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PORTFOLIO CHOICE WITH UNCERTAIN CONSUMPTION PRICES: A MEAN-VARIANCE APPROACH

by

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Abstract

This paper presents a mean-variance model of portfolio choice and asset pricing when the price of consumption goods as well as the return to assets is uncertain. The correlation of an assets return with purchases at expected prices is shown to reduce both the mean return and the variance of the return of an asset. A numerical approximation is computed to check the accuracy of the mean and variance approximation. Uncertainty of consumption prices is shown to result in long (or speculative) futures holding.

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I. Introduction

When investors realize that the returns to the assets they hold are correlated with the goods they intend to consume, they must consider the covariances of asset returns both with each other and with the prices of consumption goods. Because the covariances with the consumption prices are taken into consideration, it is possible that assets usually considered to be "speculative," such as commodity futures, will actually reduce the variance in the investors' real income. Similarly, the purchase of a house can be viewed as a hedge on the future price of shelter. This paper provides a generalized mean-variance theory of portfolio choice when the price of consumption goods is uncertain and consumables may be held as assets. Within a mean-variance framework, this paper elucidates the pricing of and demand for assets such as commodity futures and houses.

In his book, Portfolio Selection, Markowitz (1959) recommends that the analysis of portfolios be conducted in real terms; but with the very low rates of inflation of the mid-1950s, using real rather than nominal returns would have made very little difference. The effect of conducting the analysis in real terms for firms is examined by Chen and Boness (1975). For investors, Hagerman and Kim (1976) have developed the case in which there is a unique price level that is not correlated with the return on the market portfolio. But the most thorough study is contained in Grauer and Litzenberger's (1979) article on pricing futures which concludes that the pricing of a future depends on three things. The first two, the expected price of the future and the covariance of its price with the general price level, are easy to interpret. The third, the covariance of the real price of the asset with the

marginal utility of income, has the uncertainty-reducing effect of buying what one intends to consume hidden within it. By adopting a mean-variance framework, this paper is able to study asset pricing without requiring an explicit evaluation of the marginal utility of income.

In a mean-variance framework, the tastes of the decision-maker can be approximated by the bundle purchased at expected prices, which we call the anticipated bundle. The real variance of a portfolio is expressed as a function of the "distance" of a portfolio from the anticipated bundle. Portfolios that are identical to anticipated consumption (buy a house and canned corn if one plans to live in a house and eat only canned corn) have, to a first approximation, no variance in real terms. Although reducing the real variance in portfolios makes them more attractive to mean-variance decision-makers, the same factor—closeness to anticipated consumption—that makes a portfolio have low variance also diminishes its mean real return. The basic factor behind this theorem, as an example will make plain, is that a high payoff when prices are high is a low real payoff.

As a simple example, consider a world with only two states of nature; in one, the price of consumption goods, p, is 1 and, in the other, 2. The first asset, A, yields (nominal) 1 and 2 while the second asset, X, yields 2 and 1. In real terms, the expected value of the first asset is $E\ (p_A/p) = 1$, while its variance is $\sigma^2_{p_A} = 0$; and for the second asset, $E\ (p_x/p) = 5/4$, while $\sigma^2_{p_x} = 9/16$. The covariance of p_A and p is 1/4 while cov $(p_x, p) = -1/4$. In this example the asset, A, with the higher covariance with the price level has lower mean and lower variance than the other asset.

Section II states a formal model of portfolio choice when the price of consumption goods is also uncertain. With the additional restrictions that

utility is a function of mean and variance of real income and that real income is derived from a homothetic utility function, the section presents an approximation in terms of means and variances. The approximation is used to show that correlation of an asset's return with the anticipated bundle reduces the mean return and the variance of the asset.

Section III presents a numerical example using index numbers for consumption prices and asset returns. The approximation is compared with the actual solution, and the importance of including nonstock market assets is illustrated.

Section IV presents the conclusions and a discussion of the relevance of the model for fields other than finance and for futures markets.

II. Model

Under uncertainty of both asset prices and consumption goods prices, the consumer faces a two-stage, decision-making problem. In the first stage, before the state of nature is revealed, the consumer chooses the assets he will hold. After the state of nature is revealed and his disposable income becomes certain, the agent chooses his consumption goods. Since there is no uncertainty when the consumption goods are actually chosen, the second choice problem is just an ordinary consumer problem of maximizing utility subject to a budget constraint. Letting p be an n vector of prices and x an n vector of consumption goods (some of which may provide no utility) and y be the realized second-period income, then:

$$v(p, y) = \max_{x} u(x),$$

subject to $p' x = y$.

