	-896.57	0.1588	0.0502
5	-0.99	78218.75	-1981.26	0.5347	0.4312
6	-0.96	80218.75	5235.36	0.3965	0.2979
7	-1.02	74781.25	222.72	-0.5533	-0.6576
8	-0.95	74562.50	1748.44	0.0085	-0.1081
9	-0.94	72718.75	3120.72	0.0887	-0.0333
10	-0.96	69406.25	-1911.12	-0.3274	-0.4575
11	-0.94	71406.25	-1465.85	0.6101	0.4784
12	-0.91	72968.75	1849.32	0.2000	0.0729
13	-0.92	70937.50	3035.00	-0.0677	-0.1969
14	-0.93	67656.25	-550.17	0.2396	0.0983
15	-0.92	68250.00	2739.47	-0.0571	-0.1921
16	-0.92	65281.25	-2508.93	-0.0881	-0.2346
17	-0.92	68000.00	1349.32	0.8325	0.6920
18	-0.93	66531.25	2358.85	-0.0867	-0.2272
19	-0.93	64000.00	3475.01	0.2976	0.1454
20	-0.97	60250.00	2336.53	0.2619	0.1026
21	-1.00	57843.75	-2095.97	0.3152	0.1519
22	-0.98	59937.50	-5686.72	-0.1757	-0.3242
23	-0.93	65718.75	2970.32	0.6022	0.4695
24	-0.95	62531.25	828.24	0.1231	-0.0206
25	-0.95	61656.25	-384.27	-0.0385	-0.1832
26	-0.92	62062.50	-487.51	0.0111	-0.1340
27	-0.89	62593.75	-999.89	0.1197	-0.0244
28	-0.87	63718.75	378.46	0.0185	-0.1196
29	-0.89	63281.25	2090.35	0.6456	0.5056
30	-0.90	60937.50	-1773.48	0.1623	0.0146
31	-0.87	62906.25	-3285.48	0.0483	-0.0886
32	-0.81	66687.50	-3519.80	-0.2330	-0.3565
33	-0.50	71218.75	-2207.75	0.2271	0.1099
34	-0.47	75593.75	870.66	0.0715	-0.0315
35	-0.49	73750.00	0.00	0.6548	0.5501
36	-0.49	73750.00	1504.96	0.1673	0.0666
37	-0.50	70656.25	-1711.18	-0.0666	-0.1685
38	-0.46	74062.50	289.05	-0.0972	-0.1947
39	-0.45	73437.50	-1107.50	0.0618	-0.0405
40	-0.42	75906.25	1650.96	-0.0163	-0.1124
41	-0.45	72000.00	-28.19	0.4579	0.3537
42	-0.44	72062.50	2025.93	0.0432	-0.0630
43	-0.46	67406.25	42.83	0.0072	-0.1075
44	-0.44	67312.50	-1475.98	0.0259	-0.0898
45	-0.40	70656.25	960.24	0.1067	-0.0018
46	-0.41	68281.25	-281.45	-0.0120	-0.1221
47	-0.41	68968.75	532.92	0.5468	0.4367
48	-0.40	67656.25	-251.15	0.0558	-0.0561
49	-0.38	68281.25	706.71	0.0342	-0.0750
50	-0.38	66437.50	812.66	0.0239	-0.0887
51	-0.38	64312.50	502.08	0.0163	-0.1012
52	-0.38	63000.00	1773.30	0.0554	-0.0660
53	-0.42	58281.25	38.98	0.5284	0.3965
54	-0.40	58187.50	-1945.45	-0.1195	-0.2531
55	-0.34	63062.50	-370.36	0.0248	-0.1037
56	-0.31	64156.25	-620.12	0.0408	-0.0859
57	-0.29	66125.00	-958.44	0.0848	-0.0369
58	-0.26	69437.50	-225.16	0.1126	0.0013
59	-0.25	70312.50	-7.84	0.5670	0.4633
60	-0.24	70343.75	-407.67	0.0267	-0.0726
61	-0.22	72062.50	909.03	0.0270	-0.0678
62	-0.22	67843.75	-411.38	0.0032	-0.0990
63	-0.20	69718.75	-196.99	0.0791	-0.0201