The function, v, is the indirect utility function or, more exactly, the indirect felicity function because income, y, is a stochastic variable; and v(p, y) refers to only one of the many possible realizations of income, y. The first stage of the problem, that of portfolio allocation which occurs before p and y become known, is to choose a set of assets, z, at prices, s, to maximize a function of v(p, y). Since not all consumption goods can be assets and not all assets can be consumption goods, a convention for separating the two sets of goods is necessary. By convention, the first m, goods, can be held as assets; also by convention they enter the utility function in a strictly formal sense since they may convey no utility if actually consumed. To keep all vectors conformable, the assets are defined as an n vector with the last n-m elements definitionally zero: $z = (z_1, \ldots, z_m, 0, \ldots, 0)$ and their price $s = (s_1, \ldots, s_m, 0, \ldots, 0)$. With these definitions of s and z, the initial wealth (W) constraint is: $W = s'z$; and the definition of income for the second period is $y = p'z$. The remaining task is to specify an objective function.

One alternative, that most satisfying to an economic theorist, is to specify an expected utility objective function: $E\ h\ [v(p, y)]$, where E is the expectation operator and h is a twice-continuously differentiable function with h" negative. Taking the indirect utility function to be homothetic, $v = y\ f(p)$,[1] Grauer and Litzenberger (1979) found the price of an asset, s_m, relative to, say, the price of a real bond:[2],

$$\frac{s_m}{s_1} = E\ p_m\ E\ f + \text{cov}\ (p_m, f) + \frac{\text{cov}\ (h', p_m\ f)}{E\ h'},$$

where s_1 pays exactly $1/f(p)$.

Although Grauer and Litzenberger (1979) provide a good interpretation for the first two terms of this expression (expected payoff and inflation bonus), the interpretation of the last term is more difficult. The problem of interpreting (and computing) the covariance of marginal utility and payoff is often solved in finance by specifying a generalized mean-variance utility function.

A generalized mean-variance utility function is $g(M, S^2)$, where M is the mean of real income and S^2 is the variance of real income. By assumption, the objective function is twice continuously differentiable and increases in the mean of real income and decreases in the variance of real income. This class of objective functions takes its maximum on the computable efficient set. As Levy and Markowitz (1979) have shown, such functions are capable of approximating the other common utility functions of finance; the quadratic utility function is an interesting (if pathological) special case. With all of these assumptions and restrictions, the portfolio-consumption choice problem is:

$$\max_{z} \quad g(M, S^2),$$

subject to

$$W = s'z$$

$$M = E\, f(p)\, z'p$$

$$S^2 = E\, f(p)\, z'\, pp'z\, f(p) - M^2.$$

Letting L be the shadow price of wealth, the first-order conditions for an interior maximum are:

$$(g_1 - 2\, g_2)\, E[f(p)p'] + g_2\, 2\, E[f(p)^2\, z'pp'] - L\, s' = 0.$$

And when g is quadratic ($g_1 = 1$ and $g_2 = k$), the solution is:[3]

$$Z^* = \left(\frac{1}{2\,k}\right)(E\, fpp'f)^{-1}\, [L^*\, s - (1 - 2\, k)\, E\, fp]$$

$$L^* = \frac{2\,k\,W + (1 - 2\,k)s'\,(E\,fpp'f)^{-1}\,E\,fp}{s'\,(E\,fpp'f)^{-1}\,s}.$$

Since the first-order conditions include fourth moments, further approximation is in order. Approximate $v(p, y)$ by $v(\bar{p}, \bar{y}) + (v'_p + v_y\,z')\,\tilde{p}$ where $\bar{p} = E\,p$ and $p = \bar{p} + \tilde{p}$. Let $E\,\tilde{p}\tilde{p}' = C$, the variance-covariance matrix of returns and product prices. Carrying out the algebra,

$$S^2 \doteq (v'_p + v_y\,z')\,C\,(v_p + v_y\,z).$$

Using Roy's identity (Varian 1978), $(x = -v_p/v_y)$, and the definitions of $\bar{y} = \bar{p}'z$ and $v = y\,f$, and multiplying by -1 twice,

$$S^2 \doteq f(\bar{p})\,[x(\bar{p})' - z']\,C\,[x(\bar{p}) - z]\,f(\bar{p}).$$

Call the bundle purchased at expected prices the anticipated consumption. Then, $x(\bar{p}) - z$ are the anticipated net purchases, and $f(\bar{p})\,[x(\bar{p}) - z]'\,p$ is the deflated expenditure on the anticipated net purchases. To a first approximation, the variance in real income, S^2, is the variance of the deflated expenditure on the anticipated net purchases.

A first approximation of expected felicity (or real income) would be just to evaluate the real income at expected prices, but one can do somewhat better by including a covariance term,

$$E\,v(p, y) = E\,f(p)\,p'\,z = E\,f(p)\,E\,p'\,z + \mathrm{cov}\,(f, p'\,z),$$

by the definition of covariance. Approximating $f(p)$ by $f(\bar{p}) + f'_p\,\tilde{p}$, $E\,v(p, y) = f(p)\,p'z + f_p\,C\,z$, and again using Roy's identity,

$$E\ v\ (p,\ y) \doteq f(\bar{p})\ \bar{p}'\ z - f(\bar{p})\ \frac{x(\bar{p})'}{y} C\ z.$$

Apart from deflation at expected prices, mean real income is real income at expected prices less the covariance of the cost of the anticipated bundle and income, divided by expected income. Since C is positive semidefinite and x and z are nonnegative, the covariance term is <u>always</u> nonnegative, or income at expected prices always overstates expected real income (although, as we show below, this term does not always lower the value of any specific asset!).

These approximations of mean and variance can be used to approximate the first-order conditions and the asset-pricing equations. In terms of the approximation of mean and variance, the first-order conditions for the choice problem are:

$$L\ s' = g_1\ f(\bar{p}) \left[\bar{p}' - \frac{x(\bar{p})'\ C}{z'\ \bar{p}} \right] + g_2\ 2\ f(\bar{p})^2\ [z - x(\bar{p})]'\ C.$$

Letting the first asset be a (nominal) bond, s_1, that is, an asset that is not consumed and has a constant payoff of p_1 for \$1.00 invested, then the ratio of the price of the jth asset to the first asset is the ratio of their marginal utilities:

$$\frac{s_j}{s_1} = \frac{\bar{p}_j}{p_1} - \frac{\left[\text{cov}\ (x(\bar{p})\ \tilde{p},\ \tilde{p}_j\right]}{p_1 \cdot z'\ \bar{p}}$$

$$+ 2\ \frac{g_2}{g_1} \cdot \frac{f(\bar{p})}{p_1} \cdot \text{cov}\ \left\{ [z - x(\bar{p})]'\ \tilde{p},\ \tilde{p}_j \right\}.$$

The pricing equation follows from the zero variance and covariance of a nominal bond, which points up the weakness of the approximation: a nominal bond has approximately no variance and no inflation penalty. The pricing equation says that the price of an asset included in an optimal portfolio is (1) its payoff less (2) a term proportional to the covariance of its payoff and the cost of the anticipated purchases plus (3) a term proportional to the covariance of its payoff and income above that needed for the anticipated purchases.

Anticipated purchases appear twice in these pricing equations with opposite effects. So, ceteris paribus, as the covariance of an asset with the anticipated bundle increases, the mean of real income decreases and the variance of real income decreases. Thus, the effect on the equilibrium price (s_j/s_1) cannot be foretold a priori.

III. Numerical Example

A numerical example is sufficient to show that tastes in consumption goods can be quite important in the choice of broad classes of assets to be included in a portfolio and to illustrate that the approximation of the real portfolio problem in terms of means and variances can be quite precise. The felicity function was chosen as Cobb-Douglas in form. The goods for consumption purchases are the elements of the Consumer Price Index (CPI), and the weights in the utility function are the CPI weights; the prices of the goods are the components of the CPI. Assets include four broad aggregates: houses, represented by the San Francisco home price purchase index; stocks, the Dow Jones average; mortgage or bonds, at a rate of 11 percent; and futures, the Dow Jones futures average. Stocks were corrected for dividends and housing was crudely corrected for implicit rent; $100,000 was optimally allocated over

these investments, and the returns were spent on the consumption goods. Moments of the distributions of the prices, real prices, and returns were computed as the prediction error and prediction of simple autoregressive regressions.

The first experiment performed with this simple numerical example shows that changes in tastes alter the mean-variance frontier. The mean-variance frontier was computed using the CPI weights and then was recomputed with the CPI housing share reduced from .48 to .39. The weight removed from housing was redistributed proportionally to the other goods. Table I gives the frontier for these two schemes. Column 1 contains the mean return in two months' time, and columns 2 through 5 contain the portfolio. These portfolios are quite different when the consumption weights, or tastes, are changed. As the table shows, using CPI weights, the quantity of bonds purchased first increases and then decreases along the mean-variance frontier while, with the reduced housing weights, the quantity of bonds smoothly decreases. Also note that the maximum mean return portfolio contains all housing with the lowered weights while it contains all futures with the CPI weights. And there are other differences as well. The second experiment was to compute the variance of real income and compare it to its approximation in terms of anticipated bundles. These numbers are in columns 6 and 7, and they show that the approximation is usually within 20 percent of the actual variance; but it can be as bad as half the variance. When all of the information needed to compute the real variance exists, this simple and more direct procedure is obviously preferred to the approximation.