TABLE 4

ANNUALIZED ACTUAL RETURNS
Strategy 3

Period No.	Hedge Ratio b^{**}	Futures Gain	Annualized Return (R_2)	Return Deviation $R_2 - r_B$
1	−0.68	3880.24	−0.5172	−0.6318
2	−0.67	−462.82	0.0635	−0.0728
3	−0.66	−3623.57	0.8764	0.7421
4	−0.66	−622.45	0.1961	0.0875
5	−0.69	−1375.49	0.6153	0.5118
6	−0.67	3634.66	0.1882	0.0897
7	−0.71	154.63	−0.5623	−0.6667
8	−0.66	1213.86	−0.0663	−0.1829
9	−0.65	2166.57	−0.0476	−0.1696
10	−0.66	−1326.79	−0.2414	−0.3715
11	−0.65	−1017.67	0.6764	0.5447
12	−0.63	1283.90	0.1181	−0.0091
13	−0.64	2107.06	−0.2029	−0.3321
14	−0.64	−381.95	0.2652	0.1239
15	−0.64	1901.88	−0.1812	−0.3161
16	−0.64	−1741.83	0.0302	−0.1164
17	−0.64	936.77	0.7704	0.6299
18	−0.65	1637.63	−0.1948	−0.3353
19	−0.64	2412.53	0.1323	−0.0199
20	−0.67	1622.14	0.1484	−0.0109
21	−0.69	−1455.13	0.4179	0.2546
22	−0.68	−3948.02	0.0887	−0.0599
23	−0.65	2062.15	0.4717	0.3389
24	−0.66	575.01	0.0854	−0.0583
25	−0.66	−266.78	−0.0211	−0.1657
26	−0.64	−338.46	0.0332	−0.1119
27	−0.62	−694.17	0.1648	0.0206
28	−0.60	262.75	0.0018	−0.1363
29	−0.62	1451.23	0.5532	0.4132
30	−0.63	−1231.24	0.2422	0.0945
31	−0.60	−2280.95	0.1913	0.0544
32	−0.56	−2443.63	−0.0861	−0.2096
33	−0.51	−2221.97	0.2252	0.1080
34	−0.48	876.27	0.0722	−0.0307
35	−0.50	0.00	0.6548	0.5501
36	−0.49	1514.65	0.1685	0.0679
37	−0.51	−1722.20	−0.0680	−0.1699
38	−0.47	290.91	−0.0970	−0.1945
39	−0.45	−1114.63	0.0609	−0.0414
40	−0.43	1661.60	−0.0150	−0.1111
41	−0.45	−28.38	0.4579	0.3537
42	−0.44	2038.98	0.0449	−0.0613
43	−0.46	43.11	0.0072	−0.1075
44	−0.44	−1485.49	0.0247	−0.0911
45	−0.41	966.42	0.1075	−0.0010
46	−0.41	−283.26	−0.0122	−0.1223
47	−0.41	536.35	0.5472	0.4371
48	−0.40	−252.77	0.0556	−0.0563
49	−0.39	711.27	0.0348	−0.0744
50	−0.38	817.89	0.0246	−0.0880
51	−0.39	505.32	0.0167	−0.1008
52	−0.38	1784.72	0.0569	−0.0646
53	−0.42	39.23	0.5285	0.3965
54	−0.40	−1957.98	−0.1212	−0.2547
55	−0.34	−372.74	0.0245	−0.1040
56	−0.32	−624.11	0.0403	−0.0864
57	−0.29	−964.62	0.0840	−0.0376
58	−0.26	−226.61	0.1124	0.0012
59	−0.25	−7.89	0.5670	0.4633
60	−0.24	−410.30	0.0264	−0.0730
61	−0.22	914.88	0.0277	−0.0671
62	−0.22	−414.03	0.0029	−0.0993
63	−0.20	−198.26	0.0790	−0.0203

TABLE 5

HEDGING PERFORMANCE ANALYSIS

	Average Rate Deviation	Variance of Rate Deviation	Variance Reduction
Unhedged	-4.628	.14752	0
Strategy 1	3.9577	.07387	49.92
Strategy 2	8.1911	.06843	53.61

Rate deviation is defined as *ex post* return - promised return.

The average and variance of rate variation is calculated for all the periods.

Strategy 1: Continuous hedge ratio using the face value/duration only approach.

Strategy 2: Continuous hedge ratio using the face value/duration approach adjusted for volatility.