TABLE I

Real Mean and Variance of Portfolios

Real mean (thousand dollars)	Asset holdings (thousand dollars)				Variance	Approximate variance
	Stocks	Houses	Futures	Mortgages or bonds		
1	2	3	4	5	6	7
Consumer price index weights						
100.090	7.6	.7	0.0	91.7	.393	.329
100.103	6.4	1.2	0.0	92.5	.399	.326
100.216	0.0	4.5	7.1	88.4	.814	.523
100.744	0.0	25.6	74.4	0.0	14.608	6.700
Lowered housing weights						
100.180	7.3	8.5	8.0	76.3	.225	.250
100.479	0.0	29.1	17.5	53.4	.833	.809
100.890	0.0	62.0	38.1	0.0	3.820	3.250
101.065	0.0	100.0	0.0	0.0	11.950	9.500

[a]Dow Jones average.

[b]San Francisco home price purchase index.

IV. Conclusions

This paper presented a mean-variance approximate solution to the problem of choosing assets when both returns and consumption prices were uncertain. The variance of real income was shown to be proportional to the expenditure on the anticipated net purchases while the mean of real income is proportional to expected income less one over expected income times the covariance of the anticipated bundle and income. The correlation between anticipated expenditure and income decreases both real income and its variance so that it is not apparent whether the correlation increases or decreases utility. Similarly, turning to the asset-pricing equations, the covariance between the cost of the anticipated bundle and return decreases the value of an asset acting through the mean return and increases the value acting through the variance of return.

These mean-variance results are dependent on the individual's tastes. Most of the usual results of the mean-variance literature, summarized by Rubinstein (1973) or Mossin (1973), hold. But the additional requirement that each agent has the same preferences, $f(p)$, is necessary to obtain a separation theorem and make use of the Capital Asset Pricing Model of Lintner (1969) and Sharpe (1970).

While the approximations of mean and variance in terms of anticipated consumption are theoretically enlightening, they are not the preferred computational method when tastes are well specified. However, in the event that all that one knows is anticipated consumption, the approximations are very useful. For instance, developing countries face random international prices for food and other commodities; their average consumption is known, as is the distribution of international prices. By using the anticipated consumption approximations, one can easily compute the mean and variance of real income that

results from programs such as food self-sufficiency or international crop insurance. Other applications of the general framework, although not particularly of the approximation, would be to subsistence agriculture--ratio of cotton to corn in the postbellum south where corn is consumed and cotton is not--or to the demand for housing--housing is both an asset and a consumption good.

A final application of the model is to the theory of speculative financial markets such as futures. The Keynes-Hicks theory of the futures markets [with the Cootner (1960) wrinkle] is that storers of commodities can reduce the variance of their real income by hedging. Thus, the storers of commodities are willing to pay a price in terms of mean income for this hedging, and this price becomes the risk premium transferred from hedger to speculator. In this paper, we have shown that agents who do not store commodities, that is, the speculators of the Hicks-Keynes-Cootner theory, could reduce the variance in their real income by bringing their asset position closer to their anticipated consumption. The method of bringing their asset position closer to anticipated consumption would be to speculate, that is, to take a long position in the futures markets. Thus, the speculators, too, would be willing to pay a price in terms of mean income for reducing the variance in their future expected real income. With both types of agents, speculators, and hedgers willing to pay a price for taking opposite sides of the contract, theory cannot predict whether it will be the speculators who pay the hedgers or the hedgers who pay the speculators. This matter can be settled only by an empirical analysis of the supply and demand for commodity futures contracts.

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Footnotes

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[1]We will maintain throughout the paper the assumption that f is C^2; $f_{p_i} < 0$ for all i; and f is convex and homogeneous of degree -1.

[2]Grauer and Litzenberger (1979) provide a more complicated version of this model by including a nominal bond and by stating much of the discussion in terms of the price level. A simple derivation of a pricing equation like theirs is: Letting z_1 be a real bond, that is, an asset paying exactly $1/f(p)$, one can compare the price of a commodity purchased now for delivery next period with the price of a real bond. Since the expected marginal utility from buying each asset must be proportional to its price,

$$\frac{s_m}{s_1} = \frac{E\ (h'\quad p_m\quad f)}{E\ (h'\quad p_1\quad f)} = \frac{E\ (h'\quad p_m\quad f)}{E\ h'},$$

where the second equality follows from $f(p)\ p_1 \equiv 1$. Using the definition of covariance twice,

$$\frac{s_m}{s_1} = E\ P_m\ E\ f + \text{cov}\ (P_m,\ f) + \frac{\text{cov}\ (h',\ p_m\ f)}{E\ h'}.$$

[3]Providing E (f pp' f) is nonsingular.